Game Theory connections

Monday, March 1, 2021 10:09 AM

Finite, 2-person, 0-sum game : "matrix game" (not Prisoner's Dilemma)

Minimax Thm of J. von Neumann '28 (Brouwer Fixed Pt)

can use LP duality

Ex: Rock-Paper-Scissors

payoff matrix

P = paper
S = scissors
R = rock

$$\begin{array}{c|ccc} & \multicolumn{3}{c}{\boxed{\max}\ \text{Player 2}\ v} \\ \boxed{\min}\ \text{Player 1}\ u & P & S & R \\ \hline P & 0 & 1 & -1 \\ S & -1 & 0 & 1 \\ R & 1 & -1 & 0 \end{array}$$

$\underbrace{\qquad}_{P}$

$u_i = 2$ means Player 1 chooses Scissors

Payoff is

$u^T P v$

This game is "fair" (it's value is 0)

Guaranteed for all symmetric games ($P = -P^T$)

Generally,

$P \in \mathbb{R}^{n \times m}$

player 1 "pays" $P_{k\ell}$ to player 2

if player 1's move is $k \in \{1, 2, \dots, n\}$ (wants to minimize payoff)
player 2's move is $\ell \in \{1, 2, \dots, m\}$ (wants to maximize payoff)

Allow random ("mixed" strategies): u_i = Probability (move $k = i$)

v_j = Prob. (move $\ell = j$)

So expected payoff is $\sum_k \sum_\ell u_k v_\ell P_{k\ell} = u^T P v$

Define the probability simplex $\Delta = \{ u : u \geq 0, \sum u_i = 1 \}$ (dimension from context)

Case Player 2 knows player 1's strategy!

i.e., She knows u

Then easy decision: choose $v \in \underset{v \in \Delta}{\operatorname{argmax}}\ u^T P v$

which is easy to solve (let $j = \max_i (P^T u)_i$ and

$v_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$)

If Player 1 knows Player 2 will find out his strategy, then assuming Player 2 plays optimally, he should select

u to minimize her payoff: N.B. Solving P_1^* is a LP

$$p_1^* = \min_{u \in \Delta} \left(\max_{v \in \Delta} \; u^T P v \right) \qquad \min_{\substack{u \in \Delta \\ v \in \Delta \\ t \in \mathbb{R}}} \; t \quad \text{s.t. } u^T P v \le t$$

So p_1^* is best case for player 1 if player 2 can see his strategy

Case: player 1 knows player 2's strategy

...

$$p_2^* = \max_{v \in \Delta} \underbrace{\min_{u \in \Delta} \; u^T P v}_{LP}$$

Conclusion

Intuitively, for player 2, knowing player 1's strategy (CASE 1) is helpful: $p_1^* \geq p_2^*$

(or, CASE 2, also implies $p_1^* \geq p_2^*$) "WEAK DUALITY"

... the counterintuitive result is that $p_1^* = p_2^*$

i.e., no benefit (if opponent plays rationally) of knowing their strategy?

$p_1^* = p_2^*$ follows because Δ has non-empty interior ($m, n \geq 2$ else it's not a fun game)

So Slater's holds $\Rightarrow$ STRONG DUALITY