More Duality, Fri Feb 26 2021

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⚠ Notes written in realtime, even less accurate than usual

§5.1.6 Conjugate Function — we'll come back to this later

§5.2

Recall last time: Lagrange Dual Problem $d^* = \underbrace{\max_{\substack{\lambda \geq 0 \\ \nu}} g(\lambda,\nu)}_{\text{a "convex problem"}}$

where $g(\lambda,\nu) = \min_{x \in D} \underline{\mathcal{L}(x,\lambda,\nu)}$

$\mathcal{L}(x,\lambda,\nu) = f_0(x) + \sum \lambda_i f_i(x) + \sum \nu_i h_i(x)$

$p^* = \min_{x \in D} f_0(x)$ s.t. $f_i(x) \leq 0$, $h_i(x) = 0$

weak duality : $d^* \leq p^*$ ALWAYS

strong duality : $d^* = p^*$ SOMETIMES

Linear Programs (Vanderbei)

(P) $\max_{x \in \mathbb{R}^n} \langle c, x \rangle$
$x \geq 0$
$Ax \leq b : y$

(D) $\min_{y \in \mathbb{R}^m} \langle b, y \rangle$
$y \geq 0$
$A^T y \geq c : x$

A $m \times n$ matrix

Rules to transform (P) → (D) for LP's:

1) max → min + vice-versa
2) variables → constraints
   constraints → variables
3) objective and RHS flip places
4) matrices transpose

A. Benjamin SIAM Review '95 "Sensible rules for remembering the dual"
"SOB"

Primal Variable x_i:
- $x_i \geq 0$ Sensible
- no constraint Odd
- $x_i \leq 0$ Bizarre

Constraints in primal if maximizing:
- $a_i^T x \leq b_i$ Sensible
- $a_i^T x = b_i$ Odd
- $a_i^T x \geq b_i$ Bizarre

Flip if minimizing not maximizing

Rule : a dual constraint is S/O/B if primal variable is S/O/B
dual variable S/O/B " constraint S/O/B

Ex

(P) $\min_{\substack{x \geq 0 \\ x \in \mathbb{R}^2}} \quad 3x_1 + 2x_2$

sensible

$x_1 + 2x_2 \geq 5 \quad : y_1 \quad S$

$x_2 \leq 2 \quad : y_2 \quad B$

$\underbrace{\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{}}_{\text{want to } \cancel{\text{solve}} \text{ solve and certify}}$

(D) $\max_{y \in \mathbb{R}^2} \quad 5 \cdot y_1 + 2 \cdot y_2$

$y_1 \geq 0 \;\; S \qquad y_1 + 0 \cdot y_2 \leq 3$ sensible

$y_2 \leq 0 \;\; B \qquad 2y_2 + y_1 \leq 2$

$x = (2,2)$
$p^* \leq 10$

$x = (1,2)$. Check if it's feasible: $1 + 2 \cdot 2 \overset{?}{\geq} 5$ ✓, $2 \overset{?}{\leq} 2$ ✓

$f(x) = 3 \cdot 1 + 2 \cdot 2 = 7$

Conclude: $p^* \leq 7$

Observe:
$f(x) = 3x_1 + 2x_2 = \underbrace{2x_1}_{\geq 0} + (\underbrace{x_1 + 2x_2}_{\geq 5}) \geq 5$ We proved $5 \leq p^*$

$(\; x_1 + 2x_2 \geq 5 \;) \cdot 3$
$(\; -x_2 \geq -2 \;) \cdot 4$

dual variables

$3x_1 + 6x_2 \geq 15$
$-4x_2 \geq -8$
$\underbrace{3x_1 + 2x_2}_{f(x)} \geq 7$

We proved $7 \leq p^*$

Duality gap: x feasible, λ feasible

$f_0(x) - g(\lambda, \nu)$

§5.3 Geometric Interpretation

See also "min. common pt. / max crossing" problems, Bertsekas

Main strong duality results ($d^* = p^*$)

- If (P) isn't convex, strong duality is unlikely
  - Exception: Certain nonconvex QP: S-lemma / S-Procedure. See Appendix of B&V
- If (P) is convex, strong duality holds under certain constraint qualifications (CQ) Ex. Slater's

Thm If (P) is convex and $\underbrace{\text{Slater's conditions}}$ hold

(P) $\min f_0(x)$
$f_i(x) \leq 0$
$Ax = b$

hold if $\exists$ a "strictly feasible point"
$\exists x \in \text{relint}(\text{dom}(f_0))$ and
f_i is affine, $f_i(x) \leq 0$ (feasible)
f_i isn't affine, $f_i(x) < 0$ (strictly feasible)
and $Ax = b$

then ① we have strong duality, $d^* = p^* < \infty$
and ② $\exists$ an optimal solution to the dual problem.

⚠ Slater's $\not\Rightarrow$ $\exists$ optimal primal sol'n

Quick ex: $\underset{x \in \mathbb{R}}{\cancel{\min}\ \inf}\ e^x$

convex ✓
lsc ✓
proper ✓

Note often we want Slater's condition on the dual!
(since dual-of-the-dual is the primal)
$\Rightarrow \exists$ optimal primal solution

Corollary LP's are nice.
Slater's conditions hold iff problem is feasible (i.e., $p^* < \infty$)
So $p^* < \infty \Rightarrow d^* = p^*$ and dual optimal sol'n exists
$d^* > -\infty \Rightarrow d^* = p^*$ and primal " " "

hence if either p^* or $d^* \in \mathbb{R}$ (not $\pm\infty$) $\Rightarrow$ $\boxed{\text{optimal primal and dual sol'n exist}}$

Note: $d^* = -\infty$, $p^* = +\infty$ is possible ... but rare.
(not strong duality)

degenerate ARE important for SDP's