Conic Optimization Problems: LPs, SDPs, etc.

Friday, February 19, 2021 6:29 AM

§4.3 and 4.4 Boyd + Vandenberghe
and 4.6

Special (most important!) examples of convex problems
--- and easily solvable (for moderate sizes) w/ OTC solvers
off-the-shelf

"program" = "problem", not "computer code"

Linear Programs

Geometrically, linear objective and polyhedral constraints

✓ $\|x\|_1 \le 1$

✓ $\|x\|_\infty \le 1$

✗ $\|x\|_2 \le 1$

Standard forms

$$\min \langle c, x\rangle \quad G\cdot x \le h,\ Ax = b$$

or

$$\min \langle c, x\rangle \quad x \ge 0,\ Ax = b$$

standard form

or

$$\min \langle c, x\rangle \quad Ax \le b$$

inequality form

Converting between forms

Use slack variables, or tricks like $x = x^+ - x^-$, $x^+, x^- \ge 0$

Ex: $\min_x \|x\|_1$ s.t. $Ax = b$ $\longrightarrow$ $\sum_{i=1}^n |x_i|$, let $x_i = x_i^+ - x_i^-$, $x_i^+, x_i^- \ge 0$

so $|x_i| \le x_i^+ + x_i^-$

will be equality since we're going to minimize

So equivalent problem

$$\min_{x^+, x^-} \mathbb{1}^T(x^+ + x^-) \quad \text{s.t.} \quad A\cdot x^+ - Ax^- = b$$

Ex $\min \|x\|_\infty$ st $Ax = b$ $\longrightarrow$

$$\min_{\substack{x\in\mathbb{R}^n\\ t\in\mathbb{R}}} t \quad Ax = b,\ \|x\|_\infty \le t$$

"epigraph trick"

Ex

$G\cdot x \le b$, $Ax = b$ $\overset{?}{\Longrightarrow}$ $\tilde{x} \ge 0$, $\tilde{A}\tilde{x} = \tilde{b}$ use slack variables

$G\cdot x + s = b$, $s \ge 0$ "slack"

$x = x^+ - x^-$, $x^+ \ge 0$, $x^- \ge 0$

so $\tilde{x} = \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} \ge 0$ ✓

... etc

LP's:

- most well-studied kind of convex opt. problem
- for a long time, only kind we could solve well

... so {business, economics, operations research (OR), computer science ... work hard to cast modes (i.e., make approximations) as LP's

- depends, but we can solve LP's in very high dimension 10^4, 10^6, 10^9 ...

- these fields often have integer LP's. These are not convex, NP-Hard, but we do have good global solvers for dimensions $\approx 100+$ (Bertsimas claims 500,000 ... again, depends on the problem)

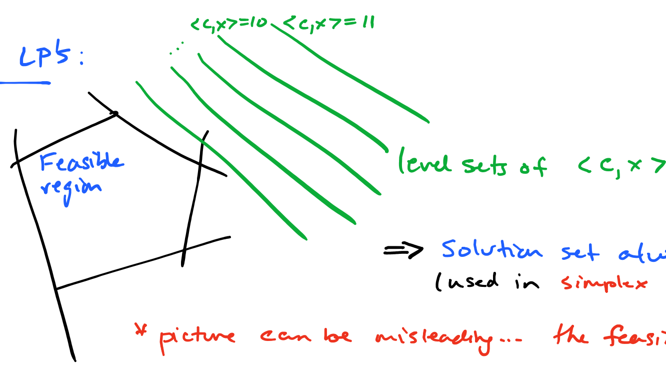
Very mature code (CPLEX, Gurobi ... MOSEK, GLPK) not as impressive

solve many LP's as subproblems

Ex: scheduling airplane flights
scheduling classrooms

} the art of turning a problem into a math problem (modeling) is often in an OR class

Geometry of LP's:



$\Rightarrow$ Solution set always includes a vertex!
(used in simplex method)

* picture can be misleading ... the feasible region can be unbounded!

Quadratic Programs

$\boxed{QP}$
$$\min_x \ \tfrac{1}{2}\langle x, Px\rangle + \langle q, x\rangle + r$$
$$\text{s.t. } Gx \le h$$
$$Ax = b$$

$P \in S^n$
$q \in \mathbb{R}^n$
$r \in \mathbb{R}$ (irrelevant)

i.e., a single quadratic term, and as many linear terms as you like

⚠ Need not be convex (convex iff $P \succeq 0$)
... though if not convex, can sometimes still be solved via S-Procedure / S-lemma (see Appendix)

Ex: Regression is a convex QP

$$\min \ \underbrace{\tfrac{1}{2}\|Ax-b\|^2}$$
$$= \tfrac{1}{2}\langle Ax, Ax\rangle - \left(\tfrac{1}{2}\langle Ax, b\rangle + \tfrac{1}{2}\langle x, A^T b\rangle\right) + \|b\|^2$$
$$= \tfrac{1}{2}\langle x, \underbrace{A^T A}_{P \succeq 0} x\rangle - \langle x, \underbrace{A^T b}_{g}\rangle + \underbrace{\|b\|^2}_{r}$$

$\boxed{\text{QCQP}}$ Quadratically constrained Quadratic Program

A QP w/ more than 1 quadratic

$$\min \ \tfrac{1}{2}\langle x, P_0 x\rangle + \langle g_0, x\rangle + r_0$$
$$\text{s.t. } Ax = b$$
$$\tfrac{1}{2}\langle x, P_i x\rangle + \langle g_i, x\rangle + r_i \le 0 \quad \forall i = 1, \dots, m$$

Generalizes QP (set $P_i = 0$, $i = 1, \dots, m$)

and LP (set $P_i = 0$, $i = 0, 1, \dots, m$)

$\boxed{\text{SOCP}}$ Second-Order Cone Program

Generalizes convex QCQP

$$\min \ \langle c_0, x\rangle$$
$$\text{s.t. } \underbrace{\|A_i x + b_i\|_2 \le \langle c_i, x\rangle}_{} + d_i$$

and $Fx = g$

affine composition w/ 2^{nd} order cone
i.e. Lorentz cone

$(\vec{y}, t) \in K$ iff $\|\vec{y}\|_2 \le t$

Efficiently solvable ...

§4.5 Geometric Programming

Polynomials w/ positive coefficients, on $x \in \mathbb{R}^n_{++}$

Not convex, but becomes convex after a log change-of-variables

Skip

§4.6 Conic Programming

For LPs, $\min \ \langle c, x\rangle$
$x \succeq 0$
$Ax = b$

i.e., $x \in \mathbb{R}^n_+$ ← non-negative orthant, example of a proper cone

1) closed
2) convex
3) solid (non-empty interior)
4) pointed (no lines)

A conic program is

$$\min \ \langle c, x\rangle \qquad\qquad \min \ \langle c, x\rangle$$
$$Fx + g \preceq_K 0 \quad \text{or} \quad x \succeq_K 0$$
$$Ax = b \qquad\qquad Ax = b$$

Standard form

where K is a proper cone and $y \succeq_K 0$ means $y \in K$

K_1, K_2 proper $\Rightarrow K_1 \times K_2$ proper

Note If we want $F_1 x + g_1 \preceq_{K_1} 0$ define $K = K_1 \times K_2$

$F_2 x + g_2 \preceq_{K_2} 0$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \cdot x + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \preceq_K 0$$

Ex.
- LPs w/ $K = \mathbb{R}^n_+$
- SOCPs w/ $K =$ second-order-cone (and direct products of these)

SDPs Semi-definite programs

Conic programs w/ $K = S^n_+$ (and direct products of these)

$$\min_{X \in S^n} \langle C, X \rangle$$

$$\langle A_i, X \rangle = b_i \quad i = 1, \dots, m$$

$$\underbrace{X \succeq 0} \quad \text{i.e. } X \succeq_K 0, \; K = S^n_+$$

Recall $\langle C, X \rangle = \mathrm{tr}(C^T X)$

$= \langle \mathrm{vec}(C), \mathrm{vec}(X) \rangle$

$\mathbb{R}^{n^2}$ inner product