Notations

Spline Function $f(x) = \sum_{i=1}^{n+k+1} c_i B_{i,k+1}(x)$

Abscissas Vector $x \in \mathbb{R}^m$: $[x_1 \quad \cdots \quad x_m]^T$

Ordinates Vector $y \in \mathbb{R}^m$: $[y_1 \quad \cdots \quad y_m]^T$

Knot Vector $t \in \mathbb{R}^{n+2k+2}$: $\left[\underbrace{0 \quad \dots \quad 0}_{k+1} \quad \underbrace{? \quad \dots \quad ?}_{n} \quad \underbrace{1 \quad \dots \quad 1}_{k+1}\right]^T$

Control Vector $c \in \mathbb{R}^{n+k+1}$: $[c_1 \quad \cdots \quad c_{n+k+1}]^T$

Data Matrix $B \in \mathbb{R}^{m \times n+k+1}$: $\begin{bmatrix} B_{1,k+1}(x_0) & \dots & B_{n+k+1,k+1}(x_0) \\ \vdots & \ddots & \vdots \\ B_{1,k+1}(x_m) & \dots & B_{n+k+1,k+1}(x_m) \end{bmatrix}$

Problem Formulation

Find c that minimizes $\ell = (1-\lambda)\delta(c) + \lambda\eta(c)$ s.t. $c_1 = y_1,\ c_m = y_m$ (clamped)

$$\delta(c) \underbrace{=}_{\substack{c_1 = y_1 \\ c_m = y_m}} \delta(\tilde{c}) = \sum_{j=2}^{m-1} \left(y_j - f(x_j)\right)^2$$

$$\eta(c) = \int_a^b \left(s''(x)\right)^2 dx$$

Solution

Throughout, denote $\tilde{a}$ as the vector a with the first and last elements removed.

We want to solve for $\tilde{c}$, since we know the values for c_1, c_m already.

The solution is given by:

$$\tilde{c} = \left[2(1-\lambda)\tilde{B}^T\tilde{B} + 2\lambda\tilde{B}^T\tilde{K}\tilde{B}\right]^{-1}\left[(1-\lambda)2\tilde{B}^T(\tilde{y} - \tilde{r}) - \lambda\tilde{B}^T\left(\widetilde{K_{*,1}}y_1 + \widetilde{K_{*,m}}y_m + \tilde{s}\right)\right]$$

$$c = (y_1, \tilde{c}, y_m)$$

Details below.

Approach 1

$$\delta(c) = \sum_{j=1}^{m} \left(y_j - s(x_j)\right)^2 = \sum_{j=1}^{m} \left(y_j - \sum_{i=1}^{n+k+1} f(x_j)\right)^2 = \|y - f\|_2^2$$

$$\eta(c) = \int_a^b |s''(x)|dx = f^T K f$$

$$f = s(x) \in \mathbb{R}^m, \qquad K \in \mathbb{R}^{m\times m}: QR^{-1}Q^T$$

$$\mathbf{Q} = \begin{bmatrix} h_1^{-1} & 0 & \cdots & 0 \\ -h_1^{-1} - h_2^{-1} & h_2^{-1} & \cdots & 0 \\ h_2^{-1} & -h_2^{-1} - h_3^{-1} & \cdots & 0 \\ 0 & h_3^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{n-1}^{-1} \end{bmatrix}_{n\times(n-2)} , h_i = x_{i+1} - x_i$$

$$\mathbf{R} = \begin{bmatrix} \frac{1}{3}(h_1 + h_3) & \frac{1}{6}h_2 & \cdots & 0 \\ \frac{1}{6}h_2 & \frac{1}{3}(h_2 + h_3) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{3}(h_{n-2} + h_{n-1}) \end{bmatrix}_{(n-2)\times(n-2)}$$

$$\ell = (1-\lambda)\|y - f\|_2^2 + \lambda f^T K f$$

$$\frac{\partial \ell}{\partial f} = (1-\lambda)(2f - 2y) + \lambda 2Kf$$

$$\frac{\partial f}{\partial c} = B^T$$

$$\frac{\partial \ell}{\partial c} = B^T[(1-\lambda)(2Bc - 2y) + 2\lambda KBc] = 0 \to 0 = (1-\lambda)(B^TBc - B^Ty) + \lambda B^TKB \to$$

$$c[(1-\lambda)B^TB - \lambda B^TKB] = (1-\lambda)B^Ty$$

Note that c, y can be multi-dimensional (and not vectors) to solve the 2D problem simultaneously for x and y.

Clamped

$$P = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} (1-\lambda)B^TB - \lambda B^TKB & P.T \\ P & 0 \end{bmatrix} \begin{bmatrix} c \\ z \end{bmatrix} = \begin{bmatrix} (1-\lambda)B^Ty \\ q \end{bmatrix}$$

Approach 2

MSE

$$\delta(\tilde{c}) = \sum_{j=1}^{m}\left(y_j - s(x_j)\right)^2 = \sum_{j=1}^{m}\left(y_j - \sum_{i=1}^{n+k+1} c_i B_{i,k+1}(x_j)\right)^2 \underbrace{=}_{\substack{y_1=s(x_1)\\ y_m=s(x_m)}} \sum_{j=2}^{m-1}\left(y_j - \sum_{i=1}^{n+k+1} c_i B_{i,k+1}(x_j)\right)^2 \underbrace{=}_{\substack{c_1=y_1\\ c_m=y_m}}$$

$$\sum_{j=2}^{m-1}\left(y_j - \underbrace{\left(y_1 B_{1,k+1}(x_j) + y_m B_{n+k+1,k+1}(x_j)\right)}_{r_j} - \sum_{i=2}^{n+k} c_i B_{i,k+1}(x_j)\right)^2$$

$$= \sum_{j=2}^{m-1}\left((y_j - r_j) - B[x_j]c\right)^2 = \left\|(\tilde{y} - \tilde{r}) - \tilde{B}\tilde{c}\right\|_2^2$$

$$r = [r_1 \quad \dots \quad r_m]^T, \qquad r_j = \left(y_1 B_{-k,k+1}(x_j) + y_m B_{n,k+1}(x_j)\right)$$

$$\tilde{c} \in \mathbb{R}^{n+k-1} = [c_2 \quad \dots \quad c_{n+k}]^T$$
$$\tilde{B} \in \mathbb{R}^{m-2\times n+k-1}: remove\ first\ and\ last\ row\ and\ column\ of\ B$$

Smoothness

$$\eta(c) = f^T K f = \sum_{j=1}^{m}\left(\sum_{i=1}^{m} f_i \cdot K_{i,j}\right) f_j = \sum_{j=1}^{m}\left[\left(f_1 \cdot K_{1,j} + f_m \cdot K_{m,j} + \sum_{i=2}^{m-1} f_i \cdot K_{i,j}\right) f_j\right] =$$

$$\sum_{j=1}^{m}\left[\left(\sum_{i=2}^{m-1} f_i \cdot K_{i,j}\right) f_j\right] + \sum_{j=1}^{m} \underbrace{\left(f_1 \cdot K_{1,j} + f_m \cdot K_{m,j}\right)}_{s_j} f_j =$$

$$\left(\sum_{i=2}^{m-1} f_i \cdot K_{i,1}\right) f_1 + \left(\sum_{i=2}^{m-1} f_i \cdot K_{i,m}\right) f_m + \sum_{j=2}^{m-1}\left[\left(\sum_{i=2}^{m-1} f_i \cdot K_{i,j}\right) f_j\right] + s^T f =$$

$$\tilde{f}^T \widetilde{K_{*,1}} f_1 + \tilde{f}^T \widetilde{K_{*,m}} f_m + \tilde{f}^T \tilde{K} \tilde{f} + s^T f = \tilde{f}^T \widetilde{K_{*,1}} f_1 + \tilde{f}^T \widetilde{K_{*,m}} f_m + \tilde{f}^T \tilde{K} \tilde{f} + s_1 f_1 + s_m f_m + \tilde{s}\tilde{f} =$$
$$\tilde{f}^T \widetilde{K_{*,1}} y_1 + \tilde{f}^T \widetilde{K_{*,m}} y_m + \tilde{f}^T \tilde{K} \tilde{f} + \tilde{s}\tilde{f} + s_1 y_1 + s_m y_m$$

Putting it together

$$\ell = (1-\lambda)\|y - f\|_2^2 + \lambda f^T K f =$$
$$(1-\lambda)\left\|(\tilde{y} - \tilde{r}) - \tilde{f}\right\|_2^2 + \lambda\left[\tilde{f}^T \widetilde{K_{*,1}} y_1 + \tilde{f}^T \widetilde{K_{*,m}} y_m + \tilde{f}^T \tilde{K} \tilde{f} + \tilde{s}^T \tilde{f} + s_1 y_1 + s_m y_m\right]$$

$$\frac{\partial \ell}{\partial \tilde{f}} = (1-\lambda)\left[-2(\tilde{y} - \tilde{r}) + 2\tilde{f}\right] + \lambda\left[\widetilde{K_{*,1}} y_1 + \widetilde{K_{*,m}} y_m + 2\tilde{K}\tilde{f} + \tilde{s}\right]$$

$$\tilde{s} = f_1 \cdot K_{1,*} + f_m \cdot K_{m,*}$$

$$\tilde{r} = y_1 \cdot \widetilde{K_{1,*}} + y_m \cdot \widetilde{K_{m,*}}$$

$$f_j = B_{j,*}c = \tilde{B}_{j,*}\tilde{c} + B_{j,1}y_1 + B_{j,m}y_m \rightarrow \tilde{f} = \tilde{B}\tilde{c} + \widetilde{B_{*,1}}y_1 + \widetilde{B_{*,m}}y_m$$

$$\frac{\partial \tilde{f}}{\partial \tilde{c}} = \tilde{B}^T$$

$$\frac{\partial \ell}{\partial \tilde{c}} = \frac{\partial \tilde{f}}{\partial \tilde{c}} \cdot \frac{\partial l}{\partial \tilde{f}} = \tilde{B}^T\left[(1-\lambda)[-2(\tilde{y}-\tilde{r}) + 2\tilde{B}c] + \lambda\left[\widetilde{K_{*,1}}y_1 + \widetilde{K_{*,m}}y_m + 2\tilde{K}\tilde{B}c + \tilde{s}\right]\right] = 0$$

$$\tilde{c} = \left[2(1-\lambda)\tilde{B}^T\tilde{B} + 2\lambda\tilde{B}^T\tilde{K}\tilde{B}\right]^{-1}\left[(1-\lambda)2\tilde{B}^T(\tilde{y}-\tilde{r}) - \lambda\tilde{B}^T\left(\widetilde{K_{*,1}}y_1 + \widetilde{K_{*,m}}y_m + \tilde{s}\right)\right]$$

$$c = (y_1, \tilde{c}, y_m)$$

Unclamped:

$$c = [(1-\lambda)B^TB + \lambda B^TKB]^{-1}[(1-\lambda)B^Ty]$$

Curvature Fixing Algorithm

Given path (x, y):

1. Construct clamped smooth B spline with midway points and set new n s.t. $N = m$ (total $2n + 2k + 1$) control points
   Implementation details below
2. Follow 2-peak solution scheme

$$c \in \mathbb{R}^{2n+2k+1} = \begin{bmatrix} c_1 & \frac{c_1 + c_3}{2} & c_3 & \frac{c_3 + c_5}{2} & c_5 & \dots & \frac{c_{n+k-1} + c_{n+k+1}}{2} & c_{n+k+1} \end{bmatrix}$$

$$\tilde{c} \in \mathbb{R}^{n+k+1} = [c_1 \quad c_3 \quad \dots \quad c_{2n+2k+1}]$$

Where $2n + 2k + 1 \le 2m - 1 \to cn = n + k + 1 \le 2m - 1 - n - k$

$$Bc = c_1 \cdot \left(B_{j,1} + \frac{B_{j,2}}{2}\right) + c_3\left(\frac{B_{j,2}}{2} + B_{j,3} + \frac{B_{j,4}}{2}\right) + \dots + c_{2n+2k-1}\left(\frac{B_{j,2n+2k-1}}{2} + B_{j,2n+2k} + \frac{B_{j,2n+2k+1}}{2}\right)$$
$$+ c_{2n+2k+1}\left(\frac{B_{j,2n+2k}}{2} + B_{j,2n+2k+1}\right) = \hat{B}\tilde{c}$$

$$\hat{B} \in \mathbb{R}^{m \times n+k+1}\!: \; B \underbrace{\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & \dots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & \ddots & 0.5 & 0 & 0 \\ 0 & 0 & 1 & \ddots & 1 & 0 & 0 \\ 0 & 0 & 0.5 & \ddots & 0.5 & 0.5 & 0 \\ 0 & \vdots & \vdots & & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}}_{2n+2k+1 \times n+k+1}$$

$$f = Bc = \hat{B}\tilde{c}$$

$$\delta(\tilde{c}) = \|y - f\|_2^2$$

$$\eta(\tilde{c}) = f^T K f$$

Same solution, just with $\hat{B}$:

$$\frac{\partial \ell}{\partial f} = (1 - \lambda)(2f - 2y) + \lambda 2Kf$$

$$\frac{\partial f}{\partial \tilde{c}} = \hat{B}^T$$

$$\tilde{c}\left[(1 - \lambda)\hat{B}^T\hat{B} + \lambda\hat{B}^T K\hat{B}\right] = (1 - \lambda)\hat{B}^T y$$

Enforce Clamping

$c_1 = y_1, c_{n+k+1} = y_m$

$m' = 2m - 1$

$\gamma \le m$ usually $\gamma = m$

$c \in \mathbb{R}^\gamma$

$\tilde{c} \in \mathbb{R}^{\gamma-2} = [c_2 \quad c_3 \quad \cdots \quad c_{\gamma-1}]$

$0 \le n = (2\gamma - 1) - k - 1 = 2\gamma - k - 2 \le 2m - k - 2$

$$\delta(\tilde{c}) = \sum_{j=1}^{m'} w_j \left(y_j - s(x_j)\right)^2 = \sum_{j=2}^{m'-1} w_j \left(y_j - \sum_{i=1}^{\gamma} \frac{c_{2i-1} + c_{2i+1}}{2} B_{2i,k+1}(x_j) - \sum_{i=2}^{\gamma-1} c_i B_{2i+1,k+1}(x_j) \right)^2$$

$$= \sum_{j=2}^{m'-1} w_j \Bigg[y_j - \Bigg(\underbrace{y_1 \cdot \left(B_{j,1} + \frac{B_{j,2}}{2}\right) + y_m \left(\frac{B_{j,2\gamma}}{2} + B_{j,2\gamma+1}\right)}_{r_j} \Bigg)$$

$$- \left(c_2 \left(\frac{B_{j,2}}{2} + B_{j,3} + \frac{B_{j,4}}{2} \right) + \cdots + c_{\gamma-1} \left(\frac{B_{j,2\gamma-2}}{2} + B_{j,2\gamma-1} + \frac{B_{j,2\gamma}}{2} \right) \right) \Bigg]^2$$

$$= \sum_{j=2}^{m-1} w_j \left[(y_j - r_j) - \tilde{B}[j]\tilde{c} \right]^2 = \left\| \widetilde{w} \circ (\tilde{y} - \tilde{r}) - \tilde{B}'\tilde{c} \right\|_2^2$$

$\widetilde{\mathrm{B}} \in \mathbb{R}^{m-2 \times n+k-1}$

$$\tilde{B}[j] = \left[\frac{1}{2} B_{\mathrm{j},2} + B_{\mathrm{j},3} + \frac{1}{2} B_{\mathrm{j},4} \quad \cdots \quad \frac{1}{2} B_{\mathrm{j},2n+2k-1} + B_{\mathrm{j},2n+2k} + \frac{1}{2} B_{\mathrm{j},2n+2k+1} \right]$$

$$\eta(\tilde{c}) = \sum_{j=1}^{n} \left(\sum_{i=1}^{2n+2k+1} a_{j,i} \cdot c_i \right)^2 = \sum_{j=1}^{n} \left(\sum_{i=2}^{n+k+1} a_{j,2i-1} \cdot c_{2i-1} + \sum_{i=1}^{n+k-1} \frac{1}{2} a_{j,2i} \cdot (c_{2i-1} + c_{2i+1}) \right)^2 =$$

$$\sum_{j=1}^{n} \Bigg(c_1 \left(a_{j,1} + \frac{1}{2} a_{j,2} \right) + c_3 \left(\frac{1}{2} a_{j,2} + a_{j,3} + \frac{1}{2} a_{j,4} \right) + \cdots + c_{2n+2k-1} \left(\frac{1}{2} a_{j,2n+2k} + a_{j,2n+2k-1} + \frac{1}{2} a_{j,2n+2k+1} \right)$$

$$+ c_{2n+2k+1} \left(\frac{1}{2} a_{j,2n+2k} + a_{j,2n+2k+1} \right) \Bigg)^2 =$$

$$\sum_{j=1}^{n}\left(\underbrace{\left(y_1\left(a_{j,1}+\frac{1}{2}a_{j,2}\right)+y_m\left(\frac{1}{2}a_{j,2n+2k}+a_{j,2n+2k+1}\right)\right)}_{s_j}+c_3\left(\frac{1}{2}a_{j,2}+a_{j,3}+\frac{1}{2}a_{j,4}\right)+\cdots\right.$$

$$\left.+c_{2n+2k-1}\left(\frac{1}{2}a_{j,2n+2k}+a_{j,2n+2k-1}+\frac{1}{2}a_{j,2n+2k+1}\right)\right)^2=\sum_{j=1}^{n}\left(s_j+\tilde{A}[j]\tilde{c}\right)^2=\left\|s+\tilde{A}\tilde{c}\right\|_2^2$$

$$c=\left[\tilde{B}'^T\tilde{B}'+\frac{\lambda}{1-\lambda}\cdot\tilde{A}^T\tilde{A}\right]^{\dagger}\left[\tilde{B}'^T\left(\widetilde{w}\circ(\tilde{y}-\tilde{r})\right)+\frac{\lambda}{1-\lambda}\cdot\tilde{A}^T s\right]$$

Enforce Initial Heading

Find $\hat{c}\in\mathbb{R}^{2p}$ comprised of p 2D control points that fit data with the initial heading constraint:

$$\hat{c}_2=\left(\hat{c}_{1.x}+\alpha cos\theta,\hat{c}_{1.y}+\alpha sin\theta\right)$$

$c^T=\left[c_{1.x},c_{3.x}\dots,c_{p.x},c_{1.y},c_{3.y}\dots,c_{p.y},\alpha\right]\in\mathbb{R}^{2p-1}$ (find c and use it to construct $\hat{c}$)

$$\underbrace{B^+}_{2m\times 2p}\underbrace{\begin{pmatrix}1&0&0&\cdots&0&\dots&0&0\\1&0&0&\dots&0&\dots&0&cos\theta\\0&1&0&\dots&0&\dots&0&0\\0&0&\dots&1&0&0&0&0\\0&0&\dots&0&1&0&0&0\\0&0&\dots&0&1&0&\dots&sin\theta\\\vdots&&&&&&\ddots&\vdots\\0&0&\dots&0&0&0&0&1\end{pmatrix}}_{2p\times 2p-1}\cdot\underbrace{c}_{2p-1\times 1}=\underbrace{y}_{2m\times 1}$$

$$B_{m\times p}\in m\times p$$

$$B_{2m\times 2p}=\begin{bmatrix}B_{m\times p}&0\\0&B_{m\times p}\end{bmatrix}$$

$$B=B_{2m\times 2p}\hat{B}$$

$$c=\left[(1-\lambda)B^TB+\lambda B^TKB\right]^{-1}\left[(1-\lambda)B^Ty\right]$$

Enforce clamping using P,q as before.

Finding nc, λ given data

Note that nc is the number of control points, where n is the interior knot length (can derived directly from nc).

1.
$$L = \sum_{i=1}^{N-1} \|d_{i+1} - d_i\|_2^2$$

Set $\lambda = 0, nc = \lceil r \cdot L \rceil + 2$ where r is defined as the ratio of maximum twists per unit of length (meter).

Advantage is that we can work with a smaller number of control points – can make computations faster and more importantly, guarantee max curvature algorithm validity.
Parameter: r

2. Redefine loss as MSE instead of SE (add average), define nc as maximum (m) and have some constant λ
$$\ell = (1-\lambda)\frac{1}{N}\sum_{i=1}^{N} \|s(z_i) - d_i\|_2^2 + \lambda \int_a^b \|s''(z)\|_2^2 dz$$
Parameter: λ
3. Take inspiration from dierckx to define s as the maximum allowed MSE value to search for i.e. search for largest λ that fits MSE constraint
Parameter: s
4. My preferred - combine 1 and 2:
Set r to be very lenient, and not a ratio that determines the minimum control points needed, but something that 'makes sense'. Then use constant λ.
Parameter: $r + \lambda$

Check validity: take some trip(s) and measure MSE and roughness for each frame. Take largest MSE frames and analyze them. A good parameter for each method is one that ensures a maximum MSE along the entire trip

1. r
2. λ
3. s
4. r, λ