

Bounded Functional Interpretation

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Proof mining:

Logical analysis of (ineffective) mathematical proofs with the aim of extracting new information.





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New information:

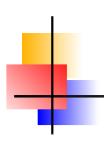
(bound on) witness for existential quantifier.





If one is looking for bounds, then bounded quantifiers shouldn't have computational content.





Bounded Quantifiers

- $\forall x^{\mathbb{N}} \leq tA(x)$ intrinsically different from $\forall x^{\mathbb{N}}A(x)$.
 - Induction on NP-predicates.
 - Bounded arithmetic.





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 - Induction on NP-predicates.
 - Bounded arithmetic.

• $\forall x^{\mathbb{N} \to \mathbb{N}} \leq tA(x)$ intrinsically different from $\forall x^{\mathbb{N} \to \mathbb{N}} A(x)$.

•
$$\forall x^{\mathbb{N} \to \mathbb{N}} \le t \dots \quad \forall x \in [0, 1] \dots$$



Bounded Quantifiers

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 - Induction on NP-predicates.
 - Bounded arithmetic.

- $\forall x^{\mathbb{N} \to \mathbb{N}} \leq tA(x)$ intrinsically different from $\forall x^{\mathbb{N} \to \mathbb{N}} A(x)$.
 - $\forall x^{\mathbb{N} \to \mathbb{N}} \leq t \dots$ $\forall x \in \text{compact Polish space.}$
 - $\forall x^{\mathbb{N} \to \mathbb{N}} \dots \forall x \in \mathsf{Polish} \; \mathsf{space}.$



- RCA₀: Basic theory of analysis.
- WKL: Every infinite binary tree has an infinite path.





An Example

- RCA₀: Basic theory of analysis.
- WKL: Every infinite binary tree has an infinite path.

Thm(Kohlenbach'92)

If
$$RCA_0 + WKL \vdash \forall x \in P; y \in K_x \exists z^{\tau} A_{\exists}(x, y, z)$$
 then

$$\exists$$
 closed term s s.t. $\forall x \in P; y \in K_x \exists z \leq s(x) A_\exists (x, y, z)$.





An Example

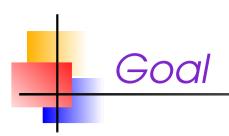
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If
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• Interpretation distinguishing $\forall x^{\rho} \leq tA(x)$ and $\forall x^{\rho}A(x)$.



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ightharpoonup Need: extended \leq to all finite types.





One Solution: Pointwise

Use pointwise less-than-equal-to relation:

$$x \leq_{\rho \to \sigma} y := \forall z^{\rho}(x(z) \leq_{\sigma} y(z)).$$





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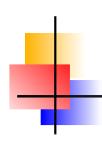
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$$x \leq_{\rho \to \sigma} y := \forall z^{\rho}(x(z) \leq_{\sigma} y(z)).$$

Problem:

 $x' \le x$ and $y' \le y$ does not imply $x'(y') \le x(y)$.





Another Solution: Monotone

- Howard/Bezem's strong majorizability relation.
- Extension of the <-relation to higher types:</p>

$$x \leq_{\rho \to \sigma}^* y := \forall v^{\rho} \forall u \leq_{\rho}^* v(\underbrace{xu \leq_{\sigma}^* yv}_{\text{above}} \land \underbrace{yu \leq_{\sigma}^* yv}_{\text{monotone}}))$$





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- **•** Example 1: for type $\mathbb{N} \to \mathbb{N}$ we have:

$$f \leq_{\mathbb{N} \to \mathbb{N}}^* g := \forall m \forall n \leq m(\underbrace{f(n) \leq g(m)}_{\text{above}} \land \underbrace{g(n) \leq g(m)}_{\text{monotone}}).$$





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• Example 2: $x \leq_{\rho \to \sigma}^* x$ means that x is monotone $x \leq_{\rho \to \sigma}^* x := \forall v^\rho \forall u \leq_{\rho}^* v (xu \leq_{\sigma}^* xv)$.



Majorizability: Some Properties

•
$$x' \le^* x \land y' \le^* y \to x'(y') \le^* x(y)$$

$$\bullet$$
 $x \leq^* y \rightarrow y \leq^* y$





Majorizability: Some Properties

•
$$x' \le^* x \land y' \le^* y \to x'(y') \le^* x(y)$$

Moreover, for each closed term t of e.g. HA^{ω} there is another closed term t^* such that, $HA^{\omega} \vdash t \leq^* t^*$.





Majorizability: A New Symbol

- Idea: Add majorizability relation \(\precedef \) to the language, functional interpretation can access the relation.
- Cannot just take:

$$\bullet$$
 $x \leq_{\mathbb{N}} y \leftrightarrow x \leq_{\mathbb{N}} y$

functional interpretation would ask for a witness for

$$x \leq_{\rho \to \sigma} y \leftarrow \forall v^{\rho} \forall u^{\rho} \leq_{\rho} v(xu \leq_{\sigma} yv \land yu \leq_{\sigma} yv)$$





Majorizability: A New Symbol

- Idea: Add majorizability relation \(\precedef \) to the language, functional interpretation can access the relation.
- One solution, use a rule instead of the implication:
 - \bullet $x \leq_{\mathbb{N}} y \leftrightarrow x \leq_{\mathbb{N}} y$

$$A_{\rm b} \to \forall u \unlhd v (su \unlhd tv \land tu \unlhd tv)$$
$$A_{\rm b} \to s \unlhd t$$





The Basic Setting

$$\mathsf{B}_\forall : \forall x \leq t A(x) \leftrightarrow \forall x (x \leq t \to A(x))$$

$$\mathsf{B}_\exists : \exists x \leq t A(x) \leftrightarrow \exists x (x \leq t \land A(x)).$$

▶ Let the theory IL^{ω}_{\leq} be intuitionistic logic (in all finite types) plus the axioms/rule for \leq , B_{\forall} and B_{\exists} .





Monotone Quantifiers

Quantify over "monotone functionals" as

$$\forall x (x \leq x \rightarrow A(x))$$

$$\exists x (x \leq x \land A(x))$$





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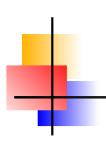
$$\forall x (x \le x \to A(x))$$

$$\exists x (x \le x \land A(x))$$

Use the following abbreviations:

$$\tilde{\forall} x A(x)$$
 instead of $\forall x (x \leq x \rightarrow A(x))$

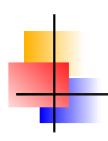
$$\exists x A(x)$$
 instead of $\exists x (x \leq x \land A(x))$



Main idea:

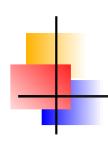
View
$$\forall x A(x)$$
 as $\tilde{\forall} b \quad \forall x \leq b A(x)$.





Main idea:

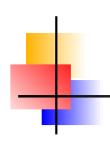




Main idea:

ullet A relativization to Bezem's model \mathcal{M} :

$$\tilde{\forall}b \forall x \leq b A(x)$$
 $\tilde{\forall}b \forall x (x \leq b \to A(x))$
 $\forall x (\tilde{\exists}b(x \leq b) \to A(x))$
 $\forall x (x \in \mathcal{M} \to A(x))$



• Associate arbitrary formulas of $\mathcal{L}^{\omega}_{\leq}$ to formulas having the form $\tilde{\exists}b\tilde{\forall}cA_{\mathrm{B}}(b,c)$.

$$A \in \mathcal{L}^{\omega}_{\triangleleft} \qquad \mapsto \qquad (A)^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c).$$





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Compare with Gödel's functional interpretation

$$A \in \mathcal{L}^{\omega} \qquad \mapsto \qquad (A)^{D} \equiv \exists x \forall y A_{\mathsf{qf}}(x, y).$$



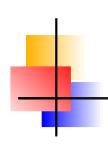
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Resulting matrix monotone on the first argument, i.e.

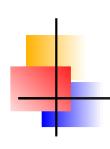
$$b \leq b' \wedge c \leq c \wedge A_{\mathcal{B}}(b,c,x) \rightarrow A_{\mathcal{B}}(b',c,x)$$





- Assume $(A(x))^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c, x)$.





- Assume $(A(x))^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c, x)$.
- $(\forall x \le t A(x))^{\mathbf{B}} \equiv \tilde{\exists} b \tilde{\forall} c \forall x \le t A_{\mathbf{B}}(b, c, x).$

$$(\forall x \leq t A(x))^{\mathrm{B}} \equiv \forall x \leq t \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c, x)$$





- Assume $(A(x))^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b,c,x)$.

$$(\forall x \le t A(x))^{\mathbf{B}} \equiv \forall x \le t \tilde{\exists} b \tilde{\forall} c A_{\mathbf{B}}(b, c, x)$$
$$\equiv \tilde{\exists} f \tilde{\forall} c \forall x \le t A_{\mathbf{B}}(fx, c, x)$$





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$$(\forall x \leq t A(x))^{B} \equiv \forall x \leq t \tilde{\exists} b \tilde{\forall} c A_{B}(b, c, x)$$
$$\equiv \tilde{\exists} f \tilde{\forall} c \forall x \leq t A_{B}(fx, c, x)$$
$$\equiv \tilde{\exists} f \tilde{\forall} c \forall x \leq t A_{B}(ft, c, x)$$





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- ▶ Assume $(A)^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b,c)$ and $(B)^{\mathrm{B}} \equiv \tilde{\exists} d \tilde{\forall} e B_{\mathrm{B}}(d,e)$.
- $(A \to B)^{B} \equiv \tilde{\exists} f, g \tilde{\forall} b, e(\tilde{\forall} c \leq gbeA_{B}(b, c) \to B_{B}(fb, e))$





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$$\equiv \tilde{\forall}b\tilde{\exists}d\tilde{\forall}e\tilde{\exists}c'(\tilde{\forall}c \leq c'A_{B}(b,c) \to B_{B}(d,e))$$





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$$\equiv \tilde{\exists}f,g\tilde{\forall}b,e(\tilde{\forall}c \leq gbeA_{B}(b,c) \to B_{B}(fb,e)).$$





The Soundness Theorem

Let $A(\underline{z})$ be an arbitrary formula of $\mathcal{L}^{\omega}_{\leq}$ and assume $(A(\underline{z}))^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b,c,\underline{z})$.

Thm(Soundness I) If

$$\mathsf{IL}^{\omega}_{\leq} \vdash A(\underline{z}),$$

then there are closed monotone terms \underline{t} of appropriate types such that

$$\mathsf{IL}^{\omega}_{\lhd} \vdash \tilde{\forall} \underline{a} \forall \underline{z} \leq \underline{a} \, \tilde{\forall} c A_{\mathrm{B}}(\underline{ta}, c, \underline{z}).$$





$$\mathsf{bAC}^{
ho, au}[\unlhd]:\ \forall x^{
ho}\exists y^{ au}A(x,y) o \tilde{\exists} f \tilde{\forall} b \forall x \unlhd b \exists y \unlhd faA(x,y)$$
,





$$\mathsf{bAC}^{\rho,\tau}[\unlhd]: \ \forall x^\rho \exists y^\tau A(x,y) \to \tilde{\exists} f \tilde{\forall} b \forall x \unlhd b \exists y \unlhd f a A(x,y)$$
,

$$\mathsf{bIP}^{\rho}_{\forall \mathrm{bd}}[\unlhd]: (\forall x A_{\mathrm{b}}(x) \to \exists y^{\rho} B(y)) \to \tilde{\exists} b(\forall x A_{\mathrm{b}}(x) \to \exists y \unlhd b B(y)),$$





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$$\mathsf{bBCC}^{\rho,\tau}_\mathrm{bd}[\unlhd]:\ \tilde{\forall} b^\tau\exists z\unlhd c^\rho\forall y\unlhd bA_\mathrm{b}(y,z)\to\exists z\unlhd c\forall yA_\mathrm{b}(y,z),$$

$$\mathsf{MAJ}^{\rho}[\unlhd] : \forall x^{\rho} \exists y (x \unlhd y).$$





Soundness: First extension

• Calling all the principles above $P[\leq]$ we have:

Thm(Soundness II) If

$$\mathsf{IL}^{\omega}_{\lhd} + \mathsf{P}[\unlhd] \vdash A(\underline{z}),$$

then there are closed monotone terms \underline{t} of appropriate types such that

$$\mathsf{IL}^{\omega}_{\leq} \vdash \widetilde{\forall}\underline{a} \forall \underline{z} \leq \underline{a} \,\widetilde{\forall} c A_{\mathrm{B}}(\underline{t}\underline{a}, c, \underline{z}).$$





Soundness: Second extension

Induction is interpreted using iteration functional.

Thm(Soundness III) If

$$\mathsf{HA}^{\omega}_{\lhd} + \mathsf{P}[\unlhd] \vdash A(\underline{z}),$$

then there are closed monotone terms \underline{t} of appropriate types such that

$$\mathsf{HA}^{\omega}_{\leq} \vdash \widetilde{\forall} \underline{a} \forall \underline{z} \leq \underline{a} \, \widetilde{\forall} c A_{\mathrm{B}}(\underline{t}\underline{a}, c, \underline{z}).$$





Interpreting Classical Theories

▶ $P_{bd}[\unlhd]$: restriction of $P[\unlhd]$ to bounded formulas.

Thm(Negative translation) If

$$\mathsf{PA}^{\omega}_{\lhd} + \mathsf{P}_{\mathrm{bd}}[\unlhd] \vdash A(\underline{z}),$$

then

$$\mathsf{HA}^{\omega}_{\leq} + \mathsf{P}_{\mathrm{bd}}[\leq] \vdash (A(\underline{z}))^{\mathrm{N}}$$



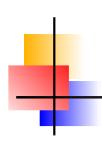


Uniform Weak König's Lemma

WKL: Every infinity binary tree has an infinite path, i.e.

$$\forall T(\mathsf{Inf}(T) \land \mathsf{Bin}(T) \to \exists p(\mathsf{Inf}(p) \land p \in T)).$$





Uniform Weak König's Lemma

WKL: Every infinity binary tree has an infinite path, i.e.

$$\forall T(\mathsf{Inf}(T) \land \mathsf{Bin}(T) \to \exists p(\mathsf{Inf}(p) \land p \in T)).$$

UWKL: Uniform version of weak König's lemma:

$$\exists \Phi \forall T (\mathsf{Inf}(T) \land \mathsf{Bin}(T) \to (\mathsf{Inf}(\Phi(T)) \land \Phi(T) \in T)).$$





Uniform Weak König's Lemma

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UWKL: Uniform version of weak König's lemma:

$$\exists \Phi \forall T (\mathsf{Inf}(T) \land \mathsf{Bin}(T) \to (\mathsf{Inf}(\Phi(T)) \land \Phi(T) \in T)).$$

Thm. $HA^{\omega} + P[\preceq] \vdash UWKL$.





Example of Meta-theorem

$$\mathsf{bAC}_0^{1,1} : \forall x^1 \exists y^1 A_0(x,y) \to \exists f \forall x \exists y \le_1 f(x) A_0(x,y).$$





Example of Meta-theorem

$$\mathsf{bAC}_0^{1,1} : \forall x^1 \exists y^1 A_0(x,y) \to \exists f \forall x \exists y \le_1 f(x) A_0(x,y).$$

Lemma.
$$PA^{\omega}_{\lhd} \vdash A(z) \Rightarrow PA^{\omega} \vdash A(z)[\leq^*/\trianglelefteq].$$





Example of Meta-theorem

$$\mathsf{bAC}_0^{1,1} : \forall x^1 \exists y^1 A_0(x,y) \to \exists f \forall x \exists y \le_1 f(x) A_0(x,y).$$

Lemma.
$$PA^{\omega}_{\lhd} \vdash A(z) \Rightarrow PA^{\omega} \vdash A(z)[\leq^*/\trianglelefteq].$$

Thm. If

$$\mathsf{PA}^{\omega} + \mathsf{bAC}_0^{1,1} + \mathsf{UWKL} \vdash \forall x^{\rho} \exists y^{\tau} A_0(x,y),$$

where A_0 is quantifier-free, then

$$\mathsf{PA}^{\omega} \vdash \tilde{\forall} a \forall x \leq^* a \exists y \leq^* q(a) A_0(x,y),$$

for some monotone closed term q.



Future work

Feasible case



Feasible case

Bounded modified realizability



Feasible case

Bounded modified realizability

Comparison with monotone functional interpretation

