# Introduction to Combinatorial Homotopy Theory

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## 1 Introduction.

*Homotopy theory* is a subdomain of topology where, instead of considering the category of topological spaces and continuous *maps*, you prefer to consider as morphisms only the continuous maps *up to homotopy*, a notion precisely defined in these notes in Section 4. Roughly speaking, you decide not to distinguish two maps which can be continuously deformed into each other; such a weakening of the notion of map is quickly identified as necessary when you intend to apply to Topology the methods of *Algebraic Topology*. Otherwise the main classification problems of topology are, except in low dimensions, out of scope.

If you want to "algebraize" the topological world, you will meet another difficulty. The traditional topological spaces, defined for example through collections of open subsets, cannot be directly processed by a computer; a computer can handle only *discrete* objects and in a sense topology is the opposite subject. A *combinatorial* intermediary notion between Topology and Algebra is required. Poincaré started Algebraic Topology about a century ago by using the polyhedra as intermediary objects, but since the fifties, the *simplicial* notions have been recognized as more appropriate. In this framework of combinatorial topology, the sensible topological spaces can be combinatorially defined, and also installed and processed on a computer. This is valid even for very complicated or abstract spaces such as classifying spaces, functional spaces; the various important functors of algebraic topology can also be implemented as functional objects.

In a sense there is a conflict between both previous observations. The homotopy relation is concerned by *continuous* deformations of maps, while combinatorial models for topological spaces *and* maps do not seem to allow enough maps to model homotopies. But we will see this apparent obstacle is easily overcome, and the so-called *Combinatorial Homotopy Theory* is now one of the standard ground theories for Algebraic Topology.

In particular, if you claim you are mainly interested by *constructive* results in Algebraic Topology, it is quickly obvious *combinatorial* topology is required. *Constructive Algebraic Topology* is a difficult but fascinating subject, and three main solutions are now available:

1. Rolf Schön's solution [14], quite elegant, unfortunately never (?) considered since his remarkable memoir, in particular from a concrete programming point of view.

2. The solution studied for years by this author and several collaborators, see the lecture notes of the previous Map Summer School at Genova [13]. The key point is that *locally effective* models for combinatorial spaces are sufficient to use standard simple Algebraic Topology and make it constructive.

3. The *operadic* solution where the algebraic world is enriched enough to make it equivalent to the topological world; more precisely the algebraic structure of chain complexes is sufficiently enriched, thanks to appropriate *operads*, to code in this way the topological spaces up to homotopy.

But whatever solution you decide to study, anyway you will have to use ingredients coming from combinatorial homotopy. For the solutions 1 and 2 above, it will even be necessary to implement on your computer the corresponding necessary simplicial objects and operators; for the solution 3, the objects of the resulting category, the $E_\infty$-operadic chain complexes, do not seem to use combinatorial homotopy, but the theoretical justifications requires by some means or other this theory. This Ictp-Map Summer School proposes an introduction to the solution 3 and the present lecture is intended to prepare the audience to the most elementary facts of combinatorial homotopy.

Section 2 describes the most elementary simplicial techniques, around the notion of simplicial *complex*. It is already possible in this simple framework to speak of combinatorial homotopy, for example it is possible to construct simplicial models for functional spaces, in particular for loop spaces. An important progress at the end of the forties was the invention (discovery?), mainly by Samuel Eilenberg, of the notion of simplicial *set*, to which the rest of these notes is devoted. An amusing paradox of this terminology must be signaled: the notion of simplicial *set* is much more complex than the notion of simplical… complex! These simplicial sets were initially called CSS-sets, an acronym for "complete-semi-simplicial"; but it was identified a little later the general notion of *simplicial object* in an arbitrary category makes sense and a CSS-set is nothing but a simplicial object in the category of sets, which explains the modern and natural terminology of *simplicial set*.

This notion of simplicial set is one of the most fascinating elementary notions in mathematics. In a sense it contains the whole richness of topology. Yet an essential drawback must immediately be pointed out: modelling a topological object as a simplicial set leads to coherent but arbitrary choices of orders (resp. orientations) for the vertices (resp. simplices). It happens these choices hide very sophisticated actions of the symmetric groups $\mathfrak{S}_n$; in a sense, elementary Algebraic Topology forgets this action and *operadic* Algebraic Topology on the contrary takes account of this action, in a totally algebraic framework, and in this way, the initial goal of Algebraic Topology, representing homotopy types as algebraic objects, is finally reached.

Once the notion of simplicial set is available, most ingredients of algebraic topology, classifying spaces, loop spaces, functional spaces, homology or cohomology groups, any sort of operators between these groups can be more or less easily described in the framework of simplicial sets. The initial essential step in this direction was the discovery by Daniel Kan [8], of a purely combinatorial definition of homotopy groups. The end of these notes shows a few typical examples of simplicial descriptions, mainly to prepare the readers to the lecture about Operadic Algebraic Topology.

# 2 Simplicial complexes.

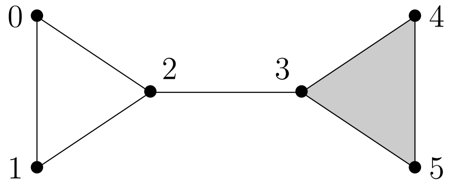
**Definition 1** — A *simplicial complex* is a pair $(V, S)$ satisfying the properties:

- $V$, the set of *vertices*, and $S$, the set of *simplices*, are... sets, possibly infinite.
- Every simplex $\sigma \in S$ is a non-empty finite set of vertices: $\sigma = \{v_0, \ldots, v_n\}$; such a simplex is called an $n$-simplex, the integer $n \geq 0$ is the *dimension* of the simplex $\sigma$. This simplex *spans* the vertices $v_0, \ldots, v_n$.
- For every vertex $v \in V$, the 0-simplex $\{v\}$ is an element of $S$.
- For every simplex $\sigma = \{v_0, \ldots, v_n\} \in S$, every $m$-sub-simplex $\{v_{i_0}, \ldots, v_{i_m}\}$ is also an element of $S$.

For example, let us consider the simplicial complex $(V, S)$ with:

- $V = \{0 \ldots 5\}$, the integers from 0 to 5.
- $S = \{0, 1, 2, 3, 4, 5, 01, 02, 12, 23, 34, 35, 45, 345\}$ where 35 for example is a shorthand for $\{3, 5\}$.

Such a simplicial complex is an "abstract" version of the geometrical object:




The triangle 012 is *hollow*, because $\{0, 1, 2\}$ is not a simplex; on the contrary, $\{3, 4, 5\}$ is a simplex and the triangle 345 is *filled*. In the simplicial complex game, you have a box with an arbitrary number of available vertices (0-simplices), edges (1-simplices), triangles, (2-simplices), tetrahedons (3-simplices) and more generally of $n$-simplices. Every vertex is labeled by the corresponding element of $V$ and the simplices of $S$ describe what collections of vertices are spanned by a simplex.

No geometry in this definition; in particular, at this level, a simplex is just an "abstract" *set* of vertices, which, when we will geometrically *realize* in a moment a simplicial complex, will finally produce an ordinary geometrical simplex.

**Definition 2** — Let $K = (V, S)$ be a simplicial complex. The *geometrical realization* $|K|$ of $K$ is defined as follows: $|K|$ is the set of the indexed families $x = (x_\sigma)_{\sigma \in S} \in [0,1]^{(S)}$ satisfying the conditions:

- $\{\sigma \ \underline{\textbf{st}}\ x_\sigma > 0\} \in S$ and in particular is finite;
- $\sum x_\sigma = 1$.

Any topology over $[0,1]^{(S)}$ defines a topology over $|K|$, but *combinatorial topology* most often is not concerned by such a topology: the combinatorial game is enough to model, *up to homotopy*, in this way most "sensible" topological spaces.

Elementary Examples. Let $V$ be an arbitrary set of vertices, possibly infinite. Then the *simplex generated by* $V$, denoted by $\Delta^V$, is the simplicial complex $(V, S)$ where $S = \mathcal{P}_{*,f}(V)$ is the set of *finite* non-empty subsets of $V$. If $V = \underline{n} := \{0, 1, \ldots, n\}$, then $\Delta^{\underline{n}}$ is usually simply denoted by $\Delta^n = (\underline{n}, \mathcal{P}_*(\underline{n}))$, it is the standard (abstract) $n$-simplex, and its realization $|\Delta^n|$ is the common geometrical $n$-simplex. If $V$ is infinite, then the simplicial complex $\Delta^V$ has simplices of arbitrary high dimension, but every simplex of $\Delta^V$ has a finite dimension.

The standard model for the $n$-sphere $S^n$ as a simplicial complex is:

$$S^n = (\underline{n+1}, \mathcal{P}_*(\underline{n+1}) - \{\underline{n+1}\}).$$

It is the standard $n+1$-simplex $\Delta^{n+1}$ from which the maximal simplex $\underline{n+1} = \{0, \ldots, n+1\}$ has been removed: think the standard simplex $\Delta^{n+1}$ is *solid* and you may so imagine our model for the $n$-sphere is on the contrary a *hollow* $(n+1)$-simplex, in other words the boundary of an $(n+1)$-simplex. Its realization is homeomorphic to the boundary of an $(n+1)$-disk (or cell, or ball), that is, a topological $n$-sphere.

Many topological constructions can be simulated in the framework of simplicial complexes. For example, if $K = (V, S)$ and $K' = (V', S')$ are two simplicial complexes *with base point*, that is, two vertices $v_0 \in V$ and $v'_0 \in V'$ are distinguished, then the *wedge* $K \vee K'$ is defined by $K'' = (V'', S'')$ with $V'' = (V \coprod V')/(v_0 \sim v'_0)$ and $S'' = (S \coprod S')/\sim$ where the last relation $\sim$ identifies any occurence of $v_0$ in an element of $S$ with any occurence of $v'_0$ in an element of $S'$. Both simplicial complexes are "attached" at their respective base vertices.

A common construction is however surprisingly difficult to be translated in the framework of simplicial complexes, namely the *product* construction. The difficulty is the following: the elementary piece in the world of simplicial complexes is a simplex, a point in dimension 0, an edge in dimension 1, a solid triangle in dimension 2, a tetrahedron in dimension 3, an $n$-simplex in dimension $n$. But the product of two edges is a square, which can be presented as the union of two triangles, if you cut this square along a diagonal; but two diagonals in a square and how to choose the right one? A little more difficult, the product of an edge by a (solid) triangle is a triangular prism which can be presented as the union of three tetrahedrons, a process neither easy nor deterministic. We will see later the product of an $m$-simplex by an $n$-simplex can be divided in $\binom{m+n}{m}$ simplices of dimension $(m+n)$, by a process not so obvious, made "automatic" if you work in the framework of simplicial *sets*.

**Definition 3** — Let $K = $ (V,S) and $K' = (V', S')$ be two simplicial complexes. A *simplicial map* $f : K \to K'$ is a set map $f : V \to V'$ satisfying the property: for every simplex $\sigma \in S$, the image $f(\sigma)$ is a simplex $f(\sigma) \in S'$.

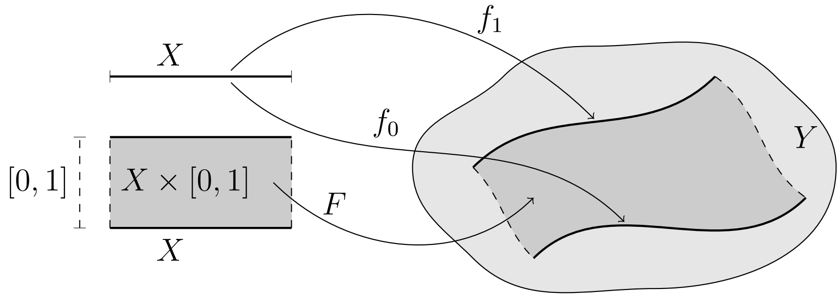
It is not required $f : V \to V'$ is injective, and the image of an $n$-simplex $\sigma \in S$ can be a simplex $f(\sigma) \in S'$ of dimension $< n$.

Is it possible to define homotopies between simplicial maps? First, let us consider the traditional notion of homotopy between continuous maps.

**Definition 4** — Two continuous maps $f_0, f_1 : X \to Y$ between the topological spaces $X$ and $Y$ are *homotopic* if there exists a (continuous) map $F : [0,1] \times X \to Y$ satisfying:

$$\begin{array}{rcl} F(0,x) & = & f_0(x) \\ F(1,x) & = & f_1(x) \end{array}$$

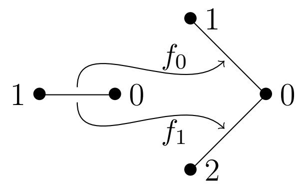
for every $x \in X$.




**Definition 5** — Let $f_0, f_1 : K = (V, s) \to K' = (V', S')$ be two simplicial maps between two simplicial complexes. The maps $f_0$ and $f_1$ are *elementarily homotopic* if the following property is satisfied: for every simplex $\sigma \in V$, the union $f_0(\sigma) \cup f_1(\sigma)$ is a simplex of $S'$.

If the required property is satisfied, you can then, at the level of the geometrical realizations, trivially interpolate the maps $f_0$ and $f_1$ by a continuous family of $f_t$'s, for $t$ running the interval $[0,1]$. Note $f_t$ cannot be simplicially implemented except for $t = 0$ or 1.

Definition 5 is natural but not at all satisfactory. Let us consider the situation with $K$ the interval $K = \Delta^1 = (\underline{1}, \mathcal{P}_*(\underline{1}))$ and $K' = (\underline{2}, S')$ with $S'$ made of the three vertices $\{0\}$, $\{1\}$ and $\{2\}$, and only two edges $\{0,1\}$ and $\{0,2\}$. Let us consider also the maps $f_0, f_1 : K \to K'$ defined by $f_0(0) = f_1(0) = 0$, $f_0(1) = 1$ and $f_1(1) = 2$. Then these maps are not elementarily homotopic though, in the topological framework, they are homotopic.

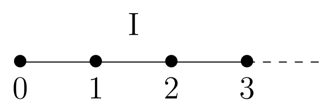


This difficulty can be overcome as follows: you decide two simplicial maps $f, g : K \to K'$ are homotopic if you can construct a *chain* $f = f_0, f_1, \ldots, f_{k-1}, f_k = g$ where two successive elements are elementarily homotopic. We let you construct the simple chain of length 2 describing how the maps of the previous example are homotopic.

It is possible also to define *functional spaces* in a combinatorial style. Because the framework of simplicial complexes will be soon given up, we show only a typical example: how to define the loop space of a based simplicial complex $(K, *)$, the *base point* $*$ being a *distinguished* vertex of $K$? Usually a loop $\gamma : [0,1] \to (X, *)$ in a based topological space is a continuous map $\gamma : [0,1] \to X$ satisfying $\gamma(0) = \gamma(1) = *$. How to copy this notion for simplicial complexes?

The interval $[0,1]$ is (the realization of) a simplicial complex and the notion of simplicial map $\gamma : [0,1] \to K$ makes sense; but combined with the condition $\gamma(0) = \gamma(1) = *$, only one such loop, the trivial constant loop at the base point, not very satisfactory!

To overcome this obstacle, instead of the simple interval $[0,1]$, let us consider the (infinite) simplicial complex $I = (\mathbb{N}, S)$ with $S = \{\{n\}\}_{n\in\mathbb{N}} \cup \{\{n, n+1\}\}_{n\in\mathbb{N}}$.



First it is natural to decide a loop $\gamma : I \to K$ is a simplicial map satisfying:

- $\gamma(0) = *$ ;
- For every $n \geq$ some $n_0$, $\gamma(n) = *$.

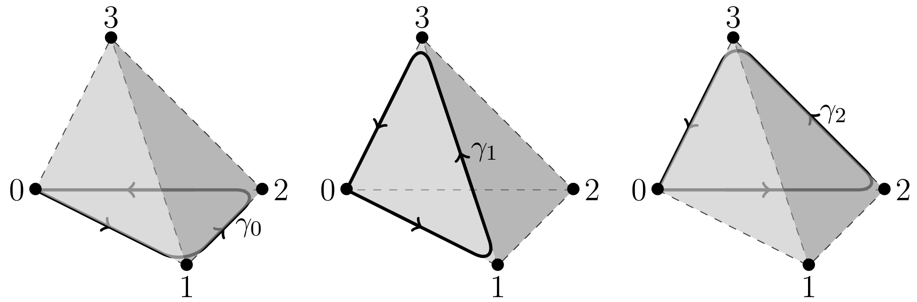
Our loop starts from the base point, runs various edges of $K$, and after the time $n_0$, remains fixed at the base point.

Then the loop space $\Omega K$ can be naturally defined *as a simplicial complex* as follows: $\Omega K = (\Lambda, S_\Lambda)$ with:

- $\Lambda$ is the set of loops as just defined;
- A finite set of loops $\{\lambda_0, \ldots, \lambda_n\}$ is an element of $S_\Lambda$, that is, a simplex of $\Omega K$, if and only if, for every integer $t > 0$, the set $\{\lambda_0(t-1), \ldots, \lambda_n(t-1)\} \cup \{\lambda_0(t), \ldots, \lambda_n(t)\}$ is a simplex of $K$.

The last condition claims that it is possible to *interpolate* in a barycentric style the loops $\lambda_0, \ldots, \lambda_n$ for every point of the "geometrical" simplex intuitively spanned by these loops; if the condition is satisfied, we therefore decide to install an "abstract" simplex between these vertices. It is an interesting exercise of topology to prove the realization $|\Omega K|$ actually has the same homotopy type as the (topological) loop space $\Omega(|K|)$, but this will not be necessary in these notes.

For example if $K = S^2$ modelled as the boundary of the standard 3-simplex: $K = (0..3, \{0, 1, 2, 3, 01, 02, 03, 12, 13, 23, 012, 013, 023, 123\})$, let us consider the loops $\gamma_0 = 0 \to 1 \to 2 \to 0$, $\gamma_1 = 0 \to 1 \to 3 \to 0$ and $\gamma_2 = 0 \to 2 \to 3 \to 0$, with notations made obvious by the figure:



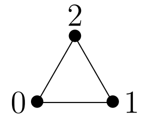

Then $\{\gamma_0, \gamma_1\}$, $\{\gamma_0, \gamma_2\}$ and $\{\gamma_1, \gamma_2\}$ are edges of $\Omega K$, and $\{\gamma_0, \gamma_1, \gamma_2\}$ is a triangle of $\Omega K$ between these edges. On the contrary, if $\gamma_1^{-1}$ is the same loop as $\gamma_1$ but run in the reverse direction (meaning?), then $\{\gamma_0, \gamma_1^{-1}, \gamma_2\}$ is not a triangle of $\Omega K$, why?

Let us decide the base point $* \in \Omega K$ is the trivial loop constant in 0. Then $* \to \gamma_0 \to \gamma_1 \to *$ is a *loop* of $\Omega K$, in other words a vertex of $\Omega\Omega K =: \Omega^2 K$. Proving the last loop is not homotopic to the trivial loop is another story.

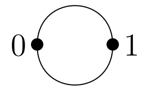
## 2.1 Simplicial *complexes* vs simplicial *sets*.

A $\underline{\Delta}$-morphism $\alpha : \underline{m} \to \underline{n}$ can in particular be a face operator $\partial_i^m : \underline{m-1} \to \underline{m}$. The corresponding $X$-operator $X_{\partial_i^m} : X_m \to X_{m-1}$ is also called the $i$-th face operator in dimension $m$ and is most often simply denoted by $\partial_i^m$ or $\partial_i$ when the underlying simplicial set $X$ is implicit. The same for a degeneracy operator $X_{\eta_i^m} : X_m \to X_{m+1}$, most often denoted by $\eta_i^m$ or $\eta_i$. Because of Corollary 11, it is enough to define the face and degeneracy operators $X_{\partial_i^m}$ and $X_{\eta_i^m}$ satisfying the required coherence properties, to define the whole collection of morphisms $\{X_\alpha\}_\alpha$.

Let us consider the simplest simplicial *complex* $X$, the realization of which is (homeomorphic to) a circle $S^1 = \{(x, y) \in \mathbb{R}^2 \ \underline{\mathbf{st}}\ x^2 + y^2 = 1\}$. Three vertices and three edges are necessary: $X = (V, S)$ with $V = \underline{2} = \{0, 1, 2\}$ and $S = \{0, 1, 2, 01, 02, 12\}$ where as usual 01 is a shorthand for $\{0, 1\}$.

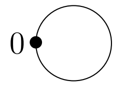


It is not possible to use only two vertices following the figure:



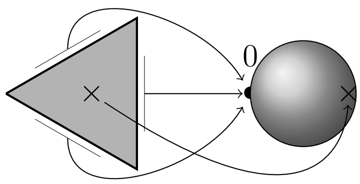

for the only possibility to *produce* an edge consists in choosing a set of vertices; so that it is possible to install only *one* edge between two given vertices and the above figure cannot correspond to a simplicial *complex*. We will see that we do not meet any problem when associating a simplicial *set* to the same figure, this will be explained soon.

We could even consider the following figure:




and observe that is is not possible in the framework of simplicial *complexes* to install a "loop" edge from a vertex to *itself*. This is also possible for simplicial *sets*.

We will see it is also possible to give a simplicial set with only two (non-degenerate) simplices, a vertex 0 and a "triangle" 012, the three edges of which being collapsed over the unix vertex.




The realization of this simplicial set will be a triangle where the whole boundary is identified to a point, that is, a 2-sphere.

More generally any $n$-sphere can be realized as a simplicial *set* with only one vertex and one $n$-simplex; more precisely only these *non-degenerate* simplices, for we will soon learn that any non-degenerate simplex generates an infinite collection of... degenerate simplices, non-visible on the figures, that is, "hidden" in the geometric realization. For example the minimal simplicial complex corresponding to a 4-sphere requires 6 vertices, 15 edges, 20 triangles, 15 tetrahedron and 6 4-simplices, while as a simplicial set, only one vertex and one 4-simplex are enough as non-degenerate simplices.

These elementary examples show in general less (non-degenerate) simplices are necessary to construct an object as a simplicial set than as a simplicial complex. You can object an infinite number of degenerate simplices is also required, but precisely these degenerate simplices will give *much more flexibility* in the construction process. It is true the underlying technology is not obvious, but thanks to this nice technology, the main parts of topology have a good translation into the combinatorial world, allowing a constructivist to easily handle topology with his computer.

# 3 Simplical sets.

Possible references for this fascinating subjects are:

- [9]: Maybe the most useful reference for the serious user; only one drawback: hardly any example, no didactic explanation! But many invaluable formulas and detailed proofs can be found only in this book.
- [11]: See in particular Section VIII.5 of this book for a short introduction to this subject, which is not the main goal of this book, but unavoidable.
- [10]: See Section §4.2.
- More modern, but also harder, references are [7], a book entirely devoted to this subject, and also [6, I.2].

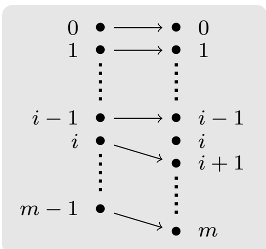
## 3.1 The category $\underline{\Delta}$.

Some strongly structured sets of indices are necessary to define the notion of *simplicial object*; they are conveniently organized as the category $\underline{\Delta}$. An object of $\underline{\Delta}$ is a set $\underline{m}$, namely the set of integers $\underline{m} := \{0, 1, \ldots, m-1, m\}$; this set is canonically *ordered* with the usual order between integers.

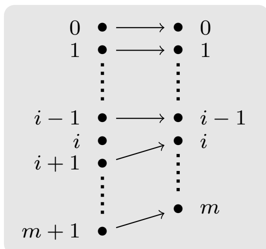
A $\underline{\Delta}$-morphism $\alpha : \underline{m} \to \underline{n}$ is an *increasing* map. Equal values are permitted; for example a $\underline{\Delta}$-morphism $\alpha : \underline{2} \to \underline{3}$ could be defined by $\alpha(0) = \alpha(1) = 1$ and $\alpha(2) = 3$. The set of $\underline{\Delta}$-morphisms from $\underline{\mathrm{m}}$ to $\underline{\mathrm{n}}$ is denoted by $\underline{\Delta}(\underline{m}, \underline{n})$; the subset of injective (resp. surjective) morphisms is denoted by $\Delta^{\mathrm{inj}}(\underline{m}, \underline{n})$ (resp. $\Delta^{\mathrm{srj}}(\underline{m}, \underline{n})$).

Some *elementary* morphisms are important, namely the simplest non-surjective and non-injective morphisms. For geometric reasons explained later, the first ones are the *face morphisms*, the second ones are the *degeneracy morphisms*.

$\partial_i^m =$




$\eta_i^m =$



**Definition 6** — The *face morphism* $\partial_i^m : \underline{m-1} \to \underline{m}$ is defined for $m \geq 1$ and $0 \leq i \leq m$ by:

$$\begin{array}{rll} \partial_i^m(j) = & j & \text{if } j < i, \\ \partial_i^m(j) = & j+1 & \text{if } j \geq i. \end{array}$$

The face morphism $\partial_i^m$ is the unique injective morphism from $\underline{\text{m-1}}$ to $\underline{\text{m}}$ such that the integer $i$ is not in the image. The face morphisms generate the injective morphisms, in fact in a unique way if a growth condition is required.

**Proposition 7** — *Any injective $\underline{\Delta}$-morphism $\alpha \in \Delta^{\text{inj}}(\underline{m}, \underline{n})$ has a unique expression:*

$$\alpha = \partial_{i_n}^n \circ \ldots \circ \partial_{i_{m+1}}^{m+1}$$

*satisfying the relation $i_n > i_{n-1} > \ldots > i_{m+1}$.*

♣ The index set $\{i_{m+1}, \ldots, i_n\}$ is exactly the difference set $\underline{n} - \alpha(\underline{m})$, that is, the set of the integers where surjectivity fails. ♣

Frequently the upper index $m$ of $\partial_i^m$ is omitted because clearly deduced from the context. For example the unique injective morphism $\alpha : \underline{2} \to \underline{5}$ the image of which is $\{0, 2, 4\}$ can be written $\alpha = \partial_5 \partial_3 \partial_1$.

If two face morphisms are composed in the wrong order, they can be exchanged: $\partial_i \circ \partial_j = \partial_{j+1} \circ \partial_i$ if $j \geq i$. Iterating this process allows you to quickly compute for example $\partial_0 \partial_2 \partial_4 \partial_6 = \partial_9 \partial_6 \partial_3 \partial_0$.

**Definition 8** — The *degeneracy morphism* $\eta_i^m : \underline{m+1} \to \underline{m}$ is defined for $m \geq 0$ and $0 \leq i \leq m$ by:

$$\begin{array}{rll} \eta_i^m(j) = & j & \text{if } j \leq i, \\ \eta_i^m(j) = & j-1 & \text{if } j > i. \end{array}$$

The degeneracy morphism $\eta_i^m$ is the unique surjective morphism from $\underline{\text{m+1}}$ to $\underline{\text{m}}$ such that the integer $i$ has two pre-images. The degeneracy morphisms generate the surjective morphisms, in fact in a unique way if a growth condition is required.

**Proposition 9** — *Any surjective $\underline{\Delta}$-morphism $\alpha \in \Delta^{\text{srj}}(\underline{m}, \underline{n})$ has a unique expression:*

$$\alpha = \eta_{i_n}^n \circ \ldots \circ \eta_{i_{m-1}}^{m-1}$$

*satisfying the relation $i_n < i_{n+1} < \ldots < i_{m-1}$.*

♣ The index set $\{i_n, \ldots, i_{m-1}\}$ is exactly the set of integers $j$ such that $\alpha(j) = \alpha(j+1)$, that is, the integers where injectivity fails. ♣

Frequently the upper index $m$ of $\eta_i^m$ is omitted because clearly deduced from the context. For example the unique surjective morphism $\alpha : \underline{5} \to \underline{2}$ such that $\alpha(0) = \alpha(1)$ and $\alpha(2) = \alpha(3) = \alpha(4)$ can be expressed $\alpha = \eta_0 \eta_2 \eta_3$.

If two degeneracy morphisms are composed in the wrong order, they can be exchanged: $\eta_i \circ \eta_j = \eta_j \circ \eta_{i+1}$ if $i \geq j$. Iterating this process allows you to quickly compute for example $\eta_3 \eta_3 \eta_2 \eta_2 = \eta_2 \eta_3 \eta_5 \eta_6$.

**Proposition 10** — *Any $\underline{\Delta}$-morphism $\alpha$ can be $\underline{\Delta}$-decomposed in a unique way:*

$$\alpha = \beta \circ \gamma$$

*with $\beta$ injective and $\gamma$ surjective.*

♣ The intermediate $\underline{\Delta}$-object $\underline{k}$ necessarily satisfies $k+1 = \mathrm{Card}(\mathbf{im}(\alpha))$. The growth condition then gives a unique choice for $\beta$ and $\gamma$. ♣

**Corollary 11** — *Any $\underline{\Delta}$-morphism $\alpha : \underline{m} \to \underline{n}$ has a unique expression:*

$$\alpha = \partial_{i_n} \circ \ldots \circ \partial_{i_{k+1}} \circ \eta_{j_k} \circ \ldots \circ \eta_{j_{m-1}}$$

*satisfying the conditions $i_n > \ldots > i_{k+1}$ and $j_k < \ldots < j_{m-1}$.* ♣

Finally if face and degeneracy morphisms are composed in the wrong order, they can be exchanged:

$$\begin{array}{rcll} \eta_i \circ \partial_j & = & \mathrm{id} & \text{if } j = i \text{ or } j = i+1; \\ & = & \partial_{j-1} \circ \eta_i & \text{if } j \geq i+2; \\ & = & \partial_j \circ \eta_{i-1} & \text{if } j < i. \end{array}$$

All these commuting relations can be used to convert an arbitrary composition of faces and degeneracies into the canonical expression:

$$\alpha = \eta_9\partial_6\eta_3\partial_7\eta_9\partial_8\eta_6\partial_2\eta_4\partial_9 = \partial_7\partial_6\partial_2\eta_2\eta_4\eta_6.$$

This relation means the image of $\alpha$ does not contain the integers 2, 6 and 7, and the relations $\alpha(2) = \alpha(3)$, $\alpha(4) = \alpha(5)$ and $\alpha(6) = \alpha(7)$ are satisfied.

The previous propositions show any functor $F$ from $\underline{\Delta}$ to another category is entirely known when the image objects $F(\underline{m})$ and the image morphisms $F(\partial_i^m)$ and $F(\eta_m)$ are given.

**Corollary 12** — *A* contravariant *functor $X : \underline{\Delta} \to \underline{\mathrm{CAT}}$ is nothing but a collection $\{X_m\}_{m\in\mathbb{N}}$ of objects of the target category $\underline{\mathrm{CAT}}$, and collections of $\underline{\mathrm{CAT}}$-morphisms $\{X(\partial_i^m) : X_m \to X_{m-1}\}_{m\geq 1,\, 0\leq i\leq m}$ and $\{X(\eta_i^m) : X_m \to X_{m+1}\}_{m\geq 0,\, 0\leq i\leq m}$ satisfying the commuting relations:*

$$\begin{array}{rcll} X(\partial_i) \circ X(\partial_j) & = & X(\partial_j) \circ X(\partial_{i+1}) & \text{if } i \geq j, \\ X(\eta_i) \circ X(\eta_j) & = & X(\eta_{j+1}) \circ X(\eta_i) & \text{if } j \geq i, \\ X(\partial_i) \circ X(\eta_j) & = & \mathrm{id} & \text{if } i = j, j+1, \\ X(\partial_i) \circ X(\eta_j) & = & X(\eta_{j-1}) \circ X(\partial_i) & \text{if } j > i, \\ X(\partial_i) \circ X(\eta_j) & = & X(\eta_j) \circ X(\partial_{i-1}) & \text{if } i > j+1. \end{array}$$

In the five last relations, the upper indices have been omitted. Such a contravariant functor is a *simplicial object* in the category $\underline{\mathrm{CAT}}$. If $\alpha$ is an arbitrary $\underline{\Delta}$-morphism, it is then sufficient to express $\alpha$ as a composition of face and degeneracy morphisms; the image $X(\alpha)$ is necessarily the composition of the images of the corresponding $X(\partial_i)$'s and $X(\eta_i)$'s; the above relations ensure the definition is coherent.

## 3.2 Terminology and notations.

**Definition 13** — Let $\underline{\text{Set}}$ be the category of sets. A *simplicial set* $X$ is a simplicial object in the $\underline{\text{Set}}$ category; that is, according to the previous section, a contravariant functor $X : \underline{\Delta} \to \underline{\text{Set}}$.

This definition is short, but, because of the rich structure of the category $\underline{\Delta}$, it is quite complex! You see defining a simplicial set $X$ requires for every non-negative integer $n$ some object $X_n$ of the $\underline{\text{Set}}$ category, in other words an ordinary set, and for every $\underline{\Delta}$-morphism $\alpha : \underline{m} \to \underline{n}$ some $\underline{\text{Set}}$-morphism, that is an ordinary map $X_\alpha : X_n \to X_m$; furthermore, the set $\{X_\alpha\}_{\alpha \in \underline{\Delta}-\text{morphisms}}$ must satisfy the coherence relations $X_\alpha X_\beta = X_{\beta\alpha}$ when the composition $\beta\alpha$ makes sense.

The geometric interpretation of this definition is not obvious, but once understood, this notion is terribly powerful. A power which deserves a little significant work to reach its marvelous possibilities. Before seriously studying this notion, let us give a few comments about the comparison between simplicial *complexes* and simplicial *sets*.

A *simplicial set* $X$ is a simplicial object in the category of sets, and therefore is given by a collection of sets $\{X(\underline{m})\}_{m\in\mathbb{N}}$ and collections of maps $\{X_\alpha\}$, the index $\alpha$ running the $\underline{\Delta}$-morphisms; the usual coherence properties must be satisfied. As explained at the end of Section 4, it is sufficient to define the $X(\partial_i^m)$'s and the $X(\eta_i^m)$'s with the corresponding commuting relations.

The set $X(\underline{m})$ is usually denoted by $X_m$ and is called the set of $m$-simplices of $X$; such a simplex has the *dimension* $m$. To be a little more precise, these simplices are sometimes called *abstract* simplices, to avoid possible confusions with the *geometric* simplices defined a little later. An (abstract) $m$-simplex is only *one* element of $X_m$.

If $\alpha \in \underline{\Delta}(\underline{n}, \underline{m})$, the corresponding morphism $X(\alpha) : X_m \to X_n$ is most often simply denoted by $\alpha^* : X_m \to X_n$ or still more simply $\alpha : X_m \to X_n$. In particular the faces and degeneracy operators are maps $\partial_i : X_m \to X_{m-1}$ and $\eta_i : X_m \to X_{m+1}$. If $\sigma$ is an $m$-simplex, the (abstract) simplex $\partial_i\sigma$ is its $i$-th face, and the simplex $\eta_i\sigma$ is its $i$-th degeneracy; we will see the last one is "particularly" abstract.

## 3.3 The structure of simplex sets.

**Definition 14** — An $m$-simplex $\sigma$ of the simplicial set $X$ is *degenerate* if there exist an integer $n < m$, an $n$-simplex $\tau \in X_n$ and a $\underline{\Delta}$-morphism $\alpha \in \underline{\Delta}(\underline{m}, \underline{n})$ such that $\sigma = \alpha(\tau)$. The set of non-degenerate simplices of dimension $m$ in $X$ is denoted by $X_m^{ND}$.

Decomposing the morphism $\alpha = \beta \circ \gamma$ with $\gamma$ surjective, we see that $\sigma = \gamma(\beta(\tau))$, with the dimension of $\beta(\tau)$ less or equal to $n$; so that in the definition of degenaracy, the connecting $\underline{\Delta}$-morphism $\alpha$ can be required to be surjective. The relation $\sigma = \alpha(\tau)$ with $\alpha$ surjective is shortly expressed by saying the $m$-simplex $\sigma$ *comes from* the $n$-simplex $\tau$.

Eilenberg's lemma explains each degenerate simplex comes from a canonical non-degenerate one, and in a unique way.

**Lemma 15** — **(Eilenberg's lemma)** *If $X$ is a simplicial set and $\sigma$ is an $m$-simplex of $X$, there exists a unique triple $T_\sigma = (n, \tau, \alpha)$ satisfying the following conditions:*

1. *The first component $n$ is a natural number $n \leq m$;*
2. *The second component $\tau$ is a non-degenerate $n$-simplex $\tau \in X_n^{ND}$;*
3. *The third component $\alpha$ is a $\underline{\Delta}$-morphism $\tau \in \Delta^{\mathrm{srj}}(\underline{m}, \underline{n})$;*
4. *The relation $\sigma = \alpha(\tau)$ is satisfied.*

**Definition 16** — This triple $T_\sigma$ is called the *Eilenberg triple* of $\sigma$.

♣ Let $\mathcal{T}$ be the set of triples $T = (n, \tau, \alpha)$ such that $n \leq m$, $\tau \in X_n$ and $\alpha \in \underline{\Delta}(\underline{m}, \underline{n})$ satisfy $\sigma = \alpha(\tau)$. The set $\mathcal{T}$ certainly contains the triple $(m, \sigma, \mathrm{id})$ and therefore is non empty. Let $(n_0, \tau_0, \alpha_0)$ be an element of $\mathcal{T}$ where the first component, the integer $n_0$, is minimal. We claim $(n_0, \tau_0, \alpha_0)$ is the Eilenberg triple.

Certainly $n_0 \leq m$. The $n_0$-simplex $\tau_0$ is non-degenerate; otherwise $\tau_0 = \beta(\tau_1)$ with the dimension $n_1$ of $\tau_1$ less than $n_0$, but then $(n_1, \tau_1, \beta\alpha_0)$ would be a triple with $n_1 < n_0$. Finally $\alpha_0$ is surjective, otherwise $\alpha_0 = \beta\gamma$ with $\gamma \in \Delta^{\mathrm{srj}}(m, n_1)$ and $n_1 < n_0$; but again the triple $(n_1, \beta(\tau_0), \gamma)$ would be a triple denying the required property of $n_0$. The existence of an Eilenberg triple is proved and uniqueness remains to be proved.

Let $(n_1, \tau_1, \alpha_1)$ be another Eilenberg triple. The morphisms $\alpha_0$ and $\alpha_1$ are surjective and respective sections $\beta_0 \in \Delta^{\mathrm{inj}}(\underline{n_0}, \underline{m})$ and $\beta_1 \in \Delta^{\mathrm{inj}}(\underline{n_1}, \underline{m})$ can be constructed: $\alpha_0\beta_0 = \mathrm{id}$ and $\alpha_1\beta_1 = \mathrm{id}$. Then $\tau_0 = (\alpha_0\beta_0)(\tau_0) = \beta_0(\alpha_0(\tau_0)) = \beta_0(\sigma) = \beta_0(\alpha_1(\tau_1)) = (\alpha_1\beta_0)(\tau_1)$; but $\tau_0$ is non-degenerate, so that $n_1 = \dim(\tau_1) \geq n_0 = \dim(\tau_0)$; the analogous relation holds when $\tau_0$ and $\tau_1$ are exchanged, so that $n_1 \leq n_0$ and the equality $n_0 = n_1$ is proved.

The relation $\tau_0 = \beta_0(\alpha_1(\tau_1))$ with $\tau_0$ non-degenerate implies $\alpha_1\beta_0 = \mathrm{id}$, otherwise $\alpha_1\beta_0 = \gamma\delta$ with $\delta \in \Delta^{\mathrm{srj}}(\underline{n_1}, \underline{n_2})$ and $n_2 < n_1 = n_0$, but this implies $\tau_0$ comes from $\gamma(\tau_1)$ of dimension $n_2$ again contradicting the non-degeneracy property of $\tau_0$; therefore $\alpha_1\beta_0 = \mathrm{id}$ but this equality implies $\tau_0 = \tau_1$.

If $\alpha_0 \neq \alpha_1$, let $i$ be an integer such that $\alpha_0(i) = j \neq \alpha_1(i)$; then the section $\beta_0$ can be chosen with $\beta_0(j) = i$; but this implies $(\alpha_1\beta_0)(j) \neq j$, so that the relation $\alpha_1\beta_0 = \mathrm{id}$ would not hold. The last required equality $\alpha_0 = \alpha_1$ is also proved. ♣

Each simplex comes from a unique non-degenerate simplex, and conversely, for any non-degenerate $m$-simplex $\sigma \in X_m^{ND}$, the collection $\{\alpha(\sigma)\,;\, \alpha \in \Delta^{\mathrm{srj}}(\underline{n}, \underline{m})\,;\, n \geq m\}$ is a perfect description of all simplices coming from $\sigma$, that is, of all degenerate simplices *above* $\sigma$. This is also expressed in the following formula, describing the structure of the simplex set of any simplicial set $X$:

$$\coprod_{m \in \mathbb{N}} X_m = \coprod_{m \in \mathbb{N}} \quad \coprod_{\sigma \in X_m^{ND}} \quad \coprod_{n \geq m} \Delta^{\mathrm{srj}}(\underline{n}, \underline{m})(\sigma).$$

In particular a 0-simplex $v \in X_0$ is always non-degenerate, it is called a *vertex*, and such a vertex generates for every positive dimension $n$ exactly one degenerate simplex $v_n = \eta^* v$ where $\eta$ is the unique element of $\Delta^{\mathrm{srj}}(\underline{n}, \underline{0})$.

## 3.4 Examples.

### 3.4.1 Discrete simplicial sets.

**Definition 17** — A simplicial set $X$ is *discrete* if $X_m = X_0$ for every $m \geq 1$, and if for every $\alpha \in \Delta(\underline{m}, \underline{n})$, the induced map $\alpha^* : X_n \to X_m$ is the identity.

The reason of this definition is that the *realization* (see Section [redacted] ) of such a simplicial set is the discrete point set $X_0$; the Eilenberg triple of any simplex $\sigma \in X_m = X_0$ is $(0, \sigma, \eta)$ where the map $\eta$ is the unique element of $\Delta(\underline{m}, \underline{0})$; the only non-degenerate simplices are the vertices, the elements of $X_0$.

### 3.4.2 The simplicial complexes.

A simplicial complex $K = (V, S)$ is a pair where the first component $V$, the *vertex set* is an arbitrary "set"; the second component $S$, the *simplex set*, is made of finite subsets of $V$ satisfying a few coherence properties, as explained in Definition 2.

The simplicial complex $K = (V, S)$ is *ordered* if the vertex set $V$ is provided with a *total* order[1]. Then a simplicial *set*, abusively again denoted by $K$, is canonically associated; the simplex set of $m$-dimensional simplices $K_m$ in this new framework is the set of *increasing* maps $\sigma : \underline{m} \to K$ such that the image of $\underline{m}$ is an element of $S$; note that such a map $\sigma$ is not necessarily injective. If $\alpha$ is a $\Delta$-morphism $\alpha \in \Delta(\underline{n}, \underline{m})$ and $\sigma$ is an $m$-simplex $\sigma \in K_m$, then $\alpha(\sigma)$ is naturally defined as $\alpha(\sigma) = \sigma \circ \alpha$. A simplex $\sigma \in K_m$ is non-degenerate if and only if $\sigma \in \Delta^{\mathrm{inj}}(\underline{m}, V)$; if $\sigma \in K_m = \Delta(\underline{m}, V)$, the Eilenberg triple $(n, \tau, \alpha)$ satisfies $\sigma = \tau \circ \alpha$ with $\alpha$ surjective and $\tau$ injective.

The non-degenerate $m$-simplices $K_m^{ND}$ is the set of *injective* increasing maps $\underline{m} \to V$ where the image is an $m$-simplex of the initial simplicial *complex*. There is so a natural 1-1 correspondance between the $m$-simplices of the *initial* simplicial *complex* and the *non-degenerate* $m$-simplices of the associated simplicial *set*. The role of the degenerate simplices will be explained later.

This in particular works for $K = (\underline{d}, \mathcal{P}(\underline{d}))$ the simplex freely generated by $\underline{d}$ provided with the canonical vertex order. We obtain in this way the canonical structure of simplicial set for the *standard d-simplex* $\Delta^d$.

This section implies the category of simplicial complexes is essentially embedded inside the category of simplicial sets, at least if you forget this matter of order over the vertices, necessary to obtain a simplicial set. The Zermelo theorem ensures such an order over the vertex set $V$ of the initial *complex* is always possible, but this matter of *order* plays a major role in the continuation of the story: such

[1] Other situations where the order is not total are also interesting but will be considered later.

an order is most often *non-natural* and the consequent punishment is not far: these non-natural orders are at the origin of the role of the symmetric groups in the operadic theories. In a sense, the simplicial set theory succeeds in hiding the essential role of the symmetric groups in our geometrical space. But the revenge of the symmetric groups will be terrible: you rejected the symmetric groups at the geometrical level? Yes, but they will appear again in the algebraic framework later: under the notion of $E_\infty$-*operad*.

The category of simplicial sets is designed to allow more flexible combinatorial construction processes than those that are possible in the the framework of simplicial complexes, as roughly explained in Section 2.1.

### 3.4.3 The spheres.

Let $d$ be a natural number. The simplest version $S = S^d$ of the $d$-sphere as a simplicial *set* is defined as follows: the set of $m$-simplices $S_m$ is $S_m = \{*_m\} \coprod \Delta^{\mathrm{srj}}(\underline{m}, \underline{d})$; if $\alpha \in \Delta(\underline{n}, \underline{m})$ and $\sigma$ is an $m$-simplex $\sigma \in S_m$, then $\alpha(\sigma)$ depends on the nature of $\sigma$:

1. If $\sigma = *_m$, then $\alpha(\sigma) = *_n$;
2. Otherwise $\sigma \in \Delta^{\mathrm{srj}}(\underline{m}, \underline{d})$ and if $\sigma \circ \alpha$ is surjective, then $\alpha(\sigma) = \sigma \circ \alpha$, else $\alpha(\sigma) = *_n$ (the emergency solution when the natural solution does not work).

This is nothing but the canonical quotient, in the simplicial *set* framework, of two simplicial *complexes* $S^d = \Delta^d / \partial \Delta^d$, at least if $d > 0$; more generally the notion of simplicial subset is naturally defined and a quotient then appears. In the case of the construction of $S^d = \Delta^d / \partial \Delta^d$, the subcomplex $\partial \Delta^d$ is made of the simplices $\alpha \in \Delta(\underline{m}, \underline{d})$ that are not surjective.

The Eilenberg triple of $*_m$ is $(0, *_0, \alpha)$ where *alpha* is the unique element of $\Delta(\underline{m}, \underline{0})$. The Eilenberg triple of $\sigma \in \Delta^{\mathrm{srj}}(\underline{m}, \underline{d}) \subset S_m$ is $(d, \mathrm{id}, \sigma)$. There are only two non-degenerate simplices, namely $*_0 \in S_0$ and $\mathrm{id}(\underline{d}) \in S_d$, even if $d = 0$.

## 3.5 Realization.

Before giving other examples of simplicial sets, it is time now to examine the notion of *realization* in the framework of the category of simplicial sets.

Let $X = (\{X_m\}_m, \{X_\alpha\}_\alpha)$ be a simplicial set; the index $m$ runs the non-negative integers $\mathbb{N}$; the index $\alpha$ runs the $\Delta$-morphisms: a possible $\alpha$ is an increasing map $\alpha : \underline{m} \to \underline{n}$.

**Definition 18** — The ("expensive") realization $|X|$ of $X$ is:

$$|X| = \coprod_{m \in \mathbb{N}} X_m \times |\Delta^m| \ / \ \approx .$$

Each component of the coproduct is the product of the *discrete* set of $m$-simplices $X_m$ by the standard *geometric* $m$-simplex $|\Delta^m|$, that is, the usual *topological* $m$-simplex; in other words, each "abstract" simplex $\sigma$ in $X_m$ gives birth to a geometric simplex $\{\sigma\} \times |\Delta^m|$, and they are attached to each other following the instructions of the equivalence relation $\approx$, to be defined. Let $\alpha \in \Delta(\underline{m}, \underline{n})$ be some $\Delta$-morphism, and let $\sigma$ be an $n$-simplex $\sigma \in X_n$ and $t \in |\Delta^m| \subset \mathbb{R}^{\underline{m}}$. Then the pairs $(\alpha^*\sigma, t)$ and $(\sigma, \alpha_* t)$ are declared equivalent. Here $\alpha_* : |\Delta^m| \to |\Delta^n|$ is the (affine) *geometrical* map covariantly induced between *geometrical* simplices by the "abstract" map $\alpha : \underline{m} \to \underline{n}$ between the vertices of these simplices, according to the usual numbering. The map $\alpha^* : X_n \to X_m$ is induced by the simplicial structure: $\alpha^* = X(\alpha)$; as usual, the *sup*-$*$ intends to recall the *contravariant* nature of the association process. Frequently we omit the sub-$*$ or the sup-$*$ when the context clearly implies it.

It is not obvious to understand what is the topological space so obtained. A description a little more explicit but also a little more complicated explains more satisfactorily what should be understood.

The *cheap* realization $\|X\|$ of the simplicial set $X$ is:

$$\|X\| = \coprod_{m \in \mathbb{N}} X_m^{ND} \times |\Delta^m| \ / \ \approx$$

where the equivalence relation $\approx$ is defined as follows. Let $\sigma$ be a non-degenerate $m$-simplex and $i$ an integer $0 \leq i \leq m$; let also $t \in |\Delta^{m-1}|$; the abstract $(m-1)$-simplex $\partial_i^* \sigma$ has a well defined Eilenberg triple $(n, \tau, \alpha)$; then we decide to declare equivalent the pairs $(\sigma, \partial_{i*}(t)) \approx (\tau, \alpha_*(t))$.

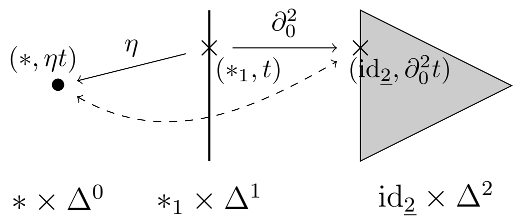
Fewer simplices are invoked in the cheap realization: only the *non*-degenerate simplices are used, but the equivalence relation assembling them to each other is more sophisticated.

For example let $S = S^d$ be the claimed simplicial version of the $d$-sphere described in Section 3.4.3. This simplical set $S$ has only two non-degenerate simplices, one in dimension 0, the other one in dimension $d$. The *cheap* realization $||S||$ needs a point $|\Delta^0| = \{*\}$ and a geometric $d$-simplex $|\Delta^d|$ corresponding to the abstract simplex $\mathrm{id} \in \Delta(\underline{d}, \underline{d})$; then if $t \in |\Delta^{d-1}|$ and $0 \leq i \leq d$, the equivalence relation asks for the Eilenberg triple of $\partial_i(\mathrm{id}) = *_{d-1}$ which is $(0, *_0, \eta)$, the map $\eta$ being the unique element of $\Delta(\underline{d-1}, \underline{0})$. Finally the initial pair $(\mathrm{id}, \partial_{i*} t)$ in the realization process must be identified with the pair $(*_0, \Delta^0)$; in other words $\|S\| = |\Delta^d| / \partial |\Delta^d|$, homeomorphic to the unit $d$-ball with the boundary collapsed to one point: the result is clearly a $(d-1)$-sphere.

You observe in this simple example about spheres the role of the degenerate simplices. Let us now consider the *expensive* realization $|S|$ of the simplicial set $S$, simplicial model of the 2-sphere $S^2$. In the rough description page 8 of a 2-sphere as a simplicial set, we explained we would like to attach the whole boundary $\partial |\Delta^2|$ of the triangle $|\Delta^2|$, for example its 0-face $\partial_0 |\Delta^2|$ to the base point '$*$'. Two $\Delta$-morphisms are invoked in the necessary attachment process:

- The unique map $\eta : \underline{1} \to \underline{0}$ is surjective non-injective. The *contravariant* functor which defines the simplicial set $S$ has in particular a map $\eta^* : S_0 \to S_1$ which associates to the base point $* \in S_0$, in fact the unique element of $S_0$, the *degenerate* 1-simplex $*_1 \in S_1$.

- The face map $\partial_0^2 : \underline{1} \to \underline{2}$, that is, the unique injective map which avoids 0 (see page 9), applied to the 2-simplex $\mathrm{id}_{\underline{2}} \in S_2$, gives again $\partial_0^{2*}(\mathrm{id}_{\underline{2}}) = *_1$, see in Section 3.4.3 the Rule 2 for the simplicial description of $S^n$.




Now let $t \in |\Delta^1|$. In the expensive realization process, we must identify:

$$S_0 \times |\Delta^0| \ni (*, 0) = (*, \eta t) \sim (\eta^* *, t) = (*_1, t) \in S_1 \times |\Delta^1|,$$

and we must also identify:

$$S_2 \times |\Delta^2| \ni (\mathrm{id}_{\underline{2}}, \partial_0^2 t) \sim (\partial_0^{2*}\mathrm{id}_{\underline{2}}, t) = (*_1, t) \in S_1 \times |\Delta^1|.$$

Finally we may forget the point $(*_1, t)$ and directly identify $(*, 0) \sim (\mathrm{id}_{\underline{2}}, \partial_1^2 t)$. This being valid for every $t \in |\Delta^1|$, and for $\partial_1^2$ and $\partial_2^2$ as well, finally the whole boundary $\partial|\Delta^2|$ is identified to the base point $*$.

In this way, any point $(\sigma, t)$ of the expensive realization, where the (abstract) simplex component $\sigma$ is degenerate, can be canonically replaced by the point $(\tau, \alpha t)$ in the same realization if $(n, \tau, \alpha)$ is the Eilenberg triple of $\sigma$, where $\tau$ is a non-degenerate simplex. The non-degenerate simplices finally *do not contribute* in the realization, but they are the necessary intermediary objects to describe the possibly sophisticated attachments.

**Proposition 19** — *Both realizations, the expensive one and the cheap one, of a simplicial set $X$ are canonically homeomorphic.*

♣ The homeomorphism $f : |X| \to \|X\|$ to be constructed maps the equivalence class of the pair $(\sigma, t) \in X_m \times \Delta^m$ to the (equivalence class of the) pair $(\tau, \alpha_*(t)) \in X_n \times \Delta^n$ if the Eilenberg triple of $\sigma$ is $(n, \tau, \alpha)$. The inverse homeomorphism $g$ is induced by the canonical inclusion $\coprod X_m^{ND} \times \Delta^m \hookrightarrow \coprod X_m \times \Delta^m$. These maps must be proved coherent with the defining equivalence relations and inverse to each other.

If $\alpha = \beta\gamma$ is a $\Delta$-morphism expressed as the composition of two other $\Delta$-morphisms, then an equivalence $(\sigma, \beta_*\gamma_* t) \approx (\gamma^*\beta^*\sigma, t)$ can be considered as a consequence of the relations $(\sigma, \beta_*\gamma_* t) \approx (\beta^*\sigma, \gamma_* t)$ and $(\beta^*\sigma, \gamma_* t) \approx (\gamma^*\beta^*\sigma, t)$, so that it is sufficient to prove the coherence of the definition of $f$ with respect to the *elementary* $\Delta$-operators, that is, the face and degeneracy operators.

Let us assume the Eilenberg triple of $\sigma \in X_m$ is $(n, \tau, \alpha)$, so that $f(\sigma, t) = (\tau, \alpha_* t)$. We must in particular prove that $f(\eta_i^* \sigma, t)$ and $f(\sigma, \eta_{i*} t)$ are coherently defined. The second image is the equivalence class of $(\tau, \alpha_* \eta_{i*} t)$; the Eilenberg triple of $\eta_i^* \sigma$ is $(n, \tau, \alpha\eta_i)$ so that the first image is the equivalence class of $(\tau, (\alpha\eta_i)_* t)$ and both image representants are even equal.

Let us do now the analogous work with the face operator $\partial_i$ instead of the degeneracy operator $\eta_i$. Two cases must be considered. If ever the composition $\alpha\partial_i \in \Delta(\underline{m-1}, \underline{n})$ is surjective, the proof is the same. The interesting case happens if $\alpha\partial_i$ is not surjective; but its image then forgets exactly one element $j$ $(0 \leq j \leq n)$ and there exists a unique surjection $\beta \in \Delta(\underline{m-1}, \underline{n-1})$ such that $\alpha\partial_i = \partial_j \beta$. The abstract simplex $\partial_j^* \tau$ gives an Eilenberg triple $(n', \tau', \alpha')$ and the unique possible Eilenberg triple for $\partial_i^* \sigma$ is $(n', \tau', \beta\alpha')$. Then, on one hand, the $f$-image of $(\sigma, \partial_{i*} t)$ is $(\tau, \alpha_* \partial_{i*} t) = (\tau, \partial_{j*} \beta_* t)$; on the other hand the $f$-image of $(\partial_i^* \sigma, t)$ is $(\tau', \alpha_* \beta_* t)$; but according to the definition of the equivalence relation $\approx$ for $\|X\|$, both $f$-images are equivalent. The coherence of $f$ is proved.

Let $\sigma \in X_m^{ND}$, $0 \leq i \leq m$, $t \in \Delta^{m-1}$ and $(n, \tau, \alpha)$ (the Eilenberg triple of $\partial_i^* \sigma$) be the ingredients in the definition of the equivalence relation for $\|X\|$; the pairs $(\sigma, \partial_{i*} t)$ and $(\tau, \alpha_* t)$ are declared equivalent in $\|X\|$; the map $g$ is induced by the canonical inclusion of coproducts, so that we must prove the same pairs are also equivalent in $|X|$. But this is a transitive consequence of $(\sigma, \partial_{i*} t) \approx (\partial_i^* \sigma, t) = (\alpha^* \tau, t) \approx (\tau, \alpha_* t)$. We see here we had only described the binary relations *generating* the equivalence relation $\approx$; the defining relation is not necessarily stable under transitivity. The coherence of $g$ is proved.

The relation $fg = \mathrm{id}$ is obvious. The other relation $gf = \mathrm{id}$ is a consequence of the equivalence in $|X|$ of $(\sigma, t) \approx (\tau, \alpha_* t)$ if the Eilenberg triple of $\sigma$ is $(n, \tau, \alpha)$. ♣

### 3.5.1 Simplicial model for the real projective spaces.

Illustrating the notion of realization with the classifying spaces of discrete groups is interesting. This construction can be extended to more general simplicial groups, see [redacted].

**Definition 20** — Let $G$ be a discrete group, possibly non commutative; the unit of $G$ is denoted 1. The *classifying space* $BG$ of $G$ is the simplicial set defined as follows:

- The simplex set $BG_m$ of $m$-dimensional simplices is $BG_m = G^m$, the elements of which are called *"m-bars"* and are traditionally denoted by $\sigma = [g_1|\cdots|g_m]$: the separator '$|$', a *bar*, is here preferred to the common comma for clarity.

- Face and degenerator operators are defined by:

$$\begin{array}{rcl}
\partial_0[g_1|\cdots|g_m] & := & [g_2|\cdots|g_m]; \\
\partial_m[g_1|\cdots|g_m] & := & [g_1|\cdots g_{m-1}]; \\
\partial_i[g_1|\cdots|g_m] & := & [g_1|\cdots|g_{i-1}|g_i g_{i+1}|g_{i+2}|\cdots|g_m]; \\
\eta_i[g_1|\cdots|g_m] & := & [g_1|\cdots|g_i|1|g_{i+1}|\cdots|g_m].
\end{array}$$

In particular $BG_0 = \{[\,]\}$ has only one element.

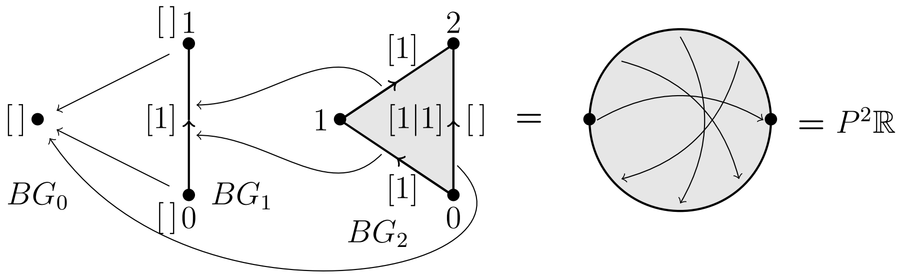
The $m$-simplex $[g_1|\ldots|g_m]$ is degenerate if and only if one of the $G$-components is the unit element.

The various commuting relations must be verified; this works but does not give obvious indications on the very nature of this construction; in fact there is a more conceptual description. Let us define the simplicial set $EG$ by $EG_m = \underline{\mathrm{Set}}(\underline{m}, G) = G^{\underline{m}} \cong G^{m+1}$, that is, the maps of $\underline{\mathrm{m}}$ to $G$ without taking account of the ordered structure of $\underline{\mathrm{m}}$ (the group $G$ is not ordered); if $\alpha \in \Delta(\underline{n}, \underline{m})$ there is a canonical way to define $\alpha : EG_m \to EG_n$; it would be fairly coherent to write $EG = G^{\Delta}$.

There is a canonical left action of the group $G$ on $EG$, and $BG$ is the natural quotient of $EG$ by this action. A simplex $\sigma \in EG_m$ is nothing but a $(m+1)$-tuple $(g_0, \ldots, g_m)$ and the action of $g$ gives the simplex $(gg_0, \ldots, gg_m)$. If two simplices are $G$-equivalent, the products $g_{i-1}^{-1} g_i$ are the same; the quotient $BG$-simplex $[g_1, \ldots, g_m]$ denotes the equivalence class of all the $EG$-simplices $(g, gg_1, gg_1g_2, \ldots)$, which can be imagined as a simplex where the *edge* between the vertices $i-1$ and $i$ $(i > 0)$ is labeled by $g_i$ to be considered as a (right) operator between the adjacent vertices. Then the boundary and degeneracy operators are clearly explained and it is even not necessary to prove the commuting relations, they can be deduced of the coherent structure of $EG$.

Let us examine what happens for the smallest non-trivial particular case, that is, the group $G$ with two elements $G = \mathbb{Z}/2\mathbb{Z} =: \mathbb{Z}_2$; it is a commutative group and we then prefer to denote 0 the “unit”. For a bar $[g_1|\cdots|g_m]$, two choices only for a component $g_i$, and the choice $g_i = 0$ implies the simplex is degenerate. So that finally exactly one non-degenerate simplex for every dimension: $G_m^{ND} = \{[1|\cdots|1]\}$.

Let us carefully examine the beginning of the construction of the realization $B\mathbb{Z}_2$. The key point is in the next figure.



Only one non-degenerate 1-simplex [1], an interval, both ends beeing identified to the unique 0-simplex [ ]: the 1-skeleton is a circle, to be understood in fact as the projective line $P^1\mathbb{R} = S^1/\mathbb{Z}_2$, that is, the circle where opposite points are identified, which is again a circle!

Only one non-degenerate 2-simplex [1|1], a triangle, with the faces $\partial_0 = [1]$, $\partial_1 = [0]$ and $\partial_2 = [1]$. The faces 0 and 2 are the non-degenerate 1-simplex, and the face 1 is degenerate, therefore collapsed over the base point, the unique vertex [ ]; after this collapsing, there "remains" in fact only two faces for our 2-simplex, both being identified with our 1-simplex [1]. Examining carefully the orientations of the faces of our triangle shows finally the 2-skeleton of our realization $|B\mathbb{Z}_2|$ is the 2-dimensional real projective space.

More generally the $m$-skeleton is the $m$-dimensional real projective space $P^m\mathbb{R}$ and the total realization $|BG|$ is the inductive limit, the infinite real projective space $P^\infty\mathbb{R}$.

In the same way, $|EG|$ is the infinite real sphere $S^\infty$ and $|BG|$ is nothing but the quotient of this sphere by the antipodal action of $\mathbb{Z}_2$.

# 4 Simplicial homology.

In Section 3.5, the strange geometric role of the *degenerate* simplices in a simplicial set has been described. It is therefore a good opportunity to introduce now the subject of *simplicial homology*, where the role, or rather the *absence* (!) of role of the degenerate simplices is also crucial.

For the most elementary notions of homological algebra, many textbooks are available. The lecture notes [13, Section 2] of another Summer School gives a careful self-contained exposition of the most elementary parts of this subject. A useful reference, for a more extended knowledge in this rich area, is [11].

**Definition 21** — Let $\mathfrak{R}$ be a unitary commutative ring, called the *coefficient ring.* Let $X = (X_n, \partial_i, \eta_i)$ be a simplicial set. The $\mathfrak{R}$-*chain complex* associated to $X$ is the object $C_*(X,\mathfrak{R}) = (C_m(X,\mathfrak{R}), d_m)_{m\in\mathbb{Z}}$ defined as follows:

- The *chain group* $C_m(X,\mathfrak{R})$ is the null group for $i < 0$, and the free $\mathfrak{R}$-module generated by the $m$-simplices $X_m$ of $X$ if $i \geq 0$.
- The *differential* $d_m : C_m(X,\mathfrak{R}) \to C_{m-1}(X,\mathfrak{R})$ is the $\mathfrak{R}$-linear map defined by:
$$d_m(\sigma) = \sum_{i=0}^{m} (-1)^i \partial_i \sigma.$$
when $\sigma \in X_m$.

In this definition, and in general in Homological Algebra, many indices, many index sets, are omitted, and the reader is assumed to be able to deduce them from context. The beginners do not like these conscious omissions, but experience shows it is necessary if you want to avoid terribly cumbersome notations, quickly making awkward formulas and diagrams. It is even an *art* in this activity to select in every situation the right indices to be displayed, and the others to keep hidden and underlying. It is also a fruitful activity for the reader to systematically elucidate what are the missing indices, to be sure of one's understanding.

For example $(X_m, \partial_i, \eta_i)$ should in principle be displayed as $(\{X_m\}_{m \in \mathbb{N}}, \{\partial_i^m\}_{m \geq 1, 0 \leq i \leq m}, \{\eta_i^m\}_{i \in \mathbb{N}, 0 \leq i \leq m})$. Taking account of the very definition of a simplicial set, the reader should admit there is a unique way to complete the first formula, a little elliptic, to obtain the second one, where everything is described. And the first formula is in fact so explicit, at least if you know the underlying definitions, that it is widely preferred.

In the most elementary situations, the coefficient ring $\mathfrak{R}$ is usually the integer ring $\mathfrak{R} = \mathbb{Z}$. Otherwise, some essentially *constant* coefficient ring is most often given, which allows to frequently omit the coefficient ring and to simply write $C_m(X, \mathfrak{R}) = C_m(X)$.

For example, let us consider the 1-circle $S^1 = S$ defined as in Section 3.4.3. Then $S_m = \{*_m\} \coprod \Delta^{\mathrm{srj}}(\underline{m}, \underline{1})$. Let us detail the chain groups $C_m(S, \mathbb{Z}) = C_m(S)$ for $m = 0, \ldots, 3$. We must firstly describe the simplices of $S_0$, $S_1$, $S_2$ and $S_2$ and their faces.

- $S_0 = \{*_0\}$, only the base point, the unique vertex of this simplicial set, no faces.
- $S_1 = \{*_1, \mathrm{id}_{\underline{1}}\}$, every face is $*_0$, no choice.
- $S_2 = \{*_2, \eta_0, \eta_1\}$, see the notations defined Section 3.1 and Proposition 3.1. For example $\partial_0(\eta_0) = \partial_1(\eta_0) = \mathrm{id}_{\underline{1}}$, but $\partial_2(\eta_0) = *_1$; this is consequence of Rule 2 in Section 3.4.3 and of the commuting relations page 10. We encourage the reader to compute in the same way the faces of $\eta_1$.
- $S_3 = \{*_3, \eta_0\eta_1, \eta_0\eta_2, \eta_1\eta_2\}$. For example, $\partial_0(\eta_0\eta_2) = \partial_1(\eta_0\eta_2) = \eta_1$ and $\partial_2(\eta_0\eta_2) = \partial_3(\eta_0\eta_2) = \eta_0$.

Knowing all these faces allows the user to compute the first terms of the chain complex canonically associated to this simplicial set $S = S^1$:

$$(C_0 = \mathbb{Z}) \xleftarrow{d_1} (C_1 = \mathbb{Z}^2) \xleftarrow{d_1} (C_2 = \mathbb{Z}^3) \xleftarrow{d_2} (C_3 = \mathbb{Z}^4)$$

where the differentials are the matrices:

$$d_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad d_3 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We observe the composition of two successive differentials are null, this is always the case.

**Proposition 22** — *If $(C_n, d)$ is the chain complex associated to a simplicial complex $X$, the composition of two successive differentials $d_q d_{q+1}$ is null. This allows to define:*

- $Z_q(X, \mathfrak{R}) := \ker d_q$ *is the group of* $q$-cycles *of* $X$.
- $B_q(X, \mathfrak{R}) := im\, d_{q+1}$ *is the group of* $q$-boundaries *of* $X$.
- *The relation* $d_q d_{q+1} = 0$*, always satisfied, is equivalent to* $B_q \subset Z_q$.
- *The quotient group* $H_q(X, \mathfrak{R}) := Z_q/B_q$ *is the* $q$-dimensional homology group with coefficients in $\mathfrak{R}$. ♣

In the example of $C_*(S)$, we can determine:

- $Z_0 = \mathbb{Z}$, $B_0 = 0$ and $H_0 = \mathbb{Z}$.
- $Z_1 = \mathbb{Z}^2$, $B_1 = \mathbb{Z}$ direct summand of $Z_1$ and $H_1 = \mathbb{Z}$.
- $Z_2 = \mathbb{Z}^2$ generated by $*_2 - \eta_0$ and $*_2 - \eta_1$, so that $B_2 = Z_2$ and $H_2 = 0$.

Writing for example $Z_0$ '=' $\mathbb{Z}$ is not correct, in fact the cycle group $Z_0$ is *isomorphic* to $\mathbb{Z}$ and *equal* only to $\mathbb{Z}*_0$, the free $\mathbb{Z}$-module generated by the unique 0-simplex $*_0$. These shorthands are common, often convenient, but can also be the source of serious drawbacks when Algebraic Topology is examined from a *constructive* point of view, see [13].

Many degenerate simplices in a simplicial set! Except for examples as simple as our circle $S^1$, it is not easy in general to compute these homology groups. In fact, from the *homological* point of view, the role of these degenerate simplices is void! The key point is the following: in general a face of a degenerate simplex can be non-degenerate, for example above $\partial_0^2 \eta_0 = \mathrm{id}_{\underline{1}}$, but the *differential*, that is, the alternate sum of faces, of a degenerate simplex is always a combination of degenerate simplices, for example $d\eta_0 = *_1$. So that we can denote by $C_*^D(X)$ the sub-chain complex generated by the degenerate simplices. It happens this chain complex is "without" homology, which, by a difference process, produces the next proposition.

**Proposition 23** — *Let* $X$ *be a simplicial set,* $C_*(X)$ *the associated chain complex,* $C_*^D(X)$ *the degenerate sub-complex and* $C_*^{ND}(X) := C_*(X)/C_*^{ND}(X)$ *the quotient chain complex. Then the canonical projection* $C_*(X) \to C_*^{ND}(X)$ *induces an isomorphism between the homology groups.*

♣ [11, VIII.6]. ♣

**Definition 24** — A *chain complex morphism* $f : C_* \to C'_*$ is a collection of linear maps $f = \{f_n : C_n \to C'_n\}$ compatible with the differentials: $df = fd$, that is, for every $n$, the relation $d_n f_n = f_{n-1} d_n$ holds.

One then says $f$ is *of degree* 0, for $f$ respects the degree. It is also possible to consider also maps of arbitrary degrees, but be careful in this case with sign coherences! Most often, the index $n$ for a component $f_n$ of a chain complex morphism is omitted, and except particular cases, elliptic formulas such as $df = fd$ are preferred.

Because of the compatibility with differentials, a chain complex morphism $f : C_* \to C'_*$ induces many natural maps, most often denoted by the same symbol $f$:

$$\begin{array}{l} f : Z_*(C_*) \to Z_*(C'_*) \\ f : B_*(C_*) \to B_*(C'_*) \\ f : H_*(C_*) \to H_*(C'_*) \end{array}$$

In fact, because of the relation $df = fd$, the image of a cycle is a cycle, the image of a boundary is a boundary, so that $f$ naturally induces a map between homology classes.

## 4.1 Homotopy and homology for simplicial complexes.

We will examine later in details the notion of *combinatorial homotopy* in the framework of simplicial sets, not so easy. Considering the particular case of simplicial *complexes* is a good introduction. Firstly, a *purely algebraic* notion of homotopy.

**Definition 25** — Let $C_*$ and $C'_*$ be two chain complexes, and $f_0, f_1 : C_* \to C'_*$ two chain complex morphisms. These morphisms are (algebraically) *homotopic* if there exists an operator $h = \{h_n : C_n \to C'_{n+1}\}_{n\in\mathbb{Z}}$ satisfying $f_1 - f_0 = dh + hd$.

$$\begin{tikzcd}
\cdots & C_{n-1} \arrow[l, dashed] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_1"] \arrow[dr, "h"] & C_n \arrow[l, "d"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_1"] \arrow[dr, "h"] & C_{n+1} \arrow[l, "d"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_1"] & \cdots \arrow[l, dashed] \\
\cdots & C'_{n-1} \arrow[l, dashed] & C'_n \arrow[l, "d"] & C'_{n+1} \arrow[l, "d"] & \cdots \arrow[l, dashed]
\end{tikzcd}$$

Our homotopy operator $h$ has *degree* $+1$. If compatible with the differentials, the relation $dh = -hd$ would be satisfied[2]; our homotopy operator is in fact not at all compatible with differentials, the "error" being just the difference $dh + hd = f_1 - f_0$.

Most topologists say such maps $f_0$ and $f_1$ are *chain equivalent*; we prefer the more coherent terminology of our definition: as illustrated later, this definition is nothing but the *algebraic* translation in the chain complex framework of the topological notion of homotopy. With a warning: we will see two maps between various sorts of topological spaces which are (topologically) homotopic induce maps *algebraically* homotopic between chain complexes, but *the converse in general is false*.

[2] Not $dh = hd$, for the "right" sign is given by the famous Koszul "rule" $dh = (-1)^{\deg(d)\cdot\deg(h)} hd$.

**Proposition 26** — *Let $f_0, f_1 : C_* \to C'_*$ be two* homotopic *chain complex morphisms. Then the induced maps $f_0, f_1 : H_*(C_*) \to H_*(C'_*)$ are* equal.

♣ Let $h \in H_*(C_*)$ be a homology class represented by some cycle $z \in Z_*(C_*)$. Then $(f_1 - f_0)(h)$ is represented by $(f_1 - f_0)(z) = (dh + hd)(z) = (dh)(z)$, for $z$ cycle means $dz = 0$. The diffference cycle $f_1(z) - f_0(z)$ therefore is the *boundary* $dh(z)$ and these cycles are homologous; in other words the homology classes $f_0(h)$ and $f_1(h)$ are *equal*. ♣

**Proposition 27** — *Let $K$ and $K'$ be two simplicial* complexes*, and $f_0, f_1 : K \to K'$ two simplicial morphisms which are (topologically)* homotopic*: see Definition 5 and the following discussion. Then the induced maps $f_0, f_1 : C_*(K) \to C_*(K')$ are (algebraically) homotopic and therefore the induced maps between homology groups $f_0, f_1 : H_*(K) \to H_*(K')$ are* equal.

This is a powerful tool for negative results: conversely, if the induced maps between homology groups are *different*, then the original continuous maps are *not* homotopic. This proposition proved here only in the simplicial complex framework in fact has a very general scope, and is at the very definition of *Algebraic* Topology: a purely algebraic observation implies topological properties.

♣ Given the hypotheses about $K$, $K'$, $f_0$ and $f_1$, we have to construct an algebraic homotopy operator $h : C_*(K) \to C_{*+1}(K')$ between both chain complex morphisms $f_0$ and $f_1$. The answer is the following:

$$h((v_0, \ldots, v_n)) := \sum_{i=0}^{n} (-1)^i (f_0 v_0, \ldots, f_0 v_i, f_1 v_i, \ldots, f_1 v_n)).$$

To save some paper and produce less $CO_2$, we verify the homotopy property only in the case $n = 1$:

$$\begin{array}{rcl}
h((v_0, v_1)) & = & (f_0v_0, f_1v_0, f_1v_1) - (f_0v_0, f_0v_1, f_1v_1); \\
dh((v_0, v_1)) & = & \quad (f_1v_0, f_1v_1)_1 - (f_0v_0, f_1v_1)_2 + (f_0v_0, f_1v_0)_3 \\
 & & -(f_0v_1, f_1v_1)_4 + (f_0v_0, f_1v_1)_2 - (f_0v_0, f_0v_1)_5; \\
d((v_0, v_1)) & = & (v_1) - (v_0); \\
hd((v_0, v_1)) & = & (f_0v_1, f_1v_1)_4 - (f_0v_0, f_1v_0)_3; \\
(f_1 - f_0)((v_0, v_1)) & = & (f_1v_0, f_1v_1)_1 - (f_0v_0, f_0v_1)_5.
\end{array}$$

where the indices after the simplex expressions show the correspondances which prove the relation $f_1 - f_0 = dh + hd$ in this particular case. The general proof is analogous, a good exercise about index handling. ♣

Is the reader really satisfied with this "proof"? He should not! We have accumulated here a terrible number of "imprecisions", let us be simple, a terrible number of *faults*. Each one is interesting and illustrates the role of the Devil of Algebraic Topology, namely the symmetric group.

The chain complex $C_*(K)$ associated to the simplicial complex $K$ is given in Definition 21, which needs in turn the explanations of Section 3.4.2, where the simplicial *set* associated to a simplicial *complex* is defined. But in the initial Definition 1 of a simplicial complex, no order over the vertices; on the contrary, when defining "the" associated simplicial set, a total order over the vertices is required. If we change this order, what happens for example for the resulting homology groups? Yes, they are the same uo to isomorphism, but the proof is not so easy.

Second difficulty, Definition 3 for a simplicial map between simplicial *complexes* does not require any compatibility conditions with vertex orders, in fact not yet considered before this definition. But after defining some orders over the vertices of $K$ and $K'$, if $f : K \to K'$ is a simplicial map, it can happen $v_0 < v_1$ in $K$ and $fv_0 > fv_1$ in $K'$, and the "induced" maps between chain groups, where the generators are made of "ordered" simplices, is then erroneously defined. Another difficulty occurs when $fv_0 = fv_1$: the image simplex is then degenerate, but we did not even mention if we preferred the total version $C_*(K)$ or the normalized one $C_*^{ND}(K)$ for "the" chain complex associated to $K$.

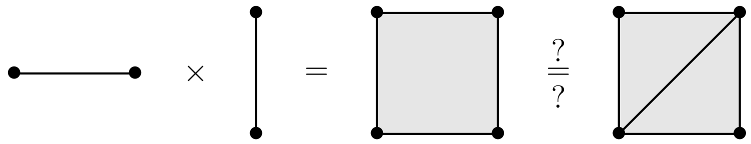
We could require the simplicial maps compatible with orders, which is very restrictive; but even with such a restriction, the mixed term $(f_0v_0, \ldots, f_0v_i, f_1v_i, \ldots, f_1v_n)$ of the formula above defining the homotopy operator $h$ can produce a simplex with vertices in a wrong order. You see this question of orders over the simplex vertices is really tough.

A really complete solution about this problem of simplex vertices can be found in [4, Chapter VI], where *two* chain complexes are associated to a simplicial complex, a big one called the *ordered* chain complex and a smaller one called the *alternating* chain complex; the last one is isomorphic to the *normalized* chain complex $C_*^{ND}(K)$ defined here through the intermediary notion of simplicial *set*; the first one accepts as generators simplices described as sequences of vertices in any order, taking account of this order; it accepts as well degenerate simplices with repetitions in the vertices. Both chain complexes have advantages and drawbacks, but their homology groups are canonically isomorphic, a frequent situation in Algebraic Topology.

# 5 Products of simplicial sets.

The general work style in Algebraic Topology consists in firstly proving results for simple spaces, next deducing analogous results for more complicated spaces constructed from these spaces. The *product* constructor is important, as in most parts of mathematics, a more sophisticated one in topology being the *twisted product* constructor, invoking fibrations.

A simple but terrible observation is to be made about products, if one works in the simplicial framework: the product of two simplices *is not* a simplex. For example a 1-simplex is an interval, the product of two intervals is a square, which cannot be naturally identified to a 2-simplex. But this square can be divided into two triangles, that is, two 2-simplices, and we must carefully organize this remark, not so easy. We will see the simplicial set structure magically gives the right solution, rather amazing!



**Definition 28** — If $X$ and $Y$ are two simplicial sets, the *simplicial product* $Z = X \times Y$ is defined by $Z_m = X_m \times Y_m$ for every natural number $m$, and $\alpha_Z^* = \alpha_X^* \times \alpha_Y^*$ if $\alpha$ is a $\Delta$-morphism.

The definition of the product of two simplicial sets is perfectly trivial and is however at the origin of several landmark problems in algebraic topology, for example the deep structure of the twisted Eilenberg-Zilber theorem, still quite mysterious, and also the enormous field around the Steenrod algebras.

Every simplex of the product $Z = X \times Y$ is a *pair* $(\sigma, \tau)$ made of one simplex in $X$ and one simplex in $Y$; both simplices must have the *same dimension*. It is tempting at this point, because of the "product" ambience, to denote by $\sigma \times \tau$ such a simplex in the product but *this would be a terrible error!* This is not at all the right point of view; the pair $(\sigma, \tau) \in Z_m$ is the unique $\boxed{m}$-simplex in $Z$ whose respective *projections* in $X$ and $Y$ are $\sigma$ and $\tau$, again some $m$-simplices, and this is the reason why the pair notation $(\sigma, \tau)$ is the only one which is possible. For example the diagonal of a square is a 1-simplex, the unique $\boxed{1}$-simplex the projections of which are both factors of the square; on the contrary, the "product" of the factors is simply the square, which does not have the dimension 1 and which is even not a simplex.

**Theorem 29** — *If $X$ and $Y$ are two simplicial sets and $Z = X \times Y$ is their simplicial product, then there exists a canonical homeomorphism between $|Z|$ and $|X| \times |Y|$, the last product being the product of $k$-spaces.*

If you consider the product $|X| \times |Y|$ as the ordinary product of topological spaces, the same accident as for CW-complexes (see [5, p.59]) can happen. The framework of $k$-spaces avoids this obstacle, reducing the problem to finite simplicial subsets. Furthermore this esoteric problem does not exist when both factors are *countable* simplicial sets (countable sets of simplices), most often the case in concrete constructive topology.

♣ There are natural simplicial projections $X \times Y \to X$ and $Y$ which define a canonical continuous map $\phi : |X \times Y| \to |X| \times |Y|$. The interesting question is to define its inverse $\psi : |X| \times |Y| \to |X \times Y|$.

First of all, let us detail the case of $X = \Delta^2$ and $Y = \Delta^1$ where the essential points are visible. The first factor $X$ has dimension 2, and the second one $Y$ has dimension 1 so that the product $Z$ shoud have dimension 3. What about the 3-simplices of $Z$? There are 3 such *non-degenerate* 3-simplices, namely $\rho_0 =$

$(\eta_0\sigma, \eta_2\eta_1\tau)$, $\rho_1 = (\eta_1\sigma, \eta_2\eta_0\tau)$ and $\rho_2 = (\eta_2\sigma, \eta_1\eta_0\tau)$, if $\sigma$ (resp. $\tau$) is the unique non-degenerate 2-simplex (resp. 1-simplex) of $\Delta^2$ (resp. $\Delta^1$). This is nothing but the decomposition of a prism $\Delta^2 \times \Delta^1$ in three tetrahedrons.

Note no non-degenerate 3-simplex is present in $X$ and $Y$ and however some 3-simplices must be produced for $Z$; this is possible thanks to the *degenerate* simplices of $X$ and $Y$ where they are again playing a quite tricky role in our workspace; in particular a pair of *degenerate* simplices in the factors can produce a *non-degenerate* simplex in the product! This happens when there is no common degeneracy in the factors.

For example the tetrahedron $\rho_0 = (\eta_0\sigma, \eta_2\eta_1\tau)$ inside $Z$ is *the* unique 3-simplex the first projection of which is $\eta_0\sigma$, and the second projection is $\eta_2\eta_1\tau$; the first projection is a tetrahedron collapsed on the triangle $\sigma$, identifying two points when the sum of barycentric coordinates of index 0 and 1 (the indices where injectivity fails in $\eta_0$) are equal; the second projection is a tetrahedron collapsed on an interval, identifying two points when the sum of barycentric coordinates of index 1, 2 and 3 are equal.

Let us take a point of coordinates $r = (r_0, r_1, r_2, r_3)$ in the simplex $\rho_0$. Its first projection is the point of $X = \Delta^2$ of barycentric coordinates $s = (s_0 = r_0 + r_1, s_1 = r_2, s_2 = r_3)$; in the same way its second projection is the point of $Y = \Delta^1$ of barycentric coordinates $t = (t_0 = r_0, t_1 = r_1 + r_2 + r_3)$. So that:

$$\phi(\rho_0, (r_0, r_1, r_2, r_3)) = ((\sigma, (r_0 + r_1, r_2, r_3)), (\tau, (r_0, r_1 + r_2 + r_3)))$$

In the same way:

$$\begin{aligned}
\phi(\rho_1, (r_0, r_1, r_2, r_3)) &= ((\sigma, (r_0, r_1 + r_2, r_3)), (\tau, (r_0 + r_1, r_2 + r_3))) \\
\phi(\rho_2, (r_0, r_1, r_2, r_3)) &= ((\sigma, (r_0, r_1, r_2 + r_3)), (\tau, (r_0 + r_1 + r_2, r_3)))
\end{aligned}$$

The challenge then consists in deciding for an arbitrary point $((\sigma, (s_0, s_1, s_2)), (\tau, (t_0, t_1))) \in |X| \times |Y|$ what simplex $\rho_i$ it comes from and what a good $\phi$-preimage $(\rho_i, r)$ could be. You obtain the solution in comparing the sums $u_0 = s_0$, $u_1 = s_0 + s_1$, $u_2 = t_0$ ; the sums $s_0 + s_1 + s_2$ and $t_0 + t_1$ are necesarily equal to 1 and do not play any role. You see in the three cases, the values of $u_i$'s are:

$$\begin{aligned}
((\eta_0\sigma, \eta_2\eta_1\tau), r) \Rightarrow \quad & u_0 = r_0 + r_1, u_1 = r_0 + r_1 + r_2, u_2 = r_0, \\
((\eta_1\sigma, \eta_2\eta_0\tau), r) \Rightarrow \quad & u_0 = r_0, u_1 = r_0 + r_1 + r_2, u_2 = r_0 + r_1, \\
((\eta_2\sigma, \eta_1\eta_0\tau), r) \Rightarrow \quad & u_0 = r_0, u_1 = r_0 + r_1, u_2 = r_0 + r_1 + r_2,
\end{aligned}$$

so that you can guess the degeneracy operators to be applied to the factors $\sigma$ and $\tau$ from the order of the $u_i$'s; more precisely, sorting the $u_i$'s puts the $u_2$ value in position 0, 1 or 2, and this gives the index for the degeneracy to be applied to $\sigma$; in the same way the $u_0$ and $u_1$ values must be installed in positions "1 and 2", or "0 and 2", or "0 and 1" and this gives the two indices (in reverse order) for the degeneracies to be applied to $\tau$. It's a question of *shuffle*. Furthermore you can find the components $r_i$ from the differences between successive $u_i$'s. Now we can construct the map $\psi$:

$$\begin{array}{rcll}\phi((\sigma,s)(\tau,t)) & = & (\rho_0,(u_2,u_0-u_2,u_1-u_0,1-u_1)) & \text{if } u_2\le u_0\le u_1,\\ & = & (\rho_1,(u_0,u_2-u_0,u_1-u_2,1-u_1)) & \text{if } u_0\le u_2\le u_1,\\ & = & (\rho_2,(u_0,u_1-u_0,u_2-u_1,1-u_2)) & \text{if } u_0\le u_1\le u_2.\end{array}$$

There seems an ambiguity occurs when there is an equality between $u_2$ and $u_0$ or $u_1$, but it is easy to see both possible preimages are in fact the same in $|Z|$.

Now this can be extended to the general case, according to the following recipe. Let $\sigma \in X_m$ and $\tau \in Y_n$ be two simplices, $s \in \Delta_m$ and $t \in \Delta^n$ two geometric points. We must define $\psi((\sigma, s),(\tau, t)) \in |Z| = |X \times Y|$. We set $u_0 = s_0$, $u_1 = s_0 + s_1$, $\ldots$, $u_{m-1} = s_0 + \ldots + s_{m-1}$, $u_m = t_0$, $u_{m+1} = t_0 + t_1$, $\ldots$, $u_{m+n-1} = t_0 + \ldots + t_{n-1}$. Then we sort the $u_i$'s according to the increasing order to obtain a sorted list $(v_0 \leq \ldots \leq v_{m+n-1})$. In particular $u_m = v_{i_0}, \ldots, u_{m+n-1} = v_{i_{n-1}}$ with $i_0 < \ldots < i_{n-1}$, and $u_0 = v_{j_0}, \ldots, u_{m-1} = v_{j_{m-1}}$ with $j_0 < \ldots < j_{m-1}$. Then:

$$\begin{array}{l}\psi((\sigma,s),(\tau,t)) = \\ ((\eta_{i_{n-1}}\ldots\eta_{i_0}\sigma,\eta_{j_{m-1}}\ldots\eta_{j_0}\tau),(v_0,v_1-v_0,\ldots,v_{m+n-1}-v_{m+n-2},1-v_{m+n-1})).\end{array}$$

Now it is easy to prove $\psi \circ \phi = \mathrm{id}_{|Z|}$ and $\phi \circ \psi = \mathrm{id}_{|X|\times|Y|}$, following the proof structure clearly visible in the case of $X = \Delta^2$ and $Y = \Delta^1$.

It is also necessary to prove the maps $\phi$ and $\psi$ are continuous. But $\phi$ is the product of the realization of two simplicial maps and is therefore continuous. The map $\psi$ is defined in a coherent way for each *cell* $\sigma \times \tau$ (this time it is really the *product* $|\sigma| \times |\tau| \subset |X| \times |Y|$) and is clearly continuous on each cell; because of the definition of the $k$-topology, the map $\psi$ is continuous. ♣

If three simplicial sets $X$, $Y$ and $Z$ are given, there is only one natural map $|X \times Y \times Z| \to |X| \times |Y| \times |Z|$ so that "both" inverses you construct by applying twice the previous construction of $\psi$, the first one going through $|X \times Y| \times |Z|$, the second one through $|X| \times |Y \times Z|$ are necessarily the same: the $\psi$-construction is *associative*, which is interesting to prove directly; it is essentially the associativity of the Eilenberg-MacLane formula.

### 5.0.1 The case of simplicial groups.

Let $G$ be a *simplicial group*. The object $G$ is a simplicial object in the group category; in other words each simplex set $G_m$ is provided with a group structure and the $\Delta$-operators $\alpha^* : G_m \to G_n$ are group homomorphisms.

This gives in particular a continuous canonical map $|G \times G| \to |G|$; then identifying $|G \times G|$ and $|G| \times |G|$, we obtain a "continuous" group structure for $|G|$; the word *continuous* is put between quotes, because this does not work in general in the topological sense: this works always only in the category of "CW-groups" where the group structure is a map $|G| \times |G| \to |G|$, the source of which being evaluated in the CW-category; because of this definition of product, it is then true that $|G| \times |G| = |G \times G|$. The composition rule so defined on $|G|$ satisfies the group axioms; in particular the associativity property comes from the considerations about the associativity of the $\psi$-construction in the previous section.

## 5.1 Kan extension condition.

Let us consider the standard simplicial model $S^1$ of the circle, with one vertex and one non-degenerate 1-simplex $\sigma$. This unique 1-simplex clearly represents a generator of $\pi_1(S^1)$, but its double cannot be so represented. This has many disadvantages and correcting this defect was elegantly solved by Kan.

**Definition 30** — A *Kan $(m, i)$-hat* (Kan hat in short) in a simplicial set $X$ is a $(m+1)$-tuple $(\sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{m+1})$ satisfying the relations $\partial_j \sigma_k = \partial_{k-1} \sigma_j$ if $j < k$, $j \neq i \neq k$.

For example the pair $(\partial_0 \mathbf{id}, \partial_1 \mathbf{id}, \partial_2 \mathbf{id},)$ is a Kan $(3,3)$-hat in the standard 3-simplex $\Delta^3$ if $\mathbf{id}$ is the unique non-degenerate 3-simplex. Also the pair $(\sigma, \sigma)$ is a Kan $(2,1)$-hat of the above $S^1$.

**Definition 31** — If $(\sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{m+1})$ is a Kan $(m, i)$-hat in the simplicial set $X$, a *filling* of this hat is a simplex $\sigma \in X_{m+1}$ such that $\partial_j \tau = \sigma_j$ for $j \neq i$.

The 3-simplex $\mathbf{id}$ of $\Delta^3$ is a filling of the example Kan hat in $\Delta^3$. The example Kan hat of $S^1$ has no filling. A Kan $(m, i)$-hat is a system of $m$-simplices arranged like all the faces except the $i$-th one of a *hypothetical* $(m+1)$-simplex.

**Definition 32** — A simplicial set $X$ satisfies the *Kan extension condition* if any Kan hat has a filling.

The standar simplex $\Delta^d$ satisfies the Kan condition. The other elementary simplicial sets in general do not.

The simplicial sets satisfying the Kan extension condition have numerous interesting properties; for example their homotopy groups can be combinatorially defined [9, Chapter 1], a canonical *minimal* version is included, also satisfying the extension condition [9, Section 9], a simple decomposition process produces a Postnikov tower [9, Section 8].

The simplicial groups are important from this point of view: in fact a simplicial group always satisfies the Kan extension condition [9, Theorem 17.1]. For example the simplicial description of $P^\infty\mathbb{R}$ (see Section **??**) is a simplicial group and therefore satisfies the Kan condition, which is not so obvious; it is even minimal. The singular complex $SX$ of a topological space $X$ also satisfies the Kan condition but in general is not minimal. These simplicial sets satisfying the Kan condition are so interesting that it is often useful to know how to *complete* an arbitrary given simplicial set $X$ and produce a new simplicial set $X'$ with the same homotopy type satisfying the Kan condition. The Kan-completed $X'$ can be constructed as follows.

Let us define first an elementary completion $\chi(X)$ for $X$. For each Kan $(m, i)$-hat of $X$, we decide to add the hypothetical $(m+1)$-simplex (even if a "solution" preexists), and the "missing" $i$-th face; such a completion operation does not change the homotopy type of $X$. Doing this completion construction for every Kan hat of $X$, we obtain the first completion $\chi(X)$. Then we can define $X_0 = X$, $X_{i+1} = \chi(X_i)$ and $X' = \lim_{\to} X_i$ is the desired Kan completion. You can also run this process in considering only the failing hats.

## 5.2 Simplicial fibrations.

A *fibration* is a map $p : E \to B$ between a *total space* $E$ and a *base space* $B$ satisfying a few properties describing more or less the total space $E$ as a *twisted product* $F \times_\tau B$. In the simplicial context, several definitions are possible. The notion of *Kan* fibration corresponds to a situation where a simplicial homotopy lifting property is satisfied; to state this property, the elementary datum is a Kan hat in the total space and a given filling of its projection in the base space; the Kan fibration property is satisfied if it is possible to fill the Kan hat in the total space in a coherent way with respect to the given filling in the base space. This notion is the simplicial version of the notion of *Serre fibration*, a projection where the homotopy lifting property is satisfied for the maps defined on polyhedra. The reference [9] contains a detailed study of the basic facts around Kan fibrations, see [9, Chapters I and II].

We will examine with a little more details the notion of *twisted cartesian product*, corresponding to the topological notion of fibre bundle. It is a key notion in topology, and the simplicial framework is particularly favourable for several reasons. In particular the Serre spectral sequence becomes well structured in this environment, allowing us to extend it up to a *constructive* version, one of the main subjects of another lecture series of this Summer School. We give here the basic necessary definitions for the notion of twisted cartesian product.

A reasonably general situation consists in considering the case where a simplicial group $G$ acts on the fiber space, a simplicial set $F$, the fiber space. As usual this means a map $\phi : F \times G \to F$ is given; source and target are simplicial sets, the first one being the product of $F$ by the simplicial set $G$, underlying the simplicial group; the map $\phi$ is a simplicial map; furthermore each component $\phi_m : (F \times G)_m = F_m \times G_m \to F_m$ must satisfy the traditional properties of the right actions of a group on a set. We will use the shorter notation $f.g$ instead of $\phi(f, g)$. Let also $B$ be our base space, some simplicial set.

**Definition 33** — A *twisting operator* $\tau : B \to G$ is a family of maps $\{\tau_m : B_m \to$

$G_{m-1}\}_{m\geq 1}$ satisfying the following properties.

1. $\partial_0\tau(b) = \tau(\partial_1 b)\tau(\partial_0 b)^{-1}$;
2. $\partial_i\tau(b) = \tau(\partial_{i+1}(b))$ if $i \leq 1$;
3. $\eta_i\tau(b) = \tau(\eta_{i+1}b)$;
4. $\tau(\eta_0 b) = e_m$ if $b \in G_{m+1}$, the unit element of $G_m$ being $e_m$.

In particular it is not required $\tau$ is a *simplicial map*, and in fact, because of the degree -1 between source and target dimensions, this does not make sense.

**Definition 34** — If a twisting operator $\tau : B \to G$ is given, the corresponding *twisted cartesian product* $E = F \times_\tau B$ is the simplicial set defined as follows. Its set of $m$-simplices $E_m$ is the same as for the non-twisted product $E_m = F_m \times B_m$; the face and degeneracy operators are also the same as for the non-twisted product with only one exception: $\partial_0(f, b) = (\partial_0 f . \tau(b), \partial_0 b)$.

The twisting operator $\tau$, the unique ingredient at the origin of a difference between the non-twisted product and the $\tau$-twisted one, acts in the following way: the twisted product is constructed in a recursive way with respect to the base dimension. Let $B^{(k)}$ be the $k$-skeleton of $B$ and let us suppose $F \times_\tau B^{(k)}$ is already constructed. Let $\sigma$ be a $(k+1)$-simplex of $B$; we must describe how the product $F \times \sigma$ is to be attached to $F \times B^{(k)}$; what is above the faces $\partial_i\sigma$ for $i \geq 1$ is naturally attached; but what is above the 0-face is shifted by the translation defined by the operation of $\tau(b)$. It is not obvious such an attachment is coherent, but the various formulas of Definition 34 are exactly the relations which must be satisfied by $\tau$ for consistency. It was not obvious, starting from scratch, to guess this is a good framework for working simplicially about fibrations; this was invented (discovered ?) by Daniel Kan [8]; the previous work by Eilenberg and MacLane [2, 3] in the particular case of the fibrations relating the elements of the Eilenberg-MacLane spectra was probably determining.

### 5.2.1 The simplest example.

Let us describe in this way the exponential fibration $\mathbf{exp} : \mathbb{R} \to S^1 : t \to e^{2\pi i t}$. We take for $S^1$ the model with one vertex $*_0$ and one non-degenerate edge $\mathrm{id}(\underline{1}) = \sigma$ (see Section **??**). For $\mathbb{R}$, we choose $\mathbb{R}_0 = \mathbb{Z}$ and $\mathbb{R}_1^{ND} = \mathbb{Z}$, that is one vertex $k_0$ and one non-degenerate edge $k_1$ for each integer $k \in \mathbb{Z}$; the faces are defined by $\partial_i(k_1) = (k+i)_0$ ($i = 0$ or $1$). The discrete (see Section **??**) simplicial group $\mathbb{Z}$ acts on the fiber; for any dimension $d$, the simplex group $\mathbb{Z}_d$ is $\mathbb{Z}$ with the natural structure, and $k_i \,.\, g = (k+g)_i$ for $i = 0$ or 1. It is then clear that the right twisting operator for the exponential fibration is $\tau(g) = 1$ for $g \in \mathbb{R}_1^{ND}$.

### 5.2.2 Fibrations between $K(\pi, n)$'s.

Let us recall (see Section **??**) $E(\pi, d)$ is the simplicial set defined by $E(\pi, d)_m = C^d(\Delta^m, \pi)$ (only *normalized* cochains) and $K(\pi, n)$ is the simplicial subset made of the *cocycles*. The maps between simplex sets to be associated to $\Delta$-morphisms are naturally defined. A simplicial projection $p : E(\pi, d) \to K(\pi, d+1)$ associating to an $m$-cochain $c$ its coboundary $\delta c$, necessarily a cocycle, is also defined. The simplicial set $\Delta^m$ is contractible, its cochain complex is acyclic and the kernel of $p$, the potential *fibre space*, is therefore the simplicial set $K(\pi, d)$. The base space is clearly the quotient of the total space by the fibre space (*principal* fibration), and a systematic examination of such a situation (see [9, Section 18]) shows $E(\pi, d)$ is necessarily a twisted cartesian product of the base space $K(\pi, d+1)$ by the fiber space $K(\pi, d)$.

It is not so easy to guess a corresponding twisting operator. A solution is described as follows; let $z \in Z^{d+1}(\Delta^m, \pi)$ a base $m$-simplex; the result $\tau(z) \in Z^d(\Delta^{m-1}, \pi)$ must be a $d$-cocyle of $\Delta^{m-1}$, that is a function defined on every $(d+1)$-tuple $(i_0, \ldots, i_d)$, with values in $\pi$, and satisfying the cocycle condition; the solution $\tau(z)(i_0, \ldots, i_d) = z(0, i_0+1, \ldots, i_d+1) - z(1, i_0+1, \ldots, i_d+1)$ works, but seems a little mysterious. The good point of view consists in considering the notion of *pseudo-section* for the studied fibration; an actual section cannot exist if the fibration is not trivial, but a pseudo-section is essentially as good as possible; see the definition of pseudo-section in [9, Section 18]. When a pseudo-section is found, a simple process produces a twisting operator; in our example, the twisting operator comes from the pseudo-section $\rho(z)(i_0, \ldots, i_d) = z(0, i_0+1, \ldots, i_d+1)$, quite natural.

The fibrations between Eilenberg-MacLane spaces are a particular case of universal fibrations associated to simplicial groups. See [9, Section 21].

### 5.2.3 Simplicial loop spaces.

A simplicial set $X$ is *reduced* if its 0-simplex set $X_0$ has only one element. We have given in Section **??** the Kan combinatorial version $GX$ of the loop space of $X$. This loop space is the fiber space of a *co-universal* fibration:

$$GX \hookrightarrow GX \times_\tau X \to X.$$

Only the twisting operator $\tau$ remains to be defined. The definition is simply… $\tau(\sigma) := \tau(\sigma)$ for both possible meanings of $\tau(\sigma)$; the first one is the value of the twisting operator to be defined for some simplex $\sigma \in X_{m+1}$ and the second one is the generator of $GX_m$ corresponding to $\sigma \in X_{m+1}$, the unit element of $GX_m$ if ever $\sigma$ is 0-degenerate (see Section **??**). The definition of the face operators for $GX$ are exactly those which are required so that the twisting operator so defined is coherent.

It is again an example of *principal fibration*, that is the fiber space is equal to the structural group and the action $GX \times GX \to GX$ is equal to the group multiplication. This fibration is co-universal, with respect to $X$; in fact, let $H \hookrightarrow H \times_{\tau'} X \xrightarrow{p} X$ another *principal* fibration above $X$ for another twisting operator $\tau' : X \to H$. Then the free group structure of $GX$ gives you a unique group homomorphism $GX \to H$ inducing a canonical morphism between both fibrations.

If the simplicial space $X$ is 1-reduced (only one vertex, no non-degenerate 1-simplex), then an important result by John Adams [?] allows one to compute the homology groups of $GX$ if the initial simplicial set $X$ is of finite type; an intermediate ingredient, the *Cobar construction*, is the key point. One of the main problems in Algebraic Topology consists in solving the analogous problem for the iterated loop spaces $G^n X$ when $X$ is $n$-reduced; it is the problem of *iterating the Cobar construction*; one of the lecture series of this Summer School is devoted to this subject, organized around a *constructive* version of Algebraic Topology.

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