First view of Lagrange Multipliers

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§4.2 Boyd & Vandenberghe

Recall

Thm x is optimal for $\min_{x \in C} f(x)$, with f differentiable and convex, C convex,

iff ① $x \in C$
② $\forall y \in C, \langle \nabla f(x), y - x \rangle \geq 0$ $\boxed{\text{Euler Inequality}}$
(an example of a Variational Inequality)

Implication #1 Unconstrained problems

$C = \mathbb{R}^n$, $\text{dom}(f) = \mathbb{R}^n$, then in ②, choose $y = x - \nabla f(x)$

so $② \iff -\|\nabla f(x)\|^2 \geq 0$
$\iff \nabla f(x) = 0$

... so x is optimal iff $\nabla f(x) = 0$ (back to (simplified) Fermat's principle)

Implication #2 Convex equality constraints $\Rightarrow$ affine constraints

$C = \{x : \underset{m \times n}{A} x = b\}$

so ② is $\forall y$ s.t. $Ay = b$, $\langle \nabla f(x), y - x \rangle \geq 0$

i.e., $y = y_p + v$, $v \in \text{Null}(A)$

any "particular" sol'n to $Ax = b$

$y = x + v$, $v \in \text{Null}(A)$ ... since "x" itself is feasible, let $y_p = x$

i.e., $\forall v \in \text{Null}(A)$, $\langle \nabla f(x), v \rangle \geq 0$

... and since $v \in \text{Null}(A)$ iff $-v \in \text{Null}(A)$,

need $\langle \nabla f(x), v \rangle = 0$

From linear algebra

So...

$\nabla f(x) \perp \text{Null}(A)$, $\nabla f(x) \in \text{Null}(A)^\perp = \text{Range}(A^T)$

$\nabla f(x) \in \text{Range}(A^T)$ means $\exists \nu \in \mathbb{R}^m$ s.t. $\nabla f(x) = A^T \nu$

or ($\nu \leftarrow -\nu$) $\boxed{\nabla f(x) + A^T \nu = 0}$ LAGRANGE MULTIPLIERS

if $A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$ this is $\nabla f(x) + \sum_{i=1}^m \nu_i \vec{a}_i = 0$.