Adaptivity, A Posteriori Error Estimation, and Applications of DGFEM

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AptoFEM

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Outline

1. Discontinuous Galerkin methods
   - DGFEMs for hyperbolic PDEs
   - DGFEMs for second–order elliptic PDEs
2. *A Posteriori* Error Estimation and Adaptivity
   - Introduction
   - High–order/*hp*–FEM
3. Energy norm *a posteriori* error estimation for CGFEMs
   - Model problem
   - Error estimation in one–dimension
   - Error estimation in higher–dimensions
4. Energy norm *a posteriori* error estimation for DGFEMs
   - Model problem and discretization
   - Energy norm error estimation
   - Applications

8. Second–Order PDEs with non-negative characteristic form
   - Model problem and IP-DG discretization
   - Error Estimation
   - Approximation Theory
   - *hp*–Adaptivity on Isotropic FE Spaces
   - Anisotropic *h*–Adaptivity
   - Anisotropic *hp*–Refinement

9. Compressible fluid flows
   - Interior penalty DGFEMs for the compressible Navier–Stokes equations
   - *hp*–Refinement
   - Anisotropic *h*–refinement
   - Anisotropic *hp*–refinement

Weak Enforcement of Boundary Conditions

- Lions 1968: Solution of PDEs with rough Dirichlet boundary conditions

Penalty Approach

Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_{\partial\Omega} \sigma(u-g)v \mathrm{d}s = \int_\Omega fv \mathrm{d}x \quad \forall v \in H^1(\Omega).$$

- Aubin 1970: FD for nonlinear problems.
- Babuška 1973: FE analysis.
- Nitsche 1971:

Penalty Approach

Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} v \mathrm{d}s - \int_{\partial\Omega} \frac{\partial v}{\partial \mathbf{n}} u \mathrm{d}s + \int_{\partial\Omega} \sigma(u-g)v \mathrm{d}s = \int_\Omega fv \mathrm{d}x \quad \forall v \in H^1(\Omega).$$

Interior Penalty Methods

- Babuška & Zlámal 1973: Weak imposition of C^1 continuity for 4th order problems using ideas from Lions 1968, Aubin 1970, & Babuška 1973.
- Douglas & Dupont 1976: Penalized C^0 elements with normal derivatives for 2nd order PDEs.
- Baker 1977: Penalized C^0 elements for 4th order PDEs.
- Wheeler 1978: IP methods for 2nd order PDES.
- Arnold 1979: Developed the analysis of IP methods.

Hybrid Methods

- Pian 1965.
- Tong & Kikuchi 1965.

Background

- Reed & Hill 1973: Proposed DG methods for the neutron–transport equation.
- Lesaint & Raviart 1974: Showed that $\|u - u_{\mathrm{DG}}\|_{L^2(\Omega)} = \mathcal{O}(h^p)$.
- Johnson, Nävert & Pitkaranta 1984, Johnson & Pitkaranta 1986: Analysis improved to show that $|||u - u_{\mathrm{DG}}||| = \mathcal{O}(h^{p+1/2})$.
- Peterson 1991: Showed that $\mathcal{O}(h^{p+1/2})$ is optimal on general meshes.
- Richter 1988: Proved $\|u - u_{\mathrm{DG}}\|_{L^2(\Omega)} = \mathcal{O}(h^{p+1})$ on structured meshes.

Nonlinear Conservation Laws

- Cockburn & Shu 1989, 1990, 1991, ...: RKDG schemes.
- Bassi & Rebay 1997, ...: Compressible Euler/Navier–Stokes equations.
- Baumann & Oden 1999: Compressible flows.

DGFEMs: Hyperbolic PDEs

hp–DGFEM

- Bey & Oden 1996: DG with SUPG stabilisation, $\delta = h/p^2$.
- H., Schwab, & Süli 2000: DG with SUPG stabilisation, $\delta = h/p$.
- H., Schwab, & Süli 2002: DG without SUPG stabilisation.

Superconvergence

- Biswas, Devine & Flaherty 1994: Observed superconvergence at the Gauss–Radau points.
- Adjerid, Aiffa & Flaherty 1998.
- Adjerid, Devine, Flaherty, & Krivodonova 2002.
- Cockburn, Luskin, Shu, & Süli 2003: Postprocessing DGFEM.

- Applications

  Linear elliptic/parabolic/hyperbolic PDEs, Fokker–Planck equations, Incompressible/Compressible fluid flows, Turbulent flows, Non-Newtonian flows, Time and frequency domain Maxwell's equations, Acoustics, MHD, Fully nonlinear PDEs.

- Advantages
  - Robustness/stability;
  - Locally conservative;
  - Ease of implementation;
  - Highly parallelizable;
  - Flexible mesh design (hybrid grids, non-matching grids, non-uniform/anisotropic polynomial degrees);
  - Wider choice of stable FE spaces for mixed problems;
  - Unified treatment of a wide range of PDEs;
  - Computational overhead/efficiency (increase in DoFs)

    $\Rightarrow$ Parallel efficiency gains: 56 Billion dofs on a GPU; Domain decomposition without overlap.

Transport Equation

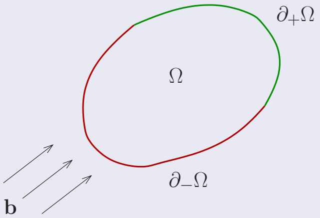
Consider the PDE problem

$$\begin{aligned} \mathbf{b} \cdot \nabla u &= f, \quad x \in \Omega, \\ u &= g, \quad x \in \partial_- \Omega. \end{aligned}$$

Inflow/Outflow boundaries

We define the inflow and outflow boundaries $\partial_- \Omega$ and $\partial_+ \Omega$, respectively, by

$$\begin{aligned} \partial_- \Omega &= \{x \in \partial\Omega : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\}, \\ \partial_+ \Omega &= \{x \in \partial\Omega : \mathbf{b}(x) \cdot \mathbf{n}(x) \geq 0\}. \end{aligned}$$



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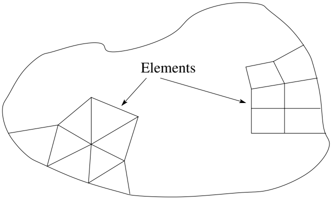
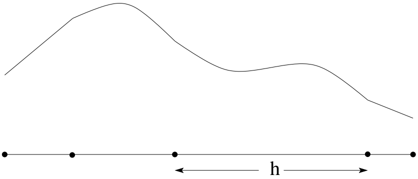
- We shall consider two variants:
  1. Standard Galerkin with strongly imposed boundary conditions;
  2. Standard Galerkin with weakly imposed boundary conditions;
- $\mathcal{T}_h$ is a non-degenerate mesh consisting of elements of granularity h.



- Finite element space:

$$V_h = \{v \in H^1(\Omega) : v|_\kappa \in \mathcal{S}_p(\kappa) \quad \forall \kappa \in \mathcal{T}_h\}.$$



- Consider the following two additional finite element spaces:

$$\begin{aligned} V_{h,E} &= \{v \in V_h : v = g \ \text{ on } \partial_-\Omega\}, \\ V_{h,E_0} &= \{v \in V_h : v = 0 \ \text{ on } \partial_-\Omega\}. \end{aligned}$$

Standard Galerkin with strongly imposed boundary conditions

Find $u_h \in V_{h,E}$ such that

$$(\mathbf{b} \cdot \nabla u_h, v_h) = (f, v_h) \quad \forall v_h \in V_{h,E_0}.$$

- Here, $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$ inner-product, i.e., given $v, w \in L^2(\Omega)$,

$$(v, w) = \int_\Omega vw \, \mathrm{d}x.$$

- Variants include the streamline–diffusion/SUPG method ($v_h \mapsto v_h + \delta \mathbf{b} \cdot \nabla v_h$, where $\delta = C_\delta h$, or more generally, $\delta = C_\delta h/p$, $C_\delta > 0$).

Standard Galerkin with weakly imposed boundary conditions

Find $u_h \in V_h$ such that

$$(\mathbf{b} \cdot \nabla u_h, v_h) - < u_h, v_h >_- = (f, v_h) - < g, v_h >_- \quad \forall v_h \in V_h,$$

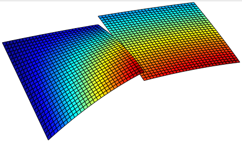
where

$$< v, w >_- = \int_{\partial_- \Omega} \mathbf{b} \cdot \mathbf{n} \, vw \, \mathrm{d}s.$$

- Finite Element Space

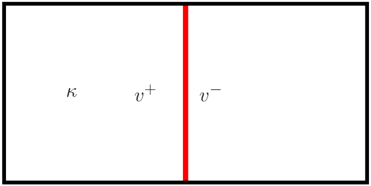
$$V_h = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{S}_p(\kappa) \ \ \forall \kappa \in \mathcal{T}_h\},$$

where $\mathcal{T}_h$ is the mesh.



- Consider a local (elementwise) FE formulation with weakly imposed bcs.
- Notation: for $v \in H^1(\kappa)$, we write

$$\begin{aligned} v^+ &= \text{interior trace of } v \text{ on } \partial\kappa \text{ taken from within } \kappa \\ v^- &= \text{exterior trace of } v \text{ on } \partial\kappa \text{ taken from outside } \kappa \end{aligned}$$



- Local FE formulation: on each $\kappa \in \mathcal{T}_h$, find $u_h \in V_h$ such that

$$\int_\kappa \mathbf{b} \cdot \nabla u_h \, v_h \mathrm{d}x - \int_{\partial_-\kappa} \mathbf{b} \cdot \mathbf{n} \, u_h^+ v_h^+ \, \mathrm{d}s = \int_\kappa f \, v_h \mathrm{d}x - \int_{\partial_-\kappa} \mathbf{b} \cdot \mathbf{n} \, \hat{g} \, v_h^+ \mathrm{d}s$$

for all $v_h \in V_h$. Here,

$$\begin{aligned} \partial_-\kappa &= \{x \in \partial\kappa : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\}, \\ \partial_+\kappa &= \{x \in \partial\kappa : \mathbf{b}(x) \cdot \mathbf{n}(x) \geq 0\}. \end{aligned}$$

Moreover, we define

$$\hat{g}(x) = \begin{cases} u_h^-(x) & x \in \partial_-\kappa \backslash \partial\Omega, \\ g(x) & x \in \partial_-\kappa \cap \partial\Omega. \end{cases}$$

- Sum over all elements $\kappa \in \mathcal{T}_h$:

DGFEM

Find $u_h \in V_h$ such that

$$\sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa \mathbf{b} \cdot \nabla u_h \, v_h \mathrm{d}x - \int_{\partial_- \kappa \setminus \partial\Omega} \mathbf{b} \cdot \mathbf{n} \, (u_h^+ - u_h^-) v_h^+ \, \mathrm{d}s \right.$$

$$\left. - \int_{\partial_- \kappa \cap \partial\Omega} \mathbf{b} \cdot \mathbf{n} \, u_h^+ v_h^+ \, \mathrm{d}s \right\} = \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa f \, v_h \mathrm{d}x - \int_{\partial_- \kappa \cap \partial\Omega} \mathbf{b} \cdot \mathbf{n} \, g v_h^+ \mathrm{d}s \right\}$$

for all $v_h \in V_h$.

Transport Equation in Conservative Form

Consider the PDE problem

$$\begin{aligned} \nabla \cdot (\mathbf{b}u) &= f, \quad x \in \Omega, \\ u &= g, \quad x \in \partial_- \Omega. \end{aligned}$$

[Assuming $\mathbf{b}$ is incompressible, i.e., $\nabla \cdot \mathbf{b} = 0$, the conservative and non-conservative forms are equivalent.]

- Local weak formulation: on each $\kappa \in \mathcal{T}_h$, find $u|_\kappa$ such that

$$-\int_\kappa (\mathbf{b}u) \cdot \nabla v \mathrm{d}x + \int_{\partial\kappa} (\mathbf{b}u^+) \cdot \mathbf{n}_\kappa v^+ \,\mathrm{d}s = 0.$$

- Inter-element continuity and bcs weakly enforced

$$-\int_\kappa (\mathbf{b}u) \cdot \nabla v \mathrm{d}x + \int_{\partial\kappa} \mathcal{H}(u^+, u^-, \mathbf{n}_\kappa) v^+ \,\mathrm{d}s = \int_\kappa f v \mathrm{d}x.$$

- $\mathcal{H}(\cdot, \cdot, \mathbf{n})$ is a numerical flux function.
- Sum over all elements $\kappa \in \mathcal{T}_h$ and restrict to the FEM space V_h:

DGFEM

Find $u_h \in V_h$ such that

$$\sum_{\kappa \in \mathcal{T}_h} \left\{ -\int_\kappa (\mathbf{b}u_h) \cdot \nabla v_h \mathrm{d}x + \int_{\partial\kappa} \mathcal{H}(u_h^+, u_h^-, \mathbf{n}_\kappa) v_h^+ \,\mathrm{d}s \right\} = \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa f \, v_h \mathrm{d}x \right\}$$

for all $v_h \in V_h$.

- Properties of the numerical flux function $\mathcal{H}(\cdot,\cdot,\cdot)$.
  1. Consistency: for each κ in $\mathcal{T}_h$ we have that
  $$\mathcal{H}(v,v,\mathbf{n}_\kappa)|_{\partial\kappa} = (\mathbf{b}v)\cdot\mathbf{n}_\kappa \quad \forall\kappa\in\mathcal{T}_h.$$
  2. Conservation: given any two neighbouring elements κ and κ' from the finite element partition $\mathcal{T}_h$, at each point $\mathbf{x}\in\partial\kappa\cap\partial\kappa'\neq\emptyset$, noting that $\mathbf{n}_{\kappa'}=-\mathbf{n}_\kappa$, we have that
  $$\mathcal{H}(v,w,\mathbf{n}_\kappa) = -\mathcal{H}(w,v,-\mathbf{n}_\kappa).$$
- Choose the upwind numerical flux:
$$\mathcal{H}(u_h^+,u_h^-,\mathbf{n}_\kappa)|_{\partial\kappa} = \mathbf{b}\cdot\mathbf{n}_\kappa \lim_{s\to 0^+} u_h(x-s\mathbf{b}) \quad \text{for } \kappa\in\mathcal{T}_h.$$
By convention, we set $u_h^-|_{\partial_-\kappa\cap\partial\Omega}=g$; thereby,
$$\mathcal{H}(u_h^+,u_h^-,\mathbf{n}_\kappa)|_{\partial_-\kappa\cap\partial\Omega} = \mathbf{b}\cdot\mathbf{n}_\kappa\, g \quad \text{for } \kappa\in\mathcal{T}_h.$$
- (Nonlinear Problems) Other Examples: Lax–Friedrichs flux, Roe's flux, Vijayasundaram flux, ...
- Note that both DGFEM formulations are equivalent (proof: exercise).

Given $\Omega \subset \mathbb{R}^d$, $d \geq 1$, and $f \in L^2(\Omega)$, find u such that

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

DG Discretization:

1. Rewrite as a first–order system.
2. Derive an elemental weak formulation.
3. Introduce numerical flux functions $\Rightarrow$ Flux Formulation.
4. Eliminate the auxiliary variables $\Rightarrow$ Primal Formulation.

Poisson's Equation

- Rewrite as a first–order system.

$$\mathbf{s} - \nabla u = 0, \qquad -\nabla \cdot \mathbf{s} = f.$$

- Elemental weak formulation: find $(\mathbf{s}, u)$ such that

$$\begin{aligned} \int_\kappa \mathbf{s} \cdot \boldsymbol{\tau} \mathrm{d}x + \int_\kappa u \nabla \cdot \boldsymbol{\tau} \mathrm{d}x - \int_{\partial\kappa} u \boldsymbol{\tau} \cdot \mathbf{n}_\kappa \mathrm{d}s &= 0, \\ \int_\kappa \mathbf{s} \cdot \nabla v \mathrm{d}x - \int_{\partial\kappa} \mathbf{s} \cdot \mathbf{n}_\kappa v \mathrm{d}s &= \int_\kappa f v \mathrm{d}x. \end{aligned}$$

- Notation: ∇_h denotes the broken gradient operator, defined elementwise.
- Numerical flux functions:
  1. $\hat{u} = \hat{u}(u_h)$,
  2. $\hat{\mathbf{s}} = \hat{\mathbf{s}}(u_h, \nabla_h u_h)$

  are approximations to u_h and $\nabla_h u_h$, respectively.
- Flux Formulation: find $u_h \in V_h$ and $\mathbf{s}_h \in \Sigma_h = [V_h]^d$ such that

$$\int_\Omega \mathbf{s}_h \cdot \boldsymbol{\tau}_h \mathrm{d}x + \int_\Omega u_h \nabla_h \cdot \boldsymbol{\tau}_h \mathrm{d}x - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{u} \boldsymbol{\tau}_h \cdot \mathbf{n}_\kappa \mathrm{d}s = 0, \quad (1)$$

$$\int_\Omega \mathbf{s}_h \cdot \nabla v_h \mathrm{d}x - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{\mathbf{s}} \cdot \mathbf{n}_\kappa v_h \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x \quad (2)$$

for all $\boldsymbol{\tau}_h \in \Sigma_h$, $v_h \in V_h$.

- Setting $\boldsymbol{\tau}_h = \nabla_h v_h$ in (1) and integrating by parts gives

$$\int_\Omega \mathbf{s}_h \cdot \nabla_h v_h \mathrm{d}x - \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \mathrm{d}x + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} (u_h^+ - \hat{u}) \nabla_h v_h^+ \cdot \mathbf{n}_\kappa \mathrm{d}s = 0. \quad (3)$$

- Inserting (3) into (2) gives the primal formulation: find $u_h \in V_h$ such that

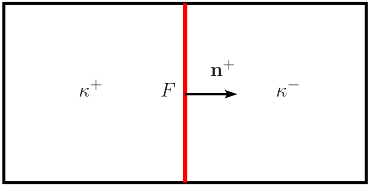
$$\int_\Omega \nabla_h u_h \cdot \nabla v_h \mathrm{d}x - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} (u_h^+ - \hat{u}) \nabla_h v_h^+ \cdot \mathbf{n}_\kappa \mathrm{d}s$$
$$- \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{\mathbf{s}} \cdot \mathbf{n}_\kappa v_h^+ \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x$$

for all $v_h \in V_h$.

Notation

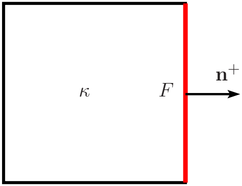
- Let $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}$ denote the set of all faces in the mesh $\mathcal{T}_h$.
- Notation:

$F \subset \mathcal{F}_h^{\mathcal{I}}$



$$
\begin{aligned}
[\![v]\!] &= v^+\mathbf{n}^+ + v^-\mathbf{n}^- \\
[\![\mathbf{q}]\!] &= \mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{q}^- \cdot \mathbf{n}^- \\
\{\!\{v\}\!\} &= (v^+ + v^-)/2
\end{aligned}
$$

$F \subset \mathcal{F}_h^{\mathcal{B}}$



$$
\begin{aligned}
[\![v]\!] &= v\mathbf{n} \\
[\![\mathbf{q}]\!] &= \mathbf{q} \cdot \mathbf{n} \\
\{\!\{v\}\!\} &= v
\end{aligned}
$$

- The following identity holds:

$$
\sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \mathbf{q}^+ \cdot \mathbf{n}^+ \, v^+ \mathrm{d}s = \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\mathbf{q}\}\!\} \cdot [\![v]\!] \mathrm{d}s + \sum_{F \in \mathcal{F}_h^{\mathcal{I}}} \int_F [\![\mathbf{q}]\!] \{\!\{v\}\!\} \mathrm{d}s.
$$

DGFEM Primal Formulation

Find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = \int_\Omega f v_h \mathrm{d}x \qquad \forall v_h \in V_h,$$

where

$$\begin{aligned} A_h(u_h, v_h) &= \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \nabla u_h \cdot \nabla v_h \mathrm{d}x \\ &+ \sum_{F \in \mathcal{F}_h} \int_F (\llbracket \hat{u} - u_h \rrbracket \cdot \{\!\{ \nabla_h v_h \}\!\} - \{\!\{ \hat{\mathbf{s}} \}\!\} \cdot \llbracket v_h \rrbracket) \mathrm{d}s \\ &+ \sum_{F \in \mathcal{F}_h^{\mathcal{I}}} \int_F (\{\!\{ \hat{u} - u_h \}\!\} \llbracket \nabla_h v_h \rrbracket - \llbracket \hat{\mathbf{s}} \rrbracket \{\!\{ v_h \}\!\}) \mathrm{d}s, \end{aligned}$$

where $\hat{u} = \hat{u}(u_h)$ and $\hat{\mathbf{s}} = \hat{\mathbf{s}}(u_h, \nabla_h u_h)$.

Numerical fluxes

- Consistency:
$$\hat{u}(v) = v, \qquad \hat{\mathbf{s}}(v, \nabla v) = \nabla v$$
for all smooth v which satisfy the Dirichlet boundary conditions.
- Galerkin Orthogonality: For $u \in H^2(\Omega)$, we have
$$A_h(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h.$$
[This automatically follows from the consistency of $\hat{u}$ and $\hat{\mathbf{s}}$.]
- Conservation: The numerical fluxes are conservative if they are single valued, i.e.,
$$[\![\hat{u}]\!] = 0, \qquad [\![\hat{\mathbf{s}}]\!] = 0.$$

Adjoint Consistency

Given $\psi \in L^2(\Omega)$, let $z \in H^2(\Omega)$ be the analytical solution of the dual/adjoint problem

$$-\Delta z = \psi \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega.$$

Then the discretization is adjoint consistent if and only if

$$A_h(v, z) = \int_\Omega \psi v \, \mathrm{d}x \quad \forall v \in H^2(\mathcal{T}_h),$$

where $H^2(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : v|_\kappa \in H^2(\kappa),\ \kappa \in \mathcal{T}_h \right\}$.

Lemma

The above discretization is adjoint consistent, if and only if, the numerical fluxes $\hat{u}$ and $\hat{\mathbf{s}}$ are conservative.

- Given $\theta \in [-1,1]$, we define

$$\hat{u} = \hat{u}(u_h) = \begin{cases} \{\!\{u_h\}\!\} + \frac{1+\theta}{2}\mathbf{n}^+ \cdot [\![u_h]\!] & \text{on} \quad F \in \mathcal{F}_h^{\mathcal{I}}, \\ (1+\theta)u_h & \text{on} \quad F \in \mathcal{F}_h^{\mathcal{B}}, \end{cases}$$

$$\hat{\mathbf{s}} = \hat{\mathbf{s}}(u_h, \nabla_h u_h) = \begin{cases} \{\!\{\nabla_h u_h\}\!\} - \sigma [\![u_h]\!] & \text{on} \quad F \in \mathcal{F}_h^{\mathcal{I}}, \\ \nabla u_h - \sigma u_h \mathbf{n} & \text{on} \quad F \in \mathcal{F}_h^{\mathcal{B}}. \end{cases}$$

- Here, σ denotes the interior penalty parameter:

$$\sigma = C_\sigma \frac{p^2}{h}, \qquad C_\sigma > 0.$$

Interior Penalty Methods

Given $\theta \in [-1,1]$, find $u_h \in V_h$ such that

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h \mathrm{d}x - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla_h u_h\}\!\} \cdot [\![v_h]\!] \mathrm{d}s$$

$$+\theta \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot [\![u_h]\!] \mathrm{d}s + \sum_{F\in\mathcal{F}_h} \int_F \sigma [\![u_h]\!] \cdot [\![u_h]\!] \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x$$

for all $v_h \in V_h$.

- $\theta = -1$ – Symmetric Interior Penalty Method (SIP).
- $\theta = 1$ – Nonsymmetric Interior Penalty Method (NIP).
- $\theta = 0$ – Incomplete Interior Penalty Method (IIP).

Arnold, Brezzi, Cockburn, & Marini 2002

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Babuška & Zlámal (1973)
(Symmetric) Interior Penalty Method; Douglas & Dupont (1976)
Bassi & Rebay (1997)
Bassi, Rebay, Mariotti, Pedinotti & Savini (1997)
Local DG Method; Cockburn & Shu (1998)
Brezzi, Manzini, Marini, Pietra and Russo (1999)
Baumann & Oden (1999)
(Non-Symmetric) IP Method; Riviere, Wheeler & Girault (1999)
Brezzi, Manzini, Marini, Pietra and Russo (2000)
WOPSIP; Brenner, Owens, & Sung (2008)

Arnold, Brezzi, Cockburn & Marini (2002)

Prudhomme, Pascal, Oden & Romkes (2000)

- Applications

  Linear elliptic/parabolic/hyperbolic PDEs, Fokker–Planck equations, Incompressible/Compressible fluid flows, Turbulent flows, Non-Newtonian flows, Time and frequency domain Maxwell's equations, Acoustics, MHD, Fully nonlinear PDEs.

- Overlapping/Fictitious Domain Methods

  Hansbo & Hansbo 2002, Becker, Hansbo, & Stenberg 2003, Hansbo, Hansbo & Larson 2003, Becker, Burman, & Hansbo 2009, Johansson & Larson 2011, . . .

- Hybridizable DG Methods

  Cockburn & Gopalakrishnan 2004, 2005, Cockburn, Dong, & Guzmán 2008, Cockburn, Gopalakrishnan, & Lazarov 2009, Cockburn, Nguyen, & Peraire 2010 . . .

- Linear Solvers

  Feng & Karakashian 2001, Gopalakrishnan & Kanschat 2003, Brenner & Wang 2005, Antonietti & Ayuso 2007, 2008, 2009, Antonietti & H. 2011, Antonietti, Ayuso, Brenner, & Sung 2011

PDE Model

Find u such that

$$\mathcal{L}u = f,$$

subject to appropriate boundary/initial conditions.

$$\Downarrow$$

Weak Formulation

Find $u \in V$ such that

$$A(u, v) = \ell(v) \quad \forall v \in V,$$

where $A(\cdot, \cdot) : V \times V \to \mathbb{R}$ is a bilinear form, $\ell(\cdot) : V \to \mathbb{R}$ is a linear functional, and V is an appropriately selected function space.

$$\Downarrow$$

Finite Element Approximation

Find $u_{h,p} \in V_{h,p}$ such that

$$A_h(u_{h,p}, v) = \ell(v) \quad \forall v \in V_{h,p},$$

where $V_{h,p}$ is a finite element space consisting of (dis)continuous piecewise polynomials of degree p represented on a mesh of granularity h.

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1. *A Priori* Error Bound

$$\|u - u_{h,p}\|_{\mathcal{V}} \leq C\, h^s\, \|u\|_{\mathcal{W}}, \quad s > 0\,.$$

- $\mathcal{V} = L^2(\Omega)$, $\mathcal{W} = H^k(\Omega)$, and

$$\mathcal{L} := -\Delta,$$

then

$$s = \min(p+1, k).$$

2. *A Posteriori* Error Bound

$$\|u - u_{h,p}\|_{\mathcal{V}} \leq C\,\|h^t(f - \mathcal{L}u_{h,p})\|_{\mathcal{W}}, \quad t \geq 0\,.$$

- The quantity

$$R(u_{h,p}) := f - \mathcal{L}u_{h,p},$$

  is referred to as the finite element residual.
- The residual measures the extent to which the finite element approximation to the analytical solution fails to satisfy the underlying PDE.
- *A posteriori* error bounds may be used in a number of ways:
  1. Assess the quality (accuracy) of a numerical solution;
  2. Drive automatic adaptive mesh refinement algorithms to deliver solutions within a desired level of accuracy.

Measurement Problem

Given a user–defined tolerance $\mathtt{TOL} > 0$, we wish to efficiently design $V_{h,p}$ such that

$$\|u - u_{h,p}\|_{\mathcal{V}} \leq \mathtt{TOL}.$$

- Exploiting the *a posteriori* error bound

$$\|u - u_{h,p}\|_{\mathcal{V}} \leq C \, \|h^t(f - \mathcal{L}u_{h,p})\|_{\mathcal{W}}, \quad t \geq 0\,,$$

amounts to designing $V_{h,p}$ such that

$$C \, \|h^t(f - \mathcal{L}u_{h,p})\|_{\mathcal{W}} \leq \mathtt{TOL}. \tag{4}$$

- If (4) is not satisfied, then we must enrich $V_{h,p}$ by *locally* refining the computational mesh, recomputing a new (improved) finite element solution $u_{h,p}^{\mathtt{new}}$, and inserting $u_{h,p}^{\mathtt{new}}$ back into (4). If (4) is still not satisfied, then we repeat the process until the desired error control is attained.

Adaptivity

- Goal:

$$\|u - u_{h,p}\|_{\mathcal{V}} \le C\,\|h^t(f - \mathcal{L}u_{h,p})\|_{\mathcal{W}} \equiv \left(\sum_{\kappa\in\mathcal{T}_h} \eta_\kappa(u_{h,p})^2\right)^{1/2} \le \mathtt{TOL}.$$

- Automatic refinement algorithm:

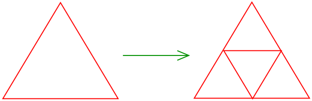
  0. Start with initial (coarse) grid $\mathcal{T}_h^{(j=0)}$.
  1. Compute the numerical solution $u_{h,p}^{(j)}$ on $\mathcal{T}_h^{(j)}$.
  2. Compute the local error indicators η_κ.
  3. If $\left(\sum_{\kappa\in\mathcal{T}_h} \eta_\kappa(u_{h,p})^2\right)^{1/2} \le \mathtt{TOL} \to$ stop. Otherwise, adapt $V_{h,p}$.
  4. $j = j + 1$, and go to step (1).

R. Verfürth
A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques,
B.G. Teubner, Stuttgart, 1996.

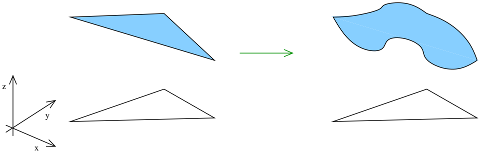
- Once the elements have been marked for refinement/derefinement, the mesh must now be modified accordingly. This may be done in a number of ways:
  1. h–/r–Refinement: Locally/globally adjust the computational mesh.
  2. p–Refinement: Locally enrich the polynomial order.
  3. hp–Refinement: Mesh and polynomial degree enrichment.

- High–order/variable order FEMs
  Babuška, Szabo, & Katz 1981, Gui & Babuška 1986, Schwab 1998

- *hp*–Refinement
  - Local mesh subdivision (*h*–refinement)



  - Local polynomial enrichment (*p*–refinement)



- Approximation theory
  - Suppose that $u|_\Omega \in H^k(\Omega)$

$$\|u - \Pi_{hp} u\|_{H^s(\Omega)} \le C\, \frac{h^{\min(p+1,k)-s}}{p^{k-s}}\, \|u\|_{H^k(\Omega)},$$

    $0 \le s \le \min(p+1, k)$.   Babuška & Suri 1987
  - If u is a real analytic function on Ω, then

$$\|u - \Pi_{hp} u\|_{H^s(\Omega)} \le C(u)\, h^{p+1-s}\, \mathrm{e}^{-bp}, \quad b > 0,$$

    $\Rightarrow$ Exponential convergence   Schwab 1998
- High–order/*hp*–FEM for PDEs with layers/shocks, etc?
  $\Rightarrow$ Solution is typically a piecewise analytic function

2D Stokes equations in an L-shaped domain

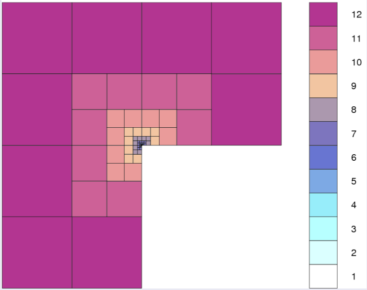
$$(\mathbf{u}, p) \in H^{1+\varepsilon}(\Omega)^2 \times H^{\varepsilon}(\Omega), \quad \varepsilon > 0.$$

Dauge 1989

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq C \exp(-\gamma\,(\text{DOF})^{1/3}), \quad \gamma > 0.$$

FEM($p, p-2$) & GLS(p, p): Schötzau & Schwab 1999
IP–DG($p, p-1$): Schötzau & Wihler 2003, H., Schötzau, & Wihler 2003

hp–Refinement



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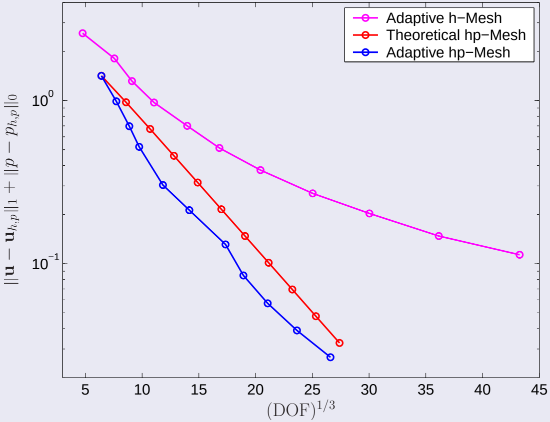
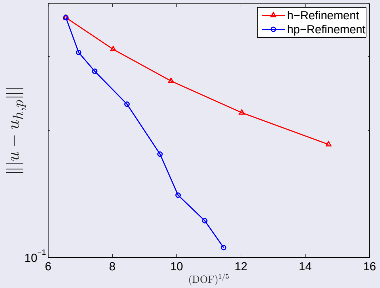
2D Stokes equations in an L-shaped domain

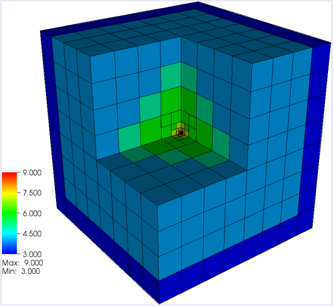


Dofs for accuracy of 10^{-1}

h–Version	100K
hp–Version	5K

hp–Refinement (Poisson's Equation)



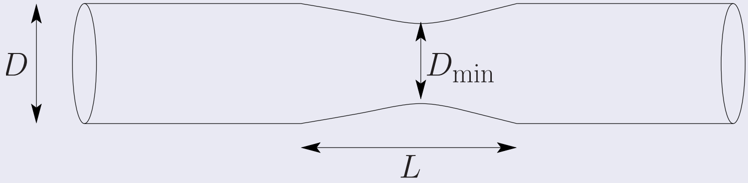


Dofs for accuracy of 10^{-1}

h–Version	NLTEC
hp–Version	$\approx$ 250K

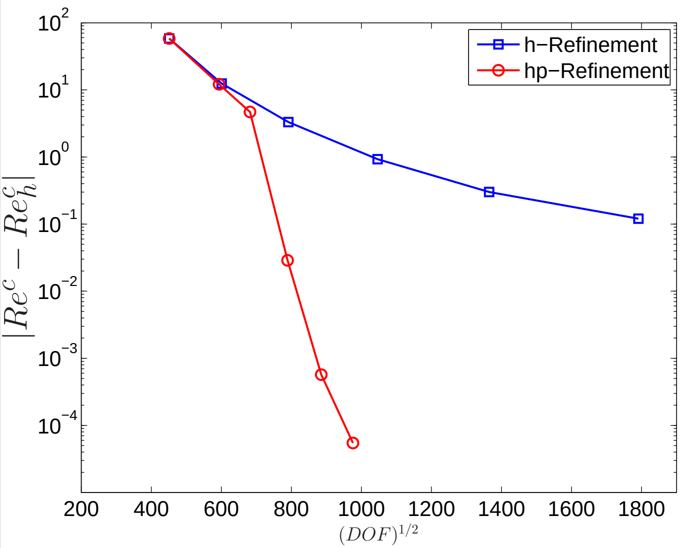
Flow Stability – Computation of the Critical Re^c

- Sherwin & Blackburn 2005: $Re^c \approx 721$.



- We set the parameters in the following ratio: $D_{\min} : D : L = 1 : 2 : 4$.

Flow Stability – Computation of the Critical Re^c



General Second–Order Elliptic PDE

Consider

$$\begin{aligned} -\Delta u + c(x)u &= f(x), \qquad x \in \Omega, \\ u &= 0, \qquad\quad\;\; x \in \partial\Omega. \end{aligned}$$

Assumptions

$$c \in C(\bar{\Omega}), \quad f \in L^2(\Omega),$$

and that

$$c(x) \geq 0, \quad x \in \bar{\Omega}.$$

Weak Formulation

Find $u \in H_0^1(\Omega)$ such that

$$A(u, v) = \ell(v) \quad \forall v \in H_0^1(\Omega).$$

$$\begin{aligned} A(u, v) &= \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega c u \, v \,\mathrm{d}x, \\ \ell(v) &= \int_\Omega f \, v \,\mathrm{d}x. \end{aligned}$$

Properties of $A(\cdot, \cdot)$ and $\ell(\cdot)$

(a) $\exists c_0 > 0 \quad \forall v \in H_0^1(\Omega) \quad A(v, v) \geq c_0 \|v\|^2_{H^1(\Omega)}$,

(b) $\exists c_1 > 0 \quad \forall v, w \in H_0^1(\Omega) \quad |A(w, v)| \leq c_1 \|w\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$,

(c) $\exists c_2 > 0 \quad \forall v \in H_0^1(\Omega) \quad |\ell(v)| \leq c_2 \|v\|_{H^1(\Omega)}$.

- Consider the boundary value problem

Find u such that

$$\begin{aligned} -u'' + c(x)u &= f(x), \quad x \in (0,1), \\ u(0) = 0, \quad u(1) &= 0. \end{aligned}$$

$$\begin{aligned} c \in C[0,1], \quad f \in L^2(0,1), \\ c(x) \geq 0, \quad x \in (0,1). \end{aligned}$$

- The weak formulation is given by:

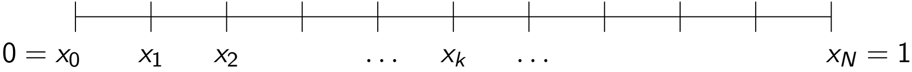
Find $u \in H_0^1(0,1)$ such that

$$A(u,v) = \ell(v) \quad \forall v \in H_0^1(0,1),$$

where

$$\begin{aligned} A(u,v) &= \int_0^1 \left(u'v' + cuv\right) \mathrm{d}x, \\ \ell(v) &= \int_0^1 fv \mathrm{d}x. \end{aligned}$$

- Mesh $\mathcal{T}_h = \{\kappa_i\}_{i=1}^N$, where $\kappa_i = (x_{i-1}, x_i)$.



- On $\mathcal{T}_h$, we define the distribution $p = (p_1, \ldots, p_N)^\top$ of polynomial degrees $p_i \geq 1$.
- Finite element space:

$$V_{h,p} = \left\{ u \in H_0^1(0,1) : u|_{\kappa_i} \in \mathcal{P}_{p_i}(\kappa_i),\ i = 1, \ldots, N \right\},$$

where $\mathcal{P}_p(I)$ denotes the set of polynomials of degree less than or equal to p on I.

- The finite element approximation is given by:

Find $u_{h,p} \in V_{h,p}$ such that

$$A(u_{h,p}, v_{h,p}) = \ell(v_{h,p}) \quad \forall v_{h,p} \in V_{h,p}.$$

- Define the *finite element residual:*

$$R(u_{h,p})(x) = f(x) - (-u''_{h,p}(x) + c(x)u_{h,p}(x)), \quad x \in (x_{i-1}, x_i),$$

for $i = 1, \ldots, N$.

- The residual measures the extent to which $u_{h,p}$ fails to satisfy the differential equation $-u'' + c(x)u = f(x)$ on $(0, 1)$.

Aim: Derive computable bounds on the error $u - u_h$ in terms of the residual $R(u_h)$.

1. $H^1(0,1)$–norm.
   $\Rightarrow$ Exploiting coercivity of $A(\cdot,\cdot)$, Galerkin orthogonality, and approximation theory.
2. $L^2(0,1)$–norm.
   $\Rightarrow$ Exploiting a duality argument, cf. Aubin–Nitsche duality argument. This involves introducing a dual problem, using Galerkin orthogonality, approximation theory, and strong stability estimates for the dual problem.

Error Estimation in the H^1–norm

- The general approach is to exploit the following three steps:
  1. Employ coercivity of the bilinear form $A(\cdot, \cdot)$.
  2. Use Galerkin Orthogonality.
  3. Employ interpolation error bounds.
- Introduce the mesh function h as follows:

$$h|_{(x_{i-1},x_i)} := h_i = x_i - x_{i-1},$$

for $i = 1, \ldots, N$.

Approximation result

Let $u \in H^1(0,1)$ such that $u|_{\kappa_i} \in H^{k_i}(\kappa_i)$, $i = 1, \ldots, N$. Then there exists $\mathcal{I}u \in V_{h,p}$ such that

$$u(x_i) = \mathcal{I}u(x_i), \qquad i = 0, \ldots, N,$$

and, for $i = 1, \ldots, N$,

$$\begin{aligned} \|u - \mathcal{I}u\|_{L^2(\kappa_i)} &\leq C_{\mathcal{I}} \frac{h_i^{\min(p_i+1,k_i)}}{p_i^{k_i}} \|u\|_{H^{k_i}(\kappa_i)}, \\ |u - \mathcal{I}u|_{H^1(\kappa_i)} &\leq C_{\mathcal{I}} \frac{h_i^{\min(p_i+1,k_i)-1}}{p_i^{k_i-1}} \|u\|_{H^{k_i}(\kappa_i)}, \end{aligned}$$

where $C_{\mathcal{I}}$ is a positive constant, independent of the mesh parameters.

Proof.

See Schwab 1998, Theorem 3.17, p.76. ■

Error Estimation in the H^1–norm

- Exploiting coercivity of the bilinear form $A(\cdot,\cdot)$ gives

$$\|u-u_{h,p}\|^2_{H^1(0,1)} \leq \frac{1}{c_0}A(u-u_{h,p},u-u_{h,p}).$$

- By Galerkin orthogonality, we have that

$$A(u-u_{h,p},v_{h,p}) = 0 \quad \forall v_{h,p} \in V_{h,p}.$$

- Selecting $v_{h,p} = \mathcal{I}(u-u_{h,p}) \in V_{h,p}$ gives

$$\|u-u_{h,p}\|^2_{H^1(0,1)} \quad \leq \quad \frac{1}{c_0}A(u-u_{h,p},u-u_{h,p}-\mathcal{I}(u-u_{h,p})).$$

- Writing $e = u - u_{h,p}$, we get

$$\begin{aligned}
\|u - u_{h,p}\|^2_{H^1(0,1)} &\leq \frac{1}{c_0} A(u - u_{h,p}, e - \mathcal{I}e) \\
&= \frac{1}{c_0} \left[A(u, e - \mathcal{I}e) - A(u_{h,p}, e - \mathcal{I}e)\right] \\
&= \frac{1}{c_0} \left[\ell(e - \mathcal{I}e) - A(u_{h,p}, e - \mathcal{I}e)\right] \\
&= \frac{1}{c_0} \left[(f, e - \mathcal{I}e) - A(u_{h,p}, e - \mathcal{I}e)\right].
\end{aligned}$$

- Now,

$$A(u_{h,p}, e - \mathcal{I}e) = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} u'_{h,p}(x)\,(e - \mathcal{I}e)'(x)\mathrm{d}x + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} c(x)\,u_{h,p}(x)\,(e - \mathcal{I}e)(x)\mathrm{d}x.$$

- Integrating by parts in each of the N integrals in the first sum on the right-hand side, we deduce that

$$A(u_{h,p}, e - \mathcal{I}e) = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[-u''_{h,p}(x) + c(x)u_{h,p}(x)\right](e - \mathcal{I}e)(x)\mathrm{d}x,$$

since $(e - \mathcal{I}e)(x_i) = 0,\ i = 0, \ldots, N$.

- Thereby, applying the Cauchy–Schwarz inequality, gives

$$
\begin{aligned}
\|u - u_{h,p}\|^2_{H^1(0,1)} &\leq \frac{1}{c_0}\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} R(u_{h,p})(x)(e - \mathcal{I}e)(x)\mathrm{d}x \\
&\leq \frac{1}{c_0}\sum_{i=1}^{N}\|(h_i/p_i)R(u_{h,p})\|_{L^2(x_{i-1},x_i)}\|(p_i/h_i)(e-\mathcal{I}e)\|_{L^2(x_{i-1},x_i)} \\
&\leq \frac{1}{c_0}\|(h/p)R(u_{h,p})\|_{L^2(0,1)}\|(p/h)(e-\mathcal{I}e)\|_{L^2(0,1)},
\end{aligned}
$$

since $h|_{(x_{i-1},x_i)} = h_i$ and $p|_{(x_{i-1},x_i)} = p_i$, $i = 1, \ldots, N$.

Error Estimation in the H^1–norm

- We now use the following interpolation error bound:

$$\|(p/h)(e - \mathcal{I}e)\|_{L^2(0,1)} \leq C_{\mathcal{I}} \|e\|_{H^1(0,1)},$$

where $C_{\mathcal{I}}$ is a positive constant, independent of the mesh size h and the polynomial degree p.

- Thereby,

$$\begin{aligned} \|u - u_{h,p}\|^2_{H^1(0,1)} &\leq \frac{C_{\mathcal{I}}}{c_0} \|(h/p)R(u_{h,p})\|_{L^2(0,1)} \|e\|_{H^1(0,1)} \\ &= \frac{C_{\mathcal{I}}}{c_0} \|(h/p)R(u_{h,p})\|_{L^2(0,1)} \|u - u_{h,p}\|_{H^1(0,1)}. \end{aligned}$$

- Dividing through both sides by $\|u - u_{h,p}\|_{H^1(0,1)}$ gives the following result.

Theorem

Given that $u \in H_0^1(0,1)$, *the following* a posteriori *error bound holds*

$$\|u - u_{h,p}\|_{H^1(0,1)} \leq \frac{C_{\mathcal{I}}}{c_0} \|(h/p)R(u_{h,p})\|_{L^2(0,1)}, \tag{5}$$

where the finite element residual $R(u_{h,p})$ *is defined by*

$$R(u_{h,p})(x) = f(x) - (-u''_{h,p}(x) + c(x)u_{h,p}(x)), \quad x \in (x_{i-1}, x_i),$$

for $i = 1, \ldots, N$.

- Note that once $u_{h,p}$ has been computed, the right–hand side of (5) is fully computable.

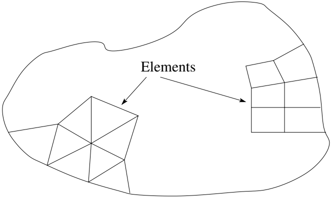
Model Problem

Second–Order Elliptic PDE

Let Ω to be a polygonal Lipschitz domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Given $f \in L^2(\Omega)$, consider

$$\begin{aligned} -\Delta u &= f(x), \quad x \in \Omega, \\ u &= 0, \qquad\quad x \in \partial\Omega. \end{aligned}$$

- $\mathcal{T}_h$ is a conforming mesh consisting of elements of granularity h.



- *hp*–finite element space:

$$V_{h,p} = \{v \in H_0^1(\Omega) : \, v|_\kappa \in \mathcal{S}_{p_\kappa}(\kappa) \quad \forall \kappa \in \mathcal{T}_h\} \, ,$$

where

$$\mathcal{S}_{p_\kappa}(\kappa) = \begin{cases} \mathcal{P}_{p_\kappa}(\kappa) & \text{if } \kappa \text{ is a simplex,} \\ \mathcal{Q}_{p_\kappa}(\kappa) & \text{if } \kappa \text{ is a hypercube.} \end{cases}$$

- For $\kappa \in \mathcal{T}_h$, we write

$$h|_\kappa = h_\kappa = \text{meas}(\kappa), \qquad p|_\kappa = p_\kappa$$

to denote the local mesh size and polynomial degree orders.

(A1) With this notation, we assume that h and p are of bounded local variation, i.e., there exists $\rho_1 \geq 1$ and $\rho_2 \geq 1$ such that

$$\rho_1^{-1} \leq h_\kappa / h_{\kappa'} \leq \rho_1, \qquad \rho_2^{-1} \leq p_\kappa / p_{\kappa'} \leq \rho_2,$$

whenever κ and κ' share a common face.

- The finite element approximation is then given by:

Find $u_{h,p} \in V_{h,p}$ such that

$$A(u_{h,p}, v_{h,p}) = \ell(v_{h,p}) \quad \forall v_{h,p} \in V_{h,p},$$

where

$$\begin{aligned} A(u,v) &= \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \\ \ell(v) &= \int_\Omega f\, v \mathrm{d}x. \end{aligned}$$

Lemma

Assume (A1) holds and that $\mathcal{T}_h$ is a conforming mesh. Given a function $v \in H_0^1(\Omega)$, there exists a (quasi-)interpolant $\mathcal{I}v \in V_{h,p}$ such that

$$\sum_{\kappa \in \mathcal{T}_h} \left(\frac{p_\kappa^2}{h_\kappa^2} \|(\mathcal{I}v - v)\|_{L^2(\kappa)}^2 + \frac{p_\kappa}{h_\kappa} \|(\mathcal{I}v - v)\|_{L^2(\partial\kappa)}^2 \right)^{1/2} \leq C_{\mathcal{I}} |v|_{H^1(\Omega)},$$

where $C_{\mathcal{I}}$ is a positive constant, independent of the mesh size h and polynomial degree p.

Proof.

See Melenk & Wohlmuth 2001, Theorem 2.2; Melenk 2005, Theorem 2.4.

■

- Proceed as in the one–dimensional setting.
- Exploiting coercivity of the bilinear form $A(\cdot,\cdot)$ gives

$$\|u - u_{h,p}\|^2_{H^1(\Omega)} \leq \frac{1}{c_0} A(u - u_{h,p}, u - u_{h,p}).$$

- By Galerkin orthogonality, we have that

$$A(u - u_{h,p}, v_{h,p}) = 0 \quad \forall v_{h,p} \in V_{h,p}.$$

- Selecting $v_{h,p} = \mathcal{I}e \in V_{h,p}$, where $e = u - u_{h,p}$, gives

$$\begin{aligned} \|u - u_{h,p}\|^2_{H^1(\Omega)} &\leq \frac{1}{c_0} A(u - u_{h,p}, e - \mathcal{I}e) \\ &= \frac{1}{c_0} \left[\ell(e - \mathcal{I}e) - A(u_{h,p}, e - \mathcal{I}e)\right]. \end{aligned}$$

- Integrating by parts gives

$$
\begin{aligned}
\|u - u_{h,p}\|^2_{H^1(\Omega)} &\leq \frac{1}{c_0}\left[(f, e - \mathcal{I}e) - A(u_{h,p}, e - \mathcal{I}e)\right] \\
&= \frac{1}{c_0}\left[\sum_{\kappa\in\mathcal{T}_h}\int_\kappa f(e - \mathcal{I}e)\mathrm{d}x - \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \nabla u_{h,p}\cdot\nabla(e - \mathcal{I}e)\mathrm{d}x\right] \\
&= \frac{1}{c_0}\left[\sum_{\kappa\in\mathcal{T}_h}\int_\kappa f(e - \mathcal{I}e)\mathrm{d}x + \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \Delta u_{h,p}(e - \mathcal{I}e)\mathrm{d}x\right. \\
&\qquad \left. - \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}\frac{\partial u_{h,p}}{\partial \mathbf{n}_\kappa}(e - \mathcal{I}e)\mathrm{d}s\right],
\end{aligned}
$$

where $\mathbf{n}_\kappa$ denotes the unit outward normal vector to element κ.

- Define the *finite element residual:*

$$R(u_{h,p})|_\kappa = f + \Delta u_{h,p}$$

for each element κ in the mesh $\mathcal{T}_h$.

- Thereby, we have that

$$\begin{aligned} \|u - u_{h,p}\|^2_{H^1(\Omega)} &\leq \frac{1}{c_0}\left[\sum_{\kappa\in\mathcal{T}_h}\int_\kappa R(u_{h,p})(e - \mathcal{I}e)\mathrm{d}x \right. \\ &\left. - \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}\frac{\partial u_{h,p}}{\partial \mathbf{n}_\kappa}(e - \mathcal{I}e)\mathrm{d}s\right] \\ &\equiv \frac{1}{c_0}\left[\mathrm{I} + \mathrm{II}\right]. \end{aligned} \tag{6}$$

- Employing the Cauchy–Schwarz inequality and the quasi-interpolation error estimates, we deduce that

$$
\begin{aligned}
|\mathrm{I}| &= \left| \sum_{\kappa \in \mathcal{T}_h} \int_\kappa R(u_{h,p})(e - \mathcal{I}e)\mathrm{d}x \right| \\
&\leq \|(h/p)R(u_{h,p})\|_{L^2(\Omega)} \|(p/h)(e - \mathcal{I}e)\|_{L^2(\Omega)} \\
&\leq C_{\mathcal{I}} \|(h/p)R(u_{h,p})\|_{L^2(\Omega)} |e|_{H^1(\Omega)} \\
&= C_{\mathcal{I}} \|(h/p)R(u_{h,p})\|_{L^2(\Omega)} |u - u_{h,p}|_{H^1(\Omega)} \\
&\leq C_{\mathcal{I}} \|(h/p)R(u_{h,p})\|_{L^2(\Omega)} \|u - u_{h,p}\|_{H^1(\Omega)}. \qquad (7)
\end{aligned}
$$

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- Recall that

$$\mathrm{II} = -\sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \frac{\partial u_{h,p}}{\partial \mathbf{n}_\kappa}(e - \mathcal{I}e)\,\mathrm{d}s.$$

- Notation:
  - Given a face $\tau = \partial\kappa_1 \cap \partial\kappa_2$ shared by the two neighbouring elements κ_1 and κ_2, we introduce the *jump operator:*

$$\left[\!\left[\frac{\partial v}{\partial \mathbf{n}_{\kappa_i}}\right]\!\right]\bigg|_\tau = \nabla v|_{\kappa_1} \cdot \mathbf{n}_{\kappa_1} + \nabla v|_{\kappa_2} \cdot \mathbf{n}_{\kappa_2},$$

    $i = 1, 2.$
  - Given that $\mathbf{n}_{\kappa_1} = -\mathbf{n}_{\kappa_2}$, $[\![\partial v/\partial \mathbf{n}_{\kappa_i}]\!]|_\tau$ is the jump across τ in the normal component of ∇v, $i = 1, 2$.

- Thereby,

$$
\begin{aligned}
\mathrm{II} &= -\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}\frac{\partial u_{h,p}}{\partial \mathbf{n}_\kappa}(e-\mathcal{I}e)\mathrm{d}s \\
&= -\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\partial\Omega}\frac{1}{2}\left[\frac{\partial u_{h,p}}{\partial \mathbf{n}_\kappa}\right](e-\mathcal{I}e)\mathrm{d}s \\
&\leq \left(\sum_{\kappa\in\mathcal{T}_h}\left\|\frac{h_\kappa^{1/2}}{2p_\kappa^{1/2}}\left[\frac{\partial u_{h,p}}{\partial \mathbf{n}_\kappa}\right]\right\|^2_{L^2(\partial\kappa\setminus\partial\Omega)}\right)^{1/2} \\
&\quad\times\left(\sum_{\kappa\in\mathcal{T}_h}\|(p_\kappa/h_k)^{1/2}(e-\mathcal{I}e)\|^2_{L^2(\partial\kappa\setminus\partial\Omega)}\right)^{1/2} \\
&\leq C_{\mathcal{I}}\left(\sum_{\kappa\in\mathcal{T}_h}\left\|\frac{h_\kappa^{1/2}}{2p_\kappa^{1/2}}\left[\frac{\partial u_{h,p}}{\partial \mathbf{n}_\kappa}\right]\right\|^2_{L^2(\partial\kappa\setminus\partial\Omega)}\right)^{1/2}\|u-u_{h,p}\|_{H^1(\Omega)}. \qquad (8)
\end{aligned}
$$

- Substituting (7) and (8) into (6) and dividing both terms by $\|u - u_{h,p}\|_{H^1(\Omega)}$, we deduce that

$$\|u - u_{h,p}\|_{H^1(\Omega)} \leq C_{\mathcal{I}}/c_0 \left[\|(h/p)R(u_{h,p})\|_{L^2(\Omega)} + \left(\sum_{\kappa \in \mathcal{T}_h} \left\| \frac{h_\kappa^{1/2}}{2p_\kappa^{1/2}} \left[\frac{\partial u_{h,p}}{\partial \mathbf{n}_\kappa} \right] \right\|^2_{L^2(\partial\kappa \setminus \partial\Omega)} \right)^{1/2} \right].$$

- Write $\Pi_{p_\kappa - 1}$ to denote the elementwise $L^2(\kappa)$ projection operator onto the space of polynomials of degree $p_\kappa - 1$.

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Theorem

Assuming that $u \in H_0^1(\Omega)$, *the following* a posteriori *error bound holds:*

$$\|u - u_{h,p}\|_{H^1(\Omega)} \leq C \left(\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 + \Theta_\kappa^2 \right)^{1/2},$$

where $\eta_\kappa^2 = \eta_{R_\kappa}^2 + \eta_{E_\kappa}^2$,

$$\begin{aligned}
\eta_{R_\kappa} &= \frac{h_\kappa}{p_\kappa} \|R_h(u_{h,p})\|_{L^2(\kappa)} \equiv \frac{h_\kappa}{p_\kappa} \|\Pi_{p_\kappa - 1} f + \Delta u_{h,p}\|_{L^2(\kappa)}, \\
\eta_{E_\kappa} &= \frac{h_\kappa^{1/2}}{2 p_\kappa^{1/2}} \left\| \left[\frac{\partial u_{h,p}}{\partial \mathbf{n}_\kappa} \right] \right\|_{L^2(\partial\kappa \setminus \partial\Omega)}, \\
\Theta_\kappa &= p_\kappa^{-1} h_\kappa \|f - \Pi_{p_\kappa - 1} f\|_{L^2(\kappa)},
\end{aligned}$$

and $C = 2C_{\mathcal{I}}/c_0$.

Theorem

For each $\kappa \in \mathcal{T}_h$*, define the set* $\omega_\kappa = \cup \{\kappa' : \kappa'$ *and* κ *share a common face*$\}$. *Then, for any* $\epsilon > 0$*, the following local lower bound holds:*

$$\eta_\kappa^2 \leq C(\epsilon) p_\kappa^{1+2\epsilon} \left(p_\kappa \|u - u_{h,p}\|_{H^1(\omega_\kappa)}^2 + \frac{h_\kappa^2}{p_\kappa^{2(1-\epsilon)}} \|f - \Pi_{p_\kappa - 1} f\|_{L^2(\omega_\kappa)}^2 \right),$$

where $C(\epsilon)$ *is a positive constant, independent of the mesh parameters* h *and* p*.*

Proof.

The proof is based on employing suitable inverse inequalities; see Melenk & Wohlmuth 2001, Theorem 3.6. ■

Remark

The lower (efficiency) bound is suboptimal with respect to p*. This can be improved by using weighted error indicators, at the expense of the p–optimality of the reliability (upper) bound.*

Model Problem

Poisson's Equation

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and $f \in L^2(\Omega)$, find u such that

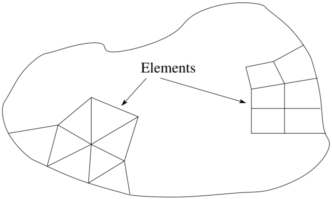
$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Weak Formulation

Find $u \in H_0^1(\Omega)$ such that

$$A(u, v) \equiv (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

- $\mathcal{T}_h$ is a non-degenerate mesh consisting of elements of granularity h.



- *hp*–finite element space:

$$V_{h,p} = \{v \in L^2(\Omega) : \ v|_\kappa \in \mathcal{S}_{p_\kappa}(\kappa) \quad \forall \kappa \in \mathcal{T}_h\} \ ,$$

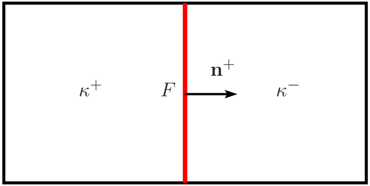
where

$$\mathcal{S}_{p_\kappa}(\kappa) = \begin{cases} \mathcal{P}_{p_\kappa}(\kappa) & \text{if } \kappa \text{ is a simplex,} \\ \mathcal{Q}_{p_\kappa}(\kappa) & \text{if } \kappa \text{ is a hypercube.} \end{cases}$$

Notation

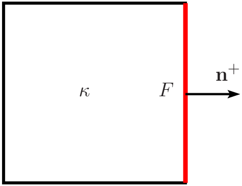
- Let $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}$ denote the set of all faces in the mesh $\mathcal{T}_h$.
- Notation:

$f \subset \mathcal{F}_h^{\mathcal{I}}$



$$\llbracket v \rrbracket = v^+\mathbf{n}^+ + v^-\mathbf{n}^-$$
$$\llbracket \mathbf{q} \rrbracket = \mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{q}^- \cdot \mathbf{n}^-$$
$$\{\!\{ v \}\!\} = (v^+ + v^-)/2$$

$f \subset \mathcal{F}_h^{\mathcal{B}}$



$$\llbracket v \rrbracket = v\mathbf{n}$$
$$\llbracket \mathbf{q} \rrbracket = \mathbf{q} \cdot \mathbf{n}$$
$$\{\!\{ v \}\!\} = v$$

- For $\kappa \in \mathcal{T}_h$, we write

$$h|_\kappa = h_\kappa = \text{meas}(\kappa), \qquad p|_\kappa = p_\kappa$$

to denote the local mesh size and polynomial degree orders.

(A1) With this notation, we assume that h and p are of bounded local variation, i.e., there exists $\rho_1 \geq 1$ and $\rho_2 \geq 1$ such that

$$\rho_1^{-1} \leq h_\kappa / h_{\kappa'} \leq \rho_1, \qquad \rho_2^{-1} \leq p_\kappa / p_{\kappa'} \leq \rho_2,$$

whenever κ and κ' share a common face.

(A2) For all elements $\kappa \in \mathcal{T}_h$, we define

$$C_\kappa = \text{card}\left\{F \in \mathcal{F}_h : F \subset \partial\kappa\right\}.$$

In the following we assume that there exists a positive constant C_F such that

$$\max_{\kappa \in \mathcal{T}_h} C_\kappa \leq C_F,$$

uniformly with respect to the mesh size.

- Interior Penalty Method: Find $u_{h,p} \in V_{h,p}$ such that

$$A_h(u_{h,p}, v) = \ell_h(v) \qquad \forall v \in V_{h,p}.$$

- Forms:

$$A_h(u, v) = \int_\Omega \nabla_h u \cdot \nabla_h v \mathrm{dx} - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h u\}\!\} \cdot [\![v]\!] \, ds$$
$$- \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v\}\!\} \cdot [\![u]\!] \mathrm{d}s + \sum_{F \in \mathcal{F}_h} \int_F \sigma [\![u]\!] \cdot [\![v]\!] \mathrm{d}s,$$
$$\ell_h(v) = \int_\Omega f v \, d\mathbf{x}.$$

- Interior penalty (stability) parameter:

$$\sigma|_F = C_\sigma \frac{\mathtt{p}^2}{\mathtt{h}} \quad \text{for } F \subset \mathcal{F}_h, \qquad C_\sigma > 0, \text{ sufficiently large } (=10),$$

where $\mathtt{p}|_F = \max(p_{\kappa_1}, p_{\kappa_2})$ and $\mathtt{h}|_F = h_F$ is a *representative* length scale associated to the face $F \subset \partial\kappa$ (see below).

- DG-norm:

$$\|u\|_{h,p}^2 = \|\nabla_h u\|_{L^2(\Omega)}^2 + \|\sqrt{\sigma}[\![u]\!]\|_{L^2(\mathcal{F}_h)}^2,$$

where

$$\|v\|_{L^2(\mathcal{F}_h)}^2 = \sum_{F\in\mathcal{F}_h} \|v\|_{L^2(F)}^2.$$

Lemma (Coercivity)

Assuming C_σ is chosen sufficiently *large, there exists $C > 0$, such that*

$$A_h(v,v) \geq C\|v\|_{h,p}^2 \quad \forall v \in V_{h,p}.$$

Lemma (Inverse inequality)

Given a face $F \in \mathcal{F}_h$ of an element $\kappa \in \mathcal{T}_h$, there exists a positive constant $C_{\rm inv}$, independent of the local mesh size and local polynomial order, such that

$$\|\nabla v\|^2_{L^2(F)} \leq C_{\rm inv} \frac{p_\kappa^2}{h_F} \|\nabla v\|^2_{L^2(\kappa)}$$

for all $v \in V_{h,p}$, where h_F is a representative *length scale associated to the face $F \subset \partial\kappa$.*

Proof.

See Schwab 1998, Theorem 4.76.
See also: Georgoulis, Hall, & H. 2007 (anisotropic h); Georgoulis 2003 (anisotropic hp) ∎

- For isotropic meshes, the above inverse inequality holds with $h_F = \text{meas}(F)$. In this setting, σ is typically selected as follows:

$$\sigma|_F = C_\sigma \frac{\mathtt{p}^2}{\mathtt{h}} \quad \text{for } F \subset \mathcal{F}_h,$$

where $\mathtt{h}|_F = \min(h_{\kappa_1}, h_{\kappa_2})$.

- On anisotropic meshes, (see later), h_F must be chosen as the dimension of the element orthogonal to the face F. A similar choice for $\mathtt{p}$ must be employed in the case when anisotropic polynomial degrees are employed.

Coercivity.

For $v \in V_{h,p}$, we note that

$$
\begin{aligned}
A_h(v,v) &= \sum_{\kappa\in\mathcal{T}_h} \|\nabla v\|_{L^2(\kappa)}^2 - 2\sum_{F\in\mathcal{F}_h}\int_F \{\!\{\nabla v\}\!\}\cdot[\![v]\!]\mathrm{d}s + \sum_{F\in\mathcal{F}_h}\left\|\sigma^{1/2}[\![v]\!]\right\|_{L^2(F)}^2, \\
&\equiv \mathrm{I}+\mathrm{II}+\mathrm{III}. \qquad (9)
\end{aligned}
$$

In order to bound term II, we first note that for $F \in \mathcal{F}_h^{\mathcal{I}}$, we have that

$$
\begin{aligned}
\int_F \{\!\{\nabla v\}\!\}\cdot[\![v]\!]\mathrm{d}s &\leq \left\|\sigma^{-1/2}\{\!\{\nabla v\}\!\}\right\|_{L^2(F)}\left\|\sigma^{1/2}[\![v]\!]\right\|_{L^2(F)} \\
&\leq \frac{1}{2}\left(\left\|\sigma^{-1/2}\nabla v^+\right\|_{L^2(F)} + \left\|\sigma^{-1/2}\nabla v^-\right\|_{L^2(F)}\right)\left\|\sigma^{1/2}[\![v]\!]\right\|_{L^2(F)} \\
&\leq \epsilon\left(\left\|\sigma^{-1/2}\nabla v^+\right\|_{L^2(F)}^2 + \left\|\sigma^{-1/2}\nabla v^-\right\|_{L^2(F)}^2\right) \\
&\quad +\frac{1}{8\epsilon}\left\|\sigma^{1/2}[\![v]\!]\right\|_{L^2(F)}^2, \qquad \epsilon>0.
\end{aligned}
$$

Coercivity.

Employing the above inverse inequality, together with the local bounded variation of p, we deduce that

$$\begin{aligned}\int_F \{\!\{\nabla v\}\!\} \cdot [\![v]\!] \mathrm{d}s &\leq C_{\mathrm{inv}} \epsilon \left(\frac{p_{\kappa^+}^2}{h_F} \left\| \sigma^{-1/2} \nabla v \right\|_{L^2(\kappa^+)}^2 + \frac{p_{\kappa^-}^2}{h_F} \left\| \sigma^{-1/2} \nabla v \right\|_{L^2(\kappa^-)}^2 \right) \\ &\quad + \frac{1}{8\epsilon} \left\| \sigma^{1/2} [\![v]\!] \right\|_{L^2(F)}^2 \\ &\leq \frac{C_{\mathrm{inv}} \rho_2^2}{C_\sigma} \epsilon \left(\|\nabla v\|_{L^2(\kappa^+)}^2 + \|\nabla v\|_{L^2(\kappa^-)}^2 \right) + \frac{1}{8\epsilon} \left\| \sigma^{1/2} [\![v]\!] \right\|_{L^2(F)}^2, \qquad (10)\end{aligned}$$

where we have used the definition of the interior penalty parameter σ.
In an analogous fashion, for $F \in \mathcal{F}_h^{\mathcal{B}}$, we have that

$$\int_F \{\!\{\nabla v\}\!\} \cdot [\![v]\!] \mathrm{d}s \leq \frac{C_{\mathrm{inv}}}{C_\sigma} \epsilon \|\nabla v\|_{L^2(\kappa^+)}^2 + \frac{1}{4\epsilon} \left\| \sigma^{1/2} [\![v]\!] \right\|_{L^2(F)}^2 . \qquad (11)$$

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Coercivity.

Thereby, exploiting Assumption (A2) above, inserting (10) and (11) into (9) gives

$$A_h(v,v) \geq \left(1 - \frac{2C_{\mathrm{inv}}C_F\rho_2^2}{C_\sigma}\epsilon\right)\sum_{\kappa\in\mathcal{T}_h}\|\nabla v\|_{L^2(\kappa)}^2 + \left(1-\frac{1}{2\epsilon}\right)\sum_{F\in\mathcal{F}_h}\left\|\sigma^{1/2}[\![v]\!]\right\|_{L^2(F)}^2.$$

Thereby, the bilinear form $A_h(\cdot,\cdot)$ is coercive over $V_{h,p}\times V_{h,p}$, assuming that $\epsilon > 1/2$ and $C_\sigma > 2C_{\mathrm{inv}}C_F\rho_2^2\epsilon$. ■

- Saturation assumptions
  Becker, Hansbo & Stenberg 2003.
- Helmholtz decompositions
  Carstensen & Funken 2001, Becker, Hansbo & Larson 2003, H., Perugia & Schötzau 2005, Bustinza, Gatica & Cockburn 2005.
- Conforming approximation results for DG functions
  Karakashian & Pascal 2003, H., Schötzau & Wihler 2005$\rightarrow$.

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Theorem (Poisson's equation)

Assuming $u \in H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^d$, with $d = 2$, and $\mathcal{T}_h$ is conforming, the following bound holds:

$$\|u - u_{h,p}\|_{h,p} \le C \left(\sum_{\kappa \in \mathcal{T}_h} (\eta_\kappa^2 + \Theta_\kappa^2) \right)^{1/2},$$

where

$$\|v\|_{h,p}^2 = \|\nabla_h v\|_{L^2(\Omega)}^2 + \|\sqrt{\sigma}\, [\![v]\!]\|_{L^2(\mathcal{F}_h)}^2,$$

and

$$\begin{aligned}
\eta_\kappa^2 &= \eta_{R_\kappa}^2 + \eta_{E_\kappa}^2 + \eta_{J_\kappa}^2, \\
\eta_{R_\kappa}^2 &= p_\kappa^{-2} h_\kappa^2 \|\Pi_{p_\kappa - 1} f + \Delta u_{h,p}\|_{L^2(\kappa)}^2, \\
\eta_{E_\kappa}^2 &= \|p_\kappa^{-1/2} h_\kappa^{1/2} [\![\nabla_h u_{h,p}]\!]\|_{L^2(\partial\kappa \setminus \partial\Omega)}^2, \\
\eta_{J_\kappa}^2 &= \|\sqrt{\sigma}\, [\![u_{h,p}]\!]\|_{L^2(\partial\kappa)}^2, \\
\Theta_\kappa^2 &= p_\kappa^{-2} h_\kappa^2 \|f - \Pi_{p_\kappa - 1} f\|_{L^2(\kappa)}^2.
\end{aligned}$$

Theorem

For each $\kappa \in \mathcal{T}_h$, define the set $\omega_\kappa = \cup \{\kappa' : \kappa' \text{ and } \kappa \text{ share a common face}\}$. Then, for any $\epsilon > 0$, the following local lower bound holds:

$$
\begin{aligned}
\eta_\kappa^2 \quad \leq \quad C(\epsilon) \Bigg[p_\kappa^{1+2\epsilon} \left(p_\kappa \|\nabla_h (u - u_{h,p})\|^2_{L^2(\omega_\kappa)} + \frac{h_\kappa^2}{p_\kappa^{2(1-\epsilon)}} \|f - \Pi_{p_\kappa - 1} f\|^2_{L^2(\omega_\kappa)} \right) \\
+ \|\sqrt{\sigma}\, [\![u_{h,p}]\!] \|^2_{L^2(\partial\kappa)} \Bigg],
\end{aligned}
$$

where $C(\epsilon)$ is a positive constant, independent of the mesh parameters h and p.

Proof.

The proof is based on employing suitable inverse inequalities; see Melenk & Wohlmuth 2001, Theorem 3.6. For details, see H., Schötzau & Wihler 2007. ∎

Remark

The lower (efficiency) bound is suboptimal with respect to p.

- Rewrite DG method in a non-consistent manner using lifting operators;
- Decompose DG finite element space

$$\text{DG space} = \text{Conforming Subspace} \oplus \text{Jumps};$$

- Approximation of DG functions by conforming ones;
- Abstract error bound;
- Estimation of the (weak) residual.

- Introduce the space

$$\mathcal{V}(h,p) = V_{h,p} + H_0^1(\Omega).$$

- For $v \in \mathcal{V}(h,p)$, we define the lifting operator $\mathbf{L} \in \mathbf{\Sigma}_{h,p}$ as follows:

$$\int_\Omega \mathbf{L}(v) \cdot \mathbf{q} \, d\mathbf{x} = \sum_{F \in \mathcal{F}_h} \int_F [\![v]\!] \cdot \{\!\{ \mathbf{q} \}\!\} \, ds \qquad \forall \mathbf{q} \in \mathbf{\Sigma}_{h,p},$$

where

$$\mathbf{\Sigma}_{h,p} = \{ \, \mathbf{q} \in L^2(\Omega)^d \; : \; \mathbf{q}|_\kappa \in \mathcal{S}_{p_\kappa}(\kappa)^d, \; \kappa \in \mathcal{T}_h \, \}.$$

Arnold, Brezzi, Cockburn, & Marini 2001, Perugia & Schötzau 2002

- For $v \in H_0^1(\Omega)$, we note that $\mathbf{L}(v) \equiv 0$.

Lemma (Stability)

There exists $C_L > 0$ *such that*

$$\|\mathbf{L}(v)\|_{L^2(\Omega)} \leq C_L \|\mathrm{p}\mathrm{h}^{-1/2}[\![v]\!]\|_{L^2(\mathcal{F}_h)}, \quad v \in \mathcal{V}(h,p).$$

where $C_L = \sqrt{2 C_{\mathrm{inv}} C_F} \rho_2$.

Proof.

First we define $\Pi_{\boldsymbol{\Sigma}_{h,p}}$ to denote the L^2–projector onto $\boldsymbol{\Sigma}_{h,p}$, i.e., given $\mathbf{v} \in [L^2(\Omega)]^d$, $\Pi_{\boldsymbol{\Sigma}_{h,p}}\mathbf{v} \in \boldsymbol{\Sigma}_{h,p}$ is defined by

$$(\Pi_{\boldsymbol{\Sigma}_{h,p}}\mathbf{v}, \mathbf{w}) = (\mathbf{v}, \mathbf{w}) \qquad \forall \mathbf{w} \in \boldsymbol{\Sigma}_{h,p}.$$

Thereby, we have that

$$\begin{aligned} \|\mathbf{L}(v)\|_{L^2(\Omega)} &= \sup_{\mathbf{w}\in[L^2(\Omega)]^d} \frac{\int_\Omega \mathbf{L}(v)\cdot\mathbf{w}\,\mathrm{d}x}{\|\mathbf{w}\|_{L^2(\Omega)}} \\ &= \sup_{\mathbf{w}\in[L^2(\Omega)]^d} \frac{\int_\Omega \mathbf{L}(v)\cdot\Pi_{\boldsymbol{\Sigma}_{h,p}}\mathbf{w}\,\mathrm{d}x}{\|\mathbf{w}\|_{L^2(\Omega)}} \\ &= \sup_{\mathbf{w}\in[L^2(\Omega)]^d} \frac{\sum_{F\in\mathcal{F}_h}\int_F [\![v]\!]\cdot\{\!\{\Pi_{\boldsymbol{\Sigma}_{h,p}}\mathbf{w}\}\!\}\,ds}{\|\mathbf{w}\|_{L^2(\Omega)}}. \end{aligned} \tag{12}$$

■

Proof.

For $F \in \mathcal{F}_h^{\mathcal{I}}$, we have that

$$\begin{aligned}
\int_F [\![v]\!] \cdot \{\!\{\Pi_{\boldsymbol{\Sigma}_{h,p}} \mathbf{w}\}\!\}\, ds &\leq \left\| \mathrm{p}\mathrm{h}^{-1/2} [\![v]\!] \right\|_{L^2(F)} \left\| \mathrm{p}^{-1}\mathrm{h}^{1/2} \{\!\{\Pi_{\boldsymbol{\Sigma}_{h,p}} \mathbf{w}\}\!\} \right\|_{L^2(F)} \\
&\leq \frac{1}{2} \left\| \mathrm{p}\mathrm{h}^{-1/2} [\![v]\!] \right\|_{L^2(F)} \\
&\quad \times \left(\left\| \mathrm{p}^{-1}\mathrm{h}^{1/2} \Pi_{\boldsymbol{\Sigma}_{h,p}} \mathbf{w}^+ \right\|_{L^2(F)} + \left\| \mathrm{p}^{-1}\mathrm{h}^{1/2} \Pi_{\boldsymbol{\Sigma}_{h,p}} \mathbf{w}^- \right\|_{L^2(F)} \right) \\
&\leq \frac{\sqrt{C_{\mathrm{inv}}}}{2} \rho_2 \left\| \mathrm{p}\mathrm{h}^{-1/2} [\![v]\!] \right\|_{L^2(F)} \\
&\quad \times \left(\left\| \Pi_{\boldsymbol{\Sigma}_{h,p}} \mathbf{w}^+ \right\|_{L^2(\kappa^+)} + \left\| \Pi_{\boldsymbol{\Sigma}_{h,p}} \mathbf{w}^- \right\|_{L^2(\kappa^-)} \right).
\end{aligned}$$

■

Proof.

Similarly, for $F \in \mathcal{F}_h^{\mathcal{B}}$, we have that

$$\int_F [\![v]\!] \cdot \{\!\{ \Pi_{\boldsymbol{\Sigma}_{h,p}} \mathbf{w} \}\!\}\, ds \quad \leq \quad \sqrt{C_{\text{inv}}} \left\| \mathrm{ph}^{-1/2} [\![v]\!] \right\|_{L^2(F)} \left\| \Pi_{\boldsymbol{\Sigma}_{h,p}} \mathbf{w}^+ \right\|_{L^2(\kappa^+)} .$$

Exploiting Assumption (A2), we deduce that

$$\begin{aligned}\sum_{F \in \mathcal{F}_h} \int_F [\![v]\!] \cdot \{\!\{ \Pi_{\boldsymbol{\Sigma}_{h,p}} \mathbf{w} \}\!\}\, ds \quad &\leq \sqrt{2 C_{\text{inv}}\, C_F} \rho_2 \| \mathrm{ph}^{-1/2} [\![v]\!] \|_{L^2(\mathcal{F}_h)} \| \Pi_{\boldsymbol{\Sigma}_{h,p}} \mathbf{w} \|_{L^2(\Omega)} \\ &\leq \sqrt{2 C_{\text{inv}}\, C_F} \rho_2 \| \mathrm{ph}^{-1/2} [\![v]\!] \|_{L^2(\mathcal{F}_h)} \| \mathbf{w} \|_{L^2(\Omega)} . \qquad (13)\end{aligned}$$

Substituting (13) into (12) gives the desired result. ∎

Introduce the extended bilinear form

$$\begin{aligned}\widetilde{A}_h(v,w) &= \int_\Omega \nabla_h v \cdot \nabla_h w \,\mathrm{d}\mathbf{x} - \int_\Omega \big(\mathbf{L}(v)\cdot\nabla_h w + \mathbf{L}(w)\cdot\nabla_h v\big)\mathrm{d}\mathbf{x} \\ &\quad + \int_{\mathcal{F}_h} \sigma [\![v]\!] \cdot [\![w]\!] \,\mathrm{d}s.\end{aligned}$$

Thereby, we note that

$$\widetilde{A}_h = A_h \quad \text{on} \quad V_{h,p}\times V_{h,p}, \qquad \widetilde{A}_h = A \quad \text{on} \quad H_0^1(\Omega)\times H_0^1(\Omega).$$

Perturbed Formulation: Find $u_{h,p}\in V_{h,p}$ such that

$$\widetilde{A}_h(u_{h,p}, v_{h,p}) = \ell_h(v_{h,p}) \quad \forall v_{h,p}\in V_{h,p}.$$

In addition, we note that the analytical solution $u\in H_0^1(\Omega)$ satisfies:

$$\widetilde{A}_h(u,v) = \ell_h(v) \quad \forall v\in H_0^1(\Omega).$$

Lemma (Continuity)

For any $u, v \in \mathcal{V}(h,p)$, we have

$$|\widetilde{A}_h(u,v)| \le C_{\text{cont}} \|u\|_{h,p} \|v\|_{h,p},$$

where $C_{\text{cont}} = \max(2, 1 + C_L^2 C_\sigma^{-1})$.

Proof.

For $u, v \in \mathcal{V}(h,p)$, we have

$$\begin{aligned}
\widetilde{A}_h(u,v) &= \int_\Omega \nabla_h u \cdot \nabla_h v \,\mathrm{d}\mathbf{x} - \int_\Omega \big(\mathbf{L}(u) \cdot \nabla_h v + \mathbf{L}(v) \cdot \nabla_h u\big)\mathrm{d}\mathbf{x} \\
&\quad + \sum_{F \in \mathcal{F}_h} \int_F \sigma [\![u]\!] \cdot [\![v]\!] \mathrm{d}s \\
&\le \|\nabla_h u\|_{L^2(\Omega)} \|\nabla_h v\|_{L^2(\Omega)} + \|\mathbf{L}(u)\|_{L^2(\Omega)} \|\nabla_h v\|_{L^2(\Omega)} \\
&\quad + \|\mathbf{L}(v)\|_{L^2(\Omega)} \|\nabla_h u\|_{L^2(\Omega)} + \|\sqrt{\sigma} u\|_{L^2(\mathcal{F}_h)} \|\sqrt{\sigma} v\|_{L^2(\mathcal{F}_h)}.
\end{aligned}$$

Proof.

Exploiting the stability of the lifting operator gives

$$\begin{aligned}
\widetilde{A}_h(u,v) &\leq \|\nabla_h u\|_{L^2(\Omega)}\|\nabla_h v\|_{L^2(\Omega)} + C_L\|\mathrm{p}\mathrm{h}^{-1/2}[\![u]\!]\|_{L^2(\mathcal{F}_h)}\|\nabla_h v\|_{L^2(\Omega)} \\
&\quad + C_L\|\mathrm{p}\mathrm{h}^{-1/2}[\![v]\!]\|_{L^2(\mathcal{F}_h)}\|\nabla_h u\|_{L^2(\Omega)} + \|\sqrt{\sigma}u\|_{L^2(\mathcal{F}_h)}\|\sqrt{\sigma}v\|_{L^2(\mathcal{F}_h)} \\
&\leq \left[2\|\nabla_h u\|^2_{L^2(\Omega)} + (1 + C_L^2 C_\sigma^{-1})\|\sqrt{\sigma}u\|^2_{L^2(\mathcal{F}_h)}\right]^{1/2} \\
&\quad \times \left[2\|\nabla_h v\|^2_{L^2(\Omega)} + (1 + C_L^2 C_\sigma^{-1})\|\sqrt{\sigma}v\|^2_{L^2(\mathcal{F}_h)}\right]^{1/2} \\
&\leq C_{\mathrm{cont}}\|u\|_{h,p}\|v\|_{h,p},
\end{aligned}$$

where $C_{\mathrm{cont}} = \max(2, 1 + C_L^2 C_\sigma^{-1})$, as required. ■

Lemma (Coercivity)

For any $u \in H_0^1(\Omega)$, the following identity holds

$$\widetilde{A}_h(u,u) = \|u\|_{h,p}^2.$$

Proof.

- For $u \in H_0^1(\Omega)$, we have that $[\![u]\!] = 0$ over all faces.
- Thereby, $\mathbf{L}(u) = 0$ and

$$A_h(u,u) = \int_\Omega |\nabla u|^2 \mathrm{d}x = \|u\|_{h,p}^2,$$

as required.

∎

Approximation by conforming functions

Assumption

Recall: we assume that the finite element mesh $\mathcal{T}_h$ is conforming, i.e., no hanging nodes exist.

Setting

$$V^c_{h,p} = V_{h,p} \cap H^1_0(\Omega),$$

we decompose $V_{h,p}$ as follows:

$$\boxed{V_{h,p} = V^c_{h,p} \oplus V^{\perp}_{h,p},}$$

where $V^{\perp}_{h,p}$ is the orthogonal complement of $V^c_{h,p}$ w.r.t. $\|\cdot\|_{h,p}$.

Approximation by conforming functions

Proposition

For every $v \in V_{h,p}$, there exists $v^c \in V^c_{h,p}$, such that

$$\| v - v^c \|_{h,p} \leq C_P \| \sqrt{\sigma} \, [\![v]\!] \|_{L^2(\mathcal{F}_h)}.$$

Proof.

- *h*–version:
  - Larson & Niklasson 2000.
  - Karakashian & Pascal 2003 (nonconforming meshes, simplices, $d = 2, 3$).
  - H., Perugia, & Schötzau 2004 (nonconforming meshes, simplices, $d = 2, 3$).
  - Burman & Ern 2007, Ern & Stephansen 2008.
- *hp*–version:
  - H., Schötzau, & Wihler 2007 (conforming meshes, quads/triangles, $d = 2$).
  - Zhu & Schötzau 2011 (nonconforming meshes, quads, $d = 2$).
  - Zhu, Giani, H. & Schötzau 2011 (nonconforming meshes, hexes, $d = 3$).

∎

Lemma

$$\|u - u_{h,p}\|_{h,p} \leq \sup_{v \in H_0^1(\Omega)} \mathcal{R}(v) + (1 + C_{\mathrm{cont}}) \|u_{h,p} - u_{h,p}^c\|_{h,p}$$

where

$$\mathcal{R}(v) = \inf_{v_{h,p} \in V_{h,p}} \frac{|\ell_h(v - v_{h,p}) - \widetilde{A}_h(u_{h,p}, v - v_{h,p})|}{\|v\|_{h,p}} \qquad \forall v \in H_0^1(\Omega).$$

Proof.

$$\|u - u_{h,p}\|_{h,p} \le \|u - u_h^c\|_{h,p} + \|u_{h,p} - u_{h,p}^c\|_{h,p}. \tag{14}$$

Set $v = u - u_{h,p}^c \in H_0^1(\Omega)$. Thereby,

$$\begin{aligned}
\|u - u_h^c\|_{h,p}^2 &= \widetilde{A}_h(u - u_{h,p}^c, v) \qquad \text{(Coercivity)} \\
&= \widetilde{A}_h(u - u_{h,p}, v) - \widetilde{A}_h(u_{h,p}^c - u_{h,p}, v) \\
&\le |\ell_h(v) - \widetilde{A}_h(u_{h,p}, v)| + C_{\mathrm{cont}} \|u_{h,p} - u_{h,p}^c\|_{h,p} \|v\|_{h,p} \\
&= |\ell_h(v - v_{h,p}) - \widetilde{A}_h(u_{h,p}, v - v_{h,p})| + C_{\mathrm{cont}} \|u_{h,p} - u_{h,p}^c\|_{h,p} \|v\|_{h,p}
\end{aligned}$$

for all $v_{h,p} \in V_{h,p}$. ■

Proof.

Hence,

$$\begin{aligned} \|u - u_h^c\|_{h,p} &\leq \sup_{v \in H_0^1(\Omega)} \inf_{v_{h,p} \in V_{h,p}} \frac{|\ell_h(v - v_{h,p}) - \widetilde{A}_h(u_{h,p}, v - v_{h,p})|}{\|v\|_{h,p}} \\ &\quad + C_{\mathrm{cont}} \|u_{h,p} - u_{h,p}^c\|_{h,p} \\ &= \sup_{v \in H_0^1(\Omega)} \mathcal{R}(v) + C_{\mathrm{cont}} \|u_{h,p} - u_{h,p}^c\|_{h,p}. \end{aligned} \tag{15}$$

Substituting (15) into (14) gives the desired result. ∎

Lemma

Assume (A1) holds and that $\mathcal{T}_h$ is a conforming mesh. Given a function $v \in H_0^1(\Omega)$, there exists a (quasi-)interpolant $\mathcal{I}v \in V_{h,p}^c$ such that

$$\sum_{\kappa \in \mathcal{T}_h} \left(\frac{p_\kappa^2}{h_\kappa^2} \|(\mathcal{I}v - v)\|_{L_2(\kappa)}^2 + \|\nabla(\mathcal{I}v - v)\|_{L_2(\kappa)}^2 + \frac{p_\kappa}{h_\kappa} \|(\mathcal{I}v - v)\|_{L_2(\partial\kappa)}^2 \right)^{1/2} \leq C_{\mathcal{I}} |v|_{H^1(\Omega)},$$

where $C_{\mathcal{I}}$ is a positive constant, independent of the mesh size h and polynomial degree p.

Proof.

See Melenk & Wohlmuth 2001, Theorem 2.2; Melenk 2005, Theorem 2.4.

■

Lemma

Given $v \in H_0^1(\Omega)$ *and* $v_{h,p} = \mathcal{I}v \in V_{h,p}^c$*, we have that*

$$\left|\ell_h(v - v_{h,p}) - \widetilde{A}_h(u_{h,p}, v - v_{h,p})\right| \leq C_A\Big(\sum_{\kappa\in\mathcal{T}_h}(\eta_\kappa^2 + \Theta_\kappa^2)\Big)^{1/2}\|\nabla v\|_{L^2(\Omega)}.$$

Here, $C_A = \sqrt{6}C_{\mathcal{I}}\max(1, C_L C_\sigma^{-1})^{1/2}$.

Proof.

Setting $\xi = v - v_{h,p} \in H_0^1(\Omega)$, we obtain

$$
\begin{aligned}
&\ell_h(v - v_{h,p}) - \widetilde{A}_h(u_{h,p}, v - v_{h,p}) \\
&= \int_\Omega f\xi \,\mathrm{d}\mathbf{x} - \int_\Omega \nabla_h u_{h,p} \cdot \nabla_h \xi \,\mathrm{d}\mathbf{x} + \int_\Omega \big(\mathbf{L}(u_{h,p}) \cdot \nabla_h \xi + \mathbf{L}(\xi) \cdot \nabla_h u_{h,p}\big) \mathrm{d}\mathbf{x} \\
&- \sum_{F \in \mathcal{F}_h} \int_F \sigma [\![u_{h,p}]\!] \cdot [\![\xi]\!] \,\mathrm{d}s \\
&= \int_\Omega f\xi \,\mathrm{d}\mathbf{x} - \int_\Omega \nabla_h u_{h,p} \cdot \nabla_h \xi \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{L}(u_{h,p}) \cdot \nabla_h \xi \,\mathrm{d}\mathbf{x},
\end{aligned}
$$

since $\xi \in H_0^1(\Omega)$. ■

Proof.

Integrating by parts gives

$$
\begin{aligned}
&\ell_h(v - v_{h,p}) - \widetilde{A}_h(u_{h,p}, v - v_{h,p}) \\
&\quad = \sum_{\kappa \in \mathcal{T}_h} \left(\int_\kappa (f + \Delta u_{h,p}) \xi \, d\mathbf{x} - \int_{\partial\kappa} \nabla_h u_{h,p} \cdot (\xi \mathbf{n}_\kappa) ds \right) \\
&\qquad + \sum_{\kappa \in \mathcal{T}_h} \int_K \mathbf{L}(u_{h,p}) \cdot \nabla \xi \, d\mathbf{x} \\
&\quad \leq C_{\mathcal{I}} \| (h/p)(f + \Delta u_{h,p}) \|_{L_2(\Omega)} |v|_{H^1(\Omega)} \\
&\qquad + C_{\mathcal{I}} \left(\sum_{\kappa \in \mathcal{T}_h} \left\| \frac{h_\kappa}{2p_\kappa} \left[\frac{\partial u_{h,p}}{\partial \mathbf{n}_\kappa} \right] \right\|^2_{L_2(\partial\kappa \setminus \partial\Omega)} \right)^{1/2} |v|_{H^1(\Omega)} \\
&\qquad + C_{\mathcal{I}} C_L \| \mathrm{ph}^{-1/2} [\![u_{h,p}]\!] \|_{L^2(\mathcal{F}_h)} |v|_{H^1(\Omega)}.
\end{aligned}
$$

∎

Proof.

Note that

$$\|\mathrm{p}\mathrm{h}^{-1/2}[\![u_{h,p}]\!]\|_{L^2(\mathcal{F}_h)} \leq \left(\sum_{\kappa \in \mathcal{T}_h} \|\mathrm{p}\mathrm{h}^{-1/2}\,[\![u_{h,p}]\!]\|^2_{L^2(\partial\kappa)} \right)^{1/2} .$$

Thereby,

$$\begin{aligned}
&\ell_h(v - v_{h,p}) - \widetilde{A}_h(u_{h,p}, v - v_{h,p}) \\
&\leq C_A \Bigg[\sum_{\kappa \in \mathcal{T}_h} \Big(p_\kappa^{-2} h_\kappa^2 \|\Pi_{p_\kappa - 1} f + \Delta u_{h,p}\|^2_{L^2(\kappa)} + p_\kappa^{-2} h_\kappa^2 \|f - \Pi_{p_\kappa - 1} f\|^2_{L^2(\kappa)} \\
&\qquad + \|p_\kappa^{-1/2} h_\kappa^{1/2} [\![\nabla_h u_{h,p}]\!]\|^2_{L^2(\partial\kappa \setminus \partial\Omega)} + \|\sqrt{\sigma}\,[\![u_{h,p}]\!]\|^2_{L^2(\partial\kappa)} \Big) \Bigg]^{1/2} |v|_{H^1(\Omega)},
\end{aligned}$$

where $C_A = \sqrt{6} C_{\mathcal{I}} \max(1, C_L C_\sigma^{-1/2})$, as required. ■

Exploiting the results on the previous slides, we deduce that:

$$
\begin{aligned}
\|u - u_{h,p}\|_{h,p} &\leq \sup_{v \in H_0^1(\Omega)} \mathcal{R}(v) + (1 + C_{\mathrm{cont}}) \|u_{h,p} - u_{h,p}^c\|_{h,p} \\
&\leq C_A \Big(\sum_{\kappa \in \mathcal{T}_h} (\eta_\kappa^2 + \Theta_\kappa^2) \Big)^{1/2} \\
&\quad + (1 + C_{\mathrm{cont}}) \, C_P \|\sqrt{\sigma} \, [\![u_{h,p}]\!] \|_{L^2(\mathcal{F}_h)} \\
&\leq (C_A + (1 + C_{\mathrm{cont}}) \, C_P) \Big(\sum_{\kappa \in \mathcal{T}_h} (\eta_\kappa^2 + \Theta_\kappa^2) \Big)^{1/2},
\end{aligned}
$$

as required.

Theorem (Poisson's equation)

Assuming $u \in H_0^1(\Omega)$, the following bound holds:

$$\|u - u_{h,p}\|_{h,p} \leq C \left(\sum_{\kappa \in \mathcal{T}_h} (\eta_\kappa^2 + \Theta_\kappa^2) \right)^{1/2},$$

where

$$\|v\|_{h,p}^2 = \|\nabla_h v\|_{L^2(\Omega)}^2 + \|\sqrt{\sigma}\, [\![v]\!]\|_{L^2(\mathcal{F}_h)}^2,$$

and

$$\begin{aligned}
\eta_\kappa^2 &= \eta_{R_\kappa}^2 + \eta_{E_\kappa}^2 + \eta_{J_\kappa}^2, \\
\eta_{R_\kappa}^2 &= p_\kappa^{-2} h_\kappa^2 \|f + \Delta u_{h,p}\|_{L^2(\kappa)}^2, \\
\eta_{E_\kappa}^2 &= \|p_\kappa^{-1/2} h_\kappa^{1/2} [\![\nabla_h u_{h,p}]\!]\|_{L^2(\partial\kappa \setminus \partial\Omega)}^2, \\
\eta_{J_\kappa}^2 &= C_\sigma^2 p_\kappa^3 h_\kappa^{-1} \|\, [\![u_{h,p}]\!]\|_{L^2(\partial\kappa)}^2, \\
\Theta_\kappa^2 &= p_\kappa^{-2} h_\kappa^2 \|f - \Pi_{p_\kappa - 1} f\|_{L^2(\kappa)}^2.
\end{aligned}$$

Outline Proof

- We assume that $\mathcal{T}_h$ is regularly reducible (Ortner & Süli, 2006, Section 7.1): there exists a shape-regular conforming (regular) mesh $\tilde{\mathcal{T}}_h$ such that
    1. The closure of each element in $\mathcal{T}_h$ is a union of closures of elements of $\tilde{\mathcal{T}}_h$;
    2. There exists a constant $C > 0$, independent of the element sizes, such that for any two elements $\kappa \in \mathcal{T}_h$ and $\tilde{\kappa} \in \tilde{\mathcal{T}}_h$ with $\tilde{\kappa} \subseteq \kappa$, we have

$$h_\kappa / h_{\tilde{\kappa}} \leq C.$$

- Based on the conforming mesh $\tilde{\mathcal{T}}_h$, we construct the corresponding DGFEM space $\tilde{V}_{h,p}$ such that $V_{h,p} \subset \tilde{V}_{h,p}$.
- The proof follows through in an analogous manner, with the interpolation operator $\mathcal{I} : H^1(\Omega) \to V_{h,p} \cap H_0^1(\Omega)$ replaced by $\tilde{\mathcal{I}} : H^1(\Omega) \to V_{h,p}$.

Find $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + (1 - 2\nu)p &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned}$$

where $0 < \nu = \frac{\lambda}{2(\lambda+\mu)} < \frac{1}{2}$ is the Poisson ratio and λ, μ are the Lamé constants [$\mathbf{f}$ has been scaled by $2(1+\nu)/E$, $E > 0$].

Linear Elasticity/Stokes problem

- *hp*-DG finite element space:

$$\mathbf{V}_{h,p} = \{\mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_\kappa \in \mathcal{S}_{p_\kappa}(\kappa)^2 \ \forall \kappa \in \mathcal{T}_h\},$$
$$Q_{h,p} = \{q \in L^2_0(\Omega) : q|_\kappa \in \mathcal{S}_{p_\kappa - 1}(\kappa) \ \forall \kappa \in \mathcal{T}_h\}.$$

(Standard) Interior Penalty Method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_{h,p} \times Q_{h,p}$ such that

$$\begin{aligned} A_h(\mathbf{u}_h, \mathbf{v}_h) + B_h(\mathbf{v}_h, p_h) &= \ell_h(\mathbf{v}_h) \\ -B_h(\mathbf{u}_h, q_h) + \ C_h(p_h, q_h) &= 0 \end{aligned}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_{h,p} \times Q_{h,p}$.

A priori error analysis: Schötzau, Schwab & Toselli 2003.

$$
\begin{aligned}
A_h(\mathbf{u},\mathbf{v}) = & \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \nabla\mathbf{u} : \nabla\mathbf{v}\, d\mathbf{x} \\
& - \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\nabla_h\mathbf{u}\}\!\} : \underline{[\![\mathbf{v}]\!]}\,\mathrm{d}s + \theta\sum_{F\in\mathcal{F}_h}\int_F \{\!\{\nabla_h\mathbf{v}\}\!\} : \underline{[\![\mathbf{u}]\!]}\,\mathrm{d}s \\
& + \sum_{F\in\mathcal{F}_h}\int_F \sigma\underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]}\,\mathrm{d}s, \\
B_h(\mathbf{v},q) = & -\sum_{\kappa\in\mathcal{T}_h}\int_\kappa q\nabla\cdot\mathbf{v}\, d\mathbf{x} + \sum_{F\in\mathcal{F}_h}\int_F \{\!\{q\}\!\}[\![\mathbf{v}]\!]\,\mathrm{d}s, \\
C_h(p,q) = & (1-2\nu)\int_\Omega pq\, d\mathbf{x}, \\
\ell_h(v) = & \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}.
\end{aligned}
$$

Jump operator: $\underline{[\![\mathbf{v}]\!]} = \mathbf{v}^+\otimes\mathbf{n}^+ + \mathbf{v}^-\otimes\mathbf{n}^-$.

Theorem

Assuming $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$, *the following bound holds:*

$$\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\|_{h,p} \leq C\left(\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa^2\right)^{1/2} \equiv C\left(\sum_{\kappa\in\mathcal{T}_h}(\eta_{R_\kappa}^2+\eta_{E_\kappa}^2+\eta_{J_\kappa}^2)\right)^{1/2},$$

where

$$\|(\mathbf{v}, q)\|_{h,p}^2 = \|\nabla_h \mathbf{v}\|_{L^2(\Omega)}^2 + \|\sqrt{\sigma}\,[\![\mathbf{v}]\!]\|_{L^2(\mathcal{F}_h)}^2 + 2(1-\nu)\|q\|_{L^2(\Omega)}^2,$$

and

$$\begin{aligned}
\eta_{R_\kappa}^2 &= p_\kappa^{-2}h_\kappa^2\|\mathbf{f}+\Delta\mathbf{u}_h-\nabla p_h\|_{L^2(\kappa)}^2 + \|\nabla\cdot\mathbf{u}_h+(1-2\nu)p_h\|_{L^2(\kappa)}^2,\\
\eta_{E_\kappa}^2 &= \|p_\kappa^{-1/2}h_\kappa^{1/2}([\![p_h]\!]-[\![\nabla_h\mathbf{u}_h]\!])\|_{L^2(\partial\kappa\backslash\Gamma)}^2,\\
\eta_{J_\kappa}^2 &= \|\sqrt{\sigma}\,[\![\mathbf{u}_h]\!]\|_{L^2(\partial\kappa)}^2.
\end{aligned}$$

- Find $\mathbf{u}$ such that

$$
\begin{aligned}
\nabla \times \nabla \times \mathbf{u} + \mathbf{u} = \mathbf{j} \qquad & \text{in } \Omega, \\
\mathbf{n} \times \mathbf{u} = \mathbf{0} \qquad & \text{on } \Gamma.
\end{aligned}
$$

- Weak Formulation: find $\mathbf{u} \in H_0(\mathrm{curl};\Omega)$ such that

$$A(u,v) \equiv (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + (\mathbf{u},\mathbf{v}) = (\mathbf{j},\mathbf{v}) \quad \forall v \in H_0(\mathrm{curl};\Omega).$$

Beck, Hiptmair, Hoppe & Wohlmuth 2000

- Energy Norm:

$$\|v\|_{h,p}^2 = \|\nabla_h \times \mathbf{v}\|_{L^2(\Omega)}^2 + \|\mathbf{v}\|_{L^2(\Omega)}^2 + \|\sqrt{\sigma}\, [\![\mathbf{v}]\!]_T\|_{L^2(\mathcal{F}_h)}^2.$$

- Jump Operator:

$$[\![\mathbf{v}]\!]_T = \mathbf{n}^+ \times \mathbf{v}^+ + \mathbf{n}^- \times \mathbf{v}^-.$$

Proposition (Approximation by conforming functions)

Set $\mathbf{V}_h^c = \mathbf{V}_h \cap H_0(\mathrm{curl};\Omega)$*; then for any* $\mathbf{v} \in \mathbf{V}_h$*, there exists* $\mathbf{v}^c \in \mathbf{V}_h^c$ *such that*

$$\|\mathbf{v} - \mathbf{v}^c\|_{h,p} \leq C \|\sqrt{\sigma}\, [\![\mathbf{v}]\!]_T\|_{L^2(\mathcal{F}_h)} = C \left(\sum_{\kappa \in \mathcal{T}_h} \eta_{J_\kappa}^2 \right)^{\frac{1}{2}} .$$

Abstract error bound:

$$\|\mathbf{u} - \mathbf{u}_h\|_{h,p} \leq \sup_{\mathbf{v} \in H_0(\mathrm{curl};\Omega)} \mathcal{R}(\mathbf{v}) + (1 + C_{\mathrm{cont}}) \|\mathbf{u}_h - \mathbf{u}_h^c\|_{h,p},$$

where

$$\mathcal{R}(\mathbf{v}) = \inf_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|\ell_h(\mathbf{v} - \mathbf{v}_h) - \widetilde{A}_h(\mathbf{u}_h, \mathbf{v} - \mathbf{v}_h)|}{\|\mathbf{v}\|_{h,p}} \quad \forall \mathbf{v} \in H_0(\mathrm{curl};\Omega).$$

Theorem

Assuming $\mathbf{u} \in H_0(\mathrm{curl};\Omega)$*, the following bound holds:*

$$\|u - u_{h,p}\|_{h,p} \leq C \left(\sum_{\kappa \in \mathcal{T}_h} (\eta_{R_\kappa}^2 + \eta_{D_\kappa}^2 + \eta_{T_\kappa}^2 + \eta_{N_\kappa}^2 + \eta_{E_\kappa}^2 + \eta_{J_\kappa}^2) \right)^{1/2},$$

where

$$\begin{aligned}
\|v\|_{h,p}^2 &= \|\nabla_h \times \mathbf{v}\|_{L^2(\Omega)}^2 + \|\mathbf{v}\|_{L^2(\Omega)}^2 + \|\sqrt{\sigma}\,[\![\mathbf{v}]\!]_T\|_{L^2(\mathcal{F}_h)}^2, \\
\eta_{R_\kappa}^2 &= h_\kappa^2 \|\mathbf{j} - \nabla \times \nabla \times \mathbf{u}_h - \mathbf{u}_h\|_{L^2(\kappa)}^2, \\
\eta_{D_\kappa}^2 &= h_\kappa^2 \|\nabla \cdot (\mathbf{j} - \mathbf{u}_h)\|_{L^2(\kappa)}^2, \\
\eta_{T_\kappa}^2 &= \|h_\kappa^{1/2} [\![\nabla \times \mathbf{u}_h]\!]_T\|_{L^2(\partial\kappa\setminus\Gamma)}^2, \\
\eta_{N_\kappa}^2 &= \|h_\kappa^{1/2} [\![\mathbf{u}_h]\!]_N\|_{L^2(\partial\kappa\setminus\Gamma)}^2, \qquad \eta_{J_\kappa}^2 = \|\sqrt{\sigma}\,[\![\mathbf{u}_h]\!]_T\|_{L^2(\partial\kappa)}^2.
\end{aligned}$$

- Fourth–order problems

  h–version: Georgoulis, Houston & Virtanen 2011

- Convection–diffusion problems

  h–version: Zhu & D. Schötzau 2009

  hp–version: Zhu & D. Schötzau 2011

- Navier–Stokes equations

  h–version: Kanschat & Schötzau 2008

- Implicit–explicit error indicators

  Ainsworth & Rankin 2010

Measurement Problem

Given a user–defined tolerance $\texttt{TOL} > 0$, we wish to efficiently design $V_{h,p}$ such that

$$\|u - u_{h,p}\|_{\mathcal{V}} \leq \texttt{TOL}.$$

- Key ingredients:
  1. Derivation of a computable *a posteriori* error bound.
     $\Rightarrow$ Stopping Criterion
  2. Design of a local error indicator.
     $\Rightarrow$ Indicates regions where the error is locally small/large.
  3. Marking Strategy
     $\Rightarrow$ Marks elements for refinement/derefinement in order to equidistribute the local error indicators.
  4. Refinement Strategy.
     $\Rightarrow$ Adjusts the local size of the elements (and/or the local polynomial degrees).

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- Automatic refinement algorithm:

  0. Start with initial (coarse) grid $\mathcal{T}_h^{(j=0)}$.
  1. Compute the numerical solution $u_{h,p}^{(j)}$ on $\mathcal{T}_h^{(j)}$.
  2. Compute the local error indicators η_κ.
  3. If the stopping criterion is satisfied, then stop. Otherwise, adapt $V_{h,p}$.
  4. $j = j + 1$, and go to step (1).

R. Verfürth
A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques,
B.G. Teubner, Stuttgart, 1996.

- On the basis of the relative size of the elemental error indicators, cf. above, elements are now marked for refinement/derefinement.
- Popular strategies:
    1. Error per cell strategy (equidistribution). In this approach the mesh objective is to equilibrate the local error indicators by refining or coarsening the elements κ in the current partition $\mathcal{T}_h$ according to the criterion
    $$\eta_\kappa \approx \frac{\mathtt{TOL}}{\sqrt{N}},$$
    where N denotes the number of elements in the current mesh $\mathcal{T}_h$.
    2. Fixed fraction strategy. At each refinement step, the elements are ordered according to the size of the local error indicators η_κ, and then a fixed partition of the elements κ with the largest η_κ are refined, while a fixed fraction of the elements κ with the smallest η_κ are derefined. For example, we may select refinement and derefinement parameters equal to 20% and 10%, respectively.

Marking Strategy

- Popular strategies (Cont...):
  3. Bulk–chasing (Dörfler 1996): Given a parameter $0 < \theta < 1$, construct a minimal subset $\widehat{\mathcal{T}}_h$ of $\mathcal{T}_h$ such that

$$\sum_{\kappa \in \widehat{\mathcal{T}}_h} \eta_\kappa^2 \geq \theta^2 \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2$$

  and mark all elements in $\widehat{\mathcal{T}}_h$ refinement.

- We note that these marking strategies are successively repeated on each (new) mesh until the stopping criterion has been achieved.

h–Refinement Strategies

- Once the elements have been marked for refinement/derefinement, the mesh must now be modified accordingly. This may be done in a number of ways:
- Popular mesh–refinement strategies:
  1. Regenerate the entire mesh using an appropriate mesh generator which targets the local parts of the computational domain, where the error indicators are large, to concentrate elements.
  2. Perform local adjustments to an initial background mesh (Subdivision).
     - Red–Green refinement;
     - Longest edge bisection;
     - Marked edge bisection;
     - Anisotropic refinement.
- Local Polynomial enrichment.

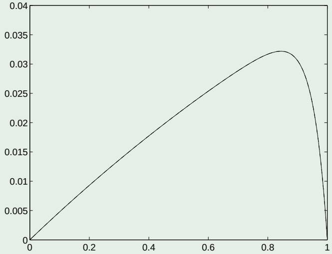
Example

Here, we consider the second–order ordinary differential equation

$$\begin{aligned} -u'' + bu' + cu &= f \quad \text{in} \quad (0,1), \\ u(0) = u(1) &= 0. \end{aligned}$$

For simplicity we select

$$b = 20, \quad c = 10 \quad \text{and} \quad f = 1.$$



Aim: Estimate the error in the $L_2(0,1)$–norm.

Example

Performance of the adaptive refinement algorithm with $\texttt{TOL} = 1 \times 10^{-4}$.

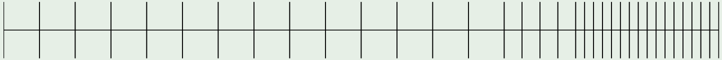
Mesh 1 with 10 elements

Mesh 2 with 12 elements

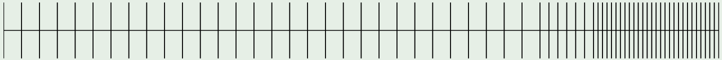
Mesh 3 with 24 elements

Example

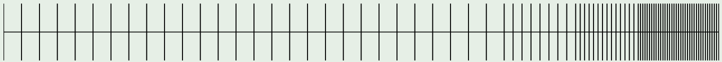
Performance of the adaptive refinement algorithm with $\mathtt{TOL} = 1 \times 10^{-4}$.



Mesh 4 with 34 elements



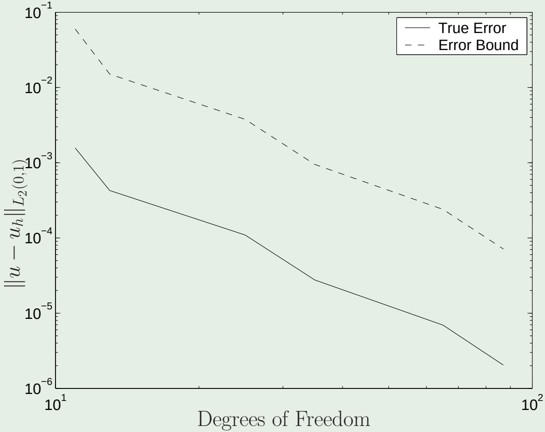
Mesh 5 with 64 elements



Mesh 6 with 86 elements

Example

Performance of the adaptive refinement algorithm with $\texttt{TOL} = 1 \times 10^{-4}$.



Basic Strategy:

	High–Error (Refinement)	Low–Error (Derefinement)
Solution Smooth	$p \to p+1$	$h \to 2h$
Solution Nonsmooth	$h \to h/2$	$p \to p-1$

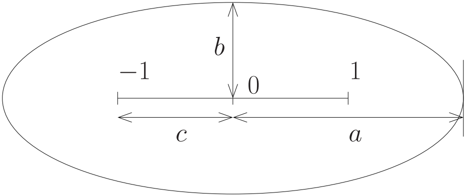
- 'Texas 3–Step'
  Oden, Patra & Feng 1992, Bey, Oden & Patra 1995, ...
- A Priori Information
  Bernardi & Raugel 1981, Valenciano & Owens 2000, Bernardi, Fiétier & Owens 2001
- Type Parameter
  Gui & Babuška 1986, Adjerid, Aiffa & Flaherty 1998
- Predicted Error Reduction
  Melenk & Wohlmuth 2001, Heuveline & Rannacher 2003
- Mesh Optimisation Strategy
  Rachowicz, Demkowicz & Oden 1989, Demkowicz, Rachowicz & Devloo 2001
- Estimate decay rates of Legendre coefficients
  Mavriplis 1994
- Local Regularity Estimation
  Ainsworth & Senior 1998, Houston & Süli 2000, Houston, Senior & Süli 2001, 2002
- Analyticity Estimation
  H., Senior & Süli 2003, H. & Süli 2005, Melenk 2002, Eibner & Melenk 2004
- Sobolev Embeddings
  Wihler 2011

Review article: Mitchell & McClain. A Comparison of *hp*-Adaptive Strategies for Elliptic Partial Differential Equations (long version, 215 pages), NISTIR 7824, 2011.

Given $u \in L_2(-1,1)$, we have that

$$u(\xi) = \sum_{i=0}^{\infty} a_i L_i(\xi), \qquad a_i = \frac{2i+1}{2} \int_{-1}^{1} u(\xi) L_i(\xi) \mathrm{d}\xi \, .$$

Bernstein Ellipse $\hat{\mathcal{E}}_\rho$
with radius $\rho = (a+b)/c$



Let u be analytic inside $\hat{\mathcal{E}}_\rho$, but not inside $\hat{\mathcal{E}}_{\rho'}$ with $\rho' > \rho$; then

$$\frac{1}{\rho} = \overline{\lim}_{i\to\infty} |a_i|^{1/i}, \qquad \rho > 1 \, . \qquad \text{(Davis 1963)}$$

The quantity

$$\theta = \frac{1}{\rho}$$

is the measure of the size of the domain of analyticity of u relative to $(-1,1)$.

- $\theta = 0$: entire analytic function
- $\theta = 1$: function with singular support in $[-1,1]$ (finite Sobolev regularity in I).

Assuming that

$$|a_i| \sim (1/\rho)^i, \quad \text{as } i \to \infty,$$

we have

$$\log |a_i| \sim i \log(1/\rho), \quad \text{as } i \to \infty.$$

Thereby, using linear regression, we may approximate θ by

$$\theta = e^{-m},$$

where

$$|\log |a_i|| = im + b,$$

and the slope m is defined by

$$m = 6 \frac{2\sum_{i=0}^{p} i|\log |a_i|| - p\sum_{i=0}^{p} |\log |a_i||}{(p+1)\left((p+1)^2 - 1\right)}.$$

hp–Extension control

- Select $0 < \theta_{\max} < 1$, say $\theta_{\max} = 1/2$.
- Given an each element κ in the mesh $\mathcal{T}_h$:
    - If $\theta > \theta_{\max}$, then $u|_\kappa$ is smooth;
    - otherwise $u|_\kappa$ has finite Sobolev regularity.

H., Senior & Süli 2003, H. & Süli 2005.
Melenk 2002, Eibner & Melenk 2004

Consider the problem:

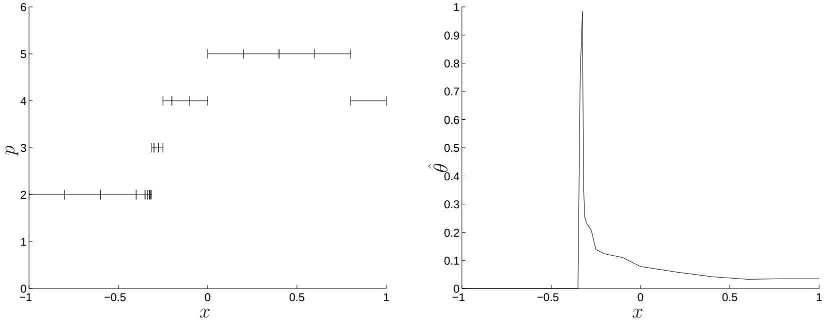
$$-u_{xx} = f \quad \text{in } (0,1),$$

where f is selected so that

$$u(x) = \begin{cases} 0 & \text{for } -1 \leq x < \beta, \\ (x-\beta)^{\alpha} & \text{for } \beta \leq x \leq 1. \end{cases}$$

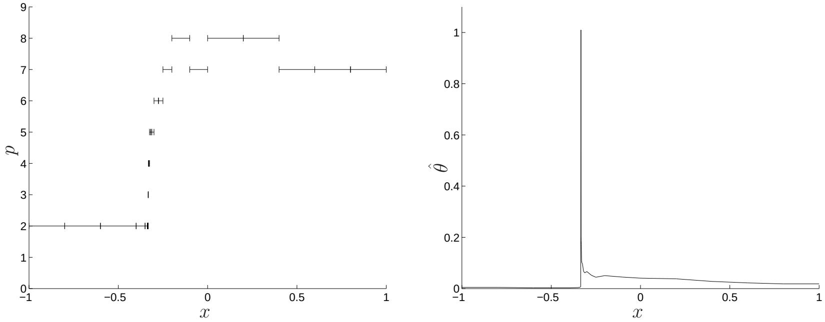
Thereby, $u \in H^{\alpha+1/2-\epsilon}(-1,1)$, $\epsilon > 0$.

Set $\alpha = 7/2$ and $\beta = -1/3$.



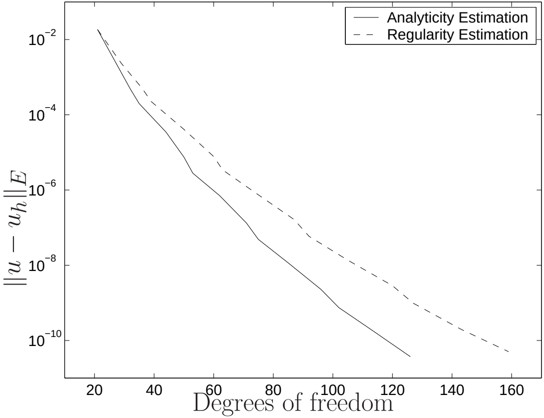
Computational mesh and computed values of $\hat{\theta}$ after 7 adaptive refinements with 62 degrees of freedom

Set $\alpha = 7/2$ and $\beta = -1/3$.



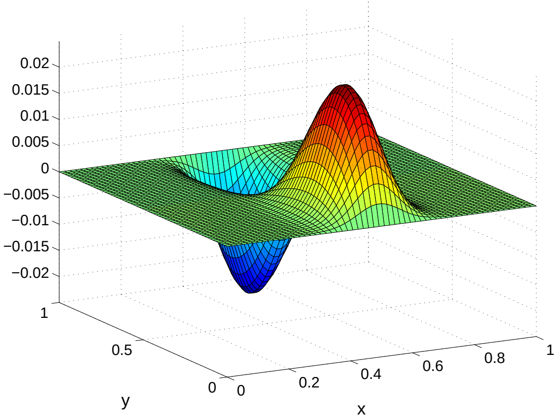
Computational mesh and computed values of $\hat{\theta}$ after 14 adaptive refinements with 126 degrees of freedom

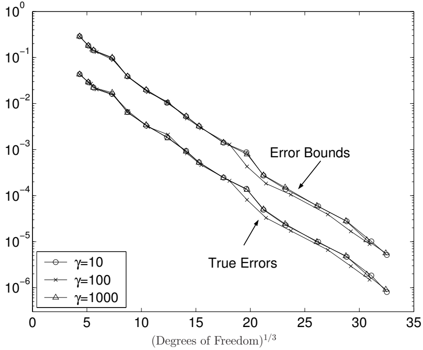
Set $\alpha = 7/2$ and $\beta = -1/3$.

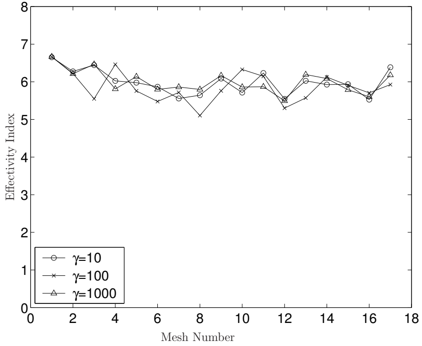


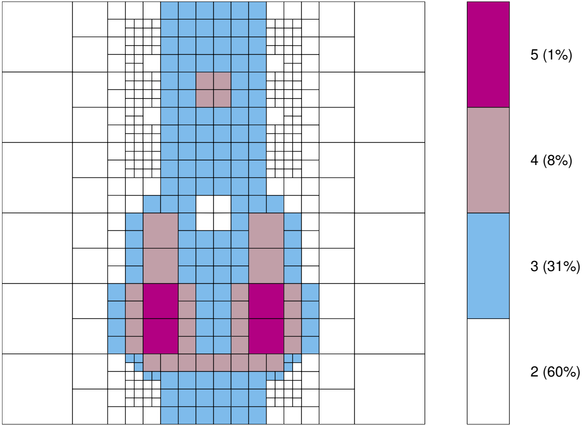
We let $\Omega = (0,1)^2$, and select f so that

$$u(x,y) = x(1-x)y(1-y)(1-2y)\mathrm{e}^{-\sigma(2x-1)^2}, \quad \text{where } \sigma = 25.$$

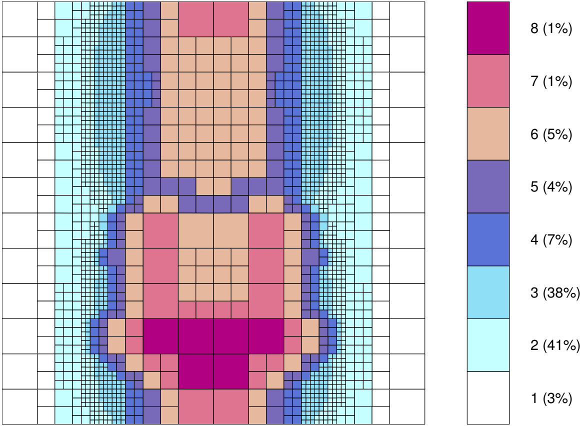



Error Bounds
True Errors
γ=10
γ=100
γ=1000
(Degrees of Freedom)$^{1/3}$


Effectivity Index
γ=10
γ=100
γ=1000
Mesh Number



hp–Mesh after 9 adaptive refinements, with 426 elements and 5392 degrees of freedom.



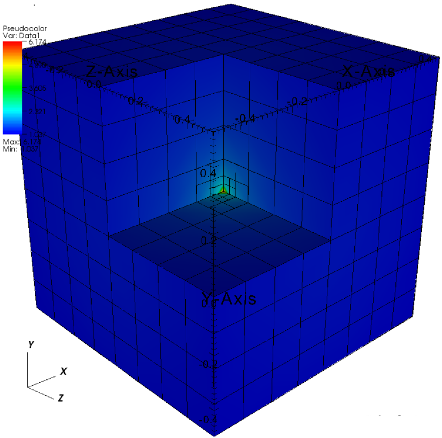
hp–Mesh after 16 adaptive refinements, with 2088 elements and 34426 degrees of freedom.

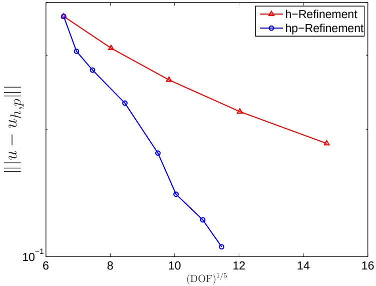
Let Ω be the Fichera corner $(-1,1)^3 \setminus [0,1)^3$, and select f so that

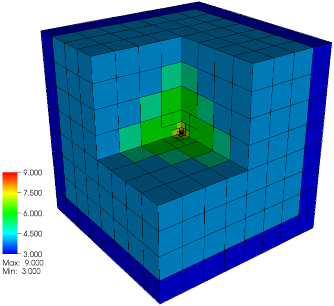
$$u(\mathbf{x}) = (x^2 + y^2 + z^2)^{q/2}, \quad q \in \mathbb{R};$$

for $q > -1/2$, $u \in H^1(\Omega)$. Here, we select $q = -1/4$.

Beilina, Korotov, Křížek 2005







Dofs for accuracy of 10^{-1}

h–Version	NLTEC
hp–Version	$\approx$ 250K

Nonlinear Problem

Given a semilinear form $\mathcal{N}(\cdot;\cdot,\cdot) : V \times V \times V \to \mathbb{R}$, find $u \in V$ such that

$$\mathcal{N}(u; u, v) = 0 \quad \forall v \in V.$$

- Assumption: $\mathcal{N}(\cdot;\cdot,\cdot)$ is *linear* with respect to the arguments right of the semi-colon.

(Standard) Discretization Method

Find $u_{h,p} \in V_{h,p}$ such that

$$\mathcal{N}_h(u_{h,p}; u_{h,p}, v_{h,p}) = 0 \quad \forall v_{h,p} \in V_{h,p}.$$

$\longrightarrow$ Leads to a potentially very large number of coupled nonlinear equations.

$\longrightarrow$ A sufficiently good guess is required to ensure convergence of, for example, a Newton-type iteration method.

Two–Grid Discretization Method

Find $u_{H,P} \in V_{H,P} \subset V_{h,p}$ such that

$$\mathcal{N}_H(u_{H,P}; u_{H,P}, v_{H,P}) = 0 \quad \forall v_{H,P} \in V_{H,P};$$

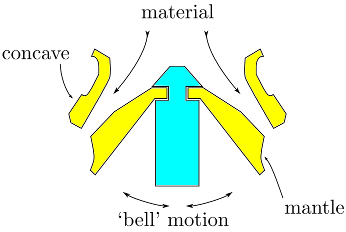
find $u_{2G} \in V_{h,p}$ such that

$$\mathcal{N}_h(u_{H,P}; u_{2G}, v_{h,p}) = 0 \quad \forall v_{h,p} \in V_{h,p}.$$

Xu 1992, 1994, 1996, Xu & Zhou 1999, Axelsson & Layton 1996, Dawson, Wheeler, & Woodward 1998, Utnes 1997, Marion & Xu 1995, Wu & Allen 1999, Bi & Ginting 2007, 2011

- This requires one nonlinear solve on the coarse space $V_{H,P}$ and one linear solve on the fine space $V_{h,p}$.
- Through this construction, multilevel preconditioners can naturally be designed for the fine grid (linear) solve.
- Applications: Non-selfadjoint PDEs, nonlinear PDEs, eigenvalue problems.

- Examples include: ketchup, custard, toothpaste, paint, blood, shampoo, ...
- Cone Crushers:



(Joint work with Oliver Bain and John Billingham)

Jop, Forterre, & Pouliquen 2006

Quasilinear Problem

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and $f \in L^2(\Omega)$: find u such that

$$-\nabla \cdot \{\mu(x, |\nabla u|)\nabla u\} = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \Gamma.$$

Assumption

(A1) *$\mu \in \mathrm{C}(\bar{\Omega} \times [0, \infty))$ and*

(A2) *there exists positive constants m_μ and M_μ such that*

$$m_\mu(t - s) \le \mu(x, t)t - \mu(x, s)s \le M_\mu(t - s), \quad t \ge s \ge 0, \quad x \in \bar{\Omega}.$$

Carreau law

$$\mu(t) = \mu_\infty + (\mu_0 - \mu_\infty)(1 + \lambda t^2)^{\frac{r-2}{2}},$$

where $\lambda > 0$, $1 < r \le 2$ and $0 < \mu_\infty < \mu_0$ (here, $m_\mu = \mu_\infty$ and $M_\mu = \mu_0$).

- $\mathcal{T}_h$ is a non-degenerate mesh consisting of elements of granularity h.
- *hp*-DG finite element space:

$$V_{h,p} = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{S}_{p_\kappa}(\kappa) \quad \forall \kappa \in \mathcal{T}_h\},$$

where

$$\mathcal{S}_{p_\kappa}(\kappa) = \begin{cases} \mathcal{P}_{p_\kappa}(\kappa) & \text{if } \kappa \text{ is a simplex,} \\ \mathcal{Q}_{p_\kappa}(\kappa) & \text{if } \kappa \text{ is a hypercube.} \end{cases}$$

- $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{B}} \cup \mathcal{F}_h^{\mathcal{I}}$ denotes the set of all faces in the mesh $\mathcal{T}_h$.

(Standard) Interior Penalty Method

Find $u_{h,p} \in V_{h,p}$ such that

$$A_{h,p}(u_{h,p}; u_{h,p}, v_{h,p}) = \ell_{h,p}(v_{h,p})$$

for all $v_{h,p} \in V_{h,p}$.

- Forms:

$$A_{h,p}(\psi; u, v) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \mu(|\nabla_h \psi|) \nabla_h u \cdot \nabla_h v \, d\mathbf{x} + \sum_{F \in \mathcal{F}_h} \int_F \sigma [\![u]\!] \cdot [\![v]\!] \, \mathrm{d}s$$

$$- \sum_{F \in \mathcal{F}_h} \int_F \left(\{\!\{\mu(|\nabla_h \psi|) \nabla_h u\}\!\} \cdot [\![v]\!] + \theta \{\!\{\mu(|[\![\psi]\!]/h_F|) \nabla_h v\}\!\} \cdot [\![u]\!] \right) \mathrm{d}s,$$

$$\ell_{h,p}(v) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa f v \, d\mathbf{x},$$

where $\theta \in [-1, 1]$.

- Interior penalty (stability) parameter:

$$\sigma = C_\sigma \frac{\mathtt{p}^2}{\mathtt{h}},$$

where $\mathtt{p}|_F = \max(p_{\kappa_1}, p_{\kappa_2})$ and $\mathtt{h}|_F = \min(h_{\kappa_1}, h_{\kappa_2})$.

- References:

Bustinza & Gatica 2004, Gatica, González, & Meddahi 2004, H., Robson, & Süli 2005, Bustinza, Cockburn, & Gatica 2005, H., Süli, & Wihler 2007, Gudi, Nataraj, & Pani 2008

DG Norm

$$\|v\|_{h,p}^2 = \|\nabla_h v\|_{L^2(\Omega)}^2 + \sum_{F \in \mathcal{F}_h} \int_F \sigma |[\![v]\!]|^2 \, \mathrm{d}s.$$

Lipschitz Continuity

The semilinear form $A_{h,p}(\cdot;\cdot,\cdot)$ is Lipschitz–continuous in the sense that

$$|A_{h,p}(w_1; w_1, v) - A_{h,p}(w_2; w_2, v)| \leq C\|w_1 - w_2\|_{h,p}\|v\|_{h,p}$$

for all $w_1, w_2, v \in V_{h,p}$, where $C > 0$ is independent of the discretization parameters.

H., Robson, & Süli 2005

Strong Monotonicity

Given $C_\sigma > C_{\sigma,\min}$, the semilinear form $A_{h,p}(\cdot;\cdot,\cdot)$ is strongly monotone, i.e.,

$$A_{h,p}(w_1; w_1, w_1 - w_2) - A_{h,p}(w_2; w_2, w_1 - w_2) \geq C\|w_1 - w_2\|_{h,p}^2$$

for all $w_1, w_2 \in V_{h,p}$, where $C > 0$ is independent of the discretization parameters.

H., Robson, & Süli 2005

Existence and Uniqueness

Given $C_\sigma > C_{\sigma,\min}$, there exists a unique $u_{h,p} \in V_{h,p}$ such that

$$A_{h,p}(u_{h,p}; u_{h,p}, v_{h,p}) = \ell_{h,p}(v_{h,p}) \qquad \forall v_{h,p} \in V_{h,p}.$$

Coercivity

Given $C_\sigma > C_{\sigma,\min}$, the semilinear form $A_{h,p}(\cdot;\cdot,\cdot)$ is coercive in the sense that

$$A_{h,p}(w_1; w_2, w_2) \geq C\|w_2\|_{h,p}^2$$

for all $w_1, w_2 \in V_{h,p}$, where $C > 0$ is independent of the discretization parameters.

Congreve, H., & Wihler (submitted)

Two-grid Approximation

1. Construct coarse and fine FE spaces $V_{H,P}$ and $V_{h,p}$, respectively, such that
$$V_{H,P} \subseteq V_{h,p}.$$
2. Compute the coarse grid approximation $u_{H,P} \in V_{H,P}$ such that
$$A_{H,P}(u_{H,P}; u_{H,P}, v_{H,P}) = \ell_{H,P}(v_{H,P})$$
for all $v_{H,P} \in V_{H,P}$.
3. Determine the fine grid solution $u_{2G} \in V_{h,p}$ such that
$$A_{h,p}(u_{H,P}; u_{2G}, v_{h,p}) = \ell_{h,p}(v_{h,p})$$
for all $v_{h,p} \in V_{h,p}$.

Lemma (Standard DGFEM)

The following bound holds:

$$\|u - u_{h,p}\|^2_{h,p} \leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{H^{k_\kappa}(\kappa)}$$

with $1 \leq s_k \leq min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_h$.

Proof.

See H., Robson, & Süli 2005. ■

Lemma (Two-grid Approximation)

The following bound holds:

$$\|u - u_{2G}\|_{h,p}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{H^{k_\kappa}(\kappa)}^2 + C \sum_{\kappa_H \in \mathcal{T}_H} \frac{H_\kappa^{2S_\kappa - 2}}{P_\kappa^{2K_\kappa - 3}} \|u\|_{H^{K_\kappa}(\kappa)}^2$$

with $1 \leq s_k \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_h$, *and* $1 \leq S_\kappa \leq \min\{P_\kappa + 1, K_\kappa\}$, $P_\kappa \geq 1$, *for* $\kappa_H \in \mathcal{T}_H$.

Proof.

Based on a generalization of the analysis in H., Robson, & Süli 2005 and Bi & Ginting 2011. ■

Remark

- *This result implies that* $V_{H,P}$ *and* $V_{h,p}$ *should grow at roughly the same rate, i.e.,* $H = \mathcal{O}(h)$ *and* $P = \mathcal{O}(p)$.
- *In contrast, when* $\mu = \mu(u)$, *we have that* $H = \mathcal{O}(h^{p/p+1})$. *Bi & Ginting 2011*

Lemma (Standard DGFEM)

The following bound holds (1–irregular meshes):

$$\|u - u_{h,p}\|_{h,p}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} (\eta_\kappa^2 + \Theta_\kappa^2),$$

Here, the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\eta_\kappa^2 = h_\kappa^2 p_\kappa^{-2} \|\Pi_{p_\kappa - 1} f + \nabla \cdot \{\mu(|\nabla u_{h,p}|)\nabla u_{h,p}\|_{L^2(\kappa)}^2 \\ + h_\kappa p_\kappa^{-1} \|[\![\mu(|u_{h,p}|)\nabla u_{h,p}]\!]\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + C_\sigma^2 h_\kappa^{-1} p_\kappa^3 \|[\![u_{h,p}]\!]\|_{L^2(\partial\kappa)}^2,$$

and the data oscillation terms Θ_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\Theta_\kappa^2 = p_\kappa^{-2} h_\kappa^2 \|f - \Pi_{p_\kappa - 1} f\|_{L^2(\kappa)}^2.$$

Proof.

See H., Süli, & Wihler 2008. ■

For simplicity, we assume that the mesh $\mathcal{T}_h$ is conforming.
Applying (A2) yields

$$\begin{aligned}\|u - u_{h,p}\|_{h,p}^2 &= \sum_{\kappa \in \mathcal{T}_h} \int_\kappa |\nabla u - \nabla u_{h,p}|^2 \, d\mathbf{x} + \sum_{F \in \mathcal{F}_h} \int_F \sigma \, [\![u - u_{h,p}]\!]^2 \mathrm{d}s \\ &\leq C \sum_{\kappa \in \mathcal{T}_h} \int_\kappa (\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{h,p}|)\nabla u_{h,p}) \cdot \nabla e_{h,p} \, d\mathbf{x} \\ &\quad + \sum_{F \in \mathcal{F}_h} \int_F \sigma \, [\![e_{h,p}]\!]^2 \mathrm{d}s,\end{aligned}$$

where $e_{h,p} = u - u_{h,p}$.

Setting
$$V_{h,p}^c = V_{h,p} \cap H_0^1(\Omega),$$
we decompose $V_{h,p}$ as follows:
$$V_{h,p} = V_{h,p}^c \oplus V_{h,p}^{\perp},$$
where $V_{h,p}^{\perp}$ is the orthogonal complement of $V_{h,p}^c$ w.r.t. $\|\cdot\|_{h,p}$.

Proposition

For every $v \in V_{h,p}$, there exists $v^c \in V_{h,p}^c$, such that
$$\|v - v^c\|_{h,p} \leq C_P \|\sqrt{\sigma}\, [\![v]\!]\|_{L^2(\mathcal{F}_h)}.$$

Corollary

Let $e_{h,p} = u - u_{h,p} = e_{h,p}^c - u_{h,p}^{\perp}$, where $e_{h,p}^c = u - u_{h,p}^c \in H_0^1(\Omega)$ and $u_{h,p}^{\perp} \in V_{h,p}^{\perp}$; then
$$\|e_{h,p}^c\|_{h,p} \leq C\|e_{h,p}\|_{h,p}.$$

Writing $e_{h,p} = u - u_{h,p} = e^c_{h,p} - u^{\perp}_{h,p}$, where $e^c_{h,p} = u - u^c_{h,p} \in H^1_0(\Omega)$ and $u^{\perp}_{h,p} \in V^{\perp}_{h,p}$, we deduce that

$$\|u - u_{h,p}\|^2_{h,p} \leq C(T_1 + T_2 + T_3), \tag{16}$$

where

$$T_1 = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa (\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{h,p}|)\nabla u_{h,p}) \cdot \nabla e^c_{h,p} \, d\mathbf{x},$$

$$T_2 = -\sum_{\kappa \in \mathcal{T}_h} \int_\kappa (\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{h,p}|)\nabla u_{h,p}) \cdot \nabla u^{\perp}_{h,p} \, d\mathbf{x},$$

$$T_3 = \sum_{F \in \mathcal{F}_h} \int_F \sigma \, [\![e_{h,p}]\!]^2 \mathrm{d}s.$$

Now

$$T_1 = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa f e^c_{h,p} \, d\mathbf{x} - \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \mu(|\nabla u_{h,p}|) \nabla u_{h,p} \cdot \nabla e^c_{h,p} \, d\mathbf{x}.$$

We now let $\varphi_{hp} \in V_{h,p}$; then, by the definition of the *hp*-DGFEM, it follows that

$$\begin{aligned}
T_1 = & \sum_{\kappa \in \mathcal{T}_h} \int_\kappa f(e^c_{h,p} - \varphi_{hp}) \, d\mathbf{x} - \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \mu(|\nabla u_{h,p}|) \nabla u_{h,p} \cdot \nabla(e^c_{h,p} - \varphi_{hp}) \, d\mathbf{x} \\
& - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{ \mu(|\nabla_h u_{h,p}|) \nabla_h u_{h,p} \}\!\} \cdot [\![\varphi_{hp}]\!] \mathrm{d}s \\
& + \theta \sum_{F \in \mathcal{F}_h} \int_F \{\!\{ \mu(h_F^{-1} |[\![u_{h,p}]\!]|) \nabla_h \varphi_{hp} \}\!\} \cdot [\![u_{h,p}]\!] \mathrm{d}s \\
& + \sum_{F \in \mathcal{F}_h} \int_F \sigma \, [\![u_{h,p}]\!] \cdot [\![\varphi_{hp}]\!] \mathrm{d}s.
\end{aligned}$$

Hence, integrating the second term on the right-hand side of the above equality by parts, leads to

$$
\begin{aligned}
T_1 = {} & \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \left(f + \nabla \cdot (\mu(|\nabla u_{h,p}|)\nabla u_{h,p})\right)(e^c_{h,p} - \varphi_{hp})\, d\mathbf{x} \\
& - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} (\mu(|\nabla u_{h,p}|)\nabla u_{h,p} \cdot \mathbf{n}_\kappa)(e^c_{h,p} - \varphi_{hp}) \mathrm{d}s \\
& - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\mu(|\nabla_h u_{h,p}|)\nabla_h u_{h,p}\}\!\} \cdot [\![\varphi_{hp}]\!] \mathrm{d}s \\
& + \theta \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\mu(h_F^{-1}|[\![u_{h,p}]\!]|)\nabla_h \varphi_{hp}\}\!\} \cdot [\![u_{h,p}]\!] \mathrm{d}s \\
& + \sum_{F \in \mathcal{F}_h} \int_F \sigma\, [\![u_{h,p}]\!] \cdot [\![\varphi_{hp}]\!] \mathrm{d}s.
\end{aligned}
$$

Term T_1

We now select φ_{hp} to be the elementwise projection of $e^c_{h,p}$; after some elementary (but lengthy) manipulations, we deduce that

$$T_1 \leq C \left(\sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \Theta_\kappa^2 \right) \right)^{\frac{1}{2}} \| e^c_{h,p} \|_{h,p}.$$

Hence, exploiting the approximation result for conforming functions, gives

$$T_1 \leq C \left(\sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \Theta_\kappa^2 \right) \right)^{\frac{1}{2}} \| e_{h,p} \|_{h,p}. \tag{17}$$

Recall:

$$T_2 = -\sum_{\kappa\in\mathcal{T}_h}\int_\kappa (\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{h,p}|)\nabla u_{h,p})\cdot\nabla u_{h,p}^\perp \, d\mathbf{x}.$$

Exploiting (A2), gives

$$\begin{aligned} T_2 &\leq \sum_{\kappa\in\mathcal{T}_h}\int_\kappa |\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{h,p}|)\nabla u_{h,p}||\nabla u_{h,p}^\perp| \, d\mathbf{x} \\ &\leq C\sum_{\kappa\in\mathcal{T}_h}\int_\kappa |\nabla e_{h,p}||\nabla u_{h,p}^\perp| \, d\mathbf{x} \\ &\leq C\|\nabla_h e_{h,p}\|_{L^2(\Omega)}\|\nabla u_{h,p}^\perp\|_{L^2(\Omega)}. \end{aligned}$$

Noting $u_{h,p}^\perp = u_{h,p} - u_{h,p}^c$, we deduce that

$$\begin{aligned} T_2 &\leq C\|e_{h,p}\|_{h,p}\|u_{h,p}^\perp\|_{h,p} \\ &\leq C\|e_{h,p}\|_{h,p}\|\sqrt{\sigma}\,[\![u_{h,p}]\!]\|_{L^2(\mathcal{F}_h)} \\ &\leq C\|e_{h,p}\|_{h,p}\left(\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa^2\right)^{\frac{1}{2}}. \qquad (18) \end{aligned}$$

Noting that $[\![e_{h,p}]\!] = [\![u - u_{h,p}]\!] = [\![u_{h,p}]\!]$, we deduce that

$$\begin{aligned} T_3 &= \sum_{F \in \mathcal{F}_h} \int_F \sigma\, [\![e_{h,p}]\!]^2 \mathrm{d}s \\ &\leq \|\sqrt{\sigma}\, [\![e_{h,p}]\!]\|_{L^2(\mathcal{F}_h)} \|\sqrt{\sigma}\, [\![u_{h,p}]\!]\|_{L^2(\mathcal{F}_h)} \\ &\leq \|e_{h,p}\|_{h,p} \left(\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 \right)^{\frac{1}{2}} . \end{aligned} \tag{19}$$

Substituting (17), (18), and (19) into (16), and dividing by $\|e_{h,p}\|_{h,p}$ gives the desired result.

Lemma (Two-grid Approximation)

The following bound holds (1–irregular meshes):

$$\|u - u_{2G}\|_{h,p}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2 + \Theta_\kappa^2\right),$$

Here, the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\eta_\kappa^2 = h_\kappa^2 p_\kappa^{-2} \|\Pi_{p_\kappa - 1} f + \nabla \cdot \{\mu(|\nabla u_{H,P}|)\nabla u_{2G}\|_{L^2(\kappa)}^2 \\ + h_\kappa p_\kappa^{-1} \|[\![\mu(|\nabla u_{H,P}|)\nabla u_{2G}]\!]\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + C_\sigma^2 h_\kappa^{-1} p_\kappa^3 \|[\![u_{2G}]\!]\|_{L^2(\partial\kappa)}^2$$

the local two-grid error indicators ξ_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\xi_\kappa^2 = \|(\mu(|\nabla u_{H,P}|) - \mu(|\nabla u_{2G}|))\nabla u_{2G}\|_{L^2(\kappa)}^2,$$

and the data oscillation terms Θ_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\Theta_\kappa^2 = p_\kappa^{-2} h_\kappa^2 \|f - \Pi_{p_\kappa - 1} f\|_{L^2(\kappa)}^2.$$

For simplicity, we assume that the mesh $\mathcal{T}_h$ is conforming.
Write $e_{h,p} = u - u_{2G} = e^c_{h,p} - u^\perp_{2G}$, where $e^c_{h,p} = u - u^c_{2G} \in H^1_0(\Omega)$ and $u^\perp_{2G} \in V^\perp_{h,p}$. Proceeding as before, gives

$$\|u - u_{2G}\|^2_{h,p} \leq C(T_1 + T_2 + T_3 + T_4), \tag{20}$$

where

$$T_1 = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa (\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{H,P}|)\nabla u_{2G}) \cdot \nabla e^c_{h,p} \, d\mathbf{x},$$

$$T_2 = -\sum_{\kappa \in \mathcal{T}_h} \int_\kappa (\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{2G}|)\nabla u_{2G}) \cdot \nabla u^\perp_{2G} \, d\mathbf{x},$$

$$T_3 = \sum_{F \in \mathcal{F}_h} \int_F \sigma \, [\![e_{h,p}]\!]^2 \mathrm{d}s,$$

$$T_4 = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \{\mu(|\nabla u_{H,P}|)\nabla u_{2G} - \mu(|\nabla u_{2G}|)\nabla u_{2G}\} \cdot \nabla e^c_{h,p} \mathrm{d}x.$$

Terms T_1, T_2, and T_3 may be bounded in a similar fashion:

$$|T_1| + |T_2| + |T_3| \leq C \left(\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 + \Theta_\kappa^2 \right)^{\frac{1}{2}} \|e_{h,p}\|_{h,p}. \tag{21}$$

We now consider the term T_4:

$$\begin{aligned} T_4 &= \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \{\mu(|\nabla u_{H,P}|)\nabla u_{2G} - \mu(|\nabla u_{2G}|)\nabla u_{2G}\} \cdot \nabla e_{h,p}^c \, d\mathbf{x} \\ &\leq \sum_{\kappa \in \mathcal{T}_h} \|(\mu(|\nabla u_{H,P}|) - \mu(|\nabla u_{2G}|))\nabla u_{2G}\|_{L^2(\kappa)} \left\|\nabla e_{h,p}^c\right\|_{L^2(\kappa)} \\ &\leq \left(\sum_{\kappa \in \mathcal{T}_h} \xi_\kappa^2 \right)^{\frac{1}{2}} \left\|e_{h,p}^c\right\|_{h,p}. \end{aligned}$$

Thereby,

$$T_4 \leq \left(\sum_{\kappa \in \mathcal{T}_h} \xi_\kappa^2 \right)^{\frac{1}{2}} \|e_{h,p}\|_{h,p}. \tag{22}$$

Inserting (21) and (22) into (20), and dividing by $\|e_{h,p}\|_{h,p}$ gives the desired result.

Non-Newtonian Fluid Problem

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and $\mathbf{f} \in L^2(\Omega)^d$, find $(\mathbf{u}, p)$ such that

$$\begin{aligned} -\nabla \cdot \{\mu(\mathbf{x}, |\mathbf{e}(\mathbf{u})|)\mathbf{e}(\mathbf{u})\} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma, \end{aligned}$$

where $\mathbf{e}(\mathbf{u})$ is the *symmetric $d \times d$ strain tensor* defined by

$$e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Assumption

1. *$\mu \in C(\bar{\Omega} \times [0, \infty))$ and*
2. *there exists positive constants m_μ and M_μ such that*

$$M_\mu(t-s) \le \mu(\mathbf{x}, t)t - \mu(\mathbf{x}, s)s \le M_\mu(t-s), \quad t \ge s \ge 0, \quad \mathbf{x} \in \bar{\Omega}.$$

- *hp*-DG finite element space:

$$\mathbf{V}_{h,p} = \{\mathbf{v} \in L^2(\Omega)^d : \mathbf{v}|_\kappa \in \mathcal{S}_{p_\kappa}(\kappa)^d, \forall \kappa \in \mathcal{T}_h\},$$
$$Q_{h,p} = \{q \in L^2_0(\Omega) : q|_\kappa \in \mathcal{S}_{p_\kappa - 1}(\kappa), \forall \kappa \in \mathcal{T}_h\}.$$

- Jump operator: $\underline{[\![\mathbf{v}]\!]} = \mathbf{v}^+ \otimes \mathbf{n}^+ + \mathbf{v}^- \otimes \mathbf{n}^-$

(Standard) Interior Penalty Method

Find $(\mathbf{u}_{h,p}, p_{h,p}) \in \mathbf{V}_{h,p} \times Q_{h,p}$ such that

$$\begin{aligned} A_{h,p}(\mathbf{u}_{h,p}; \mathbf{u}_{h,p}, \mathbf{v}_{h,p}) + B_{h,p}(\mathbf{v}_{h,p}, p_{h,p}) &= \ell_{h,p}(\mathbf{v}_{h,p}) \\ -B_{h,p}(\mathbf{u}_{h,p}, q_{h,p}) &= 0 \end{aligned}$$

for all $(\mathbf{v}_{h,p}, q_{h,p}) \in \mathbf{V}_{h,p} \times Q_{h,p}$.

$$
\begin{aligned}
A_{h,p}(\psi;\mathbf{u},\mathbf{v}) = &\sum_{\kappa\in\mathcal{T}_h}\int_\kappa \mu(|\mathbf{e}(\psi)|)\mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v})\,d\mathbf{x} \\
&- \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\mu(|\mathbf{e}(\psi)|)\mathbf{e}(\mathbf{u})\}\!\} : \underline{[\![\mathbf{v}]\!]}\,\mathrm{d}s \\
&+ \theta\sum_{F\in\mathcal{F}_h}\int_F \{\!\{\mu(h_F^{-1}|\underline{[\![\psi]\!]}|)\mathbf{e}(\mathbf{v})\}\!\} : \underline{[\![\mathbf{u}]\!]}\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}_h}\int_F \sigma\underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]}\,\mathrm{d}s, \\
B_{h,p}(\mathbf{v}.q) = &-\sum_{\kappa\in\mathcal{T}_h}\int_\kappa q\nabla\cdot\mathbf{v}\,d\mathbf{x} + \sum_{F\in\mathcal{F}_h}\int_F \{\!\{q\}\!\}[\![\mathbf{v}]\!]\,\mathrm{d}s, \\
\ell_{h,p}(v) = &\sum_{\kappa\in\mathcal{T}_h}\int_\kappa \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}.
\end{aligned}
$$

Two-Grid Approximation

1. Construct $\mathbf{V}_{H,P}$, $Q_{H,P}$, $\mathbf{V}_{h,p}$ and $Q_{h,p}$ such that
$$\mathbf{V}_{H,P} \subseteq \mathbf{V}_{h,p} \quad \text{and} \quad Q_{H,P} \subseteq Q_{h,p}$$
2. Compute $(\mathbf{u}_{H,P}, p_{H,P}) \in \mathbf{V}_{H,P} \times Q_{H,P}$ such that
$$\begin{aligned} A_{H,P}(\mathbf{u}_{H,P}; \mathbf{u}_{H,P}, \mathbf{v}_{H,P}) + B_{H,P}(\mathbf{v}_{H,P}, p_{H,P}) &= \ell_{H,P}(\mathbf{v}_{H,P}), \\ -B_{H,P}(\mathbf{u}_{H,P}, q_{H,P}) &= 0 \end{aligned}$$
for all $(\mathbf{v}_{H,P}, q_{H,P}) \in \mathbf{V}_{H,P} \times Q_{H,P}$.
3. Determine $(\mathbf{u}_{2G}, p_{2G}) \in \mathbf{V}_{h,p} \times Q_{h,p}$ such that
$$\begin{aligned} A_{h,p}(\mathbf{u}_{H,P}; \mathbf{u}_{2G}, \mathbf{v}_{h,p}) + B_{h,p}(\mathbf{v}_{h,p}, p_{2G}) &= \ell_{h,p}(\mathbf{v}_{h,p}), \\ -B_{H,P}(\mathbf{u}_{2G}, q_{h,p}) &= 0 \end{aligned}$$
for all $(\mathbf{v}_{h,p}, q_{h,p}) \in \mathbf{V}_{h,p} \times Q_{h,p}$.

DG Norms

$$\|\mathbf{v}\|_{h,p}^2 := \|\mathbf{e}_h(\mathbf{v})\|_{L^2(\Omega)}^2 + \sum_{F \in \mathcal{F}_h} \int_F \sigma |[\![\mathbf{v}]\!]|^2 \, \mathrm{d}s$$

$$\|(\mathbf{v}, q)\|_{\mathrm{DG}}^2 := \|\mathbf{v}\|_{h,p}^2 + \|q\|_{L^2(\Omega)}^2 .$$

Discrete inf-sup condition

$$\inf_{0 \neq q \in Q_{h,p}} \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}_{h,p}} \frac{B_{h,p}(\mathbf{v}, q)}{\|\mathbf{v}\|_{h,p} \|q\|_{L^2(\Omega)}} \geq c \left(\max_{\kappa \in \mathcal{T}_h} p_\kappa \right)^{-1} .$$

- for $p_\kappa \geq 2$, $\kappa \in \mathcal{T}_h$, or
- for $p \geq 1$ if $\mathcal{T}_h$ is conforming and $p_\kappa = p$ for all $\kappa \in \mathcal{T}_h$.

Schötzau, Schwab, & Toselli 2002

Lemma (Standard DGFEM)

The following bound holds:

$$\begin{aligned}
&\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\|_{\mathrm{DG}}^2 \\
&\quad \le C \max_{\kappa\in\mathcal{T}_h} p_\kappa^4 \sum_{\kappa\in\mathcal{T}_h} \left(\frac{h_\kappa^{2\min\{s_\kappa,p_\kappa\}}}{p_\kappa^{2s_\kappa-1}} \|\mathbf{u}\|_{H^{s_\kappa+1}(\kappa)}^2 + \frac{h_\kappa^{2\min\{s_\kappa,p_\kappa\}}}{p_\kappa^{2s_\kappa}} \|p\|_{H^{s_\kappa}(\kappa)}^2 \right)
\end{aligned}$$

Proof.

See Congreve, H., Süli, & Wihler (submitted). ■

Lemma (Two-grid DGFEM)

The following bound holds:

$$\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\|_{\mathrm{DG}}^2$$
$$\leq C \max_{\kappa\in\mathcal{T}_h} p_\kappa^4 \sum_{\kappa\in\mathcal{T}_h} \left(\frac{h_\kappa^{2\min\{s_\kappa,p_\kappa\}}}{p_\kappa^{2s_\kappa-1}} \|\mathbf{u}\|_{H^{s_\kappa+1}(\kappa)}^2 + \frac{h_\kappa^{2\min\{s_\kappa,p_\kappa\}}}{p_\kappa^{2s_\kappa}} \|p\|_{H^{s_\kappa}(\kappa)}^2 \right)$$
$$+ C \max_{\kappa\in\mathcal{T}_h} p_\kappa^4 \sum_{\kappa\in\mathcal{T}_H} \left(\frac{H_\kappa^{2\min\{S_\kappa,P_\kappa\}}}{P_\kappa^{2S_\kappa-1}} \|\mathbf{u}\|_{H^{S_\kappa+1}(\kappa)}^2 + \frac{H_\kappa^{2\min\{S_\kappa,P_\kappa\}}}{P_\kappa^{2S_\kappa}} \|p\|_{H^{S_\kappa}(\kappa)}^2 \right).$$

Proof.

See Congreve, H., & Wihler (in preparation). ■

Lemma (Standard Non-Newtonian DGFEM)

The following bound holds (1–irregular meshes):

$$\|(\mathbf{u} - \mathbf{u}_{h,p}, p - p_{h,p})\|_{DG}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2.$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$ as

$$\eta_\kappa^2 = \frac{h_\kappa^2}{p_\kappa^2} \|\mathbf{f} + \nabla \cdot \{\mu(|\mathbf{e}(\mathbf{u}_{h,p})|)\mathbf{e}(\mathbf{u}_{h,p})\} - \nabla p_{h,p}\|_{L^2(\kappa)}^2 + \|\nabla \cdot \mathbf{u}_{h,p}\|_{L^2(\kappa)}^2 + \frac{h_\kappa}{p_\kappa} \|[\![p_{h,p}]\!] - [\![\mu(|\mathbf{e}(\mathbf{u}_{h,p})|)\mathbf{e}(\mathbf{u}_{h,p})]\!]\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + C_\sigma^2 \frac{p_\kappa^3}{h_\kappa} \left\| \underline{[\![\mathbf{u}_{h,p}]\!]} \right\|_{L^2(\partial\kappa)}^2$$

Proof.

See Congreve, H., Süli & Wihler (submitted). ∎

Lemma (Two-Grid Non-Newtonian Approximation)

The following bound holds (1–irregular meshes):

$$\|(\mathbf{u}-\mathbf{u}_{2G}, p - p_{2G})\|_{DG}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2\right).$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$ as

$$\begin{aligned}\eta_\kappa^2 = {} & \frac{h_\kappa^2}{p_\kappa^2}\|\mathbf{f} + \nabla \cdot \{\mu(|\mathbf{e}(\mathbf{u}_{H,P})|)\mathbf{e}(\mathbf{u}_{2G})\} - \nabla p_{2G}\|_{L^2(\kappa)}^2 + \|\nabla \cdot \mathbf{u}_{2G}\|_{L^2(\kappa)}^2 \\ & + \frac{h_\kappa}{p_\kappa}\|[\![p_{2G}]\!] - [\![\mu(|\mathbf{e}(\mathbf{u}_{H,P})|)\mathbf{e}(\mathbf{u}_{2G})]\!]\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + C_\sigma^2 \frac{p_\kappa^3}{h_\kappa}\left\|\underline{[\![\mathbf{u}_{2G}]\!]}\right\|_{L^2(\partial\kappa)}^2\end{aligned}$$

and the local two-grid error indicators are defined, for all $\kappa \in \mathcal{T}_h$ as

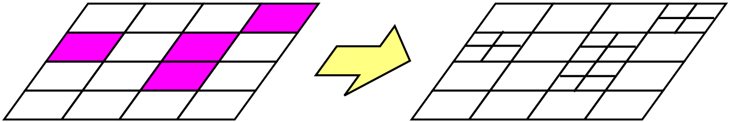
$$\xi_\kappa^2 = \|(\mu(|\nabla \mathbf{u}_{H,K}|) - \mu(|\nabla \mathbf{u}_{2G}|))\nabla \mathbf{u}_{2G}\|_{L^2(\kappa)}^2 .$$

Two-Grid Adaptivity

1. Construct coarse and fine FE spaces $V_{H,P}$ and $V_{h,p}$, respectively, such that $V_{H,P} \subseteq V_{h,p}$.
2. Compute the coarse grid approximation $u_{H,P} \in V_{H,P}$ and two grid (fine) solution $u_{2G} \in V_{h,p}$.
3. Evaluate the elemental error indicators η_κ and ξ_κ.
4. Perform h–/hp–mesh refinement to design new fine space $V_{h,p}$.
5. If $\lambda \xi_\kappa^2 \geq \eta_\kappa^2$, where $0 \leq \lambda < \infty$ is a steering parameter, then mark for refinement the coarse element $\kappa_H \in \mathcal{T}_H$ where $\kappa \subseteq \kappa_H$.
6. Perform h–/hp–mesh refinement to design new coarse space $V_{H,P}$.
7. Perform mesh smoothing to ensure $V_{H,P} \subseteq V_{h,p}$.
8. Goto 2.

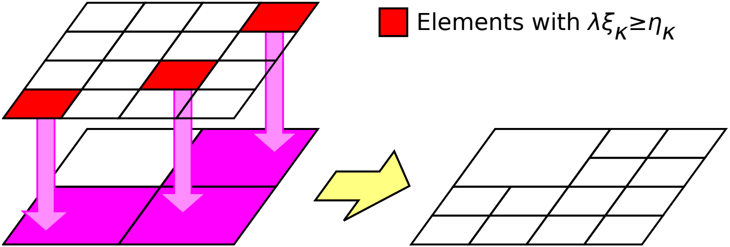
Mesh Adaptation Strategy (h–Refinement)

- Perform standard refinement on the fine mesh based on η_κ



Elements with largest η_K

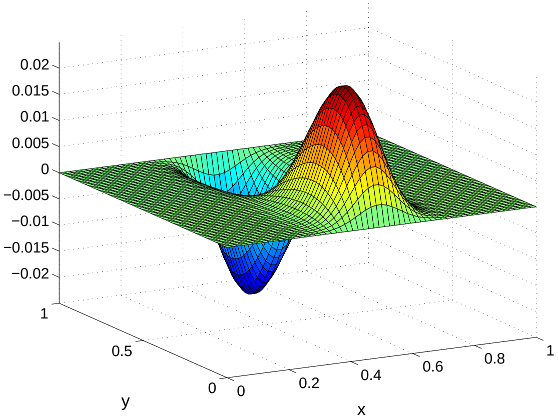
- For each fine element $\kappa \in \mathcal{T}_h$ where $\lambda\xi_\kappa \geq \eta_\kappa$, $\lambda \geq 0$ refine the coarse element $\kappa_H \in \mathcal{T}_H$ where $\kappa \subseteq \kappa_H$.

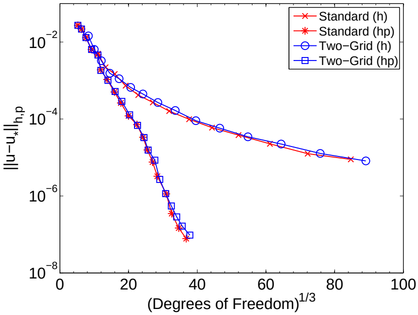


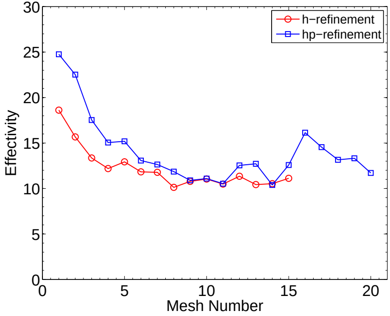
Elements with $\lambda\xi_K \geq \eta_K$

We let $\Omega = (0,1)^2$, $\mu(\mathbf{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|}$, and select f so that

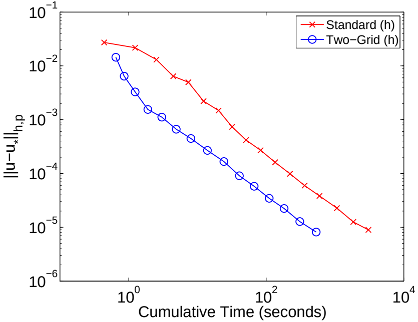
$$u(x,y) = x(1-x)y(1-y)(1-2y)\mathrm{e}^{-\sigma(2x-1)^2}, \quad \text{where } \sigma = 20.$$

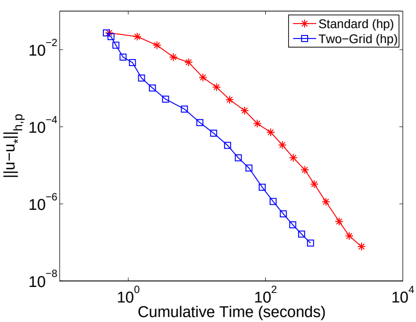



Standard (h)
Standard (hp)
Two−Grid (h)
Two−Grid (hp)
$\|u-u_*\|_{h,p}$
10^{-2}
10^{-4}
10^{-6}
10^{-8}
0
20
40
60
80
100
(Degrees of Freedom)$^{1/3}$

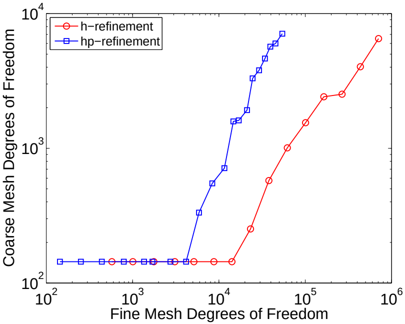

h−refinement
hp−refinement
Effectivity
30
25
20
15
10
5
0
0
5
10
15
20
Mesh Number

Quasilinear PDE: Smooth Solution

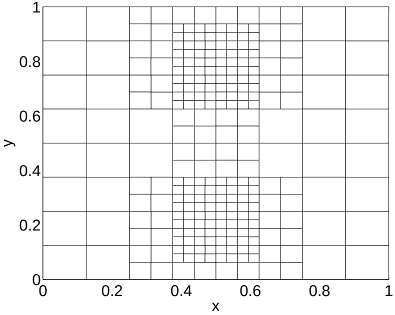




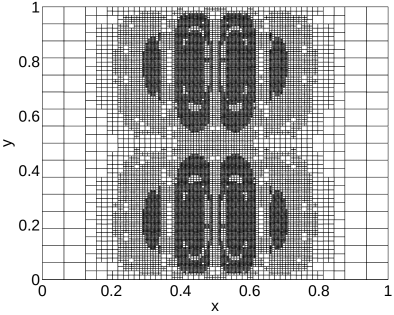
Quasilinear PDE: Smooth Solution



h–Mesh after 11 adaptive refinements

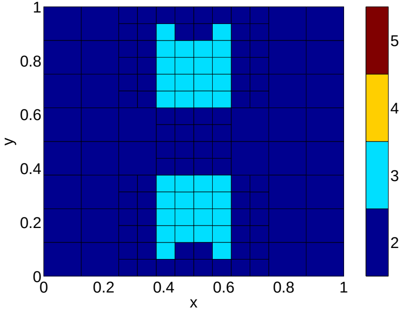


Coarse Mesh

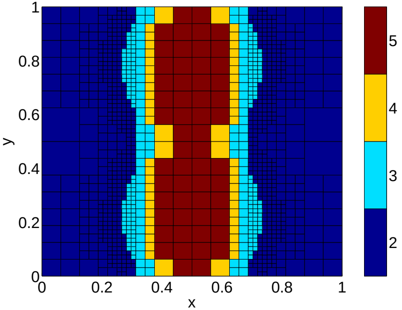


Fine Mesh

hp–Mesh after 11 adaptive refinements



Coarse Mesh



Fine Mesh

Let Ω be the Fichera corner $(-1,1)^3 \setminus [0,1)^3$,

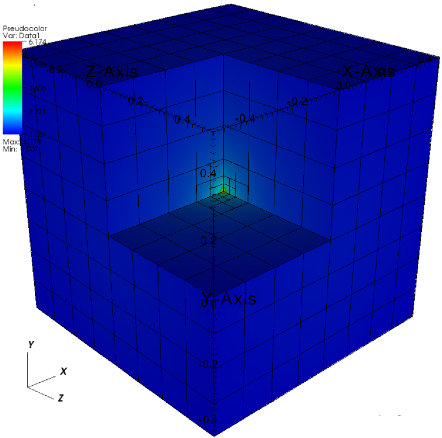
$$\mu(\mathbf{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|},$$

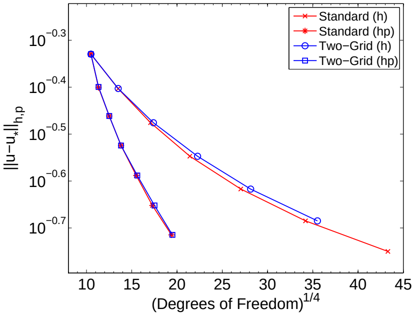
and select f so that

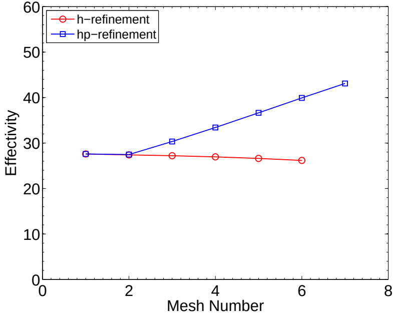
$$u(\mathbf{x}) = (x^2+y^2+z^2)^{q/2}, \quad q \in \mathbb{R};$$

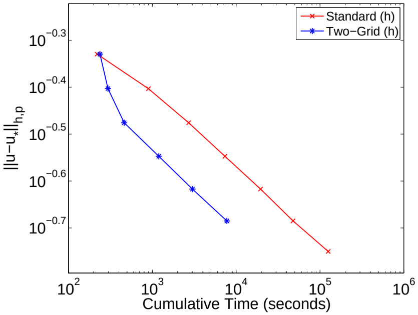
for $q > -1/2$, $u \in H^1(\Omega)$. Here, we select $q = -1/4$.

Beilina, Korotov, Křížek 2005

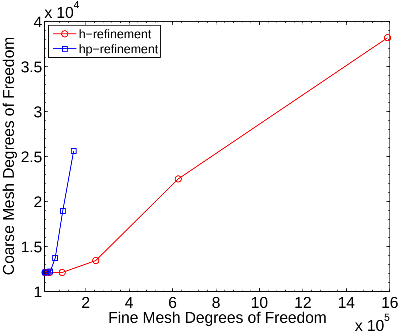



Standard (h)
Standard (hp)
Two−Grid (h)
Two−Grid (hp)
$10^{-0.3}$
$10^{-0.4}$
$10^{-0.5}$
$10^{-0.6}$
$10^{-0.7}$
$\|u-u_*\|_{h,p}$
10
15
20
25
30
35
40
45
(Degrees of Freedom)$^{1/4}$

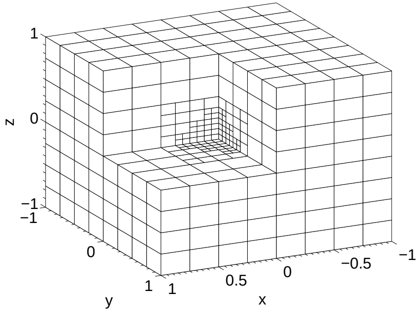

h−refinement
hp−refinement
60
50
40
30
20
10
0
Effectivity
0
2
4
6
8
Mesh Number


Standard (h)
Two−Grid (h)
$10^{-0.3}$
$10^{-0.4}$
$10^{-0.5}$
$10^{-0.6}$
$10^{-0.7}$
$||u-u_*||_{h,p}$
10^2 10^3 10^4 10^5 10^6
Cumulative Time (seconds)
Standard (hp)
Two−Grid (hp)
$10^{-0.3}$
$10^{-0.4}$
$10^{-0.5}$
$10^{-0.6}$
$10^{-0.7}$
$||u-u_*||_{h,p}$
10^2 10^3 10^4 10^5 10^6
Cumulative Time (seconds)

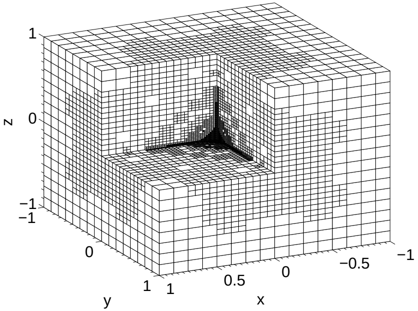
Quasilinear PDE: Singular Solution



h–Mesh after 5 adaptive refinements

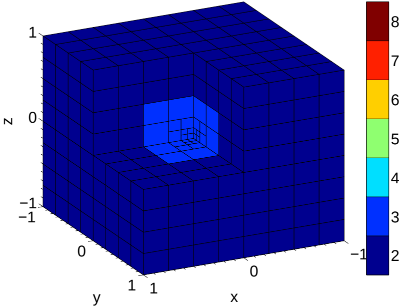


Coarse Mesh

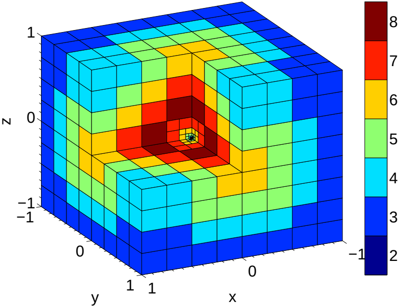


Fine Mesh

hp–Mesh after 6 adaptive refinements



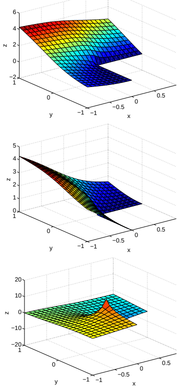
Coarse Mesh



Fine Mesh

Non-Newtonian Fluid: Singular Solution

Let $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$, $\mu = 1 + \mathrm{e}^{-|\mathbf{e}(\mathbf{u})|}$ and select $\mathbf{f}$ so that



$$\mathbf{u}(x,y) = r^{\lambda} \begin{pmatrix} (1+\lambda)\sin(\varphi)\Psi(\varphi) + \cos(\varphi)\Psi'(\varphi) \\ \sin(\varphi)\Psi'(\varphi) - (1+\lambda)\cos(\varphi)\Psi(\varphi) \end{pmatrix},$$

$$p(x,y) = -r^{\lambda-1}\left\{(1+\lambda)^2\Psi'(\varphi) + \Psi'''(\varphi)\right\}/(1-\lambda),$$

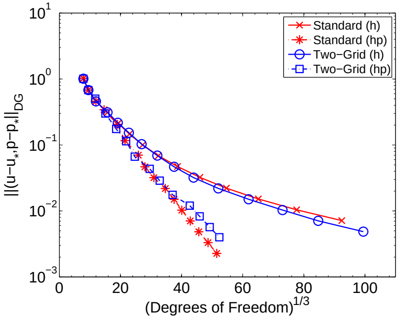
where (r,φ) denotes polar coordinates,

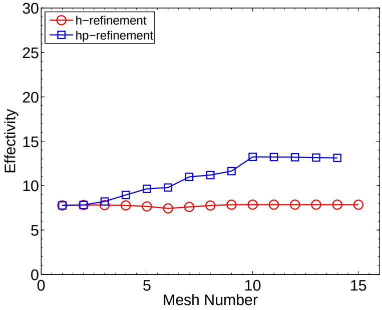
$$\Psi(\varphi) = \frac{\sin((1+\lambda)\varphi)\cos(\lambda\omega)}{1+\lambda} - \cos((1+\lambda)\varphi) - \frac{\sin((1-\lambda)\varphi)\cos(\lambda\omega)}{1-\lambda} + \cos((1-\lambda)\varphi),$$

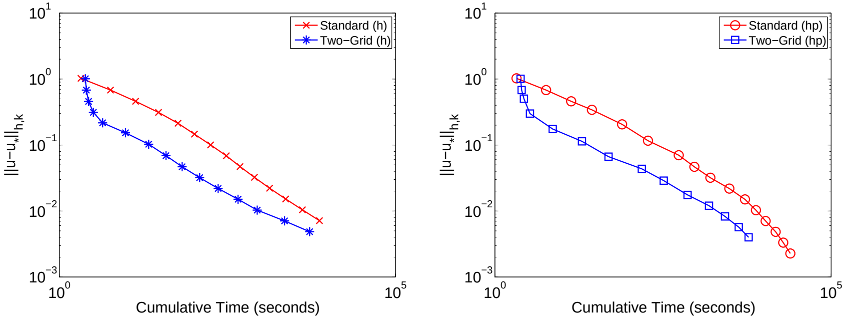
and $\omega = \frac{3\pi}{2}$. Here, the exponent λ is the smallest positive solution of

$$\sin(\lambda\omega) + \lambda\sin(\omega) = 0;$$

thereby, $\lambda \approx 0.54448373678$. Note that $\mathbf{u} \notin H^2(\Omega)^2$ and $p \notin H^1(\Omega)$.


Standard (h)
Standard (hp)
Two−Grid (h)
Two−Grid (hp)
$\|(u-u_*, p-p_*)\|_{DG}$
(Degrees of Freedom)$^{1/3}$
0
20
40
60
80
100
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}


h−refinement
hp−refinement
Effectivity
Mesh Number
0
5
10
15
20
25
30


Standard (h)
Two−Grid (h)
Standard (hp)
Two−Grid (hp)
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{0}
10^{5}
$||u-u_*||_{h,k}$
Cumulative Time (seconds)

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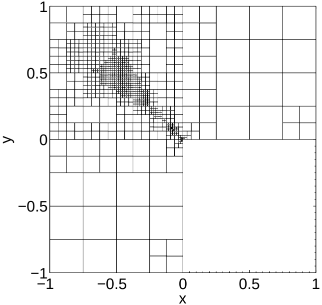
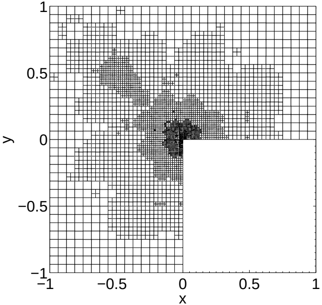
h–Mesh after 11 adaptive refinements

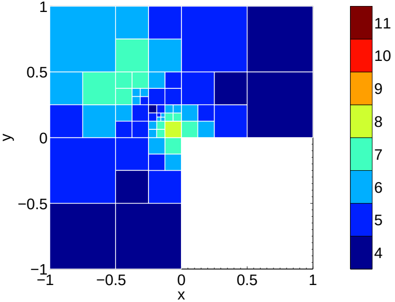


Coarse Mesh

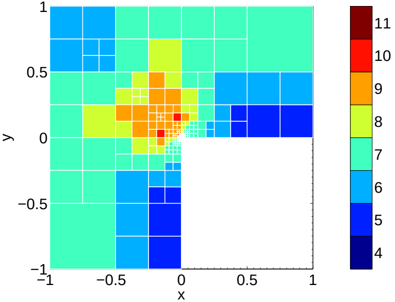


Fine Mesh

hp–Mesh after 11 adaptive refinements



Coarse Mesh



Fine Mesh

- For simplicity of notation, for $u, v \in V_{h,p}$, we write

$$A_{h,p}(u, v) = A_{h,p}(u; u, v).$$

- Gâteaux derivative of $u \to A_{h,p}(u, \cdot)$:

$$A'_{h,p}[u](\phi, \cdot) = \lim_{t \to 0} \frac{A_{h,p}(u + t\phi, \cdot) - A_{h,p}(u, \cdot)}{t}.$$

Two-grid Approximation (Incomplete Newton Method)

1. Construct coarse and fine FE spaces $V_{H,P}$ and $V_{h,p}$, respectively, such that $V_{H,P} \subseteq V_{h,p}$.
2. Compute the coarse grid approximation $u_{H,P} \in V_{H,P}$ such that

$$A_{H,P}(u_{H,P}, v_{H,P}) = \ell_{H,P}(v_{H,P})$$

for all $v_{H,P} \in V_{H,P}$.

3. Determine the fine grid solution $u_{2G} \in V_{h,p}$ such that

$$\begin{aligned} A'_{h,p}[u_{H,P}](u_{2G}, v_{h,p}) &= A'_{h,p}[u_{H,P}](u_{H,P}, v_{h,p}) \\ &\quad + \ell_{h,p}(v_{h,p}) - A_{h,p}(u_{H,P}, v_{h,p}) \end{aligned}$$

for all $v_{h,p} \in V_{h,p}$.

Xu 1996, Axelsson & Layton 1996

- We have:

$$\begin{aligned}
&A_{h,p}(u_{h,p}, u_{h,p} - u_{2G}) \\
&\qquad = A_{h,p}(u_{H,P}, u_{h,p} - u_{2G}) + A'_{h,p}[u_{H,P}](u_{h,p} - u_{H,P}, u_{h,p} - u_{2G}) \\
&\qquad\quad + \mathcal{Q}(u_{H,P}, u_{h,p}, u_{2G}),
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{Q}(u_{H,P}, u_{h,p}, u_{2G}) \\
&\qquad = \int_0^1 A''_{h,p}[\xi(t)](u_{h,p} - u_{H,P}, u_{h,p} - u_{H,P}, u_{h,p} - u_{2G})(1-t)\mathrm{d}t,
\end{aligned}$$

and $\xi(t) = u_{H,P} + t(u_{h,p} - u_{H,P})$.

Error Bound

Assumption: there exists $K(H,P,h,p)$ such that

$$|\mathcal{Q}(u_{H,P},u_{h,p},u_{2G})| \leq K(H,P,h,p)\|u_{h,p}-u_{H,P}\|_{h,p}^2\|u_{h,p}-u_{2G}\|_{h,p},$$

then the following error bound holds:

$$\|u-u_{2G}\|_{h,p} \leq \|u-u_{h,p}\|_{h,p} + CK(H,P,h,p)\|u-u_{H,P}\|_{h,p}^2.$$

Bound for $\mathcal{Q}$

$$\begin{aligned}
&|\mathcal{Q}(u_{H,P},u_{h,p},u_{2G})| \\
&\leq C(1+\|\nabla u_{h,p}\|_{L^\infty(\Omega)}\overbrace{\|\nabla u_{h,p}\|_{L^\infty(\Omega)}}^{\leq p^{3/2}} + \|\nabla u_{H,P}\|_{L^\infty(\Omega)}\overbrace{\|\nabla u_{H,P}\|_{L^\infty(\Omega)}}^{\leq P^{3/2}}) \\
&\quad \times \|\nabla(u_{h,p}-u_{H,P})\|_{L^4(\Omega)}^2 \underbrace{\|\nabla(u_{h,p}-u_{H,P})\|_{L^4(\Omega)}^2}_{\leq p^2/h\,\|u_{h,p}-u_{H,P}\|_{h,p}^2}\|u_{h,p}-u_{2G}\|_{h,p} \\
&\leq C\frac{p^{7/2}}{h}\|u_{h,p}-u_{H,P}\|_{h,p}^2\|u_{h,p}-u_{2G}\|_{h,p}.
\end{aligned}$$

Final Error Bound

$$\|u - u_{2G}\|_{h,p} \leq C\frac{h^{s-1}}{p^{k-3/2}}\|u\|_{H^k(\Omega)} + C\frac{p^{7/2}}{h}\frac{H^{2S-2}}{P^{2k-3}}\|u\|^2_{H^k(\Omega)},$$

with $1 \leq s \leq \min\{p+1,k\}$, $p \geq 1$ and $1 \leq S \leq \min\{P+1,k\}$, $P \geq 1$.

- Bound is pessimistic with respect to the polynomial degree.
- Here, we require $\mu \in \mathrm{C}^2(\bar{\Omega} \times [0,\infty))$.

Lemma (Two-grid Approximation)

The following bound holds (1–irregular meshes):

$$\|u - u_{2G}\|_{h,p}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2 + \Theta_\kappa^2\right),$$

where, for all $\kappa \in \mathcal{T}_h$, we have

$$\begin{aligned}
\eta_\kappa^2 &= h_\kappa^2 p_\kappa^{-2} \|\Pi_{p_\kappa - 1} f + \nabla \cdot \{\mu(|\nabla u_{H,P}|)\nabla u_{2G}\|_{L^2(\kappa)}^2 \\
&\quad + h_\kappa p_\kappa^{-1} \|[\![\mu(|\nabla u_{H,P}|)\nabla u_{2G}]\!]\|_{L^2(\partial\kappa \setminus \Gamma)}^2 + C_\sigma^2 h_\kappa^{-1} p_\kappa^3 \|[\![u_{2G}]\!]\|_{L^2(\partial\kappa)}^2, \\
\xi_\kappa^2 &= \|(\mu(|\nabla u_{H,P}|) - \mu(|\nabla u_{2G}|))\nabla u_{2G}\|_{L^2(\kappa)}^2 \\
&\quad + \|(\mu'[|\nabla u_{H,P}|](|\nabla u_{2G}|) - \mu'[|\nabla u_{H,P}|](|\nabla u_{H,P}|))\, \nabla u_{H,P}\|_{L^2(\kappa)}^2 \\
&\quad + h_\kappa p_\kappa^{-1} \|(\mu'[|\nabla u_{H,P}|](|\nabla u_{2G}|) - \mu'[|\nabla u_{H,P}|](|\nabla u_{H,P}|))\, \nabla u_{H,P}\|_{L^2(\partial\kappa)}^2, \\
\Theta_\kappa^2 &= p_\kappa^{-2} h_\kappa^2 \|f - \Pi_{p_\kappa - 1} f\|_{L^2(\kappa)}^2.
\end{aligned}$$

1. P. H., J. Robson, and E. Süli. Discontinuous Galerkin Finite Element Approximation of Quasilinear Elliptic Boundary Value Problems I: The Scalar Case. IMA Journal of Numerical Analysis 25:726-749, 2005.
2. P. H., E. Süli, and T.P. Wihler. A Posteriori Error Analysis of hp-Version Discontinuous Galerkin Finite Element Methods for Second-Order Quasilinear Elliptic Problems. IMA Journal of Numerical Analysis 28:245-273, 2008.
3. S. Congreve, P. H., and T.P. Wihler. Two-Grid hp-Version Discontinuous Galerkin Finite Element Methods for Second-Order Quasilinear Elliptic PDEs. Submitted for publication.
4. S. Congreve, P. H., E. Süli, and T.P. Wihler. Discontinuous Galerkin finite element approximation of quasilinear elliptic boundary value problems II: Strongly monotone quasi-Newtonian flows. Submitted for publication.
5. S. Congreve, P. H., and T.P. Wihler. Two-Grid *hp*–Version Discontinuous Galerkin Finite Element Method for Quasi-Newtonian Fluid Flows. In preparation

Given a user–defined tolerance $\mathtt{TOL} > 0$, can we efficiently design $V_{h,p}$ such that

- Norm control

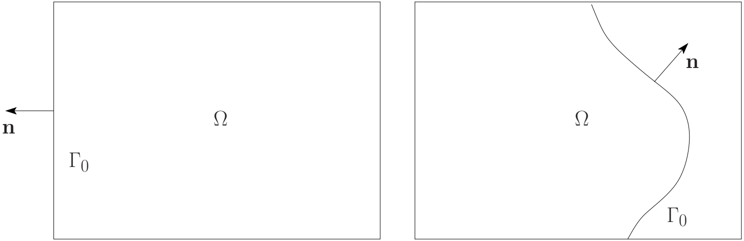
$$\|u - u_{h,p}\| \leq \mathtt{TOL};$$

- Functional control

$$|J(u) - J(u_{h,p})| \leq \mathtt{TOL}.$$

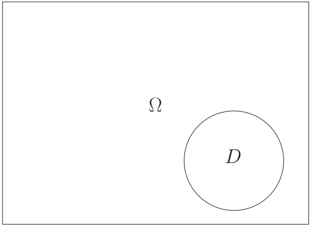
- Flux through the boundary

$$J(u) = \int_{\Gamma_0} (\mathcal{F}(u) \cdot \mathbf{n}) \psi(s) \mathrm{d}s.$$



- Mean value over subdomain

$$J(u) = \int_D u(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mathbf{x}$$



- Other examples:

Fluid dynamics:	Drag and lift coefficients.
Electromagnetics:	Far field pattern.
Elasticity theory:	Stress intensity factor.
Other examples:	Point value, critical parameters/eigenvalues, etc.

Structure of the proofs:

- Derivation of an error representation formula using duality;
- Use of Galerkin orthogonality;

$$\boxed{|J(u) - J(u_{h,p})| \leq \sum_{\kappa \in \mathcal{T}_h} |(R(u_{h,p}), z - z_{h,p})_\kappa|}$$

$\Rightarrow$ (Weighted) Type I Error Bound

Becker & Rannacher 1996, Rannacher *et al.* 1996 $\rightarrow$

- Local interpolation error estimates for the dual solution;
- Stability estimates for the dual problem.

$$\boxed{|J(u) - J(u_{h,p})| \leq C_{\mathrm{int}}\, C_{\mathrm{stab}} \|(h/p)^s R(u_{h,p})\|, \quad s > 0}$$

$\Rightarrow$ (Unweighted) Type II Error Bound

Johnson *et al.* 1995 $\rightarrow$

Consider the following problem

$$\mathcal{L}u \equiv \nabla \cdot (\mathbf{b}u) = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial_-\Omega,$$

where

$$\begin{aligned} \partial_-\Omega &= \{\mathbf{x} \in \partial\Omega : \mathbf{b} \cdot \mathbf{n} < 0\}, \\ \partial_+\Omega &= \{\mathbf{x} \in \partial\Omega : \mathbf{b} \cdot \mathbf{n} > 0\}. \end{aligned}$$

- Finite Element Space (h–version)

$$V_h = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{S}_p \quad \forall \kappa \in \mathcal{T}_h\},$$

where $\mathcal{T}_h$ is the mesh and $p \geq 0$ is the polynomial order.

DGFEM

Find $u_h \in V_h$ such that

$$\sum_{\kappa\in\mathcal{T}_h}\left\{-\int_\kappa (\mathbf{b}u_h)\cdot\nabla v_h \mathrm{d}x + \int_{\partial\kappa}\mathcal{H}(u_h^+, u_h^-, \mathbf{n}_\kappa)v_h^+ \,\mathrm{d}s\right\} = \sum_{\kappa\in\mathcal{T}_h}\left\{\int_\kappa f\, v_h \mathrm{d}x\right\}$$

for all $v_h \in V_h$.

Selecting

$$\mathcal{H}(u_h^+, u_h^-, \mathbf{n}_\kappa)|_{\partial\kappa} = \mathbf{b}\cdot\mathbf{n}_\kappa \lim_{s\to 0^+} u_h(\mathbf{x} - s\mathbf{b}) \quad \text{for } \kappa \in \mathcal{T}_h,$$

we may rewrite the DGFEM in the following manner:

DGFEM

Find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in V_h,$$

where

$$\begin{aligned} A_h(u_h, v_h) &= \sum_{\kappa\in\mathcal{T}_h} \Big\{ -\int_\kappa (\mathbf{b}u_h)\cdot\nabla v_h \mathrm{d}\mathbf{x} \\ &\quad + \int_{\partial_+\kappa} (\mathbf{b}\cdot\mathbf{n}_\kappa)\, u_h^+ v_h^+ \mathrm{d}s + \int_{\partial_-\kappa\setminus\partial\Omega} (\mathbf{b}\cdot\mathbf{n}_\kappa)\, u_h^- v_h^+ \mathrm{d}s \Big\}, \\ \ell(v_h) &= \sum_{\kappa\in\mathcal{T}_h} \Big\{ \int_\kappa f v_h \mathrm{d}\mathbf{x} - \int_{\partial_-\kappa\cap\partial\Omega} (\mathbf{b}\cdot\mathbf{n}_\kappa)\, g\, v_h^+ \mathrm{d}s \Big\}. \end{aligned}$$

Dual problem: find z such that

$$A_h(v,z) = J(v) \quad \forall v,$$

where $J(\cdot)$ is a given *linear functional*.

Examples:

1. Outflow flux: $J(u) = \int_{\partial_+\Omega} (\mathbf{b}\cdot\mathbf{n}u)\psi \,\mathrm{d}s$, $\psi \in L^2(\partial_+\Omega)$.

$$\begin{aligned} \mathcal{L}^* z \equiv -\mathbf{b}\cdot\nabla z &= 0 \quad \text{in } \Omega\,, \\ z &= \psi \quad \text{on } \partial_+\Omega\,. \end{aligned}$$

2. Meanflow functional: $J(u) = \int_\Omega u\psi \,\mathrm{d}\mathbf{x}$, $\psi \in L^2(\Omega)$.

$$\begin{aligned} \mathcal{L}^* z \equiv -\mathbf{b}\cdot\nabla z &= \psi \quad \text{in } \Omega\,, \\ z &= 0 \quad \text{on } \partial_+\Omega\,. \end{aligned}$$

Dual Problem $\neq$ Adjoint Problem

Examples:

1. Artificial Viscosity

   Discretization: find $u_h \in V_{h,p}$ such that

$$\mathcal{A}_h(u_h, v_h) \equiv A_h(u_h, v_h) + \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \varepsilon \nabla u_h \cdot \nabla v_h \mathrm{d}\mathbf{x} = \ell(v_h) \quad \forall v_h \in V_{h,p}.$$

   Dual Problem: find z such that

$$\mathcal{A}(v, z) = J(v) \quad \forall v,$$

$$\begin{aligned} -\mathbf{b} \cdot \nabla z - \nabla \cdot (\varepsilon \nabla z) &= \psi \ \text{ in } \Omega\,, \\ z + \varepsilon \mathbf{n} \cdot \nabla z = 0 \ \text{ on } \partial_+ \Omega, \quad \varepsilon \mathbf{n} \cdot \nabla z &= 0 \ \text{ on } \partial_- \Omega. \end{aligned}$$

2. SUPG Stabilization:

Discretization: find $u_h \in V_{h,p}$ such that

$$\mathcal{A}_h(u_h, v_h) \equiv A_h(u_h, v_h) + \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \delta \mathcal{L} u_h \mathcal{L} v_h \mathrm{d}\mathbf{x} = \ell(v_h) + \sum_{\kappa \in \mathcal{T}_h} \int_\kappa f \delta \mathcal{L} v_h \mathrm{d}\mathbf{x} \quad \forall v_h.$$

Dual Problem: find z such that

$$\mathcal{A}(v, z) = J(v) \quad \forall v,$$

$$\begin{aligned} \mathcal{L}^*(z + \delta \mathcal{L} z) &= \psi \ \text{ in } \Omega\,, \\ z + \delta \mathcal{L} z = 0 \ \text{ on } \partial_+\Omega, \quad \mathcal{L} z &= 0 \ \text{ on } \partial_-\Omega. \end{aligned}$$

$$
\begin{aligned}
J(u) - J(u_h) &= J(u - u_h) && \text{[Linearity]} \\
&= A_h(u - u_h, z) && \text{[Dual Problem]} \\
&= A_h(u - u_h, z - z_h) && \text{[Galerkin Orthogonality]} \\
&= \ell(z - z_h) - A_h(u_h, z - z_h) && \text{[Consistency]} \\
&= \sum_{\kappa \in \mathcal{T}_h} \Big\{ \int_\kappa (f - \nabla \cdot (\mathbf{b} u))(z - z_h) \mathrm{d}\mathbf{x} \\
&\qquad + \int_{\partial_- \kappa \setminus \partial\Omega} \mathbf{b} \cdot \mathbf{n}_\kappa (u^+ - u^-)(z - z_h)^+ \mathrm{d}s \\
&\qquad + \int_{\partial_- \kappa \cap \partial\Omega} \mathbf{b} \cdot \mathbf{n}_\kappa (u^+ - g)(z - z_h)^+ \mathrm{d}s \Big\} \\
&\equiv \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa .
\end{aligned}
$$

Proposition

Assuming the dual problem is well–posed, the following result holds:

$$|J(u) - J(u_h)| \leq \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^{(\mathrm{I})},$$

where $\eta_\kappa^{(\mathrm{I})} = |\eta_\kappa|$ *and*

$$\begin{aligned} \eta_\kappa \;\; = \;\; & \int_\kappa (f - \nabla \cdot (\mathbf{b}u))(z - z_h)\mathrm{d}\mathbf{x} \\ & + \int_{\partial_-\kappa \backslash \partial\Omega} \mathbf{b} \cdot \mathbf{n}_\kappa (u^+ - u^-)(z - z_h)^+ \mathrm{d}s, \\ & + \int_{\partial_-\kappa \cap \partial\Omega} \mathbf{b} \cdot \mathbf{n}_\kappa (u^+ - g)(z - z_h)^+ \mathrm{d}s. \end{aligned}$$

Lemma

Given $\kappa \in \mathcal{T}_h$, suppose that $v|_\kappa \in H^{k_\kappa}(\kappa)$, $0 \leq k_\kappa \leq p+1$. Then, there exists $\mathcal{I}v$ in the finite element space V_h, such that

$$\|v - \mathcal{I}v\|_{L^2(\kappa)} + h_\kappa^{1/2}\|v - \mathcal{I}v\|_{L^2(\partial\kappa)} \leq C_{\mathcal{I}} h_\kappa^{k_\kappa} \|v\|_{H^{k_\kappa}(\kappa)},$$

where $C_{\mathcal{I}}$ is a positive constant, independent of the mesh size h.

Proof.

See Babuška & Suri 1987, for example. ■

Type I *A Posteriori* Error Bound

- Set $z_h = \mathcal{I}z$ and apply the above result.
- Thereby, we have that

$$\begin{aligned}
|J(u) - J(u_h)| \quad \leq \quad & C_{\mathcal{I}} \sum_{\kappa \in \mathcal{T}_h} \Big(h_\kappa^{k_\kappa} \|(f - \nabla \cdot (\mathbf{b}u))\|_{L^2(\kappa)} \\
& + h_\kappa^{k_\kappa - 1/2} \|\mathbf{b} \cdot \mathbf{n}_\kappa (u^+ - u^-)\|_{L^2(\partial_- \kappa \setminus \partial\Omega)} \\
& + h_\kappa^{k_\kappa - 1/2} \|\mathbf{b} \cdot \mathbf{n}_\kappa (u^+ - u^-)\|_{L^2(\partial_- \kappa \cap \partial\Omega)} \Big) \|z\|_{H^{k_\kappa}(\kappa)}.
\end{aligned}$$

Type II *A Posteriori* Error Bound

Theorem

Assuming

$$\|z\|_{H^s(\Omega)} \leq C_{\mathrm{stab}}\,, \quad 1 \leq s \leq p+1\,,$$

we have that

$$|J(u) - J(u_h)| \leq C_{\mathcal{I}} C_{\mathrm{stab}} \left(\sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^{(\mathrm{II})}\right)^2 \right)^{1/2},$$

where

$$\begin{aligned} \eta_\kappa^{(\mathrm{II})} \quad = \quad & h_\kappa^s \|f - \nabla \cdot (\mathbf{b}u)\|_{L^2(\kappa)} + h_\kappa^{s-1/2} \|\mathbf{b} \cdot \mathbf{n}_\kappa (u^+ - u^-)\|_{L^2(\partial_-\kappa \setminus \partial\Omega)} \\ & + h_\kappa^{s-1/2} \|\mathbf{b} \cdot \mathbf{n}_\kappa (u^+ - u^-)\|_{L^2(\partial_-\kappa \cap \partial\Omega)}. \end{aligned}$$

- Conservation Law: Given $\Omega \subset \mathbb{R}^n$, find $\mathbf{u} : \Omega \to \mathbb{R}^m$, such that

$$\mathrm{div}\mathcal{F}(\mathbf{u}) = \mathbf{f} \quad \text{in } \Omega,$$

where $\mathcal{F}(\mathbf{u}) = (\mathcal{F}_1(\mathbf{u}), \ldots, \mathcal{F}_n(\mathbf{u}))$ and $\mathbf{f} \in \left[L^2(\Omega)\right]^m$.

- Boundary conditions: e.g., at inflow/outflow

$$B^-(\mathbf{u}, \mathbf{n})(\mathbf{u} - \mathbf{g}) = \mathbf{0} \quad \text{on } \partial\Omega$$

where

$$\begin{aligned} B(\mathbf{u}, \mathbf{n}) &= \sum_{i=1}^{n} \mathbf{n}_i \nabla_{\mathbf{u}} \mathcal{F}_i(\mathbf{u}) \\ B^{\pm}(\mathbf{u}, \mathbf{n}) &= X(\mathbf{u}, \mathbf{n})^{-1} \Lambda_{\pm} X(\mathbf{u}, \mathbf{n}). \end{aligned}$$

- Linear (scalar) advection problem

$$\nabla \cdot (\mathbf{b}u) = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial_-\Omega,$$

- Burgers' equation

$$\begin{aligned} u_t + \left((1/2)\, u^2\right)_x &= 0, \\ u(x,0) &= u_0(x)\,. \end{aligned}$$

- Compressible Euler Equations ($\Omega \subset \mathbb{R}^2$)

$$\begin{aligned} \mathbf{u} &= [\rho, \rho v_1, \rho v_2, \rho E]^\top, \\ \mathcal{F}_i(\mathbf{u}) &= [\rho v_i, \rho v_1 v_i + \delta_{1i} p, \rho v_2 v_i + \delta_{2i} p, \rho H v_i]^\top, \quad i = 1, 2, \end{aligned}$$

ρ: density, (v_1, v_2): velocity, E: energy, p: pressure, H: enthalpy.

- $\mathcal{T}_h = \{\kappa\}$ is a non-degenerate mesh;
- Finite element space

$$V_{h,p} = \{\mathbf{v} \in \left[L^2(\Omega)\right]^m : \mathbf{v}|_\kappa \in [\mathcal{S}_p]^m \quad \forall \kappa \in \mathcal{T}_h\},$$

where, for each κ, we define

$$\mathcal{S}_\ell = \begin{cases} \mathcal{P}_\ell = \operatorname{span}\{x^\alpha : 0 \leq |\alpha| \leq \ell\} & \text{(simplex)}, \\ \mathcal{Q}_\ell = \operatorname{span}\{x^\alpha : 0 \leq \alpha_i \leq \ell, \quad 1 \leq i \leq n\} & \text{(hypercube)}, \end{cases}$$

for $\ell \geq 0$.

- Local weak formulation

  On each $\kappa \in \mathcal{T}_h$: find $\mathbf{u}$ such that

$$-\int_\kappa \mathcal{F}(\mathbf{u}) : \nabla \mathbf{v} \mathrm{d}\mathbf{x} + \int_{\partial\kappa} (\mathcal{F}(\mathbf{u}^+) \cdot \mathbf{n}_\kappa) \cdot \mathbf{v}^+ \, \mathrm{d}s = \int_\kappa \mathbf{f} \cdot \mathbf{v} \mathrm{d}\mathbf{x}$$

- Weakly enforce the boundary conditions

$$-\int_\kappa \mathcal{F}(\mathbf{u}) : \nabla \mathbf{v} \mathrm{d}\mathbf{x} + \int_{\partial\kappa} \mathcal{H}(\mathbf{u}^+, \mathbf{u}^-, \mathbf{n}_\kappa) \cdot \mathbf{v}^+ \, \mathrm{d}s = \int_\kappa \mathbf{f} \cdot \mathbf{v} \mathrm{d}\mathbf{x}$$

- Add artificial viscosity

$$\begin{aligned} -\int_\kappa \mathcal{F}(\mathbf{u}) : \nabla \mathbf{v} \mathrm{d}\mathbf{x} \quad &+ \quad \int_{\partial\kappa} \mathcal{H}(\mathbf{u}^+, \mathbf{u}^-, \mathbf{n}_\kappa) \cdot \mathbf{v}^+ \, \mathrm{d}s \\ &+ \quad \int_\kappa \varepsilon_h(\mathbf{u}) \nabla \mathbf{u} : \nabla \mathbf{v} \mathrm{d}\mathbf{x} = \int_\kappa \mathbf{f} \cdot \mathbf{v} \mathrm{d}\mathbf{x} \end{aligned}$$

$$
\begin{aligned}
\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) &= \sum_{\kappa\in\mathcal{T}_h} \Big\{ -\int_\kappa \mathcal{F}(\mathbf{u}_h) : \nabla \mathbf{v}_h \mathrm{d}\mathbf{x} \\
&\quad + \int_{\partial\kappa} \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}_\kappa) \cdot \mathbf{v}_h^+ \mathrm{d}s + \int_\kappa \varepsilon_h(\mathbf{u}_h) \nabla \mathbf{u}_h : \nabla \mathbf{v}_h \mathrm{d}\mathbf{x} \Big\} \\
\ell(\mathbf{v}_h) &= \sum_{\kappa\in\mathcal{T}_h} \int_\kappa \mathbf{f} \cdot \mathbf{v}_h \mathrm{d}\mathbf{x} \\
\varepsilon_h(\mathbf{u}_h) &= C_\varepsilon\, h_\kappa^{2-\beta}\, |\mathbf{f} - \mathrm{div}\mathcal{F}(\mathbf{u}_h)|,
\end{aligned}
$$

where $C_\varepsilon \geq 0$ and $\beta = 1/10$.

Jaffre, Johnson, & Szepessy 1995, Cockburn & Gremaud 1996

DGFEM

Find $\mathbf{u}_h \in V_{h,p}$ such that

$$
\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) = \ell(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in V_{h,p}.
$$

$\mathcal{H}(\cdot,\cdot,\mathbf{n})$ is a Lipschitz continuous, consistent and conservative flux function.

- Lax–Friedrichs flux

$$\mathcal{H}(\mathbf{u}^+,\mathbf{u}^-,\mathbf{n}) = \frac{1}{2}\left(\left(\mathcal{F}(\mathbf{u}^+)\cdot\mathbf{n} + \mathcal{F}(\mathbf{u}^-)\cdot\mathbf{n}\right) - \alpha(\mathbf{u}^- - \mathbf{u}^+)\right),$$

where $\alpha = max_i|\Lambda_{ii}|$.

- Roe's flux

$$\begin{aligned}\mathcal{H}(\mathbf{u}^+,\mathbf{u}^-,\mathbf{n}) \quad &= \quad \frac{1}{2}\left(\left(\mathcal{F}(\mathbf{u}^+)\cdot\mathbf{n} + \mathcal{F}(\mathbf{u}^-)\cdot\mathbf{n}\right)\right. \\ &\qquad \left. -|B(\tilde{\mathbf{u}},\mathbf{n})|(\mathbf{u}^- - \mathbf{u}^+)\right),\end{aligned}$$

where $|B(\tilde{\mathbf{u}},\mathbf{n})| = B^+(\tilde{\mathbf{u}},\mathbf{n}) - B^-(\tilde{\mathbf{u}},\mathbf{n})$.

- Vijayasundaram flux

$$\mathcal{H}(\mathbf{u}^+,\mathbf{u}^-,\mathbf{n}) = B^+(\tilde{\mathbf{u}},\mathbf{n})\mathbf{u}^+ + B^-(\tilde{\mathbf{u}},\mathbf{n})\mathbf{u}^-.$$

Residuals

Internal Volume Residual

$$\mathbf{R}_h|_\kappa = \mathbf{f} - \nabla \cdot \mathcal{F}(\mathbf{u}_h).$$

Flux Residual

$$\boldsymbol{\sigma}_h|_{\partial\kappa} = \left(\mathcal{F}(\mathbf{u}_h^+) \cdot \mathbf{n}_\kappa - \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}_\kappa)\right)|_{\partial\kappa}.$$

AVIS Residual

$$\boldsymbol{\alpha}_h|_\kappa = \left(\varepsilon_h(\mathbf{u}_h)\nabla\mathbf{u}_h\right)|_\kappa.$$

- Gâteaux derivative of $J(\cdot)$:

$$J'[\mathbf{w}](\mathbf{v}) = \lim_{\epsilon \to 0} \frac{J(\mathbf{w} + \epsilon \mathbf{v}) - J(\mathbf{w})}{\epsilon}.$$

- Mean–value linearization of $J(\cdot)$:

$$\begin{aligned} \bar{J}(\mathbf{u}, \mathbf{u}_h; \mathbf{u} - \mathbf{u}_h) &= J(\mathbf{u}) - J(\mathbf{u}_h) \\ &= \int_0^1 J'[\theta \mathbf{u} + (1-\theta)\mathbf{u}_h](\mathbf{u} - \mathbf{u}_h)\mathrm{d}\theta. \end{aligned}$$

- Gâteaux derivative of $\mathcal{N}(\cdot,\cdot)$:

$$\mathcal{N}'_{\mathbf{u}}[\mathbf{w}](\mathbf{v},\cdot) = \lim_{\epsilon\to 0} \frac{\mathcal{N}(\mathbf{w}+\epsilon\mathbf{v},\cdot) - \mathcal{N}(\mathbf{w},\cdot)}{\epsilon}.$$

- Mean–value linearization of $\mathcal{N}(\cdot,\cdot)$:

$$\begin{aligned}\mathcal{M}(\mathbf{u},\mathbf{u}_h;\mathbf{u}-\mathbf{u}_h,\mathbf{v}) &= \mathcal{N}(\mathbf{u},\mathbf{v}) - \mathcal{N}(\mathbf{u}_h,\mathbf{v}) \\ &= \int_0^1 \mathcal{N}'_{\mathbf{u}}[\theta\mathbf{u}+(1-\theta)\mathbf{u}_h](\mathbf{u}-\mathbf{u}_h,\mathbf{v})\mathrm{d}\theta.\end{aligned}$$

- Galerkin orthogonality:

$$\mathcal{N}(\mathbf{u},\mathbf{v}_h) - \mathcal{N}(\mathbf{u}_h,\mathbf{v}_h) = \mathcal{M}(\mathbf{u},\mathbf{u}_h;\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in V_{h,p}.$$

Dual problem

Find $\mathbf{z}$ such that

$$\mathcal{M}(\mathbf{u}, \mathbf{u}_h; \mathbf{w}, \mathbf{z}) = \bar{J}(\mathbf{u}, \mathbf{u}_h; \mathbf{w}) \quad \forall \mathbf{w}.$$

Well–posedness (?)

Tadmor 1991, Bouchut & James 1998, Godlewski, Olazabal, & Raviart 1999, Ulbrich 2001, 2002, 2003

$$
\begin{aligned}
J(\mathbf{u}) - J(\mathbf{u}_h) &= \bar{J}(\mathbf{u}, \mathbf{u}_h; \mathbf{u} - \mathbf{u}_h) && \text{[Linearization]} \\
&= \mathcal{M}(\mathbf{u}, \mathbf{u}_h; \mathbf{u} - \mathbf{u}_h, \mathbf{z}) && \text{[Dual Problem]} \\
&= \mathcal{M}(\mathbf{u}, \mathbf{u}_h; \mathbf{u} - \mathbf{u}_h, \mathbf{z} - \mathbf{z}_h) && \text{[Galerkin Orthogonality]} \\
&= \ell(\mathbf{z} - \mathbf{z}_h) - \mathcal{N}(\mathbf{u}_h, \mathbf{z} - \mathbf{z}_h) && \text{[Consistency]} \\
&= \sum_{\kappa \in \mathcal{T}_h} \left(\int_\kappa \mathbf{R}_h \cdot (\mathbf{z} - \mathbf{z}_h) \mathrm{d}\mathbf{x} + \int_{\partial\kappa} \boldsymbol{\sigma}_h \cdot (\mathbf{z} - \mathbf{z}_h)^+ \mathrm{d}s \right. \\
&\qquad \left. - \int_\kappa \boldsymbol{\alpha}_h : \nabla(\mathbf{z} - \mathbf{z}_h) \mathrm{d}\mathbf{x} \right) \\
&\equiv \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa .
\end{aligned}
$$

Proposition

Assuming the dual problem is well–posed, the following result holds:

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \leq \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^{(\mathrm{I})},$$

where $\eta_\kappa^{(\mathrm{I})} = |\eta_\kappa|$,

$$\begin{aligned}
\eta_\kappa &= \int_\kappa \mathbf{R}_h \cdot (\mathbf{z} - \mathbf{z}_h) \mathrm{d}\mathbf{x} + \int_{\partial\kappa} \boldsymbol{\sigma}_h \cdot (\mathbf{z} - \mathbf{z}_h)^+ \mathrm{d}s \\
&\quad - \int_\kappa \boldsymbol{\alpha}_h : \nabla(\mathbf{z} - \mathbf{z}_h) \mathrm{d}\mathbf{x}, \\
\mathbf{R}_h|_\kappa &= \left(\mathbf{f} - \mathit{div}\mathcal{F}(\mathbf{u}_h)\right)|_\kappa, \\
\boldsymbol{\sigma}_h|_{\partial\kappa} &= \left(\mathcal{F}(\mathbf{u}_h^+) \cdot \mathbf{n}_\kappa - \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}_\kappa)\right)|_{\partial\kappa}, \\
\boldsymbol{\alpha}_h|_\kappa &= \left(\varepsilon_h(\mathbf{u}_h)\nabla \mathbf{u}_h\right)|_\kappa.
\end{aligned}$$

Lemma

Given $\kappa \in \mathcal{T}_h$, suppose that $v|_\kappa \in H^{k_\kappa}(\kappa)$, $0 \le k_\kappa \le p+1$. Then, there exists $\mathcal{I}v$ in the finite element space $V_{h,p}$, such that

$$\begin{aligned} \|v - \mathcal{I}v\|_{L^2(\kappa)} \quad &+ \quad h_\kappa \|\nabla(v - \mathcal{I}v)\|_{L^2(\kappa)} \\ &+ \quad h_\kappa^{1/2} \|v - \mathcal{I}v\|_{L^2(\partial\kappa)} \le C_{\mathcal{I}} h_\kappa^{k_\kappa} \|v\|_{H^{k_\kappa}(\kappa)}. \end{aligned}$$

Proof.

See Babuška & Suri 1987, for example. ∎

Type II *A Posteriori* Error Bound

Theorem

Assuming

$$\|\mathbf{z}\|_{H^s(\Omega)} \leq C_{\mathrm{stab}}\,, \quad 1 \leq s \leq p+1\,,$$

we have that

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \leq C_{\mathcal{I}} C_{\mathrm{stab}} \left(\sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^{(\mathrm{II})}\right)^2 \right)^{1/2},$$

where

$$\eta_\kappa^{(\mathrm{II})} = h^s \|\mathbf{R}_h\|_{L^2(\kappa)} + h^{s-1/2} \|\boldsymbol{\sigma}_h\|_{L^2(\partial\kappa)} + h^{s-1} \|\boldsymbol{\alpha}_h\|_{L^2(\kappa)}.$$

Type I bound:

- Right–hand side is not computable
  - $\mathbf{z}$ must be numerically approximated.
  - Introduction of *linearization* and *discretization* errors.

Linearization: find $\hat{\mathbf{z}}$ such that

$$\hat{\mathcal{N}}'_{\mathbf{u}}[\mathbf{u}_h](\mathbf{w}, \hat{\mathbf{z}}) = J'[\mathbf{u}_h](\mathbf{w}) \quad \forall \mathbf{w}.$$

Discretization: find $\hat{\mathbf{z}}_{h',p'} \in V_{h',p'}$ such that

$$\hat{\mathcal{N}}'_{\mathbf{u}}[\mathbf{u}_h](\mathbf{w}, \hat{\mathbf{z}}_{h',p'}) = J'[\mathbf{u}_h](\mathbf{w}_{h',p'}) \quad \forall \mathbf{w}_{h',p'} \in V_{h',p'}.$$

Numerical Approximation of the Dual Problem:

1. Keep the degree p of the approximating polynomial used to compute u_h fixed, but approximate z on a sequence of dual finite element meshes $\mathcal{T}_{\hat{h}}$ which, in general, differ from from the 'primal meshes' $\mathcal{T}_h$.
2. Compute an approximation to z using piecewise (discontinuous) polynomials of degree $\hat{p}$, $\hat{p} > p$, on the same finite element mesh $\mathcal{T}_h$ employed for the primal problem.
3. Compute the approximate dual solution using the same mesh $\mathcal{T}_h$ and polynomial degree p employed for the primal problem and patchwise extrapolate the resulting approximate dual solution $\hat{z}_{h,p} \in V_{h,p}$ to a dual solution $\hat{z}_{2h,\hat{p}} \in V_{2h,\hat{p}}$, $\hat{p} > p$.

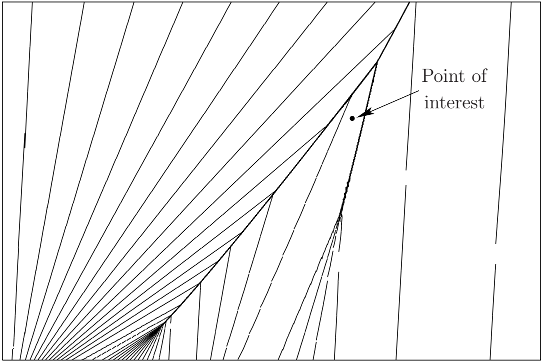
In practice, we set $V_{h',p'} = V_{h,p+1}$ (Method 2 above).

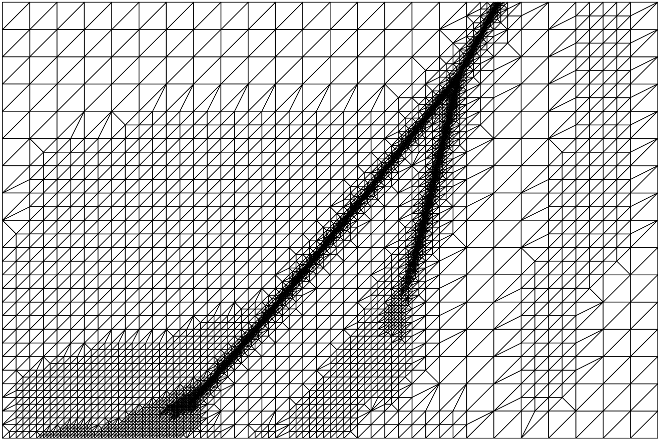
Type II bound:

- Computation of $C_{\rm int}$
    - Estimates are available from approximation theory.
- Computation of $C_{\rm stab}$
    - Theoretical bounds are typically over-pessimistic, if available at all.
    - The dual problem must be solved a (large) number of times with *typical* data.
    - Library of stability constants may be generated.

$$u_t + \left((1/2)\,u^2\right)_x = 0,$$
$$u(x,0) = 2/(1+x^3)\,\sin^2(\pi x)\;.$$

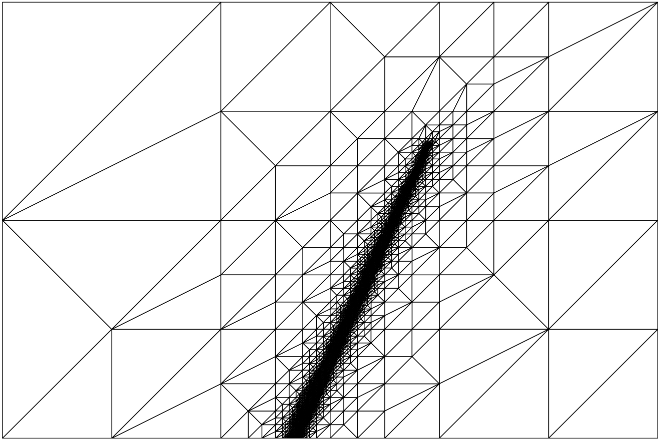
$$\boxed{J(u) \equiv u(1.95, 1.35) = 0.451408206331223\;.}$$





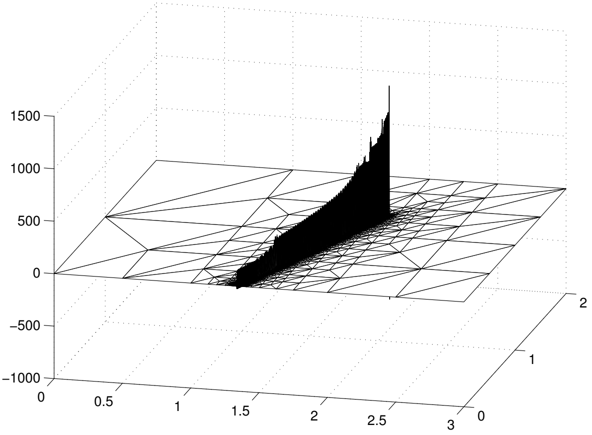
Mesh designed using Type II indicator

20186 elements, $|J(u) - J(u_h)| = 4.015 \times 10^{-4}$

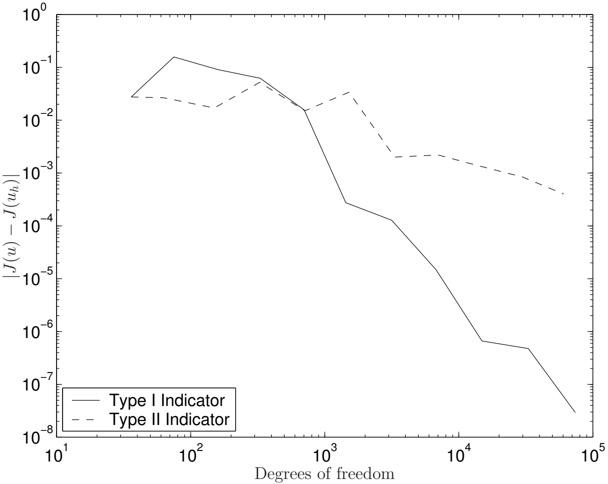


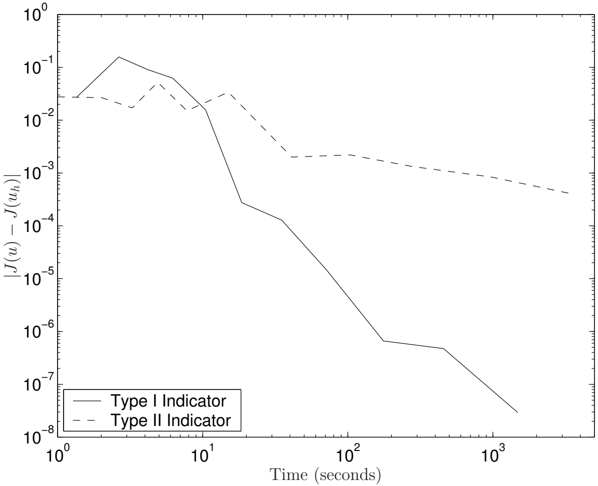
Mesh designed using Type I indicator

24662 elements, $|J(u) - J(u_h)| = 2.934 \times 10^{-8}$



Dual Solution


10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
$|J(u) - J(u_h)|$
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
Degrees of freedom
Type I Indicator
Type II Indicator



10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
$|J(u) - J(u_h)|$
10^{0}
10^{1}
10^{2}
10^{3}
Time (seconds)
Type I Indicator
Type II Indicator

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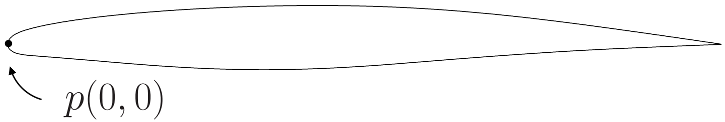
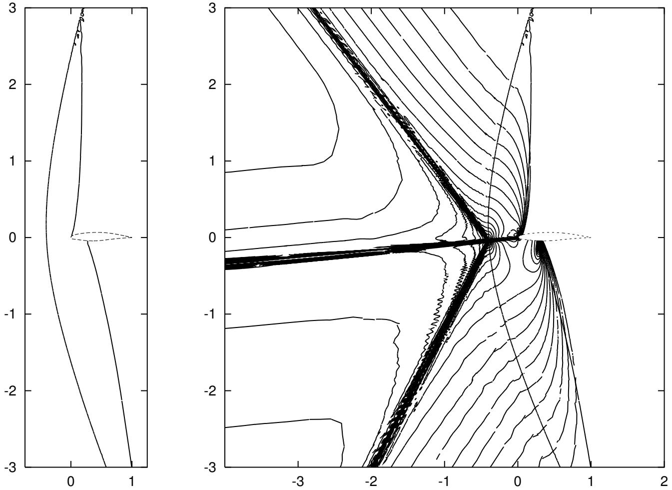
Elements	$J(u-u_h)$	$\sum_\kappa \eta_\kappa$	θ_1	$\sum_\kappa \lvert\eta_\kappa\rvert$	θ_2
53	9.09e-2	-3.19e-2	-0.35	2.24e-1	2.47
110	6.23e-2	5.20e-2	0.83	1.21e-1	1.94
234	-1.57e-2	-3.27e-3	0.21	3.88e-2	2.48
479	2.75e-4	3.30e-4	1.20	1.08e-2	39.48
1053	1.28e-4	1.21e-4	0.95	3.27e-3	25.54
2256	1.48e-5	1.46e-5	0.98	1.86e-3	125.1
4968	-6.63e-7	-7.16e-7	1.08	7.76e-4	1171
11003	-4.76e-7	-4.78e-7	1.01	3.20e-4	672.0
24662	2.93e-8	2.91e-8	0.99	1.28e-4	4357

$$\theta_1 = \frac{\sum_\kappa \eta_\kappa}{J(u-u_h)}, \qquad \theta_2 = \frac{\sum_\kappa |\eta_\kappa|}{|J(u-u_h)|}$$

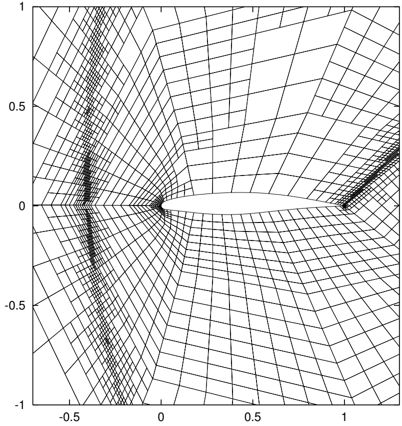
$\mathrm{Ma} = 1.2$, $\alpha = 5^\circ$, $\rho = 1$ and pressure $p = 1$.



$$\boxed{J(u) \equiv p(0,0) \approx 2.393\ .}$$

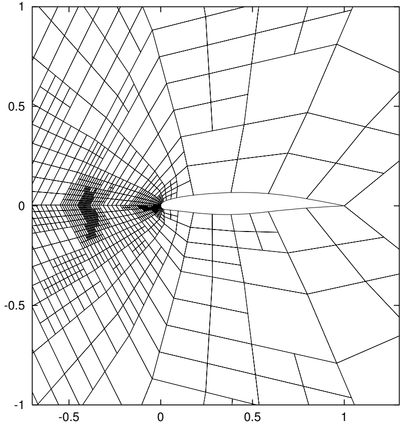

3
2
1
0
-1
-2
-3
0
1
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2

BAC3-11 Airfoil



Type II: 13719 elements

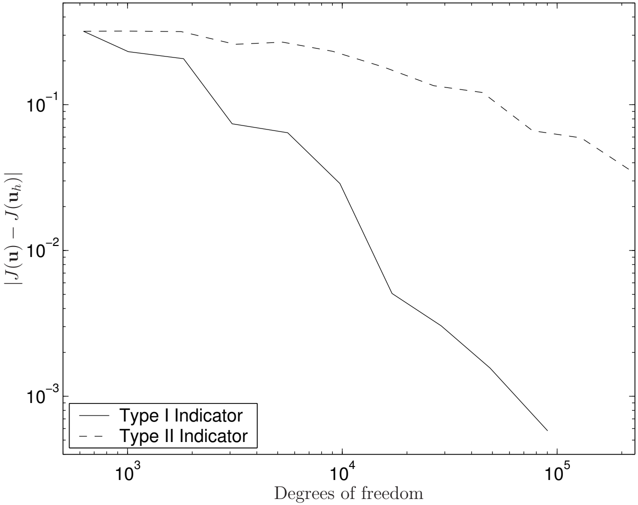
$|J(\mathbf{u}) - J(\mathbf{u}_h)| = 3.542 \times 10^{-2}$



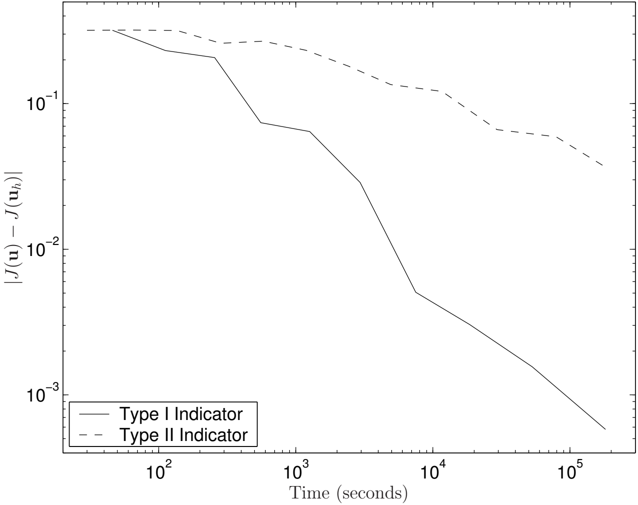
Type I: 1803 elements

$|J(\mathbf{u}) - J(\mathbf{u}_h)| = 3.042 \times 10^{-3}$

BAC3-11 Airfoil



BAC3-11 Airfoil



Elements	$J(\mathbf{u}) - J(\mathbf{u}_h)$	$\sum_\kappa \eta_\kappa$	θ_1	$\sum_\kappa \lvert\eta_\kappa\rvert$	θ_2
63	2.31e-1	-1.50e-2	-0.06	2.00e-1	0.87
114	2.07e-1	7.28e-2	0.35	3.50e-1	1.69
192	7.40e-2	5.40e-2	0.73	2.68e-1	3.62
348	6.43e-2	2.70e-2	0.42	2.12e-1	3.30
609	2.88e-2	1.39e-2	0.48	1.84e-1	6.39
1065	5.07e-3	7.60e-3	1.50	1.17e-1	23.11
1803	3.04e-3	2.87e-3	0.94	1.03e-1	33.78
3045	1.56e-3	2.80e-3	1.79	1.07e-1	68.39
5643	5.79e-4	5.79e-4	1.00	5.55e-2	95.88

$$\theta_1 = \frac{\sum_\kappa \eta_\kappa}{J(\mathbf{u}) - J(\mathbf{u}_h)}, \qquad \theta_2 = \frac{\sum_\kappa |\eta_\kappa|}{|J(\mathbf{u}) - J(\mathbf{u}_h)|}$$

Travel time functional: $J(v) : \gamma \times V \to \mathbb{R}$, defined by

$$J(v) = \int_{\gamma(v)} \frac{\phi}{\|K \nabla v\|_2} \, \mathrm{d}s,$$

where $\gamma(v)$ is the solution of the differential equation

$$\frac{dX}{dt} = -\frac{K}{\phi} \nabla v,$$

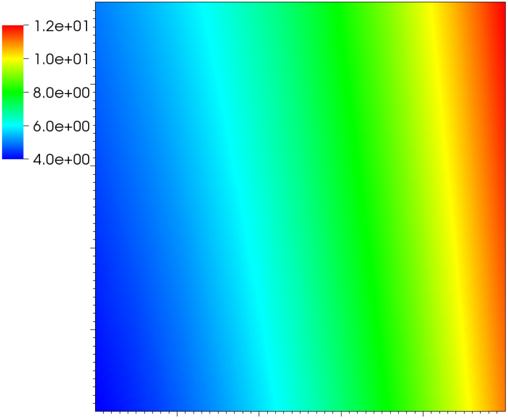
subject to the initial condition

$$X(0) = x_0,$$

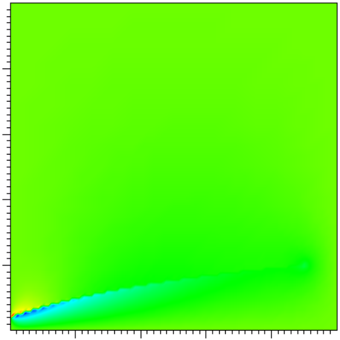
restricted to the domain Ω.

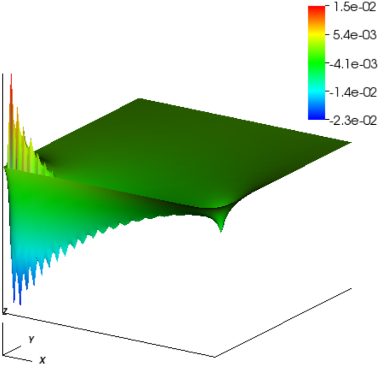
K: hydraulic conductivity.
ϕ: porosity.

Primal Solution

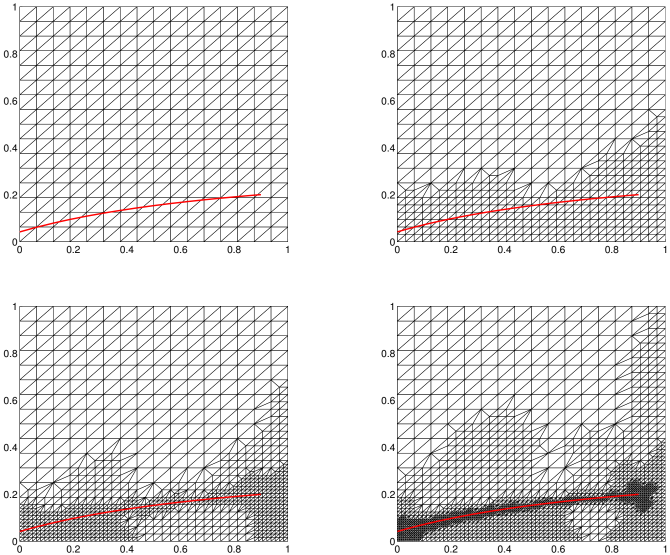


Dual Solution





Application to Nuclear Waste Disposal



Dofs	$J(u_h)$	$J(u) - J(u_h)$	$\sum_\kappa \eta_\kappa$	θ
289	0.15778338	1.114E-04	-1.18E-05	-0.11
528	0.15786326	3.148E-05	1.52E-07	0.00
999	0.15787398	2.075E-05	1.24E-05	0.60
1908	0.15788055	1.419E-05	1.21E-05	0.85
3706	0.15788641	8.327E-06	7.81E-06	0.94
7154	0.15788996	4.779E-06	4.63E-06	0.97
13896	0.15789179	2.952E-06	2.91E-06	0.98
26965	0.15789303	1.709E-06	1.68E-06	0.98

$$\theta = \frac{\sum_\kappa \eta_\kappa}{J(u) - J(u_h)}.$$

PDEs with non-negative characteristic form

$$Lu \equiv -\nabla \cdot (a(x, u, \nabla u)\nabla u) + \nabla \cdot \mathcal{F}(u) + cu = f \text{ in } \Omega \,,$$

where $\Omega \subset \mathbb{R}^d$ and a is non-negative definite, i.e.,

$$\zeta^\top a \zeta \geq 0 \quad \forall \zeta \in \mathbb{R}^d.$$

This class includes:

- Linear: Second–order (degenerate) elliptic, parabolic, first–order hyperbolic, and Fokker–Planck type equations.
- Nonlinear: p-Laplacian, parabolic, nonlinear reaction–diffusion equations.

Model Problem

Given $\Omega \subset \mathbb{R}^d$, $a \in L^\infty(\Omega)^{d\times d}_{\mathrm{sym}}$, $\mathbf{b} \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$, find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} Lu &\equiv -\nabla \cdot (a\nabla u) + \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in} \quad \Omega, \\ u &= g_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}} \cup \Gamma_-, \quad \mathbf{n} \cdot (a\nabla u) = g_{\mathrm{N}} \quad \text{on} \quad \Gamma_{\mathrm{N}}. \end{aligned}$$

Here, $\zeta^\top a(x)\zeta \geq 0 \quad \forall \zeta \in \mathbb{R}^d$, a.e. $x \in \bar{\Omega}$, and

$$\begin{aligned} \text{Elliptic:} \quad \Gamma_0 \equiv \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} &= \left\{ x \in \Gamma : \ \mathbf{n}(x)^\top a(x)\mathbf{n}(x) > 0 \right\}, \\ \text{Hyperbolic Inflow:} \quad \Gamma_- &= \{x \in \Gamma \backslash \Gamma_0 : \ \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\}, \\ \text{Hyperbolic Outflow:} \quad \Gamma_+ &= \{x \in \Gamma \backslash \Gamma_0 : \ \mathbf{b}(x) \cdot \mathbf{n}(x) \geq 0\}. \end{aligned}$$

Well–posedness

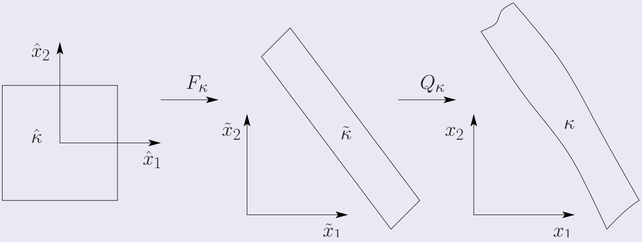
Fichera (1956, 1960), Oleĭnik & Radkevič (1973), H. & Süli (2001)

Mesh

$\mathcal{T}_h = \{\kappa\}$ is a non-degenerate (possibly anisotropic) mesh.

Element Mappings

$$F_\kappa : \hat{\kappa} \to \tilde{\kappa}, \qquad Q_\kappa : \tilde{\kappa} \to \kappa.$$

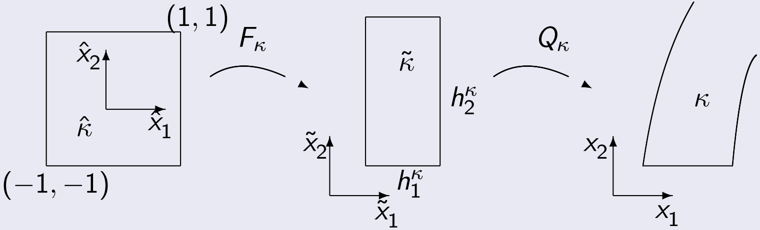


- F_κ defines the size and orientation of the element κ.
- Q_κ defines the shape of κ.

H., Schwab, & Süli 2000, Georgoulis 2003

Element Mappings – Axiparallel Setting

$$F_\kappa : \hat{\kappa} \to \tilde{\kappa}, \qquad Q_\kappa : \tilde{\kappa} \to \kappa.$$



- F_κ defines the size of the element κ.
- Q_κ defines the shape of κ.

Georgoulis 2003

Discontinuous Galerkin Method

- $\mathcal{T}_h = \{\kappa\}$ is a non-degenerate (possibly anisotropic) mesh;
- For each $\kappa \in \mathcal{T}_h$ assign a vector $\mathbf{p}_\kappa = (p_{\kappa,1}, p_{\kappa,2}, \dots, p_{\kappa,d})^\top$, with $p_{\kappa,i} \geq 1$, $i = 1, 2, \dots, d$, and define the finite element space

$$V_{h,\mathbf{p}} = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{Q}_{\mathbf{p}_\kappa} \quad \forall \kappa \in \mathcal{T}_h\},$$

where $\mathbf{p} = \{\mathbf{p}_\kappa : \kappa \in \mathcal{T}_h\}$, and

$$\mathcal{Q}_{\mathbf{p}_\kappa} := \mathrm{span}\{\Pi_{i=1}^d \hat{x}_i^j : 0 \leq j \leq p_{\kappa,i}\}.$$

$$
\begin{aligned}
A_h(u_h, v_h) &= \int_\Omega a\nabla_h u_h \cdot \nabla_h v_h \,\mathrm{d}\mathbf{x} - \sum_{F\in\mathcal{F}_h}\int_F [\![v_h]\!]\cdot\{\!\{a\nabla_h u_h\}\!\}\mathrm{d}s \\
&\quad - \sum_{F\in\mathcal{F}_h}\int_F [\![u_h]\!]\cdot\{\!\{a\nabla_h v_h\}\!\}\mathrm{d}s + \sum_{F\in\mathcal{F}_h}\int_F \sigma[\![u_h]\!]\cdot[\![v_h]\!]\mathrm{d}s \\
&\quad - \int_\Omega (u_h\mathbf{b}\cdot\nabla_h v_h - c u_h v_h)\mathrm{d}\mathbf{x} \\
&\quad + \sum_{\kappa\in\mathcal{T}_h}\left(\int_{\partial_+\kappa}(\mathbf{b}\cdot\mathbf{n}_\kappa)\,u_h^+ v_h^+\mathrm{d}s + \int_{\partial_-\kappa\setminus\Gamma}(\mathbf{b}\cdot\mathbf{n}_\kappa)\,u_h^- v_h^+\mathrm{d}s\right), \\
\ell(v_h) &= \int_\Omega f v_h \mathrm{d}\mathbf{x} - \int_{\Gamma_{\mathrm{D}}} g_{\mathrm{D}}(a\nabla_h v_h\cdot\mathbf{n} - \sigma v_h)\mathrm{d}s \\
&\quad + \int_{\partial\kappa\cap\Gamma_{\mathrm{N}}} g_{\mathrm{N}} v_h^+\mathrm{d}s - \sum_{\kappa\in\mathcal{T}_h}\int_{\partial_-\kappa\cap(\Gamma_{\mathrm{D}}\cup\Gamma_-)}(\mathbf{b}\cdot\mathbf{n})\,g_{\mathrm{D}}\,v_h^+\mathrm{d}s.
\end{aligned}
$$

Interior Penalty Method

Find $u_h \in V_{h,\mathbf{p}}$ such that

$$A_h(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_{h,\mathbf{p}}.$$

Galerkin Orthogonality

Assuming that $u \in H^2(\Omega)$, we have

$$A_h(u - u_h, v_h) = 0 \quad \forall v_h \in V_{h,\mathbf{p}}.$$

We define the DG–norm $|||\cdot|||$ by

$$
\begin{aligned}
|||w|||^2 \;=\; & \sum_{\kappa\in\mathcal{T}_h}\Bigg(\|\sqrt{a}\,\nabla w\|^2_{L^2(\kappa)} + \|c_0 w\|^2_{L^2(\kappa)} + \frac{1}{2}\|w^+\|^2_{\partial_-\kappa\cap(\Gamma_{\mathrm{D}}\cup\Gamma_-)} \\
& + \frac{1}{2}\|w^+ - w^-\|^2_{\partial_-\kappa\setminus\Gamma} + \frac{1}{2}\|w^+\|^2_{\partial_+\kappa\cap\Gamma}\Bigg) \\
& + \sum_{F\in\mathcal{F}_h}\int_F \sigma|[\![w]\!]|^2\mathrm{d}s + \sum_{F\in\mathcal{F}_h}\int_F \frac{1}{\sigma}|\{\!\{a\nabla w\}\!\}|^2\mathrm{d}s\,,
\end{aligned}
$$

where $\|\cdot\|_\tau$, $\tau\subset\partial\kappa$, denotes the (semi)norm associated with the (semi)inner-product $(v,w)_\tau = \int_\tau |\mathbf{b}\cdot\mathbf{n}_\kappa| vw\mathrm{d}s$, Here,

$$(c_0(x))^2 = c(x) + {}^1\!/_2\,\nabla\cdot\mathbf{b}(x) \;\text{ a.e. }\; x\in\Omega\,.$$

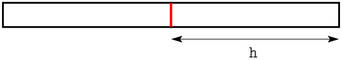
Discontinuous Galerkin Method

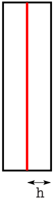
Penalty Parameter

Given $C_\sigma > 0$, we define

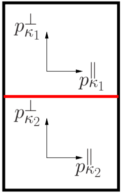
$$\sigma|_F = C_\sigma \frac{\|a\| \, \mathrm{p}^2}{\mathtt{h}} \quad \text{for } F \in \mathcal{F}_h.$$

$$\mathtt{h}|_F = \frac{\min\{\mathrm{vol}_d(\kappa_1), \mathrm{vol}_d(\kappa_2)\}}{\mathrm{vol}_{d-1}(F)}$$





$$\mathrm{p}|_F = \max\{p_{\kappa_1}^{\perp}, p_{\kappa_2}^{\perp}\}$$



Lemma (Coercivity)

Assuming C_σ *is chosen* sufficiently *large, there exists* $C > 0$*, such that*

$$A_h(v, v) \geq C|||v|||^2 \quad \forall v \in V_{h,\mathbf{p}}.$$

Georgoulis 2006, Georgoulis, Hall & H. 2007.

Adjoint problem

Find z such that

$$\boxed{A_h(v, z) = J(v) \quad \forall v.}$$

Error representation formula

$$\begin{aligned} J(u) - J(u_h) &= J(u - u_h) \\ &= A_h(u - u_h, z) \\ &= A_h(u - u_h, z - z_h) \quad \rightarrow \textit{A Priori} \\ &= \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa(u_h, z - z_h) \quad \rightarrow \textit{A Posteriori} \end{aligned}$$

for all $z_h \in V_{h,\mathbf{p}}$.

For each element κ in the mesh $\mathcal{T}_h$, we define:

- Internal residual

$$R_{\mathrm{int}}|_{\kappa} = (f - Lu_h)|_{\kappa};$$

- Boundary residuals

$$\begin{aligned} R_{\mathrm{D}}|_{\partial\kappa\cap(\Gamma_{\mathrm{D}}\cup\Gamma_{-})} &= (g_{\mathrm{D}} - u_h^{+})|_{\partial\kappa\cap(\Gamma_{\mathrm{D}}\cup\Gamma_{-})}\,, \\ R_{\mathrm{N}}|_{\partial\kappa\cap\Gamma_N} &= (g_{\mathrm{N}} - (a\nabla u_h^{+})\cdot\mathbf{n})|_{\partial\kappa\cap\Gamma_{\mathrm{N}}}\,. \end{aligned}$$

Proposition

$$|J(u) - J(u_h)| \leq \sum_{\kappa \in \mathcal{T}_h} |\eta_\kappa|,$$

$$\begin{aligned}
\eta_\kappa = & \int_\kappa R_{\rm int}(z - z_h)\mathrm{d}x - \int_{\partial_-\kappa\cap\Gamma} (\mathbf{b}\cdot\mathbf{n}_\kappa)\, R_{\rm D}(z - z_h)^+\mathrm{d}s \\
& + \int_{\partial_-\kappa\setminus\Gamma} \mathbf{b}\cdot [\![u_h]\!](z - z_h)^+\mathrm{d}s - \int_{\partial\kappa\cap\Gamma_{\rm D}} R_{\rm D}((a\nabla(z - z_h)^+)\cdot\mathbf{n}_\kappa)\mathrm{d}s \\
& + \int_{\partial\kappa\cap\Gamma_{\rm D}} \sigma R_{\rm D}(z - z_h)^+\mathrm{d}s + \int_{\partial\kappa\cap\Gamma_{\rm N}} R_{\rm N}(z - z_h)^+\mathrm{d}s \\
& + \int_{\partial\kappa\setminus\Gamma} \left\{ \frac{1}{2}[\![u_h]\!]\cdot(a\nabla(z - z_h)^+) - \frac{1}{2}[\![(a\nabla u_h)]\!](z - z_h)^+ \right\}\mathrm{d}s \\
& - \int_{\partial\kappa\setminus\Gamma} \sigma[\![u_h]\!]\cdot\mathbf{n}_\kappa(z - z_h)^+\mathrm{d}s.
\end{aligned}$$

$R_{\rm int}|_\kappa = (f - Lu_h)|_\kappa$, $R_{\rm D}|_{\Gamma_{\rm D}\cup\Gamma_-} = (g_{\rm D} - u_h^+)|_{\Gamma_{\rm D}\cup\Gamma_-}$, $R_{\rm N}|_{\Gamma_N} = (g_{\rm N} - (a\nabla u_h^+)\cdot\mathbf{n})|_{\Gamma_{\rm N}}$

- Introduce the L^2–orthogonal projection operator Π_p, and write

$$u - u_h = (u - \Pi_p u) + (\Pi_p u - u_h) \equiv \eta + \xi.$$

- Error representation formula becomes

$$J(u) - J(u_h) = A_h(u - u_h, z - z_h) = A_h(\eta, z - z_h) + A_h(\xi, z - z_h) \equiv \mathrm{I} + \mathrm{II}$$

for all $z_h \in V_{h,\mathbf{p}}$.

- Bound Terms I & II in terms of η and $z - z_h$, i.e.,

$$\begin{aligned} |\mathrm{I}| &\leq C\mathcal{E}(\eta)\,\mathcal{F}(z - z_h), \\ |\mathrm{II}| &\leq C\mathcal{G}(\eta)\,\mathcal{H}(z - z_h), \end{aligned}$$

where $z_h = \Pi_p z$.

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Term I

$$\mathrm{I} = A_h(\eta, z - z_h) \leq C\mathcal{E}(\eta)\,\mathcal{F}(z - z_h),$$

where

$$\begin{aligned}\mathcal{E}(\eta)^2 = \sum_{\kappa\in\mathcal{T}_h} \Big\{ & \|\sqrt{a}\nabla\eta\|^2_{L^2(\kappa)} + \beta_1\|\eta\|^2_{L^2(\kappa)} + \beta_2\epsilon_\kappa^{-1}\|\nabla\eta\|^2_{L^2(\kappa)} + \|[\![\eta]\!]\|^2_{\partial_-\kappa} \\ & + \|\sigma^{-1/2}\{\!\{a\nabla\eta\}\!\}\|^2_{L^2(\partial\kappa\cap(\Gamma_\mathcal{I}\cup\Gamma_\mathrm{D}))} + \|\sigma^{1/2}[\![\eta]\!]\|^2_{L^2(\partial\kappa\cap(\Gamma_\mathcal{I}\cup\Gamma_\mathrm{D}))} \Big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}(w)^2 = \sum_{\kappa\in\mathcal{T}_h} \Big\{ & \|\sqrt{a}\nabla w\|^2_{L^2(\kappa)} + \beta_1\|w\|^2_{L^2(\kappa)} + \beta_2\epsilon_\kappa\|w\|^2_{L^2(\kappa)} + \|w^+\|^2_{\partial_-\kappa} \\ & + \|\sigma^{-1/2}\{\!\{a\nabla w\}\!\}\|^2_{L^2(\partial\kappa\cap(\Gamma_\mathcal{I}\cup\Gamma_\mathrm{D}))} + \|\sigma^{1/2}[\![w]\!]\|^2_{L^2(\partial\kappa\cap(\Gamma_\mathcal{I}\cup\Gamma_\mathrm{D}))} \Big\}.\end{aligned}$$

Here, $\beta_1|_\kappa = \|c + \nabla\cdot\mathbf{b}\|_{L^\infty(\kappa)}$, $\beta_2|_\kappa = \|\mathbf{b}\|_{L^\infty(\kappa)}$ and ϵ_κ, $\kappa\in\mathcal{T}_h$, is any set of real positive numbers.

Lemma

The following inequality holds:

$$|||\xi||| \leq C\mathcal{G}(\eta),$$

where

$$\mathcal{G}(\eta)^2 = \sum_{\kappa \in \mathcal{T}_h} \left(\|\sqrt{a}\nabla\eta\|^2_{L^2(\kappa)} + \gamma_1 \|\eta\|^2_{L^2(\kappa)} + \|\eta^+\|^2_{\partial_+\kappa\cap\Gamma} + \|\eta^-\|^2_{\partial_-\kappa\setminus\Gamma} \right)$$
$$+ \int_{\Gamma_{\mathcal{I}}\cup\Gamma_{\mathrm{D}}} \frac{1}{\sigma} |\{\!\{a\nabla\eta\}\!\}|^2 \mathrm{d}s + \int_{\Gamma_{\mathcal{I}}\cup\Gamma_{\mathrm{D}}} \sigma |[\![\eta]\!]|^2 \mathrm{d}s\,,$$

and $\gamma_1|_\kappa = \|c/c_0\|^2_{L^\infty(\kappa)}$.

- Recalling the Galerkin Orthogonality property,

$$A_h(u - u_h, v_h) = 0 \quad \forall v_h \in V_{h,\mathbf{p}},$$

we deduce that

$$A_h(\eta + \xi, \xi) = 0 \quad [\xi \in V_{h,\mathbf{p}}].$$

Thereby,

$$A_h(\xi, \xi) = -A_h(\eta, \xi).$$

- Exploiting coercivity gives

$$|||\xi|||^2 \leq \frac{1}{C} A_h(\xi, \xi) = -\frac{1}{C} A_h(\eta, \xi). \quad (23)$$

$$\mathcal{B}(\eta,\xi) \leq C|||\xi||| \left(\sum_{\kappa\in\mathcal{T}_h} \left(\|\sqrt{a}\nabla\eta\|^2_{L^2(\kappa)} + \gamma_1\|\eta\|^2_{L^2(\kappa)} + \|\eta^+\|^2_{\partial_+\kappa\cap\Gamma} \right.\right.$$
$$\left.\left. +\|\eta^-\|^2_{\partial_-\kappa\setminus\Gamma} \right) + \int_{\Gamma_{\mathcal{I}}\cup\Gamma_{\mathrm{D}}} \frac{1}{\sigma}|\{\!\{a\nabla\eta\}\!\}|^2 \mathrm{d}s + \int_{\Gamma_{\mathcal{I}}\cup\Gamma_{\mathrm{D}}} \sigma|[\![\eta]\!]|^2 \mathrm{d}s \right)^{1/2}.$$

- Substituting the above inequality into (23) gives the desired result.

Term II

We have that

$$\mathrm{II} = A_h(\xi, z - z_h) \le |||\xi|||\, \mathcal{H}(z - z_h),$$

where

$$\begin{aligned}\mathcal{H}(w)^2 \;=\; & \sum_{\kappa \in \mathcal{T}_h} \Big(\|\sqrt{a}\nabla w\|^2_{L^2(\kappa)} + \gamma_2 \|w\|^2_{L^2(\kappa)} + \|w^+\|^2_{\partial_-\kappa} \\ & + \|\sigma^{1/2}[\![w]\!]\|^2_{L^2(\partial\kappa \cap (\Gamma_{\mathcal{I}} \cup \Gamma_{\mathrm{D}}))} + \|\sigma^{-1/2}\{\!\{a\nabla w\}\!\}\|^2_{L^2(\partial\kappa \cap (\Gamma_{\mathcal{I}} \cup \Gamma_{\mathrm{D}}))} \Big).\end{aligned}$$

Thereby,

$$|\mathrm{II}| \le C\mathcal{G}(\eta)\, \mathcal{H}(z - z_h),$$

as required.

General Result

The following bound on the error in the functional $J(\cdot)$ holds:

$$|J(u) - J(u_h)| \leq C\mathcal{E}(\eta)\,\mathcal{F}(z - z_h) + C\mathcal{G}(\eta)\,\mathcal{H}(z - z_h)$$

for $z_h = \Pi_p z \in V_{h,\mathbf{p}}$.

- With $z_h = \Pi_p z$, we exploit approximation error estimates to bound $\eta = u - \Pi_p u$ and $z - \Pi_p z$.

The *a priori* error analysis will be pursued in two different settings:

1. Isotropic polynomial degrees, i.e.,

$$p_{\kappa,1} = p_{\kappa,2} = \ldots = p_{\kappa,d} \equiv p_\kappa \quad \forall \kappa \in \mathcal{T}_h.$$

   Thereby, writing $p = (p_\kappa : \ \kappa \in \mathcal{T}_h)$, we have

$$V_{h,\mathbf{p}} \equiv V_{h,p}.$$

2. Anisotropic polynomial degrees on axiparallel (2D) meshes, i.e., F_κ is an affine mapping of the form

$$F_\kappa(\hat{\mathbf{x}}) = A_\kappa \hat{\mathbf{x}} + \mathbf{b}_\kappa,$$

   where $A_\kappa := \frac{1}{2}\mathrm{diag}(h_1^\kappa, h_2^\kappa)$, with h_1^κ and h_2^κ the lengths of the edges of $\tilde{\kappa}$ parallel to the $\tilde{x}_1$- and $\tilde{x}_2$-axes, respectively.

Lemma

Given $\hat{v} \in H^k(\hat{\kappa})$, there exists $C > 0$, such that

$$
\begin{aligned}
\|\hat{v} - \hat{\Pi}_p \hat{v}\|_{L^2(\hat{\kappa})} &\leq \frac{C}{p^s} |\hat{v}|_{H^s(\hat{\kappa})}, && 0 \leq s \leq \min(p+1, k), \\
|\hat{v} - \hat{\Pi}_p \hat{v}|_{H^1(\hat{\kappa})} &\leq \frac{C}{p^{s-3/2}} |\hat{v}|_{H^s(\hat{\kappa})}, && 1 \leq s \leq \min(p+1, k), \\
\|\hat{v} - \hat{\Pi}_p \hat{v}\|_{L^2(\hat{F})} &\leq \frac{C}{p^{s-1/2}} |\hat{v}|_{H^s(\hat{\kappa})}, && 1 \leq s \leq \min(p+1, k), \\
|\hat{v} - \hat{\Pi}_p \hat{v}|_{H^1(\hat{F})} &\leq \frac{C}{p^{s-5/2}} |\hat{v}|_{H^s(\hat{\kappa})}, && 2 \leq s \leq \min(p+1, k),
\end{aligned}
$$

where $\hat{\Pi}_p : L^2(\hat{\kappa}) \to \mathcal{Q}_p(\hat{\kappa})$ denotes the elemental $L^2(\hat{\kappa})$–projection.

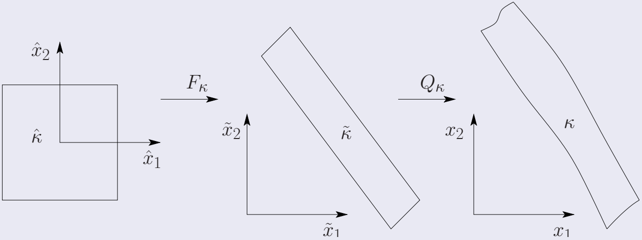
H., Schwab, & Süli 2002

References

Bänsch 1991, D'Azevedo & Simpson 1991, Rippa 1992, Apel & Dobrowolski 1992, Shenk 1994, Apel & Lube 1994, Ženíšek & Vanmaele 1995, Apel 1999→, Kunert 1999, Formaggia & Perotto 2000, 2001, Cao 2005, 2006, Huang 2005, 2006, Georgoulis 2006, . . .

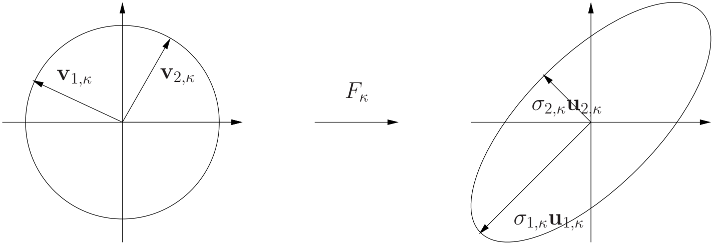
Element Mappings

$$V_{h,p} = \{u \in L^2(\Omega) \,:\, u|_\kappa \circ Q_\kappa \circ F_\kappa \in R_p(\kappa);\ \kappa \in \mathcal{T}_h\}\,.$$



- F_κ defines the size and orientation of the element κ.
- Q_κ defines the shape of κ.

H., Schwab, & Süli 2000, Georgoulis 2003



For $\kappa \in \mathcal{T}_h$, we write

$$J_{F_\kappa} = U_\kappa \Sigma_\kappa V_\kappa^\top,$$

where

$$\begin{aligned} U_\kappa &= (\mathbf{u}_{1,\kappa} \dots \mathbf{u}_{d,\kappa}), \qquad V_\kappa = (\mathbf{v}_{1,\kappa} \dots \mathbf{v}_{d,\kappa}), \\ \Sigma_\kappa &= \mathrm{diag}(\sigma_{1,\kappa}, \sigma_{2,\kappa}, \dots, \sigma_{d,\kappa}), \ \sigma_{1,\kappa} \geq \sigma_{2,\kappa} \geq \dots \geq \sigma_{d,\kappa} > 0. \end{aligned}$$

Formaggia & Perotto 2000, 2001

Observation: We note that

$$|\hat{v}|^2_{H^s(\hat{\kappa})} = \int_{\hat{\kappa}} \|\hat{\mathcal{D}}^s(\hat{v})\|_F^2 \mathrm{d}\hat{x},$$

where $\hat{\mathcal{D}}^s(\hat{v}) \in \mathbb{R}^{d\times d\times\cdots\times d}$ is the sth–order tensor

$$(\hat{\mathcal{D}}^s(\hat{v}))_{i_1,i_2,\ldots,i_s} = \frac{\partial^s \hat{v}}{\partial \hat{x}_{i_1}\cdots\partial \hat{x}_{i_s}}, \quad i_k = 1,\ldots,d, \text{ for } k = 1,\ldots,s,$$

and $\|\cdot\|_F$ denotes the Frobenius norm.

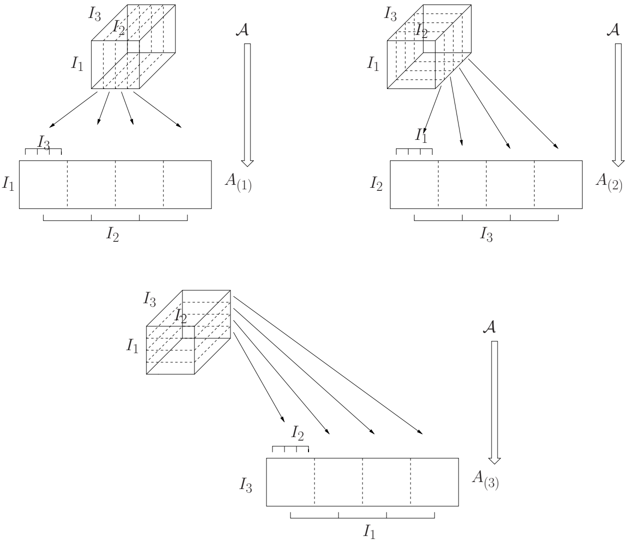
Definition (Matrix Unfolding)

$$\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N} \to A_{(n)} \in \mathbb{R}^{I_n \times (I_{n+1} I_{n+2} \ldots I_N I_1 I_2 \ldots I_{n-1})}, \quad n = 1, \ldots, N.$$

contains the element $a_{i_1 i_2 \ldots i_N}$ at the position with row number i_n and column number equal to

$$(i_{n+1} - 1) I_{n+2} I_{n+3} \ldots I_N I_2 \ldots I_{n-1} + (i_{n+2} - 1) I_{n+3} I_{n+4} \ldots I_N I_1 I_2 \ldots I_{n-1} + \ldots$$
$$+ (i_N - 1) I_1 I_2 \ldots I_{n-1} + (i_1 - 1) I_2 I_3 \ldots I_{n-1} + (i_2 - 1) I_3 I_4 \ldots I_{n-1} + \ldots + i_{n-1}.$$

Lethauwer, De Moor and Vandewalle 1999


I_3
I_2
I_1
$\mathcal{A}$
I_3
I_1
$A_{(1)}$
I_2
I_3
I_2
I_1
$\mathcal{A}$
I_1
I_2
$A_{(2)}$
I_3
I_3
I_2
I_1
$\mathcal{A}$
I_2
I_3
$A_{(3)}$
I_1

Definition (n-Mode Product)

Multiply $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ by a matrix $U \in \mathbb{R}^{J_n \times I_n}$

$$(\mathcal{A} \times_n U)_{i_1 i_2 \ldots i_{n-1} j_n i_{n+1} \ldots i_N} := \sum_{i_n=1}^{I_n} (\mathcal{A})_{i_1 i_2 \ldots i_{n-1} i_n i_{n+1} \ldots i_N} (U)_{j_n i_n}.$$

$$\underbrace{\mathcal{A} \xrightarrow{\text{Unfold}} A_{(n)} \xrightarrow{U\times} UA_{(n)} \xrightarrow{\text{Refold}} \mathcal{A} \times_n U}_{\times_n U}$$

Lethauwer, De Moor and Vandewalle 1999

For a matrix A:

$$\begin{aligned} A \times_1 U &= UA \\ A \times_2 U &= AU^\top \end{aligned}$$

Chain rule

$$\frac{\partial^s \hat{v}}{\partial \hat{x}_{i_1} \cdots \partial \hat{x}_{i_s}} = \sum_{j_1=1}^{d} \cdots \sum_{j_s=1}^{d} (J_{F_\kappa})_{j_1 i_1} \cdots (J_{F_\kappa})_{j_s i_s} \frac{\partial^s \tilde{v}}{\partial \tilde{x}_{j_1} \cdots \partial \tilde{x}_{j_s}},$$

i.e.

$$\hat{\mathcal{D}}^s(\hat{v}) = \tilde{\mathcal{D}}^s(\tilde{v}) \times_1 J_{F_\kappa}^\top \times_2 J_{F_\kappa}^\top \times_3 \ldots \times_s J_{F_\kappa}^\top.$$

$$s = 1: \ \nabla_{\hat{x}} \hat{v} = J_{F_\kappa}^\top \nabla_{\tilde{x}} \tilde{v}, \qquad s = 2: \ H_{\hat{x}}(\hat{v}) = J_{F_\kappa}^\top H_{\tilde{x}}(\tilde{v}) J_{F_\kappa}$$

Given that $J_{F_\kappa} = U_\kappa \Sigma_\kappa V_\kappa^\top$, we have

$$|\det(J_{F_\kappa})| = \Pi_{i=1}^{d} \sigma_{i,\kappa}, \quad \|J_{F_\kappa}^{-\top}\|_2 = 1/\sigma_{d,\kappa}, \quad \mathrm{vol}_{d-1}(F) \le C\,\Pi_{i=1}^{d-1}\sigma_{i,\kappa}.$$

$$|\hat{v}|^2_{H^s(\hat{\kappa})} = |\det(J_{F_\kappa}^{-1})| \int_{\tilde{\kappa}} \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \dots \sum_{i_s=1}^{d} (\sigma_{i_1,\kappa}\sigma_{i_2,\kappa}\dots\sigma_{i_s,\kappa})^2 (\tilde{\mathcal{D}}^s(\tilde{v}) \times_1 \mathbf{u}_{i_1,\kappa}^\top \times_2 \mathbf{u}_{i_2,\kappa}^\top \times_3 \dots \times_s \mathbf{u}_{i_s,\kappa}^\top)^2 \mathrm{d}\tilde{x}.$$

Thereby,

$$|\hat{v}|^2_{H^2(\hat{\kappa})} = |\det(J_{F_\kappa}^{-1})| \int_{\tilde{\kappa}} \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} (\sigma_{i_1,\kappa}\sigma_{i_2,\kappa})^2 (\mathbf{u}_{i_1,\kappa}^\top \tilde{\mathcal{H}}(\tilde{v}) \mathbf{u}_{i_2,\kappa})^2 \mathrm{d}\tilde{x}.$$

Formaggia & Perotto 2000, 2001

- Define:

$$D^s_{\tilde{\kappa}}(\tilde{v}, \Sigma_\kappa, U_\kappa) = \sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\cdots\sum_{i_s=1}^{d}(\sigma_{i_1,\kappa}\sigma_{i_2,\kappa}\ldots\sigma_{i_s,\kappa})^2 \times (\tilde{\mathcal{D}}^s(\tilde{v}) \times_1 \mathbf{u}^\top_{i_1,\kappa} \times_2 \mathbf{u}^\top_{i_2,\kappa} \times_3 \ldots \times_s \mathbf{u}^\top_{i_s,\kappa})^2.$$

- $s = 1$:

$$D^1_{\tilde{\kappa}}(\tilde{v}, \Sigma_\kappa, U_\kappa) = \sum_{i_1=1}^{d}\sigma^2_{i_1,\kappa}(\mathbf{u}^\top_{i_1,\kappa}\tilde{\nabla}\tilde{v})^2$$

- $s = 2$:

$$D^2_{\tilde{\kappa}}(\tilde{v}, \Sigma_\kappa, U_\kappa) = \sum_{i_1=1}^{d}\sum_{i_2=1}^{d}(\sigma_{i_1,\kappa}\sigma_{i_2,\kappa})^2(\mathbf{u}^\top_{i_1,\kappa}\tilde{\mathcal{H}}(\tilde{v})\mathbf{u}_{i_2,\kappa})^2$$

Lemma

Given $v \in H^k(\kappa)$, there exists $C > 0$, such that

$$\begin{aligned}
\|v - \Pi_p v\|_{L^2(\kappa)} &\leq \frac{C}{p^s}\left[\int_{\tilde{\kappa}} D^s_{\tilde{\kappa}}(\tilde{v}, \Sigma_\kappa, U_\kappa)\mathrm{d}\tilde{\mathbf{x}}\right]^{\frac{1}{2}}, \quad 0 \leq s \leq \min(p+1, k), \\
|v - \Pi_p v|_{H^1(\kappa)} &\leq \frac{C}{p^{s-3/2}}|\sigma_{d,\kappa}|^{-1} \\
&\quad \times \left[\int_{\tilde{\kappa}} D^s_{\tilde{\kappa}}(\tilde{v}, \Sigma_\kappa, U_\kappa)\mathrm{d}\tilde{\mathbf{x}}\right]^{\frac{1}{2}}, \quad 1 \leq s \leq \min(p+1, k).
\end{aligned}$$

Lemma

Given $v \in H^k(\kappa)$, there exists $C > 0$, such that

$$
\begin{aligned}
\|v - \Pi_p v\|_{L^2(F)} &\leq \frac{C}{p^{s-1/2}} |\sigma_{d,\kappa}|^{-1/2} \\
&\quad \times \left[\int_{\tilde{\kappa}} D^s_{\tilde{\kappa}}(\tilde{v}, \Sigma_\kappa, U_\kappa) \mathrm{d}\tilde{\mathbf{x}} \right]^{\frac{1}{2}}, \quad 1 \leq s \leq \min(p+1, k), \\
|v - \Pi_p v|_{H^1(F)} &\leq \frac{C}{p^{s-5/2}} \left| \frac{m_F}{m_\kappa} \right|^{\frac{1}{2}} |\sigma_{d,\kappa}|^{-1} \\
&\quad \times \left[\int_{\tilde{\kappa}} D^s_{\tilde{\kappa}}(\tilde{v}, \Sigma_\kappa, U_\kappa) \mathrm{d}\tilde{\mathbf{x}} \right]^{\frac{1}{2}}, \quad 2 \leq s \leq \min(p+1, k).
\end{aligned}
$$

Theorem

Given that $u|_\kappa \in H^{k_\kappa}(\kappa)$ *and* $z|_\kappa \in H^{l_\kappa}(\kappa)$, *we have*

$$\begin{aligned}
&|J(u) - J(u_h)|^2 \\
\leq\ & C \left(\sum_{\kappa \in \mathcal{T}_h} \left\{ \frac{\|a\|_\kappa}{\sigma_{d,\kappa}^2 p_\kappa^{2(s_\kappa - 3/2)}} + \frac{\|\mathbf{b}\|_\kappa}{\sigma_{d,\kappa} p_\kappa^{2(s_\kappa - 1/2)}} \right\} \int_{\tilde{\kappa}} D_{\tilde{\kappa}}^{s}(\tilde{u}, \Sigma_\kappa, U_\kappa) \mathrm{d}\tilde{x} \right) \\
& \times \left(\sum_{\kappa \in \mathcal{T}_h} \left\{ \frac{\|a\|_\kappa}{\sigma_{d,\kappa}^2 p_\kappa^{2(t_\kappa - 3/2)}} + \frac{\|\mathbf{b}\|_\kappa}{\sigma_{d,\kappa} p_\kappa^{2(t_\kappa - 1)}} \right\} \int_{\tilde{\kappa}} D_{\tilde{\kappa}}^{t}(\tilde{z}, \Sigma_\kappa, U_\kappa) \mathrm{d}\tilde{x} \right).
\end{aligned}$$

where $2 \leq s_\kappa \leq \min(p_\kappa + 1, k_\kappa)$ *and* $2 \leq t_\kappa \leq \min(p_\kappa + 1, l)$. *For* $m = 2$, *we have*

$$D_{\tilde{\kappa}}^{m}(\tilde{v}, \Sigma_\kappa, U_\kappa) = \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} (\sigma_{i_1,\kappa} \sigma_{i_2,\kappa})^2 (\mathbf{u}_{i_1,\kappa}^\top \tilde{\mathcal{H}}(\tilde{v}) \mathbf{u}_{i_2,\kappa})^2.$$

Georgoulis, Hall, & H. 2007

Uniform Orders

Let us now assume uniform orders, i.e.,

$$p_\kappa = p, \quad s_\kappa = s, \quad t_k = t, \quad k_\kappa = k, \quad l_\kappa = l, \qquad s,\ t,\ k,\ l \text{ integers}$$

for all $\kappa \in \mathcal{T}_h$, and uniform isotropic elements with mesh size h

1. Diffusion–dominated case (General IP Methods):

$$\begin{aligned} |J(u) - J(u_h)| \quad &\leq \quad C\,\frac{h^{s+t-2}}{p^{k+l-2}}\,p\,\|u\|_{H^k(\Omega)}\|z\|_{H^l(\Omega)} \\ &\quad + C\,(1+\theta)\,\frac{h^{s-1}}{p^{k-3/2}}\,\|u\|_{H^k(\Omega)}\|z\|_{H^2(\Omega)}. \end{aligned}$$

2. Hyperbolic case:

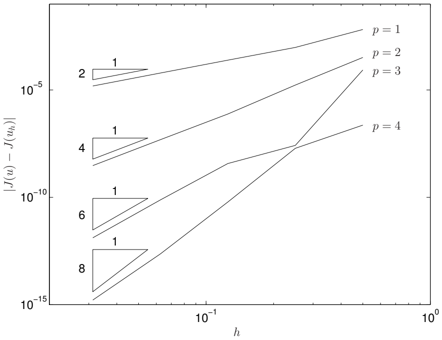
$$|J(u) - J(u_h)| \quad \leq \quad C\,\frac{h^{s+t-1}}{p^{k+l-1}}\,p^{1/2}\,\|u\|_{H^k(\Omega)}\|z\|_{H^l(\Omega)}.$$

$$u(x,y) = \frac{(1+x)^2}{4}\sin(2\pi xy)\,, \qquad J(u) \equiv M_\psi(u) = \int_\Omega u\psi \,\mathrm{d}x\,,$$

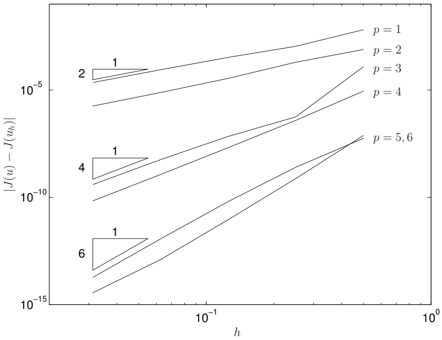
where, we define the weight function ψ by

$$\psi = \sin^2(2\pi x)\sin^2(2\pi y)\mathrm{e}^{-(x+y)}\,.$$

SIP ($\theta = -1$)



NIP ($\theta = 1$)



Theorem

For $u|_\kappa \in H^{k_\kappa}(\kappa)$ and $z|_\kappa \in H^{l_\kappa}(\kappa)$, we have

$$
\begin{aligned}
|J(u)-J(u_h)|^2 \quad \le \quad & C\Bigg(\sum_{\kappa\in\mathcal{T}_h}\sum_{i=1}^{2} \Phi(p_i^\kappa, s_i^\kappa, h_i^\kappa) \max_{(m,n)\in A} \left\{ \left(\frac{p_j^\kappa}{p_i^\kappa}\right)^m \left(\frac{h_i^\kappa}{h_j^\kappa}\right)^n \right\} \\
& \times \left(\alpha_\kappa p_i^\kappa + \frac{h_i^\kappa}{p_i^\kappa}\beta_2 + \left(\frac{h_i^\kappa}{p_i^\kappa}\right)^2 (\beta_1+\gamma_1)\right) |u|^2_{s_i^\kappa,\kappa,i} \Bigg) \\
& \times \Bigg(\sum_{\kappa\in\mathcal{T}_h}\sum_{i=1}^{2} \Phi(p_i^\kappa, s_i^\kappa, h_i^\kappa) \max_{(m,n)\in A} \left\{ \left(\frac{p_j^\kappa}{p_i^\kappa}\right)^m \left(\frac{h_i^\kappa}{h_j^\kappa}\right)^n \right\} \\
& \times \left(\alpha_\kappa p_i^\kappa + h_i^\kappa\beta_2 + \left(\frac{h_i^\kappa}{p_i^\kappa}\right)^2 (\beta_1+\gamma_2)\right) |z|^2_{t_i^\kappa,\kappa,i} \Bigg),
\end{aligned}
$$

for $2 \le s_i^\kappa \le \min(p_i^\kappa + 1, k_\kappa)$ and $2 \le t_i^\kappa \le \min(p_i^\kappa + 1, l_\kappa)$, where

$$
\Phi(p,s,h) := \left(\frac{(p-(s-1))!}{(p+(s-1))!}\right)\left(\frac{h}{2}\right)^{2(s-1)} \sim \left(\frac{h}{p}\right)^{2(s-1)},
$$

$$
|w|_{r,\kappa,i} := \left(\|\tilde\partial_i^r \tilde w\|^2_{\tilde\kappa} + \left(\frac{h_j^\kappa}{h_i^\kappa}\right)^2 \|\tilde\partial_i^{r-1}\tilde\partial_j \tilde w\|^2_{\tilde\kappa}\right)^{1/2}, \quad A = \{(0,0),(0,1),(0,2),(-1,0),(-1,1),(2,2)\}.
$$

Georgoulis, Hall, & H. 2007

Corollary

For u and z real-analytic functions

$$|J(u) - J(u_h)|^2 \leq C\Big(\sum_{\kappa\in\mathcal{T}_h}\sum_{i=1}^{2} \mathrm{e}^{-r_i p_i^\kappa} N_i^\kappa\Big)\Big(\sum_{\kappa\in\mathcal{T}_h}\sum_{i=1}^{2} \mathrm{e}^{-q_i p_i^\kappa} N_i^\kappa\Big),$$

where

$$N_i^\kappa := \left(h_i^\kappa\right)^{2s_i^\kappa} |\tilde{\kappa}| \max_{(m,n)\in A}\left\{\left(p_i^\kappa\right)^{4-m}\left(p_j^\kappa\right)^{m}\left(\frac{h_i^\kappa}{h_j^\kappa}\right)^{n}\right\}.$$

Georgoulis, Hall, & H. 2007

- Goal:

$$|J(u) - J(u_h)| \leq \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa(u_h) \leq \texttt{Tol}.$$

- Automatic refinement algorithm:

0. Start with initial (coarse) grid $\mathcal{T}_h^{(j=0)}$.
1. Compute the numerical solution $u_h^{(j)}$ on $\mathcal{T}_h^{(j)}$.
2. Compute the local error indicators η_κ.
3. If $\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa \leq \texttt{Tol} \rightarrow$ stop. Otherwise, adapt $V_{h,p}$.
4. $j = j + 1$, and go to step (1).

hp–Refinement

Basic Strategy:

	High–Error	Low–Error
Solution Smooth	$p \to p+1$	$h \to 2h$
Solution Nonsmooth	$h \to h/2$	$p \to p-1$

hp–Adaptive Strategy:

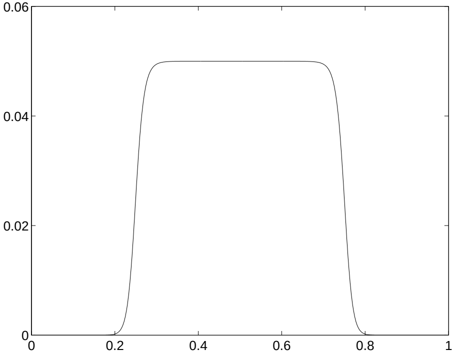
	Refinement	Derefinement
u or z smooth	$p_\kappa \to p_\kappa+1$	$h_\kappa \to 2h_\kappa$
Otherwise	$h_\kappa \to h_\kappa/2$	$p_\kappa \to p_\kappa-1$

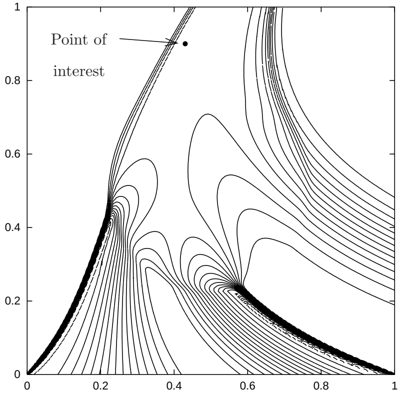
$\Rightarrow$ Exploit the analyticity smoothness algorithm to estimate the regularity of u and z .

We let $\Omega = (0,1)^2$, $f = 0$, $\mathbf{b} = (2y^2 - 4x + 1, 1 + y)$, $c = -\nabla \cdot \mathbf{b}$, and

$$
\begin{aligned}
a &= \varepsilon(x,y)I, \\
\varepsilon &= \frac{\delta}{2}(1 - \tanh((r - 1/4)(r + 1/4)/\gamma)),
\end{aligned}
$$

where $r^2 = (x - 1/2)^2 + (y - 1/2)^2$ and $\delta = 0.05$ and $\gamma = 0.01$.

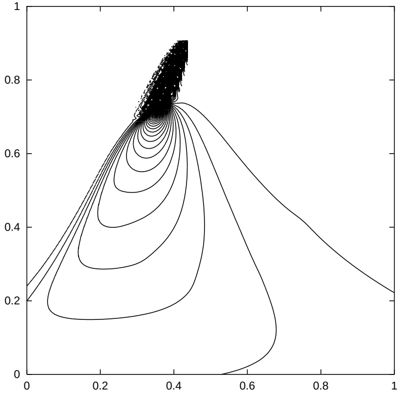




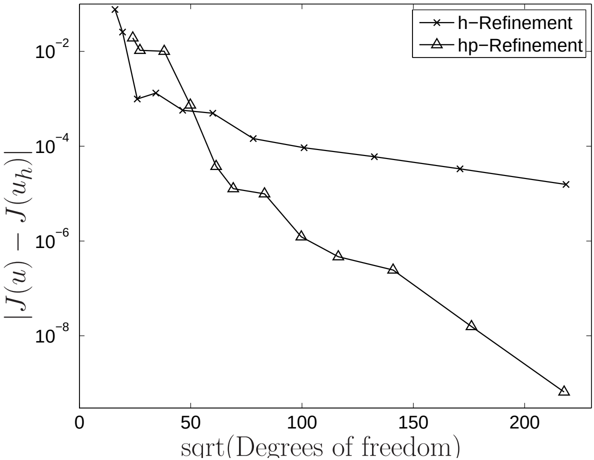
Point of
interest
0
0.2
0.4
0.6
0.8
1

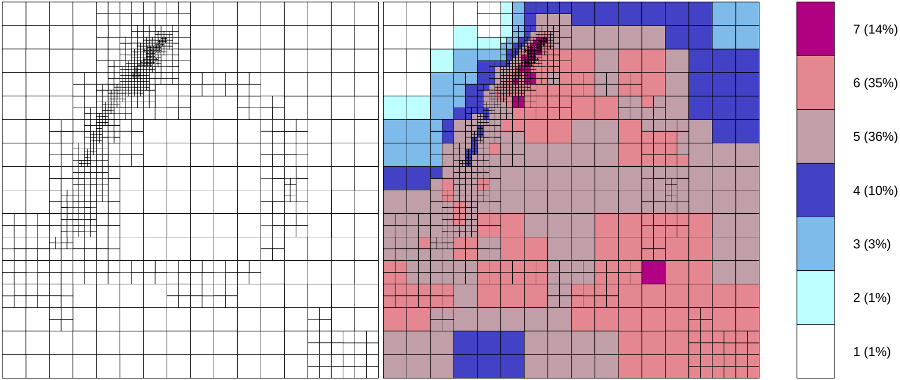
Functional:

$$J(u) = u(0.43, 0.9).$$



Dual Solution


h−Refinement
hp−Refinement
10^{-2}
10^{-4}
10^{-6}
10^{-8}
$|J(u) - J(u_h)|$
0
50
100
150
200
sqrt(Degrees of freedom)


7 (14%)
6 (35%)
5 (36%)
4 (10%)
3 (3%)
2 (1%)
1 (1%)

Anisotropic h–adaptivity

- Minimize *a priori*/interpolation error bounds
  $\Rightarrow$ Hessian(gradient)–based strategy

  Castro-Díaz, Hecht, Mohammadi, & Pironneau 1997, Habashi, Dompierre, Bourgault, Ait-Ali-Yahia, Fortin, & Vallet 2000, Formaggia & Perotto 2000, 2001, Venditti & Darmofal 2003, Apel, Grosman, Jimack, & Meyer 2004, Belhamadia, Fortin, & Chamberland 2004, Dolejší & Felcman 2004, Huang 2005, 2006

- Competitive refinement strategy exploiting local *a posteriori* error bounds.
  - Schneider & Jimack 2005 (Optimal node location)
  - Rachowicz, Demkowicz, & Oden 1989, Solin & Demkowicz 2004 $\rightarrow$ (hp–adaptivity)

Hessian-Based Strategy

- Approximation Result ($p = 1$)

$$\|v - \Pi_p v\|^2_{L^2(\kappa)} \leq C \int_\kappa \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} (\sigma_{i_1,\kappa}\sigma_{i_2,\kappa})^2 (\mathbf{u}_{i_1,\kappa}^\top \mathcal{H}(v) \mathbf{u}_{i_2,\kappa})^2 \mathrm{d}x$$

- Minimize the right-hand side

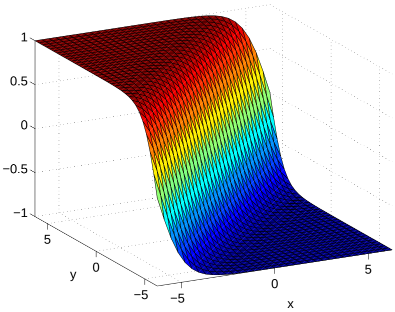
$$\mathcal{H}(v) = R\Lambda R^T, \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2), \quad \lambda_1 > \lambda_2 > 0.$$

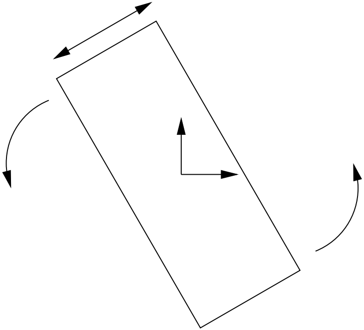
Defining the cell aspect ratio $\tau = \sigma_{1,\kappa}/\sigma_{2,\kappa}$, the minimum occurs at:

$$\tau^2 = \lambda_1/\lambda_2 \quad \text{and} \quad \mathbf{u}_{1,\kappa} \parallel \mathbf{r}_2.$$

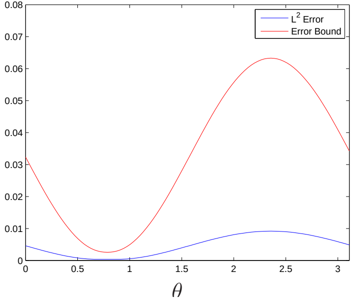
Formaggia & Perotto 2000, 2001, Hall 2007

Optimality

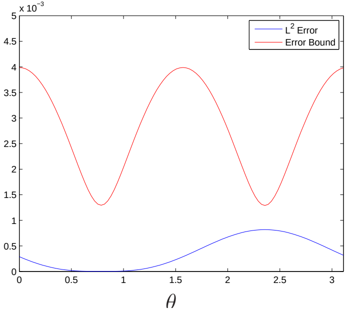




Optimality

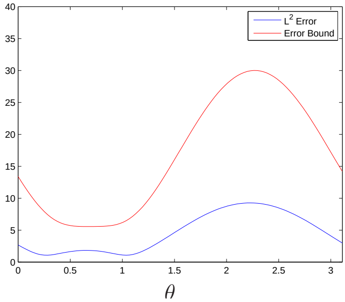


$p = 1$

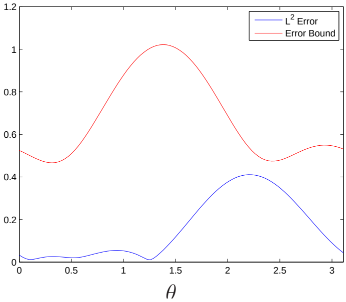


$p = 2$

$$u = \sin(\pi(\cos(\pi/6)x + \sin(\pi/6)y))\, \mathrm{e}^{10(-\sin(\pi/6)x + \cos(\pi/6)y)}$$



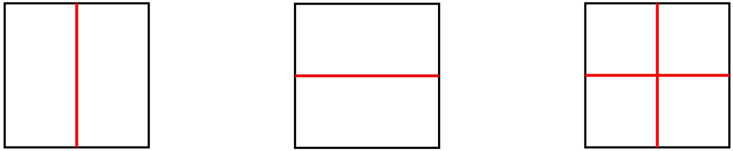
$p = 1$



$p = 2$

Adaptive Algorithm (h–Adaptivity)

- Elements marked for refinement/derefinement using the fixed–fraction strategy.
- Element refinements.



$\mathcal{R}_{\kappa,1} \equiv \sum_{\kappa' \in \mathcal{T}_{h,1}} \eta_{\kappa',1}$ $\mathcal{R}_{\kappa,2} \equiv \sum_{\kappa' \in \mathcal{T}_{h,2}} \eta_{\kappa',2}$ $\mathcal{R}_{\kappa,3} \equiv \sum_{\kappa' \in \mathcal{T}_{h,3}} \eta_{\kappa',3}$

- Solve local primal and adjoint problems on elemental patches.
- Boundary data extracted from global primal and dual solutions.

Primal: $L\tilde{u}_h = f$ in κ, $B\tilde{u}_h = Bu_h$ on $\partial\kappa \Rightarrow$ Anisotropy from cell error

H. & Süli 1998, H., Mackenzie, Süli & Warnecke 1999

Adaptive Algorithm (h–Adaptivity)

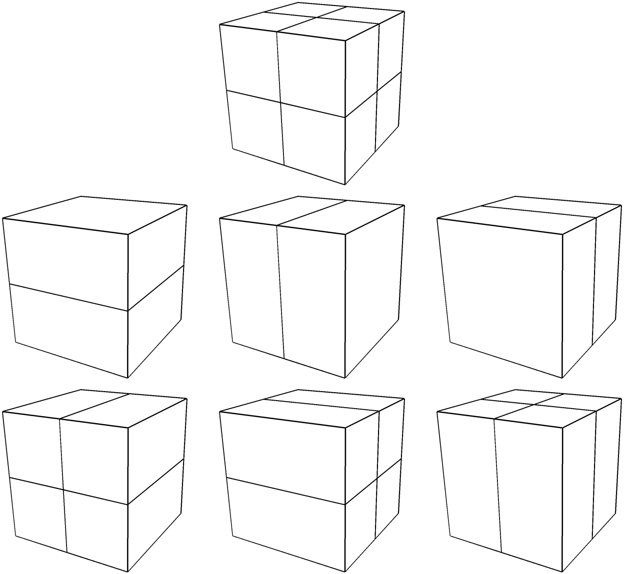
Algorithm 1

Select optimal refinement

$$\max_{i=1,2,3}(|\eta_\kappa^{\text{old}}| - |\mathcal{R}_{\kappa,i}|)/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\kappa)).$$

Algorithm 2

- Prescribe an h–anisotropy parameter $\theta_h > 1$.
- When
$$\frac{\max_{i=1,2}(|\mathcal{R}_{\kappa,i}|)}{\min_{i=1,2}(|\mathcal{R}_{\kappa,i}|)} > \theta_h,$$
perform refinement in direction with minimal $|\mathcal{R}_{\kappa,i}|$, $i = 1, 2$.
- else perform isotropic h–refinement.



We let $\Omega = (0,1)^2$, $a = \varepsilon I$, $0 < \varepsilon \ll 1$, $\mathbf{b} = (1,1)^\top$, $c = 0$.
Choose f so that

$$u(x,y) = x + y(1-x) + \left[\mathrm{e}^{-1/\varepsilon} - \mathrm{e}^{-(1-x)(1-y)/\varepsilon}\right]\left[1 - \mathrm{e}^{-1/\varepsilon}\right]^{-1}.$$

$\varepsilon = 10^{-2}$

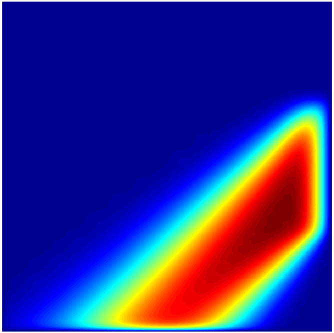
Functional: Weighted mean value of u over Ω, i.e.,

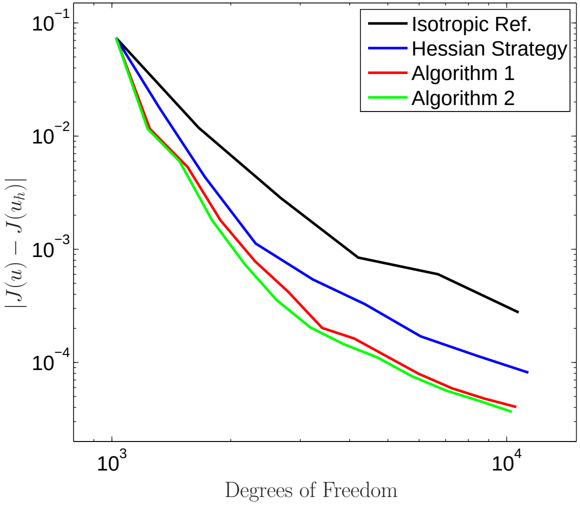
$$J(u) = \int_\Omega u\psi \mathrm{d}x,$$

where

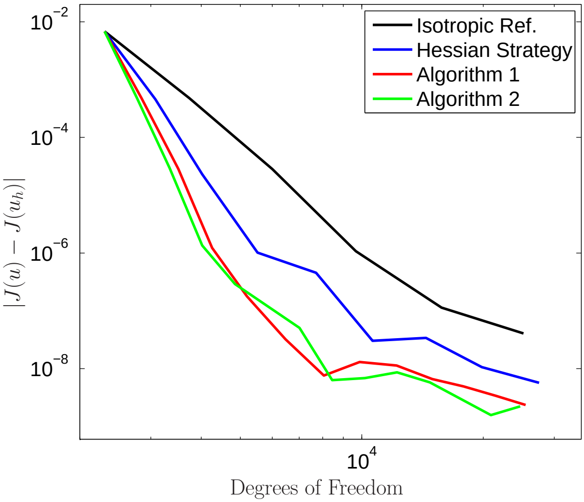
$$\psi = 100(1 - \tanh(100(r_1 - 0.01)(r_1 + 0.01)))(1 - \tanh(100(r_2 - 0.2)(r_2 + 0.2))),$$

$$r_1 = x - 1.0 \text{ and } r_2 = y - 0.5.$$



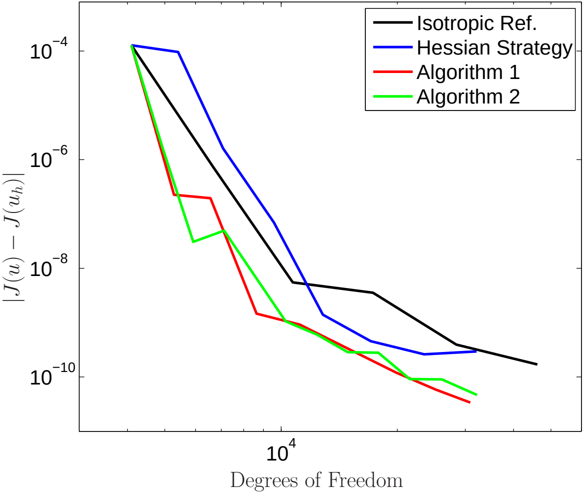

Isotropic Ref.
Hessian Strategy
Algorithm 1
Algorithm 2
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$|J(u) - J(u_h)|$
10^{3}
10^{4}
Degrees of Freedom

$$p = 1$$



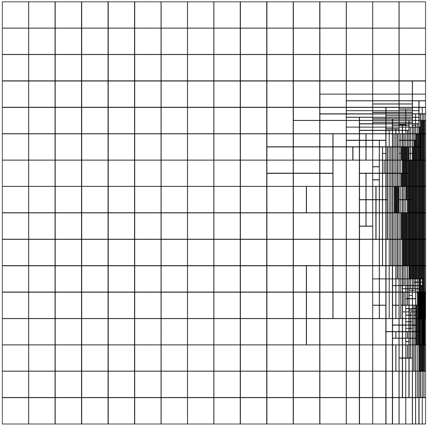
10^{-2}
10^{-4}
10^{-6}
10^{-8}
$|J(u) - J(u_h)|$
10^{4}
Degrees of Freedom
Isotropic Ref.
Hessian Strategy
Algorithm 1
Algorithm 2

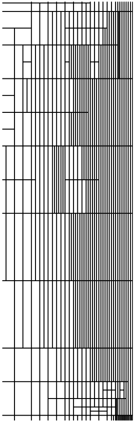
$p = 2$



Isotropic Ref.
Hessian Strategy
Algorithm 1
Algorithm 2
10^{-4}
10^{-6}
10^{-8}
10^{-10}
$|J(u) - J(u_h)|$
10^{4}
Degrees of Freedom

$p = 3$





Anisotropic mesh using Algorithm 2 after 7 adaptive refinements including detail at [0.9,.5]

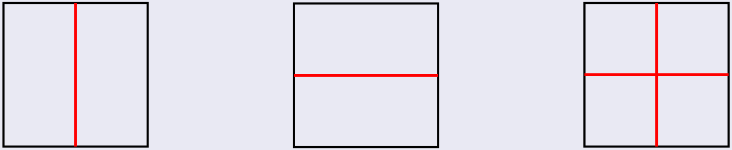
- Goal:

$$J(u) - J(u_h) \leq \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa(u_h) \leq \texttt{Tol}.$$

- Automatic *hp*–refinement algorithm:

  0. Mark the elements for refinement/derefinement according to the size of the local error indicators $|\eta_\kappa|$.
  1. Apply regularity estimation to both u and z in order to determine whether h-/p-refinement/derefinement should be undertaken.
  2. For elements marked for refinement, determine the direction of h-/p-enrichment on the basis of performing the most competitive refinement.

Local Problems



$\mathcal{R}_{\kappa,1} \equiv \sum_{\kappa' \in \mathcal{T}_{h,1}} \eta_{\kappa',1}$ $\mathcal{R}_{\kappa,2} \equiv \sum_{\kappa' \in \mathcal{T}_{h,2}} \eta_{\kappa',2}$ $\mathcal{R}_{\kappa,3} \equiv \sum_{\kappa' \in \mathcal{T}_{h,3}} \eta_{\kappa',3}$

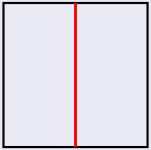
Algorithm 1

Select optimal refinement

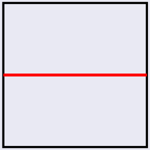
$$\max_{i=1,2,3} (|\eta_\kappa^{\text{old}}| - |\mathcal{R}_{\kappa,i}|)/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\kappa)).$$

Anisotropic h–Refinement

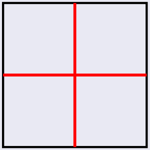
Local Problems



$\mathcal{R}_{\kappa,1} \equiv \sum_{\kappa' \in \mathcal{T}_{h,1}} \eta_{\kappa',1}$



$\mathcal{R}_{\kappa,2} \equiv \sum_{\kappa' \in \mathcal{T}_{h,2}} \eta_{\kappa',2}$



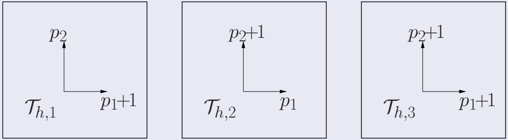
$\mathcal{R}_{\kappa,3} \equiv \sum_{\kappa' \in \mathcal{T}_{h,3}} \eta_{\kappa',3}$

Algorithm 2

- Prescribe an h–anisotropy parameter $\theta_h > 1$.
- When
$$\frac{\max_{i=1,2}(|\mathcal{R}_{\kappa,i}|)}{\min_{i=1,2}(|\mathcal{R}_{\kappa,i}|)} > \theta_h,$$
perform refinement in direction with minimal $|\mathcal{R}_{\kappa,i}|$, $i = 1, 2$.
- else perform isotropic h–refinement.

Anisotropic p–Refinement

Local Problems



$\mathcal{R}_{\kappa,1} \equiv \eta_{\kappa,1}$ $\quad \mathcal{R}_{\kappa,2} \equiv \eta_{\kappa,2}$ $\quad \mathcal{R}_{\kappa,3} \equiv \eta_{\kappa,3}$

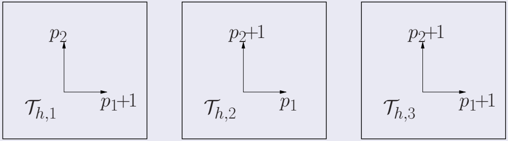
Algorithm 1

Select optimal refinement

$$\max_{i=1,2,3} (|\eta_\kappa^{\text{old}}| - |\mathcal{R}_{\kappa,i}|)/(\#\text{dofs}(V_{h,\mathbf{p}_i}(\kappa)) - \#\text{dofs}(V_{h,\mathbf{p}}(\kappa))).$$

Anisotropic p–Refinement

Local Problems



$\mathcal{R}_{\kappa,1} \equiv \eta_{\kappa,1}$ $\quad$ $\mathcal{R}_{\kappa,2} \equiv \eta_{\kappa,2}$ $\quad$ $\mathcal{R}_{\kappa,3} \equiv \eta_{\kappa,3}$

Algorithm 2

- Prescribe a p–anisotropy parameter $\theta_p > 1$
- When

$$\frac{\max_{i=1,2}(|\mathcal{R}_{\kappa,i}|/(\#\mathrm{dofs}(V_{h,\mathbf{p}_i}(\kappa)) - \#\mathrm{dofs}(V_{h,\mathbf{p}}(\kappa))))}{\min_{i=1,2}(|\mathcal{R}_{\kappa,i}|/(\#\mathrm{dofs}(V_{h,\mathbf{p}_i}(\kappa)) - \#\mathrm{dofs}(V_{h,\mathbf{p}}(\kappa))))} > \theta_p,$$

  enrich in polynomial in the direction with minimal $|\mathcal{R}_{\kappa,i}|$, $i = 1, 2$.
- else perform isotropic p–refinement.

We let $\Omega = (0,1)^2$, $a = \varepsilon I$, $0 < \varepsilon \ll 1$, $\mathbf{b} = (1,1)^\top$, $c = 0$.
Choose f so that

$$u(x,y) = x + y(1-x) + [\mathrm{e}^{-1/\varepsilon} - \mathrm{e}^{-(1-x)(1-y)/\varepsilon}]\,[1-\mathrm{e}^{-1/\varepsilon}]^{-1}.$$

Functional: Weighted mean value of u over Ω, i.e.,

$$J(u) = \int_\Omega u\psi \mathrm{d}x,$$

where

$$\begin{aligned}\psi &= 4(1-2y)(1-\mathrm{e}^{-\alpha(1-x)} - (1-\mathrm{e}^{-\alpha})(1-x)) \\ &\quad +4y(y-1)(\mathrm{e}^{-\alpha(1-x)}(\alpha - (1-\mathrm{e}^{-\alpha})).\end{aligned}$$

Here, we set $\alpha = 100$.

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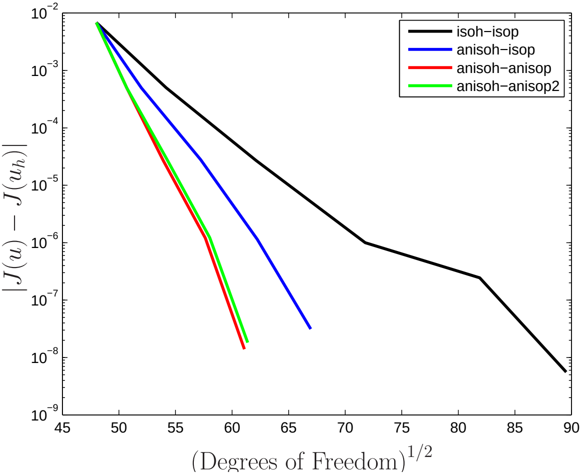
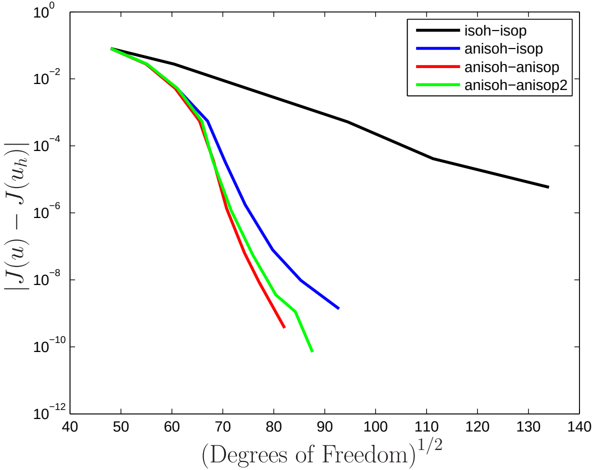
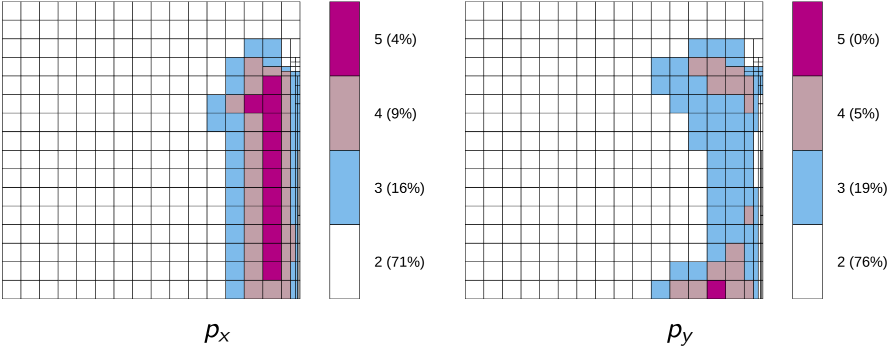


$\varepsilon = 0.01$

Convection-Diffusion Equation



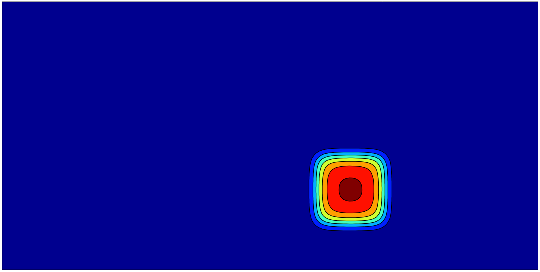
$\varepsilon = 0.001$



Anisotropic mesh using Algorithm 2 after 4 adaptive *hp*-refinements, $\varepsilon = 0.01$

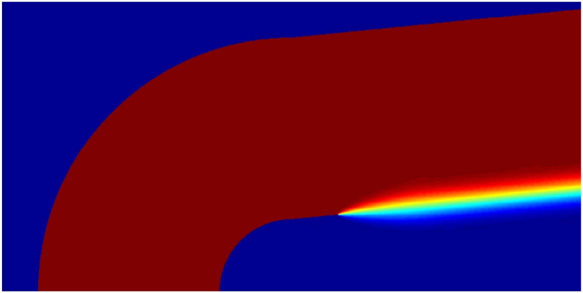
We let $\Omega = (0,2) \times (0,1)$, $f = 0$, $c = 0$,

$$\begin{aligned} a &= \varepsilon(x,y)I, \\ \varepsilon &= (1 - \tanh(100(r_1 - 0.12)(r_1 + 0.12))) \\ &\quad \times(1 - \tanh(100(r_2 - 0.12)(r_2 + 0.12)))/1000, \\ r_1 &= x - 1.3 \text{ and } r_2 = y - 0.3. \end{aligned}$$



We let $\Omega = (0,2) \times (0,1)$, $f = 0$, $c = 0$,

$$\mathbf{b} = \begin{cases} (y, 1-x)^\top & x < 1, \\ (1, 1/10)^\top & x \geq 1, \end{cases} \qquad u(x,y) = \begin{cases} 1 & y = 0,\ 1/8 < x < 3/4, \\ 0 & \text{elsewhere.} \end{cases}$$



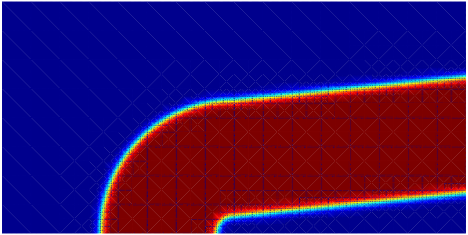
Functional:

Weighted outflow advective flux along $x = 2$, $0 \le y \le 1$, i.e.,

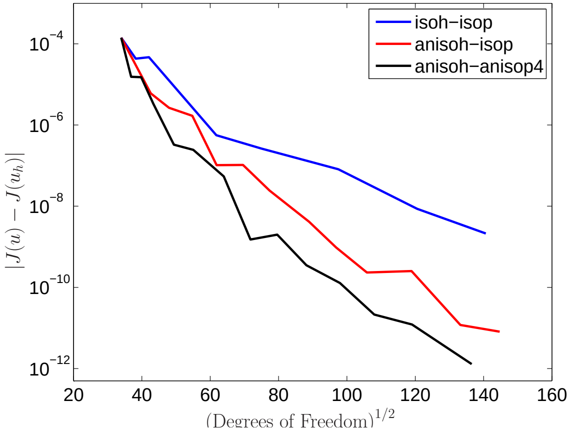
$$J(u) = \int_0^1 (\mathbf{b} \cdot \mathbf{n}) u(2, y) \psi(y) \mathrm{d}y,$$

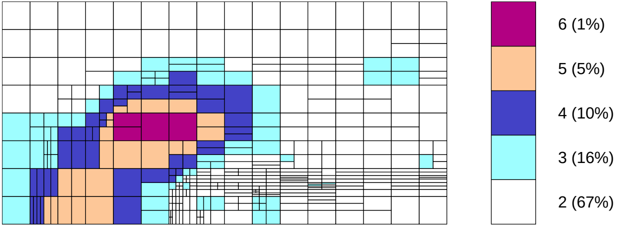
where

$$\psi(y) = \begin{cases} (\tanh(50(y - 7/40)) + 1)/2 & y < 17/40, \\ (\tanh(-50(y - 27/40)) + 1)/2 & y \ge 17/40. \end{cases}$$

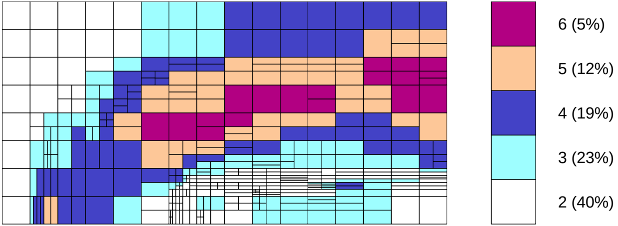


Dual Solution


isoh−isop
anisoh−isop
anisoh−anisop4
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
$|J(u) - J(u_h)|$
20
40
60
80
100
120
140
160
$(\text{Degrees of Freedom})^{1/2}$



p_x



p_y

- Given $\Omega \subset \mathbb{R}^3$, with boundary $\Gamma = \partial\Omega$, find $\mathbf{u} : \Omega \to \mathbb{R}^5$, such that

$$\operatorname{div}(\mathcal{F}^c(\mathbf{u}) - \mathcal{F}^v(\mathbf{u}, \nabla\mathbf{u})) = \mathbf{0} \quad \text{in } \Omega.$$

- Here, $\mathbf{u} = [\rho, \rho v_1, \rho v_2, \rho v_3, \rho E]^\top$.
  - ρ: density
  - $\mathbf{v} = (v_1, v_2, v_3)^\top$: velocity vector
  - p: pressure ($p = (\gamma - 1)\rho(E - \frac{1}{2}\mathbf{v}^2)$)
  - $\gamma = c_p/c_v$: ratio of specific heat capacities at constant pressure (c_p) and constant volume (c_v); for dry air, $\gamma = 1.4$.
  - E: Specific total energy
  - T: Temperature ($\mathcal{K}T = \frac{\mu\gamma}{Pr}(E - \frac{1}{2}\mathbf{v}^2)$)
  - $Pr = 0.72$: Prandtl number
  - $\mathcal{K}$: Thermal conductivity coefficient
  - H: Total enthalpy defined by ($H = E + p/\rho$)
  - μ: Dynamic viscosity coefficient

- Convective fluxes: $\mathcal{F}^c(\mathbf{u}) = (\mathbf{f}_1^c(\mathbf{u}), \mathbf{f}_2^c(\mathbf{u}), \mathbf{f}_3^c(\mathbf{u}))^\top$

$$\mathbf{f}_1^c(\mathbf{u}) = \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \\ \rho H v_1 \end{bmatrix}, \quad \mathbf{f}_2^c(\mathbf{u}) = \begin{bmatrix} \rho v_2 \\ \rho v_2 v_1 \\ \rho v_2^2 + p \\ \rho v_2 v_3 \\ \rho H v_2 \end{bmatrix}, \quad \mathbf{f}_3^c(\mathbf{u}) = \begin{bmatrix} \rho v_3 \\ \rho v_3 v_1 \\ \rho v_3 v_2 \\ \rho v_3^2 + p \\ \rho H v_3 \end{bmatrix}.$$

- Viscous fluxes: $\mathcal{F}^v(\mathbf{u}) = (\mathbf{f}_1^v(\mathbf{u}), \mathbf{f}_2^v(\mathbf{u}), \mathbf{f}_3^v(\mathbf{u}))^\top$

$$\mathbf{f}_k^v(\mathbf{u}, \nabla\mathbf{u}) = \begin{pmatrix} 0 \\ \tau_{1k} \\ \tau_{2k} \\ \tau_{3k} \\ \tau_{kl} v_l + \mathcal{K} T_{x_k} \end{pmatrix}, \qquad k = 1, 2, 3.$$

- Viscous stress tensor

$$\tau = \mu \left(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top - \tfrac{2}{3}(\nabla\cdot\mathbf{v}) I \right).$$

- Writing

$$\mathcal{F}_i^v(\mathbf{u}, \nabla \mathbf{u}) = G_{ij}(\mathbf{u}) \partial \mathbf{u} / \partial x_j, \qquad i = 1, \dots, 3,$$

where G denotes the homogeneity tensor with entries

$$G_{ij}(\mathbf{u}) = \partial \mathcal{F}_i^v(\mathbf{u}, \nabla \mathbf{u}) / \partial \mathbf{u}_{x_j},$$

$i, j = 1, \dots, 3$, gives

$$\frac{\partial}{\partial x_i}\left(\mathcal{F}_i^c(\mathbf{u}) - G_{ij}(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_j}\right) = 0 \quad \text{in } \Omega.$$

- Boundary conditions:

$$\Gamma = \Gamma_{\mathrm{D,sup}} \cup \Gamma_{\mathrm{D,sub\text{-}in}} \cup \Gamma_{\mathrm{D,sub\text{-}out}} \cup \Gamma_{\mathrm{W}},$$

where

  - $\Gamma_{\mathrm{D,sup}}$: Dirichlet (supersonic)
  - $\Gamma_{\mathrm{D,sub\text{-}in}}$: Dirichlet (subsonic-inflow)
  - $\Gamma_{\mathrm{D,sub\text{-}out}}$: Dirichlet (subsonic-outflow)
  - Γ_{W}: Solid wall boundaries

- We impose

$$\mathcal{B}(\mathbf{u}) = \mathcal{B}(\mathbf{g}_D) \quad \text{on} \quad \Gamma_{\mathrm{D,sup}} \cup \Gamma_{\mathrm{D,sub\text{-}in}} \cup \Gamma_{\mathrm{D,sub\text{-}out}},$$

where $\mathbf{g}_{\mathrm{D}}$ is a prescribed Dirichlet condition.

  - $\Gamma_{\mathrm{D,sup}}$: $\mathcal{B}(\mathbf{u}) = \mathbf{u}$.
  - $\Gamma_{\mathrm{D,sub\text{-}in}}$: $\mathcal{B}(\mathbf{u}) = (u_1, u_2, u_3, u_4, 0)^\top$.
  - $\Gamma_{\mathrm{D,sub\text{-}out}}$: $\mathcal{B}(\mathbf{u}) = \left(0, 0, 0, 0, (\gamma - 1)(u_5 - (u_2^2 + u_3^2 + u_4^2)/(2u_1))\right)^\top$.

- Wall boundary conditions:

$$\Gamma_W = \Gamma_{\text{iso}} \cup \Gamma_{\text{adia}},$$

where

- Γ_{iso}: Isothermal
- Γ_{adia}: Adiabatic

- We impose

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{W}}, \qquad T = T_{\text{wall}} \quad \text{on} \quad \Gamma_{\text{iso}}, \qquad \mathbf{n} \cdot \nabla T = 0 \quad \text{on} \quad \Gamma_{\text{adia}},$$

where T_{wall} is a given wall temperature.

- $\mathcal{T}_h$ is a non-degenerate mesh consisting of elements of granularity h.
- *hp*-DG finite element space:

$$\mathbf{V}_{h,p} = \{\mathbf{v} \in \left[L^2(\Omega)\right]^5 : \mathbf{v}|_\kappa \in [\mathcal{S}_{p_\kappa}(\kappa)]^5 \quad \forall\kappa \in \mathcal{T}_h\},$$

where

$$\mathcal{S}_{p_\kappa}(\kappa) = \begin{cases} \mathcal{P}_{p_\kappa}(\kappa) & \text{if } \kappa \text{ is a simplex,} \\ \mathcal{Q}_{p_\kappa}(\kappa) & \text{if } \kappa \text{ is a hypercube.} \end{cases}$$

- $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{B}} \cup \mathcal{F}_h^{\mathcal{I}}$ denotes the set of all faces in the mesh $\mathcal{T}_h$.

Notation

- Let $\mathbf{v}$ and $\underline{\tau}$ be vector- and matrix-valued functions, respectively.
- Average Operators:

$$\{\!\{\mathbf{v}\}\!\} = (\mathbf{v}^+ + \mathbf{v}^-)/2, \qquad \{\!\{\underline{\tau}\}\!\} = (\underline{\tau}^+ + \underline{\tau}^-)/2.$$

- Jump Operator:

$$\underline{[\![\mathbf{v}]\!]} = \mathbf{v}^+ \otimes \mathbf{n}_{\kappa^+} + \mathbf{v}^- \otimes \mathbf{n}_{\kappa^-}.$$

- For matrices $\underline{\sigma}, \underline{\tau} \in \mathbb{R}^{m\times n}$, $m, n \geq 1$,

$$\underline{\sigma} : \underline{\tau} = \sum_{k=1}^{m} \sum_{l=1}^{n} \sigma_{kl}\tau_{kl}.$$

- For vectors $\mathbf{v} \in \mathbb{R}^m, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} \otimes \mathbf{w} \in \mathbb{R}^{m\times n}$,

$$(\mathbf{v} \otimes \mathbf{w})_{kl} = v_k\, w_l.$$

$$\begin{aligned}\mathcal{N}(\mathbf{u}_h, \mathbf{v}) \;&:=\; -\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla_h \mathbf{v}\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} \mathcal{H}(\mathbf{u}^+, \mathbf{u}^-, \mathbf{n}_\kappa)\cdot \mathbf{v}^+ \mathrm{d}s \\ &+ \int_\Omega \mathcal{F}^v(\mathbf{u}_h, \nabla_h \mathbf{u}_h) : \nabla_h \mathbf{v}\,\mathrm{d}\mathbf{x} - \sum_{F\in\mathcal{F}_h^{\mathcal{I}}} \int_F \{\!\!\{\mathcal{F}^v(\mathbf{u}_h, \nabla_h \mathbf{u}_h)\}\!\!\} : \underline{[\![\mathbf{v}]\!]}\mathrm{d}s \\ &- \sum_{F\in\mathcal{F}_h^{\mathcal{I}}} \int_F \{\!\!\{G^\top(\mathbf{u}_h)\nabla_h \mathbf{v}\}\!\!\} : \underline{[\![\mathbf{u}_h]\!]}\mathrm{d}s + \sum_{F\in\mathcal{F}_h^{\mathcal{I}}} \int_F \underline{\delta}(\mathbf{u}_h) : \underline{[\![\mathbf{v}]\!]}\mathrm{d}s \\ &+ \mathcal{N}_\Gamma(\mathbf{u}_h, \mathbf{v}).\end{aligned}$$

DGFEM

Find $\mathbf{u}_h \in \mathbf{V}_{h,p}$ such that

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in \mathbf{V}_{h,p}.$$

Optimal order IPDGFEM

Three key ingredients:

1. Definition of the interior penalty terms.

   Hartmann & H. 2008

2. Adjoint consistent imposition of the boundary terms present in $\mathcal{N}_\Gamma(\cdot,\cdot)$.

   Lu & Darmofal 2006, Hartmann 2007

3. [Adjoint consistent reformulation of the target functional $J(\cdot)$.]

   Harriman, Gavaghan, & Süli 2004, Hartmann 2007

- Standard SIPG scheme

$$\underline{\delta}(\mathbf{u}_h) \equiv \underline{\delta}^{\mathrm{STSIPG}}(\mathbf{u}_h) = \; C_{\mathrm{IP}} \mu \frac{p^2}{h_F} \underline{[\![\mathbf{u}_h]\!]}.$$

- Use of lifting operators (Bassi–Rebay Scheme 2)

$$\underline{\delta}(\mathbf{u}_h) \equiv \underline{\delta}^{\mathrm{BR2}}(\mathbf{u}_h) = \; C_{\mathrm{BR2}} \{\!\{ \underline{L}(\mathbf{u}_h) \}\!\},$$

where the *lifting operator* $\underline{L}(\cdot) \in \underline{\Sigma}_{h,p}$ is defined by

$$\int_\Omega \underline{L}(\mathbf{u}_h) : \underline{\tau} \mathrm{d}\mathbf{x} = \int_F \underline{[\![\mathbf{u}_h]\!]} : \{\!\{ G^\top(\mathbf{u}_h) \underline{\tau} \}\!\} \mathrm{d}s \qquad \forall \underline{\tau} \in \underline{\Sigma}_{h,p}.$$

Here,

$$\underline{\Sigma}_{h,p} = \{ \underline{\tau}_h \in \left[L^2(\Omega) \right]^{5\times 3} : \underline{\tau}_h|_\kappa \in [\mathcal{S}_{p_\kappa}]^{5\times 3} \quad \forall \kappa \in \mathcal{T}_h \}.$$

- Modified SIPG scheme

$$\underline{\delta}(\mathbf{u}_h) \equiv \underline{\delta}^{\mathrm{SIPG}}(\mathbf{u}_h) = \; C_{\mathrm{IP}} \frac{p^2}{h_F} \{\!\{ G(\mathbf{u}_h) \}\!\} \underline{[\![\mathbf{u}_h]\!]}.$$

$$\begin{aligned}\mathcal{N}_\Gamma(\mathbf{u}_h,\mathbf{v}) = &\int_\Gamma (\mathbf{n}\cdot\mathcal{F}^v(\mathbf{u}_\Gamma(\mathbf{u}_h^+)))\cdot\mathbf{v}^+ \mathrm{d}s + \int_\Gamma \underline{\delta}_\Gamma(\mathbf{u}_h^+) : \mathbf{v}\otimes\mathbf{n}\mathrm{d}s,\\ &- \int_\Gamma \mathbf{n}\cdot\widehat{\mathcal{F}}^v(\mathbf{u}_\Gamma(\mathbf{u}_h^+),\nabla_h\mathbf{u}_h^+)\,\mathbf{v}^+\mathrm{d}s \\ &- \int_\Gamma \left(\widehat{G}^\top(\mathbf{u}_\Gamma(\mathbf{u}_h^+))\nabla_h\mathbf{v}_h^+\right) : \left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right)\otimes\mathbf{n}\mathrm{d}s.\end{aligned} \tag{24}$$

- The viscous fluxes $\widehat{\mathcal{F}}^v(\mathbf{u},\nabla\mathbf{u})$ are defined as follows

$$\widehat{\mathcal{F}}^v(\mathbf{u},\nabla\mathbf{u}) = \begin{cases} \mathcal{F}^v(\mathbf{u},\nabla\mathbf{u}) & \text{on } \Gamma\backslash\Gamma_{\text{adia}}, \\ \mathcal{F}^{v,\text{adia}}(\mathbf{u},\nabla\mathbf{u}) & \text{on } \Gamma_{\text{adia}}, \end{cases}$$

where $\mathcal{F}^{v,\text{adia}}(\mathbf{u},\nabla\mathbf{u}) = \widehat{G}(\mathbf{u})\nabla\mathbf{u}$ is defined so that

$$\mathcal{F}^{v,\text{adia}}(\mathbf{u},\nabla\mathbf{u})\cdot\mathbf{n} = (0, \tau_{1j}n_{x_j}, \tau_{2j}n_{x_j}, \tau_{3j}n_{x_j}, \tau_{ij}v_jn_{x_i})^\top;$$

thereby, enforcing the adiabatic solid wall boundary condition $\mathbf{n}\cdot\nabla T = 0$ on Γ_{adia}.

- Boundary function: $\mathbf{u}_\Gamma(\mathbf{u})$ is given according to the type of boundary condition imposed:
  - $\Gamma_{\text{D,sup}}$: $\mathbf{u}_\Gamma(\mathbf{u}) = \mathbf{g}_D$
  - $\Gamma_{\text{D,sub-in}}$: $\mathbf{u}_\Gamma(\mathbf{u}) =$ $\left((g_D)_1, (g_D)_2, (g_D)_3, (g_D)_4, \frac{p(\mathbf{u})}{\gamma-1} + ((g_D)_2^2 + (g_D)_3^2 + (g_D)_4^2)/(2(g_D)_1)\right)^\top$
  - $\Gamma_{\text{D,sub-out}}$: $\mathbf{u}_\Gamma(\mathbf{u}) = \left(u_1, u_2, u_3, u_4, \frac{p_{\text{out}}}{\gamma-1} + (u_2^2 + u_3^2 + u_4^2)/(2u_1)\right)^\top$
  - Γ_{iso}: $\mathbf{u}_\Gamma(\mathbf{u}) = (u_1, 0, 0, 0, u_1 c_v T_{\text{wall}})^\top$
  - Γ_{adia}: $\mathbf{u}_\Gamma(\mathbf{u}) = (u_1, 0, 0, 0, u_5)^\top$

- $\underline{\delta}_\Gamma(\mathbf{u}_h)$ is a boundary penalty term:

$$\underline{\delta}_\Gamma(\mathbf{u}_h) \equiv \begin{cases} \underline{\delta}_\Gamma^{\text{SIPG}}(\mathbf{u}_h) & = \; C_{\text{IP}} \frac{p^2}{h_e} G(\mathbf{u}_\Gamma(\mathbf{u}_h^+))\,(\mathbf{u}_h - \mathbf{u}_\Gamma(\mathbf{u}_h)) \otimes \mathbf{n}, \\ \underline{\delta}_\Gamma^{\text{STSIPG}}(\mathbf{u}_h) & = \; C_{\text{IP}} \mu \frac{p^2}{h_e}\,(\mathbf{u}_h - \mathbf{u}_\Gamma(\mathbf{u}_h)) \otimes \mathbf{n}, \\ \underline{\delta}_\Gamma^{\text{BR2}}(\mathbf{u}_h) & = \; C_{\text{BR2}} \underline{L}_\Gamma(\mathbf{u}_h), \end{cases}$$

Here, the local lifting operator $\mathbf{L}_\Gamma(\mathbf{u}_h) \in \underline{\Sigma}_{h,p}$ on Γ is defined by:

$$\int_\kappa \mathbf{L}_\Gamma(\mathbf{u}_h) : \underline{\tau}\,\mathrm{d}\mathbf{x} = \int_F (\mathbf{u}_h - \mathbf{u}_\Gamma(\mathbf{u}_h)) \otimes \mathbf{n} : \left(G^\top(\mathbf{u}_\Gamma(\mathbf{u}_h))\underline{\tau}\right)\mathrm{d}s \qquad \forall \underline{\tau} \in \underline{\Sigma}_{h,p}$$

We let $\Omega = (0,\pi)^2$ and select f such that

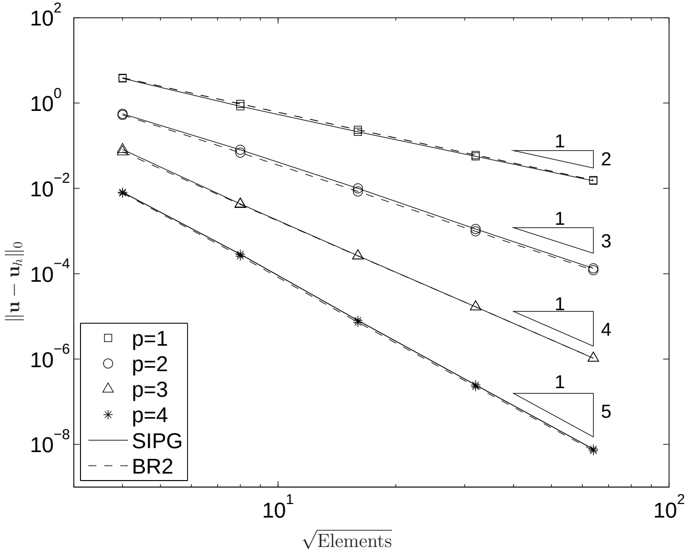
$$\mathbf{u} = (\sin(2(x+y))+4, \sin(2(x+y))/5+4, \sin(2(x+y))/5+4, (\sin(2(x+y))+4)^2)^\top.$$

Gassner, Lörcher, & Munz 2007

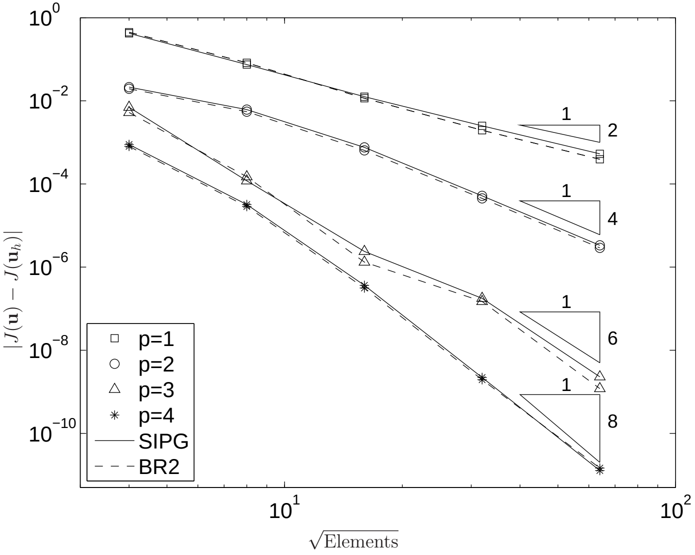
Functional:

Weighted mean-value of the density, i.e.,

$$J(\mathbf{u}) = \int_\Omega \rho\psi \mathrm{d}\mathbf{x}, \quad \psi = \sin(\pi x)\sin(\pi y).$$

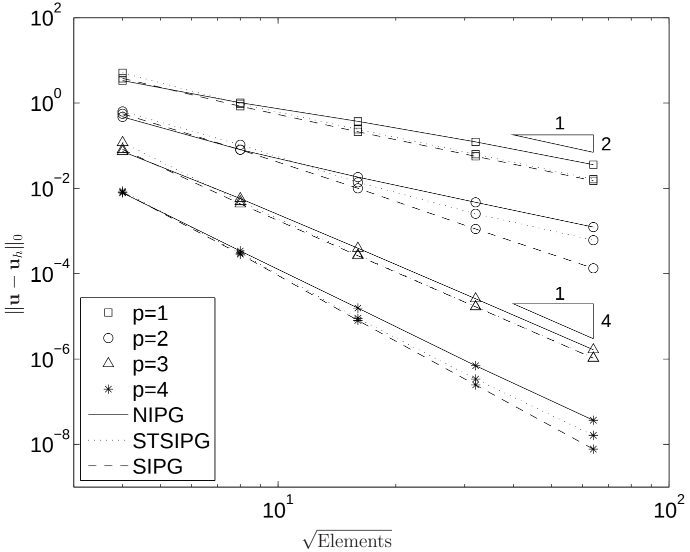


10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
$\|\mathbf{u}-\mathbf{u}_h\|_0$
10^{1}
10^{2}
$\sqrt{\text{Elements}}$
p=1
p=2
p=3
p=4
SIPG
BR2
1
2
1
3
1
4
1
5



10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
$|J(\mathbf{u}) - J(\mathbf{u}_h)|$
10^{1}
10^{2}
$\sqrt{\text{Elements}}$
1
2
1
4
1
6
1
8
p=1
p=2
p=3
p=4
SIPG
BR2

Penalty Terms



$$J(\mathbf{u}) = \frac{1}{C_\infty} \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n}) \cdot \psi \mathrm{d}s.$$

- Drag (2D):

$$\psi = (\cos(\alpha), \sin(\alpha))^\top$$

- Lift (2D):

$$\psi = (-\sin(\alpha), \cos(\alpha))^\top$$

- Here,

$$C_\infty = \tfrac{1}{2}\gamma p_\infty M_\infty^2 \bar{l} = \tfrac{1}{2}\gamma \frac{|\mathbf{v}_\infty|^2}{c_\infty^2} p_\infty \bar{l} = \tfrac{1}{2}\rho_\infty |\mathbf{v}_\infty|^2 \bar{l},$$

  where

  - M_∞: Free-stream Mach number.
  - c_∞: Free-stream speed of sound defined by $c_\infty^2 = \gamma p_\infty / \rho_\infty$.
  - p_∞: Free-stream pressure.
  - ρ_∞: Free-stream density.
  - $\bar{l}$: Reference (chord) length of the body.
  - α: Angle of attack.

- Adjoint consistent reformulation:

$$\widetilde{J}(\mathbf{u}) = J(\mathbf{u}_\Gamma(\mathbf{u})) + \int_\Gamma \underline{\delta}_\Gamma(\mathbf{u}) : \mathbf{z}_\Gamma \otimes \mathbf{n}\,\mathrm{d}s,$$

  where (2D)

$$\mathbf{z}_\Gamma = \tfrac{1}{C_\infty}(0, \psi_1, \psi_2, 0)^\top,$$

  represents the boundary values of the adjoint solution $\mathbf{z}$.
- Noting that $\underline{\delta}_\Gamma(\mathbf{u}) = 0$ holds for the analytical solution $\mathbf{u}$, assuming $\mathbf{u}$ is sufficiently regular, we have

$$\tilde{J}(\mathbf{u}) = J(\mathbf{u}).$$

Harriman, Gavaghan, & Süli 2004, Hartmann 2007

$$
\begin{aligned}
\mathbf{R}(\mathbf{u}_h)|_\kappa &= -\nabla\cdot\mathcal{F}^c(\mathbf{u}_h)+\nabla\cdot\mathcal{F}^v(\mathbf{u}_h,\nabla_h\mathbf{u}_h),\\
\mathbf{r}(\mathbf{u}_h)|_{\partial\kappa\setminus\Gamma} &= \mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}_h^+)-\mathcal{H}(\mathbf{u}_h^+,\mathbf{u}_h^-,\mathbf{n}^+)-\frac{1}{2}[\![\mathcal{F}^v(\mathbf{u}_h,\nabla_h\mathbf{u}_h)]\!]\\
&\quad -\delta(\mathbf{u}_h)\mathbf{n},\\
\rho(\mathbf{u}_h)|_{\partial\kappa\setminus\Gamma} &= \frac{1}{2}\left(G(\mathbf{u}_h)\underline{[\![\mathbf{u}_h]\!]}\right)^\top,\\
\mathbf{r}_\Gamma(\mathbf{u}_h)|_{\partial\kappa\cap\Gamma} &= \mathbf{n}\cdot\left(\mathcal{F}^c(\mathbf{u}_h^+)-\mathcal{F}^c_\Gamma(\mathbf{u}_h^+)-\mathcal{F}^v(\mathbf{u}_h^+,\nabla\mathbf{u}_h^+)+\mathcal{F}^v_\Gamma(\mathbf{u}_h^+,\nabla\mathbf{u}_h^+)\right)\\
&\quad -\delta_\Gamma(\mathbf{u}_h^+)\mathbf{n},\\
\rho_\Gamma(\mathbf{u}_h)|_{\partial\kappa\cap\Gamma} &= \left(G_\Gamma^\top(\mathbf{u}_h^+):(\mathbf{u}_h^+-\mathbf{u}_\Gamma(\mathbf{u}_h^+))\otimes\mathbf{n}\right)^\top.
\end{aligned}
$$

Proposition

Assuming the dual problem is well–posed, the following result holds:

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \leq \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^{(\mathrm{I})},$$

where $\eta_\kappa^{(\mathrm{I})} = |\eta_\kappa|$,

$$\begin{aligned}
\eta_\kappa &= \int_\Omega \mathbf{R}(\mathbf{u}_h) \cdot (\mathbf{z} - \mathbf{z}_h) \mathrm{d}\mathbf{x} \\
&\quad + \int_{\partial\kappa \setminus \Gamma} \left(\mathbf{r}(\mathbf{u}_h) \cdot (\mathbf{z} - \mathbf{z}_h)^+ + \boldsymbol{\rho}(\mathbf{u}_h) : \nabla (\mathbf{z} - \mathbf{z}_h)^+ \right) \mathrm{d}s, \\
&\quad + \int_{\partial\kappa \cap \Gamma} \left(\mathbf{r}_\Gamma(\mathbf{u}_h) \cdot (\mathbf{z} - \mathbf{z}_h)^+ + \boldsymbol{\rho}_\Gamma(\mathbf{u}_h) : \nabla (\mathbf{z} - \mathbf{z}_h)^+ \right) \mathrm{d}s.
\end{aligned}$$

$\mathrm{Ma} = 0.5$ and $\alpha = 2^\circ$.

Drag coefficient:

$$J_{C_{\mathrm{dp}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2} \int_S p\,(\mathbf{n} \cdot \psi_d)\mathrm{d}s,$$

where

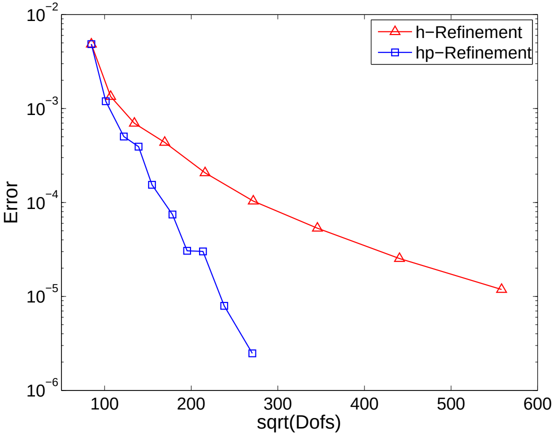
$$\psi_d = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2D Inviscid Flow – Structured Starting Mesh

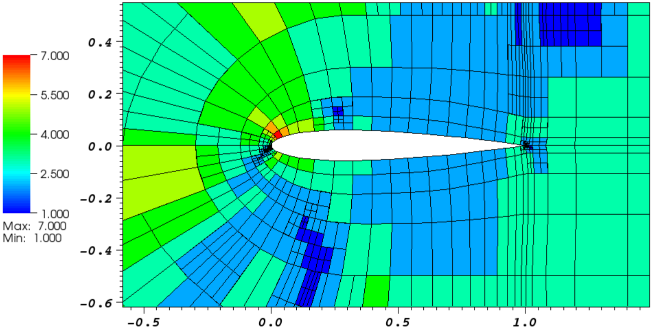
Elements	Dof	$J(\mathbf{u}) - J(\mathbf{u}_h)$	$\sum_{\kappa\in\mathcal{T}_h} \eta_\kappa$	θ_1
448	7168	-0.4844E-02	-0.4411E-02	0.910
562	10252	-0.1197E-02	-0.1111E-02	0.928
685	14912	-0.5029E-03	-0.4631E-03	0.921
784	19360	-0.3923E-03	-0.3685E-03	0.939
838	23928	-0.1541E-03	-0.1433E-03	0.930
970	31780	-0.7443E-04	-0.6990E-04	0.939
1018	38132	-0.3061E-04	-0.2893E-04	0.945
1045	45616	-0.3010E-04	-0.2770E-04	0.921
1120	56684	-0.7940E-05	-0.7772E-05	0.979
1201	73200	-0.2481E-05	-0.2341E-05	0.944

hp–Refinement algorithm based on an initial structured quadrilateral mesh.

2D Inviscid Flow – Structured Starting Mesh



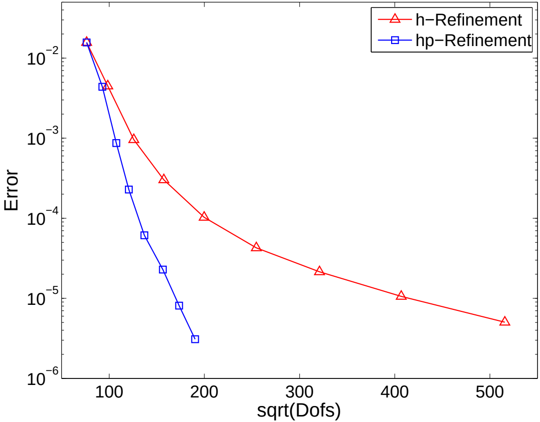
2D Inviscid Flow – Structured Starting Mesh

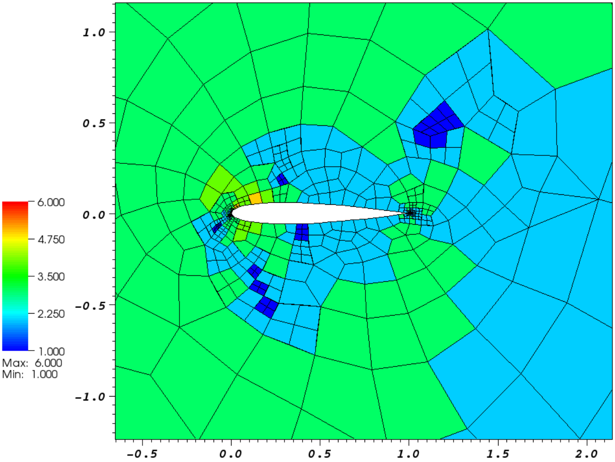


hp–Mesh distribution after 9 adaptive refinements (structured initial mesh).

Elements	Dof	$J(\mathbf{u}) - J(\mathbf{u}_h)$	$\sum_{\kappa\in\mathcal{T}_h} \eta_\kappa$	θ_1
365	5816	-0.1570E-01	-0.1276E-01	0.813
476	8612	-0.4385E-02	-0.3488E-02	0.795
530	11540	-0.8699E-03	-0.7229E-03	0.831
593	14556	-0.2288E-03	-0.2052E-03	0.897
650	18756	-0.6131E-04	-0.5476E-04	0.893
728	24456	-0.2285E-04	-0.2043E-04	0.894
809	30104	-0.8102E-05	-0.7065E-05	0.872
839	36188	-0.3086E-05	-0.2655E-05	0.860
881	45428	-0.1620E-05	-0.1456E-05	0.899
923	55592	-0.4111E-06	-0.4111E-06	1.00

hp–Refinement algorithm based on an initial unstructured hybrid mesh.


h−Refinement
hp−Refinement
Error
sqrt(Dofs)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
100
200
300
400
500



hp–Mesh distribution after 9 adaptive refinements (unstructured initial mesh).

$\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 2^\circ$ and adiabatic wall condition.

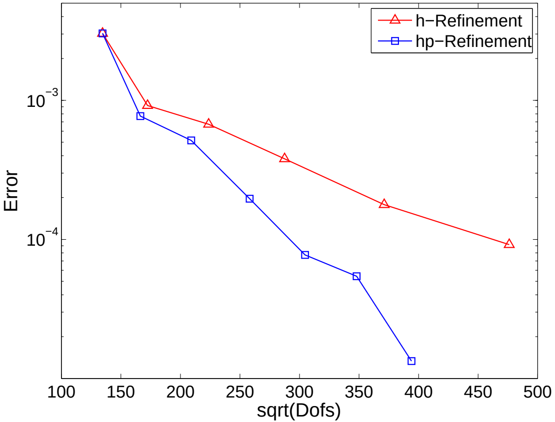
Drag coefficients:

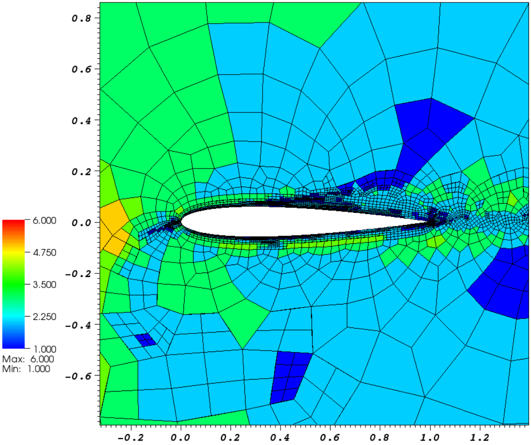
$$J_{C_{\mathrm{dp}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2}\int_S p\,(\mathbf{n}\cdot\psi_d)\mathrm{d}s, \qquad J_{C_{\mathrm{df}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2}\int_S (\boldsymbol{\tau}\,\mathbf{n})\cdot\psi_d\mathrm{d}s,$$

where

$$\psi_d = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\boxed{J_{C_{\mathrm{d}}}(\mathbf{u}) \approx 0.056084.}$$


h−Refinement
hp−Refinement
Error
10^{-3}
10^{-4}
100
150
200
250
300
350
400
450
500
sqrt(Dofs)



hp–Mesh distribution after 7 adaptive refinements (unstructured initial mesh).

$\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 2^\circ$ and adiabatic wall condition.

Drag coefficients:

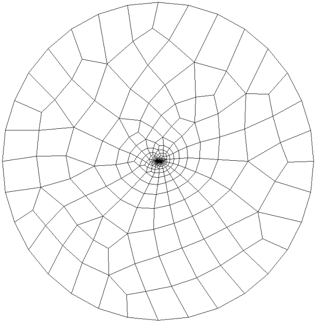
$$J_{C_{\mathrm{dp}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2} \int_S p\,(\mathbf{n} \cdot \psi_d)\mathrm{d}s, \qquad J_{C_{\mathrm{df}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2} \int_S (\boldsymbol{\tau}\,\mathbf{n}) \cdot \psi_d \mathrm{d}s,$$

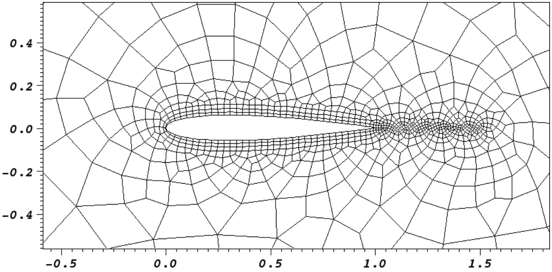
where

$$\psi_d = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

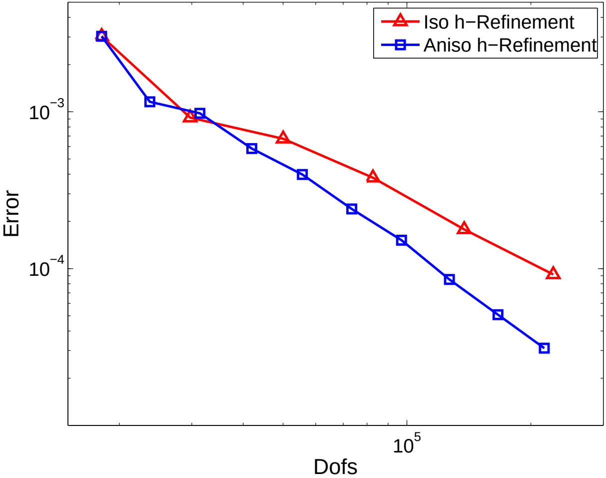
$$\boxed{J_{C_{\mathrm{d}}}(\mathbf{u}) \approx 0.056084.}$$

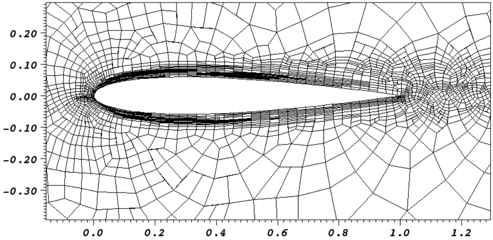
$\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 2^\circ$ and adiabatic wall condition.



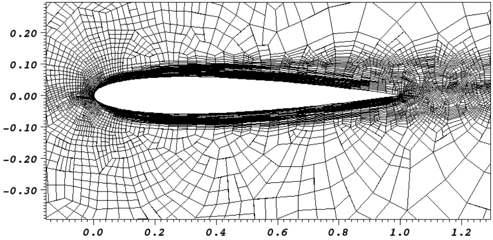


2D Laminar Flow – NACA0012 Airfoil





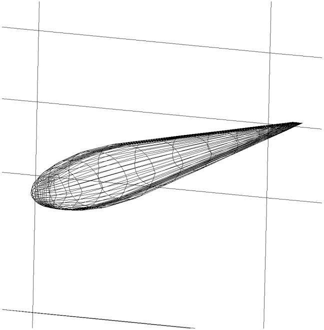
Mesh after 4 adaptive anisotropic refinements, with 3485 elements

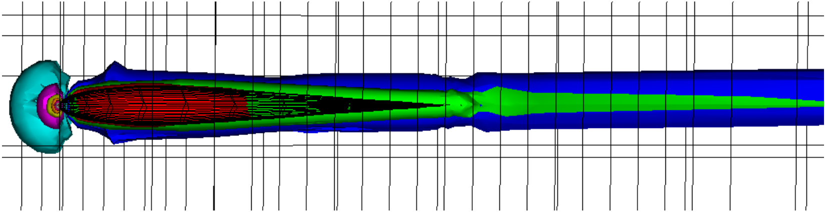


Mesh after 8 adaptive anisotropic refinements, with 10401 elements

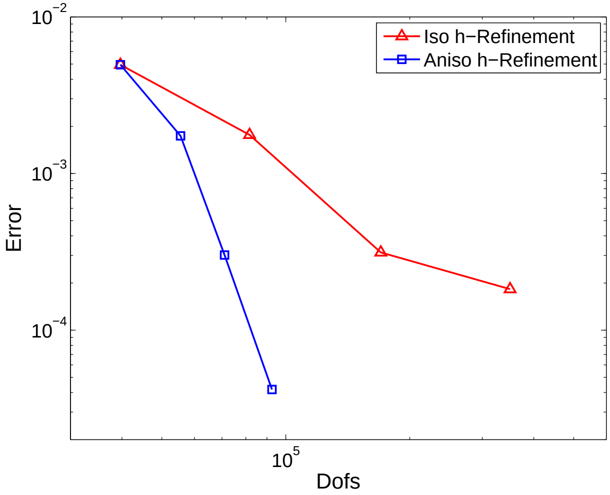
3D Laminar Flow

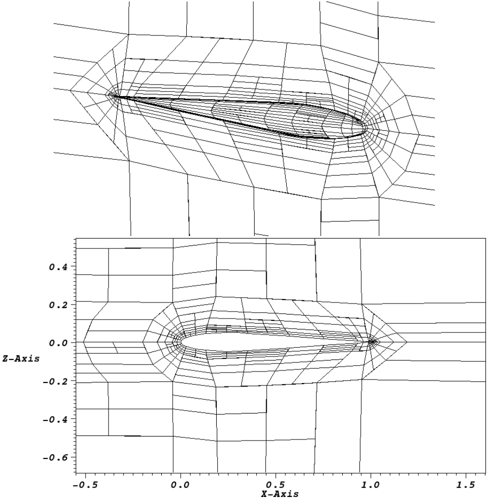
- $\mathrm{Ma} = 0.5$.
- $\mathrm{Re} = 5000$.
- $\alpha = 1^\circ$.
- Adiabatic wall condition.
- $J_{C_1}(\mathbf{u}) \approx 0.002565$





3D Laminar Flow





Mesh after 3 adaptive anisotropic refinements, with 2324 elements

$\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 2^\circ$ and adiabatic wall condition.

Drag coefficients:

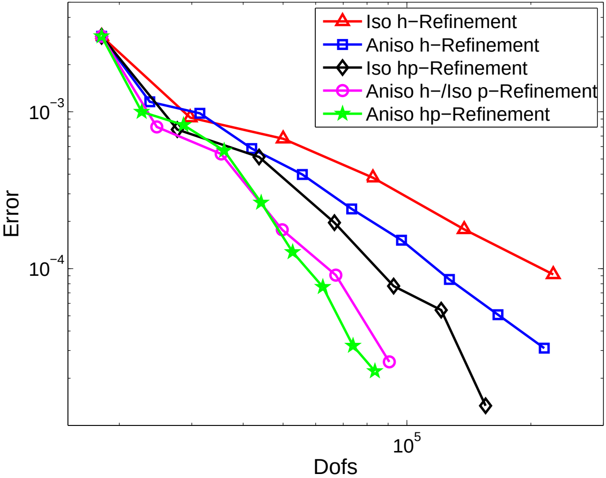
$$J_{c_{\mathrm{dp}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2}\int_S p\,(\mathbf{n}\cdot\psi_d)\mathrm{d}s, \qquad J_{c_{\mathrm{df}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2}\int_S (\boldsymbol{\tau}\,\mathbf{n})\cdot\psi_d\mathrm{d}s,$$

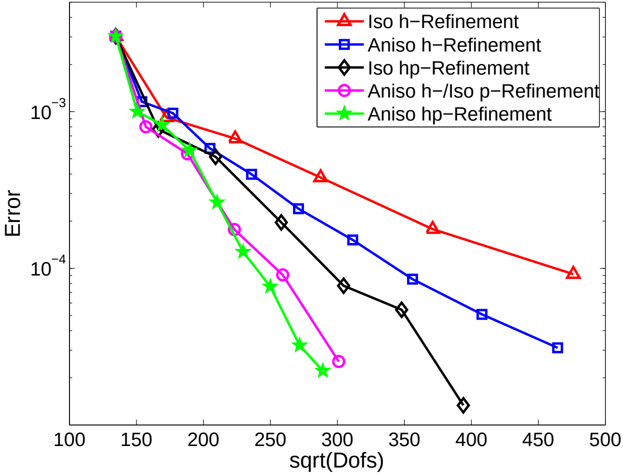
where

$$\psi_d = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

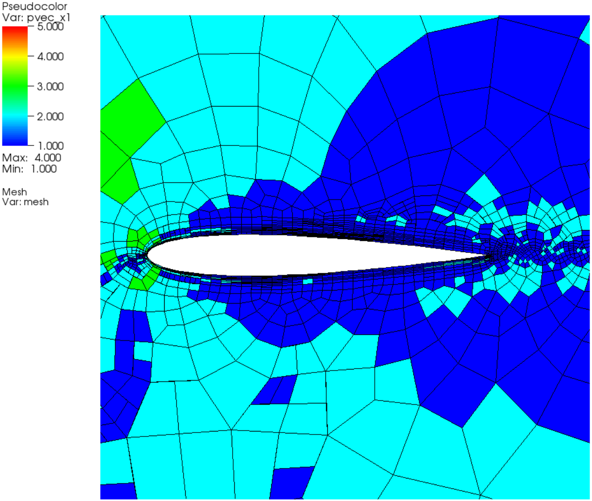
$$\boxed{J_{c_{\mathrm{d}}}(\mathbf{u}) \approx 0.056084.}$$

2D Laminar Flow – NACA0012 Airfoil

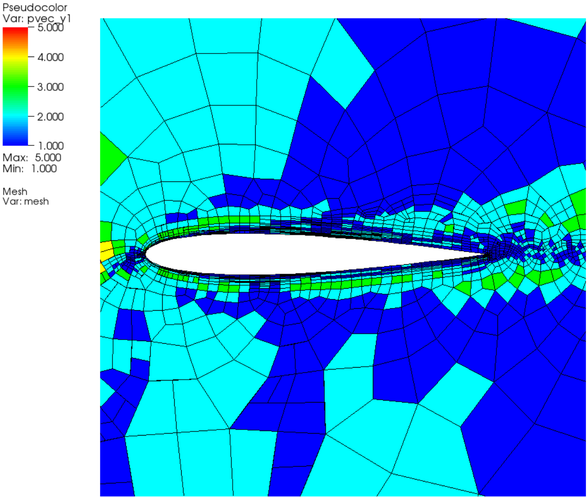




Iso h−Refinement
Aniso h−Refinement
Iso hp−Refinement
Aniso h−/Iso p−Refinement
Aniso hp−Refinement
Error
10^{-3}
10^{-4}
100
150
200
250
300
350
400
450
500
sqrt(Dofs)



h–/p_x–mesh distribution after 5 adaptive anisotropic hp–refinements, with 2200 elements and 52744 degrees of freedom



h–/p_y–mesh distribution after 5 adaptive anisotropic hp–refinements, with 2200 elements and 52744 degrees of freedom

Nonlinear Problem

Consider the solution of the following nonlinear problem:

$$(u_t +) \quad F(u, \lambda) = 0,$$

where

- u is the state variable(s);
- λ is a parameter (or set of parameters) of physical interest.

Fundamental questions include:

- How many solutions exist as some parameter of interest is varied?
- Are the solutions linearly stable?
- At what critical parameter value does a bifurcation occur?

Bratu Problem

A simple example is the 1D Bratu problem:

$$\begin{aligned} u'' + \lambda e^{u} &= 0, \quad \mathbf{x} \in (0,1), \\ u(0) = u(1) &= 0. \end{aligned}$$

In this case, the following holds:

- No solutions exist when $\lambda > \lambda_c$;
- One solution exists when $\lambda = \lambda_c$;
- Two solutions exist when $\lambda < \lambda_c$. Here, λ_c satisfies

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\,\sinh\left(\frac{\theta}{4}\right), \quad \text{and} \quad \theta = \sqrt{2\lambda_c}\,\cosh\left(\frac{\theta}{4}\right),$$

i.e.,

$$\lambda_c \approx 3.513830719125160.$$

Wazwaz 2005

Bifurcation Problems

- Locate critical parameter values at which steady solutions lose stability and bifurcations occur.

Eigenvalue Analysis

For given values of λ, solve the eigenvalue problem:

$$F_u(u, \lambda)\phi = \mu\phi$$

Larson 2000, Heuveline & Rannacher 2001, Cliffe, Hall, & H. 2010.

Extended System Formulation (Steady Bifurcation)

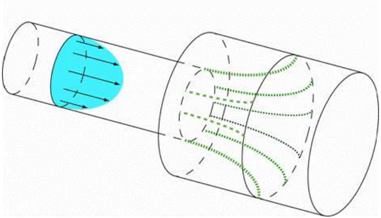
Find $\mathbf{U} = (u, \phi, \lambda)^\top$, such that

$$G(\mathbf{U}) \equiv \left(\begin{array}{c} F(u, \lambda) \\ F_u(u, \lambda)\phi \\ < \ell, \phi > -1 \end{array} \right) = \mathbf{0},$$

where ℓ is some appropriate normalization.

Brezzi, Rappaz, & Raviart 1981, Werner & Spence 1984

- Computational complexity may be reduced by exploiting the underlying structure of the problem.

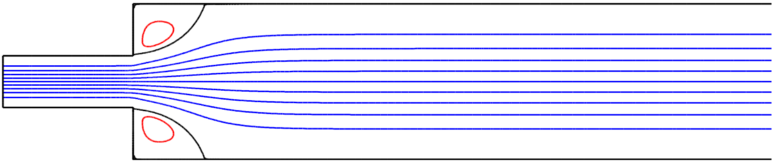


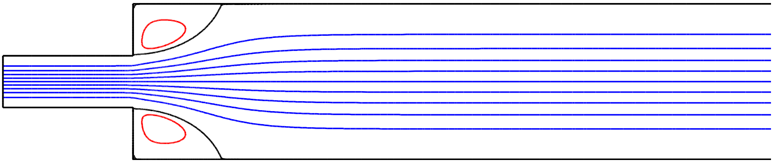
- Applications: engineering applications range from heat exchangers to combustion chambers, for example; physiological problems, such as flow past stenoses.

Aims

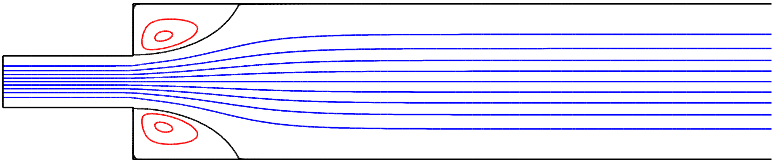
1. Accurately locate critical Reynolds numbers at which steady solutions of the three–dimensional incompressible Navier–Stokes equations lose stability and bifurcations occur.
2. Understand the structure of an open flow through a pipe with a sudden expansion.
3. Numerical ingredients: DWR *a posteriori* error analysis of *hp*–version DGFEMs.
4. Key objective: Mesh independent predictions with reliable error control.

- Flow past a cylinder in a channel.
  Jackson 1987; Cliffe and Tavener 2004.
  - Z_2 symmetry-breaking Hopf bifurcation.
    ($Re^c \approx 123$ for a $1:2$ blockage ratio)
- Flow through a sudden expansion in a channel.
  Fearn, Mullin and Cliffe 1990.
  - Steady, Z_2 symmetry-breaking bifurcation.
    ($Re^c \approx 40$ for a $1:3$ expansion ratio)
- Flow past a sphere in a pipe.
  Tavener 1994; Cliffe, Spence and Tavener 2000.
  - Steady, $O(2)$ symmetry-breaking bifurcation.

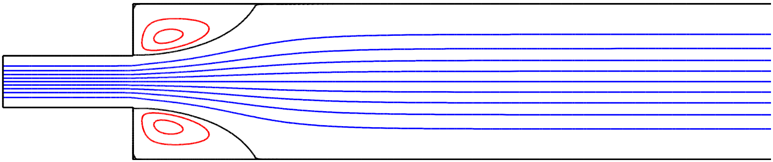




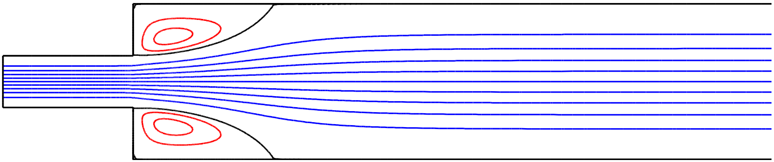
Channel with a Sudden Expansion - Re $= 30$



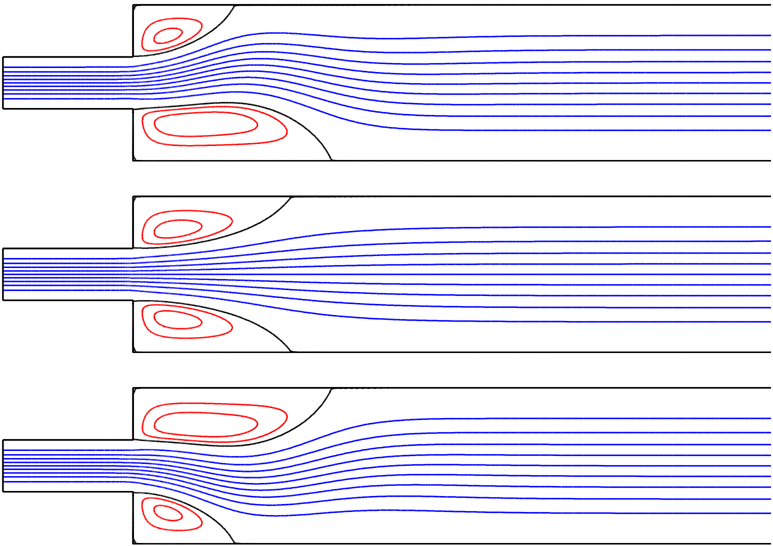
Channel with a Sudden Expansion - Re = 35



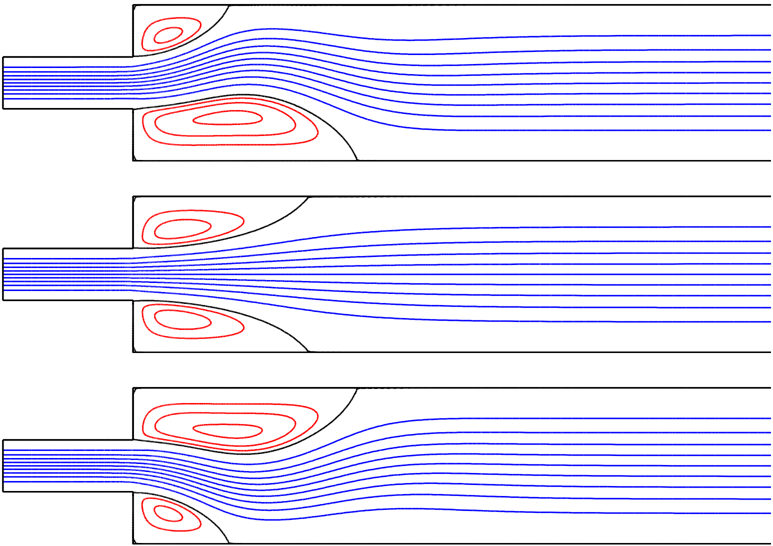
Channel with a Sudden Expansion - Re = 40



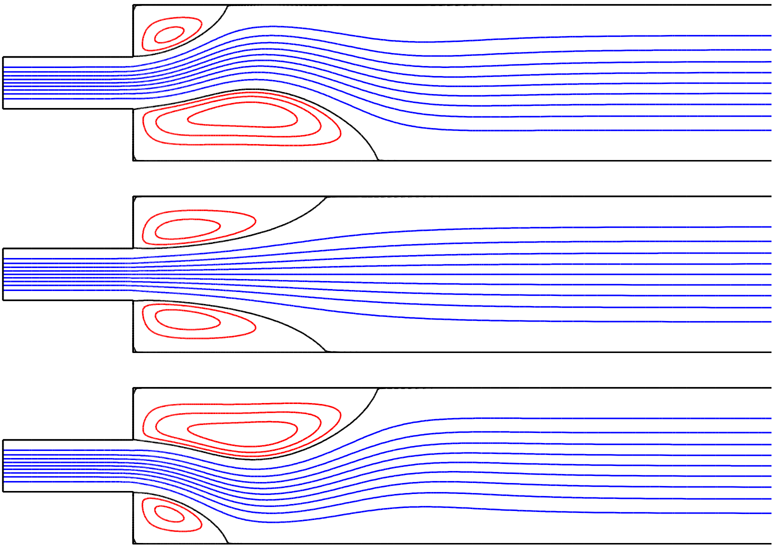
Channel with a Sudden Expansion - Re = 45



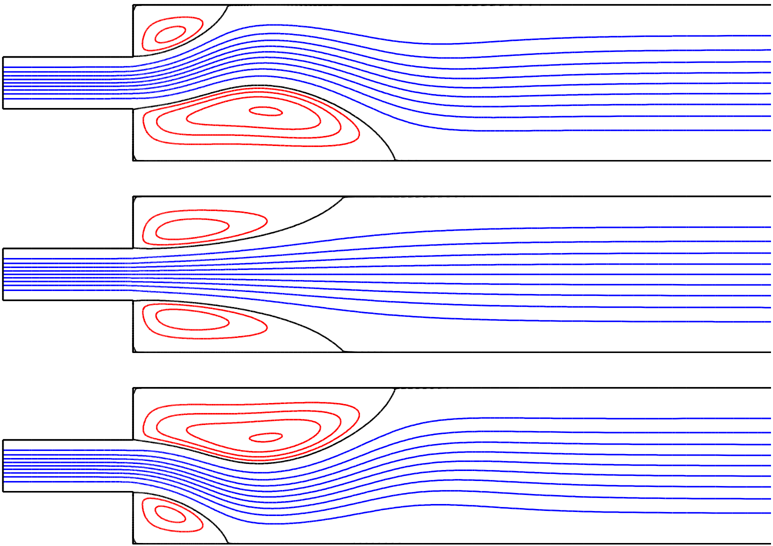
Channel with a Sudden Expansion - Re = 50



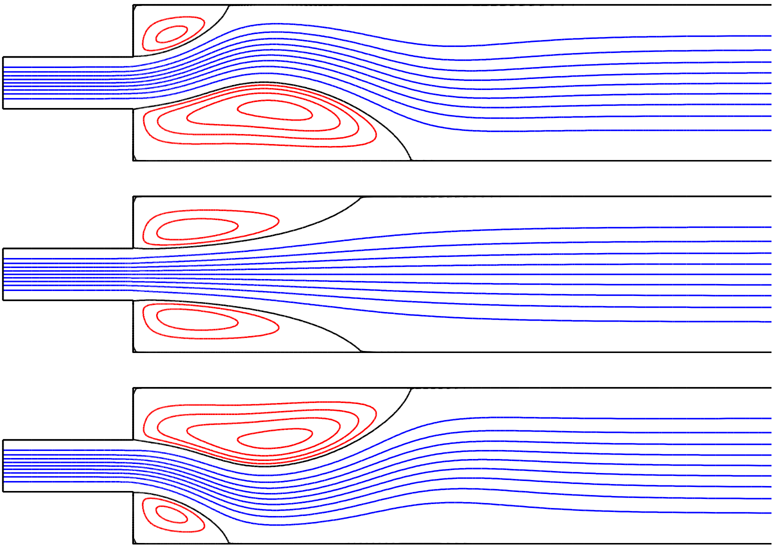
Channel with a Sudden Expansion - Re = 55



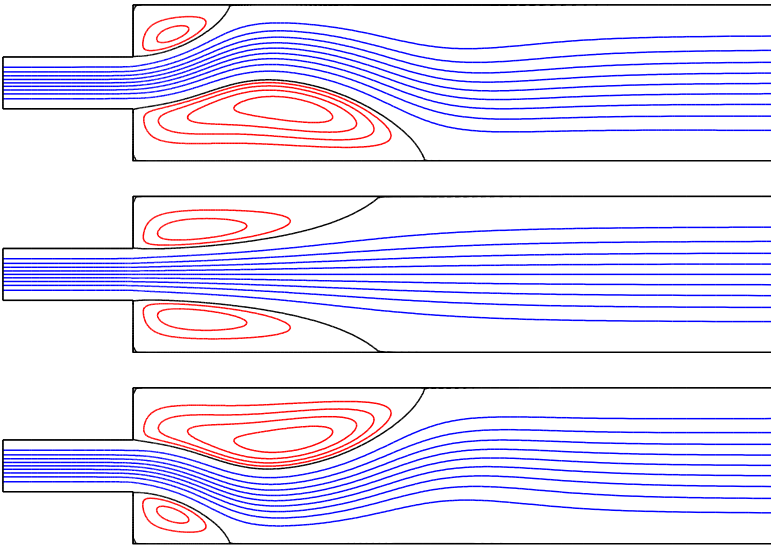
Channel with a Sudden Expansion - Re = 60

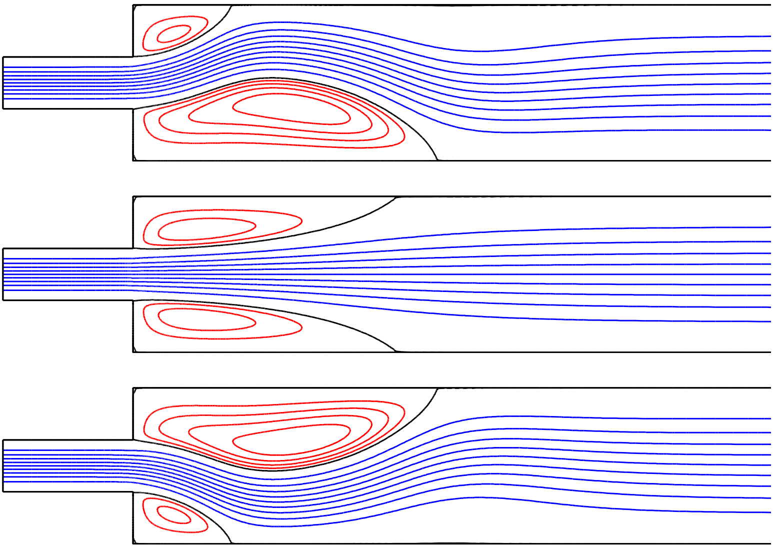


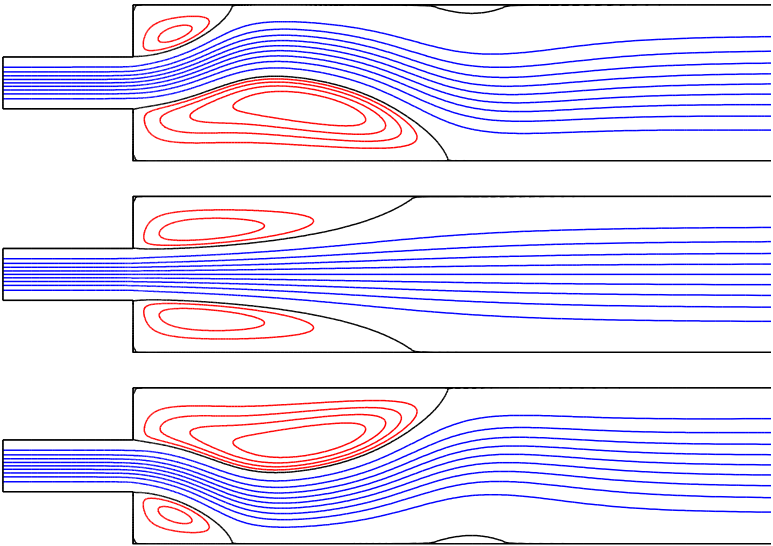
Channel with a Sudden Expansion - Re = 65



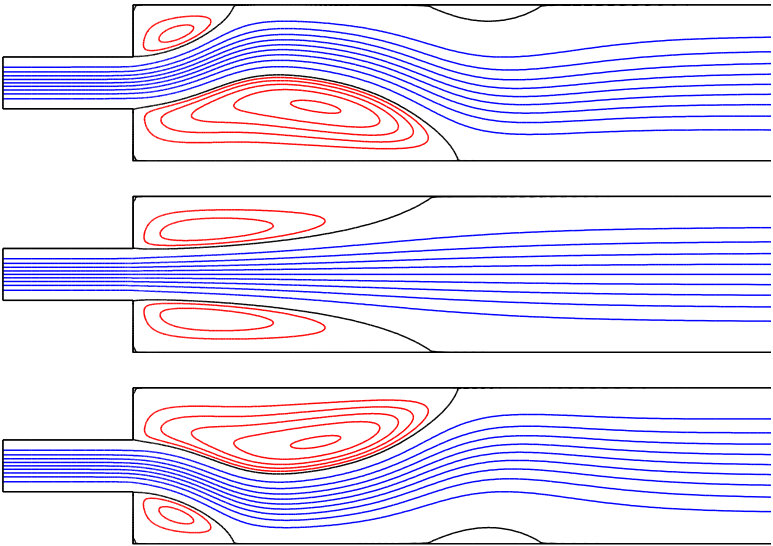
Channel with a Sudden Expansion - Re = 70

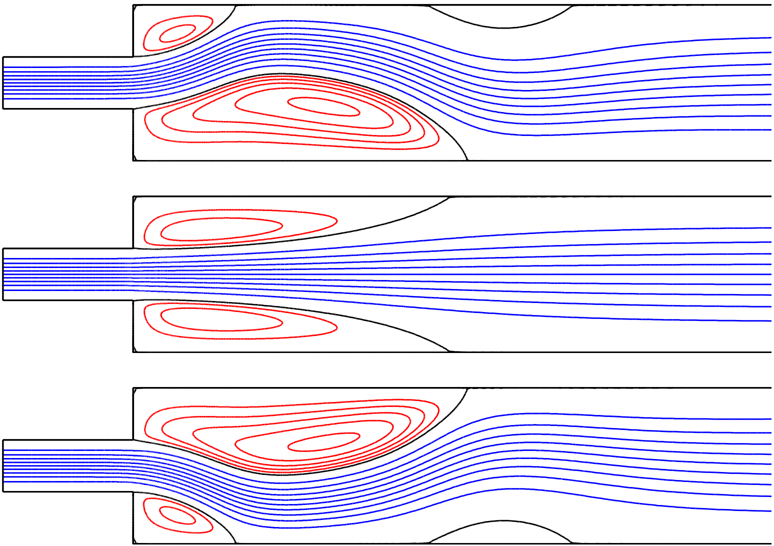


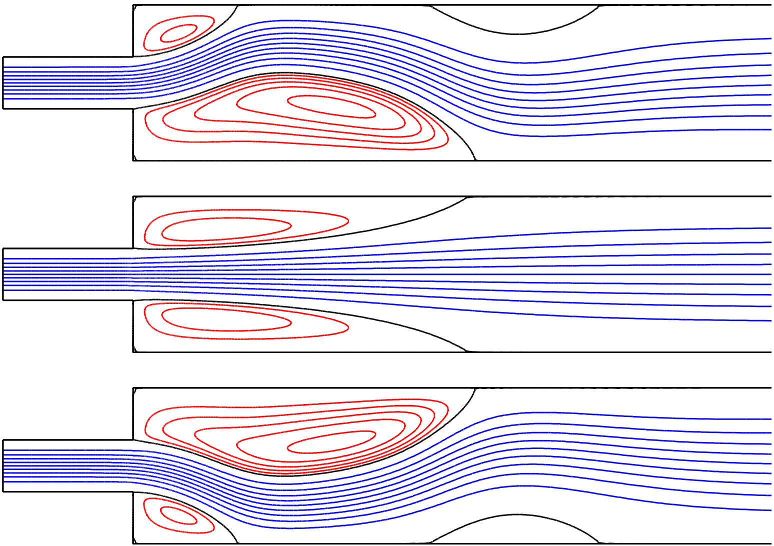


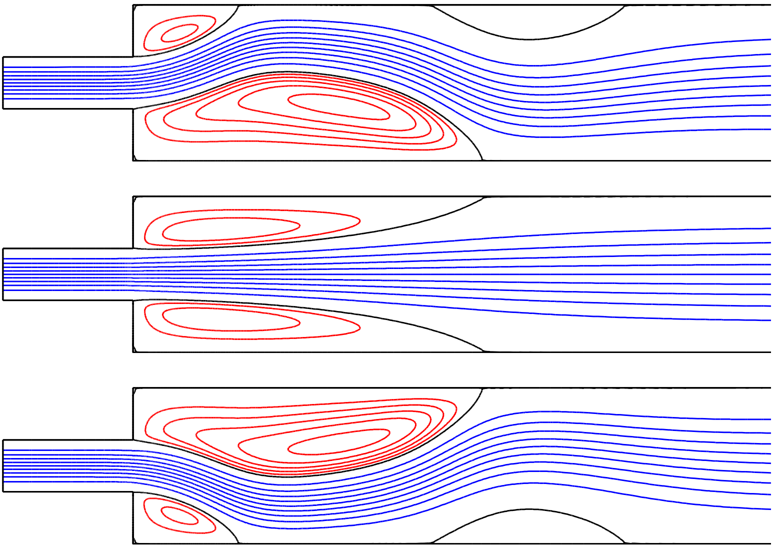


Channel with a Sudden Expansion - Re = 85









No. Elements	Base DOF	Null DOF	Re_h^c	$\lvert Re^c - Re_h^c \rvert$	τ
640	14080	14080	40.4923	6.557E-02	1.827
1129	24838	24838	40.5410	1.689E-02	2.335
2029	44638	44638	40.5480	9.832E-03	1.992
3601	79222	79222	40.5516	6.225E-03	1.626
6130	134860	134860	40.5542	3.696E-03	1.402
10501	231022	231022	40.5558	2.055E-03	1.268
17980	395560	395560	40.5568	1.103E-03	1.174
30796	677512	677512	40.5573	5.785E-04	1.093
52654	1158388	1158388	40.5576	3.064E-04	1.000

Fearn, Mullin, & Cliffe 1990: $Re_h^c = 40.45 \pm 0.17\%$, $Re_{exp}^c \approx 33$.

- Flow past a cylinder in a channel.
  Jackson 1987; Cliffe and Tavener 2004.
  - Z_2 symmetry-breaking Hopf bifurcation.
    ($Re^c \approx 123$ for a $1:2$ blockage ratio)
- Flow through a sudden expansion in a channel.
  Fearn, Mullin and Cliffe 1990.
  - Steady, Z_2 symmetry-breaking bifurcation.
    ($Re^c \approx 40$ for a $1:3$ expansion ratio)
- Flow past a sphere in a pipe.
  Tavener 1994; Cliffe, Spence and Tavener 2000.
  - Steady, $O(2)$ symmetry-breaking bifurcation.
    ($Re^c \approx 359$ for a $1:2$ blockage ratio)
- Flow in a pipe with a stenotic region.
  Sherwin & Blackburn 2005, 2007, Sherwin, Blackburn & Barkley 2008.
  - Steady, $O(2)$ symmetry-breaking bifurcation.
    ($Re^c \approx 721$ for a 75% occlusion)

- Exploit the underlying group structure within a physical system in order to rigorously justify the study of (equivalent) simplified problems; examples include, Z_2, $SO(2)$, and $O(2)$ $(= SO(2) \times Z_2)$.

  $\Rightarrow$ Leads subsequent computational savings.
- Bifurcation with $O(2)$ symmetry.

Vanderbanwhede 1982; Golubitsky & Schaeffer 1985; Golubitsky, Stewart & Schaeffer 1988; Healey & Treacy 1991; Aston 1991; Cliffe, Spence & Tavener 2000.

$O(2)$ Group

$O(2)$ is a Lie Group generated by

- Rotations r_α, $\alpha \in \mathbb{R}$;
- A Reflection s.

For any $\alpha, \beta \in \mathbb{R}$, the group actions satisfy

$$r_{\alpha+2\pi} = r_\alpha, \quad r_{\alpha+\beta} = r_\alpha r_\beta = r_\beta r_\alpha, \quad s^2 = r_0 = r_{2\pi} = I, \quad s r_\alpha = r_{-\alpha} s,$$

where I is the group identity.

Model Problem

Given a Hilbert space $(\mathbb{H}, < \cdot, \cdot >)$ and a nonlinear mapping $F : \mathbb{H} \times \mathbb{R} \mapsto \mathbb{H}$, find $(u, \lambda) \in \mathbb{H} \times \mathbb{R}$ such that

$$F(u, \lambda) = 0.$$

- Assume that the problem has $O(2)$ symmetry.
- F is $O(2)$ equivariant, i.e.,

$$\rho_\gamma F(u, \lambda) = F(\rho_\gamma(u), \lambda) \quad \forall \gamma \in O(2),$$

where ρ_γ is the representation of γ on $\mathbb{H}$.

- Define $\mathbb{H}^{O(2)} = \{v \in \mathbb{H} : v = \rho_\gamma(v)\ \forall \gamma \in O(2)\}$.

$$F|_{\mathbb{H}^{O(2)}}(u, \lambda) = 0, \quad u \in \mathbb{H}^{O(2)}.$$

- Moreover, taking the Fréchet derivative, we note that for $u \in \mathbb{H}^{O(2)}$

$$\rho_\gamma F_u(u, \lambda)\phi = F_u(u, \lambda)\rho_\gamma(\phi) \quad \forall \gamma \in O(2) \ \ \forall \phi \in \mathbb{H}.$$

- Standard decomposition

$$\mathbb{H} = \sum_{m=0}^{\infty} \oplus \mathbb{V}_m, \quad \mathbb{V}_m \perp \mathbb{V}_l, \quad m \neq l.$$

where the $\mathbb{V}_m$ are $O(2)$ invariant.

Aston 1991

Theorem (Cliffe, Spence & Tavener 2000)

Let A be any $O(2)$-equivariant linear operator on the Hilbert space $\mathbb{H}$, i.e. $\rho_\gamma A = A\rho_\gamma$ for all $\gamma \in O(2)$. Then,

$$A : \mathbb{V}_m \to \mathbb{V}_m, \quad m = 0, 1, 2, \ldots.$$

- The eigenvalue problem

$$F_u(u, \lambda)\phi = \mu\phi, \quad \phi \in \mathbb{H},$$

decouples into the infinite set of simpler eigenvalue problems

$$F_u(u, \lambda)|_{\mathbb{V}_m}\phi = \mu\phi, \quad \phi \in \mathbb{V}_m, \quad m = 0, 1, 2, \ldots$$

- Finer block structure:

$$\mathbb{V}_m = \mathbb{V}_m^s \oplus \mathbb{V}_m^a, \quad m = 1, 2, \ldots$$

where

$$\mathbb{V}_m^s = \{v \in \mathbb{V}_m : \ v = sv\}, \quad \mathbb{V}_m^a = \{v \in \mathbb{V}_m : \ v = -sv\}.$$

- This leads to a further block diagonalization of the individual eigenvalue problems:

$$\begin{pmatrix} F_u(u,\lambda)|_{\mathbb{V}_m^s} & 0 \\ 0 & F_u(u,\lambda)|_{\mathbb{V}_m^a} \end{pmatrix} \begin{pmatrix} \phi^s \\ \phi^a \end{pmatrix} = \mu \begin{pmatrix} \phi^s \\ \phi^a \end{pmatrix},$$

m = 1,2,. . . .

- In particular, we have that

$$F_u(u,\lambda)|_{\mathbb{V}_m^s} = F_u(u,\lambda)|_{\mathbb{V}_m^a},$$

i.e., the eigenvalue μ has geometric multiplicity of 2.

Extended System Formulation

Find $\mathbf{U} = (u, \phi, \lambda)^\top \in \mathbb{H}^{O(2)} \times \mathbb{V}_m^s \times \mathbb{R}$, $m = 1, 2, \ldots$, such that

$$G(\mathbf{U}) = 0,$$

where

$$G(\mathbf{U}) = \left(\begin{array}{c} F(u, \lambda) \\ F_u(u, \lambda)\phi \\ < \ell, \phi > -1 \end{array} \right)$$

and ℓ is some appropriate normalization.

Werner & Spence 1984

Remarks

- Under a certain non-degeneracy condition it can be shown that the above extended system possesses an isolated solution.
- Note that $\mathbb{V}_m^s$ may be replaced by $\mathbb{V}_m^a$.

Navier–Stokes in cylindrical coordinates

Find $\mathbf{u} = (u_r(r,\theta,z), u_\theta(r,\theta,z), u_z(r,\theta,z), p(r,\theta,z))^\top \in \mathbb{H}$ such that

$$F(\mathbf{u}, Re) = \mathbf{0},$$

where $\mathbb{H} = H^1(\Omega)^3 \times L^2(\Omega)$ and Re denotes the Reynolds number.

Action of $O(2)$ on $\mathbb{H}$

$$R_\alpha \begin{pmatrix} u_r(r,\theta,z) \\ u_\theta(r,\theta,z) \\ u_z(r,\theta,z) \\ p(r,\theta,z) \end{pmatrix} = \begin{pmatrix} u_r(r,\theta+\alpha,z) \\ u_\theta(r,\theta+\alpha,z) \\ u_z(r,\theta+\alpha,z) \\ p(r,\theta+\alpha,z) \end{pmatrix},$$

$$S \begin{pmatrix} u_r(r,\theta,z) \\ u_\theta(r,\theta,z) \\ u_z(r,\theta,z) \\ p(r,\theta,z) \end{pmatrix} = \begin{pmatrix} u_r(r,-\theta,z) \\ -u_\theta(r,-\theta,z) \\ u_z(r,-\theta,z) \\ p(r,-\theta,z) \end{pmatrix}.$$

$O(2)$ Equivariance

For all $\mathbf{u} \in \mathbb{H}$ the following relations hold:

$$R_\alpha F(\mathbf{u}, Re) = F(R_\alpha \mathbf{u}, Re), \qquad SF(\mathbf{u}, Re) = F(S\mathbf{u}, Re).$$

Decomposition of $\mathbb{H}$

The solution space $\mathbb{H}$ may be decomposed as follows:

$$\mathbb{H} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \dots$$

where, for $m > 1$, we have

$$\mathbb{V}_m = \mathrm{Span}\left\{ \begin{pmatrix} u_r^m(r,z)\cos(m\theta) \\ u_\theta^m(r,z)\sin(m\theta) \\ u_z^m(r,z)\cos(m\theta) \\ p^m(r,z)\cos(m\theta) \end{pmatrix}, \begin{pmatrix} u_r^m(r,z)\sin(m\theta) \\ u_\theta^m(r,z)\cos(m\theta) \\ u_z^m(r,z)\sin(m\theta) \\ p^m(r,z)\sin(m\theta) \end{pmatrix} \right\}.$$

Extended System

Find $\mathbf{U} = (\mathbf{u}, \phi, Re)^\top \in \mathbb{H}^{O(2)} \times \mathbb{V}_m^s \times \mathbb{R}$, $m = 1, 2, \ldots,$ such that

$$\begin{pmatrix} F(\mathbf{u}, Re) \\ F_{\mathbf{u}}(\mathbf{u}, Re)\phi \\ < \ell, \phi > -1 \end{pmatrix} = \mathbf{0}.$$

- Remarks
  - $F(\cdot, Re)|_{\mathbb{H}^{O(2)}}$ and $F_{\mathbf{u}}(\mathbf{u}, Re)|_{\mathbb{V}_m^s}$ are independent of θ.

    $\Rightarrow$ Can study stability to three dimensional disturbances using a sequence of two dimensional problems.
  - For $\mathbf{u} \in \mathbb{H}^{O(2)}$, we have $u_\theta \equiv 0$.

Weak Formulation

Find $\mathbf{U} \in \mathbb{H}^{O(2)} \times \mathbb{V}_m^s \times \mathbb{R}$, $m = 1, 2, \ldots,$

$$\mathcal{N}(\mathbf{U}, \mathbf{v}) = 0 \quad \forall \mathbf{v}.$$

- $\mathcal{T}_h = \{\kappa\}$ is a non-degenerate mesh;
- Given $p = \{p_\kappa\}$, define the finite element space

$$V_{h,p} = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{S}_{p_\kappa} \quad \forall \kappa \in \mathcal{T}_h\},$$

where $\mathcal{S}_p$ is either $\mathcal{P}_p$ or $\mathcal{Q}_p$.

DG Discretization

For $\mathbf{p} \geq 1$, define

$$\mathbf{V}_{h,p} = [V_{h,p}]^2 \times V_{h,p-1} \times [V_{h,p}]^3 \times V_{h,p-1} \times \mathbb{R}.$$

DGFEM: Find $\mathbf{U}_h \in \mathbf{V}_{h,p}$, $m = 1, 2, \ldots$, such that

$$\mathcal{N}_h(\mathbf{U}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_{h,p}.$$

Schötzau, Schwab & Toselli 2003, 2004, Cockburn, Kanschat & Schötzau 2005

- Nonlinear convective terms are treated based on employing a Lax–Friedrichs flux.

- The numerical solution $\mathbf{U}_h$ is computed using Newton's method, together with a block elimination technique: Werner & Spence 1984

$$\begin{bmatrix} F_u^{N,N} & 0 & \mathbf{F}_\lambda^{N,1} \\ F_{uu}^{M,N} & F_u^{M,M} & \mathbf{F}_{u\lambda}^{M,1} \\ \mathbf{0}^\top & \mathbf{l}^\top & 0 \end{bmatrix} \begin{bmatrix} \mathbf{du}_h \\ \mathbf{d}\phi_h \\ d\lambda_h \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ R_3 \end{bmatrix}.$$

- This requires solving a matrix with $(N+M+1)$ unknowns, where $N = \dim([V_{h,p}]^2 \times V_{h,p-1})$ and $M = \dim([V_{h,p}]^3 \times V_{h,p-1})$.
- The coefficient matrix may be written in block LU format, where

$$L = \left[\begin{array}{c|cc} F_u^{N,N} & 0 & \mathbf{0} \\ \hline F_{uu}^{M,N} & I & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{0}^\top & 1 \end{array}\right] \text{ and } U = \left[\begin{array}{c|cc} I & 0 & -\mathbf{w} \\ \hline 0 & F_u^{M,M} & F_{uu}^{M,N}\mathbf{w} + \mathbf{F}_{u\lambda}^{M,1} \\ \mathbf{0}^\top & \mathbf{l}^\top & 0 \end{array}\right],$$

and $\mathbf{w}$ satisfies $F_u^{N,N}\mathbf{w} = -\mathbf{F}_\lambda^{N,1}$.

- Thereby, this requires the solution of two $N \times N$ and one $(M+1) \times (M+1)$ matrix problems. Keller 1977

Dual problem

Find $\mathbf{z}$ such that

$$\mathcal{M}(\mathbf{U}, \mathbf{U}_h; \mathbf{w}, \mathbf{z}) = J(\mathbf{w}) \quad \forall \mathbf{w}.$$

Eriksson, Estep, Hansbo, & Johnson 1995, Becker & Rannacher 2001

- Gâteaux derivative of $\mathcal{N}_h(\cdot,\cdot)$:

$$\mathcal{N}'_{h,\mathbf{U}}[\mathbf{w}](\mathbf{v},\cdot) = \lim_{\epsilon \to 0} \frac{\mathcal{N}_h(\mathbf{w}+\epsilon\mathbf{v},\cdot) - \mathcal{N}_h(\mathbf{w},\cdot)}{\epsilon}.$$

- Mean–value linearization of $\mathcal{N}_h(\cdot,\cdot)$:

$$\begin{aligned}\mathcal{M}(\mathbf{U}, \mathbf{U}_h; \mathbf{U}-\mathbf{U}_h, \mathbf{v}) &= \mathcal{N}_h(\mathbf{U},\mathbf{v}) - \mathcal{N}_h(\mathbf{U}_h,\mathbf{v}) \\ &= \int_0^1 \mathcal{N}'_{h,\mathbf{U}}[\theta\mathbf{U} + (1-\theta)\mathbf{U}_h](\mathbf{U}-\mathbf{U}_h,\mathbf{v})\mathrm{d}\theta.\end{aligned}$$

- Bifurcation problem:

$$J(\mathbf{U}) = Re^c.$$

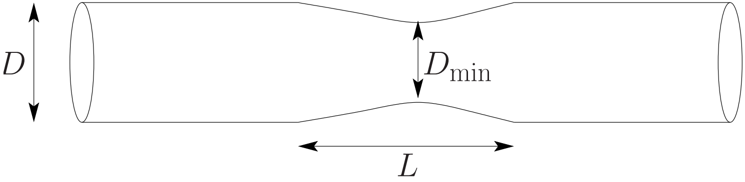
Proposition (Error Representation Formula)

Assuming the dual problem is well–posed, the following result holds:

$$Re^c - Re_h^c = -\mathcal{N}_h(\mathbf{U}_h, \mathbf{z} - \mathbf{z}_h) \equiv \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa,$$

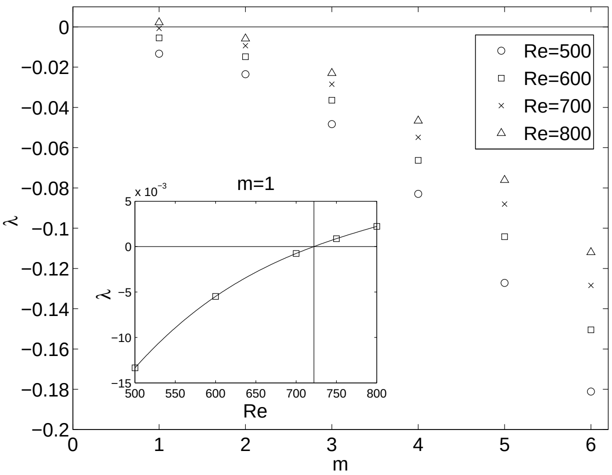
for all $\mathbf{z}_h \in \mathbf{V}_{h,p}$.

- Sherwin & Blackburn 2005: $Re^c \approx 721$.



- We set the parameters in the following ratio: $D_{\min} : D : L = 1 : 2 : 4$.

- Critical $Re^c \approx 721$.



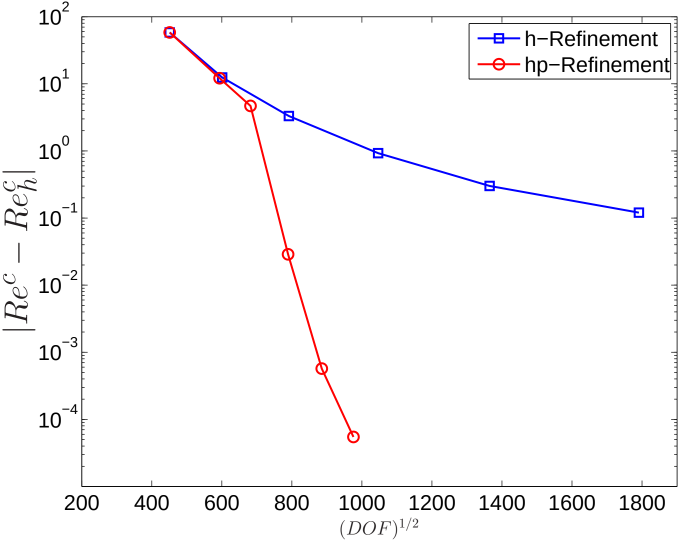
- h-adaptivity

Base DOF	Null DOF	Re_h^c	$\lvert Re^c - Re_h^c \rvert$	$\lvert \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa \rvert$	τ
84480	119040	662.66203	58.390	67.559	1.16
149886	211203	708.59280	12.460	11.781	0.95
259974	366327	717.74441	3.308	3.355	1.01
454476	640398	720.12711	9.254E-01	9.328E-01	1.01
772926	1089123	720.75181	3.007E-01	2.940E-01	0.98
1331616	1876368	720.93213	1.204E-01	9.017E-01	0.75

- hp-adaptivity

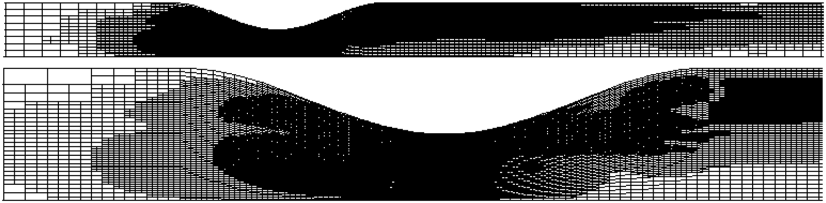
Base DOF	Null DOF	Re_h^c	$\lvert Re^c - Re_h^c \rvert$	$\lvert \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa \rvert$	τ
84480	119040	662.66203	58.390	67.559	1.16
146362	206090	708.96275	12.090	11.296	0.93
193518	271480	716.36055	4.692	4.680	1.00
259439	363362	721.02371	2.881E-02	3.575E-02	1.24
327537	456501	721.05195	5.710E-04	8.054E-04	1.41
398569	553522	721.05247	5.477E-05	5.477E-05	1.00

Effectivity Index: $\tau = \lvert \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa \rvert / \lvert Re^c - Re_h^c \rvert$.

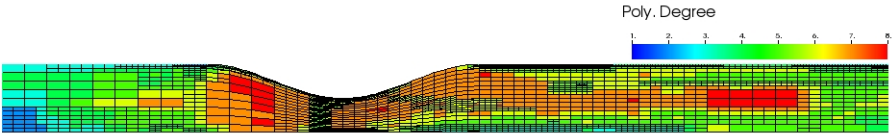


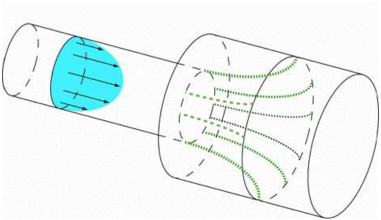
h−Refinement
hp−Refinement
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$|Re^c - Re^c_h|$
200
400
600
800
1000
1200
1400
1600
1800
$(DOF)^{1/2}$

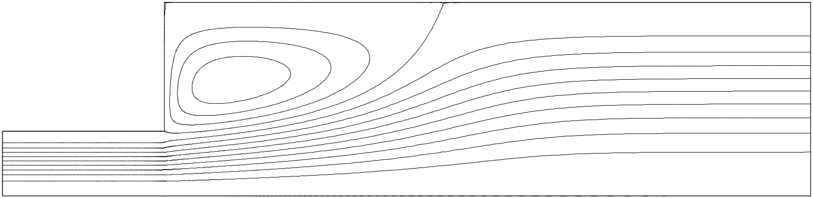
Mesh distribution after 5 h–adaptive refinements

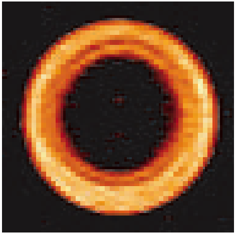


Mesh distribution after 6 hp–adaptive refinements

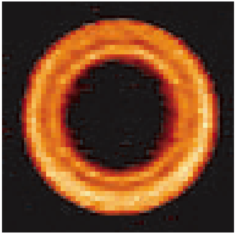




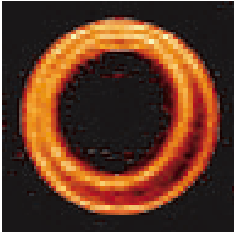




Re=372



Re=649



Re=1522

Re=1567

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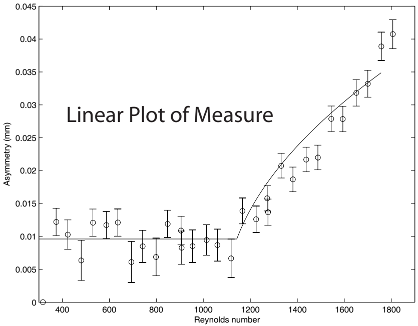

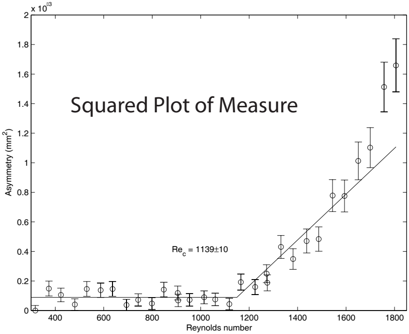

- Steady bifurcation occurs at $Re = 1139 \pm 10$.
- Onset of time dependence at $Re \approx 1500$.

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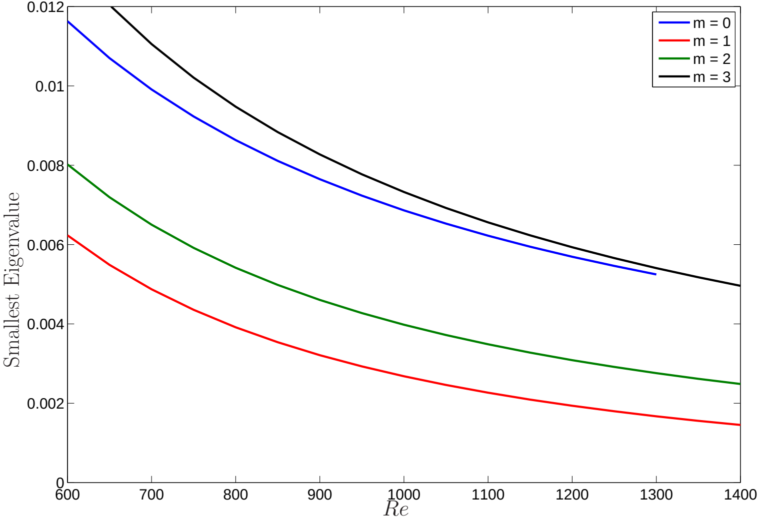
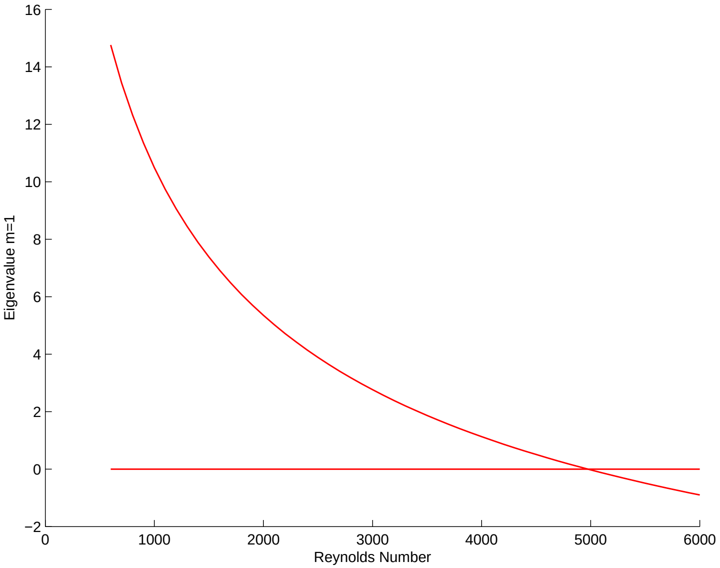
1:2 Pipe Expansion: Eigenvalues with *Re*



- $Re = 1300$, $m = 1$.

Mesh No.	No. Eles	Eig. Dofs	Eigenvalue	$\sum_{\kappa\in\mathcal{T}_h} \eta_\kappa$
1	20000	420000	0.167241E-02	1.741E-06
2	34565	725865	0.167194E-02	1.914E-06
3	65909	1384089	0.167218E-02	9.771E-07
4	111956	2351076	0.167243E-02	5.765E-07

1:2 Pipe Expansion: Eigenvalues for $m = 1$



1:2 Pipe Expansion: Bifurcation Location

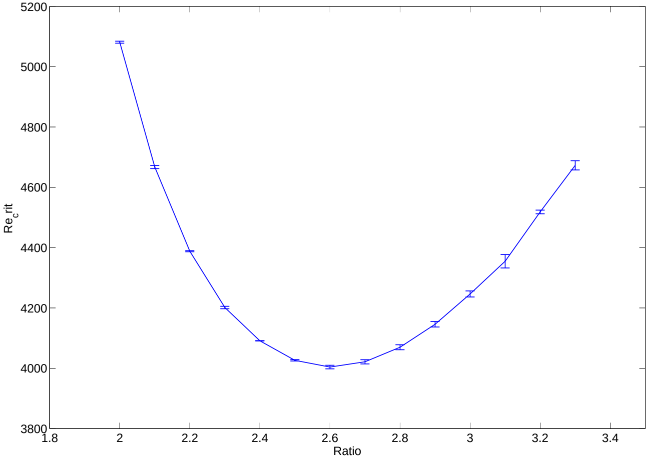
- *h*–adaptivity

Mesh No.	Base Dofs	Null Dofs	Re_h^c	$\lvert \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa \rvert$
1	232755	325857	4910.17	223.49
2	426435	597009	5106.28	11.47
3	772485	1081479	5085.25	0.77
4	1397355	1956297	5080.59	4.14
5	2467905	3455067	5082.41	–

- *hp*–adaptivity

Mesh No.	Base Dofs	Null Dofs	Re_h^c	$\lvert \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa \rvert$
1	232755	325857	4925.51	211.880
2	311300	434264	4708.29	342.372
3	448495	624696	4996.91	85.547
4	844468	1158267	5084.79	1.663
5	995544	1363010	5084.95	7.869E-02

Bifurcation Location vs. Expansion Ratio



- These techniques are generally applicable to study linear stability of a range of nonlinear problems.
  $\longrightarrow$ Bratu problem, NS in the presence of Z_2 and $O(2)$ symmetry, crystal growth, ...
- Pipe with a sudden expansion: there is a steady, supercritical, $O(2)$-symmetry-breaking bifurcation at Reynolds number approximately 5085.
- Sadly, this has nothing whatsoever to do with what is seen in the experiments. Cantwell, Barkley & Blackburn 2010

References

- K.A. Cliffe, E. Hall, and P.H. Adaptive Discontinuous Galerkin Methods for Eigenvalue Problems arising in Incompressible Fluid Flows. SIAM Journal on Scientific Computing 31(6):4607-4632, 2010.
- K.A. Cliffe, E. Hall, P.H., E.T. Phipps, and A.G. Salinger. Adaptivity and A Posteriori Error Control for Bifurcation Problems I: The Bratu Problem. Communications in Computational Physics 8(4):845-865, 2010.
- K.A. Cliffe, E. Hall, P.H., E.T. Phipps, and A.G. Salinger. Adaptivity and A Posteriori Error Control for Bifurcation Problems II: Incompressible fluid flow in Open Systems with Z_2 Symmetry. Journal of Scientific Computing 47(3):389-418, 2011.
- K.A. Cliffe, E. Hall, P.H., E.T. Phipps, and A.G. Salinger. Adaptivity and A Posteriori Error Control for Bifurcation Problems III: Incompressible fluid flow in Open Systems with O(2) Symmetry. Journal of Scientific Computing 52(1):153-179, 2012.
- K.A. Cliffe, E. Hall, and P.H. *hp*–Adaptive Discontinuous Galerkin Methods for Bifurcation Phenomena in Open Flows. Submitted for publication.

Future Challenges

- Criticality calculations for the Neutron–Transport equation.
- Fast solvers for block structured matrices.
- Error control stochastic PDEs (nuclear waste disposal).
- Non-Newtonian flows, with application to cone crushers.
- Development of (composite) DGFEMs for problems posed in complicated geometries.
- Multi-level preconditioners on overlapping meshes.
- Efficient methods for numerical modelling of chemical vapour deposition (CVD) chambers: applications to the growth of synthetic diamonds (sponsored by Element 6).
- Unified DG software for multiphysics problems:

AptoFEM