Convex Functions, part 5: Preserving convexity

Tuesday, February 2, 2021 6:45 PM

§3.2 BV '04

If you were to tattoo a math formula on your arm... don't!
But if you do, these rules would be a useful choice.

Rule 0 Non-negative weighted Sums. Used all the time.

If f is convex, so is $x \mapsto \alpha \cdot f(x)$ $\forall\, \alpha \geq 0$ important

If f, g both convex, so is $f + g$

Generally, if $f_1, \ldots, f_m$ are convex, $\alpha_i \geq 0$, then

$x \longmapsto \sum_i \alpha_i f_i(x)$ is convex too.

In fact, "infinite sums" are ok ... i.e., integrals

If $\forall\, y$, $f(\cdot, y)$ is convex, and $w(y) \geq 0$ $\forall y$,
then

$$x \longmapsto \int_\Omega f(x,y)\, w(y)\, dy \quad \text{is convex}$$

← arbitrary domain

Rule 1 / trick 1 The perspective preserves convexity

Def Let $f: \mathbb{R}^n \to \mathbb{R}$, then its perspective is $g: \mathbb{R}^{n+1} \to \mathbb{R}$

$$g(\vec{x}, t) = t \cdot f(\vec{x}/t) \quad \text{w/ domain } \{(\vec{x}, t) : \vec{x}/t \in \operatorname{dom}(f),\ t > 0\}$$

Fact $f: \mathbb{R}^n \to \mathbb{R}$ convex $\Rightarrow$ its perspective is convex

Ex $f(x) = \|x\|^2$ is convex

quadratic-over-linear that we already saw

Its perspective is $t \cdot \| \frac{x}{t} \|^2 = t \cdot \frac{\|x\|^2}{t^2} = \boxed{\frac{\|x\|^2}{t}}$ (if $t > 0$)

Ex $f(x) = -\log(x)$ is convex, $x \in \mathbb{R}_{++}$

Its perspective is $-t \cdot \log(x/t) = \underbrace{t \cdot \log(t) - t \cdot \log(x)}_{\text{relative entropy of } t,\, x}$, $x, y > 0$

More generally, the

Kullback-Leibler divergence

$$D_{KL}(u, v) := \underbrace{\sum_{i=1}^n u_i \log\left(\frac{u_i}{v_i}\right)}_{\text{also known as the relative entropy if } \sum u_i = \sum v_i = 1} - u_i + v_i$$

$u, v \in \mathbb{R}^n_{++}$

an example of a Bregman Divergence

↑ not quite a metric — often this is the weakest notion of a "distance" that makes sense.
Often used for optimization over the probability simplex

Rule 2 : special types of compositions

$f = h \circ g$ i.e. $f(x) = h(g(x))$, $\quad f: \mathbb{R}^n \to \mathbb{R}$

$g: \mathbb{R}^n \to \mathbb{R}^k, \; h: \mathbb{R}^k \to \mathbb{R}$

and if $g(x) = \infty$ is allowed,
define $h(\infty) = \infty$

Thm 1 f is convex if ① h is convex and

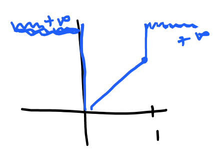
if $k=1$ $\begin{cases} \text{(2a) } h \text{ is nondecreasing}^* \text{ and } g \text{ is convex} \\ \text{or} \\ \text{(2b) } h \text{ is nonincreasing}^* \text{ and } g \text{ is concave} \end{cases}$

if $k>1$... modify (2a) so that h is nondecreasing in each argument
and if $g(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_k(x) \end{bmatrix}$ then each g_i convex

and modify (2b) similarly.

* For nonincreasing / decreasing, must take into account $\pm\infty$ values

i.e. $h(x) = x$ on $\mathrm{dom}(h) = [0,1]$ is not nondecreasing



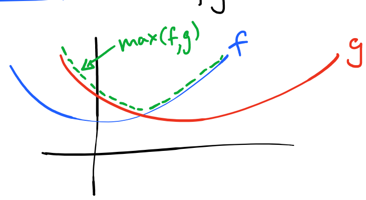
since $x=-1$, $y=0$
$f(x) = \infty$, $f(y) = 0$
$x \le y$ but $f(x) \cancel{\le} f(y)$

★ **Thm 2** $f = h \circ g$ is convex if h is convex
and g is affine.

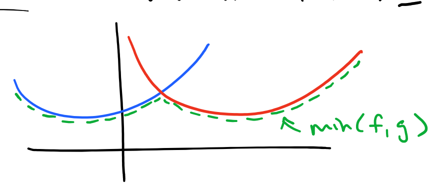
So important...
good candidate for your tattoo if you don't have any lovers / ex-lovers / political causes / favorite poems...

Rule 3 : min / max

3a : max If f, g both convex, $x \mapsto \max(f(x), g(x))$ is convex
since it's epigraph is $\mathrm{epi}(f) \cap \mathrm{epi}(g)$



NOT TRUE FOR MIN :



... in fact, works for max over more than just 2 fcn

even infinite (arbitrary! eg uncountable) max

$f(x) = \sup_{y \in \mathcal{A}} f(x;y)$ is convex as long as $f(\cdot\,;y)$ is convex $\forall y \in \mathcal{A}$

$\leftarrow$ arbitrary

ex: one way to prove convexity of the spectral norm

$$f(A) = \|A\| = \sup_{\|x\|=1} \|Ax\| \quad \text{since } \forall x,\ A \mapsto \|Ax\| \text{ is cvx}$$

3a: min

Sometimes... but need more restrictions

Thm If $f: \underbrace{\mathbb{R}^n}_{x} \times \underbrace{\mathbb{R}^m}_{y}$ is convex (i.e., $(x,y) \mapsto f(x,y)$ is jointly convex)

AND if $C \neq \emptyset$ is a convex set, then

New restrictions

$$g(x) = \inf_{y \in C} f(x,y) \quad \text{is convex.}$$

Ex: $\min(f_1(x), f_2(x))$ is not usually convex since this is like

taking $f(x,y) = \begin{cases} f_1(x) & y=1 \\ f_2(x) & y=2 \end{cases}$, and constraint $C = \{1, 2\}$ is non-convex.

Ex: The distance to a convex set is a convex function

Let $C \neq \emptyset$ be convex,

$$f(x) = \text{dist}(x, C) := \inf_{y \in C} \|x-y\|$$

jointly convex ?

convex ✓

$\boxed{Q1}$ Prove $(x,y) \mapsto \|x-y\|$ is (jointly) convex

$\boxed{A1}$ We know $z \mapsto \|z\|$ is convex.

Consider the linear operator $A(x,y) = x-y$

$$\text{i.e.,}\quad A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x - y$$

then $(x,y) \mapsto \|A(x,y)\|$ is convex

(check your tattoo)