Convex functions, part 3: Lipschitz gradient, etc.

Sunday, January 31, 2021 3:49 PM

mostly details not in BV'04

Recall a function $F: \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous if $\exists\, L > 0$ st. $\forall x, y \in \text{dom}(F)$,

$$\| F(x) - F(y) \| \le L \cdot \| x - y \|.$$

If $\underline{F'}$ exists, then $\underline{|||F'|||} \le L \implies F$ is Lipschitz continuous

appropriate operator norm, usually $|\cdot|$ if 1D or spectral norm

What do we mean by this?

The Jacobian, where if $F(x) = \begin{bmatrix} F_1(x) \\ \vdots \\ F_m(x) \end{bmatrix}$, $F_i: \mathbb{R}^n \to \mathbb{R}$,

$$J_{ij} = \frac{\partial F_i(x)}{\partial x_j} \quad \left(\text{or } \frac{\partial F_j(x)}{\partial x_i}, \text{ I can never remember, and conventions aren't consistent anyhow} \right)$$

$m \times n$ (or $n \times m$) matrix.

In optimization, "Jacobian" is often confusing, since it's unclear what "F" is.

Ex: $f: \mathbb{R}^n \to \mathbb{R}$, $f(x)$ a scalar

$\nabla f: \mathbb{R}^n \to \mathbb{R}^n$, "gradient", $\nabla f(x)$ a vector, "operates" on directions d like $\langle \nabla f(x), d \rangle$ (so a linear operator)

$\nabla^2 f: \mathbb{R}^n \to \mathbb{R}^{n \times n}$, "Hessian", $\nabla^2 f(x)$ a matrix, "operates" on d like $\langle d, \nabla^2 f(x)\, d \rangle$

(symmetric matrix)

$(\nabla f(x))_i = \frac{\partial f}{\partial x_i}$

$(\nabla^2 f(x))_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$

... so Jacobian of $F = f$ is the gradient (transposed)

... but Jacobian of $F = \nabla f$ is the Hessian

Fact Suppose $f \in C^2(\mathcal{U})$ for some open set $\mathcal{U} \subseteq \mathbb{R}^n$, and f convex, then

$$\left(\nabla f \text{ is } L\text{-Lipschitz cts. on } \mathcal{U} \right) \text{ iff } \left(\forall x \in \mathcal{U},\ \nabla^2 f(x) \preceq L I \right)$$

i.e., all eigenvalues $(\nabla^2 f(x)) \le L$

$\implies |||\nabla^2 f(x)||| \le L$

Fact Suppose $f \in C^2(\mathcal{U})$ for some open set $\mathcal{U} \subseteq \mathbb{R}^n$, then

$$\left(f \text{ is } \mu\text{-strongly convex on } \mathcal{U} \right) \text{ iff } \left(\forall x \in \mathcal{U},\ \mu I \preceq \nabla^2 f(x) \right)$$

(w/ respect to $\|\cdot\|_2$)

(need $\mu > 0$. If $\mu = 0$ this is plain old convexity)

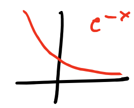
So... one of our common assumptions will be ∇f is L Lipschitz ($\nabla^2 f \preceq LI$)

and, a bit less often, also assume strong convexity ($\nabla^2 f \succeq \mu I$)

$\boxed{Q1}$ Is $f(x) = e^{-x}$ convex, strictly convex, strongly convex?
Is f' Lipschitz?

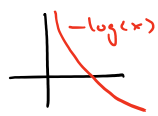
$\boxed{Q2}$ Is $f(x) = \begin{cases} -\log(x) & x > 0 \\ +\infty & x \le 0 \end{cases}$ convex....? f' Lipschitz?

$\boxed{A1}$ e^{-x} on $\mathbb{R}$ is strictly convex (hence convex) but <u>not</u> strongly cvx ($f'(x) = e^{-x} \to 0$ as $x \to \infty$). It <u>is</u> strongly convex on the domain $(-\infty, R]$ for any $R < \infty$.



Similarly, f' isn't Lipschitz on $\mathbb{R}$ but it is Lipschitz on $[-R, \infty)$ $\forall R < \infty$

$\boxed{A2}$ $-\log(x)$ is convex and strictly convex



since $f' = \frac{-1}{x}$ is monotone

... but <u>not</u> strongly convex unless we look at $(-\infty, R]$ again.

For f' Lipschitz, it isn't on $(0, \infty)$ but it <u>is</u> on $[\delta, \infty)$ $\forall \delta > 0$. $\Big\}$ Can create problems for some algorithms if they converge to $x = 0$

<u>Ex</u> $f(x) = \frac{1}{2}\|x\|_2^2$, $\nabla f(x) = x$, $\nabla^2 f(x) = I$

$\Rightarrow L = 1$, $\mu = 1$. Only function w/ this property.

<u>NICEST FUNCTION EVER</u>:

Calculus: $f(x) = e^x$

Statistics: $f(x) = e^{-x^2/2}$ or multivar. version

Optimization: $f(x) = \frac{1}{2}x^2$ (= negative log-likelihood of Gaussian!)

<u>Def</u> The condition # of f is $\kappa_f = L/\mu$. $\kappa_f \approx 1$ good

$\kappa_f \approx \infty$ bad

Why these assumptions?

Taylor's Thm: $f(y) = f(x) + f'(x)\cdot(y-x) + \frac{1}{2}f''(\xi)(y-x)^2$

for some $\xi \in [x,y]$ (or in $[y,x]$). Similar in higher-dim.

if $f''(\xi) \le L \quad \forall \xi$, then

$$f(y) = \ldots \le f(x) + f'(x)(y-x) + \frac{1}{2}L(y-x)^2.$$

so...

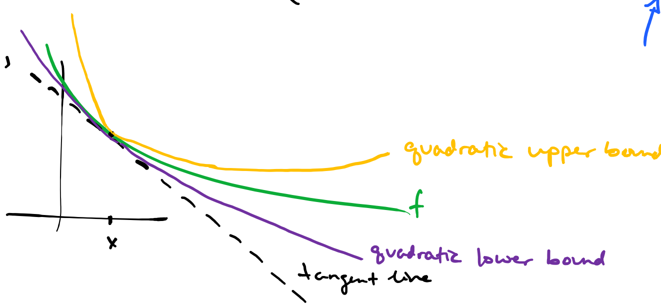
Theorem If ∇f is L-Lipschitz and f is μ-strongly convex (and take $\mu=0$ if just convex)

then $\forall x,y \in \text{dom}(f)$,

$$\frac{\mu}{2}\|y-x\|^2 \le f(y) - \left(f(x) + \langle \nabla f(x), y-x \rangle \right) \le \frac{L}{2}\|y-x\|^2$$

This inequality is sometimes called "The Descent Lemma"

Graphically,



Usually f is complicated, but now we can "sandwich" it between a quadratic upper bound and a quadratic lower bound (if strongly cvx, $\mu>0$) or a linear lower bound (if just cvx, $\mu=0$)

and quadratics are easy to work with, eg. easy to minimize in closed form, etc.

More properties

eg. f convex $\Rightarrow$ ∇f monotone, meaning $\langle x-y, \nabla f(x) - \nabla f(y) \rangle \ge 0$

These can be strengthened w/ our $\mu>0$ and L assumptions

See github class website $\boxed{\text{Handouts / StrongConvexityLipschitz.pdf}}$