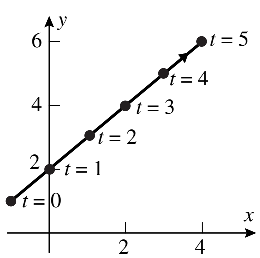
Parametric and Polar Curves; Conic Sections

Exercise Set 10.1

1. (a) $x+1=t=y-1$, $y=x+2$.

(c)

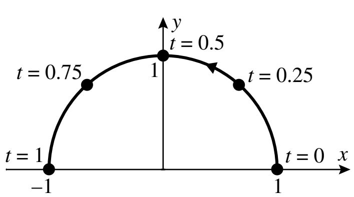
t	0	1	2	3	4	5
x	-1	0	1	2	3	4
y	1	2	3	4	5	6



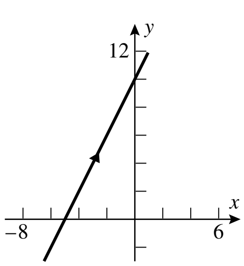
2. (a) $x^2+y^2=1$.

(c)

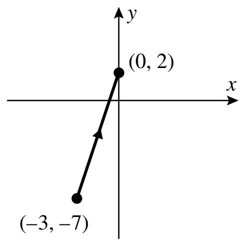
t	0	0.2500	0.50	0.7500	1
x	1	0.7071	0.00	-0.7071	-1
y	0	0.7071	1.00	0.7071	0



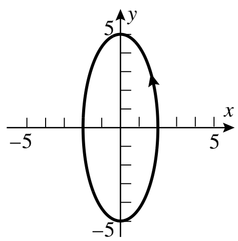
3. $t=(x+4)/3$; $y=2x+10$.



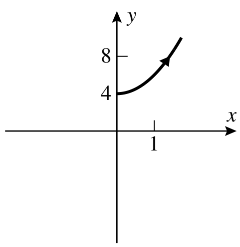
4. $t=x+3$; $y=3x+2$, $-3\le x\le 0$.



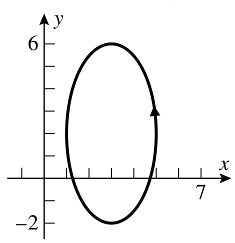
5. $\cos t = x/2$, $\sin t = y/5$; $x^2/4 + y^2/25 = 1$.



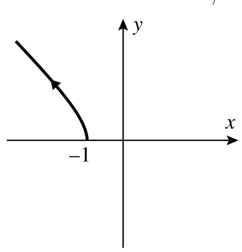
6. $t = x^2$; $y = 2x^2 + 4$, $x \geq 0$.



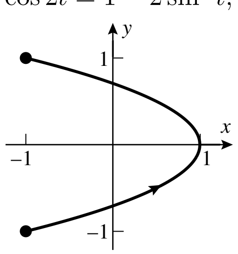
7. $\cos t = (x-3)/2$, $\sin t = (y-2)/4$; $(x-3)^2/4 + (y-2)^2/16 = 1$.



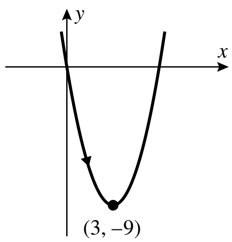
8. $\sec^2 t - \tan^2 t = 1$; $x^2 - y^2 = 1$, $x \leq -1$ and $y \geq 0$.



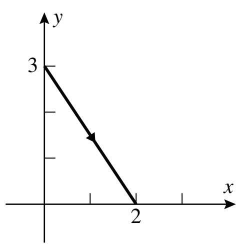
9. $\cos 2t = 1 - 2\sin^2 t$; $x = 1 - 2y^2$, $-1 \leq y \leq 1$.



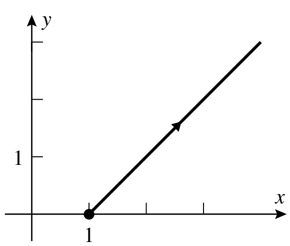
10. $t = (x-3)/4$; $y = (x-3)^2 - 9$.



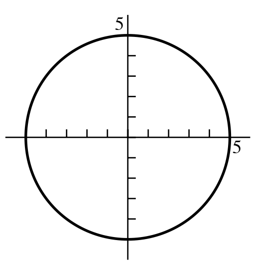
11. $x/2 + y/3 = 1$, $0 \le x \le 2$, $0 \le y \le 3$.



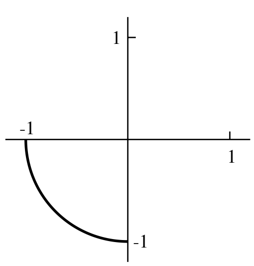
12. $y = x - 1$, $x \ge 1$, $y \ge 0$



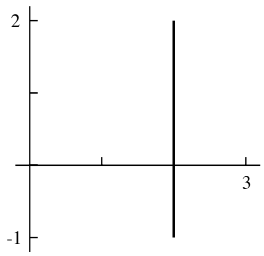
13. $x = 5\cos t$, $y = -5\sin t$, $0 \le t \le 2\pi$.



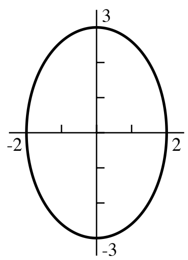
14. $x = \cos t$, $y = \sin t$, $\pi \le t \le 3\pi/2$.



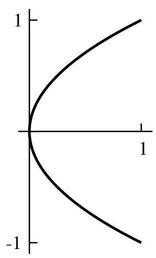
15. $x = 2$, $y = t$.



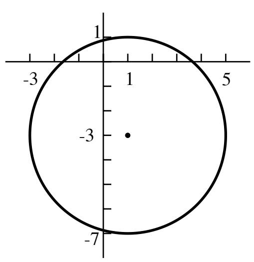
16. $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$.



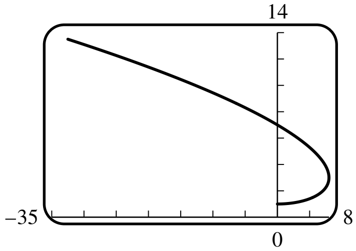
17. $x = t^2$, $y = t$, $-1 \le t \le 1$.



18. $x = 1 + 4\cos t$, $y = -3 + 4\sin t$, $0 \le t \le 2\pi$.



19. (a)

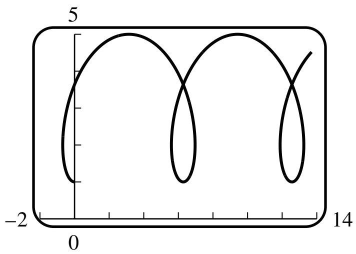


(b)

t	0	1	2	3	4	5
x	0	5.5	8	4.5	-8	-32.5
y	1	1.5	3	5.5	9	13.5

(c) $x = 0$ when $t = 0, 2\sqrt{3}$. **(d)** For $0 < t < 2\sqrt{2}$. **(e)** At $t = 2$.

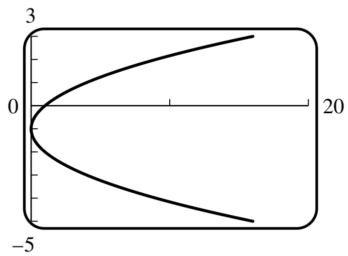
20. (a)



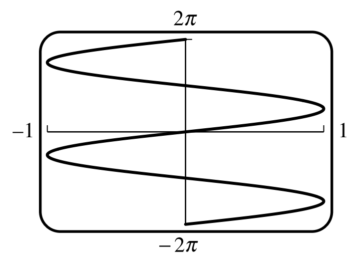
(b) y is always ≥ 1 since $\cos t \leq 1$.

(c) Greater than 5, since $\cos t \geq -1$.

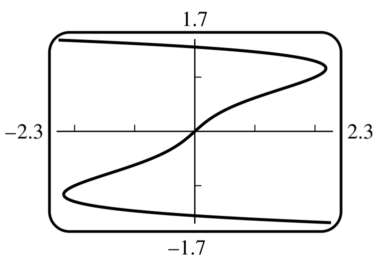
21. (a)



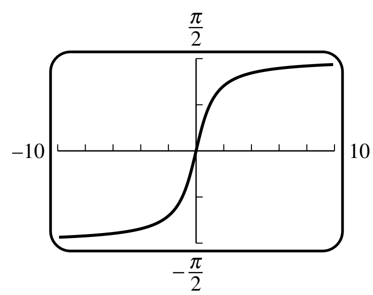
(b)



22. (a)



(b)



23. (a) IV, because x always increases whereas y oscillates.

(b) II, because $(x/2)^2 + (y/3)^2 = 1$, an ellipse.

(c) V, because $x^2 + y^2 = t^2$ increases in magnitude while x and y keep changing sign.

(d) VI; examine the cases $t < -1$ and $t > -1$ and you see the curve lies in the first, second and fourth quadrants only.

(e) III, because $y > 0$.

(f) I; since x and y are bounded, the answer must be I or II; since $x = y = 0$ when $t = \pi/2$, the curve passes through the origin, so it must be I.

24. (a) (a) (IV): from left to right. (b) (II): counterclockwise. (c) (V): counterclockwise. (d) (VI): As t travels from $-\infty$ to -1, the curve goes from (near) the origin in the fourth quadrant and travels down and right. As t travels from -1 to $+\infty$ the curve comes from way up in the second quadrant, hits the origin at $t = 0$, and then makes the loop in the first quadrant counterclockwise and finally approaches the origin again as $t \to +\infty$. (e) (III): from left to right. (f) (I): Starting, say, at $(1/2, 0)$ at $t = 0$, the curve goes up into the first quadrant, loops back through the origin and into the third quadrant, and then continues the figure-eight.

(b) The two branches corresponding to $-1 \leq t \leq 0$ and $0 \leq t \leq 1$ coincide, with opposite directions.

25. (a) $|R - P|^2 = (x - x_0)^2 + (y - y_0)^2 = t^2[(x_1 - x_0)^2 + (y_1 - y_0)^2]$ and $|Q - P|^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2$, so $r = |R - P| = |Q - P|t = qt$.

(b) $t = 1/2$. **(c)** $t = 3/4$.

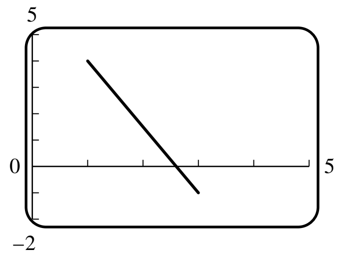
26. $x = 2 + t$, $y = -1 + 2t$.

(a) $(5/2, 0)$ **(b)** $(9/4, -1/2)$ **(c)** $(11/4, 1/2)$

27. (a) Eliminate $\dfrac{t - t_0}{t_1 - t_0}$ from the parametric equations to obtain $\dfrac{y - y_0}{x - x_0} = \dfrac{y_1 - y_0}{x_1 - x_0}$, which is an equation of the line through the 2 points.

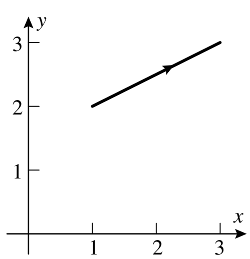
(b) From (x_0, y_0) to (x_1, y_1).

(c) $x = 3 - 2(t - 1)$, $y = -1 + 5(t - 1)$.



28. (a) If $a \neq 0$ then $t = \dfrac{x - b}{a}$ and $y = c\dfrac{x - b}{a} + d = \dfrac{c}{a}x + \left(d - \dfrac{bc}{a}\right)$ so the graph is the part of the line $y = \dfrac{c}{a}x + \left(d - \dfrac{bc}{a}\right)$ with x between $at_0 + b$ and $at_1 + b$. If $a = 0$ then $c \neq 0$ and the graph is the part of the vertical line $x = b$ with y between $ct_0 + d$ and $ct_1 + d$.

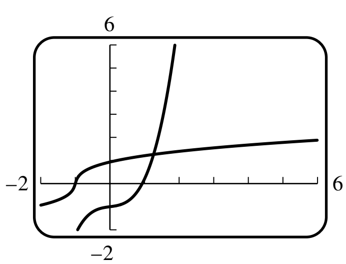
(b)



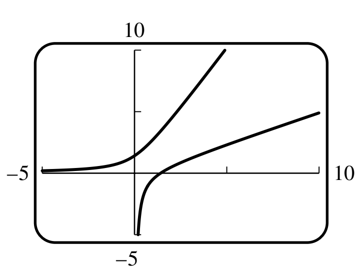
(c) If $a = 0$ the line segment is vertical; if $c = 0$ it is horizontal.

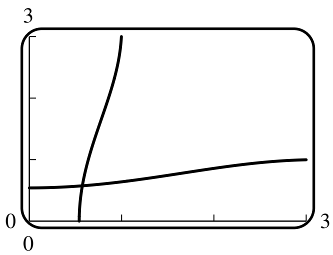
(d) The curve degenerates to the point (b, d).

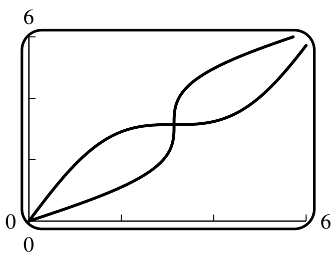
29.



30.



31. 

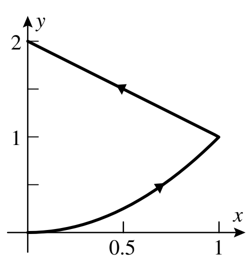
32. 

33. False. The parametric curve only gives the part of $y = 1 - x^2$ with $-1 \le x \le 1$.

34. False. It is the reflection of $y = f(x)$ across the line $y = x$.

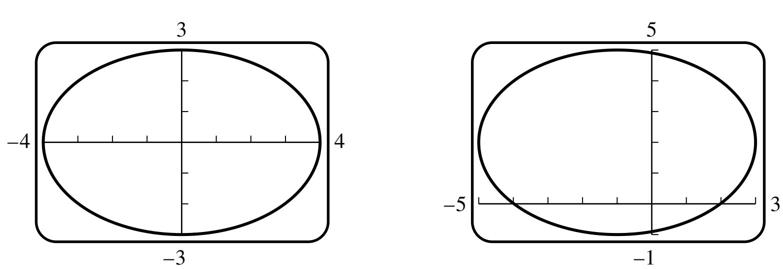
35. True. By equation (4), $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{12t^3 - 6t^2}{x'(t)}$.

36. False. $t = x^{1/3}$ so $y = x^2 + x^{1/3}$, $y' = 2x + \dfrac{1}{3}x^{-2/3}$, and $y'' = 2 - \dfrac{2}{9}x^{-5/3}$. For $t < 0$, $x < 0$ and $y'' > 2$, so the curve is concave up for $t < 0$. In fact the only part of the curve which is concave down is the part with $0 < x < 9^{-3/5}$; i.e. $0 < t < 9^{-1/5}$.

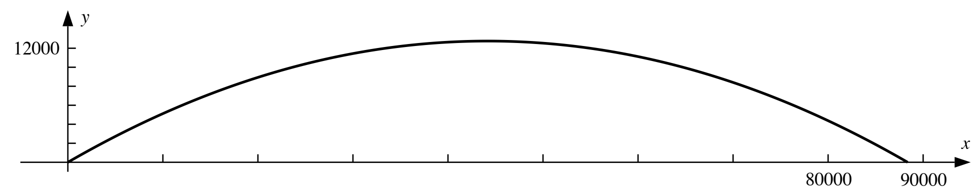
37. 

38. $x = 1/2 - 4t$, $y = 1/2$ for $0 \le t \le 1/4$; $x = -1/2$, $y = 1/2 - 4(t - 1/4)$ for $1/4 \le t \le 1/2$; $x = -1/2 + 4(t - 1/2)$, $y = -1/2$ for $1/2 \le t \le 3/4$; $x = 1/2$, $y = -1/2 + 4(t - 3/4)$ for $3/4 \le t \le 1$.

39. **(a)** $x = 4\cos t$, $y = 3\sin t$. **(b)** $x = -1 + 4\cos t$, $y = 2 + 3\sin t$.

(c) 

40. **(a)** $t = \dfrac{x}{v_0 \cos\alpha}$, so $y = -\dfrac{g}{2v_0^2\cos^2\alpha}\,x^2 + (\tan\alpha)\,x$.

(b) 

(c) The parametric equations are $x = 500\sqrt{3}t$, $y = 500t - 4.9t^2$. So $\dfrac{dy}{dt} = 500 - 9.8t = 0$ when $t = \dfrac{500}{9.8} = \dfrac{2500}{49}$. The maximum height is $y\left(\dfrac{2500}{49}\right) = \dfrac{625,000}{49} \approx 12755$ m.

(d) $y = t(500 - 4.9t) = 0$ when $t = 0$ or $t = \dfrac{500}{4.9} = \dfrac{5000}{49}$. So the horizontal distance is $x\left(\dfrac{5000}{49}\right) = \dfrac{2,500,000\sqrt{3}}{49} \approx 88370$ m.

41. (a) $dy/dx = \dfrac{2t}{1/2} = 4t$; $dy/dx\big|_{t=-1} = -4$; $dy/dx\big|_{t=1} = 4$.

(b) $y = (2x)^2 + 1, dy/dx = 8x, dy/dx\big|_{x=\pm(1/2)} = \pm 4$.

42. (a) $\dfrac{dy}{dx} = \dfrac{4\cos t}{-3\sin t} = -\dfrac{4}{3}\cot t$; $\left.\dfrac{dy}{dx}\right|_{t=\pi/4} = -\dfrac{4}{3}$, $\left.\dfrac{dy}{dx}\right|_{t=7\pi/4} = \dfrac{4}{3}$.

(b) Since $\dfrac{x^2}{9} + \dfrac{y^2}{16} = \cos^2 t + \sin^2 t = 1$, we have $y = \pm\dfrac{4}{3}\sqrt{9 - x^2}$. For $0 \le t \le \pi$, $y \ge 0$ so $y = \dfrac{4}{3}\sqrt{9 - x^2}$ and $\dfrac{dy}{dx} = -\dfrac{4x}{3\sqrt{9 - x^2}}$; at $t = \dfrac{\pi}{4}$, $x = \dfrac{3}{\sqrt{2}}$ and $\dfrac{dy}{dx} = -\dfrac{4}{3}$. For $\pi \le t \le 2\pi$, $y \le 0$ so $y = -\dfrac{4}{3}\sqrt{9 - x^2}$ and $\dfrac{dy}{dx} = \dfrac{4x}{3\sqrt{9 - x^2}}$; at $t = \dfrac{7\pi}{4}$, $x = \dfrac{3}{\sqrt{2}}$ and $\dfrac{dy}{dx} = \dfrac{4}{3}$.

43. From Exercise 41(a), $\dfrac{dy}{dx} = 4t$ so $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \Big/ \dfrac{dx}{dt} = \dfrac{4}{1/2} = 8$. The sign of $\dfrac{d^2y}{dx^2}$ is positive for all t, including $t = \pm 1$.

44. From Exercise 42(a), $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \Big/ \dfrac{dx}{dt} = \dfrac{-(4/3)(-\csc^2 t)}{-3\sin t} = -\dfrac{4}{9}\csc^3 t$; negative at $t = \pi/4$, positive at $t = 7\pi/4$.

45. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2}{1/(2\sqrt{t})} = 4\sqrt{t}$, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \Big/ \dfrac{dx}{dt} = \dfrac{2/\sqrt{t}}{1/(2\sqrt{t})} = 4$, $\left.\dfrac{dy}{dx}\right|_{t=1} = 4$, $\left.\dfrac{d^2y}{dx^2}\right|_{t=1} = 4$.

46. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t^2 - 1}{t} = t - \dfrac{1}{t}$, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \Big/ \dfrac{dx}{dt} = \left(1 + \dfrac{1}{t^2}\right) \Big/ t = \dfrac{t^2 + 1}{t^3}$, $\left.\dfrac{dy}{dx}\right|_{t=2} = \dfrac{3}{2}$, $\left.\dfrac{d^2y}{dx^2}\right|_{t=2} = \dfrac{5}{8}$

47. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\sec^2 t}{\sec t \tan t} = \csc t$, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \Big/ \dfrac{dx}{dt} = \dfrac{-\csc t \cot t}{\sec t \tan t} = -\cot^3 t$, $\left.\dfrac{dy}{dx}\right|_{t=\pi/3} = \dfrac{2}{\sqrt{3}}$, $\left.\dfrac{d^2y}{dx^2}\right|_{t=\pi/3} = -\dfrac{1}{3\sqrt{3}}$.

48. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\sinh t}{\cosh t} = \tanh t$, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \Big/ \dfrac{dx}{dt} = \dfrac{\operatorname{sech}^2 t}{\cosh t} = \operatorname{sech}^3 t$, $\left.\dfrac{dy}{dx}\right|_{t=0} = 0$, $\left.\dfrac{d^2y}{dx^2}\right|_{t=0} = 1$.

49. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\cos\theta}{1-\sin\theta}$; $\dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)\Big/\dfrac{dx}{d\theta} = \dfrac{(1-\sin\theta)(-\sin\theta)+\cos^2\theta}{(1-\sin\theta)^2}\dfrac{1}{1-\sin\theta} = \dfrac{1}{(1-\sin\theta)^2}$;

$\left.\dfrac{dy}{dx}\right|_{\theta=\pi/6} = \dfrac{\sqrt{3}/2}{1-1/2} = \sqrt{3}$; $\left.\dfrac{d^2y}{dx^2}\right|_{\theta=\pi/6} = \dfrac{1}{(1-1/2)^2} = 4$.

50. $\dfrac{dy}{dx} = \dfrac{dy/d\phi}{dx/d\phi} = \dfrac{3\cos\phi}{-\sin\phi} = -3\cot\phi$; $\dfrac{d^2y}{dx^2} = \dfrac{d}{d\phi}\left(\dfrac{dy}{dx}\right)\Big/\dfrac{dx}{d\phi} = \dfrac{-3(-\csc^2\phi)}{-\sin\phi} = -3\csc^3\phi$; $\left.\dfrac{dy}{dx}\right|_{\phi=5\pi/6} = 3\sqrt{3}$; $\left.\dfrac{d^2y}{dx^2}\right|_{\phi=5\pi/6} = -24$.

51. (a) $dy/dx = \dfrac{-e^{-t}}{e^t} = -e^{-2t}$; for $t=1$, $dy/dx = -e^{-2}$, $(x,y) = (e, e^{-1})$; $y - e^{-1} = -e^{-2}(x-e)$, $y = -e^{-2}x + 2e^{-1}$.

(b) $y = 1/x, dy/dx = -1/x^2, m = -1/e^2, y - e^{-1} = -\dfrac{1}{e^2}(x-e), y = -\dfrac{1}{e^2}x + \dfrac{2}{e}$.

52. At $t=1$, $x=6$ and $y=10$.

(a) $\dfrac{dy}{dx} = \dfrac{16t-2}{2} = 8t-1$; for $t=1$, $\dfrac{dy}{dx} = 7$. The tangent line is $y-10 = 7(x-6)$, $y = 7x-32$.

(b) $t = \dfrac{x-4}{2}$ so $y = 2x^2 - 17x + 40$. At $t=1$, $x=6$ so $\dfrac{dy}{dx} = 4x-17 = 7$ and the tangent line is $y-10 = 7(x-6)$, $y = 7x-32$.

53. $dy/dx = \dfrac{-4\sin t}{2\cos t} = -2\tan t$.

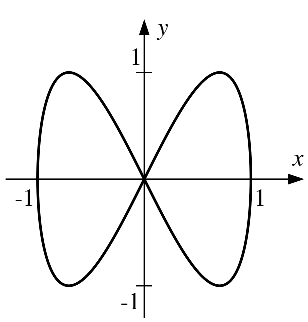
(a) $dy/dx = 0$ if $\tan t = 0$; $t = 0, \pi, 2\pi$.

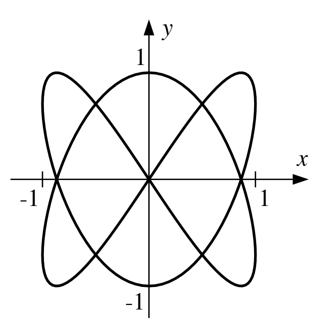
(b) $dx/dy = -\dfrac{1}{2}\cot t = 0$ if $\cot t = 0$; $t = \pi/2, 3\pi/2$.

54. $dy/dx = \dfrac{2t+1}{6t^2-30t+24} = \dfrac{2t+1}{6(t-1)(t-4)}$.

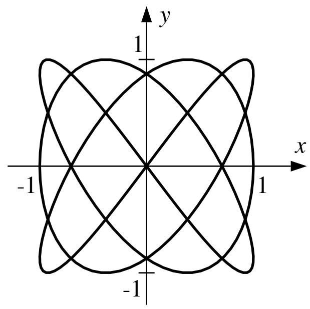
(a) $dy/dx = 0$ if $t = -1/2$.

(b) $dx/dy = \dfrac{6(t-1)(t-4)}{2t+1} = 0$ if $t = 1, 4$.

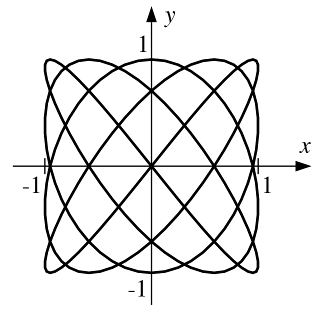




55. (a) $a=1$, $b=2$. $a=2$, $b=3$.



$a = 3,\ b = 4.$



$a = 4,\ b = 5.$

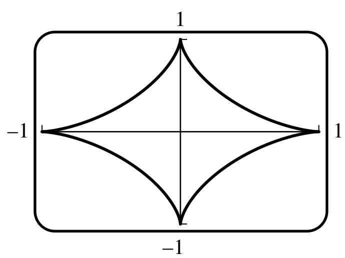
(b) $x = y = 0$ when $t = 0, \pi$; $\dfrac{dy}{dx} = \dfrac{2\cos 2t}{\cos t}$; $\left.\dfrac{dy}{dx}\right|_{t=0} = 2$, $\left.\dfrac{dy}{dx}\right|_{t=\pi} = -2$, the equations of the tangent lines are $y = -2x, y = 2x$.

56. $y(t) = 0$ has three solutions, $t = 0, \pm\pi/2$; the last two correspond to the crossing point. For $t = \pm\pi/2$, $m = \dfrac{dy}{dx} = \dfrac{2}{\pm\pi}$; the tangent lines are given by $y = \pm\dfrac{2}{\pi}(x-2)$.

57. If $x = 4$ then $t^2 = 4$, $t = \pm 2$, $y = 0$ for $t = \pm 2$ so $(4, 0)$ is reached when $t = \pm 2$. $dy/dx = (3t^2 - 4)/2t$. For $t = 2$, $dy/dx = 2$ and for $t = -2$, $dy/dx = -2$. The tangent lines are $y = \pm 2(x - 4)$.

58. If $x = 3$ then $t^2 - 3t + 5 = 3$, $t^2 - 3t + 2 = 0$, $(t-1)(t-2) = 0$, $t = 1$ or 2. If $t = 1$ or 2 then $y = 1$ so $(3, 1)$ is reached when $t = 1$ or 2. $dy/dx = (3t^2 + 2t - 10)/(2t - 3)$. For $t = 1$, $dy/dx = 5$, the tangent line is $y - 1 = 5(x - 3)$, $y = 5x - 14$. For $t = 2$, $dy/dx = 6$, the tangent line is $y - 1 = 6(x - 3)$, $y = 6x - 17$.

59. (a)



(b) $\dfrac{dx}{dt} = -3\cos^2 t \sin t$ and $\dfrac{dy}{dt} = 3\sin^2 t \cos t$ are both zero when $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$, so singular points occur at these values of t.

60. By equations (4) and (10), $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a\sin\theta}{a - a\cos\theta} = \dfrac{\sin\theta}{1-\cos\theta} = \dfrac{1+\cos\theta}{\sin\theta}$. The x-intercepts occur when $y = a - a\cos\theta = 0$, so $\cos\theta = 1$ and $\theta = 2\pi n$ for some integer n. As $\theta \to (2\pi n)^-$, $\sin\theta \to 0^-$ and $\cos\theta \to 1$, so $\dfrac{dy}{dx} = \dfrac{1+\cos\theta}{\sin\theta} \to -\infty$. As $\theta \to (2\pi n)^+$, $\sin\theta \to 0^+$ and $\cos\theta \to 1$, so $\dfrac{dy}{dx} = \dfrac{1+\cos\theta}{\sin\theta} \to +\infty$. Hence there are cusps at the x-intercepts. When $\theta = \pi + 2\pi n$ for some integer n, we have $x = \pi a(2n+1)$, which is halfway between the x-intercepts $x = 2\pi a n$ and $x = 2\pi a(n+1)$. At these points, $\dfrac{dy}{dx} = \dfrac{\sin\theta}{1-\cos\theta} = \dfrac{0}{1-(-1)} = 0$, so the tangent line is horizontal.

61. (a) From (6), $\dfrac{dy}{dx} = \dfrac{3\sin t}{1 - 3\cos t}$.

(b) At $t = 10$, $\dfrac{dy}{dx} = \dfrac{3\sin 10}{1 - 3\cos 10} \approx -0.46402$, $\theta \approx \tan^{-1}(-0.46402) = -0.4345$.

62. (a) $\dfrac{dy}{dx} = 0$ when $\dfrac{dy}{dt} = -2\cos t = 0, t = \pi/2, 3\pi/2, 5\pi/2$.

(b) $\dfrac{dx}{dt} = 0$ when $1 + 2\sin t = 0, \sin t = -1/2, t = 7\pi/6, 11\pi/6, 19\pi/6$.

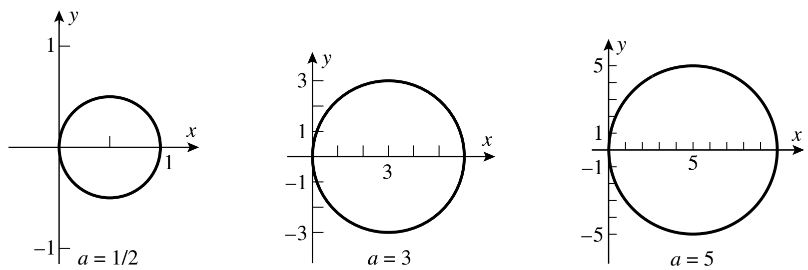
63. Eliminate the parameter to get $(x-h)^2/a^2 + (y-k)^2/b^2 = 1$, which is the equation of an ellipse centered at (h, k). Depending on the relative sizes of h and k, the ellipse may be a circle, or may have a horizontal or vertical major axis.

(a) Ellipses with a fixed center and varying shapes and sizes.

(b) Ellipses with varying centers and fixed shape and size.

(c) Circles of radius 1 with centers on the line $y = x - 1$.

64. **(a)**



(b) $(x-a)^2 + y^2 = (2a\cos^2 t - a)^2 + (2a\cos t \sin t)^2 = 4a^2\cos^4 t - 4a^2\cos^2 t + a^2 + 4a^2\cos^2 t \sin^2 t = 4a^2\cos^4 t - 4a^2\cos^2 t + a^2 + 4a^2\cos^2 t(1 - \cos^2 t) = a^2$, a circle about $(a, 0)$ of radius a.

65. $L = \displaystyle\int_0^1 \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^1 \sqrt{(2t)^2 + (t^2)^2}\, dt = \int_0^1 t\sqrt{4 + t^2}\, dt$. Let $u = 4 + t^2$, $du = 2t\, dt$. Then $L = \displaystyle\int_4^5 \frac{1}{2}\sqrt{u}\, du = \frac{1}{3}u^{3/2}\Big]_4^5 = \frac{1}{3}(5\sqrt{5} - 8)$.

66. Let $t = u^2$; the curve is also parameterized by $x = u - 2$, $y = 2u^{3/2}$, $(1 \le u \le 4)$. So $L = \displaystyle\int_1^4 \sqrt{(dx/du)^2 + (dy/du)^2}\, du = \int_1^4 \sqrt{1 + 9u}\, du = \frac{2}{27}(1 + 9u)^{3/2}\Big]_1^4 = \frac{2}{27}(37\sqrt{37} - 10\sqrt{10})$.

67. The curve is a circle of radius 1, traced one and a half times, so the arc length is $\dfrac{3}{2} \cdot 2\pi \cdot 1 = 3\pi$.

68. $L = \displaystyle\int_0^\pi \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^\pi \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2}\, dt = \int_0^\pi \sqrt{2(\cos^2 t + \sin^2 t)}\, dt =$
$= \int_0^\pi \sqrt{2}\, dt = \sqrt{2}\pi$.

69. $L = \displaystyle\int_{-1}^1 \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_{-1}^1 \sqrt{[e^{2t}(3\cos t + \sin t)]^2 + [e^{2t}(3\sin t - \cos t)]^2}\, dt = \int_{-1}^1 \sqrt{10}\, e^{2t}\, dt =$
$= \dfrac{1}{2}\sqrt{10}\, e^{2t}\Big]_{-1}^1 = \dfrac{1}{2}\sqrt{10}\,(e^2 - e^{-2})$.

70. $L = \displaystyle\int_0^{1/2} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^{1/2} \sqrt{\left(\frac{2}{\sqrt{1-t^2}}\right)^2 + \left(\frac{-2t}{1-t^2}\right)^2}\, dt = \int_0^{1/2} \frac{2}{1-t^2}\, dt = \ln\left|\frac{t+1}{t-1}\right|\Big]_0^{1/2} =$ $\ln 3$.

71. **(a)** $(dx/d\theta)^2 + (dy/d\theta)^2 = (a(1 - \cos\theta))^2 + (a\sin\theta)^2 = a^2(2 - 2\cos\theta)$, so $L = \displaystyle\int_0^{2\pi} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta =$ $a\displaystyle\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$.

(b) If you type the definite integral from (a) into your CAS, the output should be something equivalent to "8a". Here's a proof that doesn't use a CAS: $\cos\theta = 1 - 2\sin^2(\theta/2)$, so $2(1-\cos\theta) = 4\sin^2(\theta/2)$, and $L = a\int_0^{2\pi}\sqrt{2(1-\cos\theta)}\,d\theta = a\int_0^{2\pi} 2\sin(\theta/2)\,d\theta = -4a\cos(\theta/2)\Big]_0^{2\pi} = 8a$.

72. From (10), $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\sin\theta}{1-\cos\theta}$, so $\left(1+\left(\dfrac{dy}{dx}\right)^2\right)y = \left(1+\dfrac{\sin^2\theta}{(1-\cos\theta)^2}\right)(a - a\cos\theta) =$

$= a\,\dfrac{(1-\cos\theta)^2+\sin^2\theta}{1-\cos\theta} = a\,\dfrac{2-2\cos\theta}{1-\cos\theta} = 2a$.

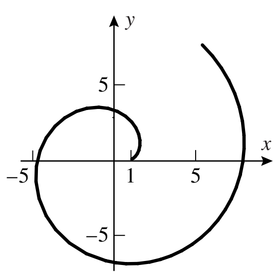
73. **(a)** The end of the inner arm traces out the circle $x_1 = \cos t, y_1 = \sin t$. Relative to the end of the inner arm, the outer arm traces out the circle $x_2 = \cos 2t, y_2 = -\sin 2t$. Add to get the motion of the center of the rider cage relative to the center of the inner arm: $x = \cos t + \cos 2t, y = \sin t - \sin 2t$.

(b) Same as part (a), except $x_2 = \cos 2t, y_2 = \sin 2t$, so $x = \cos t + \cos 2t, y = \sin t + \sin 2t$.

(c) $L_1 = \displaystyle\int_0^{2\pi}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} dt = \int_0^{2\pi}\sqrt{5-4\cos 3t}\,dt \approx 13.36489321$, $L_2 = \displaystyle\int_0^{2\pi}\sqrt{5+4\cos t}\,dt \approx 13.36489322$; L_1 and L_2 appear to be equal, and indeed, with the substitution $u = 3t - \pi$ and the periodicity of $\cos u$, $L_1 = \dfrac{1}{3}\displaystyle\int_{-\pi}^{5\pi}\sqrt{5-4\cos(u+\pi)}\,du = \int_0^{2\pi}\sqrt{5+4\cos u}\,du = L_2$.

74. **(a)** The thread leaves the circle at the point $x_1 = a\cos\theta, y_1 = a\sin\theta$, and the end of the thread is, relative to the point on the circle, on the tangent line at $x_2 = a\theta\sin\theta, y_2 = -a\theta\cos\theta$; adding, $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$.

(b) $dx/d\theta = a\theta\cos\theta, dy/d\theta = a\theta\sin\theta; dx/d\theta = 0$ has solutions $\theta = 0, \pi/2, 3\pi/2$; and $dy/d\theta = 0$ has solutions $\theta = 0, \pi, 2\pi$. At $\theta = \pi/2, dy/d\theta > 0$, so the direction is North; at $\theta = \pi$, $dx/d\theta < 0$, so West; at $\theta = 3\pi/2, dy/d\theta < 0$, so South; at $\theta = 2\pi, dx/d\theta > 0$, so East. Finally, $\displaystyle\lim_{\theta\to 0^+}\frac{dy}{dx} = \lim_{\theta\to 0^+}\tan\theta = 0$, so East.

(c)


75. $x' = 2t$, $y' = 3$, $(x')^2 + (y')^2 = 4t^2 + 9$, and $S = 2\pi\displaystyle\int_0^2 (3t)\sqrt{4t^2+9}dt = 6\pi\int_0^4 t\sqrt{4t^2+9}dt = \frac{\pi}{2}(4t^2+9)^{3/2}\Big]_0^2 = \frac{\pi}{2}(125-27) = 49\pi$.

76. $x' = e^t(\cos t - \sin t)$, $y' = e^t(\cos t + \sin t)$, $(x')^2 + (y')^2 = 2e^{2t}$, so $S = 2\pi\displaystyle\int_0^{\pi/2}(e^t\sin t)\sqrt{2e^{2t}}dt =$

$= 2\sqrt{2}\pi\displaystyle\int_0^{\pi/2} e^{2t}\sin t\,dt = 2\sqrt{2}\pi\left[\frac{1}{5}e^{2t}(2\sin t - \cos t)\right]_0^{\pi/2} = \frac{2\sqrt{2}}{5}\pi(2e^{\pi}+1)$.

77. $x' = -2\sin t\cos t$, $y' = 2\sin t\cos t$, $(x')^2 + (y')^2 = 8\sin^2 t\cos^2 t$, so $S = 2\pi\displaystyle\int_0^{\pi/2}\cos^2 t\sqrt{8\sin^2 t\cos^2 t}\;dt =$

$$4\sqrt{2}\pi \int_0^{\pi/2} \cos^3 t \sin t\, dt = -\sqrt{2}\pi \cos^4 t\Big]_0^{\pi/2} = \sqrt{2}\pi.$$

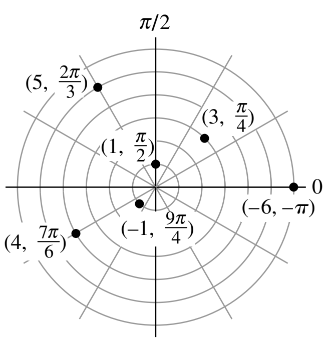
78. $x' = 6$, $y' = 8t$, $(x')^2 + (y')^2 = 36 + 64t^2$, so $S = 2\pi \displaystyle\int_0^1 6t\sqrt{36 + 64t^2}\, dt = 49\pi$.

79. $x' = -r\sin t$, $y' = r\cos t$, $(x')^2 + (y')^2 = r^2$, so $S = 2\pi \displaystyle\int_0^{\pi} r\sin t\sqrt{r^2}\, dt = 2\pi r^2 \int_0^{\pi} \sin t\, dt = 4\pi r^2$.

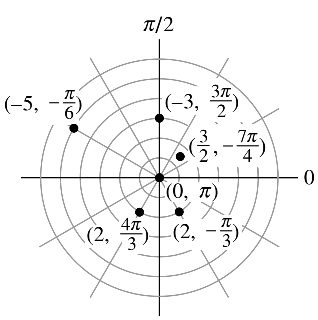
80. $\dfrac{dx}{d\phi} = a(1-\cos\phi)$, $\dfrac{dy}{d\phi} = a\sin\phi$, $\left(\dfrac{dx}{d\phi}\right)^2 + \left(\dfrac{dy}{d\phi}\right)^2 = 2a^2(1-\cos\phi)$, so $S = 2\pi \displaystyle\int_0^{2\pi} a(1-\cos\phi)\sqrt{2a^2(1-\cos\phi)}\, d\phi =$ $2\sqrt{2}\pi a^2 \displaystyle\int_0^{2\pi} (1-\cos\phi)^{3/2} d\phi$, but $1 - \cos\phi = 2\sin^2\dfrac{\phi}{2}$ so $(1-\cos\phi)^{3/2} = 2\sqrt{2}\sin^3\dfrac{\phi}{2}$ for $0 \le \phi \le \pi$ and, taking advantage of the symmetry of the cycloid, $S = 16\pi a^2 \displaystyle\int_0^{\pi} \sin^3\dfrac{\phi}{2} d\phi = 64\pi a^2/3$.

82. For some curves, we may not be able to find a formula for $f(x)$, so a parametric form may be our only option. For example, for the curve $x = t + e^t$, $y = t$ there is no elementary function f such that $y = f(x)$. Even if we can find a formula for $f(x)$, the parametric form may provide more information. For example, if the curve is the path traced out by a moving object, then expressing x and y in terms of time tells us where the object is at any given time; the $y = f(x)$ form does not.

Exercise Set 10.2



1.



2.

3. **(a)** $(3\sqrt{3}, 3)$ **(b)** $(-7/2, 7\sqrt{3}/2)$ **(c)** $(3\sqrt{3}, 3)$ **(d)** $(0, 0)$ **(e)** $(-7\sqrt{3}/2, 7/2)$ **(f)** $(-5, 0)$

4. **(a)** $(-\sqrt{2}, -\sqrt{2})$ **(b)** $(3\sqrt{2}, -3\sqrt{2})$ **(c)** $(2\sqrt{2}, 2\sqrt{2})$ **(d)** $(3, 0)$ **(e)** $(0, -4)$ **(f)** $(0, 0)$

5. **(a)** $(5, \pi), (5, -\pi)$ **(b)** $(4, 11\pi/6), (4, -\pi/6)$ **(c)** $(2, 3\pi/2), (2, -\pi/2)$ **(d)** $(8\sqrt{2}, 5\pi/4), (8\sqrt{2}, -3\pi/4)$

(e) $(6, 2\pi/3), (6, -4\pi/3)$ **(f)** $(\sqrt{2}, \pi/4), (\sqrt{2}, -7\pi/4)$

6. **(a)** $(2, 5\pi/6)$ **(b)** $(-2, 11\pi/6)$ **(c)** $(2, -7\pi/6)$ **(d)** $(-2, -\pi/6)$

7. **(a)** $(5, 0.92730)$ **(b)** $(10, -0.92730)$ **(c)** $(1.27155, 2.47582)$

8. **(a)** $(5, 2.21430)$ **(b)** $(3.44819, 2.62604)$ **(c)** $(2.06740, 0.25605)$

9. **(a)** $r^2 = x^2 + y^2 = 4$; circle.

(b) $y = 4$; horizontal line.

(c) $r^2 = 3r\cos\theta$, $x^2 + y^2 = 3x$, $(x - 3/2)^2 + y^2 = 9/4$; circle.

(d) $3r\cos\theta + 2r\sin\theta = 6$, $3x + 2y = 6$; line.

10. **(a)** $r\cos\theta = 5$, $x = 5$; vertical line.

(b) $r^2 = 2r\sin\theta$, $x^2 + y^2 = 2y$, $x^2 + (y - 1)^2 = 1$; circle.

(c) $r^2 = 4r\cos\theta + 4r\sin\theta$, $x^2 + y^2 = 4x + 4y$, $(x - 2)^2 + (y - 2)^2 = 8$; circle.

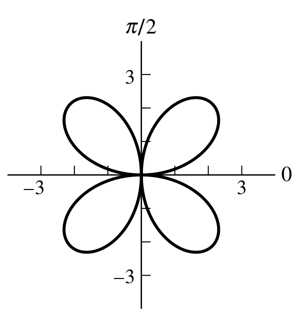
(d) $r = \dfrac{1}{\cos\theta}\dfrac{\sin\theta}{\cos\theta}$, $r\cos^2\theta = \sin\theta$, $r^2\cos^2\theta = r\sin\theta$, $x^2 = y$; parabola.

11. **(a)** $r\cos\theta = 3$. **(b)** $r = \sqrt{7}$. **(c)** $r^2 + 6r\sin\theta = 0$, $r = -6\sin\theta$.

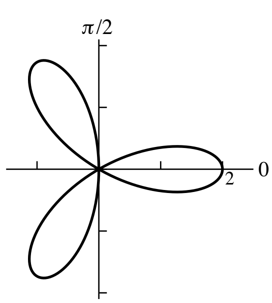
(d) $9(r\cos\theta)(r\sin\theta) = 4$, $9r^2\sin\theta\cos\theta = 4$, $r^2\sin 2\theta = 8/9$.

12. **(a)** $r\sin\theta = -3$. **(b)** $r = \sqrt{5}$. **(c)** $r^2 + 4r\cos\theta = 0$, $r = -4\cos\theta$.

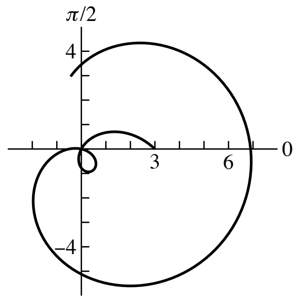
(d) $r^4\cos^2\theta = r^2\sin^2\theta$, $r^2 = \tan^2\theta$, $r = \pm\tan\theta$.



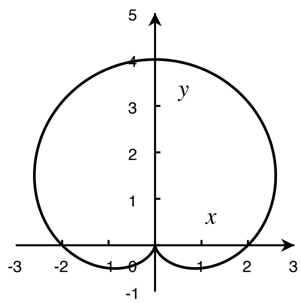
13. $r = 3\sin 2\theta$.



14. $r = 2\cos 3\theta$.



15. $r = 3 - 4\sin\left(\dfrac{\pi}{4}\theta\right)$.



16. $r = 2 + 2\sin\theta$.

17. **(a)** $r = 5$.

(b) $(x-3)^2 + y^2 = 9,\ r = 6\cos\theta$.

(c) Example 8, $r = 1 - \cos\theta$.

18. **(a)** From (8-9), $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$. The curve is not symmetric about the y-axis, so Theorem 10.2.1(b) eliminates the sine function, thus $r = a \pm b\cos\theta$. The cartesian point $(-3, 0)$ is either the polar point $(3, \pi)$ or $(-3, 0)$, and the cartesian point $(-1, 0)$ is either the polar point $(1, \pi)$ or $(-1, 0)$. A solution is $a = 1, b = -2$; we may take the equation as $r = 1 - 2\cos\theta$.

(b) $x^2 + (y + 3/2)^2 = 9/4, r = -3\sin\theta$.

(c) Figure 10.2.19, $a = 1, n = 3, r = \sin 3\theta$.

19. **(a)** Figure 10.2.19, $a = 3, n = 2, r = 3\sin 2\theta$.

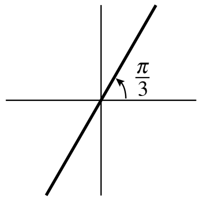
(b) From (8-9), symmetry about the y-axis and Theorem 10.2.1(b), the equation is of the form $r = a \pm b\sin\theta$. The cartesian points $(3, 0)$ and $(0, 5)$ give $a = 3$ and $5 = a + b$, so $b = 2$ and $r = 3 + 2\sin\theta$.

(c) Example 9, $r^2 = 9\cos 2\theta$.

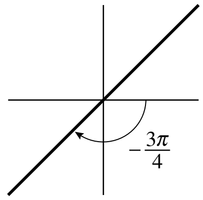
20. **(a)** Example 8 rotated through $\pi/2$ radians: $a = 3, r = 3 - 3\sin\theta$.

(b) Figure 10.2.19, $a = 1, r = \cos 5\theta$.

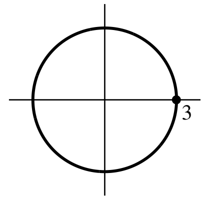
(c) $x^2 + (y-2)^2 = 4,\ r = 4\sin\theta$.

21. 

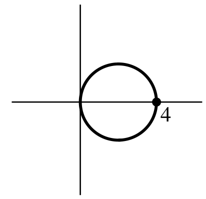
Line

22. 

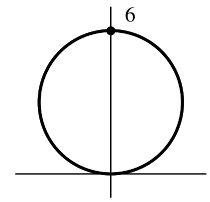
Line

23. 

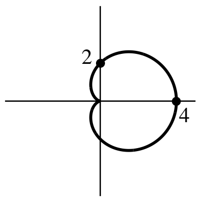
Circle

24. 

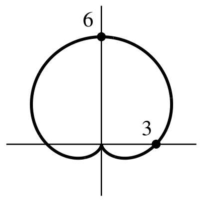
Circle

25. 

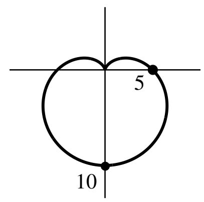
Circle

26. 

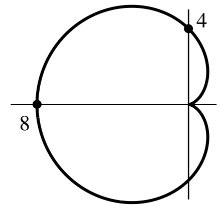
Cardioid

27. 

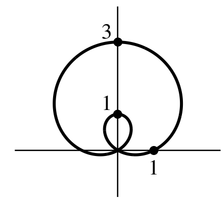
Cardioid

28. 

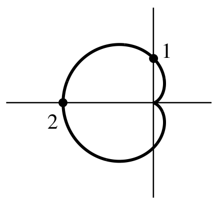
Cardioid

29. 

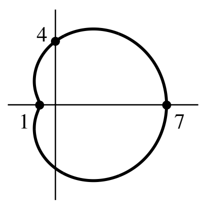
Cardioid

30. 

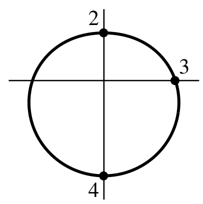
Limaçon

31. 

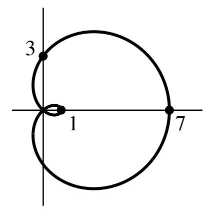
Cardioid

32. 

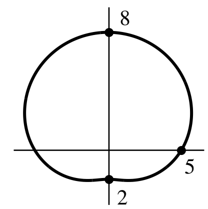
Limaçon

33. 

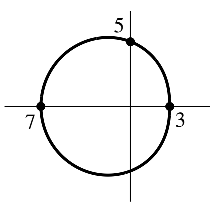
Limaçon

34. 

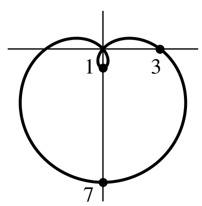
Limaçon

35. 

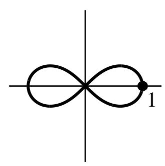
Limaçon

36. 

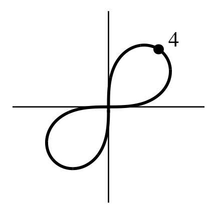
Limaçon

37. 

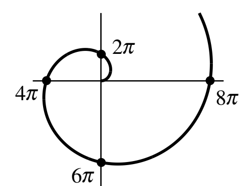
Limaçon

38. 

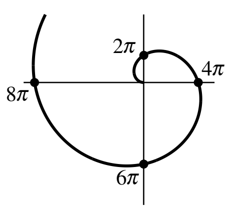
Lemniscate

39. 

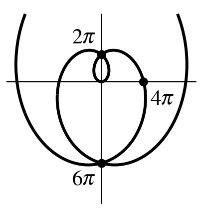
Lemniscate

40. 

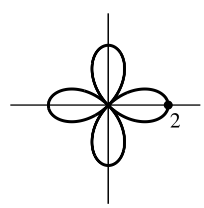
Spiral

41. 

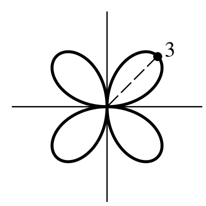
Spiral

42. 

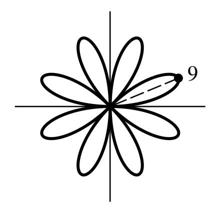
Spiral

43. 

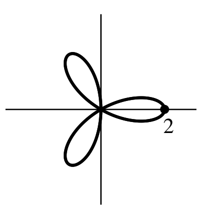
Four-petal rose

44. 

Four-petal rose

45. 

Eight-petal rose

46. 

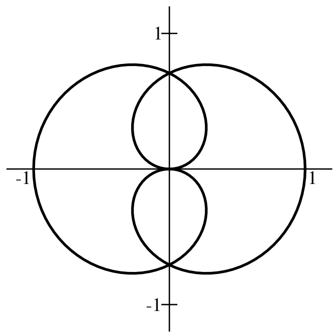
Three-petal rose

47. True. Both have rectangular coordinates $(-1/2, -\sqrt{3}/2)$.

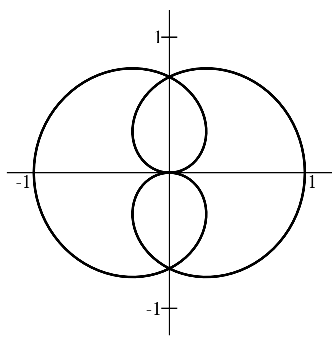
48. True. If the graph in rectangular θr-coordinates is symmetric across the r-axis, then $f(\theta) = f(-\theta)$ for all θ. So for each point $(f(\theta), \theta)$ on the graph in polar coordinates, the point $(f(-\theta), -\theta) = (f(\theta), -\theta)$ is also on the graph. But this point is the reflection of $(f(\theta), \theta)$ across the x-axis, so the graph is symmetric across the x-axis.

49. False. For $\pi/2 < \theta < \pi$, $\sin 2\theta < 0$. Hence the point with polar coordinates $(\sin 2\theta, \theta)$ is in the fourth quadrant.

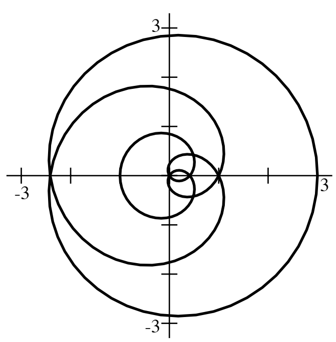
50. False. If $1 < a/b < 2$, then $a \pm b\sin\theta = b(a/b \pm \sin\theta) > 0$, and similarly $a \pm b\cos\theta > 0$. So none of the curves described by equations (8-9) pass through the origin.



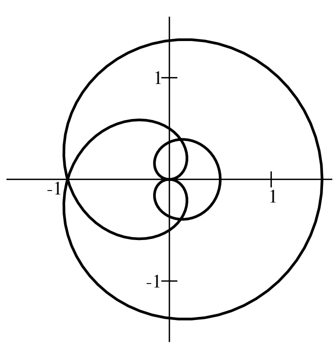
51. $0 \le \theta < 4\pi$



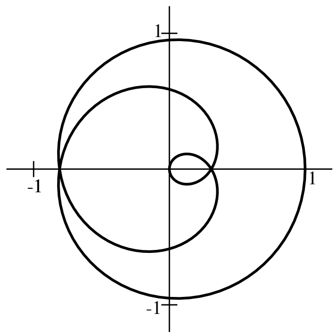
52. $0 \le \theta < 4\pi$



53. $0 \le \theta < 8\pi$



54. $0 \le \theta < 6\pi$



55. $0 \leq \theta < 5\pi$

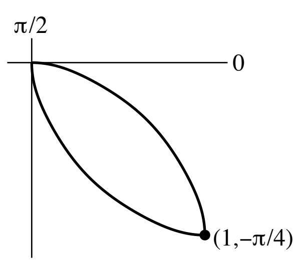
56. $0 \leq \theta < 8\pi$.

57. **(a)** $-4\pi \leq \theta \leq 4\pi$.

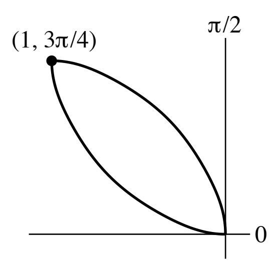
58. Family I: $x^2 + (y-b)^2 = b^2, b < 0$, or $r = 2b\sin\theta$; Family II: $(x-a)^2 + y^2 = a^2, a < 0$, or $r = 2a\cos\theta$.

59. **(a)** $r = \dfrac{a}{\cos\theta}, r\cos\theta = a, x = a$. **(b)** $r\sin\theta = b, y = b$.

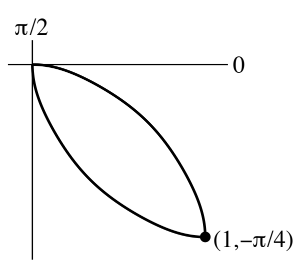
60. In I, along the x-axis, $x = r$ grows ever slower with θ. In II $x = r$ grows linearly with θ. Hence I: $r = \sqrt{\theta}$; II: $r = \theta$.



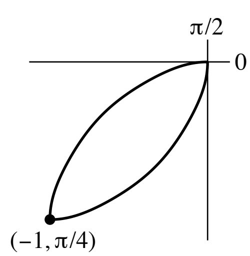
61. **(a)**



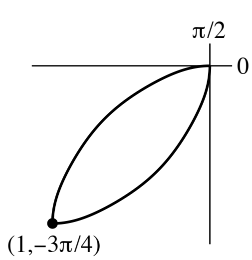
(b)



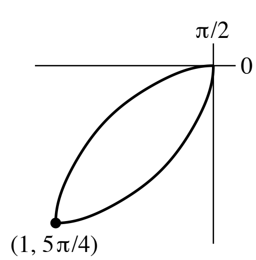
(c)



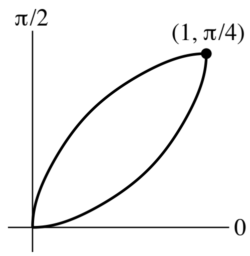
(d)



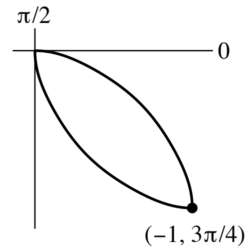
62. **(a)**



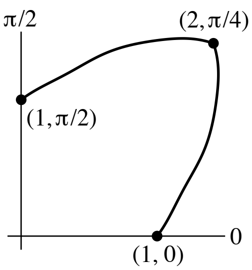
(b)



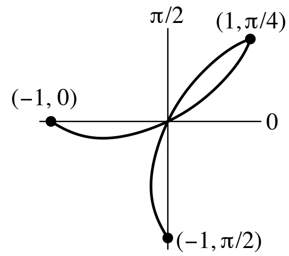
(c)



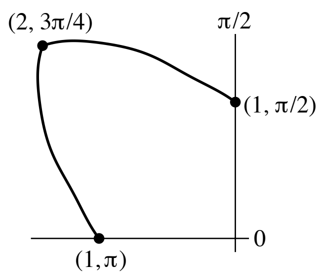
(d)



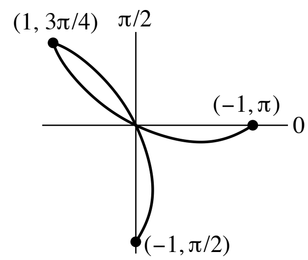
63. (a)



(b)



64. (a)



(b)

65. The image of (r_0, θ_0) under a rotation through an angle α is $(r_0, \theta_0 + \alpha)$. Hence $(f(\theta), \theta)$ lies on the original curve if and only if $(f(\theta), \theta + \alpha)$ lies on the rotated curve, i.e. (r, θ) lies on the rotated curve if and only if $r = f(\theta - \alpha)$.

66. $r^2 = 4\cos 2(\theta - \pi/2) = -4\cos 2\theta$.

67. (a) $r = 1 + \cos(\theta - \pi/4) = 1 + \dfrac{\sqrt{2}}{2}(\cos\theta + \sin\theta)$.

(b) $r = 1 + \cos(\theta - \pi/2) = 1 + \sin\theta$.

(c) $r = 1 + \cos(\theta - \pi) = 1 - \cos\theta$.

(d) $r = 1 + \cos(\theta - 5\pi/4) = 1 - \dfrac{\sqrt{2}}{2}(\cos\theta + \sin\theta)$.

68. (a) $r^2 = Ar\sin\theta + Br\cos\theta$, $x^2 + y^2 = Ay + Bx, (x - B/2)^2 + (y - A/2)^2 = (A^2 + B^2)/4$, which is a circle of radius $\dfrac{1}{2}\sqrt{A^2 + B^2}$.

(b) Formula (4) follows by setting $A = 0, B = 2a, (x - a)^2 + y^2 = a^2$, the circle of radius a about $(a, 0)$. Formula (5) is derived in a similar fashion.

69. $y = r\sin\theta = (1 + \cos\theta)\sin\theta = \sin\theta + \sin\theta\cos\theta$, $dy/d\theta = \cos\theta - \sin^2\theta + \cos^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1)$; $dy/d\theta = 0$ if $\cos\theta = 1/2$ or if $\cos\theta = -1$; $\theta = \pi/3$ or π (or $\theta = -\pi/3$, which leads to the minimum point). If $\theta = \pi/3, \pi$, then $y = 3\sqrt{3}/4, 0$ so the maximum value of y is $3\sqrt{3}/4$ and the polar coordinates of the highest point are $(3/2, \pi/3)$.

70. $x = r\cos\theta = (1+\cos\theta)\cos\theta = \cos\theta + \cos^2\theta$, $dx/d\theta = -\sin\theta - 2\sin\theta\cos\theta = -\sin\theta(1+2\cos\theta)$, $dx/d\theta = 0$ if $\sin\theta = 0$ or if $\cos\theta = -1/2$; $\theta = 0,\ 2\pi/3$, or π. If $\theta = 0,\ 2\pi/3,\ \pi$, then $x = 2, -1/4, 0$ so the minimum value of x is $-1/4$. The leftmost point has polar coordinates $(1/2, 2\pi/3)$.

71. Let (x_1, y_1) and (x_2, y_2) be the rectangular coordinates of the points (r_1, θ_1) and (r_2, θ_2) then
$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2} =$
$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1-\theta_2)}$. An alternate proof follows directly from the Law of Cosines.

72. From Exercise 71, $d = \sqrt{9 + 4 - 2\cdot 3\cdot 2\cos(\pi/6 - \pi/3)} = \sqrt{13 - 6\sqrt{3}} \approx 1.615$.

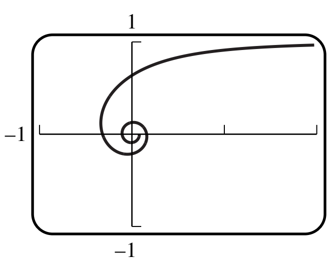
73. The tips occur when $\theta = 0, \pi/2, \pi, 3\pi/2$ for which $r = 1$: $d = \sqrt{1^2 + 1^2 - 2(1)(1)\cos(\pm\pi/2)} = \sqrt{2}$. Geometrically, find the distance between, e.g., the points $(0,1)$ and $(1,0)$.

74. The tips are located at $r = 1, \theta = \pi/6, 5\pi/6, 3\pi/2$ and, for example, $d = \sqrt{1 + 1 - 2\cos(5\pi/6 - \pi/6)} =$
$= \sqrt{2(1 - \cos(2\pi/3))} = \sqrt{3}$. By trigonometry, $d = 2\sin(\pi/3) = \sqrt{3}$.

75. **(a)** $0 = (r^2 + a^2)^2 - a^4 - 4a^2r^2\cos^2\theta = r^4 + a^4 + 2r^2a^2 - a^4 - 4a^2r^2\cos^2\theta = r^4 + 2r^2a^2 - 4a^2r^2\cos^2\theta$, so $r^2 = 2a^2(2\cos^2\theta - 1) = 2a^2\cos 2\theta$.

(b) The distance from the point (r, θ) to $(a, 0)$ is (from Exercise 73(a))
$\sqrt{r^2 + a^2 - 2ra\cos(\theta - 0)} = \sqrt{r^2 - 2ar\cos\theta + a^2}$, and to the point (a, π) is $\sqrt{r^2 + a^2 - 2ra\cos(\theta - \pi)} =$
$= \sqrt{r^2 + 2ar\cos\theta + a^2}$, and their product is
$\sqrt{(r^2 + a^2)^2 - 4a^2r^2\cos^2\theta} = \sqrt{r^4 + a^4 + 2a^2r^2(1 - 2\cos^2\theta)} = \sqrt{4a^4\cos^2 2\theta + a^4 + 2a^2(2a^2\cos 2\theta)(-\cos 2\theta)} = a^2$.

76. $\displaystyle\lim_{\theta\to0^+} y = \lim_{\theta\to0^+} r\sin\theta = \lim_{\theta\to0^+} \frac{\sin\theta}{\theta} = 1$, and $\displaystyle\lim_{\theta\to0^+} x = \lim_{\theta\to0^+} r\cos\theta = \lim_{\theta\to0^+} \frac{\cos\theta}{\theta} = +\infty$.



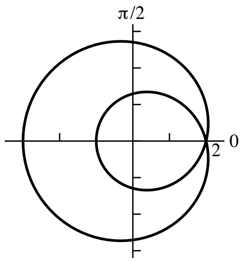
77. $\displaystyle\lim_{\theta\to0^\pm} y = \lim_{\theta\to0^\pm} r\sin\theta = \lim_{\theta\to0^\pm} \frac{\sin\theta}{\theta^2} = \lim_{\theta\to0^\pm} \frac{\sin\theta}{\theta} \lim_{\theta\to0^\pm} \frac{1}{\theta} = 1\cdot \lim_{\theta\to0^\pm} \frac{1}{\theta}$, so $\displaystyle\lim_{\theta\to0^\pm} y$ does not exist.

78. Let $r = a\sin n\theta$ (the proof for $r = a\cos n\theta$ is similar). If θ starts at 0, then θ would have to increase by some positive integer multiple of π radians in order to reach the starting point and begin to retrace the curve. Let (r, θ) be the coordinates of a point P on the curve for $0 \le \theta < 2\pi$. Now $a\sin n(\theta + 2\pi) = a\sin(n\theta + 2\pi n) = a\sin n\theta = r$ so P is reached again with coordinates $(r, \theta + 2\pi)$ thus the curve is traced out either exactly once or exactly twice for $0 \le \theta < 2\pi$. If for $0 \le \theta < \pi$, $P(r, \theta)$ is reached again with coordinates $(-r, \theta + \pi)$ then the curve is traced out exactly once for $0 \le \theta < \pi$, otherwise exactly once for $0 \le \theta < 2\pi$. But

$$a\sin n(\theta + \pi) = a\sin(n\theta + n\pi) = \begin{cases} a\sin n\theta, & n \text{ even} \\ -a\sin n\theta, & n \text{ odd} \end{cases}$$

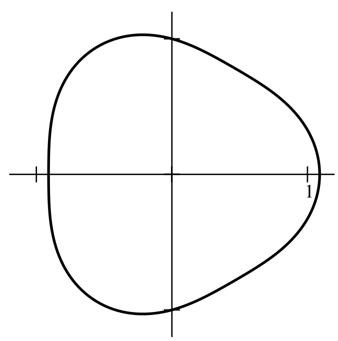
so the curve is traced out exactly once for $0 \le \theta < 2\pi$ if n is even, and exactly once for $0 \le \theta < \pi$ if n is odd.

79. **(a)**

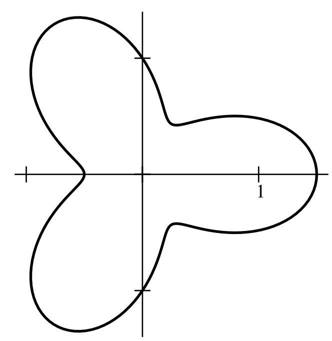


(b) Replacing θ with $-\theta$ changes $r = 2 - \sin(\theta/2)$ into $r = 2 + \sin(\theta/2)$ which is not an equivalent equation. But the locus of points satisfying the first equation, when θ runs from 0 to 4π, is the same as the locus of points satisfying the second equation when θ runs from 0 to 4π, as can be seen under the change of variables (equivalent to reversing direction of θ) $\theta \to 4\pi - \theta$, for which $2 + \sin(4\pi - \theta) = 2 - \sin\theta$.

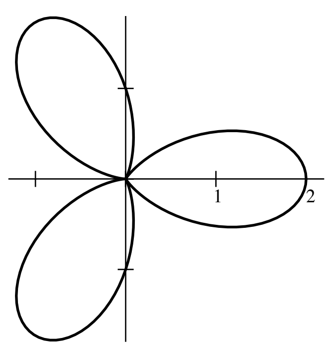
80. The curve is symmetric with respect to rotation about the origin through an angle of $2\pi/n$. If $a < 1$ it has n 'lobes' and does not pass through the origin. It can be shown that the curve is convex if $a \le \dfrac{1}{n^2+1}$; otherwise it has n 'dimples' between the lobes. If $a = 1$ it still has n lobes but each one just touches the origin. If $a > 1$ it passes through the origin and has $2n$ lobes. For n odd, half of the lobes are contained in the other half; for n even none of them are contained in others. Some examples are shown below:



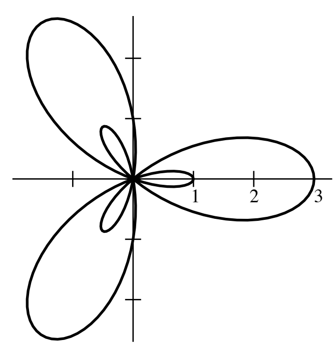
$n = 3,\ a = 0.09 < \dfrac{1}{n^2+1}$



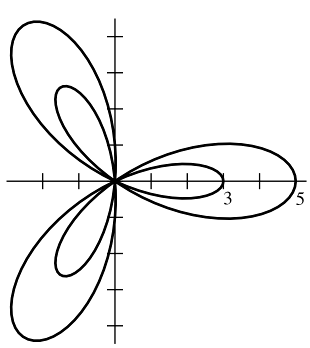
$n = 3,\ a = 0.5 > \dfrac{1}{n^2+1}$



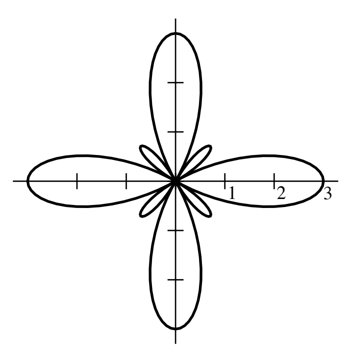
$n = 3,\ a = 1$



$n = 3,\ a = 2$



$n = 3,\ a = 4$



$n = 4,\ a = 2$

Exercise Set 10.3

1. Substituting $\theta = \pi/6$, $r = 1$, and $dr/d\theta = \sqrt{3}$ in equation (2) gives slope $m = \sqrt{3}$.

2. As in Exercise 1, $\theta = \pi/2$, $dr/d\theta = -1$, $r = 1$, $m = 1$.

3. As in Exercise 1, $\theta = 2$, $dr/d\theta = -1/4$, $r = 1/2$, $m = \dfrac{\tan 2 - 2}{2\tan 2 + 1}$.

4. As in Exercise 1, $\theta = \pi/6$, $dr/d\theta = 4\sqrt{3}a$, $r = 2a$, $m = 3\sqrt{3}/5$.

5. As in Exercise 1, $\theta = \pi/4$, $dr/d\theta = -3\sqrt{2}/2$, $r = \sqrt{2}/2$, $m = 1/2$.

6. As in Exercise 1, $\theta = \pi$, $dr/d\theta = 3$, $r = 4$, $m = 4/3$.

7. $m = \dfrac{dy}{dx} = \dfrac{r\cos\theta + (\sin\theta)(dr/d\theta)}{-r\sin\theta + (\cos\theta)(dr/d\theta)} = \dfrac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos^2\theta - \sin^2\theta}$; if $\theta = 0, \pi/2, \pi$, then $m = 1, 0, -1$.

8. $m = \dfrac{dy}{dx} = \dfrac{\cos\theta(4\sin\theta - 1)}{4\cos^2\theta + \sin\theta - 2}$; if $\theta = 0, \pi/2, \pi$ then $m = -1/2, 0, 1/2$.

9. $dx/d\theta = -a\sin\theta(1 + 2\cos\theta)$, $dy/d\theta = a(2\cos\theta - 1)(\cos\theta + 1)$.

The tangent line is horizontal if $dy/d\theta = 0$ and $dx/d\theta \neq 0$. $dy/d\theta = 0$ when $\cos\theta = 1/2$ or $\cos\theta = -1$ so $\theta = \pi/3$, $5\pi/3$, or π; $dx/d\theta \neq 0$ for $\theta = \pi/3$ and $5\pi/3$. For the singular point $\theta = \pi$ we find that $\lim\limits_{\theta\to\pi} dy/dx = 0$. There are horizontal tangent lines at $(3a/2, \pi/3), (0, \pi)$, and $(3a/2, 5\pi/3)$.

The tangent line is vertical if $dy/d\theta \neq 0$ and $dx/d\theta = 0$. $dx/d\theta = 0$ when $\sin\theta = 0$ or $\cos\theta = -1/2$ so $\theta = 0$, π, $2\pi/3$, or $4\pi/3$; $dy/d\theta \neq 0$ for $\theta = 0$, $2\pi/3$, and $4\pi/3$. The singular point $\theta = \pi$ was discussed earlier. There are vertical tangent lines at $(2a, 0), (a/2, 2\pi/3)$, and $(a/2, 4\pi/3)$.

10. $dx/d\theta = a(\cos^2\theta - \sin^2\theta) = a\cos 2\theta, dy/d\theta = 2a\sin\theta\cos\theta = a\sin 2\theta$.

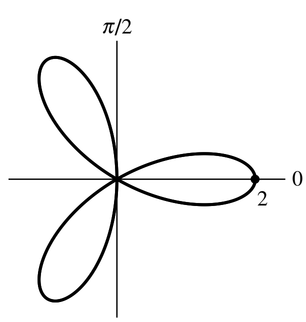
The tangent line is horizontal if $dy/d\theta = 0$ and $dx/d\theta \neq 0$. $dy/d\theta = 0$ when $\theta = 0$, $\pi/2$, π, $3\pi/2$; $dx/d\theta \neq 0$ for $(0, 0)$, $(a, \pi/2)$, $(0, \pi)$, $(-a, 3\pi/2)$; in reality only two distinct points.

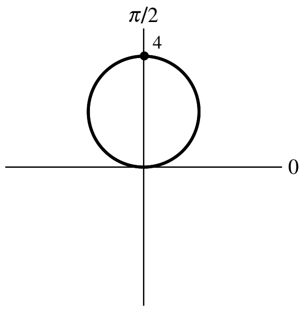
The tangent line is vertical if $dy/d\theta \neq 0$ and $dx/d\theta = 0$. $dx/d\theta = 0$ when $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$; $dy/d\theta \neq 0$ there, so vertical tangent line at $(a/\sqrt{2}, \pi/4)$, $(a/\sqrt{2}, 3\pi/4)$, $(-a/\sqrt{2}, 5\pi/4)$, $(-a/\sqrt{2}, 7\pi/4)$, only two distinct points.

11. Since $r(\theta + \pi) = -r(\theta)$, the curve is traced out once as θ goes from 0 to π. $dy/d\theta = (d/d\theta)(\sin^2\theta\cos^2\theta) = (\sin 4\theta)/2 = 0$ at $\theta = 0, \pi/4, \pi/2, 3\pi/4, \pi$. When $\theta = 0$, $\pi/2$, or π, $r = 0$, so these 3 values give the same point, and we only have 3 points to consider. $dx/d\theta = (d/d\theta)(\sin\theta\cos^3\theta) = \cos^2\theta(4\cos^2\theta - 3)$ is nonzero when $\theta = 0$, $\pi/4$, or $3\pi/4$. Hence there are horizontal tangents at all 3 of these points. (There is also a singular point at the origin corresponding to $\theta = \pi/2$.)

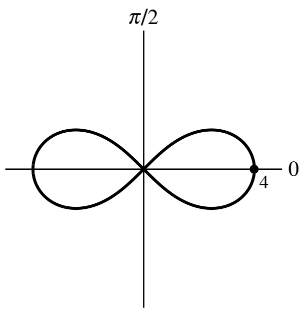
12. $dx/d\theta = 4\sin^2\theta - \sin\theta - 2$, $dy/d\theta = \cos\theta(1 - 4\sin\theta)$. $dy/d\theta = 0$ when $\cos\theta = 0$ or $\sin\theta = 1/4$ so $\theta = \pi/2$, $3\pi/2$, $\sin^{-1}(1/4)$, or $\pi - \sin^{-1}(1/4)$; $dx/d\theta \neq 0$ at these four points, so there is a horizontal tangent at each one.

13. $\theta = \pi/6, \pi/2, 5\pi/6$.

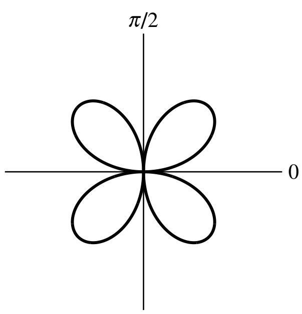




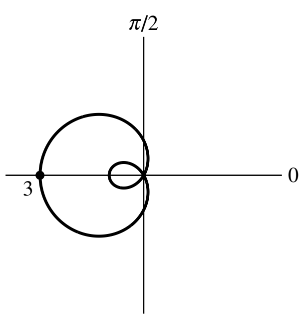
14. $\theta = 0.$



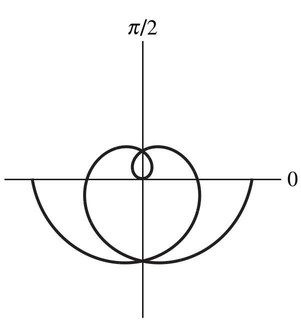
15. $\theta = \pm\pi/4.$



16. $\theta = 0, \pi/2$



17. $\theta = \pi/3, 5\pi/3$



18. $\theta = 0$

19. $r^2 + (dr/d\theta)^2 = a^2 + 0^2 = a^2$, $L = \displaystyle\int_0^{2\pi} a\, d\theta = 2\pi a.$

20. $r^2 + (dr/d\theta)^2 = (2a\cos\theta)^2 + (-2a\sin\theta)^2 = 4a^2$, $L = \int_{-\pi/2}^{\pi/2} 2a\,d\theta = 2\pi a$.

21. $r^2 + (dr/d\theta)^2 = [a(1-\cos\theta)]^2 + [a\sin\theta]^2 = 4a^2\sin^2(\theta/2)$, $L = 2\int_0^{\pi} 2a\sin(\theta/2)\,d\theta = 8a$.

22. $r^2 + (dr/d\theta)^2 = (e^{3\theta})^2 + (3e^{3\theta})^2 = 10e^{6\theta}$, $L = \int_0^2 \sqrt{10}e^{3\theta}\,d\theta = \sqrt{10}(e^6-1)/3$.

23. **(a)** $r^2 + (dr/d\theta)^2 = (\cos n\theta)^2 + (-n\sin n\theta)^2 = \cos^2 n\theta + n^2\sin^2 n\theta = (1-\sin^2 n\theta) + n^2\sin^2 n\theta = 1 + (n^2 - 1)\sin^2 n\theta$. The top half of the petal along the polar axis is traced out as θ goes from 0 to $\pi/(2n)$, so $L = 2\int_0^{\pi/(2n)} \sqrt{1+(n^2-1)\sin^2 n\theta}\,d\theta$.

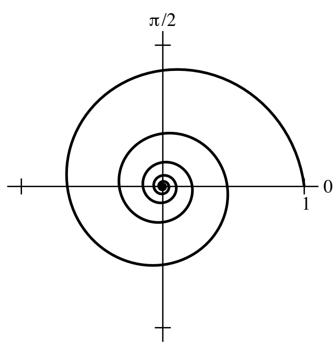
(b) $L = 2\int_0^{\pi/4} \sqrt{1+3\sin^2 2\theta}\,d\theta \approx 2.42$.

(c)

n	2	3	4	5	6	7	8	9	10	11
L	2.42211	2.22748	2.14461	2.10100	2.07501	2.05816	2.04656	2.03821	2.03199	2.02721

n	12	13	14	15	16	17	18	19	20
L	2.02346	2.02046	2.01802	2.01600	2.01431	2.01288	2.01167	2.01062	2.00971

The limit seems to be 2. This is to be expected, since as $n \to +\infty$ each petal more closely resembles a pair of straight lines of length 1.



24. **(a)**

(b) $r^2 + (dr/d\theta)^2 = (e^{-\theta/8})^2 + (-\frac{1}{8}e^{-\theta/8})^2 = \frac{65}{64}e^{-\theta/4}$, so $L = \frac{\sqrt{65}}{8}\int_0^{+\infty} e^{-\theta/8}\,d\theta$.

(c) $L = \lim_{\theta_0\to+\infty} \frac{\sqrt{65}}{8}\int_0^{\theta_0} e^{-\theta/8}\,d\theta = \lim_{\theta_0\to+\infty} \sqrt{65}(1-e^{-\theta_0/8}) = \sqrt{65}$.

25. **(a)** $\int_{\pi/2}^{\pi} \frac{1}{2}(1-\cos\theta)^2\,d\theta$. **(b)** $\int_0^{\pi/2} 2\cos^2\theta\,d\theta$. **(c)** $\int_0^{\pi/2} \frac{1}{2}\sin^2 2\theta\,d\theta$.

(d) $\int_0^{2\pi} \frac{1}{2}\theta^2\,d\theta$. **(e)** $\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1-\sin\theta)^2\,d\theta$. **(f)** $\int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos^2 2\theta\,d\theta = \int_0^{\pi/4} \cos^2 2\theta\,d\theta$.

26. $A = \int_0^{2\pi} \frac{1}{2}\theta^2\,d\theta = \frac{1}{6}\theta^3\Big]_0^{2\pi} = \frac{4\pi^3}{3}$.

27. **(a)** $A = \int_0^{\pi} \frac{1}{2}4a^2\sin^2\theta\,d\theta = \pi a^2$. **(b)** $A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}4a^2\cos^2\theta\,d\theta = \pi a^2$.

28. **(a)** $r^2 = 2r\sin\theta + 2r\cos\theta$, $x^2 + y^2 - 2y - 2x = 0$, $(x-1)^2 + (y-1)^2 = 2$.

(b) The circle's radius is $\sqrt{2}$, so its area is $\pi(\sqrt{2})^2 = 2\pi$. $A = \int_{-\pi/4}^{3\pi/4} \frac{1}{2}(2\sin\theta + 2\cos\theta)^2\, d\theta = 2\pi$.

29. $A = \int_0^{2\pi} \frac{1}{2}(2 + 2\sin\theta)^2\, d\theta = 6\pi$.

30. $A = \int_0^{\pi/2} \frac{1}{2}(1 + \cos\theta)^2\, d\theta = \frac{3\pi}{8} + 1$.

31. $A = 6\int_0^{\pi/6} \frac{1}{2}(16\cos^2 3\theta)\, d\theta = 4\pi$.

32. The petal in the first quadrant has area $\int_0^{\pi/2} \frac{1}{2} 4\sin^2 2\theta\, d\theta = \frac{\pi}{2}$, so total area $= 2\pi$.

33. $A = 2\int_{2\pi/3}^{\pi} \frac{1}{2}(1 + 2\cos\theta)^2\, d\theta = \pi - \frac{3\sqrt{3}}{2}$.

34. $A = \int_1^3 \frac{2}{\theta^2}\, d\theta = \frac{4}{3}$.

35. Area $= A_1 - A_2 = \int_0^{\pi/2} \frac{1}{2} 4\cos^2\theta\, d\theta - \int_0^{\pi/4} \frac{1}{2}\cos 2\theta\, d\theta = \frac{\pi}{2} - \frac{1}{4}$.

36. Area $= A_1 - A_2 = \int_0^{\pi} \frac{1}{2}(1 + \cos\theta)^2\, d\theta - \int_0^{\pi/2} \frac{1}{2}\cos^2\theta\, d\theta = \frac{5\pi}{8}$.

37. The circles intersect when $\cos\theta = \sqrt{3}\sin\theta, \tan\theta = 1/\sqrt{3}, \theta = \pi/6$, so $A = A_1 + A_2 = \int_0^{\pi/6} \frac{1}{2}(4\sqrt{3}\sin\theta)^2\, d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(4\cos\theta)^2\, d\theta = 2\pi - 3\sqrt{3} + \frac{4\pi}{3} - \sqrt{3} = \frac{10\pi}{3} - 4\sqrt{3}$.

38. The curves intersect when $1 + \cos\theta = 3\cos\theta, \cos\theta = 1/2, \theta = \pm\pi/3$, so the total area is $A = 2\int_0^{\pi/3} \frac{1}{2}(1 + \cos\theta)^2\, d\theta + 2\int_{\pi/3}^{\pi/2} \frac{1}{2} 9\cos^2\theta\, d\theta = 2\left(\frac{\pi}{4} + \frac{9\sqrt{3}}{16} + \frac{3\pi}{8} - \frac{9\sqrt{3}}{16}\right) = \frac{5\pi}{4}$.

39. $A = 2\int_{\pi/6}^{\pi/2} \frac{1}{2}[9\sin^2\theta - (1 + \sin\theta)^2]\, d\theta = \pi$.

40. $A = 2\int_0^{\pi} \frac{1}{2}[16 - (2 - 2\cos\theta)^2]\, d\theta = 10\pi$.

41. $A = 2\int_0^{\pi/3} \frac{1}{2}[(2 + 2\cos\theta)^2 - 9]\, d\theta = \frac{9\sqrt{3}}{2} - \pi$.

42. $A = 2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\, d\theta = \Big[2\theta - \sin 2\theta\Big]_0^{\pi/4} = \frac{\pi}{2} - 1$.

43. $A = 2\left[\int_0^{2\pi/3} \frac{1}{2}(1/2+\cos\theta)^2\,d\theta - \int_{2\pi/3}^{\pi} \frac{1}{2}(1/2+\cos\theta)^2\,d\theta\right] = \frac{\pi+3\sqrt{3}}{4}$.

44. $A = 2\int_0^{\pi/3} \frac{1}{2}\left[(2+2\cos\theta)^2 - \frac{9}{4}\sec^2\theta\right] d\theta = 2\pi + \frac{9}{4}\sqrt{3}$.

45. $A = 2\int_0^{\pi/4} \frac{1}{2}(4-2\sec^2\theta)\,d\theta = \pi - 2$.

46. $A = 8\int_0^{\pi/8} \frac{1}{2}(4a^2\cos^2 2\theta - 2a^2)\,d\theta = 2a^2$.

47. True. When $\theta = 3\pi$, $r = \cos(3\pi/2) = 0$ so the curve passes through the origin. Also, $\frac{dr}{d\theta} = -\frac{1}{2}\sin(\theta/2) = \frac{1}{2} \neq 0$. Hence, by Theorem 10.3.1, the line $\theta = 3\pi$ is tangent to the curve at the origin. But $\theta = 3\pi$ is the x-axis.

48. False. By Formula (3), the arc length is $\int_0^{\pi/2} \sqrt{(\sqrt{\theta})^2 + \left(\frac{1}{2\sqrt{\theta}}\right)^2}\,d\theta = \int_0^{\pi/2} \sqrt{\theta + \frac{1}{4\theta}}\,d\theta \approx 1.988$. The integral given in the exercise is $\int_0^{\pi/2} \sqrt{1+\frac{1}{4\theta}}\,d\theta \approx 2.104$.

49. False. The area is $\frac{\theta}{2\pi}$ times the area of the circle $= \frac{\theta}{2\pi}\cdot\pi r^2 = \frac{\theta}{2}r^2$, not θr^2.

50. True. The inner loop is traced out as θ ranges from $-\pi/4$ to $\pi/4$ and r is ≤ 0 for θ in that range. So Theorem 10.3.4 implies that the area is $\int_{-\pi/4}^{\pi/4} \frac{1}{2}(1-\sqrt{2}\cos\theta)^2\,d\theta$.

51. **(a)** r is not real for $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$.

(b) $A = 4\int_0^{\pi/4} \frac{1}{2}a^2\cos 2\theta\,d\theta = a^2$.

(c) $A = 4\int_0^{\pi/6} \frac{1}{2}\left[4\cos 2\theta - 2\right] d\theta = 2\sqrt{3} - \frac{2\pi}{3}$.

52. $A = 2\int_0^{\pi/2} \frac{1}{2}\sin 2\theta\,d\theta = 1$.

53. $A = \int_{2\pi}^{4\pi} \frac{1}{2}a^2\theta^2\,d\theta - \int_0^{2\pi} \frac{1}{2}a^2\theta^2\,d\theta = 8\pi^3 a^2$.

54. **(a)** $\frac{dr}{dt} = 2$ and $\frac{d\theta}{dt} = 1$ so $\frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt} = \frac{2}{1} = 2$, $r = 2\theta + C$, $r = 10$ when $\theta = 0$ so $10 = C, r = 2\theta + 10$.

(b) $r^2 + (dr/d\theta)^2 = (2\theta+10)^2 + 4$, during the first 5 seconds the rod rotates through an angle of $(1)(5) = 5$ radians so $L = \int_0^5 \sqrt{(2\theta+10)^2+4}\,d\theta$, let $u = 2\theta+10$ to get $L = \frac{1}{2}\int_{10}^{20} \sqrt{u^2+4}\,du = \frac{1}{2}\left[\frac{u}{2}\sqrt{u^2+4} + 2\ln|u+\sqrt{u^2+4}|\right]_{10}^{20} = \frac{1}{2}\left[10\sqrt{404} - 5\sqrt{104} + 2\ln\frac{20+\sqrt{404}}{10+\sqrt{104}}\right] \approx 75.7$ mm.

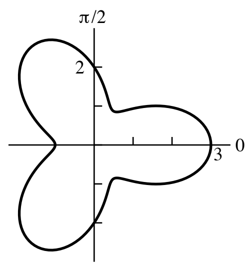
55. **(a)** $r^3\cos^3\theta - 3r^2\cos\theta\sin\theta + r^3\sin^3\theta = 0$, $r = \frac{3\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$.

(b) $A = \int_0^{\pi/2} \frac{1}{2}\left(\frac{3\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}\right)^2 d\theta = \left.\frac{2\sin^3\theta-\cos^3\theta}{2(\cos^3\theta+\sin^3\theta)}\right]_0^{\pi/2} = \frac{3}{2}$.

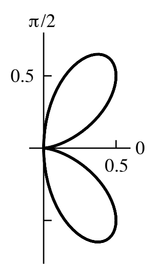
56. (a) $A = 2\int_0^{\pi/(2n)} \frac{1}{2}a^2\cos^2 n\theta\, d\theta = \frac{\pi a^2}{4n}$. **(b)** $A = 2\int_0^{\pi/(2n)} \frac{1}{2}a^2\cos^2 n\theta\, d\theta = \frac{\pi a^2}{4n}$.

(c) Total area $= 2n\cdot\frac{\pi a^2}{4n} = \frac{\pi a^2}{2}$. **(d)** Total area $= n\cdot\frac{\pi a^2}{4n} = \frac{\pi a^2}{4}$.

57. If the upper right corner of the square is the point (a,a) then the large circle has equation $r=\sqrt{2}a$ and the small circle has equation $(x-a)^2+y^2=a^2$, $r=2a\cos\theta$, so area of crescent $= 2\int_0^{\pi/4}\frac{1}{2}\left[(2a\cos\theta)^2-(\sqrt{2}a)^2\right]d\theta = a^2 =$ area of square.



58. $A = \int_0^{2\pi}\frac{1}{2}(\cos 3\theta+2)^2\, d\theta = 9\pi/2$.

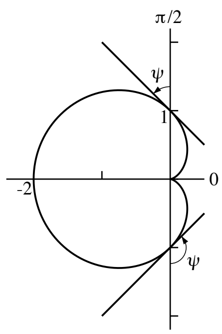


59. $A = \int_0^{\pi/2}\frac{1}{2}4\cos^2\theta\sin^4\theta\, d\theta = \pi/16$.

60. $x = r\cos\theta, y = r\sin\theta, \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta, \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$, and Formula (9) of Section 10.1 becomes $L = \int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

61. $\tan\psi = \tan(\phi-\theta) = \frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta} = \frac{\frac{dy}{dx}-\frac{y}{x}}{1+\frac{y}{x}\frac{dy}{dx}} = \frac{\frac{r\cos\theta+(dr/d\theta)\sin\theta}{-r\sin\theta+(dr/d\theta)\cos\theta}-\frac{\sin\theta}{\cos\theta}}{1+\left(\frac{r\cos\theta+(dr/d\theta)\sin\theta)}{-r\sin\theta+(dr/d\theta)\cos\theta)}\right)\left(\frac{\sin\theta}{\cos\theta}\right)} = \frac{r}{dr/d\theta}$.

62. (a) From Exercise 61, $\tan\psi = \frac{r}{dr/d\theta} = \frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$, so $\psi = \theta/2$.



(b)

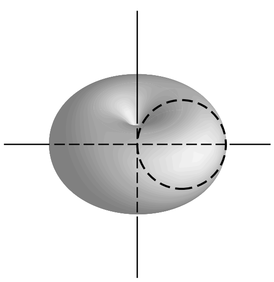
(c) At $\theta = \pi/2, \psi = \theta/2 = \pi/4$. At $\theta = 3\pi/2, \psi = \theta/2 = 3\pi/4$.

63. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{ae^{b\theta}}{abe^{b\theta}} = \dfrac{1}{b}$ is constant, so ψ is constant.

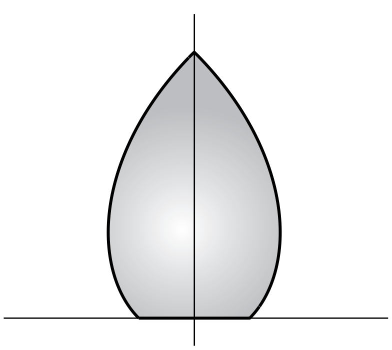
64. (a) $x = r\cos\theta, y = r\sin\theta$, $(dx/d\theta)^2 + (dy/d\theta)^2 = (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2 = f'(\theta)^2 + f(\theta)^2$; $S = \displaystyle\int_\alpha^\beta 2\pi f(\theta)\sin\theta\sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta$ if about $\theta = 0$; similarly for $\theta = \pi/2$.

(b) f' is continuous and no segment of the curve is traced more than once.

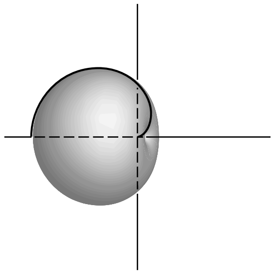
65. $r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = \cos^2\theta + \sin^2\theta = 1$, so $S = \displaystyle\int_{-\pi/2}^{\pi/2} 2\pi\cos^2\theta\,d\theta = \pi^2$.



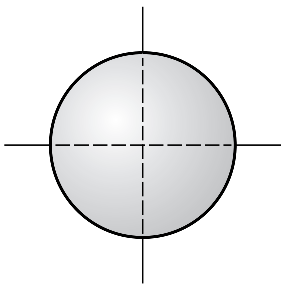
66. $S = \displaystyle\int_0^{\pi/2} 2\pi e^\theta\cos\theta\sqrt{2e^{2\theta}}\,d\theta = 2\sqrt{2}\pi\int_0^{\pi/2} e^{2\theta}\cos\theta\,d\theta = \frac{2\sqrt{2}\pi}{5}(e^\pi - 2)$.



67. $S = \displaystyle\int_0^\pi 2\pi(1-\cos\theta)\sin\theta\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\,d\theta = 2\sqrt{2}\pi\int_0^\pi \sin\theta(1-\cos\theta)^{3/2}\,d\theta = \frac{2}{5}2\sqrt{2}\pi(1-\cos\theta)^{5/2}\Big]_0^\pi =$ $32\pi/5$.



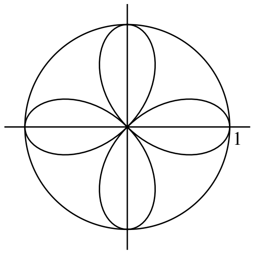
68. $S = \int_0^{\pi} 2\pi a(\sin\theta)a\,d\theta = 4\pi a^2$.



69. (a) Let P and Q have polar coordinates $(r_1, \theta_1), (r_2, \theta_2)$, respectively. Then the perpendicular from Q to OP has length $h = r_2\sin(\theta_2 - \theta_1)$ and $A = \frac{1}{2}hr_1 = \frac{1}{2}r_1 r_2\sin(\theta_2 - \theta_1)$.

(b) Define $\theta_1, \cdots, \theta_{n-1}$ and $A_1, \cdots, A_n$ as in the text's solution to the area problem. Also let $\theta_0 = \alpha$ and $\theta_n = \beta$. Then A_n is approximately the area of the triangle whose vertices have polar coordinates $(0,0)$, $(f(\theta_{n-1}), \theta_{n-1})$, and $(f(\theta_n), \theta_n)$. From part (a), $A_n \approx \frac{1}{2}f(\theta_{n-1})f(\theta_n)\sin(\theta_n - \theta_{n-1})$, so $A = \sum_{k=1}^{n} A_k \approx \sum_{k=1}^{n}\frac{1}{2}f(\theta_{n-1})f(\theta_n)\sin(\Delta\theta_n)$. If the mesh size of the partition is small, then $\theta_{n-1} \approx \theta_n$ and $\sin(\Delta\theta_n) \approx \Delta\theta_n$, so $A \approx \sum_{k=1}^{n}\frac{1}{2}f(\theta_n)^2\Delta\theta_n \approx \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2\,d\theta$.

70. Let $f(\theta) = \cos 2\theta$ and $g(\theta) = 1$. As shown in the figure, the graph of $r = f(\theta)$ is a 4-petal rose and the graph of $r = g(\theta)$ is a circle; they meet at 4 points. But $f(\theta) = g(\theta)$ when $\theta = n\pi$ for integers n; this only gives 2 of the intersection points, $(1,0)$ and $(1,\pi)$. In general, to find all of the intersection points other than the origin, we must solve the equations $f(\theta) = g(\theta + 2n\pi)$ and $f(\theta) = -g(\theta + (2n+1)\pi)$ for all integers n. Additionally, if both $f(\theta) = 0$ and $g(\theta) = 0$ have solutions, then the origin is an intersection point.



Exercise Set 10.4

1. (a) $4px = y^2$, point $(1,1), 4p = 1, x = y^2$.

(b) $-4py = x^2$, point $(3,-3), 12p = 9, -3y = x^2$.

(c) $a = 3$, $b = 2$, $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

(d) $a = 2$, $b = 3$, $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

(e) Asymptotes: $y = \pm x$, so $a = b$; point $(0, 1)$, so $y^2 - x^2 = 1$.

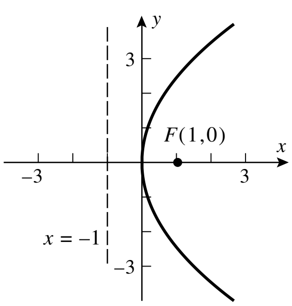
(f) Asymptotes: $y = \pm x$, so $b = a$; point $(2, 0)$, so $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$.

2. (a) Part (a): vertex $(0, 0), p = 1/4$; focus $(1/4, 0)$, directrix: $x = -1/4$. Part (b): vertex $(0, 0), p = 3/4$; focus $(0, -3/4)$, directrix: $y = 3/4$.

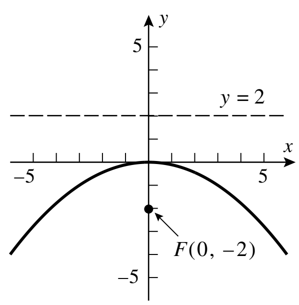
(b) Part (c): $c = \sqrt{a^2 - b^2} = \sqrt{5}$, foci $(\pm\sqrt{5}, 0)$. Part (d): $c = \sqrt{a^2 - b^2} = \sqrt{5}$, foci $(0, \pm\sqrt{5})$.

(c) Part (e): $c = \sqrt{a^2 + b^2} = \sqrt{2}$, foci at $(0, \pm\sqrt{2})$; asymptotes: $y^2 - x^2 = 0, y = \pm x$. Part (f): $c = \sqrt{a^2 + b^2} = \sqrt{8} = 2\sqrt{2}$, foci at $(\pm 2\sqrt{2}, 0)$; asymptotes: $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 0, y = \pm x$.

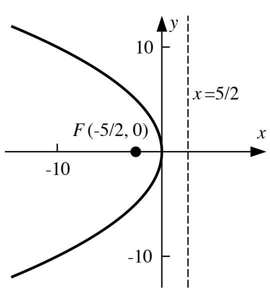
3. (a)



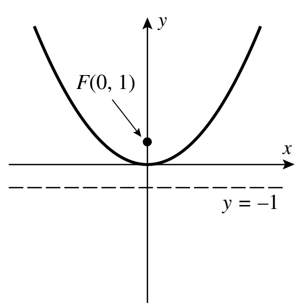
(b)



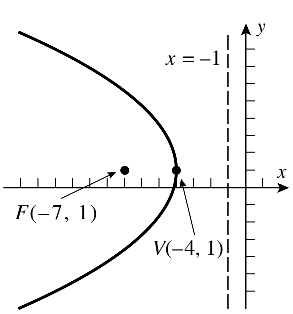
4. (a)



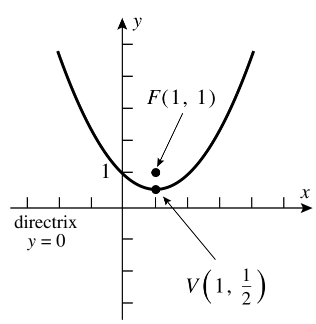
(b)

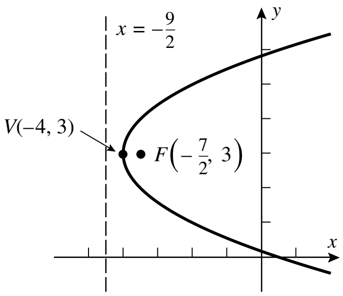


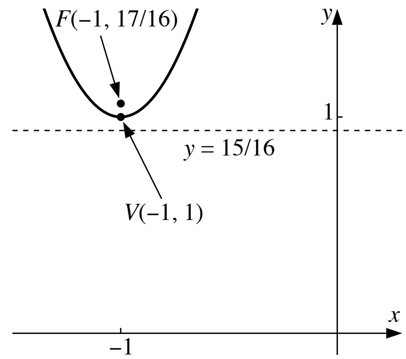
5. (a)



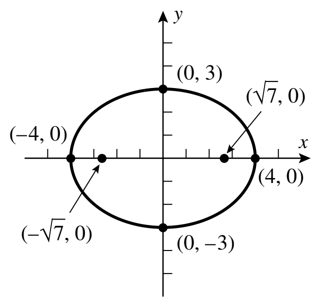
(b)



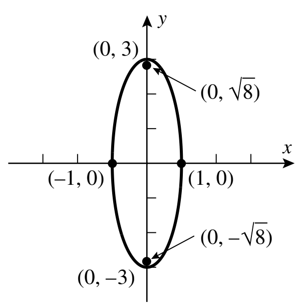




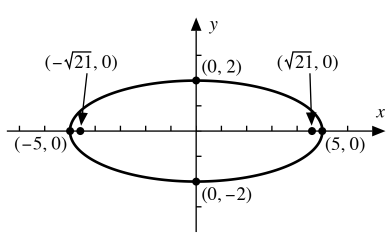
6. (a) **(b)**



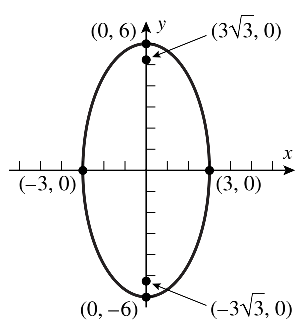
7. (a) $c^2 = 16 - 9 = 7$, $c = \sqrt{7}$.



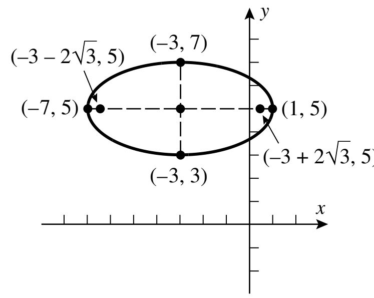
(b) $\dfrac{x^2}{1} + \dfrac{y^2}{9} = 1$, $c^2 = 9 - 1 = 8$, $c = 2\sqrt{2}$.



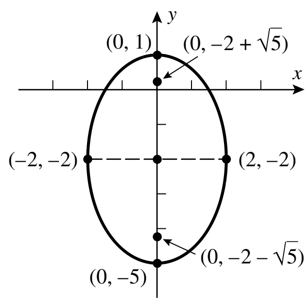
8. (a) $c^2 = 25 - 4 = 21$, $c = \sqrt{21}$.



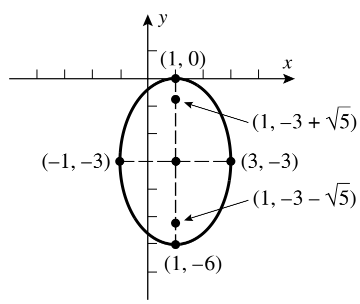
(b) $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$, $c^2 = 36 - 9 = 27$, $c = 3\sqrt{3}$.



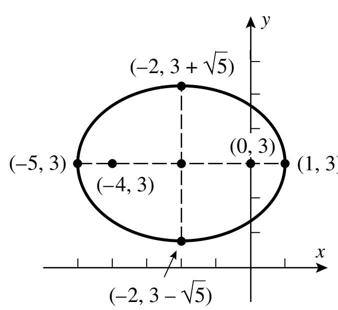
9. (a) $\dfrac{(x+3)^2}{16}+\dfrac{(y-5)^2}{4}=1$, $c^2=16-4=12$, $c=2\sqrt{3}$.



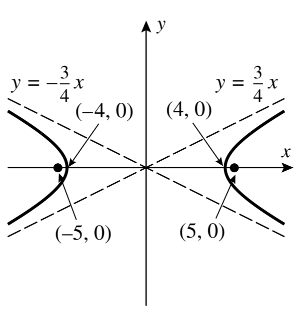
(b) $\dfrac{x^2}{4}+\dfrac{(y+2)^2}{9}=1$, $c^2=9-4=5$, $c=\sqrt{5}$.



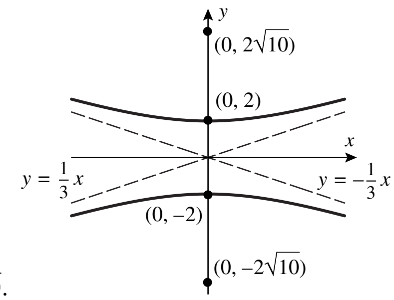
10. (a) $\dfrac{(x-1)^2}{4}+\dfrac{(y+3)^2}{9}=1$, $c^2=9-4=5$, $c=\sqrt{5}$.



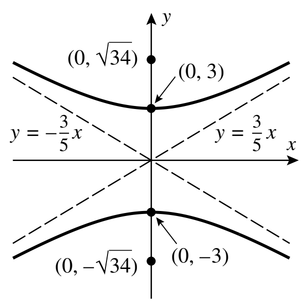
(b) $\dfrac{(x+2)^2}{9}+\dfrac{(y-3)^2}{5}=1$, $c^2=9-5=4$, $c=2$.



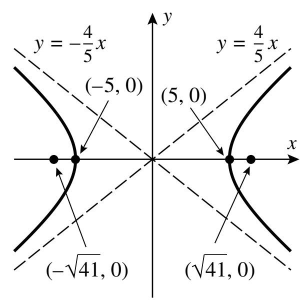
11. (a) $c^2=a^2+b^2=16+9=25$, $c=5$.



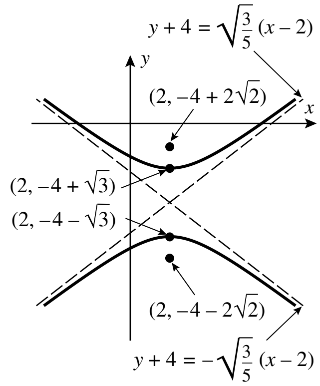
(b) $y^2/4 - x^2/36 = 1$, $c^2 = 4 + 36 = 40$, $c = 2\sqrt{10}$.



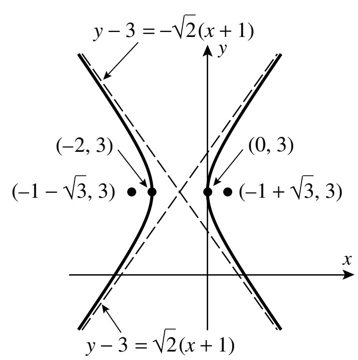
12. (a) $c^2 = a^2 + b^2 = 9 + 25 = 34$, $c = \sqrt{34}$.



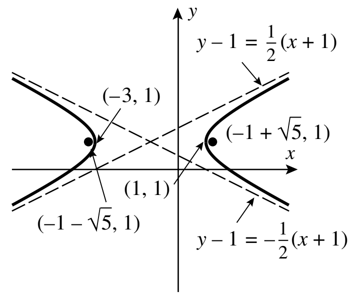
(b) $x^2/25 - y^2/16 = 1$, $c^2 = 25 + 16 = 41$, $c = \sqrt{41}$.



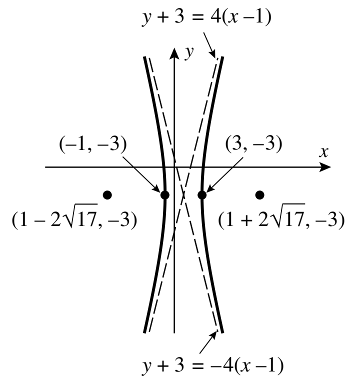
13. (a) $c^2 = 3 + 5 = 8$, $c = 2\sqrt{2}$.



(b) $(x+1)^2/1 - (y-3)^2/2 = 1$, $c^2 = 1 + 2 = 3$, $c = \sqrt{3}$.



14. (a) $(x+1)^2/4-(y-1)^2/1=1$, $c^2=4+1=5$, $c=\sqrt{5}$.



(b) $(x-1)^2/4-(y+3)^2/64=1$, $c^2=4+64=68$, $c=2\sqrt{17}$.

15. (a) $y^2=4px$, $p=3$, $y^2=12x$. **(b)** $x^2=-4py$, $p=1/4$, $x^2=-y$.

16. (a) $y^2=4px$, $p=6$, $y^2=24x$.

(b) The focus is 3 units above the directrix so $p=3/2$. The vertex is halfway between the focus and the directrix, at $(1,-1/2)$. So the equation is $(x-1)^2=6(y+1/2)$.

17. $y^2=a(x-h)$, $4=a(3-h)$ and $2=a(2-h)$, solve simultaneously to get $h=1$, $a=2$ so $y^2=2(x-1)$.

18. $(x-5)^2=a(y+3)$, $(9-5)^2=a(5+3)$ so $a=2$, $(x-5)^2=2(y+3)$.

19. (a) $x^2/9+y^2/4=1$. **(b)** $b=4$, $c=3$, $a^2=b^2+c^2=16+9=25$; $x^2/16+y^2/25=1$.

20. (a) $c=1$, $a^2=b^2+c^2=2+1=3$; $x^2/3+y^2/2=1$.

(b) $b^2=16-12=4$; either $x^2/16+y^2/4=1$ or $x^2/4+y^2/16=1$.

21. (a) $a=6$, $(-3,2)$ satisfies $\dfrac{x^2}{a^2}+\dfrac{y^2}{36}=1$ so $\dfrac{9}{a^2}+\dfrac{4}{36}=1$, $a^2=\dfrac{81}{8}$; $\dfrac{x^2}{81/8}+\dfrac{y^2}{36}=1$.

(b) The center is midway between the foci so it is at $(-1,2)$, thus $c=1$, $b=2$, $a^2=1+4=5, a=\sqrt{5}$; $(x+1)^2/4+(y-2)^2/5=1$.

22. (a) Substitute $(3,2)$ and $(1,6)$ into $x^2/A+y^2/B=1$ to get $9/A+4/B=1$ and $1/A+36/B=1$ which yields $A=10$, $B=40$; $x^2/10+y^2/40=1$.

(b) The center is at $(2,-1)$ thus $c=2$, $a=3$, $b^2=9-4=5$; $(x-2)^2/5+(y+1)^2/9=1$.

23. (a) $a=2$, $c=3$, $b^2=9-4=5$; $x^2/4-y^2/5=1$. **(b)** $a=2$, $a/b=2/3$, $b=3$; $y^2/4-x^2/9=1$.

24. (a) Vertices along x-axis: $b/a=3/2$ so $a=8/3$; $x^2/(64/9)-y^2/16=1$. Vertices along y-axis: $a/b=3/2$ so $a=6$; $y^2/36-x^2/16=1$.

(b) $c = 5$, $a/b = 2$ and $a^2 + b^2 = 25$, solve to get $a^2 = 20$, $b^2 = 5$; $y^2/20 - x^2/5 = 1$.

25. (a) Foci along the x-axis: $b/a = 3/4$ and $a^2 + b^2 = 25$, solve to get $a^2 = 16$, $b^2 = 9$; $x^2/16 - y^2/9 = 1$. Foci along the y-axis: $a/b = 3/4$ and $a^2 + b^2 = 25$ which results in $y^2/9 - x^2/16 = 1$.

(b) $c = 3$, $b/a = 2$ and $a^2 + b^2 = 9$ so $a^2 = 9/5$, $b^2 = 36/5$; $x^2/(9/5) - y^2/(36/5) = 1$.

26. (a) The center is at $(3, 6)$, $a = 3$, $c = 5$, $b^2 = 25 - 9 = 16$; $(x - 3)^2/9 - (y - 6)^2/16 = 1$.

(b) The asymptotes intersect at $(3, 1)$ which is the center, $(x - 3)^2/a^2 - (y - 1)^2/b^2 = 1$ is the form of the equation because $(0, 0)$ is to the left of both asymptotes, $9/a^2 - 1/b^2 = 1$ and $a/b = 1$ which yields $a^2 = 8$, $b^2 = 8$; $(x - 3)^2/8 - (y - 1)^2/8 = 1$.

27. False. The set described is a parabola.

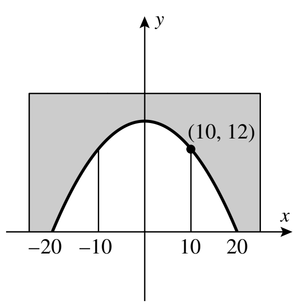
28. True, by the definition of "major axis".

29. False. The distance is $2p$, as shown in Figure 10.4.6.

30. False, unless $a = \pm 1$. The equations of the asymptotes can be found by substituting 0 for 1 in the equation of the hyperbola. So the asymptotes satisfy $\dfrac{y^2}{a^2} - x^2 = 0$; i.e. $y = \pm ax$.

31. (a) $y = ax^2 + b$, $(20, 0)$ and $(10, 12)$ are on the curve, so $400a + b = 0$ and $100a + b = 12$. Solve for b to get $b = 16$ ft = height of arch.

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $400 = a^2, a = 20$; $\dfrac{100}{400} + \dfrac{144}{b^2} = 1$, $b = 8\sqrt{3}$ ft = height of arch.

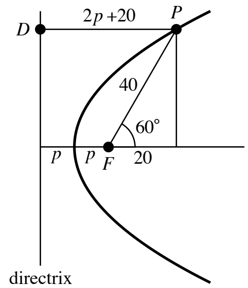


32. (a) $(x - b/2)^2 = a(y - h)$, but $(0, 0)$ is on the parabola so $b^2/4 = -ah$, $a = -\dfrac{b^2}{4h}$, $(x - b/2)^2 = -\dfrac{b^2}{4h}(y - h)$.

(b) As in part (a), $y = -\dfrac{4h}{b^2}(x - b/2)^2 + h$, $A = \displaystyle\int_0^b \left[-\frac{4h}{b^2}(x - b/2)^2 + h\right] dx = \frac{2}{3}bh$.

33. We may assume that the vertex is $(0, 0)$ and the parabola opens to the right. Let $P(x_0, y_0)$ be a point on the parabola $y^2 = 4px$, then by the definition of a parabola, PF = distance from P to directrix $x = -p$, so $PF = x_0 + p$ where $x_0 \geq 0$ and PF is a minimum when $x_0 = 0$ (the vertex).

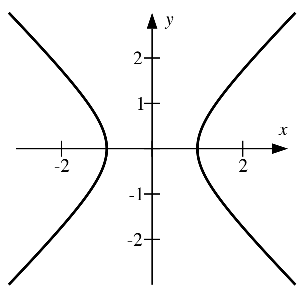
34. Let p = distance (in millions of miles) between the vertex (closest point) and the focus F. Then $PF = PD$, $40 = 2p + 40\cos(60^\circ) = 2p + 20$, and $p = 10$ million miles.



35. Use an xy-coordinate system so that $y^2 = 4px$ is an equation of the parabola. Then $(1, 1/2)$ is a point on the curve so $(1/2)^2 = 4p(1)$, $p = 1/16$. The light source should be placed at the focus which is 1/16 ft. from the vertex.

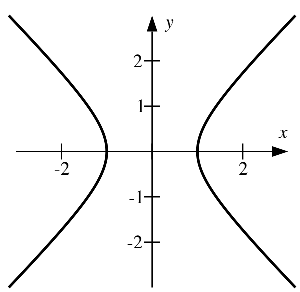
36. (a) For any point (x, y), the equation $y = b\sinh t$ has a unique solution t, $-\infty < t < +\infty$. On the hyperbola, $\dfrac{x^2}{a^2} = 1 + \dfrac{y^2}{b^2} = 1 + \sinh^2 t = \cosh^2 t$, so $x = \pm a\cosh t$.

(b)



37. (a) For any point (x, y), the equation $y = b\tan t$ has a unique solution t, $-\pi/2 < t < \pi/2$. On the hyperbola, $\dfrac{x^2}{a^2} = 1 + \dfrac{y^2}{b^2} = 1 + \tan^2 t = \sec^2 t$, so $x = \pm a\sec t$.

(b)



38. $(x-2)^2 + (y-4)^2 = y^2$, $(x-2)^2 = 8y - 16$, $(x-2)^2 = 8(y-2)$.

39. $(4, 1)$ and $(4, 5)$ are the foci so the center is at $(4, 3)$ thus $c = 2$, $a = 12/2 = 6$, $b^2 = 36 - 4 = 32$; $(x-4)^2/32 + (y-3)^2/36 = 1$.

40. From the definition of a hyperbola, $\left|\sqrt{(x-1)^2 + (y-1)^2} - \sqrt{x^2 + y^2}\right| = 1$, $\sqrt{(x-1)^2 + (y-1)^2} - \sqrt{x^2 + y^2} = \pm 1$, transpose the second radical to the right hand side of the equation and square and simplify to get $\pm 2\sqrt{x^2 + y^2} = -2x - 2y + 1$, square and simplify again to get $8xy - 4x - 4y + 1 = 0$.

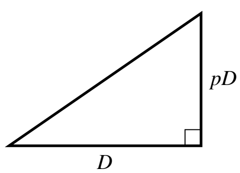
41. Let the ellipse have equation $\dfrac{4}{81}x^2 + \dfrac{y^2}{4} = 1$, then $A(x) = (2y)^2 = 16\left(1 - \dfrac{4x^2}{81}\right)$, so $V = 2\displaystyle\int_0^{9/2} 16\left(1 - \frac{4x^2}{81}\right)dx = 96$.

42. See Exercise 41, $A(y) = \sqrt{3}\,x^2 = \sqrt{3}\,\dfrac{81}{4}\left(1 - \dfrac{y^2}{4}\right)$, so $V = 2\displaystyle\int_0^2 \sqrt{3}\,\frac{81}{4}\left(1 - \frac{y^2}{4}\right) dy = 54\sqrt{3}$.

43. Assume $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, A = 4\displaystyle\int_0^a b\sqrt{1 - x^2/a^2}\,dx = \pi ab$.

44. In the $x'y'$-plane an equation of the circle is $(x')^2 + (y')^2 = r^2$ where r is the radius of the cylinder. Let $P(x, y)$ be a point on the curve in the xy-plane, then $x' = x\cos\theta$ and $y' = y$ so $x^2\cos^2\theta + y^2 = r^2$ which is an equation of an ellipse in the xy-plane.

45. $L = 2a = \sqrt{D^2 + p^2D^2} = D\sqrt{1 + p^2}$ (see figure), so $a = \dfrac{1}{2}D\sqrt{1 + p^2}$, but $b = \dfrac{1}{2}D$, $T = c = \sqrt{a^2 - b^2} = \sqrt{\dfrac{1}{4}D^2(1 + p^2) - \dfrac{1}{4}D^2} = \dfrac{1}{2}pD$.



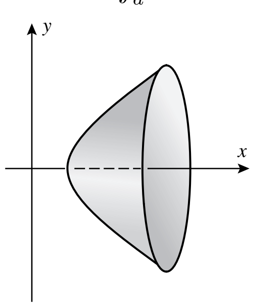
46. Let d_1 and d_2 be the distances of the first and second observers, respectively, from the point of the explosion. Then t = (time for sound to reach the second observer) − (time for sound to reach the first observer) = $d_2/v - d_1/v$ so $d_2 - d_1 = vt$. For constant v and t the difference of distances, d_2 and d_1 is constant so the explosion occurred somewhere on a branch of a hyperbola whose foci are where the observers are. Since $d_2 - d_1 = 2a, a = \dfrac{vt}{2}, b^2 = c^2 - \dfrac{v^2t^2}{4}$, and $\dfrac{x^2}{v^2t^2/4} - \dfrac{y^2}{c^2 - (v^2t^2/4)} = 1$.

47. As in Exercise 46, $d_2 - d_1 = 2a = vt = (299{,}792{,}458\text{ m/s})(100 \cdot 10^{-6}\text{ s}) \approx 29979\text{ m} = 29.979\text{ km}$. $a^2 = (vt/2)^2 \approx 224.689\text{ km}^2$; $c^2 = (50)^2 = 2500\text{ km}^2$, $b^2 = c^2 - a^2 \approx 2275.311$ km, $\dfrac{x^2}{224.688} - \dfrac{y^2}{2275.311} = 1$. But $y = 200$ km, so $x \approx 64.612$ km. The ship is located at $(64.612, 200)$.

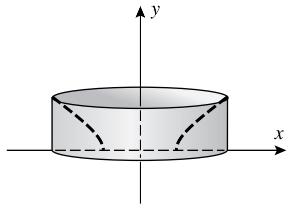
48. **(a)** $\dfrac{x^2}{225} - \dfrac{y^2}{1521} = 1$, so $V = 2\displaystyle\int_0^{h/2} 225\pi\left(1 + \frac{y^2}{1521}\right) dy = \frac{25}{2028}\pi h^3 + 225\pi h\text{ ft}^3$.

(b) $S = 2\displaystyle\int_0^{h/2} 2\pi x\sqrt{1 + (dx/dy)^2}\,dy = 4\pi\int_0^{h/2}\sqrt{225 + y^2\left(\frac{225}{1521} + \left(\frac{225}{1521}\right)^2\right)}\,dy = \frac{5\pi h}{338}\sqrt{1028196 + 194h^2} + \frac{7605\sqrt{194}}{97}\pi\ln\left[\frac{\sqrt{194}h + \sqrt{1028196 + 194h^2}}{1014}\right]\text{ ft}^2$.

49. **(a)** $V = \displaystyle\int_a^{\sqrt{a^2+b^2}} \pi\left(b^2x^2/a^2 - b^2\right) dx = \frac{\pi b^2}{3a^2}(b^2 - 2a^2)\sqrt{a^2 + b^2} + \frac{2}{3}ab^2\pi$.



(b) $V = 2\pi \int_a^{\sqrt{a^2+b^2}} x\sqrt{b^2x^2/a^2 - b^2}\, dx = (2b^4/3a)\pi$.



50. **(a)** Use $\dfrac{x^2}{9}+\dfrac{y^2}{4} = 1$, $x = \dfrac{3}{2}\sqrt{4-y^2}$. We obtain that $V = \int_{-2}^{-2+h} (2)(3/2)\sqrt{4-y^2}(18)dy = 54\int_{-2}^{-2+h} \sqrt{4-y^2}dy =$ $54\left[\dfrac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\dfrac{y}{2}\right]_{-2}^{-2+h} = 27\left(4\sin^{-1}\dfrac{h-2}{2} + (h-2)\sqrt{4h-h^2} + 2\pi\right)$ ft^3.

(b) When $h = 4$ ft, $V_{\text{full}} = 108\sin^{-1} 1 + 54\pi = 108\pi$ ft^3, so solve for h when $V = (k/4)V_{\text{full}}$, $k = 1, 2, 3$, to get $h = 1.19205, 2, 2.80795$ ft or $14.30465, 24, 33.69535$ in.

51. $y = \dfrac{1}{4p}x^2$, $dy/dx = \dfrac{1}{2p}x$, $dy/dx|_{x=x_0} = \dfrac{1}{2p}x_0$, the tangent line at (x_0, y_0) has the formula $y - y_0 = \dfrac{x_0}{2p}(x - x_0) = \dfrac{x_0}{2p}x - \dfrac{x_0^2}{2p}$, but $\dfrac{x_0^2}{2p} = 2y_0$ because (x_0, y_0) is on the parabola $y = \dfrac{1}{4p}x^2$. Thus the tangent line is $y - y_0 = \dfrac{x_0}{2p}x - 2y_0$, $y = \dfrac{x_0}{2p}x - y_0$.

52. By implicit differentiation, $\left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = -\dfrac{b^2}{a^2}\dfrac{x_0}{y_0}$ if $y_0 \neq 0$, the tangent line is $y - y_0 = -\dfrac{b^2}{a^2}\dfrac{x_0}{y_0}(x - x_0)$, $a^2y_0y - a^2y_0^2 = -b^2x_0x + b^2x_0^2$, $b^2x_0x + a^2y_0y = b^2x_0^2 + a^2y_0^2$, but (x_0, y_0) is on the ellipse so $b^2x_0^2 + a^2y_0^2 = a^2b^2$; thus the tangent line is $b^2x_0x + a^2y_0y = a^2b^2$, $x_0x/a^2 + y_0y/b^2 = 1$. If $y_0 = 0$ then $x_0 = \pm a$ and the tangent lines are $x = \pm a$ which also follows from $x_0x/a^2 + y_0y/b^2 = 1$.

53. By implicit differentiation, $\left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = \dfrac{b^2}{a^2}\dfrac{x_0}{y_0}$ if $y_0 \neq 0$, the tangent line is $y - y_0 = \dfrac{b^2}{a^2}\dfrac{x_0}{y_0}(x - x_0)$, $b^2x_0x - a^2y_0y = b^2x_0^2 - a^2y_0^2 = a^2b^2$, $x_0x/a^2 - y_0y/b^2 = 1$. If $y_0 = 0$ then $x_0 = \pm a$ and the tangent lines are $x = \pm a$ which also follow from $x_0x/a^2 - y_0y/b^2 = 1$.

54. Use $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1$ as the equations of the ellipse and hyperbola. If (x_0, y_0) is a point of intersection then $\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1 = \dfrac{x_0^2}{A^2} - \dfrac{y_0^2}{B^2}$, so $x_0^2\left(\dfrac{1}{A^2} - \dfrac{1}{a^2}\right) = y_0^2\left(\dfrac{1}{B^2} + \dfrac{1}{b^2}\right)$ and $a^2A^2y_0^2(b^2 + B^2) = b^2B^2x_0^2(a^2 - A^2)$. Since the conics have the same foci, $a^2 - b^2 = c^2 = A^2 + B^2$, so $a^2 - A^2 = b^2 + B^2$. Hence $a^2A^2y_0^2 = b^2B^2x_0^2$. From Exercises 52 and 53, the slopes of the tangent lines are $-\dfrac{b^2x_0}{a^2y_0}$ and $\dfrac{B^2x_0}{A^2y_0}$, whose product is $-\dfrac{b^2B^2x_0^2}{a^2A^2y_0^2} = -1$. Hence the tangent lines are perpendicular.

55. Use implicit differentiation on $x^2 + 4y^2 = 8$ to get $\left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = -\dfrac{x_0}{4y_0}$ where (x_0, y_0) is the point of tangency, but $-x_0/(4y_0) = -1/2$ because the slope of the line is $-1/2$, so $x_0 = 2y_0$. (x_0, y_0) is on the ellipse so $x_0^2 + 4y_0^2 = 8$ which when solved with $x_0 = 2y_0$ yields the points of tangency $(2, 1)$ and $(-2, -1)$. Substitute these into the equation of the line to get $k = \pm 4$.

56. Let (x_0, y_0) be such a point. The foci are at $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$, the lines are perpendicular if the product of their slopes is -1 so $\dfrac{y_0}{x_0 + \sqrt{5}} \cdot \dfrac{y_0}{x_0 - \sqrt{5}} = -1$, $y_0^2 = 5 - x_0^2$ and $4x_0^2 - y_0^2 = 4$. Solve these to get $x_0 = \pm 3/\sqrt{5}$, $y_0 = \pm 4/\sqrt{5}$. The coordinates are $(\pm 3/\sqrt{5}, 4/\sqrt{5})$, $(\pm 3/\sqrt{5}, -4/\sqrt{5})$.

57. Let (x_0, y_0) be one of the points; then $\left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = \dfrac{4x_0}{y_0}$, the tangent line is $y = (4x_0/y_0)x + 4$, but (x_0, y_0) is on both the line and the curve which leads to $4x_0^2 - y_0^2 + 4y_0 = 0$ and $4x_0^2 - y_0^2 = 36$, so we obtain that $x_0 = \pm 3\sqrt{13}/2$, $y_0 = -9$.

58. We may assume $A > 0$, since if $A < 0$ then we can multiply the equation by -1, and if $A = 0$ then we can exchange x with y and thus A with C (C cannot be zero if $A = 0$). Then $Ax^2 + Cy^2 + Dx + Ey + F = A\left(x + \dfrac{D}{2A}\right)^2 + C\left(y + \dfrac{E}{2C}\right)^2 + F - \dfrac{D^2}{4A} - \dfrac{E^2}{4C} = 0$.

(a) Let $AC > 0$. If $F < \dfrac{D^2}{4A} + \dfrac{E^2}{4C}$ the equation represents an ellipse (a circle if $A = C$); if $F = \dfrac{D^2}{4A} + \dfrac{E^2}{4C}$, the point $x = -D/(2A), y = -E/(2C)$; and if $F > \dfrac{D^2}{4A} + \dfrac{E^2}{4C}$ then the graph is empty.

(b) If $AC < 0$ and $F = \dfrac{D^2}{4A} + \dfrac{E^2}{4C}$, then $\left[\sqrt{A}\left(x + \dfrac{D}{2A}\right) + \sqrt{-C}\left(y + \dfrac{E}{2C}\right)\right]\left[\sqrt{A}\left(x + \dfrac{D}{2A}\right) - \sqrt{-C}\left(y + \dfrac{E}{2C}\right)\right] = 0$, a pair of lines; otherwise a hyperbola.

(c) Assume $C = 0$, so $Ax^2 + Dx + Ey + F = 0$. If $E \neq 0$, parabola; if $E = 0$ then $Ax^2 + Dx + F = 0$. If this polynomial has roots $x = x_1, x_2$ with $x_1 \neq x_2$ then a pair of parallel lines; if $x_1 = x_2$ then one line; if no roots, then graph is empty. If $A = 0, C \neq 0$ then a similar argument applies.

59. **(a)** $(x - 1)^2 - 5(y + 1)^2 = 5$, hyperbola.

(b) $x^2 - 3(y + 1)^2 = 0, x = \pm\sqrt{3}(y + 1)$, two lines.

(c) $4(x + 2)^2 + 8(y + 1)^2 = 4$, ellipse.

(d) $3(x + 2)^2 + (y + 1)^2 = 0$, the point $(-2, -1)$ (degenerate case).

(e) $(x + 4)^2 + 2y = 2$, parabola.

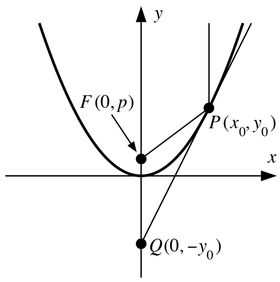
(f) $5(x + 4)^2 + 2y = -14$, parabola.

60. The distance from the point (x, y) to the focus $(0, p)$ is equal to the distance to the directrix $y = -p$, so $x^2 + (y - p)^2 = (y + p)^2$, $x^2 = 4py$.

61. The distance from the point (x, y) to the focus $(0, -c)$ plus distance to the focus $(0, c)$ is equal to the constant $2a$, so $\sqrt{x^2 + (y + c)^2} + \sqrt{x^2 + (y - c)^2} = 2a$, $x^2 + (y + c)^2 = 4a^2 + x^2 + (y - c)^2 - 4a\sqrt{x^2 + (y - c)^2}$, $\sqrt{x^2 + (y - c)^2} = a - \dfrac{c}{a}y$, and since $a^2 - c^2 = b^2$, $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$.

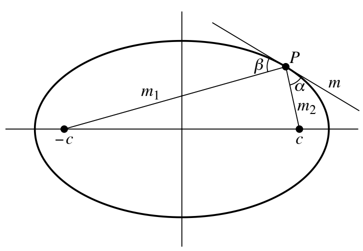
62. The distance from the point (x, y) to the focus $(-c, 0)$ less distance to the focus $(c, 0)$ is equal to $2a$, $\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a$, $(x + c)^2 + y^2 = (x - c)^2 + y^2 + 4a^2 \pm 4a\sqrt{(x - c)^2 + y^2}$, $\sqrt{(x - c)^2 + y^2} = \pm\left(\dfrac{cx}{a} - a\right)$, and, since $c^2 - a^2 = b^2$, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

63. Assume the equation of the parabola is $x^2 = 4py$. The tangent line at $P = (x_0, y_0)$ (see figure) is given by $(y - y_0)/(x - x_0) = m = x_0/2p$. To find the y-intercept set $x = 0$ and obtain $y = -y_0$. Thus the tangent line meets the y-axis at $Q = (0, -y_0)$. The focus is $F = (0, p) = (0, x_0^2/4y_0)$, so the distance from P to the focus is $\sqrt{x_0^2 + (y_0 - p)^2} = \sqrt{4py_0 + (y_0 - p)^2} = \sqrt{(y_0 + p)^2} = y_0 + p$ and the distance from the focus to Q is $p + y_0$. Hence triangle FPQ is isosceles, and angles FPQ and FQP are equal. The angle between the tangent line and the vertical line through P equals angle FQP, so it also equals angle FPQ, as stated in the theorem.

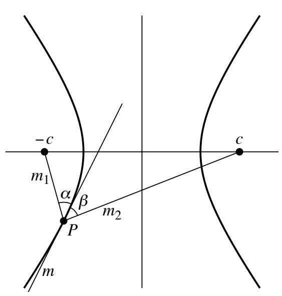


64. (a) $\tan\theta = \tan(\phi_2 - \phi_1) = \dfrac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2 \tan\phi_1} = \dfrac{m_2 - m_1}{1 + m_1 m_2}$.

(b) Let $P(x_0, y_0)$ be a point in the first quadrant on the ellipse and let m be the slope of the tangent line at P. By implicit differentiation, $m = \left.\dfrac{dy}{dx}\right|_{P(x_0,y_0)} = -\dfrac{b^2}{a^2}\dfrac{x_0}{y_0}$ if $y_0 \neq 0$. Let m_1 and m_2 be the slopes of the lines through P and the foci at $(-c, 0)$ and $(c, 0)$ respectively; then $m_1 = \dfrac{y_0}{x_0 + c}$ and $m_2 = \dfrac{y_0}{x_0 - c}$. Let α and β be the angles shown in the figure; then $\tan\alpha = \dfrac{m - m_2}{1 + m m_2} = \dfrac{-(b^2x_0)/(a^2y_0) - y_0/(x_0 - c)}{1 - (b^2x_0)/[a^2(x_0 - c)]} = \dfrac{-b^2x_0^2 - a^2y_0^2 + b^2cx_0}{[(a^2 - b^2)x_0 - a^2c]\,y_0} = \dfrac{-a^2b^2 + b^2cx_0}{(c^2x_0 - a^2c)y_0} = \dfrac{b^2}{cy_0}$, and similarly $\tan(\pi - \beta) = \dfrac{m - m_1}{1 + m m_1} = -\dfrac{b^2}{cy_0} = -\tan\beta$ so $\tan\alpha = \tan\beta$, $\alpha = \beta$. The proof for the case $y_0 = 0$ follows trivially. By symmetry, the result holds for P in the other three quadrants as well.



(c) Let $P(x_0, y_0)$ be a point in the third quadrant on the hyperbola and let m be the slope of the tangent line at P. By implicit differentiation, $m = \left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = \dfrac{b^2x_0}{a^2y_0}$ if $y_0 \neq 0$. Let m_1 and m_2 be the slopes of the lines through P and the foci at $(-c, 0)$ and $(c, 0)$ respectively; then $m_1 = \dfrac{y_0}{x_0 + c}$, $m_2 = \dfrac{y_0}{x_0 - c}$. Use $\tan\alpha = \dfrac{m_1 - m}{1 + m_1 m}$ and $\tan\beta = \dfrac{m - m_2}{1 + m m_2}$ to get $\tan\alpha = \tan\beta = -\dfrac{b^2}{cy_0}$ so $\alpha = \beta$. If $y_0 = 0$ the result follows trivially and by symmetry the result holds for P in the other three quadrants as well.



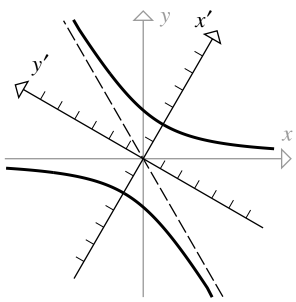
65. Assuming that the major and minor axes have already been drawn, open the compass to the length of half the major axis, place the point of the compass at an end of the minor axis, and draw arcs that cross the major axis to both sides of the center of the ellipse. Place the tacks where the arcs intersect the major axis.

Exercise Set 10.5

1. **(a)** $\sin\theta = \sqrt{3}/2$, $\cos\theta = 1/2$; $x' = (-2)(1/2) + (6)(\sqrt{3}/2) = -1 + 3\sqrt{3}$, $y' = -(-2)(\sqrt{3}/2) + 6(1/2) = 3 + \sqrt{3}$.

(b) $x = \dfrac{1}{2}x' - \dfrac{\sqrt{3}}{2}y' = \dfrac{1}{2}(x' - \sqrt{3}y')$, $y = \dfrac{\sqrt{3}}{2}x' + \dfrac{1}{2}y' = \dfrac{1}{2}(\sqrt{3}x' + y')$; $\sqrt{3}\left[\dfrac{1}{2}(x' - \sqrt{3}y')\right]\left[\dfrac{1}{2}(\sqrt{3}x' + y')\right] + \left[\dfrac{1}{2}(\sqrt{3}x' + y')\right]^2 = 6$, $\dfrac{\sqrt{3}}{4}(\sqrt{3}(x')^2 - 2x'y' - \sqrt{3}(y')^2) + \dfrac{1}{4}(3(x')^2 + 2\sqrt{3}x'y' + (y')^2) = 6$, $\dfrac{3}{2}(x')^2 - \dfrac{1}{2}(y')^2 = 6$, $3(x')^2 - (y')^2 = 12$

(c)

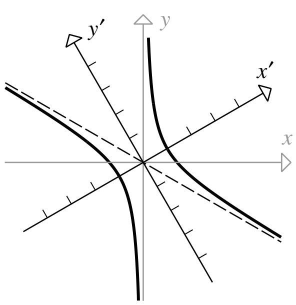


2. **(a)** $\sin\theta = 1/2$, $\cos\theta = \sqrt{3}/2$; $x' = (1)(\sqrt{3}/2) + (-\sqrt{3})(1/2) = 0$, $y' = -(1)(1/2) + (-\sqrt{3})(\sqrt{3}/2) = -2$.

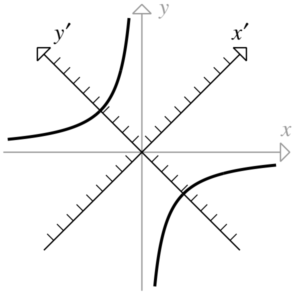
(b) $x = \dfrac{\sqrt{3}}{2}x' - \dfrac{1}{2}y' = \dfrac{1}{2}(\sqrt{3}x' - y')$, $y = \dfrac{1}{2}x' + \dfrac{\sqrt{3}}{2}y' = \dfrac{1}{2}(x' + \sqrt{3}y')$;

$2\left[\dfrac{1}{2}(\sqrt{3}x' - y')\right]^2 + 2\sqrt{3}\left[\dfrac{1}{2}(\sqrt{3}x' - y')\right]\left[\dfrac{1}{2}(x' + \sqrt{3}y')\right] = 3$, $\dfrac{1}{2}(3(x')^2 - 2\sqrt{3}x'y' + (y')^2) + \dfrac{\sqrt{3}}{2}(\sqrt{3}(x')^2 + 2x'y' - \sqrt{3}(y')^2) = 3$, $3(x')^2 - (y')^2 = 3$, $(x')^2/1 - (y')^2/3 = 1$.

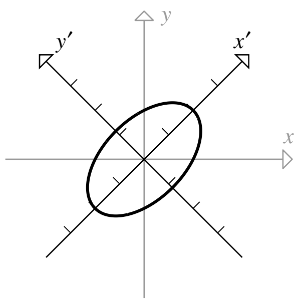
(c)



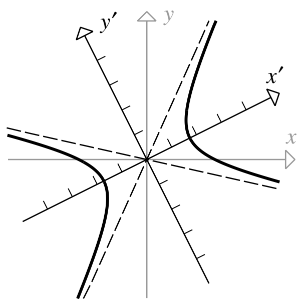
3. $\cot 2\theta = (0 - 0)/1 = 0$, $2\theta = 90°$, $\theta = 45°$, $x = (\sqrt{2}/2)(x' - y')$, $y = (\sqrt{2}/2)(x' + y')$, $(y')^2/18 - (x')^2/18 = 1$, hyperbola.



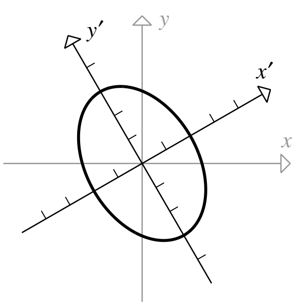
4. $\cot 2\theta = (1 - 1)/(-1) = 0$, $\theta = 45°$, $x = (\sqrt{2}/2)(x' - y')$, $y = (\sqrt{2}/2)(x' + y')$, $(x')^2/4 + (y')^2/(4/3) = 1$, ellipse.



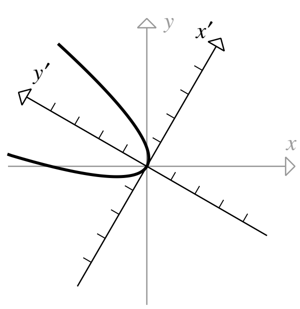
5. $\cot 2\theta = [1-(-2)]/4 = 3/4$, $\cos 2\theta = 3/5$, $\sin\theta = \sqrt{(1-3/5)/2} = 1/\sqrt{5}$, $\cos\theta = \sqrt{(1+3/5)/2} = 2/\sqrt{5}$, $x = (1/\sqrt{5})(2x'-y')$, $y = (1/\sqrt{5})(x'+2y')$, $(x')^2/3 - (y')^2/2 = 1$, hyperbola.



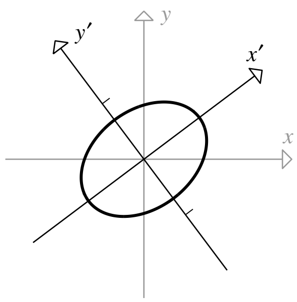
6. $\cot 2\theta = (31-21)/(10\sqrt{3}) = 1/\sqrt{3}$, $2\theta = 60°, \theta = 30°$, $x = (1/2)(\sqrt{3}x'-y')$, $y = (1/2)(x'+\sqrt{3}y')$, $(x')^2/4 + (y')^2/9 = 1$, ellipse.



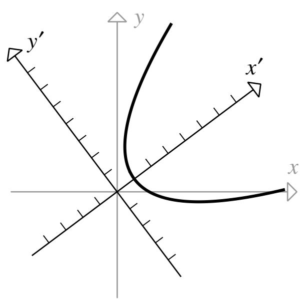
7. $\cot 2\theta = (1-3)/(2\sqrt{3}) = -1/\sqrt{3}$, $2\theta = 120°, \theta = 60°$, $x = (1/2)(x'-\sqrt{3}y')$, $y = (1/2)(\sqrt{3}x'+y')$, $y' = (x')^2$, parabola.



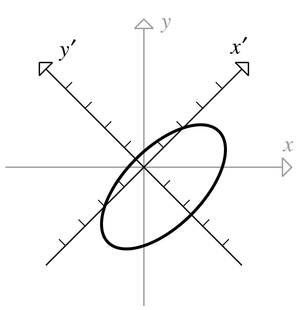
8. $\cot 2\theta = (34-41)/(-24) = 7/24$, $\cos 2\theta = 7/25$, $\sin\theta = \sqrt{(1-7/25)/2} = 3/5$, $\cos\theta = \sqrt{(1+7/25)/2} = 4/5$, $x = (1/5)(4x'-3y')$, $y = (1/5)(3x'+4y')$, $(x')^2 + (y')^2/(1/2) = 1$, ellipse.



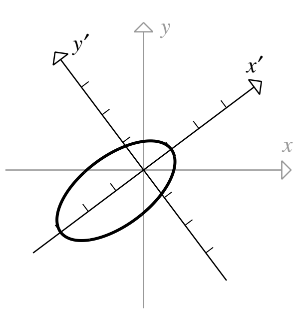
9. $\cot 2\theta = (9-16)/(-24) = 7/24$, $\cos 2\theta = 7/25$, $\sin\theta = 3/5$, $\cos\theta = 4/5$, $x = (1/5)(4x' - 3y')$, $y = (1/5)(3x' + 4y')$, $(y')^2 = 4(x' - 1)$, parabola.



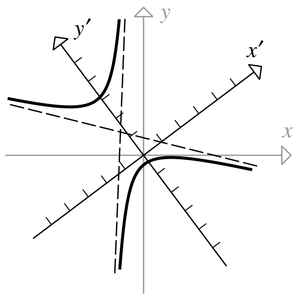
10. $\cot 2\theta = (5-5)/(-6) = 0$, $\theta = 45°$, $x = (\sqrt{2}/2)(x' - y')$, $y = (\sqrt{2}/2)(x' + y')$, $(x')^2/8 + (y'+1)^2/2 = 1$, ellipse.



11. $\cot 2\theta = (52-73)/(-72) = 7/24$, $\cos 2\theta = 7/25$, $\sin\theta = 3/5$, $\cos\theta = 4/5$, $x = (1/5)(4x' - 3y')$, $y = (1/5)(3x' + 4y')$, $(x'+1)^2/4 + (y')^2 = 1$, ellipse.



12. $\cot 2\theta = [6 - (-1)]/24 = 7/24$, $\cos 2\theta = 7/25$, $\sin\theta = 3/5$, $\cos\theta = 4/5$, $x = (1/5)(4x' - 3y')$, $y = (1/5)(3x' + 4y')$, $(y' - 7/5)^2/3 - (x' + 1/5)^2/2 = 1$, hyperbola.



13. $x' = (\sqrt{2}/2)(x+y)$, $y' = (\sqrt{2}/2)(-x+y)$ which when substituted into $3(x')^2 + (y')^2 = 6$ yields $x^2 + xy + y^2 = 3$.

14. From (5), $x = \dfrac{1}{2}(\sqrt{3}x' - y')$ and $y = \dfrac{1}{2}(x' + \sqrt{3}y')$ so $y = x^2$ becomes $\dfrac{1}{2}(x' + \sqrt{3}y') = \dfrac{1}{4}(\sqrt{3}x' - y')^2$; simplify to get $3(x')^2 - 2\sqrt{3}x'y' + (y')^2 - 2x' - 2\sqrt{3}y' = 0$.

15. Let $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$ then $x^2+y^2 = r^2$ becomes $(\sin^2\theta+\cos^2\theta)(x')^2+(\sin^2\theta+\cos^2\theta)(y')^2 = r^2$, $(x')^2+(y')^2 = r^2$. Under a rotation transformation the center of the circle stays at the origin of both coordinate systems.

16. Multiply the first equation through by $\cos\theta$ and the second by $\sin\theta$ and add to get $x\cos\theta + y\sin\theta = (\cos^2\theta + \sin^2\theta)x' = x'$. Multiply the first by $-\sin\theta$ and the second by $\cos\theta$ and add to get y'.

17. Use the Rotation Equations (5).

18. If the line is given by $Dx' + Ey' + F = 0$ then from (6), $D(x\cos\theta + y\sin\theta) + E(-x\sin\theta + y\cos\theta) + F = 0$, or $(D\cos\theta - E\sin\theta)x + (D\sin\theta + E\cos\theta)y + F = 0$, which is a line in the xy-coordinates.

19. Set $\cot 2\theta = (A-C)/B = 0, 2\theta = \pi/2, \theta = \pi/4$, $\cos\theta = \sin\theta = 1/\sqrt{2}$. Set $x = (x'-y')/\sqrt{2}$, $y = (x'+y')/\sqrt{2}$ and insert these into the equation to obtain $4y' = (x')^2$; parabola, $p = 1$. In $x'y'$-coordinates: vertex $(0,0)$, focus $(0,1)$, directrix $y' = -1$. In xy-coordinates: vertex $(0,0)$, focus $(-1/\sqrt{2}, 1/\sqrt{2})$, directrix $y = x - \sqrt{2}$.

20. $\cot 2\theta = (1-3)/(-2\sqrt{3}) = 1/\sqrt{3}$, $2\theta = \pi/3$, $\theta = \pi/6$, $\cos\theta = \sqrt{3}/2$, $\sin\theta = 1/2$. Set $x = \sqrt{3}x'/2 - y'/2, y = x'/2 + \sqrt{3}y'/2$ and obtain $4x' = (y')^2$; parabola, $p = 1$. In $x'y'$-coordinates: vertex $(0,0)$, focus $(1,0)$, directrix $x' = -1$. In xy-coordinates: vertex $(0,0)$, focus $(\sqrt{3}/2, 1/2)$, directrix $y = -\sqrt{3}x - 2$.

21. $\cot 2\theta = (9-16)/(-24) = 7/24$. Use the method of Example 4 to obtain $\cos 2\theta = \dfrac{7}{25}$, so $\cos\theta = \sqrt{\dfrac{1+\cos 2\theta}{2}} = \sqrt{\dfrac{1+\frac{7}{25}}{2}} = \dfrac{4}{5}$, $\sin\theta = \sqrt{\dfrac{1-\cos 2\theta}{2}} = \dfrac{3}{5}$. Set $x = \dfrac{4}{5}x' - \dfrac{3}{5}y'$, $y = \dfrac{3}{5}x' + \dfrac{4}{5}y'$, and insert these into the original equation to obtain $(y')^2 = 4(x'-1)$; parabola, $p = 1$. In $x'y'$-coordinates: vertex $(1,0)$, focus $(2,0)$, directrix $x' = 0$. In xy-coordinates: vertex $(4/5, 3/5)$, focus $(8/5, 6/5)$, directrix $y = -4x/3$.

22. $\cot 2\theta = (1-3)/(2\sqrt{3}) = -1/\sqrt{3}$, $2\theta = 2\pi/3$, $\theta = \pi/3$, $\cos\theta = 1/2$, $\sin\theta = \sqrt{3}/2$. Set $x = (x' - \sqrt{3}y')/2$, $y = (\sqrt{3}x' + y')/2$, and the equation is transformed into $(x')^2 = 8(y'+3)$; parabola, $p = 2$. In $x'y'$-coordinates: vertex $(0,-3)$, focus $(0,-1)$, directrix $y' = -5$. In xy-coordinates: vertex $(3\sqrt{3}/2, -3/2)$, focus $(\sqrt{3}/2, -1/2)$, directrix $y = \sqrt{3}x - 10$.

23. $\cot 2\theta = (288-337)/(-168) = 49/168 = 7/24$; proceed as in Exercise 21 to obtain $\cos\theta = 4/5$, $\sin\theta = 3/5$. Set $x = (4x'-3y')/5$, $y = (3x'+4y')/5$ to get $(x')^2/16 + (y')^2/9 = 1$; ellipse, $a = 4$, $b = 3$, $c = \sqrt{7}$. In $x'y'$-coordinates: foci $(\pm\sqrt{7}, 0)$, vertices $(\pm 4, 0)$, minor axis endpoints $(0, \pm 3)$. In xy-coordinates: foci $\pm(4\sqrt{7}/5, 3\sqrt{7}/5)$, vertices $\pm(16/5, 12/5)$, minor axis endpoints $\pm(-9/5, 12/5)$.

24. $\cot 2\theta = 0$, $2\theta = \pi/2$, $\theta = \pi/4$, $\cos\theta = \sin\theta = 1/\sqrt{2}$. Set $x = (x'-y')/\sqrt{2}$, $y = (x'+y')/\sqrt{2}$ and the equation becomes $(x')^2/16 + (y')^2/9 = 1$; ellipse, $a = 4$, $b = 3$, $c = \sqrt{7}$. In $x'y'$-coordinates: foci $(\pm\sqrt{7}, 0)$, vertices $(\pm 4, 0)$, minor axis endpoints $(0, \pm 3)$. In xy-coordinates: foci $\pm(\sqrt{7/2}, \sqrt{7/2})$, vertices $\pm(2\sqrt{2}, 2\sqrt{2})$, minor axis endpoints $\pm(-3/\sqrt{2}, 3/\sqrt{2})$.

25. $\cot 2\theta = (31-21)/(10\sqrt{3}) = 1/\sqrt{3}$, $2\theta = \pi/3$, $\theta = \pi/6$, $\cos\theta = \sqrt{3}/2$, $\sin\theta = 1/2$. Set $x = \sqrt{3}x'/2 - y'/2, y = x'/2 + \sqrt{3}y'/2$ and obtain $(x')^2/4 + (y'+2)^2/9 = 1$; ellipse, $a = 3$, $b = 2$, $c = \sqrt{9-4} = \sqrt{5}$. In $x'y'$-coordinates: foci $(0, -2 \pm \sqrt{5})$, vertices $(0, 1)$ and $(0, -5)$, ends of minor axis $(\pm 2, -2)$. In xy-coordinates: foci $\left(1 - \frac{\sqrt{5}}{2}, -\sqrt{3} + \frac{\sqrt{15}}{2}\right)$ and $\left(1 + \frac{\sqrt{5}}{2}, -\sqrt{3} - \frac{\sqrt{15}}{2}\right)$, vertices $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{5}{2}, -\frac{5\sqrt{3}}{2}\right)$, ends of minor axis $\left(1 + \sqrt{3}, 1 - \sqrt{3}\right)$ and $\left(1 - \sqrt{3}, -1 - \sqrt{3}\right)$.

26. $\cot 2\theta = 1/\sqrt{3}$, $2\theta = \pi/3$, $\theta = \pi/6$, $\cos\theta = \sqrt{3}/2$, $\sin\theta = 1/2$. Set $x = \sqrt{3}x'/2 - y'/2$, $y = x'/2 + \sqrt{3}y'/2$ and obtain $(x'-1)^2/16 + (y')^2/9 = 1$; ellipse, $a = 4$, $b = 3$, $c = \sqrt{16-9} = \sqrt{7}$. In $x'y'$-coordinates: foci $(1 \pm \sqrt{7}, 0)$, vertices $(5, 0)$ and $(-3, 0)$, ends of minor axis $(1, \pm 3)$. In xy-coordinates: foci $\left(\frac{\sqrt{3}+\sqrt{21}}{2}, \frac{1+\sqrt{7}}{2}\right)$ and $\left(\frac{\sqrt{3}-\sqrt{21}}{2}, \frac{1-\sqrt{7}}{2}\right)$, vertices $\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$ and $\left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$, ends of minor axis $\left(\frac{\sqrt{3}-3}{2}, \frac{1+3\sqrt{3}}{2}\right)$ and $\left(\frac{\sqrt{3}+3}{2}, \frac{1-3\sqrt{3}}{2}\right)$.

27. $\cot 2\theta = (1-11)/(-10\sqrt{3}) = 1/\sqrt{3}$, $2\theta = \pi/3$, $\theta = \pi/6$, $\cos\theta = \sqrt{3}/2$, $\sin\theta = 1/2$. Set $x = \sqrt{3}x'/2 - y'/2$, $y = x'/2 + \sqrt{3}y'/2$ and obtain $(x')^2/16 - (y')^2/4 = 1$; hyperbola, $a = 4$, $b = 2$, $c = \sqrt{20} = 2\sqrt{5}$. In $x'y'$-coordinates: foci $(\pm 2\sqrt{5}, 0)$, vertices $(\pm 4, 0)$, asymptotes $y' = \pm x'/2$. In xy-coordinates: foci $\pm(\sqrt{15}, \sqrt{5})$, vertices $\pm(2\sqrt{3}, 2)$, asymptotes $y = \frac{5\sqrt{3} \pm 8}{11}x$.

28. $\cot 2\theta = (17-108)/(-312) = 7/24$; proceed as in Exercise 21 to obtain $\cos\theta = 4/5$, $\sin\theta = 3/5$. Set $x = (4x'-3y')/5$, $y = (3x'+4y')/5$ to get $(y')^2/4 - (x')^2/9 = 1$; hyperbola, $a = 2, b = 3, c = \sqrt{13}$. In $x'y'$-coordinates: foci $(0, \pm\sqrt{13})$, vertices $(0, \pm 2)$, asymptotes $y = \pm 2x/3$. In xy-coordinates: foci $\pm(-\frac{3\sqrt{13}}{5}, \frac{4\sqrt{13}}{5})$, vertices $\pm(-\frac{6}{5}, \frac{8}{5})$, asymptotes $y = \frac{x}{18}$ and $y = \frac{17x}{6}$.

29. $\cot 2\theta = ((-7)-32)/(-52) = 3/4$; proceed as in Example 4 to obtain $\cos 2\theta = 3/5$, $\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \frac{2}{\sqrt{5}}$, $\sin\theta = \frac{1}{\sqrt{5}}$. Set $x = \frac{2x'-y'}{\sqrt{5}}$, $y = \frac{x'+2y'}{\sqrt{5}}$ and the equation becomes $\frac{(x')^2}{9} - \frac{(y'-4)^2}{4} = 1$; hyperbola, $a = 3, b = 2, c = \sqrt{13}$. In $x'y'$-coordinates: foci $(\pm\sqrt{13}, 4)$, vertices $(\pm 3, 4)$, asymptotes $y' = 4 \pm 2x'/3$. In xy-coordinates: foci $\left(\frac{-4+2\sqrt{13}}{\sqrt{5}}, \frac{8+\sqrt{13}}{\sqrt{5}}\right)$ and $\left(\frac{-4-2\sqrt{13}}{\sqrt{5}}, \frac{8-\sqrt{13}}{\sqrt{5}}\right)$, vertices $\left(\frac{2}{\sqrt{5}}, \frac{11}{\sqrt{5}}\right)$ and $(-2\sqrt{5}, \sqrt{5})$, asymptotes $y = \frac{7x}{4} + 3\sqrt{5}$ and $y = -\frac{x}{8} + \frac{3\sqrt{5}}{2}$.

30. $\cot 2\theta = 0$, $2\theta = \pi/2$, $\theta = \pi/4$, $\cos\theta = \sin\theta = 1/\sqrt{2}$. Set $x = (x'-y')/\sqrt{2}$, $y = (x'+y')/\sqrt{2}$ and the equation becomes $(y')^2/36 - (x'+2)^2/4 = 1$; hyperbola, $a = 6$, $b = 2$,$c = \sqrt{36+4} = 2\sqrt{10}$. In $x'y'$-coordinates: foci $(-2, \pm 2\sqrt{10})$, vertices $(-2, \pm 6)$, asymptotes $y' = \pm 3(x'+2)$. In xy-coordinates: foci $(-\sqrt{2}-2\sqrt{5}, -\sqrt{2}+2\sqrt{5})$ and $(-\sqrt{2}+2\sqrt{5}, -\sqrt{2}-2\sqrt{5})$, vertices $(-4\sqrt{2}, 2\sqrt{2})$ and $(2\sqrt{2}, -4\sqrt{2})$, asymptotes $y = -2x - 3\sqrt{2}$ and $y = -\frac{x}{2} - \frac{3}{\sqrt{2}}$.

31. $(\sqrt{x}+\sqrt{y})^2 = 1 = x + y + 2\sqrt{xy}$, $(1-x-y)^2 = x^2 + y^2 + 1 - 2x - 2y + 2xy = 4xy$, so $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$. Set $\cot 2\theta = 0$, then $\theta = \pi/4$. Change variables by the Rotation Equations to obtain $2(y')^2 - 2\sqrt{2}x' + 1 = 0$, which is the equation of a parabola. The original equation implies that x and y are in the interval $[0, 1]$, so we only get part of the parabola.

32. When (5) is substituted into (7), the term $x'y'$ will occur in the terms $A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + y'\cos\theta)^2 = (x')^2(\ldots) + x'y'(-2A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + 2C\cos\theta\sin\theta) + (y')^2(\ldots) + \ldots$, so the coefficient of $x'y'$ is $B' = B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta$.

33. It suffiices to show that the expression $B'^2 - 4A'C'$ is independent of θ. Set $g = B' = B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta$, $f = A' = (A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta)$, $h = C' = (A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta)$. It is easy to show that $g'(\theta) = -2B\sin 2\theta + 2(C - A)\cos 2\theta$, $f'(\theta) = (C - A)\sin 2\theta + B\cos 2\theta$, $h'(\theta) = (A - C)\sin 2\theta - B\cos 2\theta$ and it is a bit more tedious to show that $\dfrac{d}{d\theta}(g^2 - 4fh) = 0$. It follows that $B'^2 - 4A'C'$ is independent of θ and by taking $\theta = 0$, we have $B'^2 - 4A'C' = B^2 - 4AC$.

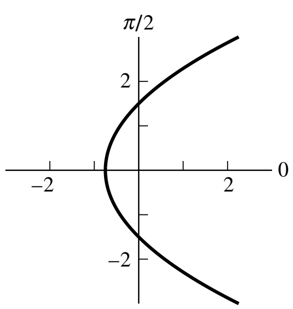
34. From equations (9), $A' + C' = A(\sin^2\theta + \cos^2\theta) + C(\sin^2\theta + \cos^2\theta) = A + C$.

35. If $A = C$ then $\cot 2\theta = (A - C)B = 0$, so $2\theta = \pi/2$, and $\theta = \pi/4$.

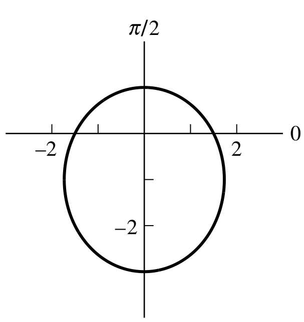
36. If $F = 0$ then $x^2 + Bxy = 0$, $x(x + By) = 0$ so $x = 0$ or $x + By = 0$ which are lines that intersect at $(0, 0)$. Suppose $F \neq 0$, rotate through an angle θ where $\cot 2\theta = 1/B$ eliminating the cross product term to get $A'(x')^2 + C'(y')^2 + F' = 0$, and note that $F' = F$ so $F' \neq 0$. From (9), $A' = \cos^2\theta + B\cos\theta\sin\theta = \cos\theta(\cos\theta + B\sin\theta)$ and $C' = \sin^2\theta - B\sin\theta\cos\theta = \sin\theta(\sin\theta - B\cos\theta)$, so $A'C' = \sin\theta\cos\theta[\sin\theta\cos\theta - B(\cos^2\theta - \sin^2\theta) - B^2\sin\theta\cos\theta] = \dfrac{1}{2}\sin 2\theta\left[\dfrac{1}{2}\sin 2\theta - B\cos 2\theta - \dfrac{1}{2}B^2\sin 2\theta\right] = \dfrac{1}{4}\sin^2 2\theta[1 - 2B\cot 2\theta - B^2] = \dfrac{1}{4}\sin^2 2\theta[1 - 2B(1/B) - B^2] = -\dfrac{1}{4}\sin^2 2\theta(1 + B^2) < 0$, thus A' and C' have unlike signs so the graph is a hyperbola.

Exercise Set 10.6

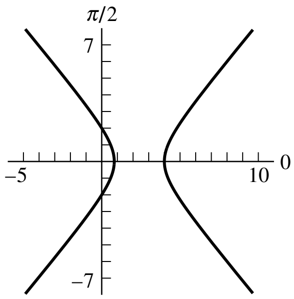
1. (a) $r = \dfrac{3/2}{1 - \cos\theta}$, $e = 1$, $d = 3/2$.



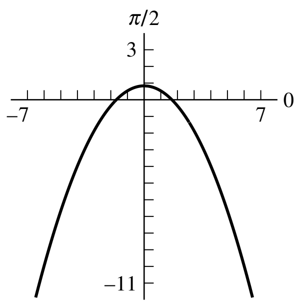
(b) $r = \dfrac{3/2}{1 + \frac{1}{2}\sin\theta}$, $e = 1/2$, $d = 3$.



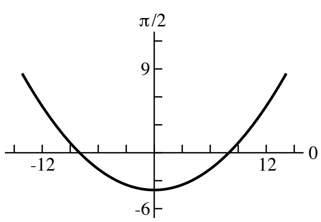
2. (a) $r = \dfrac{2}{1 + \frac{3}{2}\cos\theta}$, $e = 3/2$, $d = 4/3$.



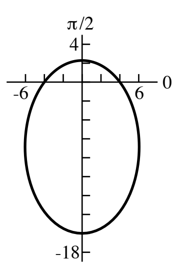
(b) $r = \dfrac{5/3}{1+\sin\theta}$, $e = 1$, $d = 5/3$.



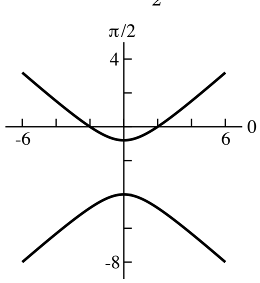
3. **(a)** $e = 1$, $d = 8$, parabola, opens up.



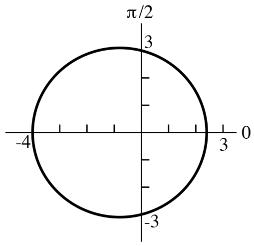
(b) $r = \dfrac{4}{1+\frac{3}{4}\sin\theta}$, $e = 3/4$, $d = 16/3$, ellipse, directrix $16/3$ units above the pole.



4. **(a)** $r = \dfrac{2}{1-\frac{3}{2}\sin\theta}$, $e = 3/2$, $d = 4/3$, hyperbola, directrix $4/3$ units below the pole.



(b) $r=\dfrac{3}{1+\frac{1}{4}\cos\theta}$, $e=1/4$, $d=12$, ellipse, directrix 12 units to the right of the pole.



5. (a) $d=2$, $r=\dfrac{ed}{1+e\cos\theta}=\dfrac{3/2}{1+\frac{3}{4}\cos\theta}=\dfrac{6}{4+3\cos\theta}$.

(b) $e=1$, $d=1$, $r=\dfrac{ed}{1+e\cos\theta}=\dfrac{1}{1+\cos\theta}$.

(c) $e=4/3$, $d=3$, $r=\dfrac{ed}{1+e\sin\theta}=\dfrac{4}{1+\frac{4}{3}\sin\theta}=\dfrac{12}{3+4\sin\theta}$.

6. (a) $r=\dfrac{ed}{1\pm e\sin\theta}$, $2=\dfrac{ed}{1\pm e}$, $6=\dfrac{ed}{1\mp e}$, $2\pm 2e=6\mp 6e$, upper sign yields $e=1/2$, $d=6$, $r=\dfrac{3}{1+\frac{1}{2}\sin\theta}=\dfrac{6}{2+\sin\theta}$.

(b) $e=1$, $r=\dfrac{d}{1-\cos\theta}$, $2=\dfrac{d}{2}$, $d=4$, $r=\dfrac{4}{1-\cos\theta}$.

(c) $e=\sqrt{2}$, $r=\dfrac{\sqrt{2}d}{1+\sqrt{2}\cos\theta}$; $r=2$ when $\theta=0$, so $d=2+\sqrt{2}$, $r=\dfrac{2+2\sqrt{2}}{1+\sqrt{2}\cos\theta}$.

7. (a) $r=\dfrac{3}{1+\frac{1}{2}\sin\theta}$, $e=1/2$, $d=6$, directrix 6 units above pole; if $\theta=\pi/2:\ r_0=2$; if $\theta=3\pi/2:\ r_1=6$, $a=(r_0+r_1)/2=4$, $b=\sqrt{r_0r_1}=2\sqrt{3}$, center $(0,-2)$ (rectangular coordinates), $\dfrac{x^2}{12}+\dfrac{(y+2)^2}{16}=1$.

(b) $r=\dfrac{1/2}{1-\frac{1}{2}\cos\theta}$, $e=1/2$, $d=1$, directrix 1 unit left of pole; if $\theta=\pi:\ r_0=\dfrac{1/2}{3/2}=1/3$; if $\theta=0:\ r_1=1$, $a=2/3$, $b=1/\sqrt{3}$, center $=(1/3,0)$ (rectangular coordinates), $\dfrac{9}{4}(x-1/3)^2+3y^2=1$.

8. (a) $r=\dfrac{6/5}{1+\frac{2}{5}\cos\theta}$, $e=2/5$, $d=3$, directrix 3 units right of pole, if $\theta=0:\ r_0=6/7$, if $\theta=\pi:\ r_1=2$, $a=10/7$, $b=2\sqrt{3}/\sqrt{7}$, center $(-4/7,0)$ (rectangular coordinates), $\dfrac{49}{100}(x+4/7)^2+\dfrac{7}{12}y^2=1$.

(b) $r=\dfrac{2}{1-\frac{3}{4}\sin\theta}$, $e=3/4$, $d=8/3$, directrix 8/3 units below pole, if $\theta=3\pi/2:\ r_0=8/7$, if $\theta=\pi/2:\ r_1=8$, $a=32/7$, $b=8/\sqrt{7}$, center: $(0,24/7)$ (rectangular coordinates), $\dfrac{7}{64}x^2+\dfrac{49}{1024}\left(y-\dfrac{24}{7}\right)^2=1$.

9. (a) $r=\dfrac{3}{1+2\sin\theta}$, $e=2$, $d=3/2$, hyperbola, directrix 3/2 units above pole, if $\theta=\pi/2:\ r_0=1$; $\theta=3\pi/2:\ r=-3$, so $r_1=3$, center $(0,2)$, $a=1$, $b=\sqrt{3}$, $-\dfrac{x^2}{3}+(y-2)^2=1$.

(b) $r = \dfrac{5/2}{1 - \frac{3}{2}\cos\theta}$, $e = 3/2$, $d = 5/3$, hyperbola, directrix 5/3 units left of pole, if $\theta = \pi : r_0 = 1$; $\theta = 0 : r = -5$, $r_1 = 5$, center $(-3, 0)$, $a = 2$, $b = \sqrt{5}$, $\dfrac{1}{4}(x+3)^2 - \dfrac{1}{5}y^2 = 1$.

10. (a) $r = \dfrac{4}{1 - 2\sin\theta}$, $e = 2$, $d = 2$, hyperbola, directrix 2 units below pole, if $\theta = 3\pi/2 : r_0 = 4/3$; $\theta = \pi/2 : r_1 = \left|\dfrac{4}{1-2}\right| = 4$, center $(0, -8/3)$, $a = 4/3$, $b = 4/\sqrt{3}$, $\dfrac{9}{16}\left(y + \dfrac{8}{3}\right)^2 - \dfrac{3}{16}x^2 = 1$.

(b) $r = \dfrac{15/2}{1 + 4\cos\theta}$, $e = 4$, $d = 15/8$, hyperbola, directrix 15/8 units right of pole, if $\theta = 0 : r_0 = 3/2$; $\theta = \pi : r_1 = \left|-\dfrac{5}{2}\right| = 5/2$, $a = 1/2$, $b = \dfrac{\sqrt{15}}{2}$, center $(2, 0)$, $4(x-2)^2 - \dfrac{4}{15}y^2 = 1$.

11. (a) $r = \dfrac{\frac{1}{2}d}{1 + \frac{1}{2}\cos\theta} = \dfrac{d}{2 + \cos\theta}$, if $\theta = 0 : r_0 = d/3$; $\theta = \pi$, $r_1 = d$, $8 = a = \dfrac{1}{2}(r_1 + r_0) = \dfrac{2}{3}d, d = 12$, $r = \dfrac{12}{2 + \cos\theta}$.

(b) $r = \dfrac{\frac{3}{5}d}{1 - \frac{3}{5}\sin\theta} = \dfrac{3d}{5 - 3\sin\theta}$, if $\theta = 3\pi/2 : r_0 = \dfrac{3}{8}d$; $\theta = \pi/2$, $r_1 = \dfrac{3}{2}d$, $4 = a = \dfrac{1}{2}(r_1 + r_0) = \dfrac{15}{16}d$, $d = \dfrac{64}{15}$, $r = \dfrac{3(64/15)}{5 - 3\sin\theta} = \dfrac{64}{25 - 15\sin\theta}$.

12. (a) $r = \dfrac{\frac{3}{5}d}{1 - \frac{3}{5}\cos\theta} = \dfrac{3d}{5 - 3\cos\theta}$, if $\theta = \pi : r_0 = \dfrac{3}{8}d$; $\theta = 0$, $r_1 = \dfrac{3}{2}d$, $4 = b = \dfrac{3}{4}d$, $d = 16/3$, $r = \dfrac{16}{5 - 3\cos\theta}$.

(b) $r = \dfrac{\frac{1}{5}d}{1 + \frac{1}{5}\sin\theta} = \dfrac{d}{5 + \sin\theta}$, if $\theta = \pi/2 : r_0 = d/6$; $\theta = 3\pi/2$, $r_1 = d/4$, $5 = c = \dfrac{1}{2}d\left(\dfrac{1}{4} - \dfrac{1}{6}\right) = \dfrac{1}{24}d$, $d = 120$, $r = \dfrac{120}{5 + \sin\theta}$.

13. For a hyperbola, both vertices and the directrix lie between the foci. So if one focus is at the origin and one vertex is at (5,0), then the directrix must lie to the right of the origin. By Theorem 10.6.2, the equation of the hyperbola has the form $r = \dfrac{ed}{1 + e\cos\theta}$. Since the hyperbola is equilateral, $a = b$, so $c = \sqrt{2}a$ and $e = c/a = \sqrt{2}$. Since $(5, 0)$ lies on the hyperbola, either $r(0) = 5$ or $r(\pi) = -5$. In the first case the equation is $r = \dfrac{5\sqrt{2} + 5}{1 + \sqrt{2}\cos\theta}$; in the second case it is $r = \dfrac{5\sqrt{2} - 5}{1 + \sqrt{2}\cos\theta}$.

14. If a hyperbola is equilateral, then $a = b$, but then $c = \sqrt{a^2 + b^2} = \sqrt{2a^2} = a\sqrt{2}$ and then $e = c/a = \sqrt{2}$. Now let $e = \sqrt{2}$, then $c = a\sqrt{2}$ and $c^2 = 2a^2$, but $c^2 = a^2 + b^2$, so $a^2 = b^2$ and then $a = b$, so the hyperbola is equilateral.

15. (a) From Figure 10.4.22, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$, $\left(1 - \dfrac{c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2$, $c^2 + x^2 + y^2 = \left(\dfrac{c}{a}x\right)^2 + a^2$, $(x - c)^2 + y^2 = \left(\dfrac{c}{a}x - a\right)^2$, $\sqrt{(x-c)^2 + y^2} = \dfrac{c}{a}x - a$ for $x > a^2/c$.

(b) From part (a) and Figure 10.6.1, $PF = \dfrac{c}{a}PD$, $\dfrac{PF}{PD} = \dfrac{c}{a}$.

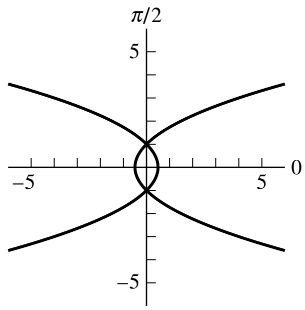
16. (a) $e = c/a = \dfrac{\frac{1}{2}(r_1 - r_0)}{\frac{1}{2}(r_1 + r_0)} = \dfrac{r_1 - r_0}{r_1 + r_0}$.

(b) $e = \dfrac{r_1/r_0 - 1}{r_1/r_0 + 1}$, $e(r_1/r_0 + 1) = r_1/r_0 - 1$, $\dfrac{r_1}{r_0} = \dfrac{1+e}{1-e}$.

17. (a) $e = c/a = \dfrac{\frac{1}{2}(r_1 + r_0)}{\frac{1}{2}(r_1 - r_0)} = \dfrac{r_1 + r_0}{r_1 - r_0}$.

(b) $e = \dfrac{r_1/r_0 + 1}{r_1/r_0 - 1}$, $e(r_1/r_0 - 1) = r_1/r_0 + 1$, $\dfrac{r_1}{r_0} = \dfrac{e+1}{e-1}$.

18. (a)



(b) $\theta = \pi/2, 3\pi/2, r = 1$.

(c) $dy/dx = \dfrac{r\cos\theta + (dr/d\theta)\sin\theta}{-r\sin\theta + (dr/d\theta)\cos\theta}$; at $\theta = \pi/2, m_1 = -1, m_2 = 1, m_1 m_2 = -1$; and at $\theta = 3\pi/2, m_1 = 1, m_2 = -1, m_1 m_2 = -1$.

19. True. A non-circular ellipse can be described by the focus-directrix characterization as shown in Figure 10.6.1, so its eccentricity satisfies $0 < e < 1$ by part (b) of Theorem 10.6.1.

20. False. The eccentricity of a parabola equals 1.

21. False. The eccentricity is determined by the ellipse's shape, not its size.

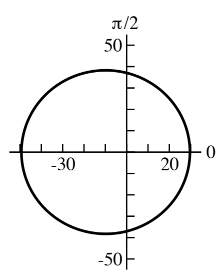
22. True. For a parabola $e = 1$, so equation (3) reduces to $r = \dfrac{d}{1 + \cos\theta}$.

23. (a) $T = a^{3/2} = 39.5^{1.5} \approx 248$ yr.

(b) $r_0 = a(1-e) = 39.5(1 - 0.249) = 29.6645$ AU $\approx$ 4,449,675,000 km, $r_1 = a(1+e) = 39.5(1 + 0.249) = 49.3355$ AU $\approx$ 7,400,325,000 km.

(c) $r = \dfrac{a(1-e^2)}{1 + e\cos\theta} \approx \dfrac{39.5(1 - (0.249)^2)}{1 + 0.249\cos\theta} \approx \dfrac{37.05}{1 + 0.249\cos\theta}$ AU.

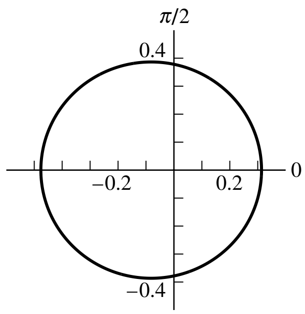
(d)



24. (a) In yr and AU, $T = a^{3/2}$; in days and km, $\dfrac{T}{365} = \left(\dfrac{a}{150 \times 10^6}\right)^{3/2}$, so $T = 365 \times 10^{-9} \left(\dfrac{a}{150}\right)^{3/2}$ days.

(b) $T = 365 \times 10^{-9} \left(\dfrac{57.95 \times 10^6}{150}\right)^{3/2} \approx 87.6$ days.

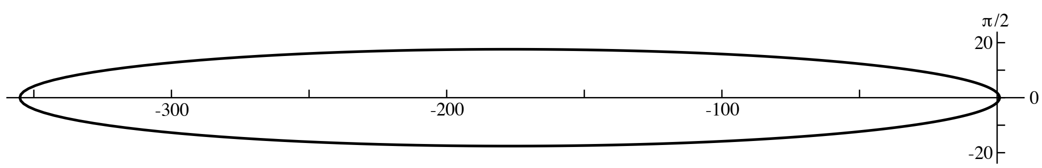
(c) From (17) the polar equation of the orbit has the form $r = \dfrac{a(1-e^2)}{1+e\cos\theta} = \dfrac{55490833.8}{1+0.206\cos\theta}$ km, or $r = \dfrac{0.3699}{1+0.206\cos\theta}$ AU.

(d) 

25. (a) $a = T^{2/3} = 2380^{2/3} \approx 178.26$ AU.

(b) $r_0 = a(1-e) \approx 0.8735$ AU, $r_1 = a(1+e) \approx 355.64$ AU.

(c) $r = \dfrac{a(1-e^2)}{1+e\cos\theta} \approx \dfrac{1.74}{1+0.9951\cos\theta}$ AU.

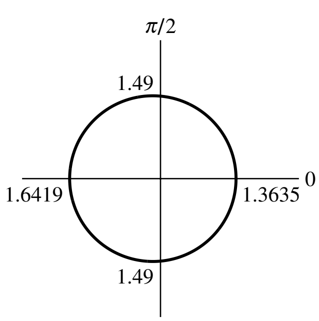


(d)

26. (a) By Exercise 15(a), $e = \dfrac{r_1 - r_0}{r_1 + r_0} \approx 0.092635$.

(b) $a = \dfrac{1}{2}(r_0 + r_1) = 225{,}400{,}000$ km ≈ 1.503 AU, so $T = a^{3/2} \approx 1.84$ yr.

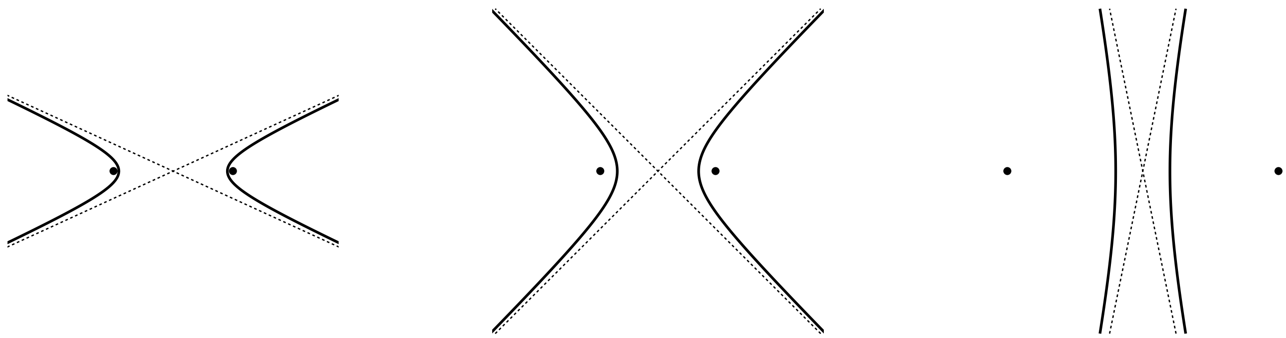
(c) $r = \dfrac{a(1-e^2)}{1+e\cos\theta} \approx \dfrac{223465774.6}{1+0.092635\cos\theta}$ km, or $\approx \dfrac{1.48977}{1+0.092635\cos\theta}$ AU.

(d) 

27. $r_0 = a(1-e) \approx 7003$ km, $h_{\min} \approx 7003 - 6440 = 563$ km, $r_1 = a(1+e) \approx 10{,}726$ km, $h_{\max} \approx 10{,}726 - 6440 = 4286$ km.

28. $r_0 = a(1-e) \approx 651{,}736$ km, $h_{\min} \approx 581{,}736$ km; $r_1 = a(1+e) \approx 6{,}378{,}102$ km, $h_{\max} \approx 6{,}308{,}102$ km.

29. Position the hyperbola so that its foci are on a horizontal line. As $e \to 1^+$, the hyperbola becomes 'pointier', squeezed between almost horizontal asymptotes. As $e \to +\infty$, it becomes more like a pair of parallel lines, with almost vertical asymptotes.



$e = 1.1$ $\qquad$ $e = \sqrt{2}$ $\qquad$ $e = 5$

30. Let x be the distance between the foci and z the distance between the center and the directrix. From Figure 10.6.11, $x = 2ae$ and $z = \dfrac{a}{e}$, so $z = \dfrac{x}{2e^2}$. If x is fixed, then $z \to +\infty$ as $e \to 0^+$.

Chapter 10 Review Exercises

1. $x(t) = \sqrt{2}\cos t, \quad y(t) = -\sqrt{2}\sin t, \quad 0 \le t \le 3\pi/2.$

2. **(a)** $x = f(1-t)$, $y = g(1-t)$.

3. **(a)** $dy/dx = \dfrac{1/2}{2t} = 1/(4t)$; $dy/dx\big|_{t=-1} = -1/4$; $dy/dx\big|_{t=1} = 1/4$.

(b) $x = (2y)^2 + 1$, $dx/dy = 8y$, $dy/dx\big|_{y=\pm(1/2)} = \pm 1/4$.

4. $\dfrac{dy}{dx} = \dfrac{t^2}{t} = t$, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\Big/\dfrac{dx}{dt} = \dfrac{1}{t}$, $\dfrac{dy}{dx}\Big|_{t=2} = 2$, $\dfrac{d^2y}{dx^2}\Big|_{t=2} = \dfrac{1}{2}$.

5. $dy/dx = \dfrac{4\cos t}{-2\sin t} = -2\cot t$.

(a) $dy/dx = 0$ if $\cot t = 0$, $t = \pi/2 + n\pi$ for $n = 0, \pm 1, \ldots$

(b) $dx/dy = -\dfrac{1}{2}\tan t = 0$ if $\tan t = 0$, $t = n\pi$ for $n = 0, \pm 1, \ldots$

6. We have $dx/dt = -20t^3$ and $dy/dt = 20t^4$, so, by Formula (9) of Section 10.1, $L = \displaystyle\int_0^1 \sqrt{(-20t^3)^2 + (20t^4)^2}\, dt = 20\int_0^1 t^3\sqrt{1+t^2}\, dt$. Let $u = 1 + t^2$, $du = 2t\,dt$. Then $L = 20\displaystyle\int_1^2 (u-1)\sqrt{u}\,\frac{1}{2}\,du = 10\int_1^2 (u^{3/2} - u^{1/2})\,du = 10\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^2 = \frac{8}{3}(\sqrt{2}+1)$.

7. **(a)** $(-4\sqrt{2}, -4\sqrt{2})$ **(b)** $(7/\sqrt{2}, -7/\sqrt{2})$ **(c)** $(4\sqrt{2}, 4\sqrt{2})$ **(d)** $(5, 0)$ **(e)** $(0, -2)$ **(f)** $(0, 0)$

8. **(a)** $(\sqrt{2}, 3\pi/4)$ **(b)** $(-\sqrt{2}, 7\pi/4)$ **(c)** $(\sqrt{2}, 3\pi/4)$ **(d)** $(-\sqrt{2}, -\pi/4)$

9. **(a)** $(5, 0.6435)$ **(b)** $(\sqrt{29}, 5.0929)$ **(c)** $(1.2716, 0.6658)$

10. **(a)** circle **(b)** rose **(c)** line **(d)** limaçon **(e)** limaçon **(f)** none **(g)** none **(h)** spiral

11. **(a)** $r = 2a/(1+\cos\theta), r + x = 2a, x^2 + y^2 = (2a - x)^2, y^2 = -4ax + 4a^2$, parabola.

(b) $r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2 = a^2$, hyperbola.

(c) $r\sin(\theta - \pi/4) = (\sqrt{2}/2)r(\sin\theta - \cos\theta) = 4, y - x = 4\sqrt{2}$, line.

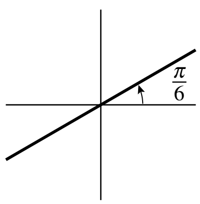
(d) $r^2 = 4r\cos\theta + 8r\sin\theta, x^2 + y^2 = 4x + 8y, (x-2)^2 + (y-4)^2 = 20$, circle.

12. **(a)** $r\cos\theta = 7$.

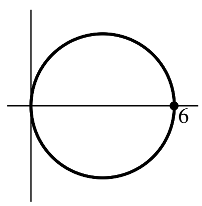
(b) $r = 3$.

(c) $r^2 - 6r\sin\theta = 0,\ r = 6\sin\theta$.

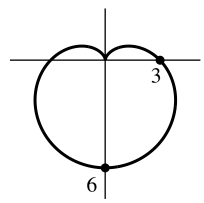
(d) $4(r\cos\theta)(r\sin\theta) = 9,\ 4r^2\sin\theta\cos\theta = 9,\ r^2\sin 2\theta = 9/2$.

13. 

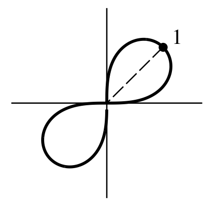
Line

14. 

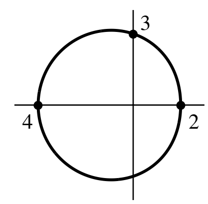
Circle

15. 

Cardioid

16. 

Lemniscate

17. 

Limaçon

18. **(a)** $y = r\sin\theta = (\sin\theta)/\sqrt{\theta}, dy/d\theta = \dfrac{2\theta\cos\theta - \sin\theta}{2\theta^{3/2}} = 0$ if $2\theta\cos\theta = \sin\theta, \tan\theta = 2\theta$ which only happens once on $(0, \pi]$. Since $\lim\limits_{\theta\to 0^+} y = 0$ and $y = 0$ at $\theta = \pi$, y has a maximum when $\tan\theta = 2\theta$.

(b) $\theta \approx 1.16556$.

(c) $y_{\max} = (\sin\theta)/\sqrt{\theta} \approx 0.85124$.

19. **(a)** $x = r\cos\theta = \cos\theta - \cos^2\theta, dx/d\theta = -\sin\theta + 2\sin\theta\cos\theta = \sin\theta(2\cos\theta - 1) = 0$ if $\sin\theta = 0$ or $\cos\theta = 1/2$, so $\theta = 0, \pi, \pi/3, 5\pi/3$; maximum $x = 1/4$ at $\theta = \pi/3, 5\pi/3$, minimum $x = -2$ at $\theta = \pi$.

(b) $y = r\sin\theta = \sin\theta - \sin\theta\cos\theta, dy/d\theta = \cos\theta + 1 - 2\cos^2\theta = 0$ at $\cos\theta = 1, -1/2$, so $\theta = 0, 2\pi/3, 4\pi/3$; maximum $y = 3\sqrt{3}/4$ at $\theta = 2\pi/3$, minimum $y = -3\sqrt{3}/4$ at $\theta = 4\pi/3$.

20. Use equation (2) of Section 10.3: $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$, then set $\theta = \pi/4$, $dr/d\theta = \sqrt{2}/2$, $r = 1 + \sqrt{2}/2$, $m = -1 - \sqrt{2}$.

21. **(a)** As t runs from 0 to π, the upper portion of the curve is traced out from right to left; as t runs from π to 2π the bottom portion is traced out from right to left, except for the bottom part of the loop. The loop is traced out counterclockwise for $\pi + \sin^{-1}\dfrac{1}{4} < t < 2\pi - \sin^{-1}\dfrac{1}{4}$.

(b) $\lim\limits_{t\to 0^+} x = +\infty, \lim\limits_{t\to 0^+} y = 1; \lim\limits_{t\to\pi^-} x = -\infty, \lim\limits_{t\to\pi^-} y = 1; \lim\limits_{t\to\pi^+} x = +\infty, \lim\limits_{t\to\pi^+} y = 1; \lim\limits_{t\to 2\pi^-} x = -\infty, \lim\limits_{t\to 2\pi^-} y = 1$; the horizontal asymptote is $y = 1$.

(c) Horizontal tangent line when $dy/dx = 0$, or $dy/dt = 0$, so $\cos t = 0, t = \pi/2, 3\pi/2$; vertical tangent line when $dx/dt = 0$, so $-\csc^2 t - 4\sin t = 0, t = \pi + \sin^{-1}\dfrac{1}{\sqrt[3]{4}}, 2\pi - \sin^{-1}\dfrac{1}{\sqrt[3]{4}}, t \approx 3.823, 5.602$.

(d) Since $\tan\theta = \dfrac{y}{x} = \tan t$, we may take $\theta = t$. $r^2 = x^2+y^2 = x^2(1+\tan^2 t) = x^2\sec^2 t = (4+\csc t)^2 = (4+\csc\theta)^2$, so $r = 4+\csc\theta$. $r = 0$ when $\csc\theta = -4$, $\sin\theta = -\dfrac{1}{4}$. The tangent lines at the pole are $\theta = \pi + \sin^{-1}\dfrac{1}{4}$ and $\theta = 2\pi - \sin^{-1}\dfrac{1}{4}$.

22. **(a)** $r = 1/\theta, dr/d\theta = -1/\theta^2, r^2 + (dr/d\theta)^2 = 1/\theta^2 + 1/\theta^4$, $L = \displaystyle\int_{\pi/4}^{\pi/2} \frac{1}{\theta^2}\sqrt{1+\theta^2}\,d\theta =$

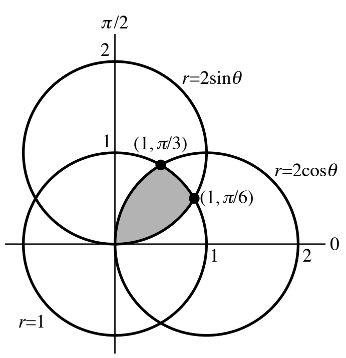
$$= \left[-\frac{\sqrt{1+\theta^2}}{\theta} + \ln(\theta + \sqrt{1+\theta^2})\right]_{\pi/4}^{\pi/2} \approx 0.9457 \text{ by Endpaper Integral Table Formula 93.}$$

(b) The integral $\displaystyle\int_1^{+\infty} \frac{1}{\theta^2}\sqrt{1+\theta^2}\,d\theta$ diverges by the comparison test (with $1/\theta$), and thus the arc length is infinite.

23. $A = 2\displaystyle\int_0^{\pi} \frac{1}{2}(2+2\cos\theta)^2 d\theta = 6\pi$.

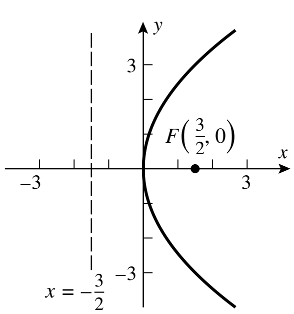
24. $A = \displaystyle\int_0^{\pi/2} \frac{1}{2}(1+\sin\theta)^2 d\theta = \frac{3\pi}{8} + 1$.

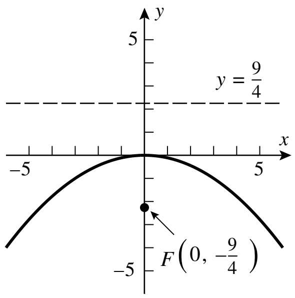
25. $A = \displaystyle\int_0^{\pi/6} \frac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\pi/6}^{\pi/3} \frac{1}{2}\cdot 1^2\,d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$. The first and third integrals are equal, by symmetry, so $A = \displaystyle\int_0^{\pi/6} 4\sin^2\theta\,d\theta + \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \int_0^{\pi/6} 2(1-\cos 2\theta)\,d\theta + \frac{\pi}{12} = (2\theta - \sin 2\theta)\Big]_0^{\pi/6} + \frac{\pi}{12} = \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\pi}{12} = \frac{5\pi}{12} - \frac{\sqrt{3}}{2}$.



26. The circle has radius $a/2$ and lies entirely inside the cardioid, so $A = \displaystyle\int_0^{2\pi} \frac{1}{2}a^2(1+\sin\theta)^2\,d\theta - \pi a^2/4 = \frac{3a^2}{2}\pi - \frac{a^2}{4}\pi = \frac{5a^2}{4}\pi$.

27.





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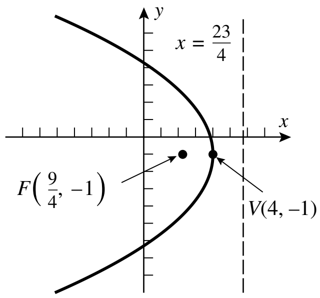

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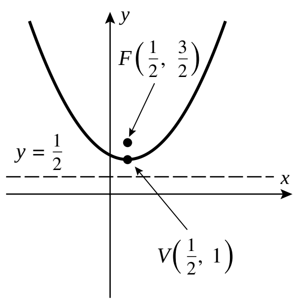
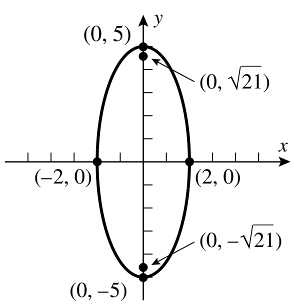
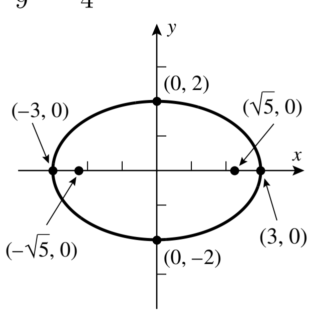


30.

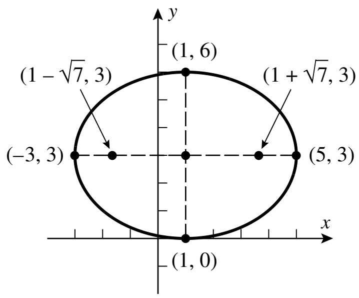
31. $c^2 = 25 - 4 = 21$, $c = \sqrt{21}$.



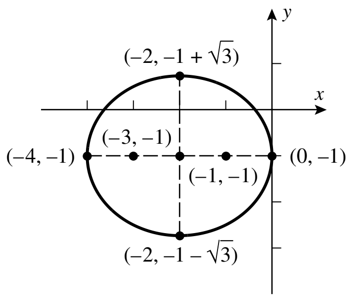
32. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, $c^2 = 9 - 4 = 5$, $c = \sqrt{5}$.



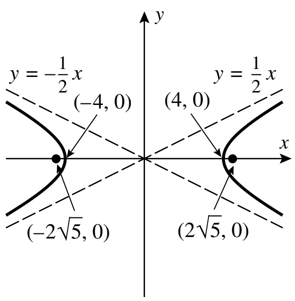
33. $\dfrac{(x-1)^2}{16} + \dfrac{(y-3)^2}{9} = 1$, $c^2 = 16 - 9 = 7$, $c = \sqrt{7}$.



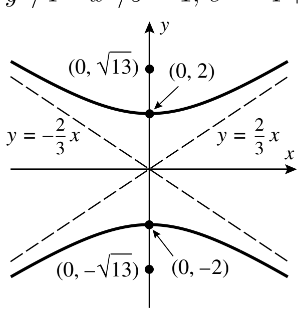
34. $\dfrac{(x+2)^2}{4}+\dfrac{(y+1)^2}{3}=1$, $c^2=4-3=1$, $c=1$.



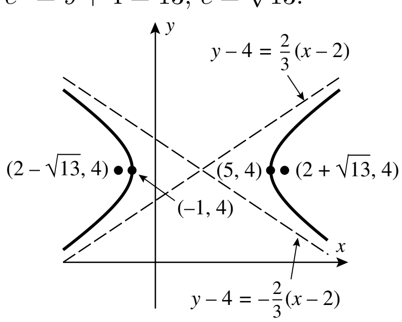
35. $c^2=a^2+b^2=16+4=20$, $c=2\sqrt{5}$.

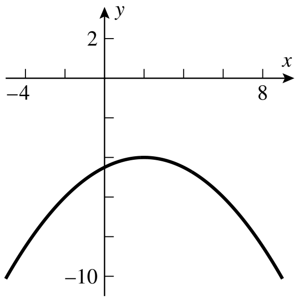
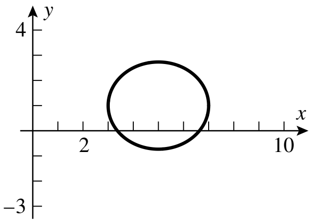
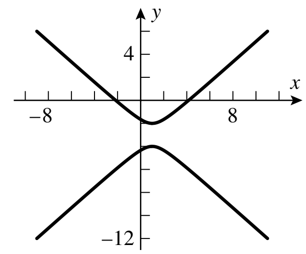


36. $y^2/4-x^2/9=1$, $c^2=4+9=13$, $c=\sqrt{13}$.



37. $c^2=9+4=13$, $c=\sqrt{13}$.



38. **(a)**  **(b)**  **(c)** 

39. $x^2 = -4py$, $p = 4$, $x^2 = -16y$.

40. $x^2 + y^2/5 = 1$.

41. $a = 3$, $a/b = 1$, $b = 3$; $y^2/9 - x^2/9 = 1$.

42. **(a)** The equation of the parabola is $y = ax^2$ and it passes through $(2100, 470)$, thus $a = \dfrac{470}{2100^2}, y = \dfrac{470}{2100^2}x^2$.

(b) $L = 2\displaystyle\int_0^{2100} \sqrt{1 + \left(2\frac{470}{2100^2}x\right)^2}\, dx = \frac{x}{220500}\sqrt{48620250000 + 2209x^2} + \frac{220500}{47}\sinh^{-1}\left(\frac{47}{220500}x\right) \approx$ ≈ 4336.3 ft.

43. **(a)** $y = y_0 + (v_0 \sin\alpha)\dfrac{x}{v_0\cos\alpha} - \dfrac{g}{2}\left(\dfrac{x}{v_0\cos\alpha}\right)^2 = y_0 + x\tan\alpha - \dfrac{g}{2v_0^2\cos^2\alpha}x^2$.

(b) $\dfrac{dy}{dx} = \tan\alpha - \dfrac{g}{v_0^2\cos^2\alpha}x, dy/dx = 0$ at $x = \dfrac{v_0^2}{g}\sin\alpha\cos\alpha, y = y_0 + \dfrac{v_0^2}{g}\sin^2\alpha - \dfrac{g}{2v_0^2\cos^2\alpha}\left(\dfrac{v_0^2\sin\alpha\cos\alpha}{g}\right)^2 =$ $y_0 + \dfrac{v_0^2}{2g}\sin^2\alpha$.

44. $\alpha = \pi/4, y_0 = 3, x = v_0t/\sqrt{2}, y = 3 + v_0t/\sqrt{2} - 16t^2$.

(a) Assume the ball passes through $x = 391, y = 50$, then $391 = v_0t/\sqrt{2}, 50 = 3 + 391 - 16t^2, 16t^2 = 344$, $t = \sqrt{21.5}$, $v_0 = \sqrt{2}x/t \approx 119.2538820$ ft/s.

(b) $\dfrac{dy}{dt} = \dfrac{v_0}{\sqrt{2}} - 32t = 0$ at $t = \dfrac{v_0}{32\sqrt{2}}$, $y_{\max} = 3 + \dfrac{v_0}{\sqrt{2}}\dfrac{v_0}{32\sqrt{2}} - 16\dfrac{v_0^2}{2^{11}} = 3 + \dfrac{v_0^2}{128} \approx 114.1053779$ ft.

(c) $y = 0$ when $t = \dfrac{-v_0/\sqrt{2} \pm \sqrt{v_0^2/2 + 192}}{-32}$, $t \approx -0.035339577$ (discard) and 5.305666365, dist $= 447.4015292$ ft.

45. $\cot 2\theta = \dfrac{A - C}{B} = 0, 2\theta = \pi/2, \theta = \pi/4, \cos\theta = \sin\theta = \sqrt{2}/2$, so $x = (\sqrt{2}/2)(x' - y')$, $y = (\sqrt{2}/2)(x' + y')$, $5(y')^2 - (x')^2 = 6$, hyperbola.

46. $\cot 2\theta = (7 - 5)/(2\sqrt{3}) = 1/\sqrt{3}, 2\theta = \pi/3, \theta = \pi/6$ then the transformed equation is $8(x')^2 + 4(y')^2 - 4 = 0$, $2(x')^2 + (y')^2 = 1$, ellipse.

47. $\cot 2\theta = (4\sqrt{5} - \sqrt{5})/(4\sqrt{5}) = 3/4$, so $\cos 2\theta = 3/5$ and thus $\cos\theta = \sqrt{(1 + \cos 2\theta)/2} = 2/\sqrt{5}$ and $\sin\theta = \sqrt{(1 - \cos 2\theta)/2} = 1/\sqrt{5}$. Hence the transformed equation is $5\sqrt{5}(x')^2 - 5\sqrt{5}y' = 0$, $y' = (x')^2$, parabola.

48. $\cot 2\theta = (17 - 108)/(-312) = 7/24$. Use the methods of Example 4 of Section 10.5 to obtain $\cos\theta = 4/5, \sin\theta = 3/5$, and the new equation is $-100(x')^2 + 225(y')^2 - 1800y' + 4500 = 0$, which, upon completing the square, becomes

$-\frac{4}{9}(x')^2+(y'-4)^2+4=0$, or $\frac{1}{9}(x')^2-\frac{1}{4}(y'-4)^2=1$. Thus center at $(0,4)$, $c^2=9+4=13, c=\sqrt{13}$, so vertices at $(-3,4)$ and $(3,4)$; foci at $(\pm\sqrt{13},4)$ and asymptotes $y'-4=\frac{2}{3}x'$.

49. **(a)** $r=\dfrac{1/3}{1+\frac{1}{3}\cos\theta}$, ellipse, right of pole, distance = 1.

(b) Hyperbola, left of pole, distance = 1/3.

(c) $r=\dfrac{1/3}{1+\sin\theta}$, parabola, above pole, distance = 1/3.

(d) Parabola, below pole, distance = 3.

50. **(a)** $\dfrac{c}{a}=e=\dfrac{2}{7}$ and $2b=6, b=3, a^2=b^2+c^2=9+\dfrac{4}{49}a^2,\ \dfrac{45}{49}a^2=9, a=\dfrac{7}{\sqrt{5}},\ \dfrac{5}{49}x^2+\dfrac{1}{9}y^2=1$.

(b) $x^2=-4py$, directrix $y=4$, focus $(-4,0), 2p=8, x^2=-16y$.

(c) For the ellipse, $a=4, b=\sqrt{3}, c^2=a^2-b^2=16-3=13$, foci $(\pm\sqrt{13},0)$; for the hyperbola, $c=\sqrt{13}, b/a=2/3, b=2a/3, 13=c^2=a^2+b^2=a^2+\dfrac{4}{9}a^2=\dfrac{13}{9}a^2,\ a=3, b=2,\ \dfrac{x^2}{9}-\dfrac{y^2}{4}=1$.

51. **(a)** $e=4/5=c/a,\ c=4a/5$, but $a=5$ so $c=4,\ b=3,\ \dfrac{(x+3)^2}{25}+\dfrac{(y-2)^2}{9}=1$.

(b) Directrix $y=2,\ p=2,\ (x+2)^2=-8y$.

(c) Center $(-1,5)$, vertices $(-1,7)$ and $(-1,3)$, $a=2,\ a/b=8,\ b=1/4,\ \dfrac{(y-5)^2}{4}-16(x+1)^2=1$.

52. $C=4\displaystyle\int_0^{\pi/2}\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{1/2}dt=4\int_0^{\pi/2}(a^2\sin^2 t+b^2\cos^2 t)^{1/2}\,dt=4\int_0^{\pi/2}(a^2\sin^2 t+(a^2-c^2)\cos^2 t)^{1/2}\,dt=$ $4a\displaystyle\int_0^{\pi/2}(1-e^2\cos^2 t)^{1/2}\,dt$. Set $u=\dfrac{\pi}{2}-t,\ C=4a\displaystyle\int_0^{\pi/2}(1-e^2\sin^2 t)^{1/2}\,dt$.

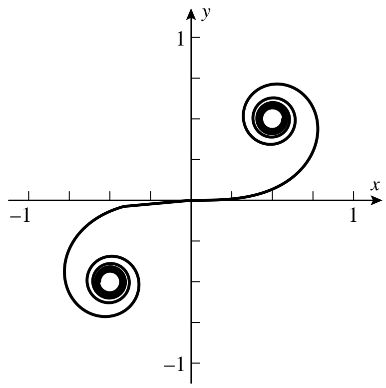
53. $a=3,\ b=2,\ c=\sqrt{5},\ C=4(3)\displaystyle\int_0^{\pi/2}\sqrt{1-(5/9)\cos^2 u}\,du\approx 15.86543959$.

54. **(a)** $\dfrac{r_0}{r_1}=\dfrac{59}{61}=\dfrac{1-e}{1+e},\ e=\dfrac{1}{60}$.

(b) $a=93\times10^6,\ r_0=a(1-e)=\dfrac{59}{60}\left(93\times10^6\right)=91{,}450{,}000$ mi.

(c) $C=4\times93\times10^6\displaystyle\int_0^{\pi/2}\left[1-\left(\frac{\cos\theta}{60}\right)^2\right]^{1/2}d\theta\approx 584{,}295{,}652.5$ mi.

Chapter 10 Making Connections



1. (a)

(b) As $t \to +\infty$, the curve spirals in toward a point P in the first quadrant. As $t \to -\infty$, it spirals in toward the reflection of P through the origin. (It can be shown that $P = (1/2, 1/2)$.)

(c) $L = \displaystyle\int_{-1}^{1} \sqrt{\cos^2\left(\frac{\pi t^2}{2}\right) + \sin^2\left(\frac{\pi t^2}{2}\right)}\, dt = 2.$

2. (a) $P : (b\cos t, b\sin t);\ Q : (a\cos t, a\sin t);\ R : (a\cos t, b\sin t).$

(b) For a circle, t measures the angle between the positive x-axis and the line segment joining the origin to the point. For an ellipse, t measures the angle between the x-axis and OPQ, not OR.

3. Let P denote the pencil tip, and let $R(x, 0)$ be the point below Q and P which lies on the line L. Then $QP + PF$ is the length of the string and $QR = QP + PR$ is the length of the side of the triangle. These two are equal, so $PF = PR$. But this is the definition of a parabola according to Definition 10.4.1.

4. Let P denote the pencil tip, and let k be the difference between the length of the ruler and that of the string. Then $QP + PF_2 + k = QF_1$, and hence $PF_2 + k = PF_1, PF_1 - PF_2 = k$. But this is the definition of a hyperbola according to Definition 10.4.3.

5. (a) Position the ellipse so its equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Then $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$, so $V = 2\displaystyle\int_0^a \pi y^2\, dx = 2\int_0^a \pi \frac{b^2}{a^2}(a^2 - x^2)\, dx = \frac{4}{3}\pi ab^2$. Also, $\dfrac{dy}{dx} = -\dfrac{bx}{a\sqrt{a^2 - x^2}}$ so $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)} = \dfrac{a^4 - c^2x^2}{a^2(a^2 - x^2)}$, where $c = \sqrt{a^2 - b^2}$. Then $S = 2\displaystyle\int_0^a 2\pi y\sqrt{1 + (dy/dx)^2}\, dx = \frac{4\pi b}{a}\int_0^a \sqrt{a^2 - x^2}\sqrt{\frac{a^4 - c^2x^2}{a^2(a^2 - x^2)}}\, dx$ $= \dfrac{4\pi bc}{a^2}\displaystyle\int_0^a \sqrt{\frac{a^4}{c^2} - x^2}\, dx = \frac{4\pi bc}{a^2}\left[\frac{x}{2}\sqrt{\frac{a^4}{c^2} - x^2} + \frac{a^4}{2c^2}\sin^{-1}\frac{cx}{a^2}\right]_0^a = 2\pi ab\left(\frac{b}{a} + \frac{a}{c}\sin^{-1}\frac{c}{a}\right)$, by Endpaper Integral Table Formula 74.

(b) Position the ellipse so its equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Then $x = \dfrac{a}{b}\sqrt{b^2 - y^2}$, so $V = 2\displaystyle\int_0^b \pi x^2\, dx = 2\int_0^b \pi \frac{a^2}{b^2}(b^2 - y^2)\, dy = \frac{4}{3}\pi a^2 b$. Also, $\dfrac{dx}{dy} = -\dfrac{ay}{b\sqrt{b^2 - y^2}}$ so $1 + \left(\dfrac{dx}{dy}\right)^2 = \dfrac{b^4 + (a^2 - b^2)y^2}{b^2(b^2 - y^2)} = \dfrac{b^4 + c^2y^2}{b^2(b^2 - y^2)}$, where $c = \sqrt{a^2 - b^2}$. Then $S = 2\displaystyle\int_0^b 2\pi x\sqrt{1 + (dx/dy)^2}\, dy = \frac{4\pi a}{b}\int_0^b \sqrt{b^2 - y^2}\sqrt{\frac{b^4 + c^2y^2}{b^2(b^2 - y^2)}}\, dy$ $= \dfrac{4\pi ac}{b^2}\displaystyle\int_0^b \sqrt{\frac{b^4}{c^2} + y^2}\, dy = 2\pi ab\left(\frac{a}{b} + \frac{b}{c}\ln\frac{a + c}{b}\right).$