VAN (086761) - Exercise 1

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Basic Probability

Question 1

Consier a random vector x with a Gaussian distribution :

$$x \sim \mathcal{N}\left(\mu_x, \Sigma_x\right)$$

(a) An explicit expression for p(x)

$$p\left(x\right) = \det\left(2\pi\Sigma\right)^{-\frac{1}{2}} e^{\left(-\frac{1}{2}\|x-\mu\|_{\Sigma}^{2}\right)}$$

where $\|\cdot\|_{\Sigma}$ is the Mahalanobis norm, defined as: $\|x\|_{\Sigma} = x^T\Sigma^{-1}x$

(b) Consider a linear transformation $y = Ax + b$, assuming A is invertible. Show that y is normally distributed and compute it's mean and covariance.

Gaussian invariant under Linear Transformations

Using that when A is invertible we have the relation:

$$f_Y\left(y\right) = \frac{f_x\left(A^{-1}\left(y-b\right)\right)}{\left|\det A\right|}$$

so,

$$
\begin{aligned}
f_Y(y) =& \frac{f_x\left(A^{-1}(y-b)\right)}{|\det A|} \\
=& \frac{exp\left(-\frac{1}{2}\left(A^{-1}(y-b)-\mu_x\right)^T \Sigma_x^{-1}\left(A^{-1}(y-b)-\mu_x\right)\right)}{\sqrt{\det(2\pi\Sigma_x)}\,|\det A|} \\
=& \frac{exp\left(-\frac{1}{2}(y-(A\mu_x+b))^T A^{-T}\Sigma_x^{-1}A^{-1}(y-(A\mu_x+b))\right)}{\sqrt{\det(2\pi\Sigma_x)\det\left(A^T A\right)}} \\
\overset{1}{=}& \frac{exp\left(-\frac{1}{2}(y-(A\mu_x+b))\left(A\Sigma_x A^T\right)^{-1}(y-(A\mu_x+b))^T\right)}{\sqrt{\det\left(2\pi A\Sigma_x A^T\right)}}
\end{aligned}
$$

1 - Because A is invertibale we have that $A^T\Sigma A$ is invertible too so we can close it with the same parentheses.

We got that Y's density function is of the with the form above (q1a) and therefore is a gaussian with mean $A\mu_x + b$ and covarivace $A^T\Sigma A$.
We can show that these are the right moments with another way:

Mean$T$

Using the linearity of the mean in 1D we get that: (a^i is A's row)

$$\mathbb{E}(y_i) \overset{1}{=} \mathbb{E}\left(\left\langle a^i, x\right\rangle + b^i\right) \overset{2}{=} \left\langle a^i, \mathbb{E}(x)\right\rangle + b^i$$

1 - Definition of y.

2 - Mean's lineartiy in every coordinate of x

Therefore, we get for y:

$$\mathbb{E}(y) = A\mathbb{E}(x) + b$$

Covariance

We'll show that:

$$Cov(y) = A^T Cov(x) A$$

for every i, j we have:

$$
\begin{aligned}
Cov(y) =& \mathbb{E}\left[(y-\mathbb{E}(y))(y-\mathbb{E}(y))^T\right] = \mathbb{E}\left[yy^T - \mathbb{E}(y)y^T + \mathbb{E}(y)\mathbb{E}(y)^T - y\mathbb{E}(y)^T\right] \\
=& \mathbb{E}\left(yy^T\right) - \mathbb{E}(y)\mathbb{E}(y)^T + \mathbb{E}(y)\mathbb{E}(y)^T - \mathbb{E}(y)\mathbb{E}(y)^T = \mathbb{E}\left(yy^T\right) - \mathbb{E}(y)\mathbb{E}(y)^T
\end{aligned}
$$

The seconed equation is just opening the parentheses and then using that mean is linear in 1D and mean of a multivariate Random vector is just the mean on its coordinates and a mean on $\mathbb{E}(y)$ is simply $\mathbb{E}(y)$ therefore for example $\mathbb{E}\left(y\mathbb{E}(y)^T\right) = \mathbb{E}(y)\mathbb{E}\left(y^T\right)$.
Now, using mean's linearity as proved before we get:

$$\begin{aligned}
\mathbb{E}\left((Ax+b)(Ax+b)^T\right) - \mathbb{E}(Ax+b)\,\mathbb{E}(Ax+b)^T &= \\
\mathbb{E}\left(Axx^TA^T + Axb^T + bx^TA^T + bb^T\right) - \left(A\mathbb{E}(x)+b\right)\left(A\mathbb{E}(x)+b\right)^T &= \\
A\mathbb{E}\left(xx^T\right)A^T + A\mathbb{E}(x)\,b^T + b\mathbb{E}\left(x^T\right)A^T + bb^T + \\
-A\mathbb{E}(x)\,\mathbb{E}(x)^T A^T - A\mathbb{E}(x)\,b^T - b\mathbb{E}(x)^T A^T - bb^T &= \\
A\mathbb{E}\left(xx^T\right)A^T - A\mathbb{E}(x)\,\mathbb{E}(x)^T A^T &= \\
A\left(\mathbb{E}\left(xx^T\right) - \mathbb{E}(x)\,\mathbb{E}(x)^T\right)A^T &= \\
A\mathrm{Cov}(x)\,A^T
\end{aligned}$$

(c)

We don't have here a complete solution but rather some thought.

Option 1 - make H "invertible"

Every matrix has a finite number of eigen values so we take λ to be one that **isn't** $H's$ eigen value. thus we have that $H-\lambda I$ is invertible. Since, if it wasn't invertible, it means there's a vector $x \neq 0$ so that $(H-\lambda I)\,x = 0 \to Hx = \lambda x$ means that x is an eigen vector and λ is an eigen value, contradiction.
so we have that:

$$y = Hx + \lambda Ix - \lambda Ix + b \to y = (H-\lambda I)\,x + \lambda Ix + b$$

but from the previous question we know that $(H-\lambda I)\,x \sim \mathcal{N}\left((H-\lambda I)\,\mu_x, (H-\lambda I)\,\Sigma_x\,(H-\lambda I)^T\right)$ and $\lambda Ix+b \sim \mathcal{N}\left(\lambda\mu_x + b, \lambda^2\Sigma_x\right)$. So, we just need to show that the sum of two normal distributed random variable is normal distributed with sum over the mean and the covariance: means :

$$\begin{aligned}
\mu_y &= (H-\lambda I)\,\mu_x + \lambda\mu_x = H\mu_x \\
\Sigma_y &= (H-\lambda I)\,\Sigma_x\,(H-\lambda I)^T + \lambda^2\Sigma_x \\
&= H\Sigma_x H^T - \lambda H\Sigma_x - \lambda\,(H\Sigma_x)^T + \lambda^2\Sigma_x
\end{aligned}$$

Option 2 - Marginalization

$$p(y) = \int_x p(y \mid x) \cdot p(x)$$

And we have from the previous question:

Question 2 |
Let p$(x) \sim \mathcal{N}(x_0, \Sigma_{x_0})$ be a prior distribution over $x \in \mathbb{R}^n$ with known mean x_0 and covariance Σ_{x_0}. Consider a measurement $z = Hx + v$ where H is known the measurement model and v is a gaussian noise with zero mean and known covriance R.

(a) A posteriori probability function

Option 1:

Using Bayes we have:

$$p(x \mid z) = p(z \mid x) \cdot p(x) \frac{1}{p(z)}$$

and by the total probability theorem we have

$$p(z) = \int_x p(z \mid x) p(x)$$

therefore:

$$(\star) : p(x \mid z) = \frac{p(z \mid x) \cdot p(x)}{\int_{x'} p(z \mid x') p(x')}$$

But we have $p(x)$ and $p(z \mid x)$:

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} e^{\left(-\frac{1}{2}\|x-\mu\|_\Sigma^2\right)}$$

and $z \mid x \sim \mathcal{N}(Hx, R)$:

$$p(z \mid x) = \det(2\pi R)^{-\frac{1}{2}} e^{\left(-\frac{1}{2}\|z-Hx\|_R^2\right)}$$

So we just need to plug it in equation $(\star)$.

(b) An expresion for the MAP and p$(x \mid z)$ mean and covariance

The MAP is :

$$\text{argmax}_x p(x \mid z)$$

We notice that

$$p(x \mid z) = p(z \mid x) \cdot p(x) \frac{1}{p(z)} = \eta \cdot p(z \mid x) \cdot p(x)$$

$$= \eta \cdot \det(2\pi\Sigma_z)^{-\frac{1}{2}} e^{\left(-\frac{1}{2}\|z-\mu_z\|^2_{\Sigma_z}\right)} \cdot \det(2\pi\Sigma_x)^{-\frac{1}{2}} e^{\left(-\frac{1}{2}\|x-\mu_x\|^2_{\Sigma_x}\right)}$$

$$= \eta \cdot \det\left(4\pi^2\Sigma_z\Sigma_x\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\left(\|z-\mu_z\|^2_{\Sigma_z}+\|x-\mu_x\|^2_{\Sigma_x}\right)}$$

It means the $p(x \mid z)$ is of the form $\eta e^{-\frac{1}{2}f}$ where f is a quadratic equation so it's a Normal distributation. Now we'll find its Mean and Covariance.
Before we do that we mention that the maximum value of a normal distribution is obtained in the mean. so the **MAP is actually the mean.**
In addition for a normal distribution

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} e^{\left(-\frac{1}{2}\|x-\mu\|^2_{\Sigma}\right)}$$

With denoting $f = -\frac{1}{2}\|x-\mu\|^2_{\Sigma}$ we have that:

$$f'(\mu) = 0$$

$$f'' = \Sigma^{-1}$$

So, we back to our density function $p(x \mid z)$ we havethe deriving the power $\|z-\mu_z\|^2_{\Sigma_z} + \|x-\mu_x\|^2_{\Sigma_x}$ once and setting it to zero will give us the mean and twice will gives us the Σ^{-1}.
First we write:

$$\|z-\mu_z\|^2_{\Sigma_z} + \|x-\mu_x\|^2_{\Sigma_x} = \|z-Hx\|^2_R + \|x-\mu_x\|^2_{\Sigma_x}$$

$$= \|R^{-1/2}(Hx-z)\|^2_2 + \|\Sigma_x^{-1/2}(x-\mu_x)\|^2_2$$

So, we drive and set to zero:

$$\|R^{-1/2}Hx - R^{-1/2}z\|^2_2 + \|\Sigma_x^{-1/2}x - \Sigma_x^{-1/2}\mu_x\|^2_2 = 0$$

$$2\left(R^{-1/2}H\right)^T\left(R^{-1/2}Hx - R^{-1/2}z\right) + 2\Sigma_x^{-T/2}\left(\Sigma_x^{-1/2}x - \Sigma_x^{-1/2}\mu_x\right) = 0$$

$$H^T R^{-1}(Hx-z) + \Sigma_x^{-1}(x-\mu_x) = 0$$

$$\Sigma_x^{-1}x + H^T R^{-1}Hx - H^T R^{-1}z - \Sigma_x^{-1}\mu_x = 0$$

$$\Sigma_x^{-1}x + H^T R^{-1}Hx = H^T R^{-1}z + \Sigma_x^{-1}\mu_x$$

$$x = \left(H^T R^{-1}H + \Sigma_x^{-1}\right)^{-1}\left(H^T R^{-1}z + \Sigma_x^{-1}\mu_x\right)$$

so

$$x = \left(H^T R^{-1}H + \Sigma_x^{-1}\right)^{-1}\left(H^T R^{-1}z + \Sigma_x^{-1}\mu_x\right)$$

And deriving it again we get:

$$\left(\Sigma_x^{-1}x + H^T R^{-1} H x - H^T R^{-1} z - \Sigma_x^{-1}\mu_x\right)' = \Sigma_x^{-1} + H^T R^{-1} H$$

So the covariance is:

$$\Sigma_x^{-1} + H^T R^{-1} H$$

Finally we have:

$$\mathcal{N}\left(\mu = \left(H^T R^{-1} H + \Sigma_x^{-1}\right)^{-1}\left(H^T R^{-1} z + \Sigma_x^{-1}\mu_x\right), \Sigma = \Sigma_x^{-1} + H^T R^{-1} H\right)$$

Hands-on Excercises

Question 1

(a) Rotations

We just need to multiply the 3 matrices:

```
def euler_to_rot_mat(yaw, pitch, roll):
    # Rz = yaw | Ry = pitch | Rx = roll
    Rx = np.array([[1., 0., 0.],
                   [0., np.cos(roll), np.sin(roll)],
                   [0., -np.sin(roll), np.cos(roll)]])

    Ry = np.array([[np.cos(pitch), 0., -np.sin(pitch)],
                   [0., 1., 0.],
                   [np.sin(pitch), 0., np.cos(pitch)]])

    Rz = np.array([[np.cos(yaw), np.sin(yaw), 0.],
                   [-np.sin(yaw), np.cos(yaw), 0.],
                   [0., 0., 1.]])

    return Rz @ Ry @ Rx
```

(b) Rotation matrix

The rotation matrix from Body to global for $\psi = \pi/7, \theta = \pi/5, \phi = \pi/4$ is:

$$R = \begin{bmatrix} 0.7288913 & 0.6812907 & -0.0676648 \\ -0.35101932 & 0.45676743 & 0.81741497 \\ 0.58778525 & -0.5720614 & 0.5720614 \end{bmatrix}$$

Our code print:

```
Question 1b:
Rotation Matrix for yaw 25.714285714285715 pitch 36.0 roll 45.0

[[ 0.72889913  0.68126907 -0.0676648 ]
 [-0.35101932  0.45674743  0.81741497]
 [ 0.58778525 -0.5720614   0.5720614 ]]
```

(c) Rotation Matrix to Euler angles

Code can be seen in function

Theory

Multiplying all the 3 matrices we get: (Note that these matrix performing a rotation clock-wise)

$$R_z(\psi)\,R_y(\theta)\,R_x(\phi) = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) \\ 0 & \cos(\phi) & \sin(\phi) \\ \sin(\theta) & -\sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\psi)\cos(\theta) & \cos(\psi)\sin(\theta)\sin(\phi)+\sin(\psi)\cos(\phi) & -\cos(\psi)\sin(\theta)\cos(\phi)+\sin(\psi)\sin(\phi) \\ -\sin(\psi)\cos(\theta) & -\sin(\psi)\sin(\theta)\sin(\phi)+\cos(\psi)\cos(\phi) & \sin(\psi)\sin(\theta)\cos(\phi)+\cos(\psi)\sin(\phi) \\ \sin(\theta) & -\sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta) \end{bmatrix}$$

And that equals to the given $R_{3\times3}$matrix.
From this we can see that:

$$\theta = \sin^{-1}\left(R_1^3\right)$$

So we have 2 options:
We have 2 options. $\theta_1 = \sin^{-1}\left(R_1^3\right)$ and $\theta_2 = -\pi - \theta_1$. We use $-\pi$ because it's clock-wise.
And now we'll seperate into 2 cases,

*$cos(\theta) \neq 0$***:**

we can compute the rest with, let's compute ϕ:

$$\begin{matrix} R_2^3 = -\sin(\phi)\cos(\theta) \\ R_3^3 = \cos(\phi)\cos(\theta) \end{matrix} \rightarrow \tan(\phi) = -\frac{R_2^3}{R_3^3} \rightarrow \phi = \arctan\left(\frac{-R_2^3}{R_3^3}\right)$$

But arctan has 2 options. We Notice that because the rotation matrix values are fixed and we have two options for θ we get that:

1. if $-\frac{R_2^3}{\cos\theta} < 0$ that we must have that $\phi \in [-\pi, 0]$
2. else: $\phi \in [0, \pi]$

To avoid devisions we can write the condition $R_2^3 \cdot \cos\theta > 0$
and we notice that $\cos(\theta) = -\cos(-\pi - \theta)$ so when for one option for θ we now what are the angles for θ_2.
so :

1. if $-\frac{R_2^3}{\cos\theta_1} < 0$ that we must have that $\phi_1 \in [-\pi, 0]$ and $\phi_2 \in [0, \pi]$
2. else: $\phi_1 \in [0, \pi]$ and $\phi_2 \in [-\pi, 0]$

We can do the same for ψ:

$$\begin{matrix} R_1^1 = \cos(\psi)\cos(\theta) \\ R_1^2 = -\sin(\psi)\cos(\theta) \end{matrix} \rightarrow \tan(\psi) = \frac{-R_1^2}{R_1^1} \rightarrow \phi = \arctan\left(\frac{-R_1^2}{R_1^1}\right)$$

1. if $-\frac{R_1^2}{\cos\theta_1} < 0$ that we must have that $\psi_1 \in [-\pi, 0]$ and $\psi_2 \in [0, \pi]$
2. else: $\psi_1 \in [0, \pi]$ and $\psi_2 \in [-\pi, 0]$

and

$cos(\theta) = 0$**:**

It means that $\theta = \frac{\pi}{2}$ OR $\frac{3\pi}{2}$.
If $\theta = \frac{\pi}{2}$:

$$= \begin{bmatrix} 0 & \cos(\psi)\sin(\phi) + \sin(\psi)\cos(\phi) & -\cos(\psi)\cos(\phi) + \sin(\psi)\sin(\phi) \\ 0 & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\phi) & \sin(\psi)\cos(\phi) + \cos(\psi)\sin(\phi) \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sin(\psi + \phi) & -\cos(\psi + \phi) \\ 0 & \cos(\psi + \phi) & \sin(\psi + \phi) \\ 1 & 0 & 0 \end{bmatrix}$$

So we have,

$$\psi + \phi = \tan^{-1}\left(\frac{R_2^1}{R_2^2}\right)$$

$$\psi + \phi = \tan^{-1}\left(-\frac{R_3^2}{R_3^1}\right)$$

So:
This linear system has an infinite solutions. So we can choose ψ to be whatever we want (including 0) and we get that $\phi_1 = \tan^{-1}\left(\frac{R_2^1}{R_2^2}\right)$ and $\phi_2 = \pi - \tan^{-1}\left(\frac{R_2^1}{R_2^2}\right)$.
and else:

$$= \begin{bmatrix} 0 & -\cos(\psi)\sin(\phi) + \sin(\psi)\cos(\phi) & \cos(\psi)\cos(\phi) + \sin(\psi)\sin(\phi) \\ 0 & \sin(\psi)\sin(\phi) + \cos(\psi)\cos(\phi) & -\sin(\psi)\cos(\phi) + \cos(\psi)\sin(\phi) \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sin(\psi - \phi) & \cos(\psi - \phi) \\ 0 & \cos(\psi - \phi) & -\sin(\psi - \phi) \\ -1 & 0 & 0 \end{bmatrix}$$

Same as above we get:
$\phi_1 = -\tan^{-1}\left(\frac{R_2^1}{R_2^2}\right)$ and $\phi_2 = \pi + \tan^{-1}\left(\frac{R_2^1}{R_2^2}\right)$

Implementation

Can be found at the implementation python file in the **rot_mat_to_euler_angles(rot_mat)** function.

(d)

For the matrix:

$$R = \begin{bmatrix} 0.813797681 & 0.440969611 & 0.378522306 \\ 0.46984631 & 0.882564119 & 0.0180283112 \\ -0.342020143 & 0.163175911 & 0.925416578 \end{bmatrix}$$

Is :

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} -10 \\ -20 \\ -30 \end{pmatrix} or \begin{pmatrix} 170 \\ -160 \\ 150 \end{pmatrix}$$

```
Quesion 1d:
Two possible angle values:
-10.000000078920193 -19.999999980143038 -29.99999998990158
169.99999992107982 -160.00000001985697 150.00000001009843
```

Question 2

We want to calculate the transformation from Global to Camera, Means we want: $R_G^C, t_{C\to G}^C$. We are given R_G^C and $t_{C\to G}^G$ and we have that:

$$t_{C\to G}^c = R_G^C t_{C\to G}^G$$

So, our desired matrix is:

$$\begin{bmatrix} \ell^C \\ 1 \end{bmatrix} = \begin{pmatrix} R_G^C & R_G^C t_{C\to G}^G \\ 0 & 1 \end{pmatrix} \begin{bmatrix} \ell^G \\ 1 \end{bmatrix}$$

Result:

$$\ell^C = \begin{pmatrix} -555.2033 \\ 351.31482 \\ 450 \end{pmatrix}$$

```
Quesiton 2:
l_cam is:
[[-555.20335297]
 [ 351.31482926]
 [ 450.        ]]
```

Question 3 | An autonomous ground vehicle (robot) is commanded to move forward by 1 meter each time step. Due to imperfect control system, the robot instead moves forward by 1.01 meter and also rotates by 1 degree.

(a)

In a 2D scenario, the pose of a robot is represented by a 3-element vector: (x, y, ϑ)T , where (x, y) represents the position coordinates, and ϑ represents the orientation angle.
Translation is relative to robot's frame, therefore $t = (0, 1)^T$ - We asume that "forward" means in the Y'th Global CS direction.
Commanded:

$$T_k^{k+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Actual (assuming angle's error is counter clock-wise):

$$T_k^{k+1} = \begin{bmatrix} \cos(1) & -\sin(1) & 0 \\ \sin(1) & \cos(1) & 1.01 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

Evolution of robot pose for 10 steps using pose composition. In terms of x-y position and orientation angle for consecutive times k, k + 1:
Notice that the heading is actually the image of e_2 under the transformation so it'll be R_1(means the first column).
We thought that "Forward" means in the Y direction.