Convex Functions, part 1

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From ch. 3 Boyd + Vandenberghe, supplemented w/ Bauschke + Combettes

Def A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if dom(f) is a convex set and $\forall\, x,y \in \text{dom}(f)$, $\forall\, 0 \le t \le 1$, $f(tx + (1-t)y) \le t f(x) + (1-t) f(y)$

and is strictly convex if dom(f) is a convex set and

$\forall\, x,y \in \text{dom}(f)$, $x \neq y$, $\forall\, 0 \le t \le 1$, $f(tx + (1-t)y) < t f(x) + (1-t) f(y)$ ← difference

and is strongly convex with respect to the norm $\|\cdot\|$ with parameter μ if dom(f) is a convex set and

$\forall\, x,y \in \text{dom}(f)$, $x \neq y$, $\forall\, 0 \le t \le 1$, $f(tx + (1-t)y) \le t f(x) + (1-t) f(y) - \frac{\mu}{2} t(1-t) \cdot \|x-y\|^2$

(strongly cvx ⟹ strictly cvx ⟹ cvx)

Simpler characterizations:

f is convex if epi(f) is convex (⟹ dom(f) is cvx too)

f strictly cvx means it always has curvature, no straight lines

f is strongly cvx w/ parameter μ and w.r.t. $\|\cdot\|_2$ Euclidean norm

iff $x \mapsto f(x) - \mu/2 \|x\|_2^2$ is convex

(if $\|\cdot\|$ not Euclidean, not true: see Amir Beck's '17 book Remark 5.18)

Convexity is a global property! Unlike continuity (local)

In most math, dom(f) is the set where f is defined.

In convex analysis, first, we allow $f: \mathcal{H} \to [-\infty, +\infty]$ → extended real line $= \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$

'generic Hilbert space' "extended value function"

or $f: \mathcal{H} \to (-\infty, +\infty]$

And if $x \notin \text{dom}(f)$,

we can pretend it is but define $f(x) = +\infty$

So now redefine

$$\text{dom}(f) = \{x : f(x) < +\infty\}$$

works well w/ minimization since we'd want to avoid $+\infty$

Ex: Define the indicator function of a set C to be

$$I_C(x) := \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases}$$

⚠ $\begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$ is the usual "indicator fcn" in other fields

So...

$$\min_{x \in C} f(x) \overset{\text{Equivalent}}{\iff} \min_{x \in \mathbb{R}^n} f(x) + I_C(x)$$

... so often you don't see any constraints, just the objective, but there may still be constraints encoded by $+\infty$ values!

~~Properness~~ Propriety

Def $f : \mathbb{R}^n \to [-\infty, +\infty]$ is **proper** if

1) it never takes value $-\infty$ (so we can write $f : \mathbb{R}^n \to (-\infty, +\infty]$)
2) $\text{dom}(f) \neq \emptyset$, i.e., not always equal to $+\infty$

... sounds pretty reasonable to me!

Needed to exclude weird cases

Ex: I_C (indicator fcn) is proper iff $C \neq \emptyset$

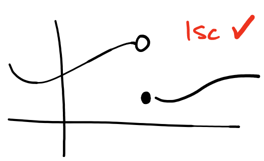
Closed / lsc

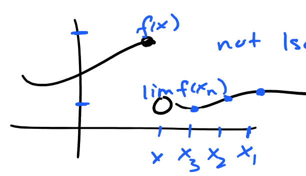
Def $f : \mathbb{R}^n \to [-\infty, +\infty]$ is **lower semi-continuous (lsc)** at $x \in \mathbb{R}^n$

if $\forall (x_n)$ s.t. $x_n \to x$, $f(x) \le \underbrace{\liminf_n f(x_n)}_{:= \lim_{n\to\infty} \inf_{k \ge n} f(x_k)}$

(so like sequential continuity but $\le \liminf$ instead of $= \lim$)

and "f is lsc" means f is lsc for all points $x \in \mathbb{R}^n$ — probably equiv. to saying $-f$ is lsc



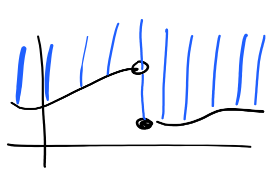


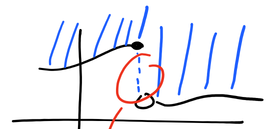
not lsc (this is "upper semi-cts")

not helpful for minimization problems

Alternatively, for domain $\mathbb{R}^n$ (in fact any Hausdorff space!)

f is lsc iff f is "closed" i.e., epi(f) is a closed set





epi(f) isn't closed here

Ex I_C is lsc iff C is a closed set

Extends classical thms C compact, f continuous $\Rightarrow$ f achieves its min and max (over C)

now: C compact, f lsc $\Rightarrow$ f achieves its min (over C)

Fact f cts iff f lsc and usc

Notation

$\Gamma(\mathbb{R}^n)$ is the set of all lsc and convex functions $f : \mathbb{R}^n \to [-\infty, +\infty]$

$\Gamma_0(\mathbb{R}^n) \subseteq \Gamma(\mathbb{R}^n)$ is the subset of $\Gamma(\mathbb{R}^n)$ that's also proper, $f:\mathbb{R}^n \to (-\infty,+\infty]$

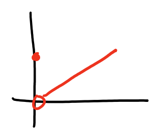
$\underbrace{\Gamma_0}_{}$ * our main class of functions for convex optimization

Ex $I_C \in \Gamma_0(\mathbb{R}^n)$ for some $C \subseteq \mathbb{R}^n$ iff C is nonempty, closed and convex.

The restriction to proper fcn is mild.
What about restricting to lsc functions? Also rather mild...
"weird things w/ convex functions can only involve boundaries (and $+\infty$)"

i.e., Thm 8.38 Bauschke + Combettes '17
If $f:\mathcal{H} \to (-\infty,+\infty]$ is proper and convex, then f is continuous at $x \in \mathrm{dom}(f)$ iff f is bounded above on a neighborhood of x.

Cor. 8.39 ... same setup...
If f is bounded above on some neighborhood, or
if f is lsc, or
if $\mathcal{H}$ is finite dimensional... then f is continuous on the interior of its domain, $\mathrm{int}(\mathrm{dom}(f))$.



A proper, convex function (not lsc), which isn't continuous (but is cts on interior)

so... if $f:\mathbb{R}^n \to \mathbb{R}$, i.e., has full-domain, then convex $\Rightarrow$ continuous