Example: Slater but primal optimal not achieved

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From Freund

Example of a "nice" convex problem (it's a SDP, and so objective is linear, constraints are convex, feasible and closed)
for which Slater's Condition holds ($\Rightarrow$ strong duality, existence of dual optimal)
but which has no primal optimal (ie., the "min" needs to really be an "inf")

$$(P)\quad \min_{X \in S^2} \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, X \right\rangle \quad \text{s.t. } X \succeq 0, \quad \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X \right\rangle = 2$$

Does Slater hold? Only nonlinear constraint is "$X \succeq 0$" (linear, non-linear)

so need $\exists\, X \succ 0$. Yes, since $X = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix} \succ 0$ is feasible

Now, analyze (P): parameterize $X = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ since symmetric.

$\left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X \right\rangle = 2 \Rightarrow 2 \cdot c = 2,\ c = 1$

so $X = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$. Let eigenvalues be λ_1, λ_2

$X \succeq 0$ means $\underline{\lambda_1, \lambda_2 \geq 0}$

Recall $tr(X) = \lambda_1 + \lambda_2 = a + b \Rightarrow a$ or b is ≥ 0 (or both)

Recall $det(X) = \lambda_1 \cdot \lambda_2 = ab - 1 \Rightarrow ab \geq 1$ $\Rightarrow$ both $a \geq 0$, $b \geq 0$

So our SDP is equivalent to

$$\min_{\substack{a \geq 0 \\ b \geq 0 \\ a \cdot b \geq 1}} a$$

by taking $a \to 0$, $b = 1/a \to \infty$ we see

$$p^* = \inf_{\substack{a \geq 0 \\ b \geq 0 \\ a \cdot b \geq 1}} a = 0$$

... but $a = 0$ isn't feasible
(since then we can't satisfy $ab \geq 1$)
and there is no optimal solution (a, b), though optimal value $p^* = 0$

Note: this means Slater's Conditions cannot hold for the dual problem, since that would imply primal optimal solution exists

The dual is

$$(D) \quad \max_{y \in \mathbb{R}} \; 2 \cdot y \quad \text{s.t.} \; \underbrace{\begin{bmatrix} 1 & -y \\ -y & 0 \end{bmatrix}}_{} \succeq 0$$

$$\lambda_1 \cdot \lambda_2 = \det(\ldots) = -y^2$$

$$\lambda_1, \lambda_2 \geq 0 \text{ by PSD}$$

but if $y \neq 0$, this means $\lambda_1 \cdot \lambda_2 < 0$, $\lambda_1, \lambda_2 \geq 0$

impossible!

... so, $y = 0$ is the only feasible point

$\Rightarrow$ (1) $d^* = 0$, confirming $p^* = d^*$

(2) Slater's conditions for the dual do <u>not</u> hold since there's no <u>strictly</u> feasible point.