Supplement: Variational Inequalities

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See "Finite-Dimensional Variational Inequalities and Complementarity Problems" vol 1 by Francisco Facchinei and Jong-Shi Pang, 2003.

Def Let $C \subseteq \mathbb{R}^n$ be closed (often assumed to be convex) and $F: C \to \mathbb{R}^n$ continuous, the Variational Inequality (VI) is

$$\text{Find } x \in C \text{ such that } \langle F(x), y-x \rangle \geq 0 \quad \forall y \in C$$

see below

*This is more general than optimization: we can recover smooth optimization problems $\min_{x \in C} f(x)$ by setting $F = \nabla f$, so the VI is Euler's inequality

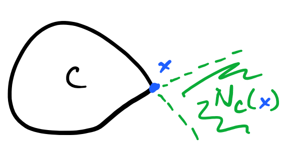
Although... in optimization, we ask for global min., and need convexity. The VI doesn't distinguish local/global min (or even min from max or other stationary points). So convexity of C matters, but not that of f (if $F = \nabla f$)

An even more general problem is that of monotone inclusions

T is monotone if $\forall x, y$ $\langle Tx - Ty, x-y \rangle \geq 0$

since the VI can be cast as finding $0 \in \underbrace{F(x) + N_C(x)}_{\text{monotone under certain conditions}}$

Normal cone $N_C(x) = \{ d \in \mathbb{R}^n : \langle d, y-x \rangle \leq 0 \ \forall y \in C \}$



An important type of VI:

Def If C is a cone, the VI can be reformulated as a "complementary problem" (CP):

$$\text{Find } x \in C \text{ s.t. } x \perp F(x) \text{ and } F(x) \in C^*$$

dual cone $\{ d : \langle d, x \rangle \geq 0 \ \forall x \in C \}$

and in particular,

Def If $C = \mathbb{R}^n_+$ and F is affine ($F(x) = \underset{\text{square matrix}}{A} \cdot x + b$) then the VI becomes a "linear complementary problem" (LCP):

reminiscent of Farkas Lemma

$$\text{Find } x \geq 0 \text{ s.t. } x \perp \underbrace{A \cdot x + b}_{y}, \text{ and } \underbrace{A \cdot x + b}_{y} \geq 0$$

i.e., Find $\underbrace{x \geq 0, y \geq 0}$ s.t. $x \perp y$ and $y = Ax + b$

$x \perp y$ means $\sum_i x_i y_i = 0$, and if $x_i \geq 0$ and $y_i \geq 0$

then $\sum_i x_i y_i = 0 \Rightarrow x_i y_i = 0$.

i.e., either $x_i = 0$ or $y_i = 0$] hence the name "complementarity"

* VI (even LCP) are a strict generalization of minimization

i.e., if $\nexists f$ s.t. $F = \nabla f$, then the VI isn't equivalent to a minimization problem

1.3.1 Facchinei / Pang

Thm If $F : \mathcal{U} \to \mathbb{R}^n$ is continuously differentiable on an open set $\mathcal{U} \subseteq \mathbb{R}^n$, then

$$\left(\exists\, f : \mathcal{U} \to \mathbb{R} \text{ s.t. } \nabla f = F \text{ on } \mathcal{U} \right) \text{ iff } \left(\underset{JF}{\text{Jacobian of}}\ F \text{ is symmetric on } \mathcal{U} \right)$$

i.e., $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}$

So... back to LCP

$F(x) = Ax + b$, A square

$JF(x) = A$, so if $A \neq A^T$, then the LCP cannot be recast as a minimization problem

Common example:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ -A_{12}^T & A_{22} \end{bmatrix}$$

skew symmetric. $A_{11}, A_{22} \succeq 0$ even sometimes

so $A \neq A^T$

(or more generally, need not be linear) Not equiv. to optimization.

This is a convex-concave saddle-point problem: a pair of (competing) optimization problems.

if $A_{11} \succeq 0$, $A_{22} \preceq 0$

Ex Nash equilibria for game theory

or

Generative Adversarial Networks (GAN) in deep learning

probably not convex-concave though

- Sometimes we write primal-dual solution to an optimization problem as a saddle-point problem (eg via KKT conditions)

--- but not all saddle-point problems can be derived from optimization.