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Homework 1
Binomial Tree

I. Binomial Model Derivation

(20%) In the binomial model, suppose that the initial stock price is S_0, and the life of the option is T. S_0 can either move up from S_0 to a new level, $S_0 u$, where $u > 1$, or down to a new level, $S_0 d$, where $0 < d < 1$. Suppose the payoff from option is f_u in the up state, and is f_d in the down state. Denote the risk-free rate by r.

Please construct a riskless portfolio in a one-step tree and show **<u>in detail</u>** that $f = e^{-rT}[pf_u + (1-p)f_d]$ where $p = \dfrac{e^{rT} - d}{u-d}$

II. Binomial Trees in Practice

Consider a non-dividend-paying stock with current stock price S_0 =\$50, volatility σ=0.3, strike price K=\$52, time to maturity T=2 years, interest rate r=5%.

Please use binomial model to price European put options. You may refer to the materials in Section18.1 of the textbook. Consider the following three alternative settings of time steps: Δt =1 month (12*T steps); 1 week (52*T steps); and 1 day (252*T steps).

(a) (10%) First compute the up step size u, the down step size d, and the probability of up move p under these three settings.

(b) (40%) Use binomial model to compute the put option prices under these three settings. Report your results and compare them with that of the Black-Scholes formula. Briefly explain your findings.

(c) (20%) Modify your program in (b) to compute the American put option values. Report your result.

(d) (10%) Change the number of time steps from 1 to 2 to 3 all the way to 252. Plot your results as well as the Black-Scholes closed form solution. Briefly explain your findings.

Bonus: (20%) For 6, 12, and 52 time steps, compute the terminal stock prices as well as their corresponding probabilities. Plot the terminal stock price distribution. Briefly explain your findings.

<u>Matlab function and syntax:</u>

1. zeros(): to create an matrix of all zeros. e.g. S = zeros(m,n)
2. sqrt(): square root
3. exp(): exponential function
4. max(): max function
5. for loop

    e.g.

```
for j=1:1:10
	statement
end
```

*You have to submit the results of your homework in a word (or pdf) file as well as **programs by e3**. Your computer program is part of this assignment. You can use either C++ or Matlab for programming.

1.

$$
\begin{tikzcd}[row sep=small]
 & \begin{matrix} S_0 u \\ f_u \end{matrix} \\
\begin{matrix} S_0 \\ f \end{matrix} \arrow[ur, "u"] \arrow[dr, "d"'] & \\
 & \begin{matrix} S_0 d \\ f_d \end{matrix}
\end{tikzcd}
$$

$$S_0 u \cdot n - f_u \cdot 1$$

$$\shortparallel$$

$$S_0 d \cdot n - f_d \cdot 1$$

$$n = \frac{f_u - f_d}{S_0 u - S_0 d}$$

$$(S_0 u \cdot n - f_u) e^{-rT} = S_0 n - f$$

$$f = S_0 \cdot n - (S_0 u \cdot n - f_u) e^{-rT}$$

$$= (S_0 \cdot n \cdot e^{rT} - S_0 \cdot u \cdot n + f_u) e^{-rT}$$

$$= (S_0 \cdot n (e^{rT} - u) + f_u) e^{-rT}$$

$$= \left(S_0 \cdot \frac{f_u - f_d}{S_0 u - S_0 d} (e^{rT} - u) + f_u \right) e^{-rT}$$

$$= \left(\frac{f_u - f_d}{u - d} (e^{rT} - u) + f_u \right) e^{-rT}$$

$$= \left(\frac{f_u e^{rT} - f_u u - f_d e^{rT} + f_d u + f_u u - f_u d}{u - d} \right) e^{-rT}$$

$$= \left[\frac{f_u (e^{rT} - d) + f_d (u - e^{rT})}{u - d} \right] e^{-rT}$$

$$= \left(\frac{e^{rT} - d}{u - d} f_u + \frac{u - e^{rT}}{u - d} f_d \right) e^{-rT}$$

$$= \left(\frac{e^{rT} - d}{u - d} f_u + \frac{u - d + d - e^{rT}}{u - d} f_d \right) e^{-rT}$$

$$= \left(\frac{e^{rT} - d}{u - d} f_u + \left(1 + \frac{d - e^{rT}}{u - d} \right) f_d \right) e^{-rT}$$

$$= \left[\frac{e^{rT} - d}{u - d} f_u + \left(1 - \frac{e^{rT} - d}{u - d} \right) f_d \right] e^{-rT}$$

$$= \left[p f_u + (1 - p) f_d \right] e^{-rT}$$

where $p = \dfrac{e^{rT} - d}{u - d}$

2.

(a)

```
(a) First compute the up step size u, the down step size d, and the probability of up move p under these three settings.

When 24 time steps, u = 1.09046, d = 0.917041, p = 0.502439.
When 104 time steps, u = 1.04248, d = 0.959251, p = 0.50116.
When 504 time steps, u = 1.01908, d = 0.981279, p = 0.500527.
```

(b)

兩者十分接近
越多step兩者的差距越小
有時binomial 比較大有時Black-Scholes比較大

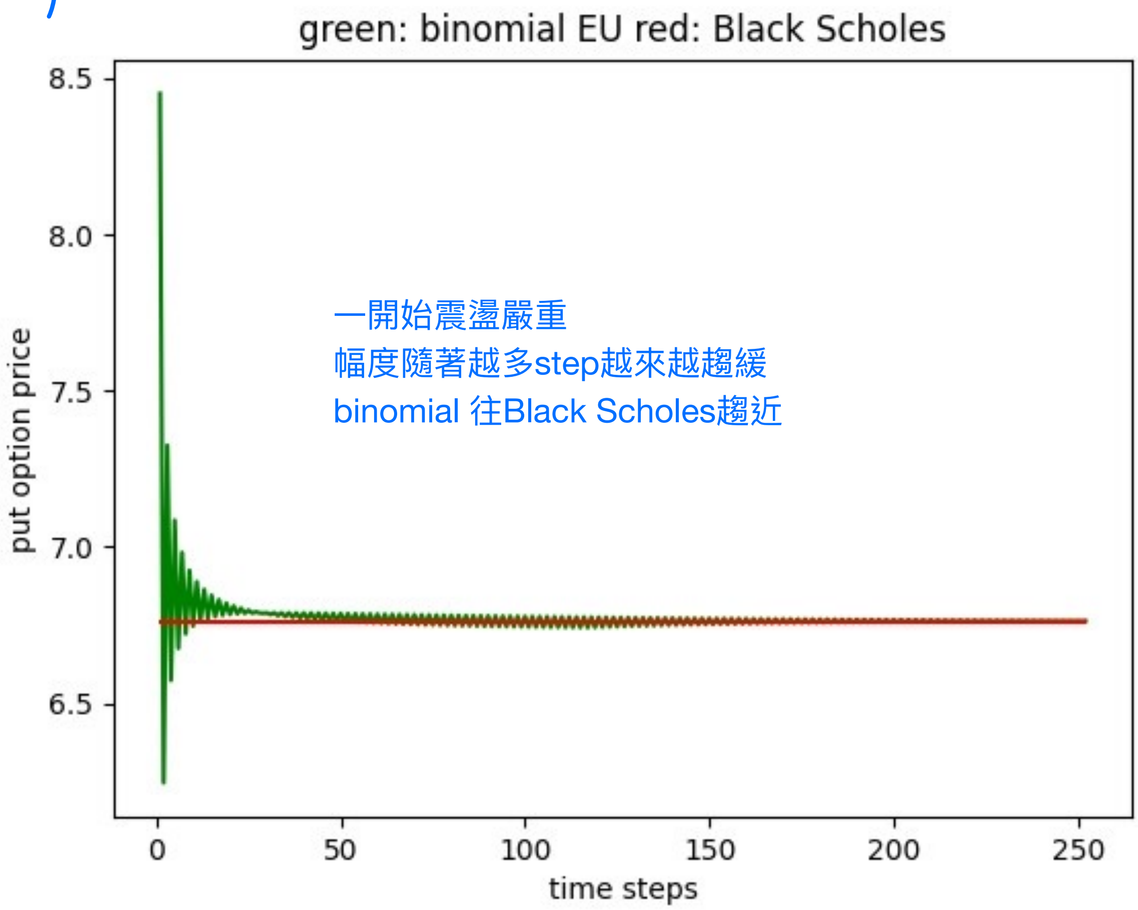
```
(b) Use binomial model to compute the put option prices under these three settings. Report your results and compare them
 with that of the Black-Scholes formula. Briefly explain your findings.
When 24 time steps:
        The put option prices of the binomial model: 6.78743
        The put option prices of the Black-Scholes formula: 6.76014
        The difference between them: -0.0272865
When 104 time steps:
        The put option prices of the binomial model: 6.77744
        The put option prices of the Black-Scholes formula: 6.76014
        The difference between them: -0.0173011
When 504 time steps:
        The put option prices of the binomial model: 6.75653
        The put option prices of the Black-Scholes formula: 6.76014
        The difference between them: 0.00360775
```

(c)

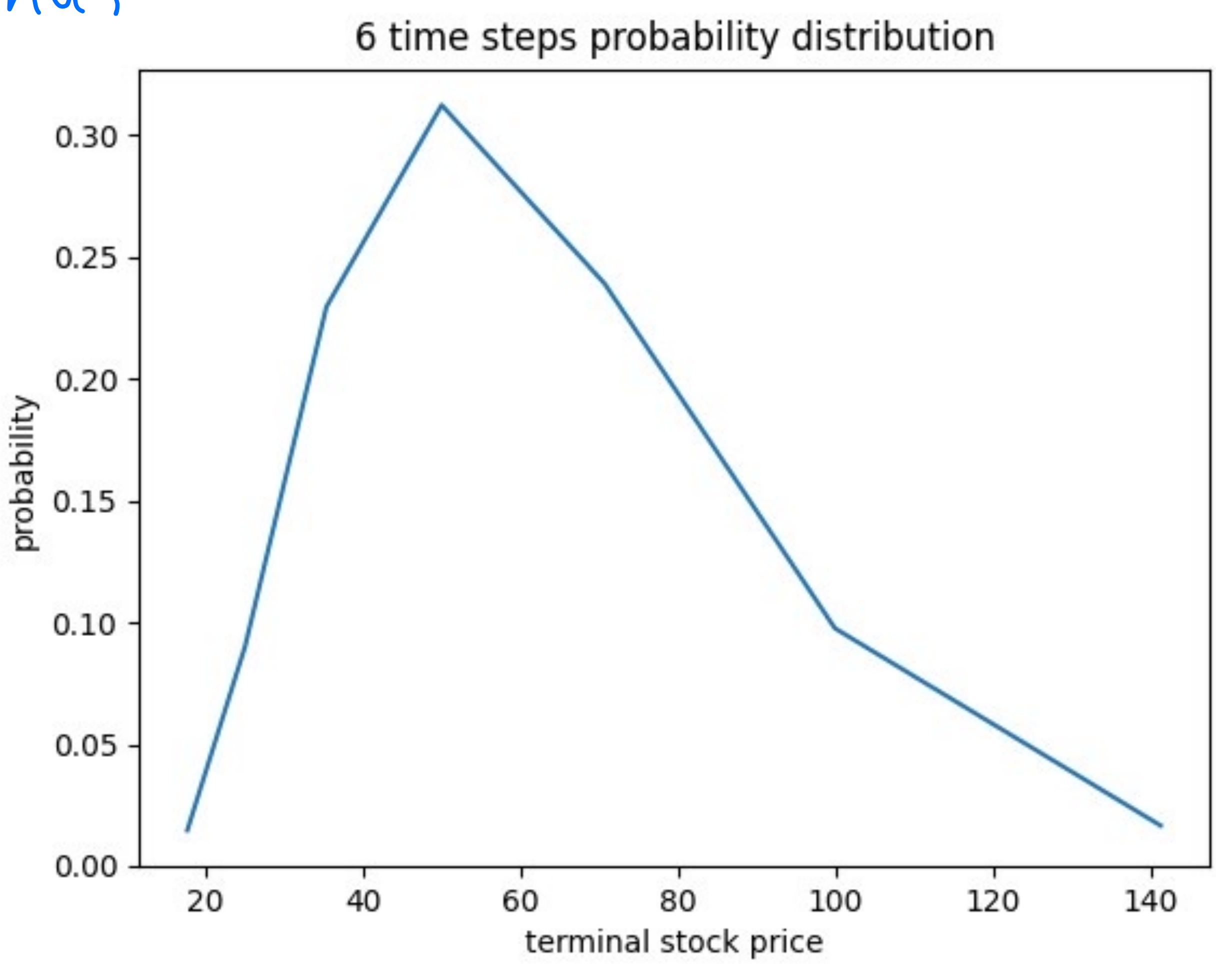
兩者十分接近
越多step兩者的差距越小
沒有early exercise 的情形發生

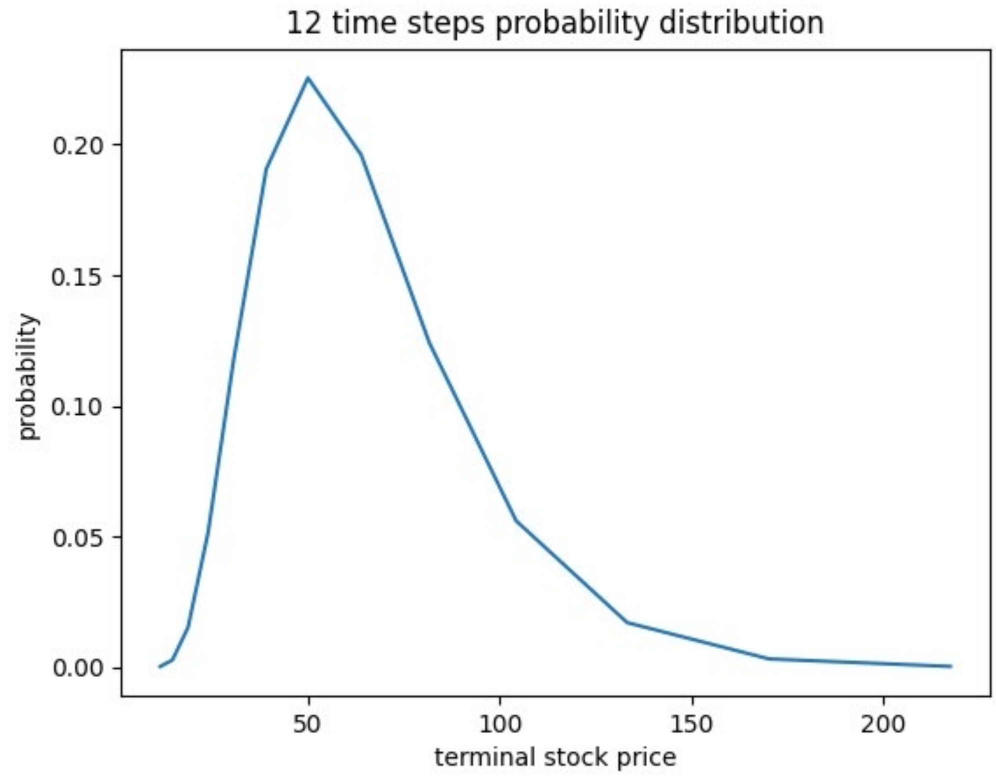
```
(c) Modify your program in (b) to compute the American put option values. Report your result.
When 24 time steps:
        The put option prices of the binomial model: 6.78743
        The put option prices of the Black-Scholes formula: 6.76014
        The difference between them: -0.0272865
When 104 time steps:
        The put option prices of the binomial model: 6.77744
        The put option prices of the Black-Scholes formula: 6.76014
        The difference between them: -0.0173011
When 504 time steps:
        The put option prices of the binomial model: 6.75653
        The put option prices of the Black-Scholes formula: 6.76014
        The difference between them: 0.00360775
```

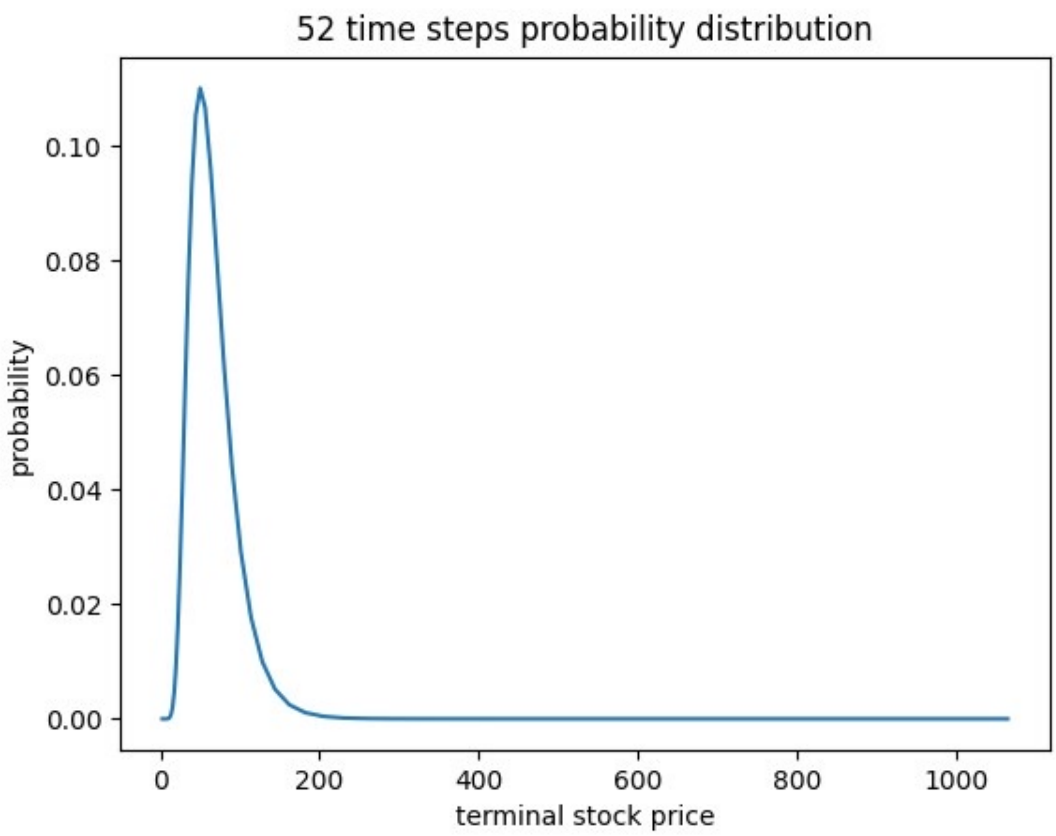
(d)


green: binomial EU red: Black Scholes
一開始震盪嚴重
幅度隨著越多step越來越趨緩
binomial 往Black Scholes趨近
put option price
time steps
8.5
8.0
7.5
7.0
6.5
0
50
100
150
200
250

Bonus


6 time steps probability distribution
probability
terminal stock price
0.30
0.25
0.20
0.15
0.10
0.05
0.00
20
40
60
80
100
120
140





step越多越趨向常態分佈 stock price範圍越廣
期望值在s0 * exp(r * T) = 55.26 附近