# Existence and Uniqueness of Minimizers

Tuesday, February 9, 2021 10:40 PM

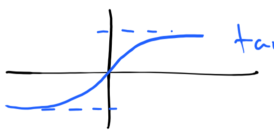
i.e., fancier versions of "the infimum of a continuous function over a compact set is achieved"

## Existence

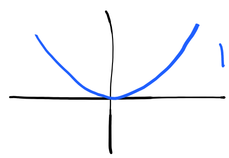
**Def** $f: \mathbb{R}^n \to [-\infty, \infty]$ is **coercive** if $\lim_{\|x\| \to \infty} f(x) = \infty$

(i.e., it grows!)

**Fact:** $f$ coercive iff all sub-level sets $(\forall \alpha)$ $\{x : f(x) \le \alpha\}$ are bounded



tanh
NOT COERCIVE



$|x|^\alpha$ $\alpha > 0$ COERCIVE

**Fact:** If $f \in \Gamma_0(\mathbb{R}^n)$, $f$ coercive iff $\exists\, \alpha$ s.t. $\{x : f(x) \le \alpha\}$ is non-empty and bounded

not true in any $\mathcal{H}$

⚠ convex + coercive need not be coercive

**Main result** Let $C$ be closed, convex, $f \in \Gamma_0(\mathbb{R}^n)$ (or $\mathcal{H}$), $C \cap \text{dom}(f) \ne \emptyset$ (feasible)

then $\min_{x \in C} f(x)$ exists (i.e., $\exists$ a minimizer $x^*$)

if either ① $f$ is coercive

or ② $C$ is bounded.

## Uniqueness

**Main result** Let $C$ be convex, $f$ is proper and convex, $C \cap \text{dom}(f) \ne \emptyset$,

then there is at most one minimizer (uniqueness) of

$\min_{x \in C} f(x)$ if either

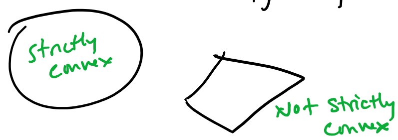
① $f$ is strictly convex (e.g., $C = \mathbb{R}^n$ is ok)

or ② $C \cap \underset{\text{(unconstrained)}}{\text{argmin}}\, f = \emptyset$ — not $\ne$

and $C$ is strictly convex

A set $C$ is strictly convex if there are no line segments on its boundary, i.e.,

$$\forall x, y \in C, \quad x \neq y, \qquad \frac{x+y}{2} \in \operatorname{int}(C)$$




## Summary

$f \in \Gamma_0(\mathbb{R}^n)$, $\quad f$ strictly convex $\Rightarrow$ at most 1 minimizer

$\quad f$ coercive $\Rightarrow$ at least 1 minimizer

If $f$ is strongly convex, $\Rightarrow$ it's also strictly convex

$\Rightarrow f(y) \geq f(x) + \langle \nabla f(x), y-x \rangle + \mu/2 \|y-x\|^2$

i.e., choose $x=0$, $\|y\| \to \infty$

$\Rightarrow f(y) \to \infty \ldots$
so coercive

$\therefore$

$$\boxed{f \text{ strongly convex} \Rightarrow \exists! \text{ minimizer.}}$$ Nice!

Convex functions are nice, strongly convex functions are very nice