HW help: proving polyhedrality

Wednesday, January 27, 2021 4:54 PM

How to prove a set $S \subseteq \mathbb{R}^n$ is polyhedral?

One, often difficult way, is to write $S = \{x : Ax \le b, \ Fx = d\}$. Usually we have tricks:

Tricks

1. Finite intersection of polyhedral sets is still polyhedral
2. "lift-and-project" $S \subseteq \mathbb{R}^n$, then "lift" to a larger space $\mathbb{R}^{n+k}$ by introducing new variables, and $S = \underset{\uparrow \text{ project}}{P}(T)$, T a polyhedral set in $\mathbb{R}^{n+k}$ $\Rightarrow$ S polyhedral

(Why does this work? T polyhedral $\overset{\text{via } 3}{\Rightarrow}$ $\exists \{z_1, \dots, z_m\} \subseteq \mathbb{R}^{n+k}$ s.t. $T = \text{conv}(\{z_1, \dots, z_m\})$

$\Rightarrow S = \text{conv}(\{Pz_1, \dots, Pz_m\}) \underset{\text{via } 3}{\Rightarrow} S$ polyhedral)

3. $\text{conv}(V)$, $\underbrace{|V|}_{\text{finite set.}} < \infty$, is polyhedral. So find vertices V

Why? $\text{conv}(\{v_1, \dots, v_m\}) = \{ \underbrace{x = \sum_{i=1}^m t_i \vec{v}_i}_{\text{encode as}} : \sum t_i = 1, \ t_i \ge 0 \}$

$$\begin{bmatrix} I & -A \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = [0] \quad \text{where } t = \begin{bmatrix} t_1 \\ \vdots \\ t_m \end{bmatrix}$$

$$A = \begin{bmatrix} v_1, \dots, v_m \end{bmatrix} \quad n \times m \text{ matrix}$$

encode as

$$\begin{bmatrix} 0 & \mathbb{1}^T \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 1$$

encode as

$$\begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \ge [0]$$

so $\begin{bmatrix} x \\ t \end{bmatrix}$ is polyhedral, so projection to just x is polyhedral (trick 2)

In my quick justifications, I used 3 to prove 2 (in fact I used an "iff" version of 3), and used 2 to prove 3 ... so not an actual proof)

From "Actually Doing it: Polyhedral Computations and its Applications"
by Jesus A. De Loera, UC Davis, Math dept

Definition 2.1.2 *The set of solutions of a system of linear inequalities is called a polyhedron. In its general form a polyhedron is then a set of the type*

$$P = \{x \in \mathbb{R}^d :< c_i, x > \leq \beta_i\}$$

$\{x : Ax \leq b, \ Fx = d\}$

for some non-zero vectors c_i in $\mathbb{R}^d$ and some real numbers β_i.

Definition 2.1.3 *A polytope is the convex hull of a finite set of points in $\mathbb{R}^d$.*

conv (V) vertices V

Theorem 2.3.1 (Weyl-Minkowski theorem) *Every polytope is a polyhedron. Every bounded polyhedron is a polytope.*

Lemma 2.2.1 *Given a bounded polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ then*

- *There is a translation of P that can be represented with in the form $\{x \in R^n : A'x \leq b' \text{ and } x \geq 0\}$.*
- *There is a polyhedron of the form $Q = \{x \in R^q : Bx = c, x \geq 0\}$ such that*
  - *the coordinate-erasing linear projection*

"lift-and-project"

$$\pi : \mathbb{R}^q \longrightarrow \mathbb{R}^n \ : \ x = (x_1, \dots, x_n, \dots x_q) \mapsto \pi(x) = (x_1, \dots, x_n)$$

  provides a bijection between Q and P.
  - *The bijection implies that $\pi(Q \cap \mathbb{Z}^q) = P \cap \mathbb{Z}^n$.*
  - *$y \in Q$ is an extreme point if and only if $\pi(x)$ is an extreme point of P*
  - *$dim(Q) = dim(P)$.*