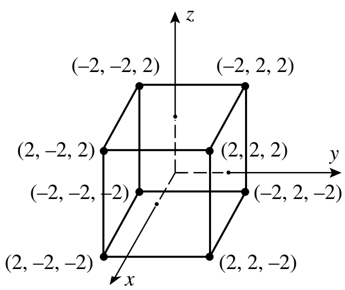
Three-Dimensional Space; Vectors

Exercise Set 11.1

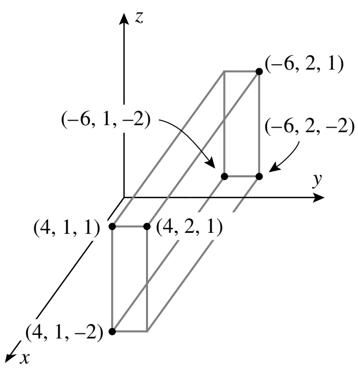
1. (a) $(0,0,0), (3,0,0), (3,5,0), (0,5,0), (0,0,4), (3,0,4), (3,5,4), (0,5,4)$.

(b) $(0,1,0), (4,1,0), (4,6,0), (0,6,0), (0,1,-2), (4,1,-2), (4,6,-2), (0,6,-2)$.

2. Corners: $(2,2,\pm 2)$, $(2,-2,\pm 2)$, $(-2,2,\pm 2)$, $(-2,-2,\pm 2)$.



3. Corners: $(4,2,-2)$, (4,2,1), (4,1,1), $(4,1,-2)$, $(-6,1,1)$, $(-6,2,1)$, $(-6,2,-2)$, $(-6,1,-2)$.



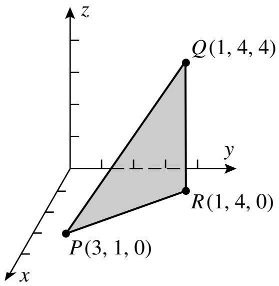
4. (a) $(x_2,y_1,z_1), (x_2,y_2,z_1), (x_1,y_2,z_1)(x_1,y_1,z_2), (x_2,y_1,z_2), (x_1,y_2,z_2)$.

(b) The midpoint of the diagonal has coordinates which are the coordinates of the midpoints of the edges. The midpoint of the edge (x_1,y_1,z_1) and (x_2,y_1,z_1) is $\left(\frac{1}{2}(x_1+x_2), y_1, z_1\right)$; the midpoint of the edge (x_2,y_1,z_1) and (x_2,y_2,z_1) is $\left(x_2, \frac{1}{2}(y_1+y_2), z_1\right)$; the midpoint of the edge (x_2,y_2,z_1) and (x_2,y_2,z_2) is $\left(x_2, y_2, \frac{1}{2}(z_1+z_2)\right)$. Thus the coordinates of the midpoint of the diagonal are $\left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2), \frac{1}{2}(z_1+z_2)\right)$.

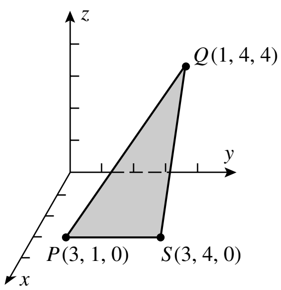
5. (a) A single point on that line. **(b)** A line in that plane. **(c)** A plane in 3−space.

6. (a) $R(1,4,0)$ and Q lie on the same vertical line, and so does the side of the triangle which connects them.

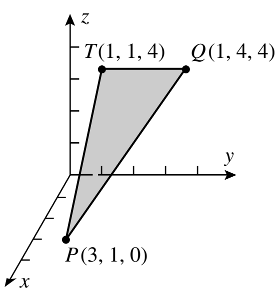
$R(1,4,0)$ and P lie in the plane $z=0$. Clearly the two sides are perpendicular, and the sum of the squares of the two sides is $|RQ|^2+|RP|^2=4^2+(2^2+3^2)=29$, so the distance from P to Q is $\sqrt{29}$.



(b) Clearly, SP is parallel to the y-axis. $S(3,4,0)$ and Q lie in the plane $y=4$, and so does SQ. Hence the two sides $|SP|$ and $|SQ|$ are perpendicular, and $|PQ|=\sqrt{|PS|^2+|QS|^2}=\sqrt{3^2+(2^2+4^2)}=\sqrt{29}$.



(c) $T(1,1,4)$ and Q lie on a line through $(1,0,4)$ and is thus parallel to the y-axis, and TQ lies on this line. T and P lie in the same plane $y=1$ which is perpendicular to any line which is parallel to the y-axis, thus TP, which lies on such a line, is perpendicular to TQ. Thus $|PQ|^2=|PT|^2+|QT|^2=(4+16)+9=29$.



7. (a) Let the base of the box have sides a and b and diagonal d_1. Then $a^2+b^2=d_1^2$, and d_1 is the base of a rectangular of height c and diagonal d, with $d^2=d_1^2+c^2=a^2+b^2+c^2$.

(b) Two unequal points (x_1,y_1,z_1) and (x_2,y_2,z_2) form diagonally opposite corners of a rectangular box with sides $x_1-x_2, y_1-y_2, z_1-z_2$, and by Part (a) the diagonal has length $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$.

8. (a) The vertical plane that passes through $(\frac{1}{2},0,0)$ and is perpendicular to the x-axis.

(b) Equidistant: $(x-\frac{1}{2})^2+y^2+z^2=x^2+y^2+z^2$, or $-2x+1=0$ or $x=\frac{1}{2}$.

9. The diameter is $d=\sqrt{(1-3)^2+(-2-4)^2+(4+12)^2}=\sqrt{296}$, so the radius is $\sqrt{296}/2=\sqrt{74}$. The midpoint $(2,1,-4)$ of the endpoints of the diameter is the center of the sphere.

10. Each side has length $\sqrt{14}$ so the triangle is equilateral.

11. (a) The sides have lengths 7, 14, and $7\sqrt{5}$; it is a right triangle because the sides satisfy the Pythagorean theorem, $(7\sqrt{5})^2=7^2+14^2$.

(b) (2,1,6) is the vertex of the 90° angle because it is opposite the longest side (the hypotenuse).

(c) Area = (1/2)(altitude)(base) = (1/2)(7)(14) = 49.

12. **(a)** 3 **(b)** 2 **(c)** 5 **(d)** $\sqrt{(2)^2+(-3)^2}=\sqrt{13}$. **(e)** $\sqrt{(-5)^2+(-3)^2}=\sqrt{34}$. **(f)** $\sqrt{(-5)^2+(2)^2}=\sqrt{29}$.

13. **(a)** $(x-7)^2+(y-1)^2+(z-1)^2=16$.

(b) $(x-1)^2+y^2+(z+1)^2=16$.

(c) $r=\sqrt{(-1-0)^2+(3-0)^2+(2-0)^2}=\sqrt{14}$, $(x+1)^2+(y-3)^2+(z-2)^2=14$.

(d) $r=\dfrac{1}{2}\sqrt{(-1-0)^2+(2-2)^2+(1-3)^2}=\dfrac{1}{2}\sqrt{5}$, center $(-1/2,2,2)$, $(x+1/2)^2+(y-2)^2+(z-2)^2=5/4$.

14. $r=|[\text{distance between }(0,0,0)\text{ and }(3,-2,4)]\pm 1|=\sqrt{29}\pm 1$, $x^2+y^2+z^2=r^2=\left(\sqrt{29}\pm 1\right)^2=30\pm 2\sqrt{29}$.

15. $(x-2)^2+(y+1)^2+(z+3)^2=r^2$, so

(a) $(x-2)^2+(y+1)^2+(z+3)^2=9$. **(b)** $(x-2)^2+(y+1)^2+(z+3)^2=1$. **(c)** $(x-2)^2+(y+1)^2+(z+3)^2=4$.

16. **(a)** The sides have length 1, so the radius is $\dfrac{1}{2}$; hence $(x+2)^2+(y-1)^2+(z-3)^2=\dfrac{1}{4}$.

(b) The diagonal has length $\sqrt{1+1+1}=\sqrt{3}$ and is a diameter, so $(x+2)^2+(y-1)^2+(z-3)^2=\dfrac{3}{4}$.

(c) Radius: $(6-2)/2=2$, center: $(\dfrac{6+2}{2},\dfrac{5+9}{2},\dfrac{4+0}{2})$, so $(x-4)^2+(y-7)^2+(z-2)^2=4$.

(d) Center is the same, radius is half the diagonal, $r=2\sqrt{3}$, so $(x-4)^2+(y-7)^2+(z-2)^2=12$.

17. Let the center of the sphere be (a,b,c). The height of the center over the x-y plane is measured along the radius that is perpendicular to the plane. But this is the radius itself, so height = radius, i.e. $c=r$. Similarly $a=r$ and $b=r$.

18. If r is the radius of the sphere, then the center of the sphere has coordinates (r,r,r) (see Exercise 17). Thus the distance from the origin to the center is $\sqrt{r^2+r^2+r^2}=\sqrt{3}r$, from which it follows that the distance from the origin to the sphere is $\sqrt{3}r-r$. Equate that with $3-\sqrt{3}$: $\sqrt{3}r-r=3-\sqrt{3}, r=\sqrt{3}$. The sphere is given by the equation $(x-\sqrt{3})^2+(y-\sqrt{3})^2+(z-\sqrt{3})^2=3$.

19. False; need be neither right nor circular, see "extrusion".

20. False, it is a right circular cylinder.

21. True; $y=z=0$.

22. False, the sphere satisfies the equality, not the inequality.

23. $(x+5)^2+(y+2)^2+(z+1)^2=49$; sphere, $C(-5,-2,-1)$, $r=7$.

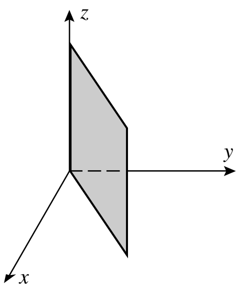
24. $x^2+(y-1/2)^2+z^2=1/4$; sphere, $C(0,1/2,0)$, $r=1/2$.

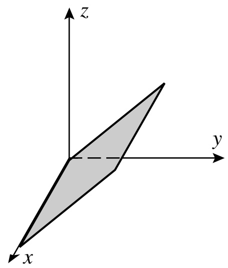
25. $(x-1/2)^2+(y-3/4)^2+(z+5/4)^2=54/16$; sphere, $C(1/2,3/4,-5/4)$, $r=3\sqrt{6}/4$.

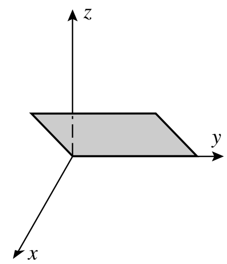
26. $(x+1)^2+(y-1)^2+(z+1)^2=0$; the point $(-1,1,-1)$.

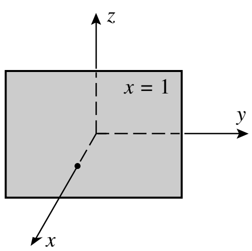
27. $(x-3/2)^2+(y+2)^2+(z-4)^2=-11/4$; no graph.

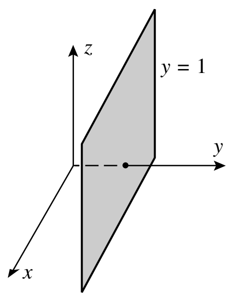
28. $(x-1)^2+(y-3)^2+(z-4)^2=25$; sphere, $C(1,3,4)$, $r=5$.

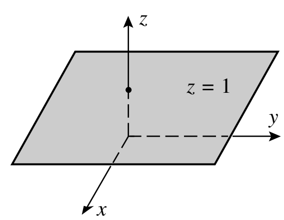
29. **(a)** 

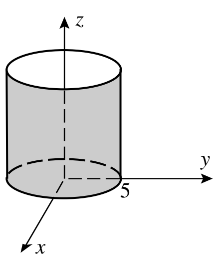
(b) 

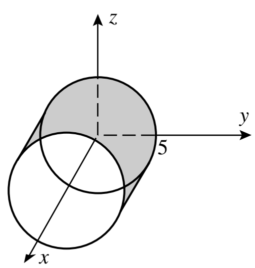
(c) 

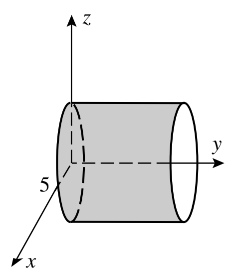
30. **(a)** 

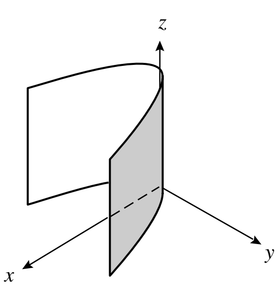
(b) 

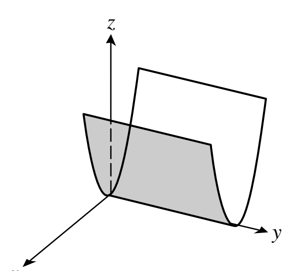
(c) 

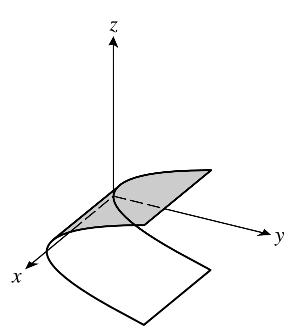
31. **(a)** 

(b) 

(c) 

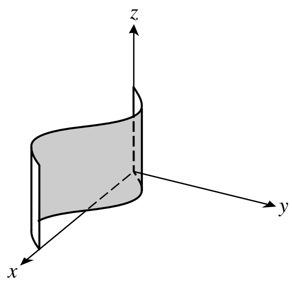
32. **(a)** 

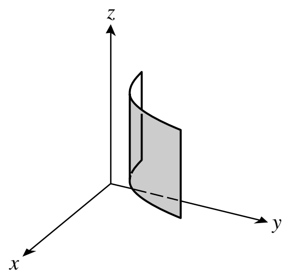
(b) 

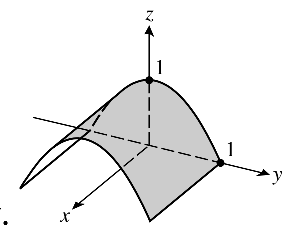
(c) 

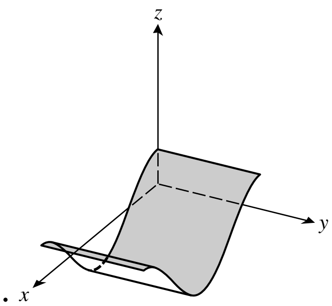
33. **(a)** $-2y+z=0$. **(b)** $-2x+z=0$. **(c)** $(x-1)^2+(y-1)^2=1$. **(d)** $(x-1)^2+(z-1)^2=1$.

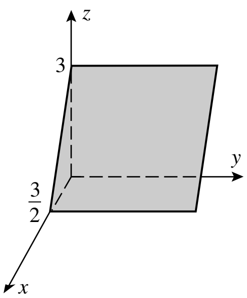
34. **(a)** $(x-a)^2+(z-a)^2=a^2$. **(b)** $(x-a)^2+(y-a)^2=a^2$. **(c)** $(y-a)^2+(z-a)^2=a^2$.

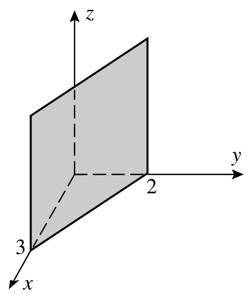
35. 

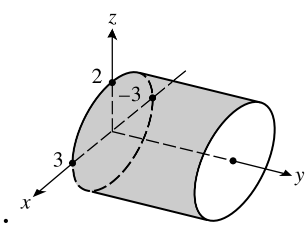
36. 

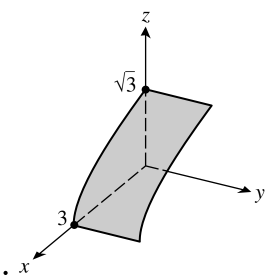
37. 

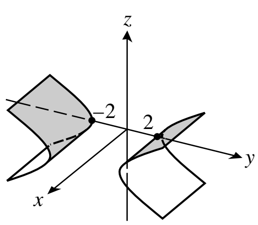
38. 

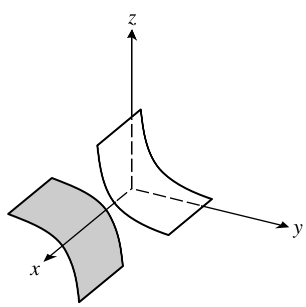
39. 

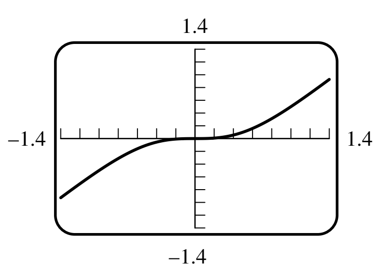
40. 

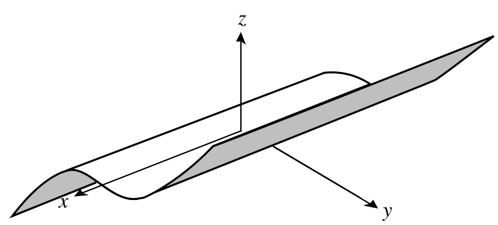
41. 

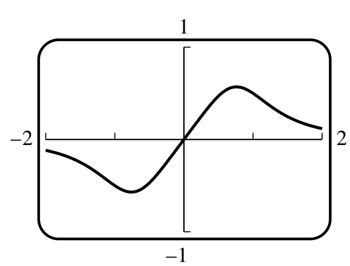
42. 

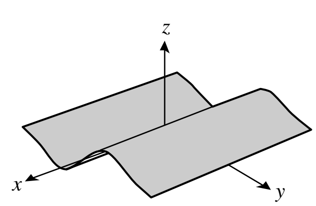
43. 

44. 

45. (a) 

(b) 

46. (a) 

(b) 

47. Complete the squares to get $(x+1)^2 + (y-1)^2 + (z-2)^2 = 9$; center $(-1, 1, 2)$, radius 3. The distance between the origin and the center is $\sqrt{6} < 3$ so the origin is inside the sphere. The largest distance is $3 + \sqrt{6}$, the smallest is $3 - \sqrt{6}$.

48. $(x-1)^2 + y^2 + (z+4)^2 \le 25$; all points on and inside the sphere of radius 5 with center at $(1, 0, -4)$.

49. $(y+3)^2 + (z-2)^2 > 16$; all points outside the circular cylinder $(y+3)^2 + (z-2)^2 = 16$.

50. $\sqrt{(x-1)^2+(y+2)^2+z^2} = 2\sqrt{x^2+(y-1)^2+(z-1)^2}$, square and simplify to get $3x^2+3y^2+3z^2+2x-12y-8z+3=0$, then complete the squares to get $(x+1/3)^2+(y-2)^2+(z-4/3)^2 = 44/9$; center $(-1/3, 2, 4/3)$, radius $2\sqrt{11}/3$.

51. Let r be the radius of a styrofoam sphere. The distance from the origin to the center of the bowling ball is equal to the sum of the distance from the origin to the center of the styrofoam sphere nearest the origin and the distance between the center of this sphere and the center of the bowling ball so $\sqrt{3}R = \sqrt{3}r + r + R$, $(\sqrt{3}+1)r = (\sqrt{3}-1)R$, $r = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}R = (2-\sqrt{3})R$.

52. (a) Complete the squares to get $(x+G/2)^2+(y+H/2)^2+(z+I/2)^2 = K/4$, so the equation represents a sphere when $K>0$, a point when $K=0$, and no graph when $K<0$.

(b) $C(-G/2, -H/2, -I/2)$, $r = \sqrt{K}/2$.

53. (a) At $x=c$ the trace of the surface is the circle $y^2+z^2 = [f(c)]^2$, so the surface is given by $y^2+z^2 = [f(x)]^2$.

(b) $y^2+z^2 = e^{2x}$.

(c) $y^2+z^2 = 4-\dfrac{3}{4}x^2$, so let $f(x) = \sqrt{4-\dfrac{3}{4}x^2}$.

54. (a) Permute x and y in Exercise 53a: $x^2+z^2 = [f(y)]^2$.

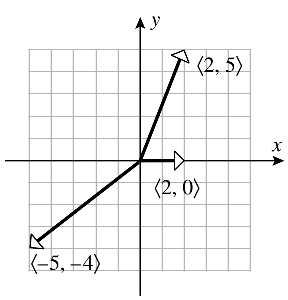
(b) Permute x and z in Exercise 53a: $x^2+y^2 = [f(z)]^2$.

(c) Permute y and z in Exercise 53a: $y^2+z^2 = [f(x)]^2$.

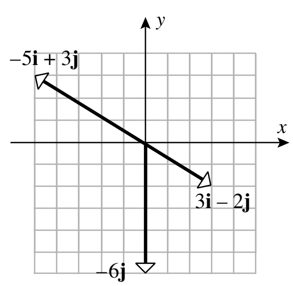
55. $(a\sin\phi\cos\theta)^2+(a\sin\phi\sin\theta)^2+(a\cos\phi)^2 = a^2\sin^2\phi\cos^2\theta + a^2\sin^2\phi\sin^2\theta + a^2\cos^2\phi = a^2\sin^2\phi(\cos^2\theta+\sin^2\theta) + a^2\cos^2\phi = a^2\sin^2\phi + a^2\cos^2\phi = a^2(\sin^2\phi+\cos^2\phi) = a^2$.

Exercise Set 11.2

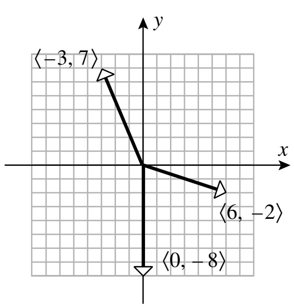
1. (a-c)



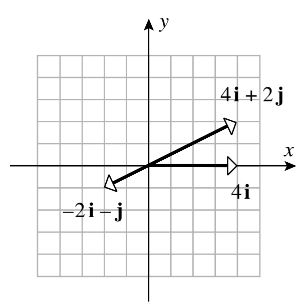
(d-f)

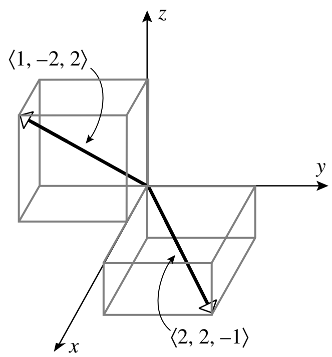


2. (a-c)

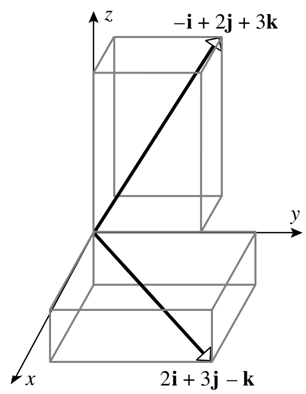


(d-f)

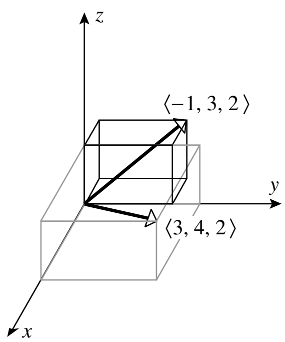




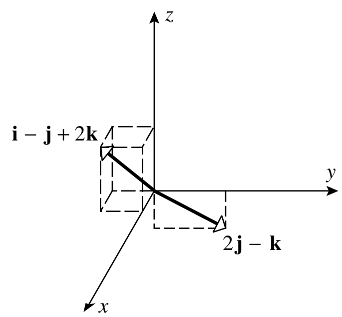
3. (a-b)



(c-d)

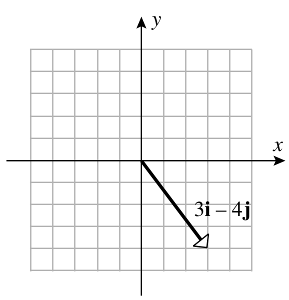


4. (a-b)

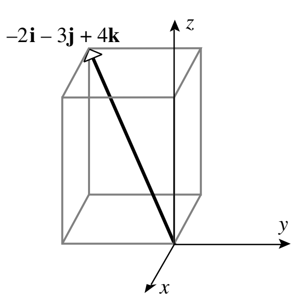


(c-d)

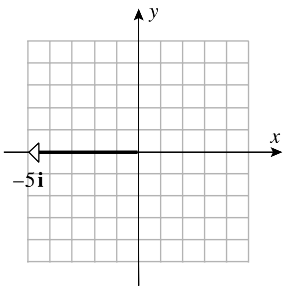
5. (a) $\langle 4-1,\, 1-5\rangle = \langle 3,\, -4\rangle$



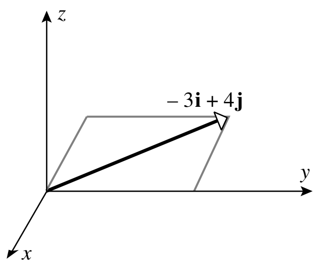
(b) $\langle 0-2,\, 0-3,\, 4-0\rangle = \langle -2,\, -3,\, 4\rangle$



6. (a) $\langle -3-2,\, 3-3\rangle = \langle -5,\, 0\rangle$



(b) $\langle 0-3,\ 4-0,\ 4-4\rangle = \langle -3,\ 4,\ 0\rangle$



7. (a) $\langle 2-3,\ 8-5\rangle = \langle -1, 3\rangle$ **(b)** $\langle 0-7,\ 0-(-2)\rangle = \langle -7, 2\rangle$ **(c)** $\langle -3, 6, 1\rangle$

8. (a) $\langle -4-(-6),\ -1-(-2)\rangle = \langle 2, 1\rangle$ **(b)** $\langle -1, 6, 1\rangle$ **(c)** $\langle 5, 0, 0\rangle$

9. (a) Let (x, y) be the terminal point, then $x-1=3$, $x=4$ and $y-(-2)=-2$, $y=-4$. The terminal point is $(4, -4)$.

(b) Let (x, y, z) be the initial point, then $5-x=-3$, $-y=1$, and $-1-z=2$ so $x=8$, $y=-1$, and $z=-3$. The initial point is $(8, -1, -3)$.

10. (a) Let (x, y) be the terminal point, then $x-2=7$, $x=9$ and $y-(-1)=6$, $y=5$. The terminal point is (9,5).

(b) Let (x, y, z) be the terminal point, then $x+2=1$, $y-1=2$, and $z-4=-3$ so $x=-1$, $y=3$, and $z=1$. The terminal point is $(-1, 3, 1)$.

11. (a) $-\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ **(b)** $18\mathbf{i}+12\mathbf{j}-6\mathbf{k}$ **(c)** $-\mathbf{i}-5\mathbf{j}-2\mathbf{k}$ **(d)** $40\mathbf{i}-4\mathbf{j}-4\mathbf{k}$ **(e)** $-2\mathbf{i}-16\mathbf{j}-18\mathbf{k}$ **(f)** $-\mathbf{i}+13\mathbf{j}-2\mathbf{k}$

12. (a) $\langle 1, -2, 0\rangle$ **(b)** $\langle 28, 0, -14\rangle + \langle 3, 3, 9\rangle = \langle 31, 3, -5\rangle$ **(c)** $\langle 3, -1, -5\rangle$

(d) $3(\langle 2, -1, 3\rangle - \langle 28, 0, -14\rangle) = 3\langle -26, -1, 17\rangle = \langle -78, -3, 51\rangle$ **(e)** $\langle -12, 0, 6\rangle - \langle 8, 8, 24\rangle = \langle -20, -8, -18\rangle$

(f) $\langle 8, 0, -4\rangle - \langle 3, 0, 6\rangle = \langle 5, 0, -10\rangle$

13. (a) $\|\mathbf{v}\| = \sqrt{1+1} = \sqrt{2}$ **(b)** $\|\mathbf{v}\| = \sqrt{1+49} = 5\sqrt{2}$ **(c)** $\|\mathbf{v}\| = \sqrt{21}$ **(d)** $\|\mathbf{v}\| = \sqrt{14}$

14. (a) $\|\mathbf{v}\| = \sqrt{9+16} = 5$ **(b)** $\|\mathbf{v}\| = \sqrt{2+7} = 3$ **(c)** $\|\mathbf{v}\| = 3$ **(d)** $\|\mathbf{v}\| = \sqrt{3}$

15. (a) $\|\mathbf{u}+\mathbf{v}\| = \|2\mathbf{i}-2\mathbf{j}+2\mathbf{k}\| = 2\sqrt{3}$ **(b)** $\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{14}+\sqrt{2}$ **(c)** $\|-2\mathbf{u}\| + 2\|\mathbf{v}\| = 2\sqrt{14}+2\sqrt{2}$

(d) $\|3\mathbf{u}-5\mathbf{v}+\mathbf{w}\| = \|-12\mathbf{j}+2\mathbf{k}\| = 2\sqrt{37}$ **(e)** $(1/\sqrt{6})\mathbf{i} + (1/\sqrt{6})\mathbf{j} - (2/\sqrt{6})\mathbf{k}$ **(f)** 1

16. Yes, it is possible. Consider $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$.

17. False; only if one vector is a positive scalar multiple of the other. If one vector is a positive multiple of the other, say $\mathbf{u} = \alpha\mathbf{v}$ with $\alpha > 0$, then $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are parallel and $\|\mathbf{u}+\mathbf{v}\| = (1+\alpha)\|\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$.

18. True.

19. True (assuming they have the same initial point), namely $\pm x/||x||$.

20. True, $\mathbf{a} = \dfrac{1}{c}(\mathbf{d} - \mathbf{b})$.

21. **(a)** $\|-\mathbf{i} + 4\mathbf{j}\| = \sqrt{17}$ so the required vector is $\left(-1/\sqrt{17}\right)\mathbf{i} + \left(4/\sqrt{17}\right)\mathbf{j}$.

(b) $\|6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}\| = 2\sqrt{14}$ so the required vector is $(-3\mathbf{i} + 2\mathbf{j} - \mathbf{k})/\sqrt{14}$.

(c) $\overrightarrow{AB} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\|\overrightarrow{AB}\| = 3\sqrt{2}$ so the required vector is $(4\mathbf{i} + \mathbf{j} - \mathbf{k})/\left(3\sqrt{2}\right)$.

22. **(a)** $\|3\mathbf{i} - 4\mathbf{j}\| = 5$ so the required vector is $-\dfrac{1}{5}(3\mathbf{i} - 4\mathbf{j}) = -\dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$.

(b) $\|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}\| = 3$ so the required vector is $\dfrac{2}{3}\mathbf{i} - \dfrac{1}{3}\mathbf{j} - \dfrac{2}{3}\mathbf{k}$.

(c) $\overrightarrow{AB} = 4\mathbf{i} - 3\mathbf{j}$, $\|\overrightarrow{AB}\| = 5$ so the required vector is $\dfrac{4}{5}\mathbf{i} - \dfrac{3}{5}\mathbf{j}$.

23. **(a)** $-\dfrac{1}{2}\mathbf{v} = \langle -3/2, 2\rangle$. **(b)** $\|\mathbf{v}\| = \sqrt{85}$, so $\dfrac{\sqrt{17}}{\sqrt{85}}\mathbf{v} = \dfrac{1}{\sqrt{5}}\langle 7, 0, -6\rangle$ has length $\sqrt{17}$.

24. **(a)** $3\mathbf{v} = -6\mathbf{i} + 9\mathbf{j}$. **(b)** $-\dfrac{2}{\|v\|}\mathbf{v} = \dfrac{6}{\sqrt{26}}\mathbf{i} - \dfrac{8}{\sqrt{26}}\mathbf{j} - \dfrac{2}{\sqrt{26}}\mathbf{k}$.

25. **(a)** $\mathbf{v} = \|\mathbf{v}\|\langle\cos(\pi/4), \sin(\pi/4)\rangle = \langle 3\sqrt{2}/2, 3\sqrt{2}/2\rangle$. **(b)** $\mathbf{v} = \|\mathbf{v}\|\langle\cos 90^\circ, \sin 90^\circ\rangle = \langle 0, 2\rangle$.

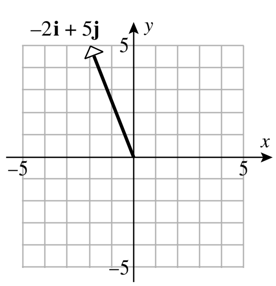
(c) $\mathbf{v} = \|\mathbf{v}\|\langle\cos 120^\circ, \sin 120^\circ\rangle = \langle -5/2, 5\sqrt{3}/2\rangle$. **(d)** $\mathbf{v} = \|\mathbf{v}\|\langle\cos\pi, \sin\pi\rangle = \langle -1, 0\rangle$.

26. From (12), $\mathbf{v} = \langle\cos(\pi/6),\ \sin(\pi/6)\rangle = \langle\sqrt{3}/2,\ 1/2\rangle$ and $\mathbf{w} = \langle\cos(3\pi/4),\ \sin(3\pi/4)\rangle = \langle -\sqrt{2}/2,\ \sqrt{2}/2\rangle$, so $\mathbf{v} + \mathbf{w} = ((\sqrt{3} - \sqrt{2})/2,\ (1 + \sqrt{2})/2)$, $\mathbf{v} - \mathbf{w} = ((\sqrt{3} + \sqrt{2})/2, (1 - \sqrt{2})/2)$.

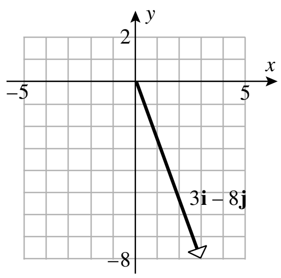
27. From (12), $\mathbf{v} = \langle\cos 30^\circ, \sin 30^\circ\rangle = \langle\sqrt{3}/2, 1/2\rangle$ and $\mathbf{w} = \langle\cos 135^\circ, \sin 135^\circ\rangle = \langle -\sqrt{2}/2, \sqrt{2}/2\rangle$, so $\mathbf{v} + \mathbf{w} = ((\sqrt{3} - \sqrt{2})/2, (1 + \sqrt{2})/2)$.

28. $\mathbf{w} = \langle 1, 0\rangle$, and from (12), $\mathbf{v} = \langle\cos 120^\circ, \sin 120^\circ\rangle = \langle -1/2, \sqrt{3}/2\rangle$, so $\mathbf{v} + \mathbf{w} = \langle 1/2, \sqrt{3}/2\rangle$.

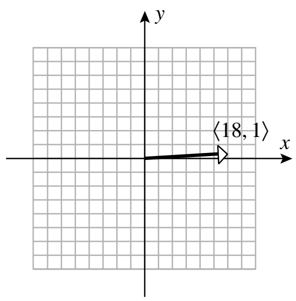
29. **(a)** The initial point of $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is the origin and the endpoint is $(-2, 5)$, so $\mathbf{u} + \mathbf{v} + \mathbf{w} = \langle -2, 5\rangle$.



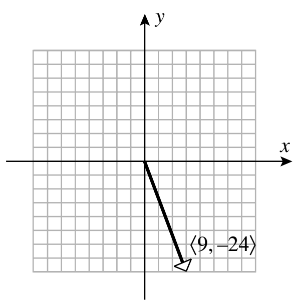
(b) The initial point of $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is $(-5, 4)$ and the endpoint is $(-2, -4)$, so $\mathbf{u} + \mathbf{v} + \mathbf{w} = \langle 3, -8\rangle$.



30. (a) $\mathbf{v} = \langle -10, 2 \rangle$ by inspection, so $\mathbf{u} - \mathbf{v} + \mathbf{w} = \mathbf{u} + \mathbf{v} + \mathbf{w} - 2\mathbf{v} = \langle -2, 5 \rangle + \langle 20, -4 \rangle = \langle 18, 1 \rangle$.



(b) $\mathbf{v} = \langle -3, 8 \rangle$ by inspection, so $\mathbf{u} - \mathbf{v} + \mathbf{w} = \mathbf{u} + \mathbf{v} + \mathbf{w} - 2\mathbf{v} = \langle 3, -8 \rangle + \langle 6, -16 \rangle = \langle 9, -24 \rangle$.



31. $6\mathbf{x} = 2\mathbf{u} - \mathbf{v} - \mathbf{w} = \langle -4, 6 \rangle, \mathbf{x} = \langle -2/3, 1 \rangle$.

32. $\mathbf{u} - 2\mathbf{x} = \mathbf{x} - \mathbf{w} + 3\mathbf{v}$, $3\mathbf{x} = \mathbf{u} + \mathbf{w} - 3\mathbf{v}$, $\mathbf{x} = \dfrac{1}{3}(\mathbf{u} + \mathbf{w} - 3\mathbf{v}) = \langle 2/3, 2/3 \rangle$.

33. $\mathbf{u} = \dfrac{5}{7}\mathbf{i} + \dfrac{2}{7}\mathbf{j} + \dfrac{1}{7}\mathbf{k}$, $\mathbf{v} = \dfrac{8}{7}\mathbf{i} - \dfrac{1}{7}\mathbf{j} - \dfrac{4}{7}\mathbf{k}$.

34. $3\mathbf{u} + 2\mathbf{v} - 2(\mathbf{u} + \mathbf{v}) = \mathbf{u} = \langle -5, 8 \rangle, \mathbf{v} = \mathbf{u} + \mathbf{v} - \mathbf{u} = \langle 7, -11 \rangle$.

35. $\|(\mathbf{i} + \mathbf{j}) + (\mathbf{i} - 2\mathbf{j})\| = \|2\mathbf{i} - \mathbf{j}\| = \sqrt{5}, \|(\mathbf{i} + \mathbf{j}) - (\mathbf{i} - 2\mathbf{j})\| = \|3\mathbf{j}\| = 3$.

36. Let A, B, C be the vertices (0,0), (1,3), (2,4) and D the fourth vertex (x, y). For the parallelogram ABCD, $\overrightarrow{AD} = \overrightarrow{BC}$, $\langle x, y \rangle = \langle 1, 1 \rangle$ so $x = 1$, $y = 1$ and D is at (1,1). For the parallelogram ACBD, $\overrightarrow{AD} = \overrightarrow{CB}$, $\langle x, y \rangle = \langle -1, -1 \rangle$ so $x = -1$, $y = -1$ and D is at $(-1, -1)$. For the parallelogram $ABDC$, $\overrightarrow{AC} = \overrightarrow{BD}$, $\langle x - 1, y - 3 \rangle = \langle 2, 4 \rangle$, so $x = 3, y = 7$ and D is at $(3, 7)$.

37. (a) $5 = \|k\mathbf{v}\| = |k|\|\mathbf{v}\| = \pm 3k$, so $k = \pm 5/3$.

(b) $6 = \|k\mathbf{v}\| = |k|\|\mathbf{v}\| = 2\|\mathbf{v}\|$, so $\|\mathbf{v}\| = 3$.

38. If $\|k\mathbf{v}\| = 0$ then $|k|\|\mathbf{v}\| = 0$ so either $k = 0$ or $\|\mathbf{v}\| = 0$; in the latter case, by (9) or (10), $\mathbf{v} = \mathbf{0}$.

39. (a) Choose two points on the line, for example $P_1(0,2)$ and $P_2(1,5)$; then $\overrightarrow{P_1P_2} = \langle 1,3\rangle$ is parallel to the line, $\|\langle 1,3\rangle\| = \sqrt{10}$, so $\langle 1/\sqrt{10}, 3/\sqrt{10}\rangle$ and $\langle -1/\sqrt{10}, -3/\sqrt{10}\rangle$ are unit vectors parallel to the line.

(b) Choose two points on the line, for example $P_1(0,4)$ and $P_2(1,3)$; then $\overrightarrow{P_1P_2} = \langle 1,-1\rangle$ is parallel to the line, $\|\langle 1,-1\rangle\| = \sqrt{2}$ so $\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle$ and $\langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$ are unit vectors parallel to the line.

(c) Pick any line that is perpendicular to the line $y = -5x+1$, for example $y = x/5$; then $P_1(0,0)$ and $P_2(5,1)$ are on the line, so $\overrightarrow{P_1P_2} = \langle 5,1\rangle$ is perpendicular to the line, so $\pm\dfrac{1}{\sqrt{26}}\langle 5,1\rangle$ are unit vectors perpendicular to the line.

40. (a) $\pm\mathbf{k}$ **(b)** $\pm\mathbf{j}$ **(c)** $\pm\mathbf{i}$

41. (a) The circle of radius 1 about the origin.

(b) The closed disk of radius 1 about the origin.

(c) All points outside the closed disk of radius 1 about the origin.

42. (a) The circle of radius 1 about the tip of $\mathbf{r}_0$.

(b) The closed disk of radius 1 about the tip of $\mathbf{r}_0$.

(c) All points outside the closed disk of radius 1 about the tip of $\mathbf{r}_0$.

43. (a) The (hollow) sphere of radius 1 about the origin.

(b) The closed ball of radius 1 about the origin.

(c) All points outside the closed ball of radius 1 about the origin.

44. The sum of the distances between (x,y) and the points (x_1,y_1), (x_2,y_2) is the constant k, so the set consists of all points on the ellipse with foci at (x_1,y_1) and (x_2,y_2), and major axis of length k.

45. Since $\phi = \pi/2$, from (14) we get $\|\mathbf{F}_1+\mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2 + \|\mathbf{F}_2\|^2 = 3600 + 900$, so $\|\mathbf{F}_1+\mathbf{F}_2\| = 30\sqrt{5}$ lb, and $\sin\alpha = \dfrac{\|\mathbf{F}_2\|}{\|\mathbf{F}_1+\mathbf{F}_2\|}\sin\phi = \dfrac{30}{30\sqrt{5}}, \alpha \approx 26.57^\circ, \theta = \alpha \approx 26.57^\circ$.

46. $\|\mathbf{F}_1+\mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2+\|\mathbf{F}_2\|^2+2\|\mathbf{F}_1\|\|\mathbf{F}_2\|\cos\phi = 14{,}400+10{,}000+2(120)(100)\dfrac{1}{2} = 36{,}400$, so $\|\mathbf{F}_1+\mathbf{F}_2\| = 20\sqrt{91}$ N, $\sin\alpha = \dfrac{\|\mathbf{F}_2\|}{\|\mathbf{F}_1+\mathbf{F}_2\|}\sin\phi = \dfrac{100}{20\sqrt{91}}\sin 60^\circ = \dfrac{5\sqrt{3}}{2\sqrt{91}}, \alpha \approx 27.00^\circ, \theta = \alpha \approx 27.00^\circ$.

47. $\|\mathbf{F}_1+\mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2+\|\mathbf{F}_2\|^2+2\|\mathbf{F}_1\|\|\mathbf{F}_2\|\cos\phi = 160{,}000+160{,}000-2(400)(400)\dfrac{\sqrt{3}}{2}$, so $\|\mathbf{F}_1+\mathbf{F}_2\| \approx 207.06$ N, and $\sin\alpha = \dfrac{\|\mathbf{F}_2\|}{\|\mathbf{F}_1+\mathbf{F}_2\|}\sin\phi \approx \dfrac{400}{207.06}\left(\dfrac{1}{2}\right), \alpha = 75.00^\circ, \theta = \alpha - 30^\circ = 45.00^\circ$.

48. $\|\mathbf{F}_1+\mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2+\|\mathbf{F}_2\|^2+2\|\mathbf{F}_1\|\|\mathbf{F}_2\|\cos\phi = 16+4+2(4)(2)\cos 77^\circ$, so $\|\mathbf{F}_1+\mathbf{F}_2\| \approx 4.86$ lb, and $\sin\alpha = \dfrac{\|\mathbf{F}_2\|}{\|\mathbf{F}_1+\mathbf{F}_2\|}\sin\phi = \dfrac{2}{4.86}\sin 77^\circ, \alpha \approx 23.64^\circ, \theta = \alpha - 27^\circ \approx -3.36^\circ$.

49. Let $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ be the forces in the diagram with magnitudes $40, 50, 75$ respectively. Then $\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3 = (\mathbf{F}_1+\mathbf{F}_2)+\mathbf{F}_3$. Following the examples, $\mathbf{F}_1+\mathbf{F}_2$ has magnitude 45.83 N and makes an angle 79.11° with the positive x-axis. Then $\|(\mathbf{F}_1+\mathbf{F}_2)+\mathbf{F}_3\|^2 \approx 45.83^2 + 75^2 + 2(45.83)(75)\cos 79.11^\circ$, so $\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3$ has magnitude ≈ 94.995 N and makes an angle $\theta = \alpha \approx 28.28^\circ$ with the positive x-axis.

50. Let $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ be the forces in the diagram with magnitudes $150, 200, 100$ respectively. Then $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (\mathbf{F}_1 + \mathbf{F}_2) + \mathbf{F}_3$. Following the examples, $\mathbf{F}_1 + \mathbf{F}_2$ has magnitude 279.34 N and makes an angle 91.24° with the positive x-axis. Then $\|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\|^2 \approx 279.34^2 + 100^2 + 2(279.34)(100)\cos(270 - 91.24)^\circ$, and $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ has magnitude ≈ 179.37 N and makes an angle 91.94° with the positive x-axis.

51. Let $\mathbf{F}_1, \mathbf{F}_2$ be the forces in the diagram with magnitudes $8, 10$ respectively. Then $\|\mathbf{F}_1 + \mathbf{F}_2\|$ has magnitude $\sqrt{8^2 + 10^2 + 2 \cdot 8 \cdot 10 \cos 120^\circ} = 2\sqrt{21} \approx 9.165$ lb, and makes an angle $60^\circ + \sin^{-1} \dfrac{\|\mathbf{F}_1\|}{\|\mathbf{F}_1 + \mathbf{F}_2\|} \sin 120 \approx 109.11^\circ$ with the positive x-axis, so $\mathbf{F}$ has magnitude 9.165 lb and makes an angle -70.89° with the positive x-axis.

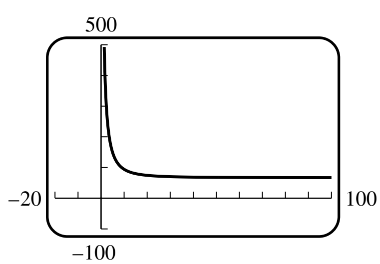
52. $\|\mathbf{F}_1 + \mathbf{F}_2\| = \sqrt{120^2 + 150^2 + 2 \cdot 120 \cdot 150 \cos 75^\circ} = 214.98$ N and makes an angle 92.63° with the positive x-axis, and $\|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| = 232.90$ N and makes an angle 67.23° with the positive x-axis, hence $\mathbf{F}$ has magnitude 232.90 N and makes an angle -112.77° with the positive x-axis.

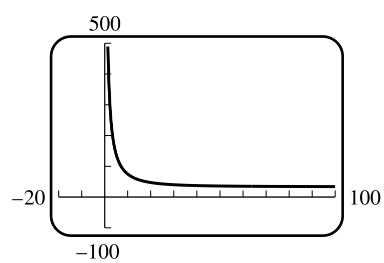
53. $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F} = \mathbf{0}$, where $\mathbf{F}$ has magnitude 250 and makes an angle -90° with the positive x-axis. Thus $\|\mathbf{F}_1 + \mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2 + \|\mathbf{F}_2\|^2 + 2\|\mathbf{F}_1\|\|\mathbf{F}_2\| \cos 105^\circ = 250^2$ and $45^\circ = \alpha = \sin^{-1}\left(\dfrac{\|\mathbf{F}_2\|}{250} \sin 105^\circ\right)$, so $\dfrac{\sqrt{2}}{2} \approx \dfrac{\|\mathbf{F}_2\|}{250} 0.9659$, $\|\mathbf{F}_2\| \approx 183.02$ lb, $\|\mathbf{F}_1\|^2 + 2(183.02)(-0.2588)\|\mathbf{F}_1\| + (183.02)^2 = 62{,}500$, $\|\mathbf{F}_1\| = 224.13$ lb.

54. Similar to Exercise 53, $\|\mathbf{F}_1\| = 100\sqrt{3}$ N, $\|\mathbf{F}_2\| = 100$ N.

55. Three forces act on the block: its weight $-300\mathbf{j}$; the tension in cable A, which has the form $a(-\mathbf{i} + \mathbf{j})$; and the tension in cable B, which has the form $b(\sqrt{3}\mathbf{i} + \mathbf{j})$, where a, b are positive constants. The sum of these forces is zero, which yields $a = 450 - 150\sqrt{3}$, $b = 150\sqrt{3} - 150$. Thus the forces along cables A and B are, respectively, $\|150(3 - \sqrt{3})(\mathbf{i} - \mathbf{j})\| = 450\sqrt{2} - 150\sqrt{6}$ lb, and $\|150(\sqrt{3} - 1)(\sqrt{3}\mathbf{i} - \mathbf{j})\| = 300\sqrt{3} - 300$ lb.

56. **(a)** Let $\mathbf{T}_A$ and $\mathbf{T}_B$ be the forces exerted on the block by cables A and B. Then $\mathbf{T}_A = a(-10\mathbf{i} + d\mathbf{j})$ and $\mathbf{T}_B = b(20\mathbf{i} + d\mathbf{j})$ for some positive a, b. Since $\mathbf{T}_A + \mathbf{T}_B - 100\mathbf{j} = \mathbf{0}$, we find that $a = \dfrac{200}{3d}$, $b = \dfrac{100}{3d}$, $\mathbf{T}_A = -\dfrac{2000}{3d}\mathbf{i} + \dfrac{200}{3}\mathbf{j}$, and $\mathbf{T}_B = \dfrac{2000}{3d}\mathbf{i} + \dfrac{100}{3}\mathbf{j}$. Thus $\mathbf{T}_A = \dfrac{200}{3}\sqrt{1 + \dfrac{100}{d^2}}$, $\mathbf{T}_B = \dfrac{100}{3}\sqrt{1 + \dfrac{400}{d^2}}$, and the graphs are:





(b) An increase in d will decrease both forces.

(c) The inequality $\|\mathbf{T}_A\| \le 150$ is equivalent to $d \ge \dfrac{40}{\sqrt{65}}$, and $\|\mathbf{T}_B\| \le 150$ is equivalent to $d \ge \dfrac{40}{\sqrt{77}}$. Hence we must have $d \ge \dfrac{40}{\sqrt{65}}$.

57. **(a)** $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = (2c_1 + 4c_2)\,\mathbf{i} + (-c_1 + 2c_2)\,\mathbf{j} = 4\mathbf{j}$, so $2c_1 + 4c_2 = 0$ and $-c_1 + 2c_2 = 4$, which gives $c_1 = -2$, $c_2 = 1$.

(b) $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \langle c_1 - 2c_2,\, -3c_1 + 6c_2 \rangle = \langle 3, 5 \rangle$, so $c_1 - 2c_2 = 3$ and $-3c_1 + 6c_2 = 5$ which has no solution.

58. **(a)** Equate corresponding components to get the system of equations $c_1 + 3c_2 = -1$, $2c_2 + c_3 = 1$, and $c_1 + c_3 = 5$. Solve to get $c_1 = 2$, $c_2 = -1$, and $c_3 = 3$.

(b) Equate corresponding components to get the system of equations $c_1 + 3c_2 + 4c_3 = 2$, $-c_1 - c_3 = 1$, and $c_2 + c_3 = -1$. From the second and third equations, $c_1 = -1 - c_3$ and $c_2 = -1 - c_3$; substitute these into the first equation to get $-4 = 2$, which is false so the system has no solution.

59. Place $\mathbf{u}$ and $\mathbf{v}$ tip to tail so that $\mathbf{u}+\mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$. The shortest distance between two points is along the line joining these points so $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

60. (a): $\mathbf{u}+\mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j}) + (v_1\mathbf{i} + v_2\mathbf{j}) = (v_1\mathbf{i} + v_2\mathbf{j}) + (u_1\mathbf{i} + u_2\mathbf{j}) = \mathbf{v}+\mathbf{u}$.

(c): $\mathbf{u}+\mathbf{0} = (u_1\mathbf{i} + u_2\mathbf{j}) + 0\mathbf{i} + 0\mathbf{j} = u_1\mathbf{i} + u_2\mathbf{j} = \mathbf{u}$.

(e): $k(l\mathbf{u}) = k(l(u_1\mathbf{i} + u_2\mathbf{j})) = k(lu_1\mathbf{i} + lu_2\mathbf{j}) = klu_1\mathbf{i} + klu_2\mathbf{j} = (kl)\mathbf{u}$.

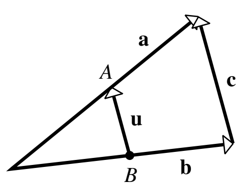
61. (d): $\mathbf{u}+(-\mathbf{u}) = (u_1\mathbf{i} + u_2\mathbf{j}) + (-u_1\mathbf{i} - u_2\mathbf{j}) = (u_1 - u_1)\mathbf{i} + (u_1 - u_1)\mathbf{j} = \mathbf{0}$.

(g): $(k+l)\mathbf{u} = (k+l)(u_1\mathbf{i} + u_2\mathbf{j}) = ku_1\mathbf{i} + ku_2\mathbf{j} + lu_1\mathbf{i} + lu_2\mathbf{j} = k\mathbf{u} + l\mathbf{u}$.

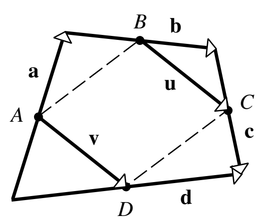
(h): $1\mathbf{u} = 1(u_1\mathbf{i} + u_2\mathbf{j}) = 1u_1\mathbf{i} + 1u_2\mathbf{j} = u_1\mathbf{i} + u_2\mathbf{j} = \mathbf{u}$.

62. Draw the triangles with sides formed by the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$, $k\mathbf{v}$, $k\mathbf{u}+k\mathbf{v}$. By similar triangles, $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u} + k\mathbf{v}$.

63. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be vectors along the sides of the triangle and A,B the midpoints of $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{u} = \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a}-\mathbf{b}) = \frac{1}{2}\mathbf{c}$ so $\mathbf{u}$ is parallel to $\mathbf{c}$ and half as long.



64. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be vectors along the sides of the quadrilateral and A, B, C, D the corresponding midpoints, then $\mathbf{u} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$ and $\mathbf{v} = \frac{1}{2}\mathbf{d} - \frac{1}{2}\mathbf{a}$ but $\mathbf{d} = \mathbf{a}+\mathbf{b}+\mathbf{c}$ so $\mathbf{v} = \frac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c}) - \frac{1}{2}\mathbf{a} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} = \mathbf{u}$ thus $ABCD$ is a parallelogram because sides AD and BC are equal and parallel.



Exercise Set 11.3

1. (a) $(1)(6) + (2)(-8) = -10$; $\cos\theta = (-10)/[(\sqrt{5})(10)] = -1/\sqrt{5}$.

(b) $(-7)(0) + (-3)(1) = -3$; $\cos\theta = (-3)/[(\sqrt{58})(1)] = -3/\sqrt{58}$.

(c) $(1)(8) + (-3)(-2) + (7)(-2) = 0$; $\cos\theta = 0$.

(d) $(-3)(4) + (1)(2) + (2)(-5) = -20$; $\cos\theta = (-20)/[(\sqrt{14})(\sqrt{45})] = -20/(3\sqrt{70})$.

2. (a) $\mathbf{u}\cdot\mathbf{v} = (1)(2)\cos(\pi/6) = \sqrt{3}$ **(b)** $\mathbf{u}\cdot\mathbf{v} = (2)(3)\cos 135^\circ = -3\sqrt{2}$.

3. (a) $\mathbf{u}\cdot\mathbf{v} = -34 < 0$, obtuse. **(b)** $\mathbf{u}\cdot\mathbf{v} = 6 > 0$, acute. **(c)** $\mathbf{u}\cdot\mathbf{v} = -1 < 0$, obtuse. **(d)** $\mathbf{u}\cdot\mathbf{v} = 0$, orthogonal.

4. Let the points be P, Q, R in order, then $\overrightarrow{PQ} = \langle 2-(-1), -2-2, 0-3\rangle = \langle 3, -4, -3\rangle$, $\overrightarrow{QR} = \langle 3-2, 1-(-2), -4-0\rangle = \langle 1, 3, -4\rangle$, $\overrightarrow{RP} = \langle -1-3, 2-1, 3-(-4)\rangle = \langle -4, 1, 7\rangle$; since $\overrightarrow{QP} \cdot \overrightarrow{QR} = -3(1)+4(3)+3(-4) = -3 < 0$, $\angle PQR$ is obtuse; since $\overrightarrow{RP} \cdot \overrightarrow{RQ} = -4(-1)+(-3)+7(4) = 29 > 0$, $\angle PRQ$ is acute; since $\overrightarrow{PR} \cdot \overrightarrow{PQ} = 4(3)-1(-4)-7(-3) = 37 > 0$, $\angle RPQ$ is acute.

5. Since $\mathbf{v}_0 \cdot \mathbf{v}_i = \cos\phi_i$, the answers are, in order, $\sqrt{2}/2, 0, -\sqrt{2}/2, -1, -\sqrt{2}/2, 0, \sqrt{2}/2$.

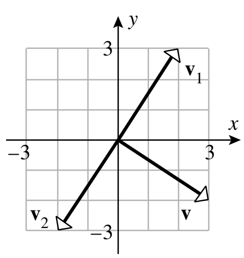
6. Proceed as in Exercise 5; $25/2, -25/2, -25, -25/2, 25/2$.

7. **(a)** $\overrightarrow{AB} = \langle 1, 3, -2\rangle$, $\overrightarrow{BC} = \langle 4, -2, -1\rangle$, $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$ so $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are orthogonal; it is a right triangle with the right angle at vertex B.

(b) Let A, B, and C be the vertices $(-1,0)$, $(2,-1)$, and (1,4) with corresponding interior angles α, β, and γ, then $\cos\alpha = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AC}\|} = \dfrac{\langle 3,-1\rangle \cdot \langle 2,4\rangle}{\sqrt{10}\sqrt{20}} = 1/(5\sqrt{2})$, so $\alpha \approx 82^\circ$, $\cos\beta = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BA}\|\,\|\overrightarrow{BC}\|} = \dfrac{\langle -3,1\rangle \cdot \langle -1,5\rangle}{\sqrt{10}\sqrt{26}} = 4/\sqrt{65}$, so $\beta \approx 60^\circ$, $\cos\gamma = \dfrac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\|\overrightarrow{CA}\|\,\|\overrightarrow{CB}\|} = \dfrac{\langle -2,-4\rangle \cdot \langle 1,-5\rangle}{\sqrt{20}\sqrt{26}} = 9/\sqrt{130}$, so $\gamma \approx 38^\circ$.

8. **(a)** $\mathbf{v} \cdot \mathbf{v}_1 = -ab + ba = 0$; $\mathbf{v} \cdot \mathbf{v}_2 = ab + b(-a) = 0$.

(b) Let $\mathbf{v}_1 = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v}_2 = -2\mathbf{i} - 3\mathbf{j}$; take $\mathbf{u}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \dfrac{2}{\sqrt{13}}\mathbf{i} + \dfrac{3}{\sqrt{13}}\mathbf{j}$, $\mathbf{u}_2 = -\mathbf{u}_1$.



9. **(a)** The dot product of a vector $\mathbf{u}$ and a scalar $\mathbf{v} \cdot \mathbf{w}$ is not defined.

(b) The sum of a scalar $\mathbf{u} \cdot \mathbf{v}$ and a vector $\mathbf{w}$ is not defined.

(c) $\mathbf{u} \cdot \mathbf{v}$ is not a vector.

(d) The dot product of a scalar k and a vector $\mathbf{u} + \mathbf{v}$ is not defined.

10. **(a)** A scalar $\mathbf{u} \cdot \mathbf{v}$ times a vector $\mathbf{w}$. **(b)** A scalar $\mathbf{u} \cdot \mathbf{v}$ times a scalar $\mathbf{v} \cdot \mathbf{w}$.

(c) A scalar $\mathbf{u} \cdot \mathbf{v}$ plus a scalar k. **(d)** A dot product of a vector $k\mathbf{u}$ with a vector $\mathbf{v}$.

11. (b): $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (6\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot ((2\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}) + (\mathbf{i} + \mathbf{j} - 3\mathbf{k})) = (6\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (3\mathbf{i} + 8\mathbf{j} + \mathbf{k}) = 12$; $\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = (6\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}) + (6\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 13 - 1 = 12$. (c): $k(\mathbf{u} \cdot \mathbf{v}) = -5(13) = -65$; $(k\mathbf{u}) \cdot \mathbf{v} = (-30\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}) \cdot (2\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}) = -65$; $\mathbf{u} \cdot (k\mathbf{v}) = (6\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (-10\mathbf{i} - 35\mathbf{j} - 20\mathbf{k}) = -65$.

12. **(a)** $\langle 1,2\rangle \cdot (\langle 28,-14\rangle + \langle 6,0\rangle) = \langle 1,2\rangle \cdot \langle 34,-14\rangle = 6$. **(b)** $\|6\mathbf{w}\| = 6\|\mathbf{w}\| = 36$. **(c)** $24\sqrt{5}$ **(d)** $24\sqrt{5}$

13. $\overrightarrow{AB} \cdot \overrightarrow{AP} = [2\mathbf{i} + \mathbf{j} + 2\mathbf{k}] \cdot [(r-1)\mathbf{i} + (r+1)\mathbf{j} + (r-3)\mathbf{k}] = 2(r-1) + (r+1) + 2(r-3) = 5r - 7 = 0, r = 7/5$.

14. By inspection, $3\mathbf{i}-4\mathbf{j}$ is orthogonal to and has the same length as $4\mathbf{i}+3\mathbf{j}$, so $\mathbf{u}_1=(4\mathbf{i}+3\mathbf{j})+(3\mathbf{i}-4\mathbf{j})=7\mathbf{i}-\mathbf{j}$ and $\mathbf{u}_2=(4\mathbf{i}+3\mathbf{j})+(-1)(3\mathbf{i}-4\mathbf{j})=\mathbf{i}+7\mathbf{j}$ each make an angle of 45° with $4\mathbf{i}+3\mathbf{j}$; unit vectors in the directions of $\mathbf{u}_1$ and $\mathbf{u}_2$ are $(7\mathbf{i}-\mathbf{j})/\sqrt{50}$ and $(\mathbf{i}+7\mathbf{j})/\sqrt{50}$.

15. (a) $\|\mathbf{v}\|=\sqrt{3}$, so $\cos\alpha=\cos\beta=1/\sqrt{3}$, $\cos\gamma=-1/\sqrt{3}$, $\alpha=\beta\approx 55°$, $\gamma\approx 125°$.

(b) $\|\mathbf{v}\|=3$, so $\cos\alpha=2/3$, $\cos\beta=-2/3$, $\cos\gamma=1/3$, $\alpha\approx 48°$, $\beta\approx 132°$, $\gamma\approx 71°$.

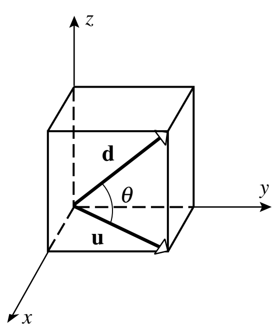
16. (a) $\|\mathbf{v}\|=7$, so $\cos\alpha=3/7$, $\cos\beta=-2/7$, $\cos\gamma=-6/7$, $\alpha\approx 65°$, $\beta\approx 107°$, $\gamma\approx 149°$.

(b) $\|\mathbf{v}\|=5$, so $\cos\alpha=3/5$, $\cos\beta=0$, $\cos\gamma=-4/5$, $\alpha\approx 53°$, $\beta=90°$, $\gamma\approx 143°$.

17. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\dfrac{v_1^2}{\|\mathbf{v}\|^2}+\dfrac{v_2^2}{\|\mathbf{v}\|^2}+\dfrac{v_3^2}{\|\mathbf{v}\|^2}=\left(v_1^2+v_2^2+v_3^2\right)/\|\mathbf{v}\|^2=\|\mathbf{v}\|^2/\|\mathbf{v}\|^2=1.$

18. Let $\mathbf{v}=\langle x,y,z\rangle$, then $x=\sqrt{x^2+y^2}\cos\theta$, $y=\sqrt{x^2+y^2}\sin\theta$, $\sqrt{x^2+y^2}=\|\mathbf{v}\|\cos\lambda$, and $z=\|\mathbf{v}\|\sin\lambda$, so $x/\|\mathbf{v}\|=\cos\theta\cos\lambda$, $y/\|\mathbf{v}\|=\sin\theta\cos\lambda$, and $z/\|\mathbf{v}\|=\sin\lambda$.

19. (a) Let k be the length of an edge and introduce a coordinate system as shown in the figure, then $\mathbf{d}=\langle k,k,k\rangle$, $\mathbf{u}=\langle k,k,0\rangle$, $\cos\theta=\dfrac{\mathbf{d}\cdot\mathbf{u}}{\|\mathbf{d}\|\,\|\mathbf{u}\|}=\dfrac{2k^2}{\left(k\sqrt{3}\right)\left(k\sqrt{2}\right)}=2/\sqrt{6}$, so $\theta=\cos^{-1}(2/\sqrt{6})\approx 35°$.

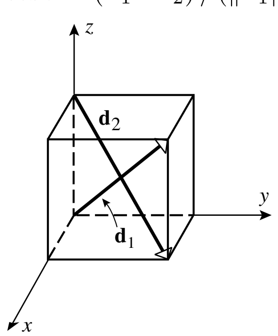


(b) $\mathbf{v}=\langle -k,0,k\rangle, \cos\theta=\dfrac{\mathbf{d}\cdot\mathbf{v}}{\|\mathbf{d}\|\,\|\mathbf{v}\|}=0$, so $\theta=\pi/2$ radians.

20. Let $\mathbf{u}_1=\|\mathbf{u}_1\|\langle\cos\alpha_1,\cos\beta_1,\cos\gamma_1\rangle, \mathbf{u}_2=\|\mathbf{u}_2\|\langle\cos\alpha_2,\cos\beta_2,\cos\gamma_2\rangle$, $\mathbf{u}_1$ and $\mathbf{u}_2$ are perpendicular if and only if $\mathbf{u}_1\cdot\mathbf{u}_2=0$ so $\|\mathbf{u}_1\|\,\|\mathbf{u}_2\|(\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2)=0, \cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=0$.

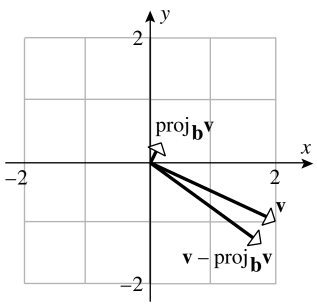
21. $\cos\alpha=\dfrac{\sqrt{3}}{2}\dfrac{1}{2}=\dfrac{\sqrt{3}}{4}$, $\cos\beta=\dfrac{\sqrt{3}}{2}\dfrac{\sqrt{3}}{2}=\dfrac{3}{4}$, $\cos\gamma=\dfrac{1}{2}$; $\alpha\approx 64°,\beta\approx 41°,\gamma=60°$.

22. With the cube as shown in the diagram, and a the length of each edge, $\mathbf{d}_1=a\mathbf{i}+a\mathbf{j}+a\mathbf{k},\mathbf{d}_2=a\mathbf{i}+a\mathbf{j}-a\mathbf{k}$, $\cos\theta=(\mathbf{d}_1\cdot\mathbf{d}_2)/(\|\mathbf{d}_1\|\,\|\mathbf{d}_2\|)=1/3,\theta\approx 71°$.

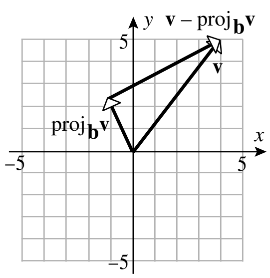


23. Take **i**, **j**, and **k** along adjacent edges of the box, then $10\mathbf{i}+15\mathbf{j}+25\mathbf{k}$ is along a diagonal, and a unit vector in this direction is $\dfrac{2}{\sqrt{38}}\mathbf{i}+\dfrac{3}{\sqrt{38}}\mathbf{j}+\dfrac{5}{\sqrt{38}}\mathbf{k}$. The direction cosines are $\cos\alpha = 2/\sqrt{38}$, $\cos\beta = 3/\sqrt{38}$, and $\cos\gamma = 5/\sqrt{38}$ so $\alpha \approx 71°$, $\beta \approx 61°$, and $\gamma \approx 36°$.

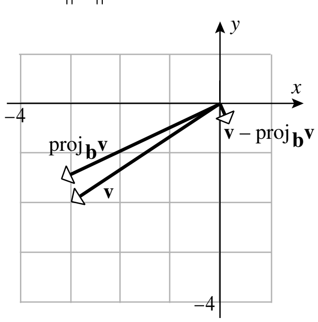
24. **(a)** $\dfrac{\mathbf{b}}{\|\mathbf{b}\|} = \langle 3/5, 4/5\rangle$, so $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle 6/25, 8/25\rangle$ and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = \langle 44/25, -33/25\rangle$.



(b) $\dfrac{\mathbf{b}}{\|\mathbf{b}\|} = \langle 1/\sqrt{5}, -2/\sqrt{5}\rangle$, so $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle -6/5, 12/5\rangle$ and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = \langle 26/5, 13/5\rangle$.



(c) $\dfrac{\mathbf{b}}{\|\mathbf{b}\|} = \langle 2/\sqrt{5}, 1/\sqrt{5}\rangle$, so $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle -16/5, -8/5\rangle$ and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = \langle 1/5, -2/5\rangle$.



25. **(a)** $\dfrac{\mathbf{b}}{\|\mathbf{b}\|} = \langle 1/3, 2/3, 2/3\rangle$, so $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle 2/3, 4/3, 4/3\rangle$ and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = \langle 4/3, -7/3, 5/3\rangle$.

(b) $\dfrac{\mathbf{b}}{\|\mathbf{b}\|} = \langle 2/7, 3/7, -6/7\rangle$, so $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle -74/49, -111/49, 222/49\rangle$ and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = \langle 270/49, 62/49, 121/49\rangle$.

26. **(a)** $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle -1, -1\rangle$, so $\mathbf{v} = \langle -1, -1\rangle + \langle 3, -3\rangle$.

(b) $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle 16/5, 0, -8/5\rangle$, so $\mathbf{v} = \langle 16/5, 0, -8/5\rangle + \langle -1/5, 1, -2/5\rangle$.

(c) $\mathbf{v} = -2\mathbf{b} + \mathbf{0}$.

27. **(a)** $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle 1, 1\rangle$, so $\mathbf{v} = \langle 1, 1\rangle + \langle -4, 4\rangle$.

(b) $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle 0, -8/5, 4/5\rangle$, so $\mathbf{v} = \langle 0, -8/5, 4/5\rangle + \langle -2, 13/5, 26/5\rangle$.

(c) $\mathbf{v}\cdot \mathbf{b} = 0$, hence $\text{proj}_{\mathbf{b}}\mathbf{v} = \mathbf{0}, \mathbf{v} = \mathbf{0} + \mathbf{v}$.

28. False, for example $\mathbf{a} = \langle 1, 2\rangle, \mathbf{b} = \langle -1, 0\rangle, \mathbf{c} = \langle 5, -3\rangle$.

29. True, because $(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \neq 0$.

30. True, by Theorem 11.3.3, $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = 1\cdot\|\mathbf{v}\|\cdot(\pm 1) = \pm\|\mathbf{v}\|$.

31. True, $\text{proj}_{\mathbf{b}}(\mathbf{v}) = \dfrac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}$ is a scalar multiple of the vector $\mathbf{b}$ and is therefore parallel to $\mathbf{b}$.

32. $\overrightarrow{AP} = -\mathbf{i} + 3\mathbf{j}, \overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j}, \|\text{proj}_{\overrightarrow{AB}}\overrightarrow{AP}\| = |\overrightarrow{AP}\cdot\overrightarrow{AB}|/\|\overrightarrow{AB}\| = 9/5$, $\|\overrightarrow{AP}\| = \sqrt{10}$, $\sqrt{10 - 81/25} = 13/5$.

33. $\overrightarrow{AP} = -4\mathbf{i} + 2\mathbf{k}$, $\overrightarrow{AB} = -3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, $\|\text{proj}_{\overrightarrow{AB}}\overrightarrow{AP}\| = |\overrightarrow{AP}\cdot\overrightarrow{AB}|/\|\overrightarrow{AB}\| = 4/\sqrt{29}$. $\|\overrightarrow{AP}\| = \sqrt{20}$, $\sqrt{20 - 16/29} = \sqrt{564/29}$.

34. Let $\mathbf{e}_1 = -\langle\cos 27°, \sin 27°\rangle$ and $\mathbf{e}_2 = \langle \sin 27°, -\cos 27°\rangle$ be the forces parallel to and perpendicular to the slide, and let $\mathbf{F}$ be the downward force of gravity on the child. Then $\|\mathbf{F}\| = 34(9.8) = 333.2$ N, and $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = (\mathbf{F}\cdot\ \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{F}\cdot\ \mathbf{e}_2)\mathbf{e}_2$. The force parallel to the slide is therefore $\|\mathbf{F}\|\cos 63° \approx 151.27$ N, and the force against the slide is $\|\mathbf{F}\|\cos 27° \approx 296.88$ N, so it takes a force of 151.27 N to prevent the child from sliding.

35. Let x denote the magnitude of the force in the direction of $\mathbf{Q}$. Then the force $\mathbf{F}$ acting on the child is $\mathbf{F} = x\mathbf{i} - 333.2\mathbf{j}$. Let $\mathbf{e}_1 = -\langle\cos 27°, \sin 27°\rangle$ and $\mathbf{e}_2 = \langle\sin 27°, -\cos 27°\rangle$ be the unit vectors in the directions along and against the slide. Then the component of $\mathbf{F}$ in the direction of $\mathbf{e}_1$ is $\mathbf{F}\cdot\ \mathbf{e}_1 = -x\cos 27° + 333.2\sin 27°$ and the child is prevented from sliding down if this quantity is negative, i.e. $x > 333.2\tan 27° \approx 169.77$ N.

36. We will obtain the work in two different ways. First, it is simply $4\cdot 151.27 = 605.08$ J. (Force times displacement.) Second, it is the same as the change in potential energy, so it is $mgh = 34\cdot 9.8\cdot 4\sin(27°) = 605.08$ J.

37. $W = \mathbf{F}\cdot 15\mathbf{i} = 15\cdot 50\cos 60° = 375$ ft $\cdot$ lb.

38. Let P and Q be the points (1,3) and (4,7) then $\overrightarrow{PQ} = 3\mathbf{i} + 4\mathbf{j}$ so $W = \mathbf{F}\cdot\ \overrightarrow{PQ} = -12$ ft $\cdot$ lb.

39. $W = \mathbf{F}\cdot(15/\sqrt{3})(\mathbf{i}+\mathbf{j}+\mathbf{k}) = -15/\sqrt{3}$ N $\cdot$ m $= -5\sqrt{3}$ J.

40. $W = \mathbf{F}\cdot\ \overrightarrow{PQ} = \|\mathbf{F}\|\,\|\overrightarrow{PQ}\|\cos 45° = (500)(100)\left(\sqrt{2}/2\right) = 25{,}000\sqrt{2}$ N $\cdot$ m $= 25{,}000\sqrt{2}$ J.

41. $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are vectors along the diagonals, $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ so $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = 0$ if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

42. The diagonals have lengths $\|\mathbf{u}+\mathbf{v}\|$ and $\|\mathbf{u}-\mathbf{v}\|$ but $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$, and $\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$. If the parallelogram is a rectangle then $\mathbf{u}\cdot\mathbf{v} = 0$ so $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2$; the diagonals are equal. If the diagonals are equal, then $4\mathbf{u}\cdot\mathbf{v} = 0$, $\mathbf{u}\cdot\mathbf{v} = 0$ so $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and hence the parallelogram is a rectangle.

43. $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$ and $\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$, add to get $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$. The sum of the squares of the lengths of the diagonals of a parallelogram is equal to twice the sum of the squares of the lengths of the sides.

44. $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$ and $\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$, subtract to get $\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = 4\mathbf{u}\cdot\mathbf{v}$, the result follows by dividing both sides by 4.

45. $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ so $\mathbf{v}\cdot\mathbf{v}_i = c_i\mathbf{v}_i\cdot\mathbf{v}_i$ because $\mathbf{v}_i\cdot\mathbf{v}_j = 0$ if $i \neq j$, thus $\mathbf{v}\cdot\mathbf{v}_i = c_i\|\mathbf{v}_i\|^2, c_i = \mathbf{v}\cdot\mathbf{v}_i/\|\mathbf{v}_i\|^2$ for $i = 1, 2, 3$.

46. $\mathbf{v}_1\cdot\mathbf{v}_2 = \mathbf{v}_1\cdot\mathbf{v}_3 = \mathbf{v}_2\cdot\mathbf{v}_3 = 0$ so they are mutually perpendicular. Let $\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, then $c_1 = \dfrac{\mathbf{v}\cdot\mathbf{v}_1}{\|\mathbf{v}_1\|^2} = \dfrac{3}{7}$, $c_2 = \dfrac{\mathbf{v}\cdot\mathbf{v}_2}{\|\mathbf{v}_2\|^2} = -\dfrac{1}{3}$, and $c_3 = \dfrac{\mathbf{v}\cdot\mathbf{v}_3}{\|\mathbf{v}_3\|^2} = \dfrac{1}{21}$.

47. **(a)** $\mathbf{u} = x\mathbf{i} + (x^2+1)\mathbf{j}, \mathbf{v} = x\mathbf{i} - (x+1)\mathbf{j}, \theta = \cos^{-1}[(\mathbf{u}\cdot\mathbf{v})/(\|\mathbf{u}\|\|\mathbf{v}\|)]$. Use a CAS to solve $d\theta/dx = 0$ to find that the minimum value of θ occurs when $x \approx -0.53567$ so the minimum angle is about $40°$. NB: Since $\cos^{-1} u$ is a decreasing function of u, it suffices to maximize $(\mathbf{u}\cdot\mathbf{v})/(\|\mathbf{u}\|\|\mathbf{v}\|)$, or, what is easier, its square.

(b) Solve $\mathbf{u}\cdot\mathbf{v} = 0$ for x to get $x \approx -0.682328$.

48. **(a)** $\mathbf{u} = \cos\theta_1\mathbf{i} \pm \sin\theta_1\mathbf{j}, \mathbf{v} = \pm\sin\theta_2\mathbf{j} + \cos\theta_2\mathbf{k}$, $\cos\theta = \mathbf{u}\cdot\mathbf{v} = \pm\sin\theta_1\sin\theta_2$.

(b) $\cos\theta = \pm\sin^2 45° = \pm 1/2,\ \theta = 60°$.

(c) Let $\theta(t) = \cos^{-1}(\sin t\sin 2t)$; solve $\theta'(t) = 0$ for t to find that $\theta_{\max} \approx 140°$ (reject, since θ is acute) when $t \approx 2.186276$ and that $\theta_{\min} \approx 40°$ when $t \approx 0.955317$; for $\theta_{\max}$ check the endpoints $t = 0, \pi/2$ to obtain $\theta_{\max} = \cos^{-1}(0) = \pi/2$.

49. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle, \mathbf{v} = \langle v_1, v_2, v_3\rangle, \mathbf{w} = \langle w_1, w_2, w_3\rangle$. Then $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = u_1(v_1+w_1)+u_2(v_2+w_2)+u_3(v_3+w_3) = u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + u_3v_3 + u_3w_3 = u_1v_1 + u_2v_2 + u_3v_3 + u_1w_1 + u_2w_2 + u_3w_3 = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$, also $\mathbf{0}\cdot\mathbf{v} = 0\cdot v_1 + 0\cdot v_2 + 0\cdot v_3 = 0$.

Exercise Set 11.4

1. **(a)** $\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -\mathbf{j}+\mathbf{k}$.

(b) $\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = (\mathbf{i}\times\mathbf{i}) + (\mathbf{i}\times\mathbf{j}) + (\mathbf{i}\times\mathbf{k}) = -\mathbf{j}+\mathbf{k}$.

2. **(a)** $\mathbf{j}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i}-\mathbf{k}$; $\mathbf{j}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = (\mathbf{j}\times\mathbf{i}) + (\mathbf{j}\times\mathbf{j}) + (\mathbf{j}\times\mathbf{k}) = \mathbf{i}-\mathbf{k}$.

(b) $\mathbf{k}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -\mathbf{i}+\mathbf{j}$; $\mathbf{k}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = (\mathbf{k}\times\mathbf{i}) + (\mathbf{k}\times\mathbf{j}) + (\mathbf{k}\times\mathbf{k}) = \mathbf{j}-\mathbf{i}+\mathbf{0} = -\mathbf{i}+\mathbf{j}$.

3. $\langle 7, 10, 9\rangle$

4. $-\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$

5. $\langle -4, -6, -3\rangle$

6. $\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

7. **(a)** $\mathbf{v}\times\mathbf{w} = \langle -23, 7, -1\rangle, \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \langle -20, -67, -9\rangle$.

(b) $\mathbf{u}\times\mathbf{v} = \langle -10, -14, 2\rangle, (\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \langle -78, 52, -26\rangle$.

(c) $(\mathbf{u}\times\mathbf{v})\times(\mathbf{v}\times\mathbf{w}) = \langle -10, -14, 2\rangle\times\langle -23, 7, -1\rangle = \langle 0, -56, -392\rangle$.

(d) $(\mathbf{v}\times\mathbf{w})\times(\mathbf{u}\times\mathbf{v}) = \langle 0, 56, 392\rangle$.

9. $\mathbf{u}\times\mathbf{v} = (\mathbf{i}+\mathbf{j})\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \mathbf{k}-\mathbf{j}-\mathbf{k}+\mathbf{i} = \mathbf{i}-\mathbf{j}$, the direction cosines are $\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, 0$.

10. $\mathbf{u}\times\mathbf{v} = 12\mathbf{i}+30\mathbf{j}-6\mathbf{k}$, so $\pm\left(\dfrac{2}{\sqrt{30}}\mathbf{i}+\dfrac{\sqrt{5}}{\sqrt{6}}\mathbf{j}-\dfrac{1}{\sqrt{30}}\mathbf{k}\right)$.

11. $\mathbf{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \langle 1,1,-3\rangle\times\langle -1,3,-1\rangle = \langle 8,4,4\rangle$, unit vectors are $\pm\dfrac{1}{\sqrt{6}}\langle 2,1,1\rangle$.

12. A vector parallel to the yz-plane must be perpendicular to $\mathbf{i}$; $\mathbf{i}\times(\mathbf{3i}-\mathbf{j}+\mathbf{2k}) = -2\mathbf{j}-\mathbf{k}$, $\|-2\mathbf{j}-\mathbf{k}\| = \sqrt{5}$, the unit vectors are $\pm(2\mathbf{j}+\mathbf{k})/\sqrt{5}$.

13. True.

14. False; $(\mathbf{i}\times\mathbf{j})\times\mathbf{j} = -\mathbf{k}\times\mathbf{j} = \mathbf{i}$, $\mathbf{i}\times(\mathbf{j}\times\mathbf{j}) = \mathbf{0}$.

15. False; let $\mathbf{v} = \langle 2,1,-1\rangle, \mathbf{u} = \langle 1,3,-1\rangle, \mathbf{w} = \langle -5,0,2\rangle$, then $\mathbf{v}\times\mathbf{u} = \mathbf{v}\times\mathbf{w} = \langle 2,1,5\rangle$, but $\mathbf{u}\neq\mathbf{w}$.

16. True; by Theorem 11.4.6(b); if one row of a determinant is a linear combination of the other two rows, then the determinant is zero. Equivalently, if $\mathbf{u} = a\mathbf{v}+b\mathbf{w}$ then $\mathbf{u}$ lies in the plane of $\mathbf{v}$ and $\mathbf{w}$ and is thus perpendicular to their cross product.

17. $A = \|\mathbf{u}\times\mathbf{v}\| = \|-7\mathbf{i}-\mathbf{j}+3\mathbf{k}\| = \sqrt{59}$.

18. $A = \|\mathbf{u}\times\mathbf{v}\| = \|-6\mathbf{i}+4\mathbf{j}+7\mathbf{k}\| = \sqrt{101}$.

19. $A = \dfrac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \dfrac{1}{2}\|\langle -1,-5,2\rangle\times\langle 2,0,3\rangle\| = \dfrac{1}{2}\|\langle -15,7,10\rangle\| = \sqrt{374}/2$.

20. $A = \dfrac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \dfrac{1}{2}\|\langle -1,4,8\rangle\times\langle 5,2,12\rangle\| = \dfrac{1}{2}\|\langle 32,52,-22\rangle\| = 9\sqrt{13}$.

21. $(2\mathbf{i}-3\mathbf{j}+\mathbf{k})\cdot(8\mathbf{i}-20\mathbf{j}+4\mathbf{k}) = 80$.

22. $\langle 1,-2,2\rangle\cdot\langle -11,-8,12\rangle = 29$.

23. $\langle 2,1,0\rangle\cdot\langle -3,3,12\rangle = -3$.

24. $\mathbf{i}\cdot(\mathbf{i}-\mathbf{j}) = 1$.

25. $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |-16| = 16$.

26. $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |45| = 45$.

27. **(a)** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 0$, yes. **(b)** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 0$, yes. **(c)** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 245$, no.

28. **(a)** $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = -\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -3$. **(b)** $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 3$. **(c)** $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 3$.

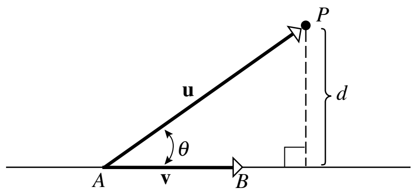
(d) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = \mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = -3$. **(e)** $(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = -3$. **(f)** $\mathbf{v}\cdot(\mathbf{w}\times\mathbf{w}) = \mathbf{v}\cdot\mathbf{0} = 0$.

29. **(a)** $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |-9| = 9$.

(b) $A = \|\mathbf{u}\times\mathbf{w}\| = \|3\mathbf{i}-8\mathbf{j}+7\mathbf{k}\| = \sqrt{122}$.

(c) $\mathbf{v}\times\mathbf{w} = -3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ is perpendicular to the plane determined by $\mathbf{v}$ and $\mathbf{w}$; let θ be the angle between $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$, then $\cos\theta = \dfrac{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}{\|\mathbf{u}\|\,\|\mathbf{v}\times\mathbf{w}\|} = \dfrac{-9}{\sqrt{14}\sqrt{14}} = -9/14$, so the acute angle ϕ that $\mathbf{u}$ makes with the plane determined by $\mathbf{v}$ and $\mathbf{w}$ is $\phi = \theta - \pi/2 = \sin^{-1}(9/14)$.

30. From the diagram, $d = \|\mathbf{u}\|\sin\theta = \dfrac{\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta}{\|\mathbf{v}\|} = \dfrac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{v}\|}$.



31. **(a)** $\mathbf{u} = \overrightarrow{AP} = -4\mathbf{i}+2\mathbf{k}$, $\mathbf{v} = \overrightarrow{AB} = -3\mathbf{i}+2\mathbf{j}-4\mathbf{k}$, $\mathbf{u}\times\mathbf{v} = -4\mathbf{i}-22\mathbf{j}-8\mathbf{k}$; distance $= \|\mathbf{u}\times\mathbf{v}\|/\|\mathbf{v}\| = 2\sqrt{141/29}$.

(b) $\mathbf{u} = \overrightarrow{AP} = 2\mathbf{i}+2\mathbf{j}$, $\mathbf{v} = \overrightarrow{AB} = -2\mathbf{i}+\mathbf{j}$, $\mathbf{u}\times\mathbf{v} = 6\mathbf{k}$; distance $= \|\mathbf{u}\times\mathbf{v}\|/\|\mathbf{v}\| = 6/\sqrt{5}$.

32. Take $\mathbf{v}$ and $\mathbf{w}$ as sides of the (triangular) base, then area of base $= \dfrac{1}{2}\|\mathbf{v}\times\mathbf{w}\|$ and height $= \|\mathrm{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\| = \dfrac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|}$ so $V = \dfrac{1}{3}$ (area of base) (height) $= \dfrac{1}{6}|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$.

33. $\overrightarrow{PQ} = \langle 3,-1,-3\rangle$, $\overrightarrow{PR} = \langle 2,-2,1\rangle$, $\overrightarrow{PS} = \langle 4,-4,3\rangle$, $V = \dfrac{1}{6}|\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS})| = \dfrac{1}{6}|-4| = 2/3$.

34. **(a)** $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = -\dfrac{23}{49}$.

(b) $\sin\theta = \dfrac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{\|36\mathbf{i}-24\mathbf{j}\|}{49} = \dfrac{12\sqrt{13}}{49}$.

(c) $\dfrac{23^2}{49^2} + \dfrac{144\cdot 13}{49^2} = \dfrac{2401}{49^2} = 1$.

35. Since $\overrightarrow{AC}\cdot(\overrightarrow{AB}\times\overrightarrow{AD}) = \overrightarrow{AC}\cdot(\overrightarrow{AB}\times\overrightarrow{CD}) + \overrightarrow{AC}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) = \mathbf{0}+\mathbf{0} = \mathbf{0}$, the volume of the parallelepiped determined by $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ is zero, thus A, B, C, and D are coplanar (lie in the same plane). Since $\overrightarrow{AB}\times\overrightarrow{CD} \neq \mathbf{0}$, the lines are not parallel. Hence they must intersect.

36. The points P lie on the plane determined by A, B and C.

37. From Theorems 11.3.3 and 11.4.5a it follows that $\sin\theta = \cos\theta$, so $\theta = \pi/4$.

38. $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$.

39. **(a)** $\mathbf{F} = 10\mathbf{j}$ and $\overrightarrow{PQ} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, so the vector moment of $\mathbf{F}$ about P is $\overrightarrow{PQ}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 10 & 0 \end{vmatrix} = -10\mathbf{i}+10\mathbf{k}$, and the scalar moment is $10\sqrt{2}$ lb·ft. The direction of rotation of the cube about P is counterclockwise looking along $\overrightarrow{PQ}\times\mathbf{F} = -10\mathbf{i}+10\mathbf{k}$ toward its initial point.

(b) $\mathbf{F} = 10\mathbf{j}$ and $\overrightarrow{PQ} = \mathbf{j}+\mathbf{k}$, so the vector moment of $\mathbf{F}$ about P is $\overrightarrow{PQ}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 0 & 10 & 0 \end{vmatrix} = -10\mathbf{i}$, and the scalar moment is 10 lb·ft. The direction of rotation of the cube about P is counterclockwise looking along $-10\mathbf{i}$ toward its initial point.

(c) $\mathbf{F} = 10\mathbf{j}$ and $\overrightarrow{PQ} = \mathbf{j}$, so the vector moment of $\mathbf{F}$ about P is $\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 10 & 0 \end{vmatrix} = \mathbf{0}$, and the scalar moment is 0 lb·ft. Since the force is parallel to the direction of motion, there is no rotation about P.

40. (a) $\mathbf{F} = \dfrac{1000}{\sqrt{2}}(-\mathbf{i}+\mathbf{k})$ and $\overrightarrow{PQ} = 2\mathbf{j}-\mathbf{k}$, so the vector moment of $\mathbf{F}$ about P is $\overrightarrow{PQ} \times \mathbf{F} = 500\sqrt{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{vmatrix} =$ $500\sqrt{2}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$, and the scalar moment is $1500\sqrt{2}$ N·m.

(b) The direction angles of the vector moment of $\mathbf{F}$ about the point P are $\cos^{-1}(2/3) \approx 48°, \cos^{-1}(1/3) \approx 71°$, and $\cos^{-1}(2/3) \approx 48°$.

41. Take the center of the bolt as the origin of the plane. Then $\mathbf{F}$ makes an angle 72° with the positive x-axis, so $\mathbf{F} =$ $200\cos 72°\mathbf{i} + 200\sin 72°\mathbf{j}$ and $\overrightarrow{PQ} = 0.2\,\mathbf{i} + 0.03\,\mathbf{j}$. The scalar moment is given by $\left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.2 & 0.03 & 0 \\ 200\cos 72° & 200\sin 72° & 0 \end{matrix} \right\| =$ $\left| 40\dfrac{1}{4}(\sqrt{5}-1) - 6\dfrac{1}{4}\sqrt{10+2\sqrt{5}} \right| \approx 36.1882$ N·m.

42. Part (b): let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$; show that $\mathbf{u} \times (\mathbf{v}+\mathbf{w})$ and $(\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w})$ are the same.

Part (c): $(\mathbf{u}+\mathbf{v}) \times \mathbf{w} = -[\mathbf{w} \times (\mathbf{u}+\mathbf{v})]$ (from Part (a)) $= -[(\mathbf{w}\times\mathbf{u}) + (\mathbf{w}\times\mathbf{v})]$ (from Part (b)) $= (\mathbf{u}\times\mathbf{w}) + (\mathbf{v}\times\mathbf{w})$ (from Part (a)).

43. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$; show that $k(\mathbf{u}\times\mathbf{v})$, $(k\mathbf{u})\times\mathbf{v}$, and $\mathbf{u}\times(k\mathbf{v})$ are all the same; Part (e) is proved in a similar fashion.

44. Suppose the first two rows are interchanged. Then by definition,

$$\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} - b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + b_3\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = b_1(a_2c_3 - a_3c_2) - b_2(a_1c_3 - a_3c_1) + b_3(a_1c_2 - a_2c_1),$$

which is the negative of the right hand side of (2) after expansion. If two other rows were to be exchanged, a similar proof would hold. Finally, suppose Δ were a determinant with two identical rows. Then the value is unchanged if we interchange those two rows, yet $\Delta = -\Delta$ by Part (b) of Theorem 12.4.1. Hence $\Delta = -\Delta$, $\Delta = 0$.

45. $-8\mathbf{i} - 8\mathbf{k}, -8\mathbf{i} - 20\mathbf{j} + 2\mathbf{k}$. In the first triple, $\mathbf{u}$ is 'outer' because it's not inside the parentheses, $\mathbf{v}$ is 'adjacent' because it lies next to $\mathbf{u}$ and $\mathbf{w}$ (typographically speaking), and $\mathbf{w}$ is 'remote' because it's inside the parentheses far from $\mathbf{u}$. In the second triple product, $\mathbf{w}$ is 'outer', $\mathbf{u}$ is 'remote' and $\mathbf{v}$ is 'adjacent'.

46. (a) From the first formula in Exercise 45, it follows that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$ and hence lies in the plane determined by them, and from the second formula it follows that $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and hence lies in their plane.

(b) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ is orthogonal to $\mathbf{v}\times\mathbf{w}$ and hence lies in the plane of $\mathbf{v}$ and $\mathbf{w}$; similarly for $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$.

47. (a) Replace $\mathbf{u}$ with $\mathbf{a}\times\mathbf{b}$, $\mathbf{v}$ with $\mathbf{c}$, and $\mathbf{w}$ with $\mathbf{d}$ in the first formula of Exercise 41.

(b) From the second formula of Exercise 41, $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{b}\times\mathbf{c})\times\mathbf{a} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b} = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a} + (\mathbf{a}\cdot\mathbf{b})\mathbf{c} -$ $(\mathbf{a}\cdot\mathbf{c})\mathbf{b} + (\mathbf{b}\cdot\mathbf{c})\mathbf{a} - (\mathbf{b}\cdot\mathbf{a})\mathbf{c} = \mathbf{0}$.

48. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ lie in the same plane then $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ are parallel so $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=\mathbf{0}$.

49. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors from a point on the curve to the points $(2,-1,0)$ and $(3,2,2)$, respectively. Then $\mathbf{u}=(2-x)\mathbf{i}+(-1-\ln x)\mathbf{j}$ and $\mathbf{v}=(3-x)\mathbf{i}+(2-\ln x)\mathbf{j}+2\mathbf{k}$. The area of the triangle is given by $A=(1/2)\|\mathbf{u}\times\mathbf{v}\|$; solve $dA/dx=0$ for x to get $x=2.091581$. The minimum area is 1.887850.

50. $\overrightarrow{PQ'}\times\mathbf{F}=\overrightarrow{PQ}\times\mathbf{F}+\overrightarrow{QQ'}\times\mathbf{F}=\overrightarrow{PQ}\times\mathbf{F}$, since $\mathbf{F}$ and $\overrightarrow{QQ'}$ are parallel.

Exercise Set 11.5

In many of the exercises in this section other answers are also possible.

1. (a) L_1: $P(1,0),\mathbf{v}=\mathbf{j},x=1,y=t$; L_2: $P(0,1),\mathbf{v}=\mathbf{i},x=t,y=1$; L_3: $P(0,0),\mathbf{v}=\mathbf{i}+\mathbf{j},x=t,y=t$.

(b) L_1: $P(1,1,0),\mathbf{v}=\mathbf{k},x=1,y=1,z=t$; L_2: $P(0,1,1),\mathbf{v}=\mathbf{i},x=t,y=1,z=1$; L_3: $P(1,0,1),\mathbf{v}=\mathbf{j},x=1,y=t,z=1$; L_4: $P(0,0,0),\mathbf{v}=\mathbf{i}+\mathbf{j}+\mathbf{k},x=t,y=t,z=t$.

2. (a) L_1: $x=t,y=1,0\le t\le 1$; L_2: $x=1,y=t,0\le t\le 1$; L_3: $x=t,y=t,0\le t\le 1$.

(b) L_1: $x=1,y=1,z=t,0\le t\le 1$; L_2: $x=t,y=1,z=1,0\le t\le 1$; L_3: $x=1,y=t,z=1,0\le t\le 1$; L_4: $x=t,y=t,z=t,0\le t\le 1$.

3. (a) $\overrightarrow{P_1P_2}=\langle 2,3\rangle$ so $x=3+2t$, $y=-2+3t$ for the line; for the line segment add the condition $0\le t\le 1$.

(b) $\overrightarrow{P_1P_2}=\langle -3,6,1\rangle$ so $x=5-3t$, $y=-2+6t$, $z=1+t$ for the line; for the line segment add the condition $0\le t\le 1$.

4. (a) $\overrightarrow{P_1P_2}=\langle -3,-5\rangle$ so $x=-3t$, $y=1-5t$ for the line; for the line segment add the condition $0\le t\le 1$.

(b) $\overrightarrow{P_1P_2}=\langle 0,0,-3\rangle$ so $x=-1$, $y=3$,$z=5-3t$ for the line; for the line segment add the condition $0\le t\le 1$.

5. (a) $x=2+t,y=-3-4t$. **(b)** $x=t,y=-t,z=1+t$.

6. (a) $x=3+2t,y=-4+t$. **(b)** $x=-1-t,y=3t,z=2$.

7. (a) $\mathbf{r}_0=2\mathbf{i}-\mathbf{j}$ so $P(2,-1)$ is on the line, and $\mathbf{v}=4\mathbf{i}-\mathbf{j}$ is parallel to the line.

(b) At $t=0,P(-1,2,4)$ is on the line, and $\mathbf{v}=5\mathbf{i}+7\mathbf{j}-8\mathbf{k}$ is parallel to the line.

8. (a) At $t=0,P(-1,5)$ is on the line, and $\mathbf{v}=2\mathbf{i}+3\mathbf{j}$ is parallel to the line.

(b) $\mathbf{r}_0=\mathbf{i}+\mathbf{j}-2\mathbf{k}$ so $P(1,1,-2)$ is on the line, and $\mathbf{v}=\mathbf{j}$ is parallel to the line.

9. (a) $\langle x,y\rangle=\langle -3,4\rangle+t\langle 1,5\rangle;\mathbf{r}=-3\mathbf{i}+4\mathbf{j}+t(\mathbf{i}+5\mathbf{j})$.

(b) $\langle x,y,z\rangle=\langle 2,-3,0\rangle+t\langle -1,5,1\rangle;\mathbf{r}=2\mathbf{i}-3\mathbf{j}+t(-\mathbf{i}+5\mathbf{j}+\mathbf{k})$.

10. (a) $\langle x,y\rangle=\langle 0,-2\rangle+t\langle 1,1\rangle;\mathbf{r}=-2\mathbf{j}+t(\mathbf{i}+\mathbf{j})$.

(b) $\langle x,y,z\rangle=\langle 1,-7,4\rangle+t\langle 1,3,-5\rangle;\mathbf{r}=\mathbf{i}-7\mathbf{j}+4\mathbf{k}+t(\mathbf{i}+3\mathbf{j}-5\mathbf{k})$.

11. False; $x=t,y=0,z=0$ is not parallel to $x=0,y=1+t,z=0$, nor do they intersect.

12. True: $\mathbf{v_0}$ is parallel to L_0 is parallel to L_1 is parallel to $\mathbf{v_1}$, so $\mathbf{v_0}$ is parallel to $\mathbf{v_1}$. Since they are nonzero vectors, each is a scalar multiple of the other.

13. False; if (x, y, z) is the point of intersection then there exists t_0 such that $x = x_0 + a_0t_0, y = y_0 + b_0t_0, z = z_0 + c_0t_0$ and there exists t_1 such that $x = x_0 + a_0t_1, y = y_0 + b_0t_1, z = z_0 + c_0t_1$, but it is not necessary that $t_0 = t_1$.

14. True; for some $t_0, 0 = x_0 + a_0t_0, 0 = y_0 + b_0t_0, 0 = z_0 + c_0t_0$, so $\langle x_0, y_0, z_0\rangle = -t_0\langle a_0, b_0, c_0\rangle$.

15. $x = -5 + 2t$, $y = 2 - 3t$.

16. $x = t$, $y = 3 - 2t$.

17. $2x + 2yy' = 0$, $y' = -x/y = -(3)/(-4) = 3/4$, $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$; $x = 3 + 4t$, $y = -4 + 3t$.

18. $y' = 2x = 2(-2) = -4$, $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$; $x = -2 + t$, $y = 4 - 4t$.

19. $x = -1 + 3t$, $y = 2 - 4t$, $z = 4 + t$.

20. $x = 2 - t$, $y = -1 + 2t$, $z = 5 + 7t$.

21. The line is parallel to the vector $\langle 2, -1, 2\rangle$ so $x = -2 + 2t$, $y = -t$, $z = 5 + 2t$.

22. The line is parallel to the vector $\langle 1, 1, 0\rangle$ so $x = t$, $y = t$, $z = 0$.

23. **(a)** $y = 0, 2 - t = 0, t = 2, x = 7$. **(b)** $x = 0, 1 + 3t = 0, t = -1/3, y = 7/3$.

(c) $y = x^2, 2 - t = (1 + 3t)^2, 9t^2 + 7t - 1 = 0, t = \dfrac{-7 \pm \sqrt{85}}{18}, x = \dfrac{-1 \pm \sqrt{85}}{6}, y = \dfrac{43 \mp \sqrt{85}}{18}$.

24. $(4t)^2 + (3t)^2 = 25, 25t^2 = 25, t = \pm 1$, the line intersects the circle at $\pm\langle 4, 3\rangle$.

25. **(a)** $z = 0$ when $t = 3$ so the point is $(-2, 10, 0)$. **(b)** $y = 0$ when $t = -2$ so the point is $(-2, 0, -5)$.

(c) x is always -2 so the line does not intersect the yz-plane.

26. **(a)** $z = 0$ when $t = 4$ so the point is $(7, 7, 0)$. **(b)** $y = 0$ when $t = -3$ so the point is $(-7, 0, 7)$.

(c) $x = 0$ when $t = 1/2$ so the point is $(0, 7/2, 7/2)$.

27. $(1 + t)^2 + (3 - t)^2 = 16$, $t^2 - 2t - 3 = 0$, $(t + 1)(t - 3) = 0$; $t = -1, 3$. The points of intersection are $(0, 4, -2)$ and $(4, 0, 6)$.

28. $2(3t) + 3(-1 + 2t) = 6$, $12t = 9$; $t = 3/4$. The point of intersection is $(5/4, 9/4, 1/2)$.

29. The lines intersect if we can find values of t_1 and t_2 that satisfy the equations $2 + t_1 = 2 + t_2$, $2 + 3t_1 = 3 + 4t_2$, and $3 + t_1 = 4 + 2t_2$. Solutions of the first two of these equations are $t_1 = -1$, $t_2 = -1$ which also satisfy the third equation so the lines intersect at $(1, -1, 2)$.

30. Solve the equations $-1 + 4t_1 = -13 + 12t_2$, $3 + t_1 = 1 + 6t_2$, and $1 = 2 + 3t_2$. The third equation yields $t_2 = -1/3$ which when substituted into the first and second equations gives $t_1 = -4$ in both cases; the lines intersect at $(-17, -1, 1)$.

31. The lines are parallel, respectively, to the vectors $\langle 7, 1, -3\rangle$ and $\langle -1, 0, 2\rangle$. These vectors are not parallel so the lines are not parallel. The system of equations $1 + 7t_1 = 4 - t_2$, $3 + t_1 = 6$, and $5 - 3t_1 = 7 + 2t_2$ has no solution so the lines do not intersect.

32. The vectors $\langle 8,-8,10\rangle$ and $\langle 8,-3,1\rangle$ are not parallel so the lines are not parallel. The lines do not intersect because the system of equations $2+8t_1=3+8t_2$, $6-8t_1=5-3t_2$, $10t_1=6+t_2$ has no solution.

33. The lines are parallel, respectively, to the vectors $\mathbf{v}_1=\langle -2,1,-1\rangle$ and $\mathbf{v}_2=\langle -4,2,-2\rangle$; $\mathbf{v}_2=2\mathbf{v}_1$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel so the lines are parallel.

34. The lines are not parallel because the vectors $\langle 3,-2,3\rangle$ and $\langle 9,-6,8\rangle$ are not parallel.

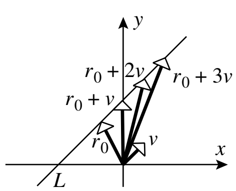
35. $\overrightarrow{P_1P_2}=\langle 3,-7,-7\rangle, \overrightarrow{P_2P_3}=\langle -9,-7,-3\rangle$; these vectors are not parallel so the points do not lie on the same line.

36. $\overrightarrow{P_1P_2}=\langle 2,-4,-4\rangle, \overrightarrow{P_2P_3}=\langle 1,-2,-2\rangle; \overrightarrow{P_1P_2}=2\ \overrightarrow{P_2P_3}$ so the vectors are parallel and the points lie on the same line.

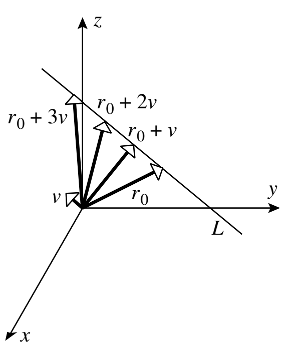
37. The point $(3,1)$ is on both lines ($t=0$ for L_1, $t=4/3$ for L_2), as well as the point $(-1,9)$ ($t=4$ for L_1, $t=0$ for L_2). An alternative method: if t_2 gives the point $\langle -1+3t_2, 9-6t_2\rangle$ on the second line, then $t_1=4-3t_2$ yields the point $\langle 3-(4-3t_2), 1+2(4-3t_2)\rangle=\langle -1+3t_2, 9-6t_2\rangle$ on the first line, so each point of L_2 is a point of L_1; the converse is shown with $t_2=(4-t_1)/3$.

38. The point $(1,-2,0)$ is on both lines ($t=0$ for L_1, $t=1/2$ for L_2), as well as the point $(4,-1,2)$ ($t=1$ for L_1, $t=0$ for L_2). An alternative method: if t_1 gives the point $\langle 1+3t_1, -2+t_1, 2t_1\rangle$ on L_1, then $t_2=(1-t_1)/2$ gives the point $\langle 4-6(1-t_1)/2, -1-2(1-t_1)/2, 2-4(1-t_1)/2\rangle=\langle 1+3t_1,-2+t_1,2t_1\rangle$ on L_2, so each point of L_1 is a point of L_2; the converse is shown with $t_1=1-2t_2$.

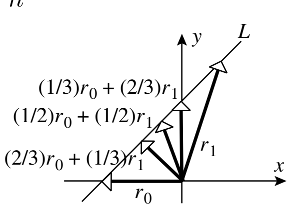
39. L passes through the tips of the vectors. $\langle x,y\rangle=\langle -1,2\rangle+t\langle 1,1\rangle$.



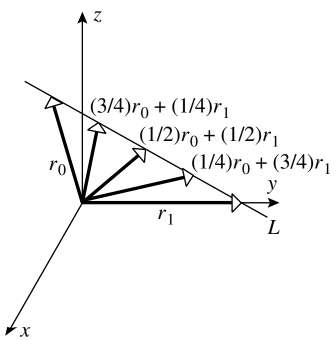
40. It passes through the tips of the vectors. $\langle x,y,z\rangle=\langle 0,2,1\rangle+t\langle 1,0,1\rangle$.



41. $\dfrac{1}{n}$ of the way from $\langle -2,0\rangle$ to $\langle 1,3\rangle$.



42. $\dfrac{1}{n}$ of the way from $\langle 2,0,4\rangle$ to $\langle 0,4,0\rangle$.



43. The line segment joining the points (1,0) and $(-3,6)$.

44. The line segment joining the points $(-2,1,4)$ and (7,1,1).

45. Let the desired point be $P(x_0, y_0)$; then $\overrightarrow{P_1P} = (2/5)\ \overrightarrow{P_1P_2}$, $\langle x_0 - 3, y_0 - 6\rangle = (2/5)\langle 5, -10\rangle = \langle 2, -4\rangle$, so $x_0 = 5, y_0 = 2$.

46. Let the desired point be $P(x_0, y_0, z_0)$, then $\overrightarrow{P_1P} = (2/3)\ \overrightarrow{P_1P_2}$, $\langle x_0 - 1, y_0 - 4, z_0 + 3\rangle = (2/3)\langle 0, 1, 2\rangle = \langle 0, 2/3, 4/3\rangle$; equate corresponding components to get $x_0 = 1$, $y_0 = 14/3$, $z_0 = -5/3$.

47. $A(3, 0, 1)$ and $B(2, 1, 3)$ are on the line, and (with the method of Exercise 11.3.32) $\overrightarrow{AP} = -5\mathbf{i} + \mathbf{j}$, $\overrightarrow{AB} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\|\text{proj}_{\overrightarrow{AB}}\ \overrightarrow{AP}\ \| = |\ \overrightarrow{AP} \cdot \overrightarrow{AB}\ |/\|\ \overrightarrow{AB}\ \| = \sqrt{6}$ and $\|\ \overrightarrow{AP}\ \| = \sqrt{26}$, so distance $= \sqrt{26 - 6} = 2\sqrt{5}$. Using the method of Exercise 11.4.30, distance $= \dfrac{\|\ \overrightarrow{AP} \times \overrightarrow{AB}\ \|}{\|\ \overrightarrow{AB}\ \|} = 2\sqrt{5}$.

48. $A(2, -1, 0)$ and $B(3, -2, 3)$ are on the line, and (with the method of Exercise 11.3.32) $\overrightarrow{AP} = -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $\overrightarrow{AB} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\|\text{proj}_{\overrightarrow{AB}}\ \overrightarrow{AP}\ \| = |\ \overrightarrow{AP} \cdot \overrightarrow{AB}\ |/\|\ \overrightarrow{AB}\ \| = \dfrac{15}{\sqrt{11}}$ and $\|\ \overrightarrow{AP}\ \| = \sqrt{35}$, so distance $= \sqrt{35 - 225/11} = 4\sqrt{10/11}$. Using the method of Exercise 11.4.30, distance $= \dfrac{\|\ \overrightarrow{AP} \times \overrightarrow{AB}\ \|}{\|\ \overrightarrow{AB}\ \|} = 4\sqrt{10/11}$.

49. The vectors $\mathbf{v}_1 = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v}_2 = 2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ are parallel to the lines, $\mathbf{v}_2 = -2\mathbf{v}_1$ so $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel. Let $t = 0$ to get the points $P(2, 0, 1)$ and $Q(1, 3, 5)$ on the first and second lines, respectively. Let $\mathbf{u} = \overrightarrow{PQ} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{v}_2 = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$; $\mathbf{u} \times \mathbf{v} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$; by the method of Exercise 30 of Section 11.4, distance $= \|\mathbf{u} \times \mathbf{v}\|/\|\mathbf{v}\| = \sqrt{35/6}$.

50. The vectors $\mathbf{v}_1 = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$ and $\mathbf{v}_2 = 3\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}$ are parallel to the lines, $\mathbf{v}_2 = (3/2)\mathbf{v}_1$ so $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel. Let $t = 0$ to get the points $P(0, 3, 2)$ and $Q(1, 0, 0)$ on the first and second lines, respectively. Let $\mathbf{u} = \overrightarrow{PQ} = \mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{v}_1 = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$; $\mathbf{u} \times \mathbf{v} = 13\mathbf{i} + \mathbf{j} + 5\mathbf{k}$, distance $= \|\mathbf{u} \times \mathbf{v}\|/\|\mathbf{v}\| = \sqrt{195/14}$ (Exercise 30, Section 11.4).

51. **(a)** The line is parallel to the vector $\langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$ so $x = x_0 + (x_1 - x_0)\,t$, $y = y_0 + (y_1 - y_0)\,t$, $z = z_0 + (z_1 - z_0)\,t$.

(b) The line is parallel to the vector $\langle a, b, c\rangle$ so $x = x_1 + at$, $y = y_1 + bt$, $z = z_1 + ct$.

52. Solve each of the given parametric equations (2) for t to get $t = (x - x_0)/a$, $t = (y - y_0)/b$, $t = (z - z_0)/c$, so (x, y, z) is on the line if and only if $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$.

53. **(a)** It passes through the point $(1, -3, 5)$ and is parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

(b) $\langle x,y,z\rangle = \langle 1+2t, -3+4t, 5+t\rangle$.

54. (a) Perpendicular, since $\langle 2,1,2\rangle \cdot \langle -1,-2,2\rangle = 0$.

(b) L_1: $\langle x,y,z\rangle = \langle 1+2t, -\frac{3}{2}+t, -1+2t\rangle$; L_2: $\langle x,y,z\rangle = \langle 4-t, 3-2t, -4+2t\rangle$.

(c) Solve simultaneously $1+2t_1 = 4-t_2, -\frac{3}{2}+t_1 = 3-2t_2, -1+2t_1 = -4+2t_2$, solution $t_1 = \frac{1}{2}, t_2 = 2, x = 2, y = -1, z = 0$.

55. (a) Let $t = 3$ and $t = -2$, respectively, in the equations for L_1 and L_2.

(b) $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$ are parallel to L_1 and L_2, $\cos\theta = \mathbf{u}\cdot\mathbf{v}/(\|\mathbf{u}\|\,\|\mathbf{v}\|) = 1/(3\sqrt{11}), \theta \approx 84^\circ$.

(c) $\mathbf{u}\times\mathbf{v} = 7\mathbf{i} + 7\mathbf{k}$ is perpendicular to both L_1 and L_2, and hence so is $\mathbf{i}+\mathbf{k}$, thus $x = 7+t$, $y = -1$, $z = -2+t$.

56. (a) Let $t = 1/2$ and $t = 1$, respectively, in the equations for L_1 and L_2.

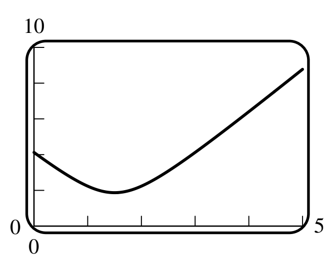
(b) $\mathbf{u} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$ are parallel to L_1 and L_2, $\cos\theta = \mathbf{u}\cdot\mathbf{v}/(\|\mathbf{u}\|\,\|\mathbf{v}\|) = 14/\sqrt{432}, \theta \approx 48^\circ$.

(c) $\mathbf{u}\times\mathbf{v} = -6\mathbf{i} - 14\mathbf{j} - 2\mathbf{k}$ is perpendicular to both L_1 and L_2, and hence so is $3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$, thus $x = 2+3t$, $y = 7t$, $z = 3+t$.

57. $Q(0,1,2)$ lies on the line L $(t=0)$ so $\mathbf{u} = \mathbf{j} - \mathbf{k}$ is a vector from Q to the point $P(0,2,1)$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ is parallel to the given line (set $t = 0, 1$). Next, $\mathbf{u}\times\mathbf{v} = -2\mathbf{j} - 2\mathbf{k}$, and hence $\mathbf{w} = \mathbf{j} + \mathbf{k}$, are perpendicular to both lines, so $\mathbf{v}\times\mathbf{w} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, and hence $\mathbf{i} + \mathbf{j} - \mathbf{k}$, is parallel to the line we seek. Thus $x = t$, $y = 2+t$, $z = 1-t$ are parametric equations of the line. $Q(-2/3, 4/3, 5/3)$ lies on both lines, so distance $= |PQ| = 2\sqrt{3}/3$.

58. $(-2,4,2)$ is on the given line $(t=0)$ so $\mathbf{u} = 5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$ is a vector from this point to the point $(3,1,-2)$, and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is parallel to the given line. Hence $\mathbf{u}\times\mathbf{v} = 5\mathbf{i} - 13\mathbf{j} + 16\mathbf{k}$ is perpendicular to both lines so $\mathbf{v}\times(\mathbf{u}\times\mathbf{v}) = 45\mathbf{i} - 27\mathbf{j} - 36\mathbf{k}$, and hence $5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$ is parallel to the line we seek. Thus $x = 3+5t$, $y = 1-3t$, $z = -2-4t$ are parametric equations of the line. FInally $Q(-2,4,2)$ lies on both lines, so the distance between the lines is $|PQ| = 5\sqrt{2}$.

59. (a) When $t = 0$ the bugs are at $(4,1,2)$ and $(0,1,1)$ so the distance between them is $\sqrt{4^2+0^2+1^2} = \sqrt{17}$ cm.

(b) 

(c) The distance has a minimum value.

(d) Minimize D^2 instead of D (the distance between the bugs). $D^2 = [t-(4-t)]^2 + [(1+t)-(1+2t)]^2 + [(1+2t)-(2+t)]^2 = 6t^2 - 18t + 17$, $d(D^2)/dt = 12t - 18 = 0$ when $t = 3/2$; the minimum distance is $\sqrt{6(3/2)^2 - 18(3/2) + 17} = \sqrt{14}/2$ cm.

60. The line intersects the xz-plane when $t = -1$, the xy-plane when $t = 3/2$. Along the line, $T = 25t^2(1+t)(3-2t)$ for $-1 \le t \le 3/2$. Solve $dT/dt = 0$ for t to find that the maximum value of T is about 50.96 when $t \approx 1.073590$.

Exercise Set 11.6

1. $P_1: \ z = 5, \ P_2: \ x = 3, \ P_3: \ y = 4$.

2. $P_1: \ z = z_0, \ P_2: \ x = x_0, \ P_3: \ y = y_0$.

3. $(x-2) + 4(y-6) + 2(z-1) = 0$, $x + 4y + 2z = 28$.

4. $-(x+1) + 7(y+1) + 6(z-2) = 0$, $-x + 7y + 6z = 6$.

5. $0(x-1) + 0(y-0) + 1(z-0) = 0$, i.e. $z = 0$.

6. $2x - 3y - 4z = 0$.

7. $\mathbf{n} = \mathbf{i} - \mathbf{j}$, $P(0,0,0)$, $x - y = 0$.

8. $\mathbf{n} = \mathbf{i} + \mathbf{j}$, $P(1,0,0)$, $(x-1) + y = 0$, $x + y = 1$.

9. $\mathbf{n} = \mathbf{j} + \mathbf{k}$, $P(0,1,0)$, $(y-1) + z = 0$, $y + z = 1$.

10. $\mathbf{n} = \mathbf{j} - \mathbf{k}$, $P(0,0,0)$, $y - z = 0$.

11. $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \langle 2,1,2 \rangle \times \langle 3,-1,-2 \rangle = \langle 0,10,-5 \rangle$, for convenience choose $\langle 0,2,-1 \rangle$ which is also normal to the plane. Use any of the given points to get $2y - z = 1$.

12. $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \langle -1,-1,-2 \rangle \times \langle -4,1,1 \rangle = \langle 1,9,-5 \rangle, x + 9y - 5z = 16$.

13. **(a)** Parallel, because $\langle 2,-8,-6 \rangle$ and $\langle -1,4,3 \rangle$ are parallel.

(b) Perpendicular, because $\langle 3,-2,1 \rangle$ and $\langle 4,5,-2 \rangle$ are orthogonal.

(c) Neither, because $\langle 1,-1,3 \rangle$ and $\langle 2,0,1 \rangle$ are neither parallel nor orthogonal.

14. **(a)** Neither, because $\langle 3,-2,1 \rangle$ and $\langle 6,-4,3 \rangle$ are neither parallel nor orthogonal.

(b) Parallel, because $\langle 4,-1,-2 \rangle$ and $\langle 1,-1/4,-1/2 \rangle$ are parallel.

(c) Perpendicular, because $\langle 1,4,7 \rangle$ and $\langle 5,-3,1 \rangle$ are orthogonal.

15. **(a)** Parallel, because $\langle 2,-1,-4 \rangle$ and $\langle 3,2,1 \rangle$ are orthogonal.

(b) Neither, because $\langle 1,2,3 \rangle$ and $\langle 1,-1,2 \rangle$ are neither parallel nor orthogonal.

(c) Perpendicular, because $\langle 2,1,-1 \rangle$ and $\langle 4,2,-2 \rangle$ are parallel.

16. **(a)** Parallel, because $\langle -1,1,-3 \rangle$ and $\langle 2,2,0 \rangle$ are orthogonal.

(b) Perpendicular, because $\langle -2,1,-1 \rangle$ and $\langle 6,-3,3 \rangle$ are parallel.

(c) Neither, because $\langle 1,-1,1 \rangle$ and $\langle 1,1,1 \rangle$ are neither parallel nor orthogonal.

17. **(a)** $3t - 2t + t - 5 = 0$, $t = 5/2$ so $x = y = z = 5/2$, the point of intersection is $(5/2, 5/2, 5/2)$.

(b) $2(2-t) + (3+t) + t = 1$ has no solution so the line and plane do not intersect.

18. **(a)** $2(3t) - 5t + (-t) + 1 = 0$, $1 = 0$ has no solution so the line and the plane do not intersect.

(b) $(1+t) - (-1+3t) + 4(2+4t) = 7$, $t = -3/14$ so $x = 1 - 3/14 = 11/14$, $y = -1 - 9/14 = -23/14$, $z = 2 - 12/14 = 8/7$, the point is $(11/14, -23/14, 8/7)$.

19. $\mathbf{n}_1 = \langle 1,0,0\rangle, \mathbf{n}_2 = \langle 2,-1,1\rangle, \mathbf{n}_1 \cdot \mathbf{n}_2 = 2$, so $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \dfrac{2}{\sqrt{1}\sqrt{6}} = 2/\sqrt{6}, \theta = \cos^{-1}(2/\sqrt{6}) \approx 35°$.

20. $\mathbf{n}_1 = \langle 1,2,-2\rangle, \mathbf{n}_2 = \langle 6,-3,2\rangle, \mathbf{n}_1 \cdot \mathbf{n}_2 = -4$, so $\cos\theta = \dfrac{(-\mathbf{n}_1)\cdot \mathbf{n}_2}{\|-\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \dfrac{4}{(3)(7)} = 4/21, \theta = \cos^{-1}(4/21) \approx 79°$. (Note: $-\mathbf{n}_1$ is used instead of $\mathbf{n}_1$ to get a value of θ in the range $[0, \pi/2]$.)

21. True. $(\pm)$

22. True.

23. True.

24. True, see Theorem 11.6.2.

25. $\langle 4,-2,7\rangle$ is normal to the desired plane and $(0,0,0)$ is a point on it; $4x - 2y + 7z = 0$.

26. $\mathbf{v} = \langle 3,2,-1\rangle$ is parallel to the line and $\mathbf{n} = \langle 1,-2,1\rangle$ is normal to the given plane so $\mathbf{v}\times\mathbf{n} = \langle 0,-4,-8\rangle$ is normal to the desired plane. Let $t = 0$ in the line to get $(-2,4,3)$ which is also a point on the desired plane, use this point and (for convenience) the normal $\langle 0,1,2\rangle$ to find that $y + 2z = 10$.

27. Find two points P_1 and P_2 on the line of intersection of the given planes and then find an equation of the plane that contains P_1, P_2, and the given point $P_0(-1,4,2)$. Let (x_0, y_0, z_0) be on the line of intersection of the given planes; then $4x_0 - y_0 + z_0 - 2 = 0$ and $2x_0 + y_0 - 2z_0 - 3 = 0$, eliminate y_0 by addition of the equations to get $6x_0 - z_0 - 5 = 0$; if $x_0 = 0$ then $z_0 = -5$, if $x_0 = 1$ then $z_0 = 1$. Substitution of these values of x_0 and z_0 into either of the equations of the planes gives the corresponding values $y_0 = -7$ and $y_0 = 3$ so $P_1(0,-7,-5)$ and $P_2(1,3,1)$ are on the line of intersection of the planes. $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \langle 4,-13,21\rangle$ is normal to the desired plane whose equation is $4x - 13y + 21z = -14$.

28. $\langle 1,2,-1\rangle$ is parallel to the line and hence normal to the plane $x + 2y - z = 10$.

29. $\mathbf{n}_1 = \langle 2,1,1\rangle$ and $\mathbf{n}_2 = \langle 1,2,1\rangle$ are normals to the given planes, $\mathbf{n}_1\times\mathbf{n}_2 = \langle -1,-1,3\rangle$ so $\langle 1,1,-3\rangle$ is normal to the desired plane whose equation is $x + y - 3z = 6$.

30. $\mathbf{n} = \langle 4,-1,3\rangle$ is normal to the given plane, $\overrightarrow{P_1P_2} = \langle 3,-1,-1\rangle$ is parallel to the line through the given points, $\mathbf{n}\times\overrightarrow{P_1P_2} = \langle 4,13,-1\rangle$ is normal to the desired plane whose equation is $4x + 13y - z = 1$.

31. $\mathbf{n}_1 = \langle 2,-1,1\rangle$ and $\mathbf{n}_2 = \langle 1,1,-2\rangle$ are normals to the given planes, $\mathbf{n}_1\times\mathbf{n}_2 = \langle 1,5,3\rangle$ is normal to the desired plane whose equation is $x + 5y + 3z = -6$.

32. Let $t = 0$ and $t = 1$ to get the points $P_1(-1,0,-4)$ and $P_2(0,1,-2)$ that lie on the line. Denote the given point by P_0, then $\overrightarrow{P_0P_1}\times\overrightarrow{P_0P_2} = \langle 7,-1,-3\rangle$ is normal to the desired plane whose equation is $7x - y - 3z = 5$.

33. The plane is the perpendicular bisector of the line segment that joins $P_1(2,-1,1)$ and $P_2(3,1,5)$. The midpoint of the line segment is $(5/2,0,3)$ and $\overrightarrow{P_1P_2} = \langle 1,2,4\rangle$ is normal to the plane so an equation is $x + 2y + 4z = 29/2$.

34. $\mathbf{n}_1 = \langle 2,-1,1\rangle$ and $\mathbf{n}_2 = \langle 0,1,1\rangle$ are normals to the given planes, $\mathbf{n}_1\times\mathbf{n}_2 = \langle -2,-2,2\rangle$ so $\mathbf{n} = \langle 1,1,-1\rangle$ is parallel to the line of intersection of the planes. $\mathbf{v} = \langle 3,1,2\rangle$ is parallel to the given line, $\mathbf{v}\times\mathbf{n} = \langle -3,5,2\rangle$ so $\langle 3,-5,-2\rangle$ is normal to the desired plane. Let $t = 0$ to find the point $(0,1,0)$ that lies on the given line and hence on the desired plane. An equation of the plane is $3x - 5y - 2z = -5$.

35. The line is parallel to the line of intersection of the planes if it is parallel to both planes. Normals to the given planes are $\mathbf{n}_1 = \langle 1, -4, 2\rangle$ and $\mathbf{n}_2 = \langle 2, 3, -1\rangle$ so $\mathbf{n}_1 \times \mathbf{n}_2 = \langle -2, 5, 11\rangle$ is parallel to the line of intersection of the planes and hence parallel to the desired line whose equations are $x = 5 - 2t$, $y = 5t$, $z = -2 + 11t$.

36. (a) The equation of the plane is satisfied by the points on the line: $2(3t + 1) + (-5t) - (t) = 2$.

(b) The vector $\langle 3, -5, 1\rangle$ is a direction vector for the line and $\langle 1, 1, 2\rangle$ is a normal to the plane; $\langle 3, -5, 1\rangle \cdot \langle 1, 1, 2\rangle = 0$, so the line is parallel to the plane. Fix t; then the point $\langle 3t + 1, -5t, t\rangle$ satisfies $3t + 1 - 5t + 2t = 1$, i.e. it lies in the plane $x + y + 2z = 1$ which in turn lies above the given plane.

37. $\mathbf{v}_1 = \langle 1, 2, -1\rangle$ and $\mathbf{v}_2 = \langle -1, -2, 1\rangle$ are parallel, respectively, to the given lines and to each other so the lines are parallel. Let $t = 0$ to find the points $P_1(-2, 3, 4)$ and $P_2(3, 4, 0)$ that lie, respectively, on the given lines. $\mathbf{v}_1 \times \overrightarrow{P_1P_2} = \langle -7, -1, -9\rangle$ so $\langle 7, 1, 9\rangle$ is normal to the desired plane whose equation is $7x + y + 9z = 25$.

38. The system $4t_1 - 1 = 12t_2 - 13$, $t_1 + 3 = 6t_2 + 1$, $1 = 3t_2 + 2$ has the solution (Exercise 30, Section 11.5) $t_1 = -4$, $t_2 = -1/3$ so $(-17, -1, 1)$ is the point of intersection. $\mathbf{v}_1 = \langle 4, 1, 0\rangle$ and $\mathbf{v}_2 = \langle 12, 6, 3\rangle$ are (respectively) parallel to the lines, $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 3, -12, 12\rangle$ so $\langle 1, -4, 4\rangle$ is normal to the desired plane whose equation is $x - 4y + 4z = -9$.

39. Denote the points by A, B, C, and D, respectively. The points lie in the same plane if $\overrightarrow{AB} \times \overrightarrow{AC}$ and $\overrightarrow{AB} \times \overrightarrow{AD}$ are parallel (method 1). $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0, -10, 5\rangle, \overrightarrow{AB} \times \overrightarrow{AD} = \langle 0, 16, -8\rangle$, these vectors are parallel because $\langle 0, -10, 5\rangle = (-10/16)\langle 0, 16, -8\rangle$. The points lie in the same plane if D lies in the plane determined by A, B, C (method 2), and since $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0, -10, 5\rangle$, an equation of the plane is $-2y + z + 1 = 0$, $2y - z = 1$ which is satisfied by the coordinates of D.

40. The intercepts correspond to the points $A(a, 0, 0), B(0, b, 0)$, and $C(0, 0, c)$. $\overrightarrow{AB} \times \overrightarrow{AC} = \langle bc, ac, ab\rangle$ is normal to the plane so $bcx + acy + abz = abc$ or $x/a + y/b + z/c = 1$.

41. $\mathbf{n}_1 = \langle -2, 3, 7\rangle$ and $\mathbf{n}_2 = \langle 1, 2, -3\rangle$ are normals to the planes, $\mathbf{n}_1 \times \mathbf{n}_2 = \langle -23, 1, -7\rangle$ is parallel to the line of intersection. Let $z = 0$ in both equations and solve for x and y to get $x = -11/7$, $y = -12/7$ so $(-11/7, -12/7, 0)$ is on the line, a parametrization of which is $x = -11/7 - 23t$, $y = -12/7 + t$, $z = -7t$.

42. Similar to Exercise 41 with $\mathbf{n}_1 = \langle 3, -5, 2\rangle$, $\mathbf{n}_2 = \langle 0, 0, 1\rangle$, $\mathbf{n}_1 \times \mathbf{n}_2 = \langle -5, -3, 0\rangle$. $z = 0$ so $3x - 5y = 0$, let $x = 0$ then $y = 0$ and $(0, 0, 0)$ is on the line, a parametrization of which is $x = -5t$, $y = -3t$, $z = 0$.

43. $D = |2(1) - 2(-2) + (3) - 4|/\sqrt{4 + 4 + 1} = 5/3$.

44. $D = |3(0) + 6(1) - 2(5) - 5|/\sqrt{9 + 36 + 4} = 9/7$.

45. $(0, 0, 0)$ is on the first plane so $D = |6(0) - 3(0) - 3(0) - 5|/\sqrt{36 + 9 + 9} = 5/\sqrt{54}$.

46. $(0, 0, 1)$ is on the first plane so $D = |(0) + (0) + (1) + 1|/\sqrt{1 + 1 + 1} = 2/\sqrt{3}$.

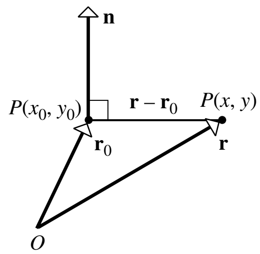
47. $(1, 3, 5)$ and $(4, 6, 7)$ are on L_1 and L_2, respectively. $\mathbf{v}_1 = \langle 7, 1, -3\rangle$ and $\mathbf{v}_2 = \langle -1, 0, 2\rangle$ are, respectively, parallel to L_1 and L_2, $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 2, -11, 1\rangle$ so the plane $2x - 11y + z + 51 = 0$ contains L_2 and is parallel to L_1, $D = |2(1) - 11(3) + (5) + 51|/\sqrt{4 + 121 + 1} = 25/\sqrt{126}$.

48. $(3, 4, 1)$ and $(0, 3, 0)$ are on L_1 and L_2, respectively. $\mathbf{v}_1 = \langle -1, 4, 2\rangle$ and $\mathbf{v}_2 = \langle 1, 0, 2\rangle$ are parallel to L_1 and L_2, $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 8, 4, -4\rangle = 4\langle 2, 1, -1\rangle$ so $2x + y - z - 3 = 0$ contains L_2 and is parallel to L_1, $D = |2(3) + (4) - (1) - 3|/\sqrt{4 + 1 + 1} = \sqrt{6}$.

49. The distance between $(2, 1, -3)$ and the plane is $|2 - 3(1) + 2(-3) - 4|/\sqrt{1 + 9 + 4} = 11/\sqrt{14}$ which is the radius of the sphere; an equation is $(x - 2)^2 + (y - 1)^2 + (z + 3)^2 = 121/14$.

50. The vector $2\mathbf{i}+\mathbf{j}-\mathbf{k}$ is normal to the plane and hence parallel to the line so parametric equations of the line are $x=3+2t$, $y=1+t$, $z=-t$. Substitution into the equation of the plane yields $2(3+2t)+(1+t)-(-t)=0$, $t=-7/6$; the point of intersection is $(2/3,-1/6,7/6)$.

51. $\mathbf{v}=\langle 1,2,-1\rangle$ is parallel to the line, $\mathbf{n}=\langle 2,-2,-2\rangle$ is normal to the plane, $\mathbf{v}\cdot\mathbf{n}=0$ so $\mathbf{v}$ is parallel to the plane because $\mathbf{v}$ and $\mathbf{n}$ are perpendicular. $(-1,3,0)$ is on the line so $D=|2(-1)-2(3)-2(0)+3|/\sqrt{4+4+4}=5/\sqrt{12}$.

52. (a) 

(b) $\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0)=a(x-x_0)+b(y-y_0)=0$.

(c) See the proof of Theorem 11.6.1. Since a and b are not both zero, there is at least one point (x_0,y_0) that satisfies $ax+by+c=0$, so $ax_0+by_0+c=0$. If (x,y) also satisfies $ax+by+c=0$ then, subtracting, $a(x-x_0)+b(y-y_0)=0$, which is the equation of a line with $\mathbf{n}=\langle a,b\rangle$ as normal.

(d) Let $Q(x_1,y_1)$ be a point on the line, and position the normal $\mathbf{n}=\langle a,b\rangle$, with length $\sqrt{a^2+b^2}$, so that its initial point is at Q. The distance is the orthogonal projection of $\overrightarrow{QP_0}=\langle x_0-x_1,y_0-y_1\rangle$ onto $\mathbf{n}$. Then $D=\|\text{proj}_{\mathbf{n}}\overrightarrow{QP_0}\|=\left\|\dfrac{\overrightarrow{QP_0}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}\right\|=\dfrac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$.

(e) $D=|2(-3)+(5)-1|/\sqrt{4+1}=2/\sqrt{5}$.

53. (a) If $\langle x_0,y_0,z_0\rangle$ lies on the second plane, so that $ax_0+by_0+cz_0+d_2=0$, then by Theorem 11.6.2, the distance between the planes is $D=\dfrac{|ax_0+by_0+cz_0+d_1|}{\sqrt{a^2+b^2+c^2}}=\dfrac{|-d_2+d_1|}{\sqrt{a^2+b^2+c^2}}$.

(b) The distance between the planes $-2x+y+z=0$ and $-2x+y+z+\dfrac{5}{3}=0$ is $D=\dfrac{|0-5/3|}{\sqrt{4+1+1}}=\dfrac{5}{3\sqrt{6}}$.

Exercise Set 11.7

1. (a) Elliptic paraboloid, $a=2,b=3$. **(b)** Hyperbolic paraboloid, $a=1,b=5$.

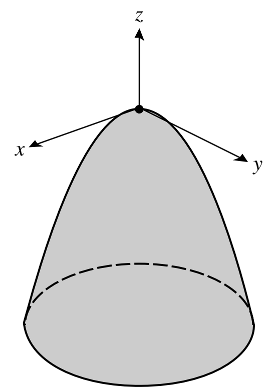
(c) Hyperboloid of one sheet, $a=b=c=4$. **(d)** Circular cone, $a=b=1$.

(e) Elliptic paraboloid, $a=2,b=1$. **(f)** Hyperboloid of two sheets, $a=b=c=1$.

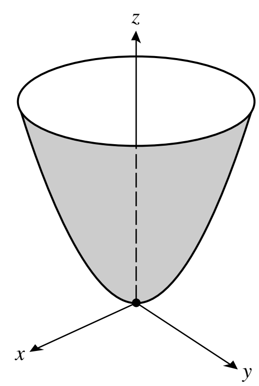
2. (a) Ellipsoid, $a=\sqrt{2},b=2,c=\sqrt{3}$. **(b)** Hyperbolic paraboloid, $a=b=1$.

(c) Hyperboloid of one sheet, $a=1,b=3,c=1$. **(d)** Hyperboloid of two sheets, $a=1,b=2,c=1$.

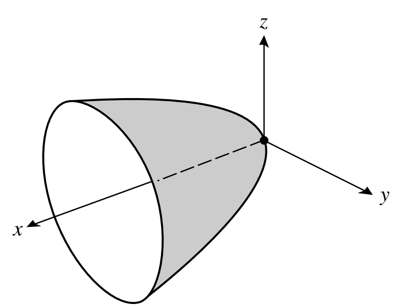
(e) Elliptic paraboloid, $a=\sqrt{2},b=\sqrt{2}/2$. **(f)** Elliptic cone, $a=2,b=\sqrt{3}$.



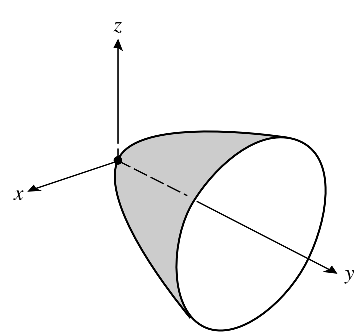
3. (a) $-z = x^2 + y^2$, circular paraboloid opening down the negative z-axis.



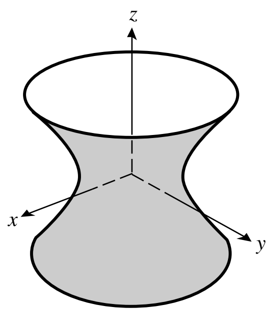
(b,c,d) $z = x^2 + y^2$, circular paraboloid, no change.



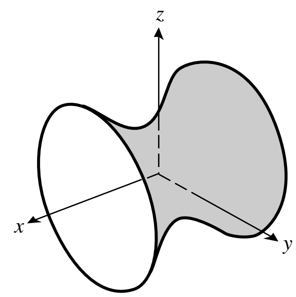
(e) $x = y^2 + z^2$, circular paraboloid opening along the positive x-axis.



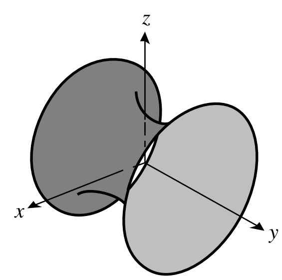
(f) $y = x^2 + z^2$, circular paraboloid opening along the positive y-axis.



4. (a,b,c,d) $x^2 + y^2 - z^2 = 1$, no change.



(e) $-x^2 + y^2 + z^2 = 1$, hyperboloid of one sheet with x-axis as axis.



(f) $x^2 - y^2 + z^2 = 1$, hyperboloid of one sheet with y-axis as axis.

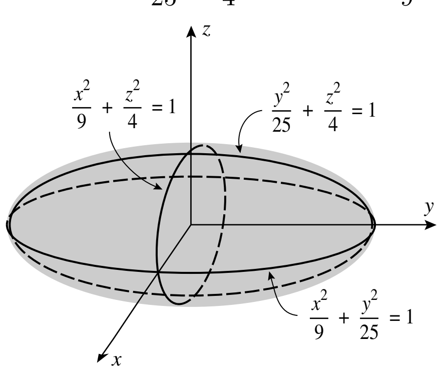
5. (a) Hyperboloid of one sheet, axis is y-axis. **(b)** Hyperboloid of two sheets separated by yz-plane.

(c) Elliptic paraboloid opening along the positive x-axis. **(d)** Elliptic cone with x-axis as axis.

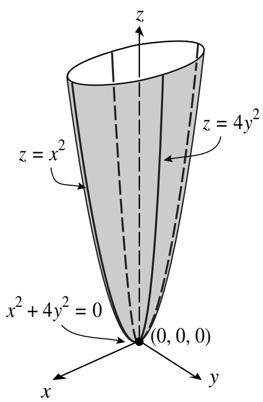
(e) Hyperbolic paraboloid straddling the z-axis. **(f)** Paraboloid opening along the negative y-axis.

6. (a) Same. **(b)** Same. **(c)** Same. **(d)** Same. **(e)** $y = \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2}$. **(f)** $y = \dfrac{x^2}{a^2} + \dfrac{z^2}{c^2}$.

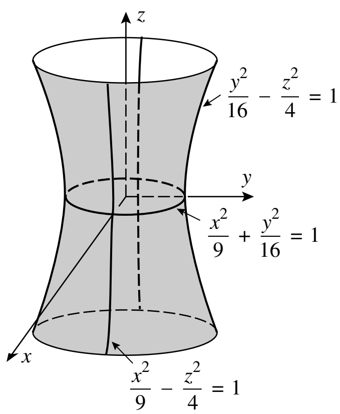
7. (a) $x = 0 : \dfrac{y^2}{25} + \dfrac{z^2}{4} = 1; y = 0 : \dfrac{x^2}{9} + \dfrac{z^2}{4} = 1; z = 0 : \dfrac{x^2}{9} + \dfrac{y^2}{25} = 1.$



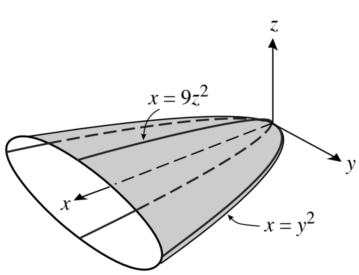
(b) $x = 0 : z = 4y^2; y = 0 : z = x^2; z = 0 : x = y = 0.$



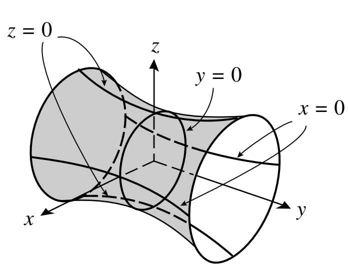
(c) $x = 0 : \dfrac{y^2}{16} - \dfrac{z^2}{4} = 1; y = 0 : \dfrac{x^2}{9} - \dfrac{z^2}{4} = 1; z = 0 : \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1.$



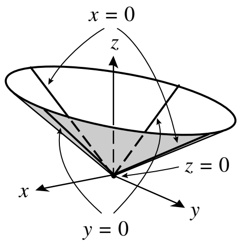
8. (a) $x = 0 : y = z = 0; y = 0 : x = 9z^2; z = 0 : x = y^2.$



(b) $x = 0 : -y^2 + 4z^2 = 4; y = 0 : x^2 + z^2 = 1; z = 0 : 4x^2 - y^2 = 4$



(c) $x = 0 : z = \pm\dfrac{y}{2}; y = 0 : z = \pm x; z = 0 : x = y = 0$



9. **(a)** $4x^2 + z^2 = 3$; ellipse. **(b)** $y^2 + z^2 = 3$; circle. **(c)** $y^2 + z^2 = 20$; circle.

(d) $9x^2 - y^2 = 20$; hyperbola. **(e)** $z = 9x^2 + 16$; parabola. **(f)** $9x^2 + 4y^2 = 4$; ellipse.

10. **(a)** $y^2 - 4z^2 = 27$; hyperbola. **(b)** $9x^2 + 4z^2 = 25$; ellipse. **(c)** $9z^2 - x^2 = 4$; hyperbola.

(d) $x^2 + 4y^2 = 9$; ellipse. **(e)** $z = 1 - 4y^2$; parabola. **(f)** $x^2 - 4y^2 = 4$; hyperbola.

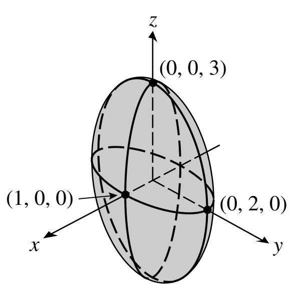
11. False; 'quadric' surfaces are of second degree.

12. False; $(x-1)^2 + y^2 + z^2 = 1/4$ has no solution if $x = y = 0$.

13. False.

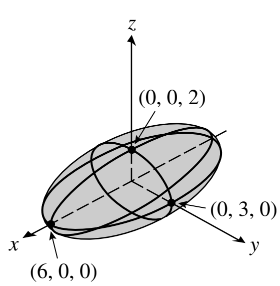
14. True: $y = \pm(b/a)x$.

15.



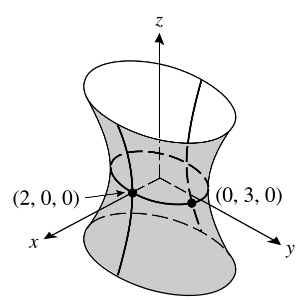
Ellipsoid

16.



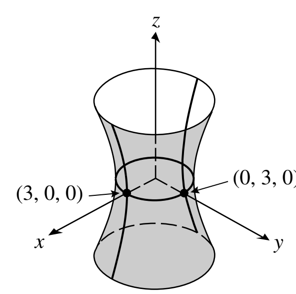
Ellipsoid

17.



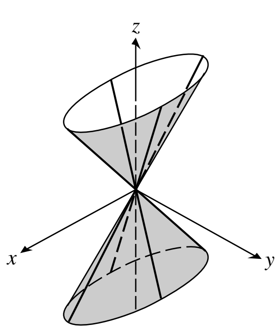
Hyperboloid of one sheet

18.



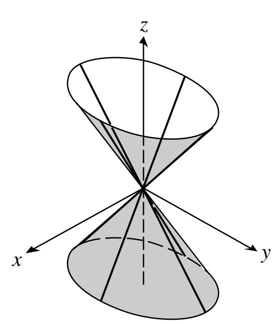
Hyperboloid of one sheet

19.



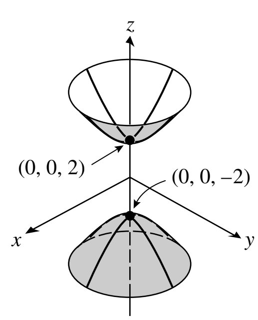
Elliptic cone

20.



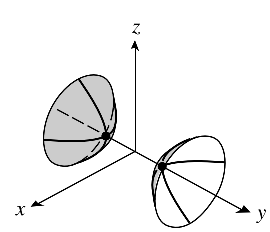
Elliptic cone

21.

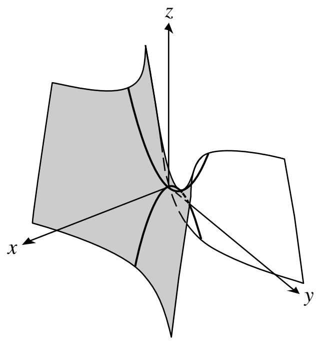


Hyperboloid of two sheets

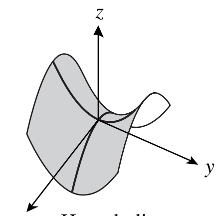
22.



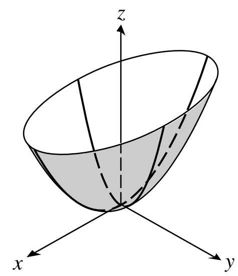
Hyperboloid of two sheets

23. 

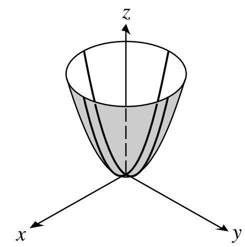
Hyperbolic paraboloid

24. 

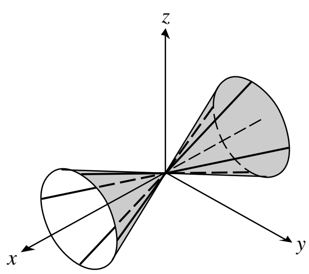
Hyperbolic paraboloid

25. 

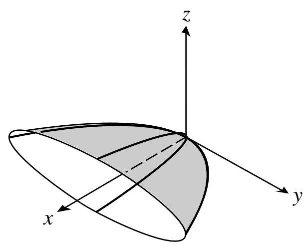
Elliptic paraboloid

26. 

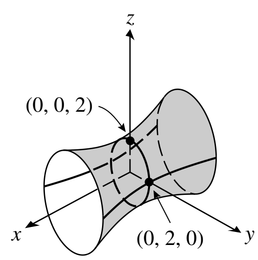
Circular paraboloid

27. 

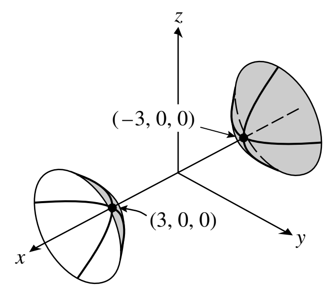
Circular cone

28. 

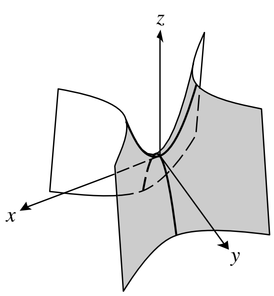
Elliptic paraboloid

29. 

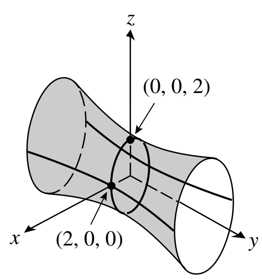
Hyperboloid of one sheet

30. 

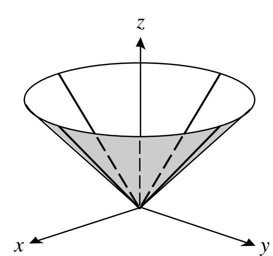
Hyperboloid of two sheets

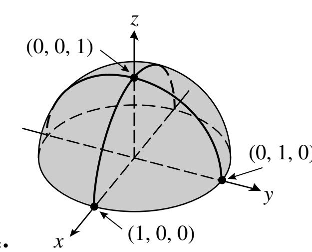
31. 

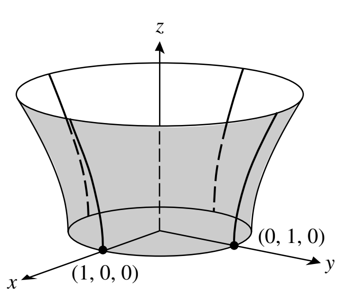
Hyperbolic paraboloid

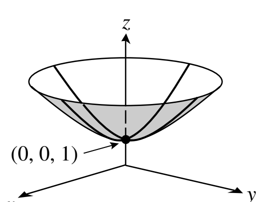
32. 

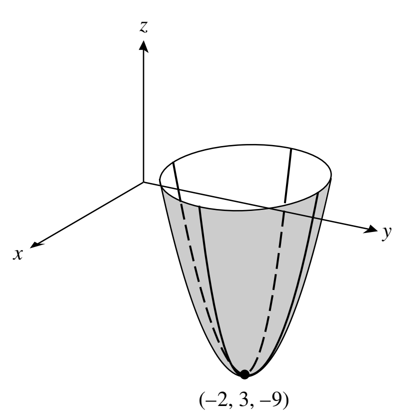
Hyperboloid of one sheet

33. 

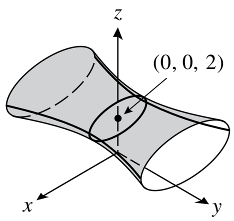
34. 

35. 

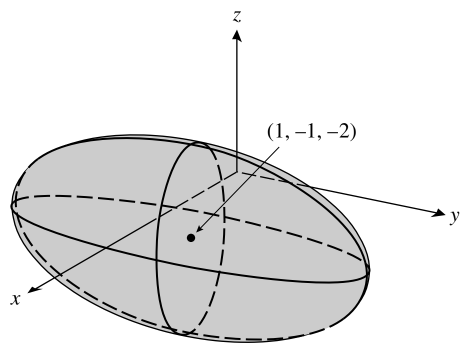
36. 

37. 

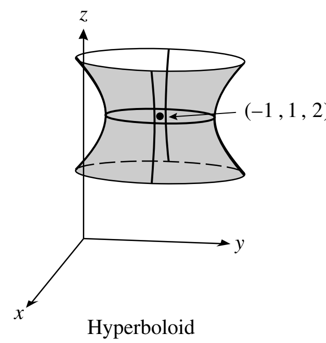
Circular paraboloid



38. Hyperboloid of one sheet



39. Ellipsoid



40. Hyperboloid of one sheet

41. **(a)** $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1.$ **(b)** $6, 4.$ **(c)** $(\pm\sqrt{5}, 0, \sqrt{2}).$ **(d)** The focal axis is parallel to the x-axis.

42. **(a)** $\dfrac{y^2}{4}+\dfrac{z^2}{2}=1.$ **(b)** $4, 2\sqrt{2}.$ **(c)** $(3, \pm\sqrt{2}, 0).$ **(d)** The focal axis is parallel to the y-axis.

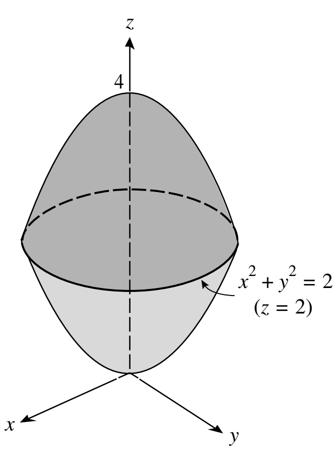
43. **(a)** $\dfrac{y^2}{4}-\dfrac{x^2}{4}=1.$ **(b)** $(0, \pm 2, 4).$ **(c)** $(0, \pm 2\sqrt{2}, 4).$ **(d)** The focal axis is parallel to the y-axis.

44. **(a)** $\dfrac{x^2}{4}-\dfrac{y^2}{4}=1.$ **(b)** $(\pm 2, 0, -4).$ **(c)** $(\pm 2\sqrt{2}, 0, -4).$ **(d)** The focal axis is parallel to the x-axis.

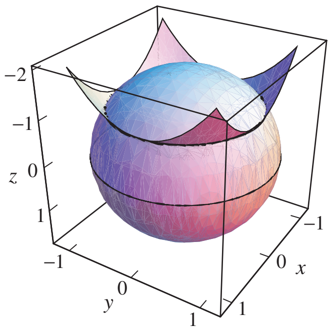
45. **(a)** $z+4=y^2.$ **(b)** $(2, 0, -4).$ **(c)** $(2, 0, -15/4).$ **(d)** The focal axis is parallel to the z-axis.

46. **(a)** $z-4=-x^2.$ **(b)** $(0, 2, 4).$ **(c)** $(0, 2, 15/4).$ **(d)** The focal axis is parallel to the z-axis.

47. $x^2+y^2=4-x^2-y^2, x^2+y^2=2$; circle of radius $\sqrt{2}$ in the plane $z=2$, centered at $(0, 0, 2)$.



48. $3=2(x^2+y^2)+z^2=2z+z^2, (z+3)(z-1)=0$; circle $x^2+y^2=1$ in the plane $z=1$ (the root $z=-3$ is extraneous).



49. $y = 4(x^2 + z^2)$.

50. $y^2 = 4(x^2 + z^2)$.

51. $|z - (-1)| = \sqrt{x^2 + y^2 + (z-1)^2}$, $z^2 + 2z + 1 = x^2 + y^2 + z^2 - 2z + 1$, $z = (x^2 + y^2)/4$; circular paraboloid.

52. $|z + 1| = 2\sqrt{x^2 + y^2 + (z-1)^2}$, $z^2 + 2z + 1 = 4\left(x^2 + y^2 + z^2 - 2z + 1\right)$, $4x^2 + 4y^2 + 3z^2 - 10z + 3 = 0$, $\dfrac{x^2}{4/3} + \dfrac{y^2}{4/3} + \dfrac{(z - 5/3)^2}{16/9} = 1$; ellipsoid, center at $(0, 0, 5/3)$.

53. If $z = 0$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$; if $y = 0$ then $\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = 1$; since $c < a$ the major axis has length $2a$, the minor axis has length $2c$.

54. $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} = 1$, where $a = 6378.1370, b = 6356.5231$.

55. Each slice perpendicular to the z-axis for $|z| < c$ is an ellipse whose equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{c^2 - z^2}{c^2}$, or $\dfrac{x^2}{(a^2/c^2)(c^2 - z^2)} + \dfrac{y^2}{(b^2/c^2)(c^2 - z^2)} = 1$, the area of which is $\pi\left(\dfrac{a}{c}\sqrt{c^2 - z^2}\right)\left(\dfrac{b}{c}\sqrt{c^2 - z^2}\right) = \pi\dfrac{ab}{c^2}\left(c^2 - z^2\right)$ so $V = 2\displaystyle\int_0^c \pi\frac{ab}{c^2}\left(c^2 - z^2\right) dz = \frac{4}{3}\pi abc$.

Exercise Set 11.8

1. **(a)** $(8, \pi/6, -4)$ **(b)** $\left(5\sqrt{2}, 3\pi/4, 6\right)$ **(c)** $(2, \pi/2, 0)$ **(d)** $(8, 5\pi/3, 6)$

2. **(a)** $(2, 7\pi/4, 1)$ **(b)** $(1, \pi/2, 1)$ **(c)** $(4\sqrt{2}, 3\pi/4, -7)$ **(d)** $(2\sqrt{2}, 7\pi/4, -2)$

3. **(a)** $\left(2\sqrt{3}, 2, 3\right)$ **(b)** $\left(-4\sqrt{2}, 4\sqrt{2}, -2\right)$ **(c)** $(5, 0, 4)$ **(d)** $(-7, 0, -9)$

4. **(a)** $\left(3, -3\sqrt{3}, 7\right)$ **(b)** $(0, 1, 0)$ **(c)** $(0, 3, 5)$ **(d)** $(0, 4, -1)$

5. **(a)** $\left(2\sqrt{2}, \pi/3, 3\pi/4\right)$ **(b)** $(2, 7\pi/4, \pi/4)$ **(c)** $(6, \pi/2, \pi/3)$ **(d)** $(10, 5\pi/6, \pi/2)$

6. **(a)** $\left(8\sqrt{2}, \pi/4, \pi/6\right)$ **(b)** $\left(2\sqrt{2}, 5\pi/3, 3\pi/4\right)$ **(c)** $(2, 0, \pi/2)$ **(d)** $(4, \pi/6, \pi/6)$

7. **(a)** $(5\sqrt{6}/4, 5\sqrt{2}/4, 5\sqrt{2}/2)$ **(b)** $(7, 0, 0)$ **(c)** $(0, 0, 1)$ **(d)** $(0, -2, 0)$

8. **(a)** $\left(-\sqrt{2}/4, \sqrt{6}/4, -\sqrt{2}/2\right)$ **(b)** $\left(3\sqrt{2}/4, -3\sqrt{2}/4, -3\sqrt{3}/2\right)$ **(c)** $(2\sqrt{6}, 2\sqrt{2}, 4\sqrt{2})$ **(d)** $(0, 2\sqrt{3}, 2)$

9. **(a)** $\left(2\sqrt{3}, \pi/6, \pi/6\right)$ **(b)** $\left(\sqrt{2}, \pi/4, 3\pi/4\right)$ **(c)** $(2, 3\pi/4, \pi/2)$ **(d)** $\left(4\sqrt{3}, 1, 2\pi/3\right)$

10. **(a)** $\left(4\sqrt{2}, 5\pi/6, \pi/4\right)$ **(b)** $\left(2\sqrt{2}, 0, 3\pi/4\right)$ **(c)** $(5, \pi/2, \tan^{-1}(4/3))$ **(d)** $(2\sqrt{10}, \pi, \tan^{-1} 3)$

11. **(a)** $\left(5\sqrt{3}/2, \pi/4, -5/2\right)$ **(b)** $(0, 7\pi/6, -1)$ **(c)** $(0, 0, 3)$ **(d)** $(4, \pi/6, 0)$

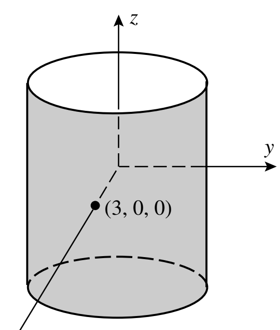
12. **(a)** $(0, \pi/2, 5)$ **(b)** $(3\sqrt{2}, 0, -3\sqrt{2})$ **(c)** $(0, 3\pi/4, -\sqrt{2})$ **(d)** $(5/2, 2\pi/3, -5\sqrt{3}/2)$

15. True.

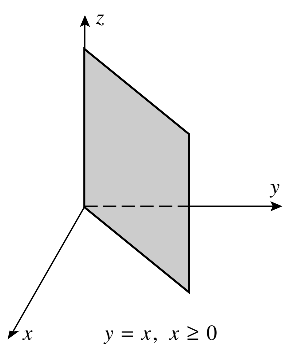
16. True.

17. True.

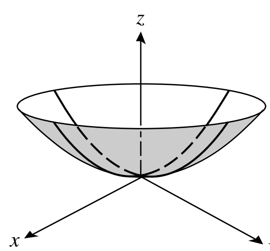
18. True.



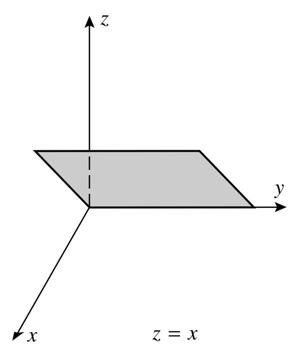
19. $x^2 + y^2 = 9$



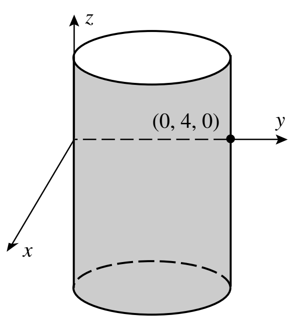
20. $y = x, \; x \geq 0$



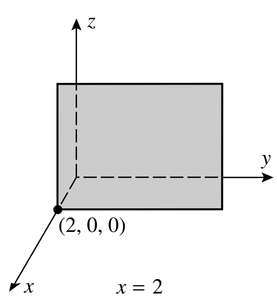
21. $z = x^2 + y^2$



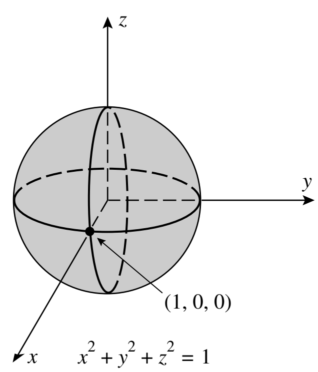
22. $z = x$



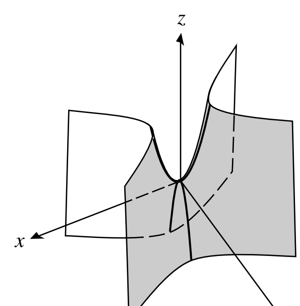
23. $x^2 + (y-2)^2 = 4$



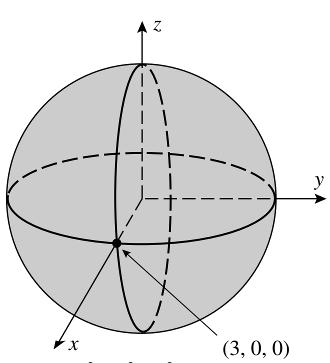
24. $x = 2$



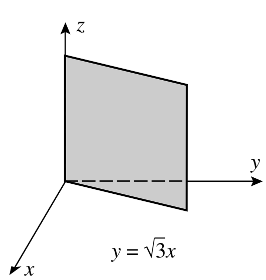
25. $x^2 + y^2 + z^2 = 1$



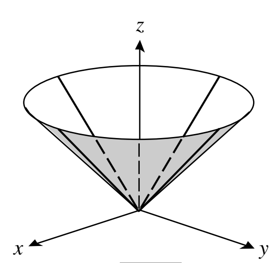
26. $x^2 - y^2 = z$



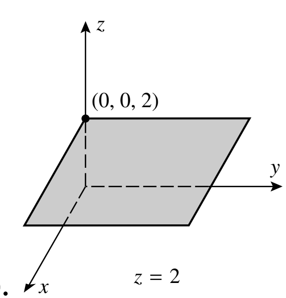
27. $x^2 + y^2 + z^2 = 9$



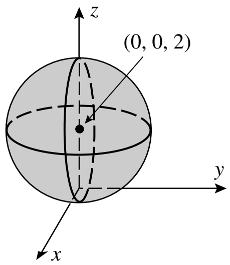
28. $y = \sqrt{3}x$



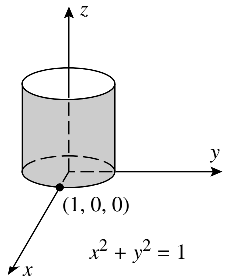
29. $z = \sqrt{x^2 + y^2}$



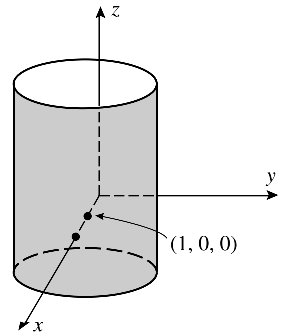
30. $z = 2$



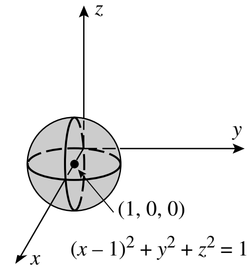
31. $x^2 + y^2 + (z-2)^2 = 4$



32. $x^2 + y^2 = 1$



33. $(x-1)^2 + y^2 = 1$



34. $(x-1)^2 + y^2 + z^2 = 1$

35. **(a)** $z = 3$. **(b)** $\rho\cos\phi = 3, \rho = 3\sec\phi$.

36. **(a)** $r\sin\theta = 2, r = 2\csc\theta$. **(b)** $\rho\sin\phi\sin\theta = 2, \rho = 2\csc\phi\csc\theta$.

37. **(a)** $z = 3r^2$. **(b)** $\rho\cos\phi = 3\rho^2\sin^2\phi, \rho = \dfrac{1}{3}\csc\phi\cot\phi$.

38. **(a)** $z = \sqrt{3}r$. **(b)** $\rho\cos\phi = \sqrt{3}\rho\sin\phi, \tan\phi = \dfrac{1}{\sqrt{3}}, \phi = \dfrac{\pi}{6}$.

39. **(a)** $r = 2$. **(b)** $\rho\sin\phi = 2, \rho = 2\csc\phi$.

40. **(a)** $r^2 - 6r\sin\theta = 0$, $r = 6\sin\theta$. **(b)** $\rho\sin\phi = 6\sin\theta, \rho = 6\sin\theta\csc\phi$.

41. **(a)** $r^2 + z^2 = 9$. **(b)** $\rho = 3$.

42. **(a)** $z^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta)$, $z^2 = r^2\cos 2\theta$.

(b) Use the result in part (a) with $r = \rho\sin\phi$, $z = \rho\cos\phi$ to get $\rho^2\cos^2\phi = \rho^2\sin^2\phi\cos 2\theta$, $\cot^2\phi = \cos 2\theta$.

43. **(a)** $2r\cos\theta + 3r\sin\theta + 4z = 1$. **(b)** $2\rho\sin\phi\cos\theta + 3\rho\sin\phi\sin\theta + 4\rho\cos\phi = 1$.

44. **(a)** $r^2 - z^2 = 1$.

(b) Use the result of part (a) with $r = \rho\sin\phi$, $z = \rho\cos\phi$ to get $\rho^2\sin^2\phi - \rho^2\cos^2\phi = 1$, $\rho^2\cos 2\phi = -1$.

45. **(a)** $r^2\cos^2\theta = 16 - z^2$.

(b) $x^2 = 16 - z^2$, $x^2 + y^2 + z^2 = 16 + y^2$, $\rho^2 = 16 + \rho^2\sin^2\phi\sin^2\theta$, $\rho^2\left(1 - \sin^2\phi\sin^2\theta\right) = 16$.

46. **(a)** $r^2 + z^2 = 2z$. **(b)** $\rho^2 = 2\rho\cos\phi$, $\rho = 2\cos\phi$.

47. All points on or above the paraboloid $z = x^2 + y^2$, that are also on or below the plane $z = 4$.

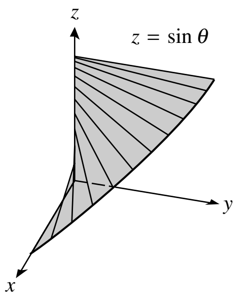
48. A right circular cylindrical solid of height 3 and radius 1 whose axis is the line $x = 0, y = 1$.

49. All points on or between concentric spheres of radii 1 and 3 centered at the origin.

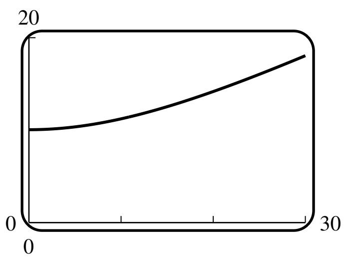
50. All points on or above the cone $\phi = \pi/6$, that are also on or below the sphere $\rho = 2$.

51. $\theta = \pi/6$, $\phi = \pi/6$, spherical $(4000, \pi/6, \pi/6)$, rectangular $\left(1000\sqrt{3}, 1000, 2000\sqrt{3}\right)$.

52. **(a)** $y = r\sin\theta = a\sin\theta$, but $az = a\sin\theta$ so $y = az$, which is a plane that contains the curve of intersection of $z = \sin\theta$ and the circular cylinder $r = a$. The curve of intersection of a plane and a circular cylinder is an ellipse.

(b) 

53. **(a)** $(10, \pi/2, 1)$ **(b)** $(0, 10, 1)$ **(c)** $(\sqrt{101}, \pi/2, \tan^{-1} 10)$

54. 

Chapter 11 Review Exercises

1. **(b)** $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

(c) $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} = a\mathbf{v}$ or $\mathbf{v} = b\mathbf{u}$.

(d) $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in the same plane if and only if $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

2. **(a)** E.g. $\mathbf{i}$ and $\mathbf{j}$.

(b) For points A, B, C and D consider the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$, then apply Exercise 1.(d).

(c) $\mathbf{F} = -\mathbf{i} - \mathbf{j}$.

(d) $\|\langle 1, -2, 2\rangle\| = 3$, so $\|\mathbf{r} - \langle 1, -2, 2\rangle\| = 3$, or $(x-1)^2 + (y+2)^2 + (z-2)^2 = 9$.

3. **(b)** $x = \cos 120^\circ = -1/2, y = \pm\sin 120^\circ = \pm\sqrt{3}/2$.

(d) True: $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\sin(\theta)| = 1$.

4. **(b)** Area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

(c) Volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

(d) $x + 2y - z = 0$.

5. $(x+3)^2 + (y-5)^2 + (z+4)^2 = r^2$, so

(a) $(x+3)^2+(y-5)^2+(z+4)^2 = 16$. **(b)** $(x+3)^2+(y-5)^2+(z+4)^2 = 25$. **(c)** $(x+3)^2+(y-5)^2+(z+4)^2 = 9$.

6. The sphere $x^2 + (y-1)^2 + (z+3)^2 = 16$ has center $Q(0, 1, -3)$ and radius 4, and $\|\overrightarrow{PQ}\| = \sqrt{1^2 + 4^2} = \sqrt{17}$, so minimum distance is $\sqrt{17} - 4$, maximum distance is $\sqrt{17} + 4$.

7. $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS} = 3\mathbf{i} + 4\mathbf{j} + \overrightarrow{QR} = 3\mathbf{i} + 4\mathbf{j} + (4\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 5\mathbf{j}$.

8. **(a)** $\langle 16, 0, 13\rangle$ **(b)** $\langle 2/\sqrt{17}, -2/\sqrt{17}, 3/\sqrt{17}\rangle$ **(c)** $\sqrt{35}$ **(d)** $\sqrt{66}$

9. **(a)** $\mathbf{a} \cdot \mathbf{b} = 0$, $4c + 3 = 0$, $c = -3/4$.

(b) Use $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta$ to get $4c+3 = \sqrt{c^2+1}(5)\cos(\pi/4)$, $4c+3 = 5\sqrt{c^2+1}/\sqrt{2}$. Square both sides and rearrange to get $7c^2 + 48c - 7 = 0$, $(7c-1)(c+7) = 0$ so $c = -7$ (invalid) or $c = 1/7$.

(c) Proceed as in (b) with $\theta = \pi/6$ to get $11c^2 - 96c + 39 = 0$ and use the quadratic formula to get $c = \left(48 \pm 25\sqrt{3}\right)/11$.

(d) $\mathbf{a}$ must be a scalar multiple of $\mathbf{b}$, so $c\mathbf{i} + \mathbf{j} = k(4\mathbf{i} + 3\mathbf{j}), k = 1/3, c = 4/3$.

10. **(a)** The plane through the origin which is perpendicular to $\mathbf{r}_0$.

(b) The plane through the tip of $\mathbf{r}_0$ which is perpendicular to $\mathbf{r}_0$.

11. $\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = 2(1 - \cos\theta) = 4\sin^2(\theta/2)$, so $\|\mathbf{u} - \mathbf{v}\| = 2\sin(\theta/2)$.

12. $5\langle \cos 60°, \cos 120°, \cos 135°\rangle = \langle 5/2, -5/2, -5\sqrt{2}/2\rangle$.

13. $\overrightarrow{PQ} = \langle 1, -1, 6\rangle$, and $W = \mathbf{F} \cdot \overrightarrow{PQ} = 13$ lb·ft.

14. $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \overrightarrow{PQ} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, W = \mathbf{F} \cdot \overrightarrow{PQ} = -11$ N·m $= -11$ J.

15. **(a)** $\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{AC} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\overrightarrow{AB} \times \overrightarrow{AC} = -4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, area $= \dfrac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{26}/2$.

(b) Area $= \dfrac{1}{2}h\|\overrightarrow{AB}\| = \dfrac{3}{2}h = \dfrac{1}{2}\sqrt{26}$, $h = \sqrt{26}/3$.

16. **(a)** False; perhaps they are orthogonal.

(b) False; perhaps they are parallel.

(c) True; $0 = \|\mathbf{u}\| \cdot \|\mathbf{v}\|\cos\theta = \|\mathbf{u}\| \cdot \|\mathbf{v}\|\sin\theta$, so either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ since $\cos\theta = \sin\theta = 0$ is impossible.

17. $\overrightarrow{AB} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}, \overrightarrow{AC} = -2\mathbf{i} - \mathbf{j} - 2\mathbf{k}, \overrightarrow{AD} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

(a) From Theorem 11.4.6 and formula (9) of Section 11.4, $\begin{vmatrix} 1 & -2 & -2 \\ -2 & -1 & -2 \\ 1 & 2 & -3 \end{vmatrix} = 29$, so $V = 29$.

(b) The plane containing A, B, and C has normal $\overrightarrow{AB} \times \overrightarrow{AC} = 2\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$, so the equation of the plane is $2(x-1) + 6(y+1) - 5(z-2) = 0, 2x + 6y - 5z = -14$. From Theorem 11.6.2, $D = \dfrac{|2(2) + 6(1) - 5(-1) + 14|}{\sqrt{65}} = \dfrac{29}{\sqrt{65}}$.

18. **(a)** $\mathbf{F} = -6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$.

(b) $\overrightarrow{OA} = \langle 5, 0, 2\rangle$, so the vector moment is $\overrightarrow{OA} \times \mathbf{F} = -6\mathbf{i} + 18\mathbf{j} + 15\mathbf{k}$.

19. $x = 4 + t$, $y = 1 - t$, $z = 2$.

20. **(a)** $\langle 2, 1, -1\rangle \times \langle 1, 2, 1\rangle = \langle 3, -3, 3\rangle$, so the line is parallel to $\mathbf{i} - \mathbf{j} + \mathbf{k}$. To find one common point, choose $x = 0$ and solve the system of equations $y - z = 3$, $2y + z = 3$ to obtain that $(0, 2, -1)$ lies on both planes, so the line has an equation $\mathbf{r} = 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - \mathbf{j} + \mathbf{k})$, that is, $x = t, y = 2 - t, z = -1 + t$.

(b) $\cos\theta = \dfrac{\langle 2, 1, -1\rangle \cdot \langle 1, 2, 1\rangle}{\|\langle 2, 1, -1\rangle\|\|\langle 1, 2, 1\rangle\|} = 1/2$, so $\theta = \pi/3$.

21. A normal to the plane is given by $\langle 1, 5, -1\rangle$, so the equation of the plane is of the form $x + 5y - z = D$. Insert $(1, 1, 4)$ to obtain $D = 2, x + 5y - z = 2$.

22. $(\mathbf{i}+\mathbf{k}) \times (2\mathbf{j}-\mathbf{k}) = -2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ is a normal to the plane, so an equation of the plane is of the form $-2x+y+2z = D, -2(4) + (3) + 2(0) = -5, -2x + y + 2z = -5$.

23. The normals to the planes are given by $\langle a_1, b_1, c_1\rangle$ and $\langle a_2, b_2, c_2\rangle$, so the condition is $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

24. **(b)** $(y, x, z), (x, z, y), (z, y, x)$.

(c) The set of points $\{(5, \theta, 1)\}, 0 \le \theta \le 2\pi$.

(d) The set of points $\{(\rho, \pi/4, 0)\}, 0 \le \rho < +\infty$.

25. **(a)** $(x - 3)^2 + 4(y + 1)^2 - (z - 2)^2 = 9$, hyperboloid of one sheet.

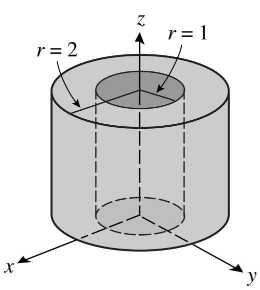
(b) $(x + 3)^2 + (y - 2)^2 + (z + 6)^2 = 49$, sphere.

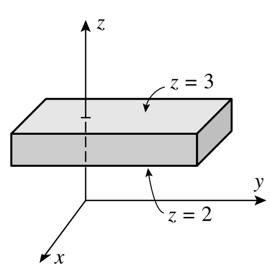
(c) $(x - 1)^2 + (y + 2)^2 - z^2 = 0$, circular cone.

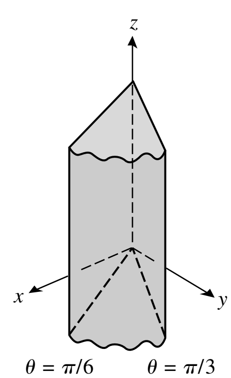
26. **(a)** $r^2 = z; \rho^2 \sin^2\phi = \rho\cos\phi, \rho = \cot\phi\csc\phi$.

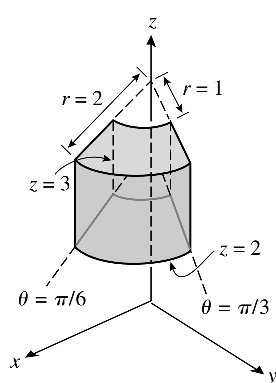
(b) $r^2(\cos^2\theta - \sin^2\theta) - z^2 = 0, z^2 = r^2\cos 2\theta; \rho^2\sin^2\phi\cos^2\theta - \rho^2\sin^2\phi\sin^2\theta - \rho^2\cos^2\phi = 0, \cos 2\theta = \cot^2\phi$.

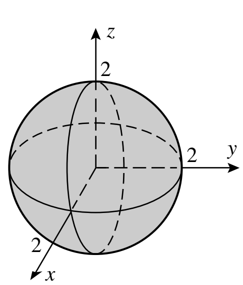
27. **(a)** $z = r^2\cos^2\theta - r^2\sin^2\theta = x^2 - y^2$. **(b)** $(\rho\sin\phi\cos\theta)(\rho\cos\phi) = 1, xz = 1$.

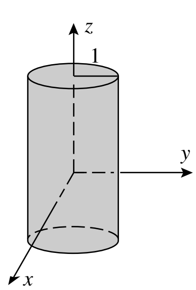
28. **(a)**


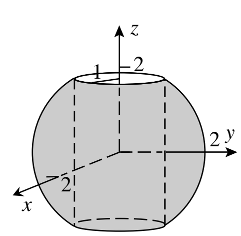
(b)


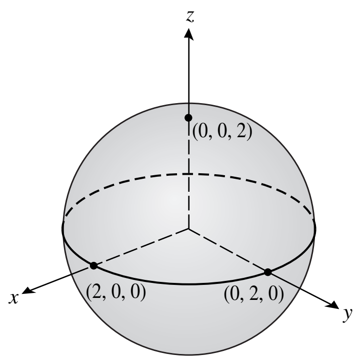
(c)


(d)


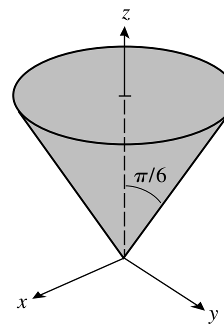
29. **(a)**


(b)


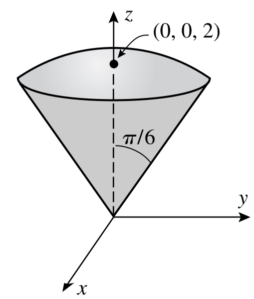
(c)




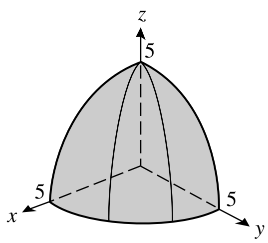
30. **(a)**



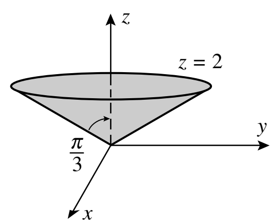
(b)



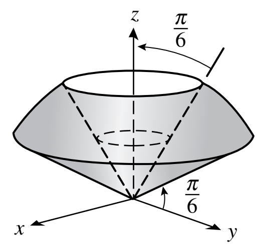
(c)



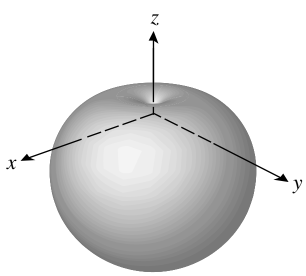
31. **(a)**



(b)



(c)



32.

Chapter 11 Making Connections

1. **(a)** $R(x\mathbf{i}+y\mathbf{j})\cdot(x\mathbf{i}+y\mathbf{j})=-yx+xy=0$, so they are perpendicular. From $R(\mathbf{i})=\mathbf{j}$ and $R(\mathbf{j})=-\mathbf{i}$ it follows that R rotates vectors counterclockwise.

(b) If $\mathbf{v}=x\mathbf{i}+y\mathbf{j}$ and $\mathbf{w}=r\mathbf{i}+s\mathbf{j}$, then $R(c\mathbf{v})=R(c[x\mathbf{i}+y\mathbf{j}])=R((cx)\mathbf{i}+(cy)\mathbf{j})=-cy\mathbf{i}+cx\mathbf{j}=c[-y\mathbf{i}+x\mathbf{j}]=cR(x\mathbf{i}+y\mathbf{j})=cR(\mathbf{v})$ and $R(\mathbf{v}+\mathbf{w})=R([x\mathbf{i}+y\mathbf{j}]+[r\mathbf{i}+s\mathbf{j}])=R((x+r)\mathbf{i}+(y+s)\mathbf{j})=-(y+s)\mathbf{i}+(x+r)\mathbf{j}=(-y\mathbf{i}+x\mathbf{j})+(-s\mathbf{i}+r\mathbf{j})=R(x\mathbf{i}+y\mathbf{j})+R(r\mathbf{i}+s\mathbf{j})=R(\mathbf{v})+R(\mathbf{w})$.

2. Although the problem is a two-dimensional one, we add a dimension in order to use the cross-product. Let the triangle be part of the x-y plane, and introduce the z-direction with the unit vector $\mathbf{k}$. Let the vertices of the triangle be A, B, C taken in a counter-clockwise fashion. Then under a right-handed system, $\mathbf{k}\times\overrightarrow{AB}$ is a vector $\mathbf{n}_1$ which is perpendicular to $\mathbf{k}$, and therefore in the plane, and perpendicular to $\overrightarrow{AB}$, so it is normal to the side AB, and in fact it is an exterior normal because of the right-handedness of the system $\{\overrightarrow{AB},\mathbf{n_1},\mathbf{k}\}$, and, finally, $\|\mathbf{n_1}\|=\|\mathbf{k}\|\|\overrightarrow{AB}\|\sin\theta=\|\overrightarrow{AB}\|$. Similarly, we define $\mathbf{n_2}=\overrightarrow{BC}\times\mathbf{k}$ and $\mathbf{n_3}=\overrightarrow{CA}\times\mathbf{k}$.

(a) With the definitions above we have $\mathbf{n_1}+\mathbf{n_2}+\mathbf{n_3}=(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA})\times\mathbf{k}=\overrightarrow{AA}\times\mathbf{k}=\mathbf{0}\times\mathbf{k}=\mathbf{0}$.

(b) Given a polygon with vertices $A_1, A_2, \ldots, A_k$ (define $A_0=A_k$) we define normal vectors $\mathbf{n_1},\mathbf{n_2},\ldots,\mathbf{n_k}$ in the manner described above; and then $\displaystyle\sum_{i=1}^{k}\mathbf{n_i}=\left(\sum_{i=1}^{k}\overrightarrow{A_{i-1}A_i}\right)\times\mathbf{k}=(\overrightarrow{A_1A_1})\times\mathbf{k}=\mathbf{0}$.

3. **(a)** Suppose one face lies in the x-y plane and has vertices A, B, C taken in counter-clockwise order as one traverses the boundary of the triangle looking down. Then the outer normal to triangle ABC points down. One normal to triangle ABC is given by $\overrightarrow{CB} \times \overrightarrow{BA}$. The length of this vector is twice the area of the triangle (Theorem 11.4.5), so we take $\mathbf{n_1} = \frac{1}{2}(\overrightarrow{CB} \times \overrightarrow{BA})$. Similarly, $\mathbf{n_2} = \frac{1}{2}(\overrightarrow{BC} \times \overrightarrow{CD})$, $\mathbf{n_3} = \frac{1}{2}(\overrightarrow{AD} \times \overrightarrow{DC})$, $\mathbf{n_4} = \frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{BD})$. Then $2(\mathbf{n_1} + \mathbf{n_2} + \mathbf{n_3} + \mathbf{n_4}) = \overrightarrow{AB} \times (\overrightarrow{BD} + \overrightarrow{CB}) + \overrightarrow{CD} \times (\overrightarrow{CB} + \overrightarrow{AD}) = \overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{CD} \times (\overrightarrow{CB} + \overrightarrow{AD}) = \overrightarrow{CD} \times (\overrightarrow{BA} + \overrightarrow{CB} + \overrightarrow{AD}) = \overrightarrow{CD} \times \overrightarrow{CD} = \mathbf{0}$.

(b) Use the hint and note that everything works out except the two normal vectors on the face which actually divides the larger tetrahedron (pyramid, four-sided base) into the smaller ones with triangular bases. But the normal vectors point in opposite directions and have the same magnitude (the area of the common face) and thus cancel in all the calculations.

(c) Consider a polyhedron each face of which is a triangle, save possibly one which is an arbitrary polygon. Then this last face can be broken into triangles and the results of parts (a) and (b) can be applied, with the same conclusion, that the sum of the exterior normals is the zero vector.

4. The three faces that meet at the chosen vertex are called A, B and C; let the fourth face be D, with area d. Using Exercise 3 choose a vector n_A which is normal to face A, likewise n_B, n_C, n_D. Each vector is assumed to have length equal to the area of the corresponding face.

(a) Since the sum of the normal vectors is the zero vector, we have $0 = (n_D + n_A + n_B + n_C) \cdot (n_D - n_A - n_B - n_C) = d^2 - a^2 - b^2 - c^2 - 2ab\cos\alpha - 2bc\cos\beta - 2ac\cos\gamma$.

(b) If all of the angles formed at the chosen vertex are right angles, then $d^2 = a^2 + b^2 + c^2$. (Note that such a tetrahedron could be considered a corner cut from a rectangular solid).

5. Let P and Q have spherical coordinates $(\rho, \theta_i, \phi_i), i = 1, 2$. Then Cartesian coordinates are given by $(\rho\sin\phi_i\cos\theta_i, \rho\sin\phi_i\sin\theta_i, \rho\cos\phi_i), i = 1, 2$, and the distance between the two points as taken on the great circle is $\rho\cos\alpha$, where α is the angle between the vectors that go from the origin to the points P and Q. Taking dot products, we have $\rho^2\cos\alpha = \rho^2(\sin\phi_1\sin\phi_2\cos\theta_1\cos\theta_2 + \sin\phi_1\sin\phi_2\sin\theta_1\sin\theta_2 + \cos\phi_1\cos\phi_2) = \rho^2(\sin\phi_1\sin\phi_2\cos(\theta_1 - \theta_2) + \cos\phi_1\cos\phi_2)$, and the great circle distance is given by $d = \rho\cos^{-1}(\sin\phi_1\sin\phi_2\cos(\theta_1 - \theta_2) + \cos\phi_1\cos\phi_2)$.

6. Using spherical coordinates: for point A, $\theta_A = 360° - 60° = 300°, \phi_A = 90° - 40° = 50°$; for point B, $\theta_B = 360° - 40° = 320°, \phi_B = 90° - 20° = 70°$. Unit vectors directed from the origin to the points A and B, respectively, are $\mathbf{u}_A = \sin 50°\cos 300°\mathbf{i} + \sin 50°\sin 300°\mathbf{j} + \cos 50°\mathbf{k}$, and $\mathbf{u}_B = \sin 70°\cos 320°\mathbf{i} + \sin 70°\sin 320°\mathbf{j} + \cos 70°\mathbf{k}$. The angle α between $\mathbf{u}_A$ and $\mathbf{u}_B$ is $\alpha = \cos^{-1}(\mathbf{u}_A \cdot \mathbf{u}_B) \approx 0.459486$ so the shortest distance is $6370\alpha \approx 2927$ km.