A Survey on Combinatorial Duality Approach to Zero-dimensional Ideals

Teo Mora
DIMA and DISI, Università di Genova
theomora@dima.unige.it

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1 Gröbner Technology

$\mathbb{F}$ denotes an arbitrary field, $\overline{\mathbb{F}}$ denotes its algebraic closure and $\mathbb{F}_q$ denotes a finite field of size q (so q is implicitly understood to be a power of a prime) and $\mathcal{P} := \mathbb{F}[X] := \mathbb{F}[x_1, \ldots, x_n]$ the polynomial ring over the field $\mathbb{F}$.

For any ideal $\mathsf{I} \subset \mathcal{P}$ and any extension field E of $\mathbb{F}$, let $\mathcal{V}_E(\mathsf{I})$ be the rational points of I over E. We also write $\mathcal{V}(\mathsf{I}) = \mathcal{V}_{\overline{\mathbb{F}}}(\mathsf{I})$.

Let $\mathcal{T}$ be the set of terms in $\mathcal{P}$, *id est*

$$\mathcal{T} := \{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \ldots, a_n) \in \mathbb{N}^n\},$$

which is a multiplicative version of the additive semigroup $\mathbb{N}^n$, the relation between these notations being obvious: given

$$\alpha := (a_1, \ldots, a_n), \quad \beta := (b_1, \ldots, b_n), \quad \gamma := (c_1, \ldots, c_n)$$

and the terms

$$\tau_a := X^\alpha = x_1^{a_1} \cdots x_n^{a_n}, \quad \tau_b := X^\beta = x_1^{b_1} \cdots x_n^{b_n}, \quad \tau_c := X^\gamma = x_1^{c_1} \cdots x_n^{c_n},$$

we have

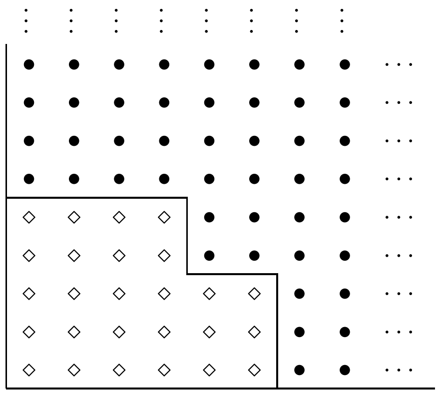
$$\begin{array}{rcccc} \tau_a \cdot \tau_b = \tau_c & \iff & a_i + b_i = c_i \text{ for each } i & \iff & \alpha + \beta = \gamma \\ \tau_a \mid \tau_b & \iff & a_i \le b_i \text{ for each } i & \iff & \alpha \le_P \beta \end{array}$$

where $<_P$ is the natural partial ordering over $\mathbb{N}^n$.

The assignement of a finite set of terms

$$G := \{\tau_1, \ldots, \tau_\nu\} \subset \mathcal{T}, \tau_i = x_1^{a_1^{(i)}} \cdots x_n^{a_n^{(i)}}$$

Figure 1:



— or, equivalently of a finite set of integer vectors

$$\{a^{(1)},\ldots,a^{(\nu)}\}\subset\mathbb{N}^n, a^{(i)}=(a_1^{(i)},\ldots,a_n^{(i)})\in\mathbb{N}^n,$$

defines a partition of $\mathcal{T}$ (resp. $\mathbb{N}^n$) in two parts (see Figure 1 where $G := \left\{x_1^7, x_1^5x_2^3, x_2^5\right\} \subset \mathcal{T}$):

- $T := \{\tau\tau_i : \tau \in \mathcal{T}, 1 \le i \le \nu\} \cong \{\alpha + a^{(i)} : \alpha \in \mathbb{N}^n, 1 \le i \le \nu\} =: \Sigma$ which is a *semigroup ideal*, *id est* a subset $T \subset \mathcal{T}$(resp. $\Sigma \subset \mathbb{N}^n$) such that

$$\tau\in T, t\in\mathcal{T} \implies t\tau\in T, \text{ resp. } a\in\Sigma, b\in\mathbb{N}^n, a\le_P b \implies b\in\Sigma;$$

$\diamond$ $N := \mathcal{T} \setminus T \cong \mathbb{N}^n \setminus \Sigma =: \Delta$ which is an *order ideal*, *id est* a subset $N \subset \mathcal{T}$ (resp. $\Delta \subset \mathbb{N}^n$) such that

$$\tau\in N, t\in\mathcal{T}, t\mid\tau \implies t\in N, \text{ resp. } a\in\Delta, b\in\mathbb{N}^n, a\ge_P b \implies b\in\Delta.$$

Remark that the assignement of

- a finite monomial set $G \subset \mathcal{T}$,
- a semigroup ideal $T \subset \mathcal{T}$,
- an order ideal $N \subset \mathcal{T}$

uniquely characterize the other data: in fact

- N and T are related by their being complementary in $\mathcal{T}$,
- each semigroup ideal $T \subset \mathcal{T}$ has a unique minimal basis $G \subset T$ such that $T := \{\tau\tau_i : \tau \in \mathcal{T}, \tau_i \in G\}$; the fact, whose proof is quite involved, that G is finite is known as Dickson's Lemma but actually was already proved by Gordan [29].

We recall that the well-orderings on $\mathcal{T}$ which are a *semigroup ordering*, *id est* satysfy

$$\tau_1 < \tau_2 \implies \tau\tau_1 < \tau\tau_2 \text{ for each } \tau, \tau_1, \tau_2 \in \mathcal{T}$$

are called *term orderings*, even if the old-fashioned notion of *admissible ordering* can still be found somewhere.

For a free-module $\mathcal{P}^m$, $m \in \mathbb{N}$, denote $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ its canonical basis,

$$\begin{aligned} \mathcal{T}^{(m)} &= \{t\mathbf{e}_i, t \in \mathcal{T}, 1 \leq i \leq m\} = \\ &= \{x_1^{a_1} \cdots x_n^{a_n}\mathbf{e}_i, (a_1, \ldots, a_n) \in \mathbb{N}^n, 1 \leq i \leq m\} \end{aligned}$$

its monomial $\mathbb{F}$-basis and $\prec$ a well-ordering on $\mathcal{T}^{(m)}$ which is compatible with the term-ordering $<$ on $\mathcal{T}$, that is, satisfying

$$t_1 \leq t_2, \tau_1 \preceq \tau_2 \implies t_1\tau_1 \preceq t_2\tau_2$$

for each $t_1, t_2 \in \mathcal{T}, \tau_1, \tau_2 \in \mathcal{T}^{(m)}$.

Note that $\mathcal{T}^{(1)} = \mathcal{T}$.

For each $f = \sum_{\tau \in \mathcal{T}^{(m)}} \mathsf{c}(f, \tau)\tau \in \mathcal{P}^m$, its *support* is

$$\operatorname{supp}(f) := \{\tau \in \mathcal{T}^{(m)} : \mathsf{c}(f, \tau) \neq 0\},$$

its *leading term* is the term $\mathbf{T}_\prec(f) := \max_\prec(\operatorname{supp}(f))$, its *leading coefficient* is $\operatorname{lc}_\prec(f) := \mathsf{c}(f, \mathbf{T}_\prec(f))$ and its *leading monomial* is $\mathbf{M}_\prec(f) := \operatorname{lc}_\prec(f)\mathbf{T}_\prec(f)$.

When $\prec$ is understood we will drop the subscript, as in $\mathbf{T}(f) = \mathbf{T}_\prec(f)$.

For any set $F \subset \mathcal{P}^m$, write

- $\mathbf{T}\{F\} := \mathbf{T}_\prec\{F\} := \{\mathbf{T}(f) : f \in F\}$;
- $\mathbf{M}\{F\} := \mathbf{M}_\prec\{F\} := \{\mathbf{M}(f) : f \in F\}$;
- $\mathbf{T}(F) := \mathbf{T}_\prec(F) := \{\tau\mathbf{T}(f) : \tau \in \mathcal{T}, f \in F\}$, a *monomial module*[1];
- $\mathbf{N}(F) := \mathbf{N}_\prec(F) := \mathcal{T}^{(m)} \setminus \mathbf{T}_\prec(F)$, an *order module*[2];
- $\mathbb{I}(F) = \langle F \rangle$ the module generated by F.

Remark that, if $m = 1$, the assignment of $\mathbf{T}\{F\}$ gives the partition $\mathcal{T} = \mathbf{T}(F) \sqcup \mathbf{N}(F)$ discussed above, that the related semigroup ideal $\mathbf{T}(F)$ is also denoted $\Sigma(F)$ while the related order ideal $\mathbf{N}(F)$ is also denoted $\Delta(F)$ and labelled Δ-*set* or *footprint*. When F is the Gröbner basis of the module $\mathbb{I}(F)$ it generates, $\mathbf{N}(F)$ is called the *Gröbner éscalier*[26] of $\mathbb{I}(F)$.

We can now however induce a finer partition of $\mathcal{T}^{(m)}$ in terms of a module $\mathsf{M} \subset \mathcal{P}^m$ and a term-ordering $\prec$, by defining (see Figure 2 where again $\mathsf{M} = \mathbb{I}(x_1^7, x_1^5x_2^3, x_2^5) \subset \mathcal{P}$)

$\diamond$ $\mathbf{N}_\prec(\mathsf{M}) = \mathcal{T}^{(m)} \setminus \mathbf{T}_<(\mathsf{M})$ its *Gröbner éscalier*;

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[1] *Id est* a subset $T \subset \mathcal{T}^{(m)}$ such that $\tau \in T, t \in \mathcal{T} \implies t\tau \in T$.

[2] *Id est* a subset $T \subset \mathcal{T}^{(m)}$ such that $t\tau \in T, t \in \mathcal{T} \implies \tau \in T$.

$\circ$ $\mathbf{B}_\prec(\mathsf{M}) := \{x_h\tau : 1 \leq h \leq n, \tau \in \mathbf{N}_\prec(\mathsf{M})\} \setminus \mathbf{N}_\prec(\mathsf{M})$, its *border set*;

$\bullet$ $\mathbf{J}_\prec(\mathsf{M}) := \mathbf{T}_\prec(\mathsf{M}) \setminus \mathbf{B}_\prec(\mathsf{M})$,

$*$ $\mathbf{G}_\prec(\mathsf{M}) \subset \mathbf{B}_\prec(\mathsf{M})$ the unique minimal basis of $\mathbf{T}_\prec(\mathsf{M})$,

$\cdot$ $\mathbf{C}_\prec(\mathsf{M}) := \{\tau \in \mathbf{N}_\prec(\mathsf{M}) : x_h\tau \in \mathbf{T}_\prec(\mathsf{M}), \forall h\}$ its *corner set*.

Under this notation, the following properties are trivially satisfied:

Lemma 1 *It holds*

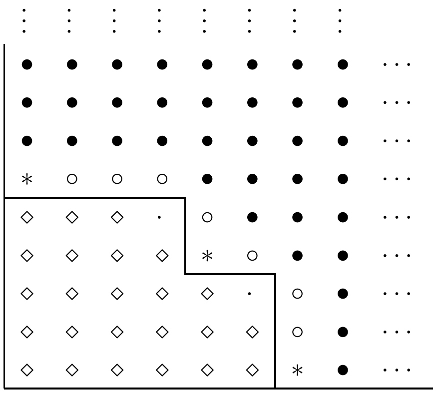
1. $\mathbf{T}_\prec(\mathsf{M}) = \{\tau \in \mathcal{T} : \exists g \in \mathsf{M} : \mathbf{T}_\prec(g) = \tau\}$;
2. $\mathbf{J}_\prec(\mathsf{M}) = \left\{\tau \in \mathbf{T}_\prec(\mathsf{M}) : x_i \mid \tau \implies \frac{\tau}{x_i} \in \mathbf{T}_\prec(\mathsf{M})\right\}$;
3. $\mathbf{B}_\prec(\mathsf{M}) = \left\{\tau \in \mathbf{T}_\prec(\mathsf{M}) : \exists x_i \mid \tau, \frac{\tau}{x_i} \in \mathbf{N}_\prec(\mathsf{M})\right\}$;
4. $\mathbf{G}_\prec(\mathsf{M}) = \left\{\tau \in \mathbf{T}_\prec(\mathsf{M}) : \forall x_i \mid \tau, \frac{\tau}{x_i} \in \mathbf{N}_\prec(\mathsf{M})\right\}$;
5. $\mathbf{C}_\prec(\mathsf{M}) = \{\tau \in \mathbf{N}_\prec(\mathsf{M}) : \forall i, x_i\tau \in \mathbf{B}_\prec(\mathsf{M})\}$;
6. $\mathbf{N}_\prec(\mathsf{M}) = \{\tau \in \mathcal{T} : \nexists g \in \mathsf{M} : \mathbf{T}_\prec(g) = \tau\}$;
7. $\mathbf{C}_\prec(\mathsf{M}) \cup \mathbf{T}_\prec(\mathsf{M})$ *is a monomial module;*
8. $\mathbf{N}_\prec(\mathsf{M}) \cup \mathbf{G}_\prec(\mathsf{M})$ *and* $\mathbf{N}_\prec(\mathsf{M}) \cup \mathbf{B}_\prec(\mathsf{M})$ *are order modules.*
9. $\tau \in \mathbf{J}_\prec(\mathsf{M}) \iff \forall x_i \mid \tau, \frac{\tau}{x_i} \in \mathbf{T}_\prec(\mathsf{M})$;
10. $\tau \in \mathbf{B}_\prec(\mathsf{M}) \setminus \mathbf{G}_\prec(\mathsf{M}) \iff \exists h, H : \frac{\tau}{x_h} \in \mathbf{N}_\prec(\mathsf{M}), \frac{\tau}{x_H} \in \mathbf{B}_\prec(\mathsf{M}) \subset \mathbf{T}_\prec(\mathsf{M})$;
11. $\tau \in \mathbf{B}_\prec(\mathsf{M}) \setminus \mathbf{G}_\prec(\mathsf{M}) \implies \forall x_i \mid \tau, \frac{\tau}{x_i} \in \mathbf{N}_\prec(\mathsf{M}) \cup \mathbf{B}_\prec(\mathsf{M})$;
12. $\tau \in \mathbf{N}_\prec(\mathsf{M}) \cup \mathbf{G}_\prec(\mathsf{M}) \iff \forall x_i \mid \tau, \frac{\tau}{x_i} \in \mathbf{N}_\prec(\mathsf{M})$;
13. $\tau \in \mathbf{T}_\prec(\mathsf{M}) \cup \mathbf{C}_\prec(\mathsf{M}) \iff \forall i, x_i\tau \in \mathbf{T}_\prec(\mathsf{M})$;
14. $\tau \in \mathbf{N}_\prec(\mathsf{M}) \setminus \mathbf{C}_\prec(\mathsf{M}) \iff \exists h : x_h\tau \in \mathbf{N}_\prec(\mathsf{M})$. $\square$

Lemma 2 *Let* N *be a finitely generated* $\mathcal{P}$*-module,* $\Phi : \mathcal{P}^m \mapsto \mathsf{N}$ *be any surjective morphism and set* $\mathsf{M} := \ker(\Phi)$. *Then*

1. $\mathcal{P}^m \cong \mathsf{M} \oplus \mathrm{Span}_{\mathbb{F}}(\mathbf{N}(\mathsf{M}))$;
2. $\mathsf{N} \cong \mathrm{Span}_{\mathbb{F}}(\mathbf{N}(\mathsf{M}))$;
3. *for each* $f \in \mathcal{P}^m$, *there is a unique* $g := \mathrm{Can}(f, \mathsf{M}, \prec) \in \mathrm{Span}_{\mathbb{F}}(\mathbf{N}(\mathsf{M}))$ *such that* $f - g \in \mathsf{M}$.

   Such g *is called the* canonical form *of* f *w.r.t.* M *and satisfies also:*

Figure 2:



(a) $\mathrm{Can}(f_1, \mathsf{M}, \prec) = \mathrm{Can}(f_2, \mathsf{M}, \prec) \iff f_1 - f_2 \in \mathsf{M}$;

(b) $\mathrm{Can}(f, \mathsf{M}, \prec) = 0 \iff f \in \mathsf{M}$. □

Definition 3 *Let* N *be a finitely generated* $\mathcal{P}$*-module,* $\Phi : \mathcal{P}^m \mapsto \mathsf{N}$ *be any surjective morphism and set* $\mathsf{M} := \ker(\Phi)$.

Let $G \subset \mathsf{M}$, $f, h, f_1, f_2 \in \mathcal{P}^m$*. Then*

1. *G will be called a* Gröbner basis *of* M *if*

$$\mathbf{T}(G) = \mathbf{T}(\mathsf{M}),$$

that is, $\mathbf{T}\{G\} := \{\mathbf{T}(g) : g \in G\}$ *generates* $\mathbf{T}(\mathsf{M}) = \mathbf{T}\{\mathsf{M}\}$.

2. *For each* $f_1, f_2 \in \mathcal{P}^m$ *such that*

$$\mathbf{T}(f_1) = t_1 \mathbf{e}_{i_1}, \mathbf{T}(f_2) = t_2 \mathbf{e}_{i_2},$$

the S-polynomial of f_1 and f_2 *exists only if* $\mathbf{e}_{i_1} = \mathbf{e}_{i_2} := \epsilon$*, in which case it is*

$$S(f_1, f_2) := \mathrm{lc}(f_2)^{-1} \frac{\delta(f_1, f_2)}{t_2} f_2 - \mathrm{lc}(f_1)^{-1} \frac{\delta(f_1, f_2)}{t_1} f_1,$$

where $\delta := \delta(f_1, f_2) := \mathrm{lcm}(t_1, t_2)$*;* $\delta\epsilon$ *is called the* formal term *of* $S(f_1, f_2)$.

3. *f has a* Gröbner representation $\sum_{i=1}^{\mu} p_i g_i$ *in terms of* G *if* [3]

$$f = \sum_{i=1}^{\mu} p_i g_i, p_i \in \mathcal{P}, g_i \in G, \mathbf{T}(p_i)\mathbf{T}(g_i) \preceq \mathbf{T}(f), \text{ for each } i.$$

[3]note that here, unilike in (4), we are not assuming $i \neq j \implies \mathbf{T}(p_i)\mathbf{T}(g_i) \neq \mathbf{T}(p_j)\mathbf{T}(g_j)$; moreover both here, in (4) and in (5) a same element of G can repeatedly appear.

4. *f has the* (strong) Gröbner representation $\sum_{i=1}^{\mu} c_i t_i g_i$ *in terms of G if*

$$f = \sum_{i=1}^{\mu} c_i t_i g_i, c_i \in \mathbb{F} \setminus \{0\}, t_i \in \mathcal{T}, g_i \in G,$$

with $\mathbf{T}(f) = t_1\mathbf{T}(g_1) \succ \cdots \succ t_i\mathbf{T}(g_i) \succ \cdots$.

5. *f has the* weak Gröbner representation $\sum_{i=1}^{\mu} c_i t_i g_i$ *in terms of G if*

$$f = \sum_{i=1}^{\mu} c_i t_i g_i, c_i \in \mathbb{F} \setminus \{0\}, t_i \in \mathcal{T}, g_i \in G,$$

with $\mathbf{T}(f) = t_1\mathbf{T}(g_1) \succeq \cdots \succeq t_i\mathbf{T}(g_i) \succeq \cdots$.

6. *For any $f_1, f_2 \in \mathcal{P}^m$, whose S-polynomial exists and has $\delta\epsilon$ as formal term, we say that $S(f_1, f_2)$ has a* quasi-Gröbner representation *in terms of G if it can be written as $S(g, f) = \sum_{k=1}^{\mu} p_k g_k$, with $p_k \in \mathcal{P}, g_k \in G$* and $\mathbf{T}(p_k)\mathbf{T}(g_k) \prec \delta\epsilon$ for each k.

7. *$h := \mathrm{NF}_{\prec}(f, G)$ is called a* normal form *of f w.r.t. G, if*
   - *$f - h \in \mathbb{I}(G)$ has a strong Gröbner representation in terms of G and*
   - *$h \neq 0 \implies \mathbf{T}(h) \notin \mathbf{T}(G)$.*

8. *The* reduced Gröbner basis *of M wrt $\prec$ is the set*

$$\{\tau - \mathrm{Can}(\tau, \mathsf{M}, \prec) : \tau \in \mathbf{G}_{\prec}(\mathsf{M})\}.$$

9. *The* border basis *of M w.r.t. $\prec$ is the set*

$$\{\tau - \mathrm{Can}(\tau, \mathsf{M}, \prec) : \tau \in \mathbf{B}_{\prec}(\mathsf{M})\}.$$

10. *A* Gröbner representation *of M is the assignment of*
    - *a linearly independent set $\mathbf{q} = \{q_1, \ldots, q_s\}$ ($q_1 = 1$), where $s = \#(\mathbf{N}(\mathsf{M}))$, such that $\mathcal{P}^m/\mathsf{M} = \mathrm{Span}_{\mathbb{F}}(\mathbf{q})$,*
    - *the set*
    $$\mathcal{M} = \mathcal{M}(\mathbf{q}) := \left\{\left(a_{lj}^{(h)}\right) \in \mathbb{F}^{s^2}, 1 \leq h \leq n\right\}$$
    of the $s \times s$ square matrices $\left(a_{lj}^{(h)}\right)$ defined by the equalities
    $$x_h q_l = \sum_j a_{lj}^{(h)} q_j, \forall l, j, h, 1 \leq l, j \leq s, 1 \leq h \leq n$$
    in $\mathcal{P}^m/\mathsf{M} = \mathrm{Span}_{\mathbb{F}}(\mathbf{q})$.

11. *For each $f \in \mathcal{P}$ the* Gröbner description *of f in terms of a Gröbner representation $(\mathbf{q}, \mathcal{M})$ is the unique vector*

$$\mathbf{Rep}(f, \mathbf{q}) := (\gamma(f, q_1, \mathbf{q}), \dots, \gamma(f, q_s, \mathbf{q})) \in \mathbb{F}^s$$

such that $f - \sum_j \gamma(f, q_j, \mathbf{q}) q_j \in \mathsf{M}$.

12. *The* linear representation *of M w.r.t. $\prec$ is the Gröbner representation $(\mathbf{N}_\prec(\mathsf{M}), \mathcal{M}(\mathbf{N}_\prec(\mathsf{M})))$ where $\mathbf{q} = \mathbf{N}_\prec(\mathsf{M})$.* □

With these definitions, if $\mathbf{N}_\prec(\mathsf{M}) = \{\tau_1, \dots, \tau_s\}$, *the Gröbner description*

$$\mathbf{Rep}(f, \mathbf{N}_\prec(\mathsf{M})) := (\gamma(f, \tau_1, \mathbf{N}_\prec(\mathsf{M})), \dots, \gamma(f, \tau_s, \mathbf{N}_\prec(\mathsf{M})))$$

of f in terms of the linear representation of M *w.r.t.* $\prec$ is a convoluted synonym of the notion of the canonical form

$$\mathrm{Can}(f, \mathsf{M}, \prec) = \sum_{j=1}^{s} \gamma(f, \tau_j, \prec) \tau_j = \sum_{j=1}^{s} \gamma(f, \tau_j, \mathbf{N}_\prec(\mathsf{M})) \tau_j$$

of f in terms of $\prec$.

2 Duality (1)

Denote $\mathcal{P}^* := \mathrm{Hom}_{\mathbb{F}}(\mathcal{P}, \mathbb{F})$ the $\mathbb{F}$-vector space of all $\mathbb{F}$-linear functionals $\ell : \mathcal{P} \mapsto \mathbb{F}$ and remark that it holds $f \in \mathcal{P}, \ell \in \mathcal{P}^* \implies \ell(f) = \sum_{\tau \in \mathcal{T}} \mathsf{c}(f, \tau) \ell(\tau)$ and that $\mathcal{P}^*$ is made a $\mathcal{P}$-module defining $\forall \ell \in \mathcal{P}^*, f \in \mathcal{P}$, $\ell \cdot f \in \mathcal{P}^*$ as $(\ell \cdot f)(g) := \ell(fg) \forall g \in \mathcal{P}$.

Two sets $\mathbb{L} = \{\ell_1, \dots, \ell_r\} \subset \mathcal{P}^*$ and $\mathbf{q} = \{q_1, \dots, q_s\} \subset \mathcal{P}$ are said to be

- *triangular* if $r = s$ and $\ell_i(q_j) = 0$, for each $i < j$;
- *biorthogonal* if $r = s$ and $\ell_i(q_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

For each $\mathbb{F}$-vector subspace $L \subset \mathcal{P}^*$, let

$$\mathfrak{P}(L) := \{g \in \mathcal{P} : \ell(g) = 0, \forall \ell \in L\}$$

and, for each $\mathbb{F}$-vector subspace $P \subset \mathcal{P}$, let

$$\mathfrak{L}(P) := \{\ell \in \mathcal{P}^* : \ell(g) = 0, \forall g \in P\}.$$

Lemma 4 *For each $\mathbb{F}$-vector subspaces $P, P_1, P_2 \subset \mathcal{P}$ and each $\mathbb{F}$-vector subspaces $L, L_1, L_2 \subset \mathcal{P}^*$ it holds*

1. *if P is an ideal then $\mathfrak{L}(P)$ is a $\mathcal{P}$-module.*
2. *if L is a $\mathcal{P}$-module then $\mathfrak{P}(L)$ is an ideal.*

3. $P_1 \subset P_2 \implies \mathfrak{L}(P_1) \supset \mathfrak{L}(P_2)$;
4. $L_1 \subset L_2 \implies \mathfrak{P}(L_1) \supset \mathfrak{P}(L_2)$;
5. $\mathfrak{L}(P_1 \cap P_2) \supset \mathfrak{L}(P_1) + \mathfrak{L}(P_2)$;
6. $\mathfrak{P}(L_1 \cap L_2) \supset \mathfrak{P}(L_1) + \mathfrak{P}(L_2)$;
7. $\mathfrak{L}(P_1 + P_2) = \mathfrak{L}(P_1) \cap \mathfrak{L}(P_2)$;
8. $\mathfrak{P}(L_1 + L_2) = \mathfrak{P}(L_1) \cap \mathfrak{P}(L_2)$.
9. $P = \mathfrak{P}\mathfrak{L}(P)$.
10. $L \subset \mathfrak{L}\mathfrak{P}(L)$;
11. $\dim_{\mathbb{F}}(L) < \infty \implies L = \mathfrak{L}\mathfrak{P}(L)$; $\square$

id est $\mathfrak{P}$ and $\mathfrak{L}$ define a dulaity between finite dimensional $\mathcal{P}$-modules of functionals and zero-dimensional ideals.

3 Möller's Algorithm

Let $\mathbb{L} = \{\ell_1, \dots, \ell_s\} \subset \mathcal{P}^*$ be a (not necessarily linearly indipendent) set of $\mathbb{F}$-linear functionals such that $L := \mathrm{Span}_{\mathbb{F}}(\mathbb{L})$ is a $\mathcal{P}$-module, and let us denote, for each $f \in \mathcal{P}$,

$$v(f, \mathbb{L}) := (\ell_1(f), \dots, \ell_s(f)) \in \mathbb{F}^s.$$

Since $\dim_{\mathbb{F}}(L) < \infty$ then $\mathsf{I} := \mathfrak{P}(L)$ is a zero-dimensional ideal and

$$\#(\mathbf{N}(\mathsf{I})) = \deg(\mathsf{I}) = \dim_{\mathbb{F}}(L) =: r \leq s;$$

therefore, denoting

$$\mathbf{N}(\mathsf{I}) = \{t_1, \dots, t_r\}, \quad 1 = t_1 < \dots < t_i < t_{i+1} < \dots < t_r,$$

we can consider the $s \times r$ matrix $\ell_i(t_j)$ whose columns are the vectors $v(t_j, \mathbb{L})$ and are linearly independent, since any relation $\sum_j c_j v(t_j, \mathbb{L}) = 0$ would imply

$$\ell_i(\sum_j c_j t_j) = \sum_j c_j \ell_i(t_j) = 0 \text{ and } \sum_j c_j t_j \in \mathfrak{P}(L) = \mathsf{I}$$

contradicting the definition of $\mathbf{N}(\mathsf{I})$.

The matrix $\ell_i(t_j)$ has rank $r \leq s$ and it is possible to extract an ordered subset $\Lambda := \{\lambda_1, \dots, \lambda_r\} \subset \mathbb{L}$, satisfying $\mathrm{Span}_{\mathbb{F}}\{\Lambda\} = \mathrm{Span}_{\mathbb{F}}\{\mathbb{L}\}$ and to re-enumerate the terms in $\mathbf{N}(\mathsf{I})$ in such a way that each principal minor $\lambda_i(t_j), 1 \leq i, j \leq \sigma \leq r$ is invertible. Therefore, if we consider a set

$$\mathbf{q} := \{q_1, \dots, q_r\} \subset \mathcal{P}$$

which is triangular w.r.t. $\mathbb{L}$, and (a_{ij}) denotes the invertible matrix such that $q_i = \sum_{j=1}^r a_{ij} t_j, \forall i \leq r$, then for each $\sigma \leq r$

- $\{q_1, \dots, q_\sigma\}$ and $\{\lambda_1, \dots, \lambda_\sigma\}$ are triangular;
- $\mathrm{Span}_{\mathbb{F}}\{t_1, \dots, t_\sigma\} = \mathrm{Span}_{\mathbb{F}}\{q_1, \dots, q_\sigma\}$;
- (a_{ij}) is lower triangular.

If we now further assume that

1. $\dim_{\mathbb{F}}(L) = r = s$ and
2. each subvectorspace $L_\sigma := \mathrm{Span}_{\mathbb{F}}(\{\ell_1, \dots, \ell_\sigma\})$ is a $\mathcal{P}$-module

so that each $\mathsf{I}_\sigma = \mathfrak{P}(L_\sigma)$ is a zero-dimensional ideal and there is a chain

$$\mathsf{I}_1 \supset \mathsf{I}_2 \supset \cdots \supset \mathsf{I}_s = \mathsf{I},$$

then we have

- $\lambda_\sigma = \ell_\sigma, \forall\sigma$
- $\mathbf{N}(\mathsf{I}_\sigma) = \{t_1, \dots, t_\sigma\}$ is an order ideal $\forall\sigma$
- $\mathsf{I}_\sigma \oplus \mathrm{Span}_{\mathbb{F}}\{q_1, \dots, q_\sigma\} = \mathcal{P}, \forall\sigma$
- $\mathbf{T}(q_\sigma) = t_\sigma, \forall\sigma$.

In conclusion we have proved

Theorem 5 (Möller) *Let $\mathcal{P} := \mathbb{F}[x_1, \dots, x_n]$, and $<$ be any termordering. Let $\mathbb{L} = \{\ell_1, \dots, \ell_s\} \subset \mathcal{P}^*$ be a set of $\mathbb{F}$-linear functionals such that $\mathfrak{P}(\mathrm{Span}_{\mathbb{F}}(\mathbb{L}))$ is a zero-dimensional ideal.*

Then there are

- *an integer $r \in \mathbb{N}$,*
- *an order ideal $\mathbf{N} := \{t_1, \dots, t_r\} \subset \mathcal{T}$,*
- *an ordered subset $\Lambda := \{\lambda_1, \dots, \lambda_r\} \subset \mathbb{L}$,*
- *an ordered set $\mathbf{q} := \{q_1, \dots, q_r\} \subset \mathcal{P}$,*

such that, denoting $L := \mathrm{Span}_{\mathbb{F}}(\mathbb{L})$ and $\mathsf{I} := \mathfrak{P}(L)$, it holds:

1. $r = \deg(\mathsf{I}) = \dim_{\mathbb{F}}(\mathbb{L})$,
2. $\mathbf{N}(\mathsf{I}) = \mathbf{N}$,
3. $\mathrm{Span}_{\mathbb{F}}(\Lambda) = \mathrm{Span}_{\mathbb{F}}(\mathbb{L})$,
4. $\mathrm{Span}_{\mathbb{F}}\{t_1, \dots, t_\sigma\} = \mathrm{Span}_{\mathbb{F}}\{q_1, \dots, q_\sigma\}, \forall\sigma \le r$,
5. $\{q_1, \dots, q_\sigma\}$, $\{\lambda_1, \dots, \lambda_\sigma\}$ *are triangular,* $\forall\sigma \le r$.

If, moreover, we have

- $\dim_{\mathbb{F}}(L) = r = s$ *and*
- $L_\sigma := \mathrm{Span}_{\mathbb{F}}(\{\ell_1, \dots, \ell_\sigma\})$ *is a* $\mathcal{P}$*-module,* $\forall\sigma$*,*

then it further holds

6. $\lambda_\sigma = \ell_\sigma$*,*
7. $\mathbf{N}(\mathsf{I}_\sigma) = \{t_1, \dots, t_\sigma\}$ *is an order ideal,*
8. $\mathsf{I}_\sigma \oplus \mathrm{Span}_{\mathbb{F}}\{q_1, \dots, q_\sigma\} = \mathcal{P}$*,*
9. $\mathbf{T}(q_\sigma) = t_\sigma$

for each $\sigma \leq r$*, where* $\mathsf{I}_\sigma = \mathfrak{P}(L_\sigma)$*.* □

Corollary 6 (Lagrange Interpolation Formula) *Let* $\mathcal{P} := \mathbb{F}[x_1, \dots, x_n]$*,* $<$ *be any termordering.* $\mathbb{L} = \{\ell_1, \dots, \ell_s\} \subset \mathcal{P}^*$ *be a set of* $\mathbb{F}$*-linear functionals such that* $\mathsf{I} := \mathfrak{P}(\mathrm{Span}_{\mathbb{F}}(\mathbb{L}))$ *is a 0-dim. ideal.*

There exists a set $\mathbf{q} = \{q_1, \dots, q_s\} \subset \mathcal{P}$ *such that*

1. $q_i = \mathrm{Can}(q_i, \mathsf{I}) \in \mathrm{Span}_{\mathbb{F}}(\mathbf{N}(\mathsf{I}))$;
2. $\mathbb{L}$ *and* $\mathbf{q}$ *are triangular;*
3. $\mathcal{P}/\mathsf{I} \cong \mathrm{Span}_{\mathbb{F}}(\mathbf{q})$.

There exists a set $\mathbf{q}' = \{q'_1, \dots, q'_s\} \subset \mathcal{P}$ *such that*

1. $q'_i = \mathrm{Can}(q'_i, \mathsf{I}) \in \mathrm{Span}_{\mathbb{F}}(\mathbf{N}(\mathsf{I}))$;
2. $\mathbb{L}$ *and* $\mathbf{q}'$ *are biorthogonal;*
3. $\mathcal{P}/\mathsf{I} \cong \mathrm{Span}_{\mathbb{F}}(\mathbf{q}')$.

Let $c_1, \dots, c_s \in \mathbb{F}$ *and let* $q := \sum_i c_i q'_i \in \mathcal{P}$*. Then, if* $\{g_1, \dots, g_t\}$ *denotes a Gröbner basis of* I*, one has*

1. q *is the unique polynomial in* $\mathrm{Span}_{\mathbb{F}}(\mathbf{N}(\mathsf{I}))$ *such that* $\ell_i(q) = c_i$*, for each* i;
2. *for each* $p \in \mathcal{P}$ *it is equivalent*
   - (a) $\ell_i(p) = c_i$*, for each* i*,*
   - (b) $q = \mathrm{Can}(p, \mathsf{I})$*,*
   - (c) *exist* $h_j \in \mathcal{P}$ *such that*

$$p = q + \sum_{j=1}^{t} h_j g_j, \mathbf{T}(h_j)\mathbf{T}(g_j) \leq \mathbf{T}(p - q).$$

□

Möller's Algorithm [45, 23, 41, 2] is a procedure which, given a set of $\mathbb{F}$-linear functionals $\mathbb{L} = \{\ell_1, \ldots, \ell_s\} \subset \mathcal{P}^*$ such that $\mathfrak{P}(\mathrm{Span}_{\mathbb{F}}(\mathbb{L}))$ is a zero-dimensional ideal, allows to compute the data whose existence is stated in Theorem 5. The stronger version of the algorithm (Figure 3) assumes that, for each $\sigma \leq s$ $L_\sigma := \mathrm{Span}_{\mathbb{F}}(\{\ell_1, \ldots, \ell_\sigma\})$ is a $\mathcal{P}$-module, is performed by induction on σ and gives the complete structure of each ideal $\mathsf{I}_\sigma = \mathfrak{P}(L_\sigma)$.

Its correctness is based on the following

Lemma 7 *Let*

$\mathcal{P} := \mathbb{F}[x_1, \ldots, x_n]$,

$<$ *be any termordering;*

$\mathbb{L} = \{\ell_1, \ldots, \ell_r\} \subset \mathcal{P}^*$ *be a set of linearly independent* $\mathbb{F}$*-linear functionals such that* $\mathsf{I} := \mathfrak{P}(\mathrm{Span}_{\mathbb{F}}(\mathbb{L}))$ *is a zero-dimensional ideal*

and let

$\mathbf{N} := \{t_1, \ldots, t_r\} \subset \mathcal{T}$,

$\mathbf{q} := \{q_1, \ldots, q_r\} \subset \mathcal{P}$,

$G := \{g_1, \ldots, g_t\} \subset \mathcal{P}$,

be such that

- $\mathbf{N}$ *is an order ideal,*
- $\mathrm{Span}_{\mathbb{F}}\{t_1, \ldots, t_r\} = \mathrm{Span}_{\mathbb{F}}\{q_1, \ldots, q_r\}$,
- $\{q_1, \ldots, q_r\}$ *and* $\{\ell_1, \ldots, \ell_r\}$ *are triangular,*
- $\ell(g) = 0$ *for each* $g \in G$ *and each* $\ell \in \mathbb{L}$,
- $\mathbf{N} \sqcup \mathbf{T}_<(G) = \mathcal{T}$,
- *for each* $g \in G, g - \mathrm{lc}(g)\mathbf{T}_<(g) \in \mathrm{Span}_{\mathbb{F}}(\mathbf{N})$,

then G *is a reduced Gröbner basis of* $\mathfrak{P}(\mathrm{Span}_{\mathbb{F}}(\mathbb{L}))$ *w.r.t.* $<$.

The assumption that for each $\sigma \leq s$, $L_\sigma := \mathrm{Span}_{\mathbb{F}}(\{\ell_1, \ldots, \ell_\sigma\})$ can be satisfied if for instance the 0-dimensional ideal $\mathsf{I} = \mathfrak{P}(\mathrm{Span}_{\mathbb{F}}(\mathbb{L}))$ is described in terms of a *Macaulay representation* (cf. [3]), but often [4] is not satisfied, thus requiring an alternative version (Figure 4) performed on inductions on the terms and not on the functionals and which returns also a basis of $\mathrm{Span}_{\mathbb{F}}(\mathbb{L})$.

Remark 8 If, in the algorithm of Figure 3, we define p in instruction $\diamond$ as $p := x_h \mathsf{f}$ instead of $p := x_h t$, we have two counterbalancing effects:

[4] mainly in the solution of the FGLM-Problem, where in any case the fuctionals are properly reordered so they satisfy such property

Figure 3: Möller's Algorithm (1)

$(G_1, \dots, G_s, \mathbf{N}, \mathbf{q}) :=$ **G-basis**$(\mathbb{L}, <)$

where

$\mathbb{L} = \{\ell_1, \dots, \ell_s\} \subset \mathcal{P}^*$ is s.t.

$$L_\sigma := \mathrm{Span}_{\mathbb{F}}(\{\ell_1, \dots, \ell_\sigma\})$$

is a $\mathcal{P}$-module, for each $\sigma \le s$,

$\mathsf{I}_\sigma = \mathfrak{P}(L_\sigma)$, for each $\sigma \le s$,

$G_\sigma \subset \mathsf{I}_\sigma$ is the red. Gröbner basis of $\mathsf{I}_\sigma, \forall \sigma \le s$,

$\mathbf{N} := \{t_1, \dots, t_s\}$ is an order ideal,

$\mathbf{q} := \{q_1, \dots, q_s\} \subset \mathcal{P}$ is a set triangular to $\mathbb{L}$,

$\mathbf{N}_\sigma := \{t_1, \dots, t_\sigma\} = \mathbf{N}(\mathsf{I}_\sigma)$, $\forall \sigma \le s$,

$q_\sigma \in \mathrm{Span}_{\mathbb{F}}\{\mathbf{N}_\sigma\}$, and $\mathbf{T}(q_\sigma) = t_\sigma, \forall \sigma \le s$,

$\mathrm{Span}_{\mathbb{F}}\{t_1, \dots, t_\sigma\} = \mathrm{Span}_{\mathbb{F}}\{q_1, \dots, q_\sigma\}, \forall \sigma \le s$,

$\{q_1, \dots, q_\sigma\}$ and $\{\ell_1, \dots, \ell_\sigma\}$ are triangular $\forall \sigma$.

$\sigma := 1, t_1 := 1, \mathbf{N} := \{t_1\}$, $q_1 := \ell_1(1)^{-1}(t_1)t_1$,

$\mathbf{q} := \{q_1\}, G_1 := \{x_h - \ell_1(x_h), 1 \le h \le n\}$,

%% $\mathbf{N}_\sigma \sqcup \mathbf{T}(G_\sigma) = \mathcal{T}$.

%% $\ell_j(f) = 0$ for all $f \in G_\sigma, 1 \le j \le \sigma$.

For $\sigma := 2..s$ **do**

- ◦ $t := \min\{\mathbf{T}(f) : f \in G_\sigma, \ell_\sigma(f) \ne 0\}$,
  Let $\mathsf{f} \in G_\sigma : \mathbf{T}(\mathsf{f}) = t$,
  $t_\sigma := t, q_\sigma := \ell_\sigma(\mathsf{f})^{-1}\mathsf{f}$, $\mathbf{N} := \mathbf{N} \cup \{t_\sigma\}$,
- • $\mathbf{q} := \mathbf{q} \cup \{q_\sigma\}$,
- ⋆ $G_\sigma := \{f - \ell_\sigma(f)q_\sigma : f \in G_{\sigma-1}\}$.
  For each $h = 1..n : x_h t \notin \mathbf{T}(G_\sigma)$ **do**
  - ⋄ $p := x_h t$,
  - ∗ **For** $i = 1..\sigma$ **do** $p := p - \ell_i(p)q_i$,
    $G_\sigma := G_\sigma \cup \{p\}$;

  %% $\mathbf{N}_\sigma \sqcup \mathbf{T}(G_\sigma) = \mathcal{T}$,
  %% $\ell_j(f) = 0$ for all $f \in G_\sigma, 1 \le j \le \sigma$.

- the final output, while still a Gröbner basis, is not, in principle, reduced;
- since $\mathsf{f} \in \mathsf{I}_\sigma$, we have $x_h\mathsf{f} \in \mathsf{I}_\sigma$ and $\ell_i(p) = 0$ for each $i \leq \sigma$ so that one can perform the instruction $*$ for the single value $i := \sigma$.

Equivalently, defining, in the algorithm of Figure 3, p in instruction $\diamond$ as

$$p := x_h\mathsf{f} - \ell_\sigma(x_h\mathsf{f})q_\sigma = \left(x_h - \ell_\sigma(x_h\mathsf{f})\ell_\sigma(\mathsf{f})^{-1}\right)\mathsf{f} \tag{1}$$

we can simply remove the instruction $*$.

Finally note that the algorithm discussed in [31] is the module generalization of the version of the algorithm of Figure 3 in which p is defined as in (1) in instruction $\diamond$ and the instructions $*$ and $\bullet$ are removed. $\square$

4 The FGLM Problem

For its elimination property, the *lex* ordering is a good tool for solving [Gianni–Kalkbener Algorithm [29, 30], Lazard's trianglar sets[35, 34, 4, 5]] or for applications [see the CRHT-like algoithms in BCH codes[51]] but both practical experience and theoretical argument show that, in general, *lex* is a very bad choice for applying Buchberger Algorithm. On the other side the *degrevlex ordering* is the *optimal* choice for applying Buchberger Algorithm [8].

This suggests[23] the

Problem 9 (FGLM Problem) *Given*

- *a termordering $<$ on the polynomial ring $\mathcal{P} := \mathbb{F}[x_1, \dots, x_n]$,*
- *a zero-dimensional ideal $\mathsf{I} \subset \mathcal{P}$ and*
- *its reduced Gröbner basis $G_\prec$ w.r.t. the term-ordering $\prec$,*

to deduce the Gröbner basis $G_<$ of I w.r.t. $<$. $\square$

5 The FGLM Matrix

Let $\prec$ be a termordering and $\mathbf{N}_\prec(\mathsf{I}) = \{\tau_1, \dots, \tau_s\}$; in order to apply Möller Algoriothm to the FGLM Problem, we just need to choose as functionals $\mathbb{L} := \{\ell_1, \dots, \ell_s\}$ the coefficients of the canonical forms $\ell_i(\cdot) := \gamma(\cdot, \tau_i, \mathbf{N}_\prec(\mathsf{I}))$ so that we need to compute

$$\mathbf{Rep}(f, \mathbf{N}_\prec(\mathsf{I})) := (\gamma(f, \tau_1, \mathbf{N}_\prec(\mathsf{I})), \dots, \gamma(f, \tau_s, \mathbf{N}_\prec(\mathsf{I})))$$

for each $f \in \mathsf{B} := \{x_i\tau_j, 1 \leq i \leq n, 1 \leq j \leq s\}$.

Figure 4: Möller'a Algorithm (2)

$(G, r, \mathbf{N}, \Lambda, \mathbf{q}) := \textbf{G-basis}(\mathbb{L}, <)$

where

$\mathbb{L} = \{\ell_1, \ldots, \ell_s\} \subset \mathcal{P}^*$ is s.t. $\mathsf{I} := \mathfrak{P}(\mathrm{Span}_{\mathbb{F}}(\mathbb{L}))$ is a zero-dimensional ideal;

$G \subset \mathsf{I}$ is the reduced Gröbner basis of I w.r.t. $<$;

$r = \deg(\mathsf{I}) = \dim_{\mathbb{F}}(\mathrm{Span}_{\mathbb{F}}(\mathbb{L}))$;

$\mathbf{N} := \{t_1, \ldots, t_r\} = \mathbf{N}(\mathsf{I})$;

$1 = t_1 < t_2 < \ldots < t_i < t_{i+1} < \ldots < t_r$,

$\Lambda := \{\lambda_1, \ldots, \lambda_r\} \subset \mathbb{L}$, is a linearly indipendent basis of $\mathrm{Span}_{\mathbb{F}}(\mathbb{L})$;

$\mathbf{q} := \{q_1, \ldots, q_r\} \subset \mathcal{P}$ is a set triangular to Λ;

$q_i \in \mathrm{Span}_{\mathbb{F}}\{t_1, \ldots, t_i\}, \mathbf{T}(q_i) = t_i$, for each $i \leq r$;

$\mathrm{Span}_{\mathbb{F}}\{t_1, \ldots, t_i\} = \mathrm{Span}_{\mathbb{F}}\{q_1, \ldots, q_i\}$, for each $i \leq r$;

$\{q_1, \ldots, q_i\}$ and $\{\lambda_1, \ldots, \lambda_i\}$ are triangular, for each $i \leq r$.

$G := \emptyset, r := 1, t_1 := 1, \mathbf{N} := \{t_1\}$,

$v := (\ell_1(t_1), \ldots, \ell_s(t_1))$,

$\mu := \min\{j : \ell_j(1) \neq 0\}$,

$\lambda_1 := \ell_\mu$, $\Lambda := \{\lambda_1\}$,

$q_1 := \lambda_1(1)^{-1} t_1, \mathbf{q} := \{q_1\}$, $\mathrm{vect}(1) := \lambda_1(1)^{-1} v$,

%% $\mathrm{vect}(1) = (\ell_1(q_1), \ldots, \ell_s(q_1))$,

While $\mathbf{N} \sqcup \mathbf{T}(G) \neq \mathcal{T}$ **do**

$t := \min_<\{\tau \in \mathcal{T}, \tau \notin \mathbf{N} \sqcup \mathbf{T}(G)\}$,

$q := t, v := (\ell_1(q), \ldots, \ell_s(q))$

For $j = 1..r$ **do**

$v := v - \lambda_j(q)\,\mathrm{vect}(j), q := q - \lambda_j(q) q_j$,

%% $v = (\ell_1(q), \ldots, \ell_s(q))$.

If $v = 0$ **then**

$G := G \cup \{q\}$,

else

$r := r + 1$

$t_r := t, \mathbf{N} := \mathbf{N} \cup \{t_r\}$,

$\mu := \min\{j : \ell_j(q) \neq 0\}$,

$\lambda_r := \ell_\mu, \Lambda := \Lambda \cup \{\lambda_r\}$,

$q_r := \lambda_r(q)^{-1} q, \mathbf{q} := \mathbf{q} \cup \{q_r\}, \mathrm{vect}(r) := \lambda_r(q)^{-1} v$

%% $\mathrm{vect}(i) = (\ell_1(q_i), \ldots, \ell_s(q_i))$ for each $i, 1 \leq i \leq r$

$G, r, \mathbf{N}, \Lambda, \mathbf{q}$

If such elements are treated by $\prec$-incresing ordering, when the loop is treating a term $x_h\tau_l$, we have previously managed the term τ_l so that we have previously computed $\mathbf{Rep}(\tau_l, \mathbf{N}_\prec(\mathsf{I}))$ which satisfies the relation

$$\tau_l - \sum_{j=1}^{s} \gamma(\tau_l, \tau_j, \mathbf{N}_\prec(\mathsf{I}))\tau_j = \tau_l - \mathrm{Can}(\tau_l, \mathsf{I}, \prec) \in \mathsf{I},$$

so that $x_h\tau_l - \sum_{j=1}^{s} \gamma(\tau_l, \tau_j, \mathbf{N}_\prec(\mathsf{I}))x_h\tau_j \in \mathsf{I}$, and

$$\begin{aligned}
\mathrm{Can}(x_h\tau_l, \mathsf{I}, \prec) &= \sum_{j=1}^{s} \gamma(\tau_l, \tau_j, \mathbf{N}_\prec(\mathsf{I}))\,\mathrm{Can}(x_h\tau_j, \mathsf{I}, \prec) \\
&= \sum_{i=1}^{s} \left(\sum_{j=1}^{s} \gamma(\tau_l, \tau_j, \mathbf{N}_\prec(\mathsf{I}))\gamma(x_h\tau_j, \tau_i, \mathbf{N}_\prec(\mathsf{I})) \right) \tau_i.
\end{aligned}$$

For the $\prec$-minimal $\omega := x_h\tau_l \in \mathsf{B}$ under consideration we have the following three cases:

- if $\omega \notin \mathbf{T}_\prec(\mathsf{I})$ then $\omega \in \mathbf{N}_\prec(\mathsf{I})$, so that we add ω to $\mathbf{N}$ and $\{\omega x_h : 1 \leq h \leq n\}$ to B;
- if there is $g \in G_\prec$ such that

  $$\mathbf{T}_\prec(g) = \omega \text{ and } g = \omega - \sum_{\tau \in \mathbf{N}_\prec(\mathsf{I})} \gamma(\omega, \tau, \mathbf{N}_\prec(\mathsf{I}))\tau,$$

  since the procedure iterates on $\prec$-increasing values of ω, we have

  $$\gamma(\omega, \tau, \mathbf{N}_\prec(\mathsf{I})) \neq 0 \implies \tau \prec \omega \implies \tau \in \mathbf{N};$$

- if there is $H, 1 \leq H \leq n, \tau \in \mathbf{T}_\prec(\mathsf{I})$ such that $\omega = x_H\tau$; thus $\tau \prec \omega$ has been already treated so that we have obtained a representation

  $$\mathrm{Can}(\tau, \mathsf{I}, \prec) = \sum_{j=1}^{s} \gamma(\tau, \tau_j, \mathbf{N}_\prec(\mathsf{I}))\tau_j;$$

  since in such representation we have

  $$\gamma(\tau, \tau_j, \mathbf{N}_\prec(\mathsf{I})) \neq 0 \implies \tau_j \prec \tau \implies \tau_j \in \mathbf{N}, x_H\tau_j \prec x_H\tau = \omega = x_h\tau_l$$

  and $\tau = X_h\tau_\iota$ for $\tau_\iota := \frac{\tau_l}{X_H}$, we also have the representation

  $$\mathrm{Can}(x_H\tau, \mathsf{I}, \prec) = \sum_{j=1}^{s} \gamma(\tau, \tau_j, \mathbf{N}_\prec(\mathsf{I}))\,\mathrm{Can}(x_H\tau_j, \mathsf{I}, \prec)$$

and we can use the same formula as above to derive

$$
\begin{aligned}
& \gamma(x_h\tau_l, \tau_i, \mathbf{N}_\prec(\mathsf{I})) = \gamma(x_H\tau, \tau_i, \mathbf{N}_\prec(\mathsf{I})) \\
= & \sum_{j=1}^{s} \gamma(\tau, \tau_j, \mathbf{N}_\prec(\mathsf{I}))\gamma(x_H\tau_j, \tau_i, \mathbf{N}_\prec(\mathsf{I})) \\
= & \sum_{j=1}^{s} \gamma(x_h\tau_\iota, \tau_j, \mathbf{N}_\prec(\mathsf{I}))\gamma(x_H\tau_j, \tau_i, \mathbf{N}_\prec(\mathsf{I})).
\end{aligned}
$$

These remarks can be formalized in the algorithm descriped in Figure 5; Figure 6 proposes the instanciacion of Möller's Algoriothm (Figure 4) to the setting of the FGLM Problem.

6 Pointers

Remark (Compare [31]) that the Berlekamp-Massey Algorithm can be interpreted as a sort of FGLM Algorithm on modules with functionals depending on the state of the computation[5].

However, the earliest instance of the FGLM Algorithm goes back to 1936: in fact, Todd-Coxeter Algorithm [54] can be easily read [52] as a re-formulation of **FGLM-Matrix** (Figure 5) over groups view as quotients of a non-commutative polynomial rings modulo a bimonomial ideal.

The FGLM Problem was already solved essentially by means of the FGLM Algorithm in [15].

Möller's Algorithm was introduced for the first time in [45]: in that setting the considered functionals were point evaluation, the aim being multivariate interpolation; the same procedure was proposed in [28] as a tool for efficiently perform change of coordinate into a 0-dimensional ideal.

[23] introduced the FGLM Problem and solved it by means of Figure 6; the paper gives also a precise complexity analysis and introduced both the FGLM Matrix and the efficient algorithm (Figure 5) computing it.

[5] in fact, with Berlekamp's [9] notation we assume to have found the basis

$$\left\{(\sigma^{(k)}, \omega^{(k)}), (\tau^{(k)}, \gamma^{(k)})\right\}$$

of the module

$$M_k := \left\{(a(z), b(z)) \in \mathbb{Z}_2[z]^2 : (1+S)a(z) \equiv b(z) \bmod z^{k+1}\right\} \subset \mathbb{Z}_2[z]^2$$

and we consider the new functional $\lambda_{k+1} : \mathbb{Z}_2[z]^2 \to \mathbb{Z}_2$ defined by $\lambda_{k+1}(a(z), b(z)) := \Delta_1^{(k)}$ where $\Delta_1^{(k)} \in \mathbb{Z}_2$ is the value for which $(1+S)a(z) - b(z) \equiv \Delta_1^{(k)} z^{k+1} \bmod z^{k+2}$

In other words we can consider the functionals $\lambda_k : \mathbb{Z}_2[z]^2 \to \mathbb{Z}_2, 0 \leq k \leq 2t$ defined by $\lambda_{k+1}(a(z), b(z)) := c_k$ where $\sum_k c_k z^k = (1+S)a(z) - b(z) \in \mathbb{Z}_2[[z]]$ and each module M_k satisfies

$$M_k := \left\{(a(z), b(z)) \in \mathbb{Z}_2[z]^2 : \lambda_i(a(z), b(z)) = 0, 0 \leq i \leq k\right\} \subset \mathbb{Z}_2[z]^2.$$

For this interpretatyion I am strongly indepted to [25, 55].

Figure 5: The FGLM Matrix

$(\mathbf{N}_\prec, \mathcal{M}) :=$ **FGLM-Matrix**$(G_\prec)$

where

$G_\prec \subset \mathsf{I}$ is the reduced Gröbner basis of I w.r.t. $\prec$;

$s = \deg(\mathsf{I})$,

$\mathbf{N}_\prec := \{\tau_1, \dots, \tau_s\} = \mathbf{N}_\prec(\mathsf{I})$,

$1 = \tau_1 \prec \tau_2 \prec \dots \prec \tau_j \prec \tau_{j+1} \prec \dots \prec \tau_s$,

$\mathcal{M} = \mathcal{M}(\mathbf{N}_\prec) = \left\{ \left(a_{lj}^{(h)} \right) \in \mathbb{F}^{s^2}, 1 \leq h \leq n \right\}$ is the set of the square matrices defined by the equalities $x_h \tau_l = \sum_j a_{lj}^{(h)} \tau_j$ in $\mathcal{P}/\mathsf{I} = \mathrm{Span}_{\mathbb{F}}(\mathbf{N}_\prec)$;

$r := 1$, $\tau_1 := 1, \mathbf{N}_\prec := \{\tau_1\}$, $\mathsf{B} := \{x_h : 1 \leq h \leq n\}$,

While $\mathsf{B} \neq \emptyset$ **do**

$\omega := \min_\prec(\mathsf{B})$, $\mathsf{B} := \mathsf{B} \setminus \{\omega\}$,

$h, l : \omega := x_h \tau_l$

If $\omega \notin \mathbf{T}_\prec(\mathsf{I})$ **then**

$r := r + 1$

$\tau_r := \omega, \mathbf{N}_\prec := \mathbf{N}_\prec \cup \{\tau_r\}$, $\mathsf{B} := \mathsf{B} \cup \{x_h \tau_r : 1 \leq h \leq n\}$,

$a_{lr}^{(k)} := 1$;

else

if $\exists g := \mathbf{T}_\prec(g) - \sum_{j=1}^r \gamma(\omega, \tau_j, \mathbf{N}_\prec)\tau_j \in G_\prec : \mathbf{T}_\prec(g) = \omega = x_h \tau_l$ **then**

For $j = 1..r$ **do** $a_{lj}^{(h)} := \gamma(\omega, \tau_j, \mathbf{N}_\prec)$

else

Let $H, \iota : 1 \leq H \leq n, 1 \leq \iota \leq r : x_h \tau_\iota \in \mathbf{T}_\prec(G_\prec)$, $\tau_l = x_H \tau_\iota$;

For $i = 1..r$ **do** $a_{li}^{(h)} := \sum_{j=1}^r a_{\iota j}^{(h)} a_{ji}^{(H)}$

For each $H, i : x_H \tau_i = \omega$ **do**

For $j = 1..r$ **do** $a_{ij}^{(H)} := a_{lj}^{(h)}$;

$\mathbf{N}_\prec, \mathcal{M}$

Figure 6: The FGLM Algorithm

$(G, \mathbf{N}, \mathbf{q}) := \mathbf{FGLM}(G_{\prec}, <)$

where

$<$ and $\prec$ are termorderings on $\mathcal{P}$,

$\mathsf{I} \subset \mathcal{P}$ is a zero-dimensional ideal,

$G_{\prec} \subset \mathsf{I}$ is the reduced Gröbner basis of I w.r.t. $\prec$;

$s = \deg(\mathsf{I})$,

$\mathbf{N}_{\prec} := \{\tau_1, \ldots, \tau_s\} = \mathbf{N}_{\prec}(\mathsf{I})$,

$1 = \tau_1 \prec \tau_2 \prec \ldots \prec \tau_j \prec \tau_{j+1} \prec \ldots \prec \tau_s$,

$\mathcal{M} = \mathcal{M}(\mathbf{N}_{\prec}) = \{\left(a_{lj}^{(h)}\right) \in \mathbb{F}^{s^2}, 1 \le h \le n\}$ is the set of the square matrices defined by the equalities $x_h \tau_l = \sum_j a_{lj}^{(h)} \tau_j$ in $\mathcal{P}/\mathsf{I} = \mathrm{Span}_{\mathbb{F}}(\mathbf{N}_{\prec})$;

$G \subset \mathsf{I}$ is the reduced Gröbner basis of I w.r.t. $<$,

$\mathbf{N} := \{t_1, \ldots, t_s\} = \mathbf{N}_{<}(\mathsf{I})$,

$1 = t_1 < t_2 < \ldots < t_j < t_{j+1} < \ldots < t_s$,

$\mu : \{1, \ldots, s\} \mapsto \{1, \ldots, s\}$ is a permutation,

$\mathbf{q} := \{q_1, \ldots, q_s\} \subset \mathcal{P}$ is a set triangular to $\{\gamma(\cdot, \tau_{\mu(1)}, \mathbf{N}_{\prec}), \ldots, \gamma(\cdot, \tau_{\mu(s)}, \mathbf{N}_{\prec})\}$

$q_i \in \mathrm{Span}_{\mathbb{F}}\{t_1, \ldots, t_i\}, \mathbf{T}_{<}(q_i) = t_i$, for each $i \le s$,

$\{q_1, \ldots, q_i\}$ and $\{\gamma(\cdot, \tau_{\mu(1)}, \mathbf{N}_{\prec}), \ldots, \gamma(\cdot, \tau_{\mu(i)}, \mathbf{N}_{\prec})\}$ are triangular for all $i \le s$.

$(\mathbf{N}_{\prec}, \mathcal{M}) := \mathbf{FGLM\text{-}Matrix}(G_{\prec})$

$G := \emptyset, r := 1, t_1 := 1, \mathbf{N} := \{t_1\}, q_1 := 1, \mathbf{q} := \{q_1\}$,

$\mathsf{B} := \{x_h, 1 \le h \le n\}$

$\mathrm{vect}(1) := (1, 0, \ldots, 0), \mu(1) := 1$,

%% $\mathrm{vect}(1) = \mathbf{Rep}(q_1, \mathbf{N}_{\prec}), \mu(1) = \min\{j : \gamma(q_1, \tau_j, \mathbf{N}_{\prec}) \ne 0\}$

Let $\mathsf{B} := \{(x_h, h, 1), 1 \le h \le n\}$

While $\mathsf{B} \ne \emptyset$ **do**

  $t := \min_{<}(\mathsf{B}), \mathsf{B} := \mathsf{B} \setminus \{t\}$,
  $l, h : t = x_h t_l = x_h \mathbf{T}_{<}(q_l)$
  If $t \notin \mathbf{T}_{<}(G)$ **then**
    $q := x_h t_l$
    For $i = 1..s$ **do** $v_i := \sum_{j=1}^{s} \gamma(q_l, \tau_j, \mathbf{N}_{\prec}) a_{ji}^{(h)}$;
    $v := (v_1, \ldots, v_s)$
    %% $v = \mathbf{Rep}(q, \mathbf{N}_{\prec})$
    For $j = 1..r$ **do**
      $v := v - \gamma(q, \tau_{\mu(j)}, \mathbf{N}_{\prec})\,\mathrm{vect}(j), q := q - \gamma(q, \tau_{\mu(j)}, \mathbf{N}_{\prec}) q_j$,
      %% $v = \mathbf{Rep}(q, \mathbf{N}_{\prec})$
    If $v = 0$ **then**
      $G := G \cup \{q\}$,
    else
      $r := r + 1$
      $t_r := t, \mathbf{N} := \mathbf{N} \cup \{t_r\}$,
      $\mu(r) := \min\{j : \gamma(q, \tau_j, \mathbf{N}_{\prec}) \ne 0\}$,
      $q_r := \gamma(q, \tau_{\mu(r)}, \mathbf{N}_{\prec})^{-1} q, \mathrm{vect}(r) := \gamma(q, \tau_{\mu(r)}, \mathbf{N}_{\prec})^{-1} v$
      %%$\mathrm{vect}(i) = \mathbf{Rep}(q_i, \mathbf{N}_{\prec}), \forall i, 1 \le i \le r$
      $\mathbf{q} := \mathbf{q} \cup \{q_r\}$,
      $\mathsf{B} := \mathsf{B} \cup \{x_h t_r, 1 \le h \le n\}$,

$G, \mathbf{N}, \mathbf{q}$

[41] reconsidered Möller's and FGLM Algorithms, merging them and interepreting them in the setting of functionls; [2] is a survey which discusses also Macaulay's Algorithm to describe the structure of the canonical module $\mathfrak{L}(\mathsf{I})$.

The FGLM Algorithm *proper* solves the FGLM Problem only for a 0-dimensional ideal; [37] explains how to extend it to a multi-dimensional ideal; the corresponding algorithm is however far to be fast. The same weakness is shared by the Gröbner Walk Algorithm [21].

The most efficient algorithm for the solution of the FGLM-Problem, at least in the multidimensional case, is the Hilbert Driven Algorithm [56]: assuming wlog that I is homogeneous, the knowledge of the basis $G_\prec$ allows to compute the Hilbert function of I and thus, at each step, to predict how many new generators of a fixed degree are needed in the basis $G_<$; when such generators are produced, all other S-pairs of same degree are discarded and the Hilbert function of the monomial ideal $(\mathbf{T}_<(g) : g \in G_<)$ is re-evaluated and the computation is performed in higher degree.

Recently new ideas have been proposed which promise to be more efficient than the FGLM and the Hilbert Driven Algorithms [7, 53].

Möller's Algorithm has been generalized to projectiive spaces [1] and to non-commutative setting [10].

[11, 12, 13] use an improved version of the FGLM algorithm for binomial ideals in order to correct binary linear codes (see [14]).

7 Duality (2)

Let us begin by remarking that a Gröbner representation of a 0-dimensional ideal $\mathsf{I} \subset \mathcal{P} := k[X_1, \ldots, X_n]$ allows to deduce easily the $\mathcal{P}$-module structure of its canonical module $\mathfrak{L}(\mathsf{I})$.

In fact

Lemma 10 *Let*

$\mathbb{L} := \{\ell_1, \ldots, \ell_r\} \subset \mathcal{P}^*$ *be a linearly indipendent set of k-linear functionals such that*

$L := \mathrm{Span}_k(\mathbb{L})$ *is a $\mathcal{P}$-module so that*

$\mathsf{I} := \mathfrak{P}(L)$ *is a zero-dimensional ideal;*

$\mathbf{N}(\mathsf{I}) := \{t_1, \ldots, t_r\}$,

$\mathbf{q} := \{q_1, \ldots, q_r\} \subset \mathcal{P}$ *the set triangular to $\mathbb{L}$, obtained via Möller's Algorithm;*

$\left(q_{ij}^{(h)}\right) \in k^{r^2}, 1 \leq k \leq r$ *be the matrices defined by* $X_h q_i = \sum_j q_{ij}^{(h)} q_j \bmod \mathsf{I}$,

$\Lambda := \{\lambda_1, \ldots, \lambda_r\}$ *be the set biorthogonal to $\mathbf{q}$, which can be trivially deduced by Gaussian reduction.*

Then

$$X_h\lambda_j = \sum_{i=1}^{r} q_{ij}^{(h)}\lambda_i, \forall i, j, h.$$

$\square$

Denoting $\mathsf{m} := (X_1, \ldots, X_n)$ the maximal at the origin we recalled that, given an ideal $\mathsf{I} \subset \mathcal{P}$, its m-*closuse* is the ideal $\bigcap_d \mathsf{I} + \mathsf{m}^d$, and I is called m-*closed* iff $\mathsf{I} = \bigcap_d \mathsf{I} + \mathsf{m}^d$.

We can produce a natural representation of $\mathcal{P}^*$, if we associate, to each term $\tau \in \mathcal{T}$, the functional $M(\tau) : \mathcal{P} \to k$ defined by

$$M(\tau) = c(f, \tau), \forall f = \sum_{t\in\mathcal{T}} c(f, t)t \in \mathcal{P}.$$

; in fact, denoting $\mathbb{M} := \{M(\tau) : \tau \in \mathcal{T}\}$, we obtain $\mathcal{P}^* \cong k[[\mathbb{M}]]$.

Remark that, with this notation, for all

$$f := \sum_{t\in\mathcal{T}} a_t t \in \mathcal{P} \text{ and } \ell := \sum_{\tau\in\mathcal{T}} c_\tau M(\tau) \in k[[\mathbb{M}]] \cong \mathcal{P}^*$$

it holds $\ell(f) = \sum_{t\in\mathcal{T}} a_t c_t$.

The $\mathcal{P}$-module structure of $\mathcal{P}^* \cong k[[\mathbb{M}]]$ is described by

$$\forall\tau \in \mathcal{T}, X_i \cdot M(\tau) = \begin{cases} M(\frac{\tau}{X_i}) & \text{if } X_i \mid \tau \\ 0 & \text{if } X_i \nmid \tau \end{cases}.$$

We will say that a k-vector subspace $\Lambda \subset \mathrm{Span}_k(\mathbb{M})$ is *stable* if $\lambda \in \Lambda \implies X_i \cdot \lambda \in \Lambda$ i.e. Λ is a $\mathcal{P}$-module.

Clearly $\mathcal{P}^* \cong k[[\mathbb{M}]]$; however in order to have reasonable duality[6] we must restrict ourselves to $\mathrm{Span}_k(\mathbb{M}) \cong k[\mathbb{M}]$.

Under this restriction, for each k-vector subspace $\Lambda \subset \mathrm{Span}_k(\mathbb{M})$ we denote

$$\mathfrak{I}(\Lambda) := \mathfrak{P}(\Lambda) = \{f \in \mathcal{P} : \ell(f) = 0, \forall\ell \in \Lambda\}$$

and for each k-vector subspace $P \subset \mathcal{P}$ we denote

$$\begin{aligned} \mathfrak{M}(P) &:= \mathfrak{L}(P) \cap \mathrm{Span}_K(\mathbb{M}) \\ &= \{\ell \in \mathrm{Span}_K(\mathbb{M}) : \ell(f) = 0, \forall f \in P\} \end{aligned}$$

and we obtain

Lemma 11 *[38, 39, 32, 46, 41, 3] The mutually inverse maps $\mathfrak{I}(\cdot)$ and $\mathfrak{M}(\cdot)$ give a biunivocal, inclusion reversing, correspondence between the set of the m-closed ideals $\mathsf{I} \subset \mathcal{P}$ and the set of the stable k-vector subspaces $\Lambda \subset \mathrm{Span}_k(\mathbb{M})$.*

They are the restriction of, respectively, $\mathfrak{P}(\cdot)$ to m-closed ideals $\mathsf{I} \subset \mathcal{P}$, and $\mathfrak{L}(\cdot)$ to stable k-vector subspaces $\Lambda \subset \mathrm{Span}_k(\mathbb{M})$.

[6]Recall that $\mathfrak{L}P(L) = L$ holds only if $\dim_k(L) < \infty$.

Moreover, for any m*-primary ideal* $\mathsf{q} \subset \mathcal{P}$, $\mathfrak{M}(\mathsf{q})$ *is finite* k*-dimensional and we have*

$$\deg(\mathsf{q}) = \dim_K(\mathfrak{M}(\mathsf{q}));$$

conversely for any finite k*-dim. stable* k*-vector subspace* $\Lambda \subset \mathrm{Span}_K(\mathbb{M})$, $\mathfrak{I}(\Lambda)$ *is an* m*-primary ideal and we have*

$$\dim_k(\Lambda) = \deg(\mathfrak{I}(\Lambda)).$$

$\square$

8 Macaulay Bases

Let $<$ be a semigroup ordering on $\mathcal{T}$ and $\mathsf{I} \subset \mathcal{P}$ an m-closed ideal. We have

$$\mathrm{Can}(t, \mathsf{I}, <) =: \sum_{\tau \in \mathbf{N}_<(\mathsf{I})} \gamma(t, \tau, <)\tau \in k[[\mathbf{N}_<(\mathsf{I})]] \subset k[[X_1, \ldots, X_n]]$$

so that

$t - \sum_{\tau \in \mathbf{N}_<(\mathsf{I})} \gamma(t, \tau, <)\tau \in \mathsf{I}$,

$t < \tau \implies \gamma(t, \tau, <) = 0$.

Define, for each $\tau \in \mathbf{N}_<(\mathsf{I})$,

$$\ell(\tau) := M(\tau) + \sum_{t \in \mathbf{T}_<(\mathsf{I})} \gamma(t, \tau, <)M(t) \in k[[\mathbb{M}]]$$

and remark that $\ell(\tau) \in \mathfrak{M}(\mathsf{I})$ requires $\ell(\tau) \in k[\mathbb{M}]$ which holds iff $\{t : \gamma(t, \tau, <) \neq 0\}$ is finite and is granted if $\{t : t > \tau\}$ is finite.

To obtain this, we must choose as $<$ a *standard ordering* i.e. a semigroup ordering such that

- $X_i < 1, \forall i$,
- for each infinite decreasing sequence in $\mathcal{T}$

  $$\tau_1 > \tau_2 > \cdots \tau_\nu > \cdots$$

  and each $\tau \in \mathcal{T}$ there is $\nu : \tau > \tau_n$.

In this setting the generalization of the notion of Gröbner basis is called Hironoka/standard basis and deals with *series* instead of polynomials. The choice of this setting is natural, since a Hironaka basis of an ideal I returns its m-closure.

Thus let $<$ be a standard ordering on $\mathcal{T}$ and $\mathsf{I} \subset \mathcal{P}$ an m-closed ideal; denoting

$$\mathrm{Can}(t, \mathsf{I}, <) =: \sum_{\tau \in \mathbf{N}_<(\mathsf{I})} \gamma(t, \tau, <)\tau \in k[[\mathbf{N}_<(\mathsf{I})]]$$

and, for each $\tau \in \mathbf{N}_<(\mathsf{I})$,

$$\ell(\tau) := M(\tau) + \sum_{t \in \mathbf{T}_<(\mathsf{I})} \gamma(t, \tau, <) M(t) \in k[\mathbb{M}],$$

we have

$$\mathfrak{M}(\mathsf{I}) = \mathrm{Span}_k\{\ell(\tau), \tau \in \mathbf{N}_<(\mathsf{I})\}.$$

Definition 12 *[3]*
The set $\{\ell(\tau), \tau \in \mathbf{N}_<(\mathsf{I})\}$ *is called the* Macaulay Basis *of* I. □

There is an algorithm [41, 3] which, given a finite basis (not necessarily Gröbner/standard) of an m-primary ideal I, computes its Macaulay Basis. Such algorithm becomes an infinite procedure which, given a finite basis of an ideal $\mathsf{I} \subset \mathsf{m}$, returns the infinite Macaulay Basis of its m-closure.

Definition 13 *[44] Let*

$\mathsf{I} \subset \mathcal{P}$ *be a 0-dimensional ideal*

$\mathsf{Z} := \{\mathsf{a} \in k^n : f(\mathsf{a}) = 0, \forall f \in \mathsf{I}\}$

for each $\mathsf{a} \in \mathsf{Z}$

- $\lambda_\mathsf{a} : \mathcal{P} \mapsto \mathcal{P}$ *the translation* $\lambda_\mathsf{a}(X_i) = X_i + a_i, \forall i$,
- $\mathfrak{m}_\mathsf{a} = (X_1 - a_1, \dots, X_n - a_n)$,
- $\mathfrak{q}_\mathsf{a}$ *the* $\mathfrak{m}_\mathsf{a}$*-primary component of* I,
- $\Lambda_\mathsf{a} := \mathfrak{M}(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a})) \subset \mathrm{Span}_K(\mathbb{M})$,
- $\ell_{\upsilon\mathsf{a}}$, *for each* $\upsilon \in \mathbf{N}_<(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a}))$, *the Macaulay equation* $\ell_{\upsilon\mathsf{a}} := \ell(\upsilon)$ *so that*
- $\{\ell_{\upsilon\mathsf{a}} : \upsilon \in \mathbf{N}_<(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a}))\}$ *is the Macaulay basis of* Λ_a.

A Macaualy representation *of* $\mathsf{I} = \bigcup_{\mathsf{a} \in \mathsf{Z}} \mathfrak{q}_\mathsf{a}$ *is the data*

- $\mathsf{Z} := \{\mathsf{a} \in k^n : f(\mathsf{a}) = 0, \forall f \in \mathsf{I}\}$,
- *for each* $\mathsf{a} \in \mathsf{Z}$ *the Macaulay basis* $\{\ell_{\upsilon\mathsf{a}} : \upsilon \in \mathbf{N}_<(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a}))\}$ *is the Macaulay basis of* Λ_a

so that the lineraly independent set

$$\mathbb{L} := \{\ell_{\upsilon\mathsf{a}}\lambda_\mathsf{a} : \upsilon \in \mathbf{N}_<(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a})), \mathsf{a} \in \mathsf{Z}\} \subset \mathcal{P}^*$$

satisfies $\mathrm{Span}_k(\mathbb{L}) = \mathfrak{L}(\mathbb{L})$. □

9 Cerlienco–Mureddu Correspondence

Cerlienco and Mureddu [16, 17, 18] solve the following

Problem 14 *Given a finite set of points,*

$$\{\mathsf{a}_1, \ldots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \ldots, a_{in}),$$

to compute $\mathbf{N}_<(\mathsf{I})$ *w.r.t. the lexicographical ordering* $<$ *induced by* $X_1 < \cdots < X_n$ *where*

$$\mathsf{I} := \{f \in \mathcal{P} : f(\mathsf{a}_i) = 0, 1 \leq i \leq s\}.$$

$\square$

by means of an efficient combinatorial algorithm which to each *ordered* finite set of points

$$\mathsf{X} := \{\mathsf{a}_1, \ldots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \ldots, a_{in}),$$

associates

- an order ideal $\mathbf{N} := \mathbf{N}(\mathsf{X})$ and
- a bijection $\Phi := \Phi(\mathsf{X}) : \mathsf{X} \mapsto \mathbf{N}$

satisfying

Theorem 15 *[16]* $\mathbf{N}(\mathsf{I}) = \mathbf{N}(\mathsf{X})$ *holds for each finite set of points* $\mathsf{X} \subset k^n$. $\square$

Since they do so by induction on $s = \#(\mathsf{X})$ let us consider the subset $\mathsf{X}' := \{\mathsf{a}_1, \ldots, \mathsf{a}_{s-1}\}$, and the corresponding order ideal $\mathbf{N}' := \mathbf{N}(\mathsf{X}')$ and bijection $\Phi' := \Phi(\mathsf{X}')$.

If $s = 1$ the only possible solution is $\mathbf{N} = \{1\}, \Phi(\mathsf{a}_1) = 1$.

Denoting

$$\begin{aligned} \mathcal{T}[1,m] &:= \mathcal{T} \cap k[X_1, \ldots, X_m] \\ &= \{X_1^{a_1} \cdots X_m^{a_m} : (a_1, \ldots, a_m) \in \mathbb{N}^m\}, \end{aligned}$$

$\pi_m : k^n \mapsto k^m, \quad \pi_m(x_1, \ldots, x_n) = (x_1, \ldots, x_m),$

$\pi_m : \mathcal{T} \cong \mathbb{N}^n \mapsto \mathbb{N}^m \cong \mathcal{T}[1,m],$

$\quad \pi_m(X_1^{a_1} \cdots X_n^{a_n}) = X_1^{a_1} \cdots X_m^{a_m}.$

Cerlinco–Mureddu Algorithm set

$m := \max(j : \exists i < s : \pi_j(\mathsf{a}_i) = \pi_j(\mathsf{a}_s));$

$d := \#\{\mathsf{a}_i, i < s : \pi_m(\mathsf{a}_i) = \pi_m(\mathsf{a}_s)\};$

$\mathsf{W} := \{\mathsf{a}_i : \Phi'(\mathsf{a}_i) = \tau_i X_{m+1}^d, \tau_i \in \mathcal{T}[1,m]\} \cup \{\mathsf{a}_s\};$

$\mathsf{Z} := \pi_m(\mathsf{W})$;

$\tau := \Phi(\mathsf{Z})(\pi_m(\mathsf{a}_s))$;

$t_s := \tau X_{m+1}^d$;

$\mathbf{N} := \mathbf{N}' \cup \{t_s\}$,

$$\Phi(\mathsf{a}_i) := \begin{cases} \Phi'(\mathsf{a}_i) & i < s \\ t_s & i = s \end{cases}$$

where $\mathbf{N}(\mathsf{Z})$ and $\Phi(\mathsf{Z})$ are the result of the application of the present algorithm to Z, which can be inductively applied since $\#(\mathsf{Z}) \leq s - 1$.

Example 16 *For the following sequence of points we iteratively obtain*

$\mathsf{a}_1 := (0, 0, 1)$,

$\Phi(\mathsf{a}_1) := t_1 := 1$;

$\mathsf{a}_2 := (0, 1, -2)$,

$m = 1, d = 1, \mathsf{W} = \{(0, 1)\}, \tau = 1, \Phi(\mathsf{a}_2) := t_2 := X_2$,

$\mathsf{a}_3 := (2, 0, 2)$,

$m = 0, d = 1, \mathsf{W} = \{(2, 0)\}, \tau = 1, \Phi(\mathsf{a}_3) := t_3 := X_1$,

$\mathsf{a}_4 := (0, 2, -2)$,

$m = 1, d = 2, \mathsf{W} = \{(0, 2)\}, \tau = 1, \quad \Phi(\mathsf{a}_4) := t_4 := X_2^2$,

$\mathsf{a}_5 := (1, 0, 3)$,

$m = 0, d = 2, \mathsf{W} = \{(1, 0)\}, \tau = 1, \Phi(\mathsf{a}_5) := t_5 := X_1^2$,

$\mathsf{a}_6 := (1, 1, 3)$,

$m = 1, d = 1, \mathsf{W} = \{(0, 1), (1, 1)\}, \tau = X_1, \Phi(\mathsf{a}_6) := t_6 := X_1 X_2$.

$\mathsf{a}_7 := (1, 1, 1)$,

$m = 2, d = 1, \mathsf{W} = \{(1, 1, 1)\}, \tau = 1, \Phi(\mathsf{a}_7) := t_7 := X_3$.

$\mathsf{a}_8 := (2, 0, 1)$,

$m = 2, d = 1, \mathsf{W} = \{(1, 1, 1), (2, 0, 1)\}, \tau = X_1, \Phi(\mathsf{a}_8) := t_8 := X_1 X_3$,

$\mathsf{a}_9 := (2, 0, 0)$,

$m = 2, d = 2, \mathsf{W} = \{(2, 0, 0))\}, \tau = 1, \Phi(\mathsf{a}_9) := t_9 := X_3^2$,

$(0, 2, -2)$		
$(0, 1, -2)$	$(1, 1, 3)$	
$(0, 0, 1)$	$(2, 0, 2)$	$(1, 0, 3)$

□

[27] and [24] give a combinatorial reformulation of Cerlienco–Mureddu Algorithm which

– builds a tree on the basis of the point coordinates,

– cominatorially recombines the tree,

– reeds on this tree the monomial structure.

Their formulation returns $\mathbf{N}$ but not Φ; more important,apparently it is *not* iterative.

[44] extends Cerlienco–Mureddu Algorithm to multiple points described via Macaulay representation.

10 Macaulay's Algorithm

Let

$<$ be a standard-ordering on $\mathcal{T}$,

$\mathsf{I} \subset \mathcal{P}$ an m-closed ideal,

$\mathbf{C}_<(\mathsf{I}) := \{\omega_1, \dots, \omega_s\}$ the finite corner set of of I wrt $<$,

$\{\ell(\tau) : \tau \in \mathbf{N}_<(\mathsf{I})\}$, the (not-necessarily finite) Macaulay basis of I,

the k-vectorspace $\Lambda \subset \mathrm{Span}_k(\mathbb{M})$ generated by it,

$\forall j, 1 \leq j \leq s, \Lambda_j := \mathrm{Span}_k\{\upsilon \cdot \ell(\omega_j) : \upsilon \in \mathcal{T}\}$.

$\forall j, 1 \leq j \leq s, \mathfrak{q}_j := \mathfrak{I}(\Lambda_j)$.

$\forall j, 1 \leq j \leq s, \Lambda_j := \mathrm{Span}_k\{\upsilon \cdot \ell(\omega_j) : \upsilon \in \mathcal{T}\}$.

$\forall j, 1 \leq j \leq s, \mathfrak{q}_j := \mathfrak{I}(\Lambda_j)$.

Let $J \subset \{1, \dots, s\}$ be the set such that $\{\mathfrak{q}_j : j \in J\}$ is the set of the minimal elements of $\{\mathfrak{q}_j : 1 \leq j \leq s\}$ and remark that $\mathfrak{q}_i \subset \mathfrak{q}_j \iff \Lambda_i \supset \Lambda_j$.

Lemma 17 (Macaulay) *[38, 39] With the notation above, for each j, denoting*

$$\Lambda'_j := \mathrm{Span}_K\{\upsilon \cdot \ell(\omega_j) : \upsilon \in \mathcal{T} \cap \mathsf{m}\}$$

we have

$\dim_K(\Lambda'_j) = \dim_K(\Lambda_j) - 1$,

$\ell(\omega_j) \notin \Lambda'_j = \mathfrak{M}(\mathfrak{q}_j : \mathsf{m})$,

$\mathfrak{q}' \supset \mathfrak{q}_j \implies \mathfrak{M}(\mathfrak{q}') \subseteq \Lambda'_j$. $\square$

Corollary 18 (Macaulay) *[38, 39] Let* I *be a zero-dimensional ideal,* $\deg(\mathsf{I}) = s$ *Then the Macaulay representation* $\mathbb{L} = \{\ell_1, \ldots, \ell_s\}$ *of* I *can be properly ordered so that*

$L := \mathrm{Span}_k(\mathbb{L}) = \mathfrak{L}(\mathsf{I})$,

each subvectorspace $L_\sigma := \mathrm{Span}_k(\{\ell_1, \ldots, \ell_\sigma\})$ *is a* $\mathcal{P}$*-module so that*

each $\mathsf{I}_\sigma = \mathfrak{P}(L_\sigma)$ *is a zero-dimensional ideal and*

there is a chain $\mathsf{I}_1 \supset \mathsf{I}_2 \supset \cdots \supset \mathsf{I}_s = \mathsf{I}$. □

Macaulay's construction allowsw, as it was remarked by Gröbner[32, 50], to compute an irreducible decomposition of primaries ideals[7]:

Theorem 19 (Gröbner) *If* I *is* m*-primary, then:*

1. *each* Λ_j *is a finite-dim. stable vectorspace;*
2. *each* q_j *is an* m*-primary ideal,*
3. *is reduced*
4. *and irreducible.*
5. $\mathsf{I} := \cap_{j \in J} \mathsf{q}_j$ *is a reduced representation of* I.

11 Reduced Irreducible Decomposition

It is well known [Lasker-Noether Decomposition Theorem] that

- each ideal $\mathsf{I} \subset \mathcal{P}$ is the finite intersection of irreducible ideals;
- irreducible ideals are primaries, but the converse, in general, is false;
- if, into such a representation, each primaries associated to a same prime are substituted by their intersection, then $\mathsf{I} \subset \mathcal{P}$ has a representation as intersection of finite primary[8] ideals;
- the primes associated to such primaries are unique as well as the isolated primaries.

It is instead less known that this formulation given by Noether [49] is an adfapatation of a preliminary formulation with respect to which irreduciility and *reduceness* are sacrified in order to obtain uniqueness.

In fact Noether introduced the following

Definition 20 (Noether) *[49]*

[7] For the definitions see the section below

[8] but not necessarily irreducible

A representation $\mathfrak{a} = \cap_{j=1}^{r} \mathfrak{i}_j$ *of an ideal* $\mathfrak{a}$ *in a noetherian ring* R *as intersection of finitely many irreducible ideals is called a* reduced representation *if*

- $\forall j \in \{1, \ldots, r\}$, $\mathfrak{i}_j \not\supset \bigcap_{\substack{h=1 \\ j \neq h}}^{r} \mathfrak{i}_h$ *and*
- *there is no irreducible ideal* $\mathfrak{i}_j{}' \supset \mathfrak{i}_j$ *such that* $\mathfrak{a} = \left(\bigcap_{\substack{h=1 \\ j \neq h}}^{r} \mathfrak{i}_h \right) \cap \mathfrak{i}_j{}'$.

A primary component $\mathfrak{q}_j$ *of an ideal* $\mathfrak{a}$ *contained in a noetherian ring* R, *is called* reduced *if there is no primary ideal* $\mathfrak{q}_j{}' \supset \mathfrak{q}_j$ *such that* $\mathfrak{a} = \left(\bigcap_{\substack{i=1 \\ j \neq i}}^{r} \mathfrak{q}_i \right) \cap \mathfrak{q}_j{}'$. □

and proved that

Theorem 21 (Noether) *[49]) In a noetherian ring* R, *each ideal* $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$ $\mathfrak{a} \subset R$ *has a reduced representation as intersection of finitely many irreducible ideals.*

In an irredundant primary decomposition of an ideal of a noetherian ring, each primary component can be chosen to be reduced. □

Example 22 *The decomposition*

$$(X^2, XY) = (X) \cap (X^2, XY, Y^\lambda), \forall \lambda \in \mathbb{N}, \lambda \geq 1,$$

where $\sqrt{(X^2, XY, Y^\lambda)} = (X, Y) \supset (X)$, *shows that embedded components are not unique; however,*

$$(X^2, XY, Y) = (X^2, Y) \supseteq (X^2, XY, Y^\lambda), \forall \lambda > 1,$$

shows that (X^2, Y) *is a reduced embedded irreducible component and that*

$$(X^2, XY) = (X) \cap (X^2, Y)$$

is a reduced representation. □

Example 23 *The decompositions*

$$(X^2, XY) = (X) \cap (X^2, Y + aX), \forall a \in \mathbb{Q},$$

where $\sqrt{(X^2, Y + aX)} = (X, Y) \supset (X)$ *and, clearly, each* $(X^2, Y + aX)$ *is reduced, show that also reduced representations are not unique; remark that, setting* $a = 0$, *we find again the previous one* $(X^2, XY) = (X) \cap (X^2, Y)$. □

For an m-primary ideal, Theorem refGr give an algorithm to compute its reduced representation.

If I is not m-primary, its reduced representation can be obtained in the following way: let

$\nabla_\rho := \{M(\omega) : \omega \in |calT, \deg(\omega) < \rho\}$

$\mathbf{C}_<(\mathsf{I}) := \{\omega_1, \dots, \omega_t\}$,

$\rho := \max\{\deg(\omega_j) + 1 : \omega_j \in \mathbf{C}_<(\mathsf{I})\} + 1$ so that

$\mathsf{q}' := \mathsf{I} + \mathsf{m}^\rho$ is an m-primary component of I,

$\Lambda \cap \nabla_\rho = \mathfrak{M}(\mathsf{q}')$;

$\mathsf{I} = \cap_{i=1}^r \mathsf{q}_i$ be an irredundant primary representation of I where $\sqrt{\mathsf{q}_1} = \mathsf{m}$,

$\mathsf{J} := \cap_{i=2}^r \mathsf{q}_i$,

$\mathsf{J} = \cap_{i=1}^u \mathfrak{i}_i$, a reduced representation of J;

$\mathbf{C}_<(\mathsf{q}') := \{\omega_1, \dots, \omega_t, \omega_{t+1}, \dots, \omega_s\} \supset \mathbf{C}_<(\mathsf{I})$

for each $j, 1 \le j \le s$ $\Lambda_j := \mathrm{Span}_K\{v\ell(\omega_j) : v \in \mathcal{T}\}$

and $\mathsf{q}_j := \mathfrak{I}(\Lambda_j)$;

$\mathsf{q} := \cap_{j=1}^t \mathsf{q}_j$.

Then

Corollary 24 *With the notation above, it holds:*

1. $\mathsf{J} := \mathsf{I} : \mathsf{m}^\infty = \cap_{i=2}^r \mathsf{q}_i$,
2. $\mathsf{q} \subset \mathsf{q}'$ *is a reduced* m*-primary component of* I
3. $\mathsf{q}' := \cap_{j=1}^s \mathsf{q}_j$ *is a reduced representation of* q,
4. $\mathsf{q} := \cap_{j=1}^t \mathsf{q}_j$ *is a reduced representation of* q,
5. $\mathsf{q}_i \supset \mathsf{J} \iff i > t$,
6. $\mathsf{I} = \cap_{i=1}^u \mathfrak{i}_i \bigcap \cap_{j=1}^t \mathsf{q}_j$ *is a reduced representation of* I. $\square$

For $\mathsf{I} := (X^2, XY)$ we have

$\Lambda = \mathrm{Span}_K\{M(1), M(X)\} \cup \{M(Y^i), i \in \mathbb{N}\}$,

$\mathbf{C}_<(\mathsf{I}) = \{X\}$;

$\mathsf{I} : \mathsf{m}^\infty = (X)$

$\rho = 3$, $\mathsf{q}' = \mathsf{I} + \mathsf{m}^3 = (X^2, XY, Y^3)$ $\mathbf{C}_<(\mathsf{q}') = \{X, Y^2\}$;

$\omega_1 := X$, $\Lambda_1 = \mathrm{Span}_K\{M(1), M(X)\}, \mathsf{q}_1 = (X^2, Y)$;

$\omega_2 := Y^2, \Lambda_2 = \{M(1), M(Y), M(Y^2)\}, \mathsf{q}_2 = (X, Y^3) \supset (X)$;

whence $(X^2, XY) = (X) \cap (X^2, Y)$.

Both the reduced representation and the notion of Macaulay basis strongly depend on the choice of a frame of coordinates.
In fact, considering, for each $a \in \mathbb{Q}, a \neq 0$,

$$\Lambda = \mathrm{Span}_k\{M(1), M(X) - aM(Y)\} \cup \{M(Y^i), i \in \mathbb{N}\},$$

we obtain

$\rho = 3$, $\Lambda \cap \nabla_\rho = \{M(1), M(X) - aM(Y), M(Y), M(Y^2)\}$,

$\omega_1 := X, \Lambda_1 = \{M(1), M(X) - aM(Y)\}, \mathfrak{q}_1 = (X^2, Y + aX)$;

$\omega_2 := Y^2, \Lambda_2 = \{M(1), M(Y)\}, \mathfrak{q}_2 = (X, Y^3) \supset (X)$;

whence $(X^2, XY) = (X) \cap (X^2, Y + aX)$.

Let us now discuss deeply the same example by performing the generic change of coordinate

$$\Phi : \mathbb{Q}[X, Y] \mapsto \mathbb{Q}[X, Y] : \Phi(X) = aX + bY, \Phi(Y) = cX + dY, ad - bc \neq 0 \neq a :$$

for $\mathsf{I} := (X^2, XY)$, we obtain

$\Phi(\mathsf{I}) = \big(aXY + bY^2, a^2X^2 - bY^2\big)$,

$\Lambda := \mathrm{Span}_K\{M(1), M(X), M(Y), a^2M(Y^2) - abM(XY) + b^2M(X^2), \cdots\}$

$\mathsf{J} = \mathsf{I} : \mathsf{m}^\infty = (aX + bY)$,

$\rho = 3$, $\mathbf{C}_<(\mathfrak{q}') = \{X, Y^2\}$;

$\Lambda \cap \nabla_\rho = \mathrm{Span}_K\{M(1), M(X), M(Y), a^2M(Y^2) - abM(XY) + b^2M(X^2)\}$;

$\omega_1 := X, \Lambda_1 = \{M(1), M(X)\}, \mathfrak{q}_1 = (X^2, Y)$;

$\omega_2 := Y^2, \Lambda_2 = \{M(1), aM(Y) - bM(X), a^2M(Y^2) - abM(XY) + b^2M(X^2)\}$,

$\mathfrak{q}_2 = (aX + bY, Y^3) \supset (aX + bY)$;

whence $\Phi(\mathsf{I}) = (aX + bY) \cap (X^2, Y)$.

We have chosen $\{M(1), M(X), M(Y)\}$ as basis of ∇_2; however, what we have to do is to extend the basis $\{M(1), aM(Y) - bM(X)\}$ of $\mathfrak{M}(\mathsf{J}) \cap \nabla_2$, in order to obtain a basis of ∇_2.

Any choice $eM(Y) + fM(X), af + be \neq 0$ is acceptable giving the reduced primary

$$\mathfrak{I}(\{M(1), eM(Y) + fM(X)\}) = (X^2, eX - fY)$$

and the decomposition $\Phi(\mathsf{I}) = (aX + bY) \cap (X^2, eX - fY)$.

12 Lazard Structural Theorem

Lazard Structural Theorem [33] is one of earlies important results within Gröbner Theory; it describes the structure of the lex Gröbner basis of a generic ideal in 2 variables; Gianni–Kalkbrenner's Theorem can be seen as its ultimate generalization.

Theorem 25 (Lazard) *Let $\mathcal{P} := k[X_1, X_2]$ and let $<$ be the lex. ordering induced by $X_1 < X_2$.*

Let $\mathsf{I} \subset \mathcal{P}$ be an ideal and let $\{f_0, f_1, \dots, f_k\}$ be a Gröbner basis of I ordered so that

$$\mathbf{T}(f_0) < \mathbf{T}(f_1) < \cdots < \mathbf{T}(f_k).$$

Then

- $f_0 = PG_1 \cdots G_{k+1}$,
- $f_j = PH_jG_{j+1} \cdots G_{k+1}, 1 \leq j < k$,
- $f_k = PH_kG_{k+1}$,

where

P is the primitive part of $f_0 \in k[X_1][X_2]$;

$G_i \in k[X_1], 1 \leq i \leq k+1$;

$H_i \in k[X_1][X_2]$ is a monic polynomial of degree $d(i)$, for each i;

$d(1) < d(2) < \cdots < d(k)$;

$H_{i+1} \in (G_1 \cdots G_i, \dots, H_jG_{j+1} \cdots G_i, \dots, H_{i-1}G_i, H_i), \forall i$. □

13 Axis-of-Evil Theorem

The Axis-of-Evil Theorem [42, 43, 44] describes the combinatorial structure [Gröbner and border basis, linear and Gröbner representation] wrt the lex ordering of a 0-dimensional ideal $\mathsf{I} \subset \mathcal{P}$, in terms of its Macaulay representation.

Such description is "algorithmical" in terms of elementary combinatorial tools and linear interpolation and extends Cerlienco–Mureddu Correspondence and Lazard's Structural Theorem; the proof is essentially a direct application of Möller's Algorithm [45, 23].

It is summarized into 22[9] statements.

We report here one of its extreme statements:

Theorem 26 *Let*

$<$ the lex ordering induced by $X_1 < \cdots , X_n$,

[9]in honour of Trythemius, the founder of cryptography (*Steganographia* [1500], *Polygraphia* [1508]) which introdiced in german the 22^{th} letter **W** in order to perform german gematria.

$\mathsf{I} \subset \mathcal{P}$ *be a zero-dimensional radical ideal;*

$\mathsf{Z} := \{\mathbf{a}_1, \ldots, \mathbf{a}_s\} \subset k^n$ *its roots;*

$\mathbf{N} := \mathbf{N}_<(\mathsf{I})$*;*

$\mathbf{G}_<(\mathsf{I}) := \{\mathsf{t}_1, \ldots, \mathsf{t}_r\}, \mathsf{t}_1 < \mathsf{t}_2 < \ldots < \mathsf{t}_r$, $\mathsf{t}_i := X_1^{d_1^{(i)}} \cdots X_n^{d_n^{(i)}}$ *the minimal basis of its associated monomial ideal* $\mathbf{T}_<(\mathsf{I})$*;*

$G := \{f_1, \ldots, f_r\}, \mathbf{T}(f_i) = \mathsf{t}_i \forall i$, *the unique reduced lexicographical Gröbner basis of* I.

There is a combinatorial algorithm which, <u>given</u> Z, *<u>returns</u> sets of points*

$$\mathsf{Z}_{m\delta i} \subset k^m, \forall m, \delta, i : 1 \leq i \leq r, 1 \leq m \leq n, 1 \leq \delta \leq d_m^{(i)},$$

thus allowing to compute

- *by means of Cerlienco–Mureddu Algorithm the corresponding order ideal*

$$F_{m\delta i} := \mathbf{N}(\mathsf{Z}_{m\delta i}) \subset \mathcal{T} \cap k[X_1, \ldots, X_{m-1}]$$

- *and, by interpolation*[10] *unique polynomials*

$$\gamma_{m\delta i} := X_m - \sum_{\omega \in F_{m\delta i}} c_\omega \omega$$

which satisfy the relation

$$f_i = \prod_m \prod_\delta \gamma_{m\delta i} \pmod{(f_1, \ldots, f_{i-1})} \forall i.$$

Moreover, setting

ν *the maximal value such that* $d_\nu^{(i)} \neq 0, d_m^{(i)} = 0, m > \nu$ *so that* $f_i \in k[X_1, \ldots, X_\nu] \setminus k[X_1, \ldots, X_{\nu-1}]$,

$L_i := \prod_{m=1}^{\nu-1} \prod_\delta \gamma_{m\delta i}$ *and*

$P_i := \prod_\delta \gamma_{\nu\delta i}$

we have $f_i = L_i P_i$ *where* L_i *is the* leading polynomial *of* f_i. $\square$

Example 27 *For the nine points considered in Example 16 the corresponding Gröbner basis is* $G = \{g_1, g_2, g_3, g_4, f_1, f_2, f_3, f_4\}$ *where*

$$\begin{array}{rcl cl}
g_1 & := & X_1^3 - 3X_1^2 + 2X_1 & = & (X_1 - 2)(X_1 - 1)X_1 \\
g_2 & := & X_1^2 X_2 - X_1 X_2 & = & X_2(X_1 - 1)X_1, \\
g_3 & := & X_1 X_2^2 - X_1 X_2 & = & X_2(X_2 - 1)X_1, \\
g_4 & := & X_2^3 - 3X_2^2 + 2X_2 & = & X_2(X_2 - 1)(X_2 - 2),
\end{array}$$

[10] $X_m(\mathsf{a}) = \sum_{\omega \in F_{m\delta i}} c_\omega \omega(\mathsf{a}), \mathsf{a} \in Z_{m\delta i}$.

perfectly illustrating Lazard Structural Theorem, and

$$\begin{array}{lcl} f_1 & := & X_3X_1^2 - 3X_3X_1 + 2X_3 - 3X_2^2 - 6X_2X_1 + 9X_2 - X_1^2 + 3X_1 - 2, \\ f_2 & := & X_3X_2 + X_3X_1 - 2X_3 + 3X_2^2 + X_2X_1 - 7X_2 - 2X_1^2 + 3X_1 + 2, \\ f_3 & := & X_3^2X_1 - 2X_3^2 - 4X_3X_1 + 8X_3 - 15X_2^2 - 30X_2X_1 + 45X_2 + 3X_1 - 6, \\ f_4 & := & X_3^3 - 3X_3^2 + 3X_3X_1 - 4X_3 - 3X_2^2 - 6X_2X_1 + 9X_2 - 3X_1 + 6, \end{array}$$

satisfy (mod $(g_1, \dots, g_4)$)

$$\begin{array}{lcl} f_1 & = & (X_1 - 2)(X_1 - 1)(X_3 - \frac{3}{2}X_2^2 + \frac{9}{2}X_2 - 1) \\ f_2 & = & (X_2 + X_1 - 2)(X_3 + 3X_2 - 2X_1 - 1) \\ f_3 & = & (X_1 - 2)(X_3 - 1)(X_3 - 5X_1 + 2) \\ f_4 & = & (X_3 - 1)X_3(X_3 + 3X_1^2 - 8X_1 + 2) \end{array}$$

where

- $(X_1^2 - 3X_1 + 2, X_2 + X_1 - 2, X_3 - 1)$ *is the Gröbner basis of the ideal whose roots are* $\{\pi_2(\mathsf{a}_7), \pi_2(\mathsf{a}_8)\}$,
- $\{\mathsf{a} \in \mathsf{X} : (X_1^2 - 3X_1 + 2)(\mathsf{a}) \neq 0\} = \{\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_4\}$ *to which Cerlienco–Mureddu Correspondence associates* $\{1, X_2, X_2^2\}$
- $\{\mathsf{a} \in \mathsf{X} : (X_2 + X_1 - 2)(\mathsf{a}) \neq 0\} = \{\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_5\}$ *to which Cerlienco–Mureddu Correspondence associates* $\{1, X_1, X_2\}$
- $\{\mathsf{a} \in \mathsf{X} : (X_1 - 2)(X_3 - 1)(\mathsf{a}) \neq 0\} = \{\mathsf{a}_2, \mathsf{a}_4, \mathsf{a}_5, \mathsf{a}_6\}$ *to which Cerlienco–Mureddu Correspondence associates* $\{1, X_1, X_2, X_1X_2\}$.
- $\{\mathsf{a} \in \mathsf{X} : (X_3^2 - X_3))(\mathsf{a}) \neq 0\} = \{\mathsf{a}_2, \mathsf{a}_3, \mathsf{a}_4, \mathsf{a}_5, \mathsf{a}_6\}$ *to which Cerlienco–Mureddu Correspondence associates* $\{1, X_1, X_1^2, X_2, X_1X_2\}$.

□

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