More on SDPs

Recall a semi-definite program (SDP) in standard form is:

$$\min_{X \in S^n} \ \langle C, X \rangle \quad = tr(C^T X) \ \text{aka} \ C \cdot X$$

$$\text{s.t.} \ \langle A_i, X \rangle = b_i, \quad i = 1, \dots, p$$

$$X \succeq 0 \quad \text{aka} \ X \in S^n_+$$

S^n = symm. or Hermitian $n \times n$ matrices

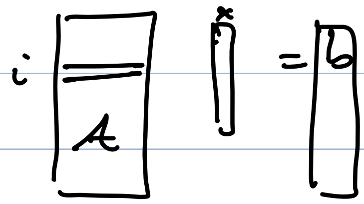
Another way to write this:

How to characterize a linear function on $\mathbb{R}^{n \times n}$?

Simple: "vectorize" input $\underbrace{X}_{\mathbb{R}^{n\times n}} \mapsto \underbrace{vec(X)}_{\mathbb{R}^{n^2 \times 1}} =: x$ lowercase

then let $\mathcal{A}$ be a $p \times n^2$ matrix:

$$\mathcal{A}(vec(X)) = b$$



means

$$[\underbrace{\mathcal{A}(vec(X))}]_i = b_i$$

$$= a_i^T \cdot vec(x)$$

$$= \langle mat(a_i), X \rangle$$

$$= \langle A_i, X \rangle$$

each row of $\mathcal{A}$ is $vec(A_i)$

Linear Matrix Inequality (LMI)

Closely related to SDPs (we'll see later that they are the duals of SDPs)

$$\min_{x \in \mathbb{R}^p} \ \langle c, x \rangle \quad \text{s.t.} \ \sum_{i=1}^{p} x_i \cdot A_i + G \preceq 0$$

vector

$\in S^n$

and $A_0 \cdot x = b$

if A_i $(i \in [p])$ and G are diagonal, this is a LP!

In a SDP, if we force X to be diagonal
(can be enforced via linear constraints), it's a LP!
Since $\text{diag}(x) \succeq 0$ iff $x \geq 0$

Can also cast SOCPs (hence QPs, QCQPs)
as SDPs using the Schur Complement
(cf. problem 4.40 and § A.5.5 in Boyd + Vandenberghe)
i.e.

$A \in S^r_{++}$, $C \in S^s$, $B \in \mathbb{R}^{r \times s}$
then

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \quad \text{iff} \quad C - B^T A^{-1} B \succeq 0$$

Product cones

Often we want $X \succeq 0$ and $Y \succeq 0$

So can combine as $Z = \begin{bmatrix} X & W \\ W^T & Y \end{bmatrix}$

and enforce $Z \succeq 0$, $W = 0$

(mathematically OK, but computationally a bad idea.
Good software can handle multiple conic constraints
explicitly)

ex: $\underbrace{\text{SeDuMi, SDPT3, Mosek}}_{\text{SDPs}}$, Gurobi

Aside (optional)

Matrix Inversion Lemma

aka Sherman-Morrison-Woodbury Formula

Very useful!

$$\big(\underbrace{A + UCV^T}_{n\times n}\big)^{-1} = A^{-1} - A^{-1}U\big(\underbrace{C^{-1} + V^TA^{-1}U}_{p\times p}\big)^{-1}V^TA^{-1}$$

n×n

via Schur complement on

$$\begin{array}{c|cc|c} & n & p & \\ \hline n & A & U & n \\ p & V^T & -C^{-1} & p \\ \hline & n & p & \end{array}$$

often $p=1$

Aside (optional) Saddle pt. systems

Block linear systems of the form

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} \quad \text{i.e.} \quad \begin{aligned} Ax + By &= c \\ B^Tx &= d \end{aligned}$$

One way to solve: $x = A^{-1}(c - By)$

$$\underbrace{B^TA^{-1}B}_{\text{often positive definite}}\, y = B^TA^{-1}c - d$$

often A is diagonal

usually $A \in S^n$