Mathematical Formulations

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This documents defines mathematical formulations referenced in the paper:

1 Mixed-Integer Linear Program for Joint

Minimizing Maximum Link Utilization

Minimize MLU

Constants and Variables

$$\begin{cases} M := \max\{\sum_{(s,t,d)\in\mathcal{D}} d, 2|E|\} & \triangleright \text{"Large M"} \\ \mathcal{S} := \{(p,q) \in V^2 \mid p \neq q\} & \triangleright \text{Segments} \end{cases} \tag{1}$$

$$\begin{cases} S_\delta^{(p,q)}, x_\ell^t \in \{0,1\}, \\ f_\ell^{(p,q)}, D^{(p,q)}, dist_p^q \geq 0, \\ 0 < \omega_\ell \leq 1 \end{cases} \qquad \begin{cases} (p,q) \in \mathcal{S}, \ell \in E, \\ \delta = (s,t,d) \in \mathcal{D} \end{cases} \tag{2}$$

Flow Conservation

$$\sum_{\ell=(*,v)\in E} f_\ell^{(p,q)} - \sum_{\ell=(v,*)\in E} f_\ell^{(p,q)} = \begin{cases} 0 & v \neq p,q \\ -D^{(p,q)} & v = p \\ D^{(p,q)} & v = q \end{cases} \qquad \begin{cases} (p,q) \in \mathcal{S}, \\ v \in V \end{cases} \tag{3}$$

$$\sum_{(p,q)\in\mathcal{S}} f_\ell^{(p,q)} \leq MLU \cdot C_\ell \qquad \ell \in ELP : capacity \tag{4}$$

Waypoint Setting

$$D^{(p,q)} = \sum_{\delta=(s,t,d)\in\mathcal{D}} S_\delta^{(p,q)} \cdot d \qquad (p,q) \in \mathcal{S} \tag{5}$$

$$\sum_{(p,q)\in\mathcal{S}, q=v} S_D^{(p,q)} - \sum_{(p,q)\in\mathcal{S}, p=v} S_D^{(p,q)} = \begin{cases} 0 & v \neq s,t \\ -1 & v = s \\ 1 & v = t \end{cases} \qquad \begin{cases} \delta = (s,t,d) \in \mathcal{D}, \\ v \in V \end{cases} \tag{6}$$

$$\sum_{(p,q)\in\mathcal{S}} S_D^{(p,q)} \leq W + 1 \qquad \delta = (s,t,d) \in \mathcal{D} \tag{7}$$

Even Splitting

$$f_\ell^{(p,q)} \leq M \cdot x_\ell^q \qquad (p,q) \in \mathcal{S}, \ell \in E \tag{8}$$

$$\sum_{\ell=(v,*)\in E} x_\ell^t \leq \text{SF} \qquad v,t \in V, v \neq t \tag{9}$$

$$\begin{cases} \sum_{p:(p,t)\in\mathcal{S}} f_\ell^{(p,t)} \leq f_v^t \\ f_v^t - \sum_{p:(p,t)\in\mathcal{S}} f_\ell^{(p,t)} \leq M \cdot (1 - x_\ell^t) \end{cases} \qquad \begin{cases} v,t \in V, v \neq t, \\ \ell = (v,*) \end{cases} \tag{10}$$

Weight Setting

$$\begin{cases} dist_u^t \leq dist_v^t + \omega_\ell \\ dist_v^t - dist_u^t + \omega_\ell \leq M \cdot (1 - x_\ell^t) \\ 1 - x_\ell^t \leq M \cdot (dist_v^t - dist_u^t + \omega_\ell) \end{cases} \qquad \begin{cases} \ell = (u,v), \\ t \in V, t \neq u \end{cases} \tag{11}$$

Variables

Line (**??**) defines two LP constants. The constant M is a large number used in place of ∞. The set $\mathcal{S}$ consists of all node pairs, where each pair represents a possible segment. The variables defined at (**??**) are described as follows.

SF: the splitting factor. A flow splits into at most SF equal size parts.

W: maximum number of waypoints allowed per demand.

M: a constant large enough.

x_ℓ^t: binary variables indicating whether link ℓ is on a shortest path to node t.

$f_\ell^{(p,q)}$: the fractional amount of flow of the segment (p, q) assigned to the link ℓ.

$f_v^{(p,q)}$: the total flow of segment (p, q) leaving node v.

$S_\delta^{(p,q)}$: binary variable indicating whether the segment (p, q) is active for the demand $\delta = (s, t, d) \in \mathcal{D}$.

$dist_v^t$: shortest path weight from v to t.

ω_ℓ: the weight of the link ℓ.

Constraints

- (**??**)-(**??**) enforce flow conservation and capacity rules for the flow each segment separately. The flow size depends on sizes of demands for which the segment is active, hence a variable.
- (**??**) for each segment $(p, q) \in \mathcal{S}$, determines the total flow size that must traverse between the endpoints of the segment. The size of a demand is added to this quantity only if the segment is active for that demand.

  (**??**) ensures segments active for any demand are consecutive and connect the endpoints of that demand. (**??**) limits the number of active segments for a demand, i.e., the number of waypoints. If $W = 0$ then the MILP is the formulation of WPO.
- (**??**) ensures a link will be on a shortest path to q if it is assigned a positive flow for any segment ending at q.
- (**??**) limits the splitting factor, that is, the number of parts a flow is allowed to split into. E.g., if SF $= 1$ then no flow splits in any feasible solution. (**??**) ensures a flow splits evenly on all adjacent links that are set to be on a shortest path to t.
- (**??**) determines the weight of each link that is set be on a shortest path, by determining a weighted distance ("node potentials") for its endpoints.

2 Maximal Multi-commodity Flow (MCF)

Assume a set of source-terminal pairs is given in $\mathcal{D}$. The following LP computes a demand size for each pair such that the sum of all demands is maximized.

$$\text{Maximize} \sum_{(s,t)\in\mathcal{D}} d^{(s,t)} \tag{12}$$

$$f_\ell^{(s,t)}, d^{(s,t)} \geq 0 \qquad (s,t) \in \mathcal{D}, \ell \in E \tag{13}$$

$$\sum_{\ell=(*,v)\in E} f_\ell^{(s,t)} - \sum_{\ell=(v,*)\in E} f_\ell^{(s,t)} = \begin{cases} 0 & v \neq s,t \\ -d^{(s,t)} & v = s \\ d^{(s,t)} & v = t \end{cases} \qquad \begin{cases} (s,t) \in \mathcal{D}, \\ v \in V \end{cases} \tag{14}$$

$$\sum_{(s,t)\in\mathcal{D}} f_\ell^{(s,t)} \leq C_\ell \qquad \ell \in E \tag{15}$$

3 Maximum Concurrent Multi-commodity Flow

Assume demands are given and $d^{(s,t)}$ denotes the size of the demand from s to t for each source-terminal pair (s,t). The next LP computes a scale factor F that when we multiply to every demand size, the sum of new demand sizes equals the size of the maximum concurrent MCF.

$$\text{Maximize } F \tag{16}$$

$$f_\ell^{(s,t)}, F \geq 0 \qquad (s,t) \in \mathcal{D}, \ell \in E \tag{17}$$

$$\sum_{\ell=(*,v)\in E} f_\ell^{(s,t)} - \sum_{\ell=(v,*)\in E} f_\ell^{(s,t)} = \begin{cases} 0 & v \neq s,t \\ -d^{(s,t)} \cdot F & v = s \\ d^{(s,t)} \cdot F & v = t \end{cases} \qquad \begin{cases} (s,t) \in \mathcal{D}, \\ v \in V \end{cases} \tag{18}$$

$$\sum_{(s,t)\in\mathcal{D}} f_\ell^{(s,t)} \leq C_\ell \qquad \ell \in E \tag{19}$$