Metarules
APPM 5630 Spring 2023
Advanced Convex Optimization

Caveat: most of these rules implicitly assume that the minimizers exist. All mistakes due to Stephen Becker.

1. - $\boxed{\min_{x\in C} f(x) = -\max_{x\in C} -f(x)}$
   - $\boxed{\operatorname{argmin}_{x\in C} f(x) = \operatorname{argmax}_{x\in C} -f(x)}$
   - $\boxed{\operatorname{argmin}_{x\in C} f(x) = \operatorname{argmin}_{x\in C} \varphi(f(x))}$ if φ is a monotonic increasing function on the range of f.
     - What can you say if φ is simply monotonic non-decreasing? e.g., take $\varphi(\beta) \equiv 0$.
     - Example: $\operatorname{argmin}_x \|Ax-b\| = \operatorname{argmin}_x \frac{1}{2}\|Ax-b\|^2$

2. $\boxed{\min_{x\in C} f(x) \le \min_{x\in D} \text{ if } D \subset C}$
   - Caveat: existence. e.g., take $f(x) = x$ and $D = (0,1)$ and $C = [0,1]$, then $\min_{x\in D} x$ does not exist but $\min_{x\in C} x$ does. Or vice-versa: take $C = (0,1)$ and $D = \{0.5\}$.

3. $\boxed{\min_x[f(x)+g(x)] \ne [\min_x f(x)] + [\min_y f(y)]}$
   - In fact, you can say the following: $\min_x[f(x)+g(x)] \ge [\min_x f(x)] + [\min_y f(y)]$

4. $\boxed{\min_x \min_y f(x,y) = \min_y \min_y f(x,y)}$ (hence $\min_{x,y} f(x,y)$ is well-defined). [Caveat: minima need to exist, e.g., $\min_{x\ge 0}\min_y x\cdot e^{-y}$, where the joint minimum value is 0 (e.g., at $(0,2)$) but for an arbitrary $x \ne 0$ the problem $\min_y x\cdot e^{-y}$ is il-defined since there is no minimizer]
   - *Proof* Let $f(x^\star, y^\star) = \min_x \min_y f(x,y) \equiv A$, and let $B \equiv \min_y \min_x$. Then
   $$\begin{aligned} B = \min_y \min_x f(x,y) &\le \min_x f(x,y^\star) \\ &\le f(x^\star,y^\star) = A \end{aligned}$$
   hence $B \le A$, and similarly you could show $A \le B$, hence $A = B$.
   - A similar statement (probably?) holds with inf − inf (and of course with max − max and sup − sup too).
   - On the other hand, if you have a **mixed** min/max or inf/sup, now it is not always true, and you need to satisfy the conditions of a saddle point theorem. What is always true is an **inequality**, $\boxed{\sup_y \inf_x f(x,y) \le \inf_x \sup_y f(x,y)}$.

Myths

See Harvey Greenberg's nice document from about 2010. I only point out a few here:

1. Myth: $\min_{x,y} g(x) + h(y) \;:\; y = g(x)$ is equivalent to $\min_y y + h(x)$ (via direct substitution)
   - This is true sometimes (e.g., under some conditions), but in general, it is not true, and you need to do a proper Lagrange multiplier treatment. Here's a counter-example. Take $\min_{x,y} x^2+(y-5)^2$ subject to $16y = x^2$. So substitute in the x^2 and get $\min_y 16y + (y-5)^2$, which is unconstrained and has just one critical point at $y = -3$, but at this point, there is no real value of x, so we don't have a solution to the original problem! The issue is we needed to divide into cases of whether $x \ge 0$ or $x < 0$. The true minimum is at $(x,y) = (0,0)$.