DSP Final

Task 1

RPCA Algorithm

- Augmented Lagrange Multiplier (ALM)

Reference: [The Augmented Lagrange Multiplier Method for Exact Recovery of Corrupted Low-Rank Matrices]

Algorithm 4 (RPCA via the Exact ALM Method)

Input: Observation matrix $D \in \mathbb{R}^{m \times n}$, λ.

1: $Y_0^* = \text{sgn}(D)/J(\text{sgn}(D))$; $\mu_0 > 0$; $\rho > 1$; $k = 0$.
2: **while** not converged **do**
3: // Lines 4-12 solve $(A_{k+1}^*, E_{k+1}^*) = \arg\min_{A,E} L(A, E, Y_k^*, \mu_k)$.
4: $A_{k+1}^0 = A_k^*$, $E_{k+1}^0 = E_k^*$, $j = 0$;
5: **while** not converged **do**
6: // Lines 7-8 solve $A_{k+1}^{j+1} = \arg\min_A L(A, E_{k+1}^j, Y_k^*, \mu_k)$.
7: $(U, S, V) = \text{svd}(D - E_{k+1}^j + \mu_k^{-1} Y_k^*)$;
8: $A_{k+1}^{j+1} = U \mathcal{S}_{\mu_k^{-1}}[S] V^T$;
9: // Line 10 solves $E_{k+1}^{j+1} = \arg\min_E L(A_{k+1}^{j+1}, E, Y_k^*, \mu_k)$.
10: $E_{k+1}^{j+1} = \mathcal{S}_{\lambda\mu_k^{-1}}[D - A_{k+1}^{j+1} + \mu_k^{-1} Y_k^*]$;
11: $j \leftarrow j + 1$.
12: **end while**
13: $Y_{k+1}^* = Y_k^* + \mu_k (D - A_{k+1}^* - E_{k+1}^*)$.
14: Update μ_k to μ_{k+1}.
15: $k \leftarrow k + 1$.
16: **end while**

Output: (A_k^*, E_k^*).

```
def ALM(D, lmbda=0.01, mu=None, rho=6, sv=10., tol_inner=1e-6, tol_outer=1e-7,
maxIter=1000, max_mu=1e7):
    """
    D: Data matrix
    A: Low-ramk matrix component
    E: Error matrix, Sparse component
    Y: Lagrange multiplier
    J: Dual norm
    j: Inner iteration
    k: Outer iteration
    sv: Predicted dimension
    svp:  # singular values in the sv singular values that are larger than 1 / μ
    primal_error: D - (A + E)
    """

    # Line 1 - Initialize variables
    dual_norm = J(D, lmbda)
```

```
    Y = D / dual_norm
    A_prev = np.zeros_like(Y)
    E_prev = np.zeros_like(Y)
    D_norm = np.linalg.norm(D, 'fro')
    mu = 0.5 / np.linalg.norm(np.sign(D), 2) if mu is None else mu
    d = Y.shape[1]
    # Line 2 - Line 16
    for k in range(maxIter):
        # Line 5 - Line 12
        for j in range(maxIter):
            A = A_prev
            E = E_prev

            # Line 7
            U, S, V = np.linalg.svd(D - E + (1 / mu) * Y, full_matrices=False)
            svp = get_svp(S, mu)
            # Line 8
            A = np.dot(np.dot(U[:, :svp], np.diag(S[:svp] - 1 / mu)), V[:svp, :])
            # Line 10
            E = soft_threshold(D - A + (1 / mu) * Y, lmbda / mu)
            sv = update_sv(sv, svp, d)

            # Check convergence
            stop_criterion_inner = ((np.linalg.norm(A - A_prev, 'fro') / D_norm)
< tol_inner) or ((np.linalg.norm(E - E_prev, 'fro') / D_norm) < tol_inner)
            if stop_criterion_inner:
                break

        sv = np.min([svp + np.round(0.1 * d), d])
        # Line 13
        primal_error = D - (A + E)
        Y = Y + mu * primal_error
        # Line 14
        mu = np.min([mu * rho, max_mu])

        # Check convergence
        stop_criterion_outer = (np.linalg.norm(primal_error, 'fro') / D_norm) <
tol_outer
        if stop_criterion_outer:
            break

    return A, E
```

- Inexact Augmented Lagrange Multiplier (IALM)

Reference: ChatGPT, [Robust PCA—Inexect ALM](), [The Augmented Lagrange Multiplier Method for Exact Recovery of Corrupted Low-Rank Matrices]()

Algorithm 5 (RPCA via the Inexact ALM Method)

Input: Observation matrix $D \in \mathbb{R}^{m \times n}$, λ.

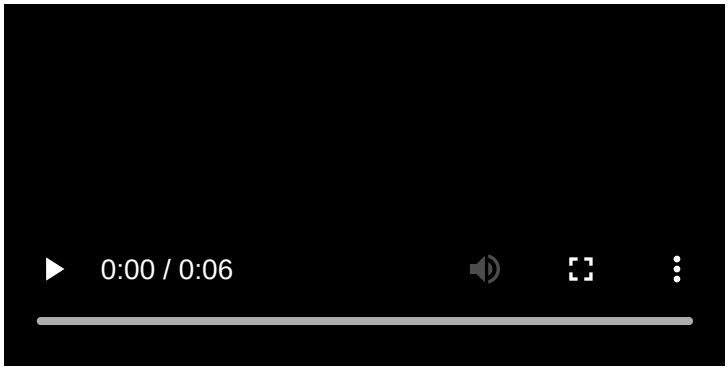
$Y_0 = D/J(D)$; $E_0 = 0$; $\mu_0 > 0$; $\rho > 1$; $k = 0$.
while not converged **do**
  // Lines 4-5 solve $A_{k+1} = \arg\min_A L(A, E_k, Y_k, \mu_k)$.
  $(U, S, V) = \text{svd}(D - E_k + \mu_k^{-1} Y_k)$;
  $A_{k+1} = U\mathcal{S}_{\mu_k^{-1}}[S]V^T$.
  // Line 7 solves $E_{k+1} = \arg\min_E L(A_{k+1}, E, Y_k, \mu_k)$.
  $E_{k+1} = \mathcal{S}_{\lambda\mu_k^{-1}}[D - A_{k+1} + \mu_k^{-1} Y_k]$.
  $Y_{k+1} = Y_k + \mu_k(D - A_{k+1} - E_{k+1})$.
  Update μ_k to μ_{k+1}.
  $k \leftarrow k + 1$.
end while

Output: (A_k, E_k).

```
def IALM(D, lmbda=0.01, mu=None,  rho=1.6, sv=10., tol=1e-7, maxIter=1000,
max_mu=1e7):
    """
    D: Data matrix
    A: Low-ramk matrix component
    E: Error matrix, Sparse component
    Y: Lagrange multiplier
    J: Dual norm
    k: Iteration
    sv: Predicted dimension
    svp:  # singular values in the sv singular values that are larger than 1 / μ
    primal_error: D - (A + E)
    """

    # Line 1 - Initialize variables
    dual_norm = J(D, lmbda)
    Y = D / dual_norm
    A = np.zeros_like(Y)
    E = np.zeros_like(Y)
    D_norm = np.linalg.norm(D, 'fro')
    D_norm_two = np.linalg.norm(D, 2)
    mu = 1.25 / D_norm_two if mu is None else mu
    d = Y.shape[1]

    # Line 2 - Line 11
    for k in range(maxIter):
        # Line 4
        U, S, V = np.linalg.svd(D - E + (1 / mu) * Y, full_matrices=False)
        svp = get_svp(S, mu)
        # Line 5
        A = np.dot(np.dot(U[:, :svp], np.diag(S[:svp] - 1 / mu)), V[:svp, :])
        # Line 7
        E = soft_threshold(D - A + (1 / mu) * Y, lmbda / mu)
        sv = update_sv(sv, svp, d)
```

```
        # Line 8
        primal_error = D - (A + E)
        Y = Y + mu * primal_error
        # Line 9
        mu = np.min([mu * rho, max_mu])

        # Check convergence
        stop_criterion = (np.linalg.norm(primal_error, 'fro') / D_norm) < tol
        if stop_criterion:
            break
    # Line 11
    return A, E
```

Video



Experiment

Algorithm	lambda	max iter	score
ALM	1	1000	48.05
IALM	0.01	1000	46.92
IALM	1	1000	48.05

Conclusion

- A smaller lambda is preferable.
- Increasing the maximum iteration limit is beneficial when the stop condition is not met.
- RPCA is a powerful enough method for denoising.

- It is evident that the denoising process greatly enhances the clarity of the data, causing the presence of unwanted elements to gradually diminish, similar to how a train on the rail gradually disappears into the distance.

Task 2

RPCA Algorithm

Same as Task 1

Methodology

Once we have obtained the low-rank component (L) and the sparse component (S) through Robust Principal Component Analysis (RPCA), the next step involves calculating the residual by subtracting L from the original data matrix D. We then sort the residuals based on their norms. Subsequently, we select the kth smallest norm as a threshold to determine the indices that are considered suspicious.

Experiment

Algorithm	lambda	max iter	score
ALM	0.0001	900	0.9619
ALM	0.001	10000	0.9619
ALM	0.01	10000	0.9625
IALM	0.01	10000	0.9613

Conclusion

- While a smaller lambda is generally preferred, it is important to note that making it too small can also lead to worsened results.
- Increasing the maximum iteration limit is beneficial when the stop condition is not met.
- RPCA is a powerful enough method for anomaly detection.