HW help: proving polyhedrality

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How to prove a set $S \subseteq \mathbb{R}^n$ is polyhedral?

One, often difficult way, is to write $S = \{x : Ax \le b,\ Fx = d\}$. Usually we have tricks:

Tricks

1. Finite intersection of polyhedral sets is still polyhedral
2. "lift-and-project" $S \subseteq \mathbb{R}^n$, then "lift" to a larger space $\mathbb{R}^{n+k}$ by introducing new variables, and $S = \underset{\uparrow \text{ project}}{P}(T)$, T a polyhedral set in $\mathbb{R}^{n+k}$ $\Rightarrow$ S polyhedral

   (Why does this work? T polyhedral $\xRightarrow{\text{via } 3}$ $\exists \{z_1, \ldots, z_m\} \subseteq \mathbb{R}^{n+k}$ s.t. $T = \mathrm{conv}(\{z_1, \ldots, z_m\})$ $\Rightarrow$ $S = \mathrm{conv}(\{Pz_1, \ldots, Pz_m\})$ $\xRightarrow{\text{via } 3}$ S polyhedral)

3. $\mathrm{conv}(V)$, $\underbrace{|V|}_{\text{finite set.}} < \infty$, is polyhedral. So find vertices V

Why? $\mathrm{conv}(\{v_1, \ldots, v_m\}) = \{ \underbrace{x = \sum_{i=1}^m t_i \vec{v}_i}_{\text{encode as}} : \sum t_i = 1,\ t_i \ge 0 \}$

$$\begin{bmatrix} I & -A \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = [0] \quad \text{where } t = \begin{bmatrix} t_1 \\ \vdots \\ t_m \end{bmatrix}$$

$$A = \begin{bmatrix} v_1, \ldots, v_m \end{bmatrix} \quad n \times m \text{ matrix}$$

encode as

$$\begin{bmatrix} 0 & \mathbb{1}^T \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 1$$

encode as

$$\begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \ge [0]$$

So $\begin{bmatrix} x \\ t \end{bmatrix}$ is polyhedral, so projection to just x is polyhedral (trick 2)

In my quick justifications, I used 3 to prove 2 (in fact I used an "iff" version of 3), and used 2 to prove 3 ... so not an actual proof)