§5.1 B+V: Lagrange Dual Function

(P) Primal problem

$$\min f_0(x)$$
$$f_i(x) \le 0 \quad i \in [m]$$
$$h_i(x) = 0 \quad i \in [p]$$

Assume non-empty domains

Let p^* be the optimal value, $p^* = +\infty$ is possible (means infeasible)

= Not assuming convexity =

Def The Lagrangian of (P) is a function $\mathcal{L}$

$$\mathcal{L}(x, \nu, \lambda) := f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i \cdot h_i(x)$$

primal variable

Lagrange multipliers or "dual variables"

⚠ Note: you can keep some constraints in f_0 (i.e., as indicator functions). This will lead to a different dual problem (still valid), sometimes helpful

Ex: $\min f(x) + g(x)$

OR $\min f(x) + g(y)$
s.t. $x = y$

Def The dual function is

$$g(\lambda, \nu) := \inf_x \mathcal{L}(x, \lambda, \nu) \quad (-\infty \text{ is possible})$$

Properties of g

Recall: $h(y) = \sup_{x \in \Omega} f(y,x)$ is convex if f is convex in y (Ω arbitrary)

$h(y) = \inf_{x \in \Omega} f(y,x)$ is convex if f is jointly convex in x and y, and Ω is convex

or...

$g(y) = \inf_{x \in \Omega} f(y,x)$

is concave if f is concave in y

this is the form of our dual function

$\mathcal{L}$ is affine in λ, ν, hence concave, hence g is concave even if f isn't convex.

Def The dual problem is

$$(D) \quad d^* = \max_{\substack{\lambda \geq 0 \\ \nu}} g(\lambda, \nu)$$

and this is a convex optimization problem!

(even if (P) wasn't)

Why would you solve the dual?

Take any dual feasible points λ, ν
$\underbrace{\text{feasible}}_{\text{i.e. } \lambda \geq 0}$

Let $\tilde{x}$ be any primal feasible point
(eg. the optimal one!)

$$g(\lambda,\nu) := \inf_x \mathcal{L}(x,\lambda,\nu) \leq \mathcal{L}(\tilde{x},\lambda,\nu)$$

$$= f_0(\tilde{x}) + \underbrace{\sum_i \underbrace{\lambda_i}_{\geq 0} \underbrace{f_i(\tilde{x})}_{\leq 0}}_{\leq 0} + \sum_i \nu_i \underbrace{h_i(\tilde{x})}_{=0}$$

via feasibility

$$\leq f_0(\tilde{x}).$$

ie. $\forall$ feasible λ, ν
and $\forall$ feasible x,

$$g(\lambda,\nu) \leq f_0(x)$$

so...

$$\lambda, \nu \text{ feasible}^* \Rightarrow g(\lambda,\nu) \leq p^*$$

The dual problem is looking for the largest lower bound on p^*.

We just proved the weak duality theorem: $d^* \leq p^*$

* Sometimes we call them feasible if $\lambda \geq 0$ and it's a useful bound, meaning $g(\lambda,\nu) \neq -\infty$

We can "sandwich" p^*: d^* below (a convex problem),
and $f_0(\tilde{x})$ for any feasible $\tilde{x}$

Ex : Dual problem of a LP

(P) $\min \langle c, x\rangle$
$x \geq 0 \quad \leftarrow$ i.e. $f_i(x) = -x_i \leq 0$
$Ax = b \quad \leftarrow h_i(x) = a_i^T x - b_i$

then the Lagrangian is

$$\mathcal{L}(x, \lambda, \nu) = \langle c, x\rangle + \sum_i \lambda_i f_i(x) + \sum_i \nu_i h_i(x)$$
$$= \langle c, x\rangle - \sum_i \lambda_i x_i + \nu^T(Ax - b)$$

and

$$g(\lambda, \nu) = -\nu^T b + \underbrace{\min_x \langle \underbrace{c + A^T\nu - \lambda}_{\text{is } 0}, x\rangle}_{\min = -\infty \text{ unless}}$$

$$= -\nu^T b$$
w/ the constraint $c + A^T\nu - \lambda = 0$

so "dual feasible" means
① $\lambda \geq 0$
② $c + A^T\nu - \lambda = 0$

can eliminate λ and just write $c + A^T\nu \; (= \lambda) \geq 0$

so

(D) $\max\limits_{\nu, \lambda} \; -\langle b, \nu\rangle$
$c + A^T\nu = \lambda$
$\lambda \geq 0$

OR EQUIVALENTLY

$-\min\limits_{\nu} \langle b, \nu\rangle$
$A^T\nu + c \geq 0$

Also a LP!

DUAL OF A LP IS A LP!

Explicit example of duality

(P) $\min_{\substack{x \geq 0 \\ x \in \mathbb{R}^2}} \quad 3x_1 + 2x_2 =: f(x)$

s.t. $x_1 + 2x_2 \geq 5$

$x_2 \leq 2$

I can prove $p^* \leq 7$. How? Observe $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is feasible and $f(x) = 7$.

Can you do better? (How to prove you can or can't?)

Observe:

$$f(x) = 3x_1 + 2x_2 = \underbrace{2x_1}_{\geq 0} + \underbrace{(x_1 + 2x_2)}_{\geq 5} \geq 5$$

if feasible

So $p^* \geq 5$

Even better:

$(x_1 + 2x_2 \geq 5) \times 3$

$(x_2 \leq 2) \times -4$

$\Rightarrow$

$$\begin{array}{r} 3x_1 + 6x_2 \geq 15 \\ + \quad -4x_2 \geq -8 \\ \hline \underbrace{3x_1 + 2x_2}_{f(x)} \geq 7 \end{array}$$

combine

turns out these are the optimal dual variables

so $p^* \geq 7$

hence $7 \leq p^* \leq 7$ so $p^* = 7$ and $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is optimal!

The dual is

(D) $\max_{\substack{y_1 \geq 0 \\ y_2 \leq 0}} \quad 5y_1 + 2y_2$

s.t. $y_1 \leq 3$

$2y_1 + y_2 \leq 2$

How to easily find dual of a LP

Standard form is done for you: A is $m \times n$

(P) $\max_{x \in \mathbb{R}^n} \langle c, x \rangle$, $x \geq 0$, $Ax = b$

(D) $\min_{y \in \mathbb{R}^m} \langle b, y \rangle$, $y \geq 0$, $A^T y = c$

--- but what if you don't feel like putting it in standard form?

Arthur Benjamin's "SOB" method, SIAM Review 1995

Rules:

max → min

min → max

variables → constraints

constraints → variables

objective and RHS switch places

and transpose equations

and match "type" of constraint:

primal: direct constraints on variable

$x_i \geq 0$	S	(sensible)
no constraint	O	(odd)
$x_i \leq 0$	B	(bizarre)

and other constraints

$a_i^T x \leq b_i$	S
$a_i^T x = b_i$	O
$a_i^T x \geq b_i$	B

flip if you are minimizing

Business motivation: maximize profit, non-neg. variables, budget constraints ("sensible" case)

Then when you create the dual, the dual constraint is the same (S-O-B) type as primal variable

and dual variable is the same type as primal constraint

S-O-B example

$$\min_{x \in \mathbb{R}^3} \quad 3x_1 + 5x_2 + 1x_3$$

$n = 3$ dimensions

(P) s.t.

$$x_1 + x_2 + x_3 = 2 : y_1 \quad O$$

$$2x_1 \quad - 3x_3 \le 0 : y_2 \quad B$$

$m = 2$ constraints

$$\underbrace{x_1 \ge 0}_{S} \quad \underbrace{x_2 \ge 0}_{S} \quad \underbrace{x_3 \in \mathbb{R}}_{O}$$

$$m \begin{array}{c} n \\ \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -3 \end{bmatrix} \end{array}$$

So

(D)

$$\max_{y \in \mathbb{R}^2} \quad 2y_1 \; (+ 0 \cdot y_2)$$

s.t.

$$y_1 + 2y_2 \le 3 \quad S$$

$$y_1 \le 5 \quad S$$

$$y_1 - 3y_2 = 1 \quad O$$

$$n \begin{array}{c} m \\ \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -3 \end{bmatrix} \end{array}$$

$$\underbrace{y_1 \in \mathbb{R}}_{O}, \quad \underbrace{y_2 \le 0}_{B}$$

Back to earlier example

(P)

$$\min \quad 3x_1 + 2x_2$$

$$x_1 + 2x_2 \ge 5 : S$$

$$x_2 \le 2 : B$$

$x_1 \ge 0$ S

$x_2 \ge 0$ S

(D)

$$\max \quad 5y_1 + 2y_2$$

$$y_1 \le 3 : S$$

$$2y_1 + y_2 \le 2 : S$$

$y_1 \ge 0$ S

$y_2 \le 0$ B