Conjugate Functions

Tuesday, February 2, 2021 9:38 PM

aka Fenchel-Legendre conjugate*

* when speaking, I sometimes say "dual" when I mean "conjugate", and vice-versa. They are distinct, though related... Sorry in advance.

or, Fenchel-Legendre Transform, which reduces to the Legendre-Transform when you're differentiable.

Def The F.-L.- conjugate of f is

$$f^*(y) = \sup_x \; \langle y, x \rangle - f(x)$$

BV'04 says $y^T x$ but that's just specializing to Eucl. space. For, e.g., matrices, use $\mathrm{tr}(Y^T X)$

Prop f^* is convex (whether f is or not)

proof $y \mapsto \langle y, x \rangle - f(x)$ is convex $\forall\, y$, and arbitrary supremem preserve convexity $\square$

When f is differentiable and full domain, the supremum occurs when $\nabla_x(\langle y, x \rangle - f(x)) = 0$

i.e., $y = \nabla f(x)$, so $x^* = (\nabla f)^{-1}(y)$

$$\left.\begin{aligned} f^*(y) &= \langle y, x^* \rangle - f(x^*) \\ &= \langle \nabla f(x^*), x^* \rangle - f(x^*) \quad \text{w/} \quad x^* = (\nabla f)^{-1}(y) \end{aligned}\right\} \text{ Legendre Transform}$$

Legendre Transform in 1D ... to give us intuition.