- Applications

  Linear elliptic/parabolic/hyperbolic PDEs, Fokker–Planck equations, Incompressible/Compressible fluid flows, Turbulent flows, Non-Newtonian flows, Time and frequency domain Maxwell's equations, Acoustics, MHD, Fully nonlinear PDEs.

- Advantages

  - Robustness/stability;
  - Locally conservative;
  - Ease of implementation;
  - Highly parallelizable;
  - Flexible mesh design (hybrid grids, non-matching grids, non-uniform/anisotropic polynomial degrees);
  - Wider choice of stable FE spaces for mixed problems;
  - Unified treatment of a wide range of PDEs;
  - Computational overhead/efficiency (increase in DoFs)

    $\Rightarrow$ Parallel efficiency gains: 56 Billion dofs on a GPU; Domain decomposition without overlap.

Transport Equation

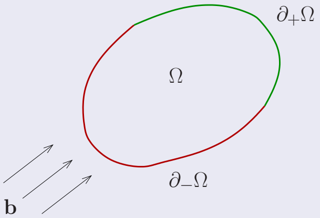
Consider the PDE problem

$$\begin{aligned} \mathbf{b} \cdot \nabla u &= f, \quad x \in \Omega, \\ u &= g, \quad x \in \partial_- \Omega. \end{aligned}$$

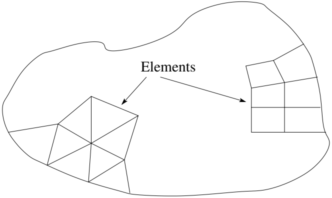
Inflow/Outflow boundaries

We define the inflow and outflow boundaries $\partial_- \Omega$ and $\partial_+ \Omega$, respectively, by

$$\begin{aligned} \partial_- \Omega &= \{x \in \partial\Omega : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\}, \\ \partial_+ \Omega &= \{x \in \partial\Omega : \mathbf{b}(x) \cdot \mathbf{n}(x) \geq 0\}. \end{aligned}$$

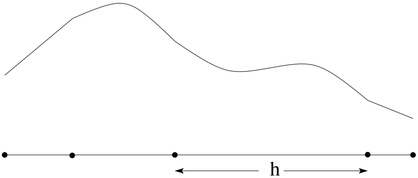


- We shall consider two variants:
  1. Standard Galerkin with strongly imposed boundary conditions;
  2. Standard Galerkin with weakly imposed boundary conditions;
- $\mathcal{T}_h$ is a non-degenerate mesh consisting of elements of granularity h.



- Finite element space:

$$V_h = \{v \in H^1(\Omega) : v|_\kappa \in \mathcal{S}_p(\kappa) \quad \forall \kappa \in \mathcal{T}_h\}.$$



- Consider the following two additional finite element spaces:

$$\begin{aligned} V_{h,E} &= \{v \in V_h : v = g \ \text{ on } \partial_-\Omega\}, \\ V_{h,E_0} &= \{v \in V_h : v = 0 \ \text{ on } \partial_-\Omega\}. \end{aligned}$$

Standard Galerkin with strongly imposed boundary conditions

Find $u_h \in V_{h,E}$ such that

$$(\mathbf{b} \cdot \nabla u_h, v_h) = (f, v_h) \quad \forall v_h \in V_{h,E_0}.$$

- Here, $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$ inner-product, i.e., given $v, w \in L^2(\Omega)$,

$$(v, w) = \int_\Omega vw \,\mathrm{d}x.$$

- Variants include the streamline–diffusion/SUPG method ($v_h \mapsto v_h + \delta \mathbf{b} \cdot \nabla v_h$, where $\delta = C_\delta h$, or more generally, $\delta = C_\delta h/p$, $C_\delta > 0$).

Standard Galerkin with weakly imposed boundary conditions

Find $u_h \in V_h$ such that

$$(\mathbf{b} \cdot \nabla u_h, v_h) - < u_h, v_h >_- = (f, v_h) - < g, v_h >_- \quad \forall v_h \in V_h,$$

where

$$< v, w >_- = \int_{\partial_- \Omega} \mathbf{b} \cdot \mathbf{n}\, v w \,\mathrm{d}s.$$

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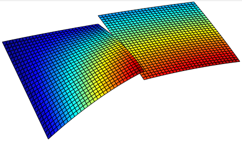
- Finite Element Space

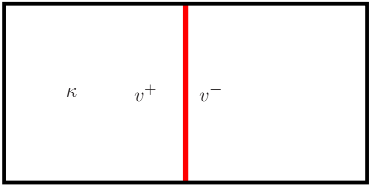


$$V_h = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{S}_p(\kappa) \ \ \forall \kappa \in \mathcal{T}_h\},$$

where $\mathcal{T}_h$ is the mesh.

- Consider a local (elementwise) FE formulation with weakly imposed bcs.
- Notation: for $v \in H^1(\kappa)$, we write

$$\begin{aligned} v^+ &= \text{interior trace of } v \text{ on } \partial\kappa \text{ taken from within } \kappa \\ v^- &= \text{exterior trace of } v \text{ on } \partial\kappa \text{ taken from outside } \kappa \end{aligned}$$



- Local FE formulation: on each $\kappa \in \mathcal{T}_h$, find $u_h \in V_h$ such that

$$\int_\kappa \mathbf{b} \cdot \nabla u_h \, v_h \mathrm{d}x - \int_{\partial_-\kappa} \mathbf{b} \cdot \mathbf{n} \, u_h^+ v_h^+ \, \mathrm{d}s = \int_\kappa f \, v_h \mathrm{d}x - \int_{\partial_-\kappa} \mathbf{b} \cdot \mathbf{n} \, \hat{g} \, v_h^+ \mathrm{d}s$$

for all $v_h \in V_h$. Here,

$$\begin{aligned} \partial_-\kappa &= \{x \in \partial\kappa : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\}, \\ \partial_+\kappa &= \{x \in \partial\kappa : \mathbf{b}(x) \cdot \mathbf{n}(x) \geq 0\}. \end{aligned}$$

Moreover, we define

$$\hat{g}(x) = \begin{cases} u_h^-(x) & x \in \partial_-\kappa \backslash \partial\Omega, \\ g(x) & x \in \partial_-\kappa \cap \partial\Omega. \end{cases}$$

- Sum over all elements $\kappa \in \mathcal{T}_h$:

DGFEM

Find $u_h \in V_h$ such that

$$\sum_{\kappa\in\mathcal{T}_h}\left\{\int_\kappa \mathbf{b}\cdot\nabla u_h\, v_h \mathrm{d}x - \int_{\partial_-\kappa\setminus\partial\Omega} \mathbf{b}\cdot\mathbf{n}\,(u_h^+ - u_h^-)v_h^+\,\mathrm{d}s \right.$$
$$\left. - \int_{\partial_-\kappa\cap\partial\Omega} \mathbf{b}\cdot\mathbf{n}\, u_h^+ v_h^+\,\mathrm{d}s\right\} = \sum_{\kappa\in\mathcal{T}_h}\left\{\int_\kappa f\, v_h \mathrm{d}x - \int_{\partial_-\kappa\cap\partial\Omega}\mathbf{b}\cdot\mathbf{n}\, g v_h^+ \mathrm{d}s\right\}$$

for all $v_h \in V_h$.

Flux Formulation

Transport Equation in Conservative Form

Consider the PDE problem

$$\begin{aligned} \nabla \cdot (\mathbf{b}u) &= f, \quad x \in \Omega, \\ u &= g, \quad x \in \partial_- \Omega. \end{aligned}$$

[Assuming $\mathbf{b}$ is incompressible, i.e., $\nabla \cdot \mathbf{b} = 0$, the conservative and non-conservative forms are equivalent.]

- Local weak formulation: on each $\kappa \in \mathcal{T}_h$, find $u|_\kappa$ such that

$$-\int_\kappa (\mathbf{b}u) \cdot \nabla v \mathrm{d}x + \int_{\partial\kappa} (\mathbf{b}u^+) \cdot \mathbf{n}_\kappa v^+ \, \mathrm{d}s = 0.$$

- Inter-element continuity and bcs weakly enforced

$$-\int_\kappa (\mathbf{b}u) \cdot \nabla v \mathrm{d}x + \int_{\partial\kappa} \mathcal{H}(u^+, u^-, \mathbf{n}_\kappa) v^+ \, \mathrm{d}s = \int_\kappa f v \mathrm{d}x.$$

- $\mathcal{H}(\cdot, \cdot, \mathbf{n})$ is a numerical flux function.
- Sum over all elements $\kappa \in \mathcal{T}_h$ and restrict to the FEM space V_h:

DGFEM

Find $u_h \in V_h$ such that

$$\sum_{\kappa \in \mathcal{T}_h} \left\{ -\int_\kappa (\mathbf{b}u_h) \cdot \nabla v_h \mathrm{d}x + \int_{\partial\kappa} \mathcal{H}(u_h^+, u_h^-, \mathbf{n}_\kappa) v_h^+ \, \mathrm{d}s \right\} = \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa f \, v_h \mathrm{d}x \right\}$$

for all $v_h \in V_h$.

- Properties of the numerical flux function $\mathcal{H}(\cdot,\cdot,\cdot)$.
  1. Consistency: for each κ in $\mathcal{T}_h$ we have that
  $$\mathcal{H}(v, v, \mathbf{n}_\kappa)|_{\partial\kappa} = (\mathbf{b}v)\cdot\mathbf{n}_\kappa \quad \forall\kappa\in\mathcal{T}_h.$$
  2. Conservation: given any two neighbouring elements κ and κ' from the finite element partition $\mathcal{T}_h$, at each point $\mathbf{x}\in\partial\kappa\cap\partial\kappa'\neq\emptyset$, noting that $\mathbf{n}_{\kappa'} = -\mathbf{n}_\kappa$, we have that
  $$\mathcal{H}(v, w, \mathbf{n}_\kappa) = -\mathcal{H}(w, v, -\mathbf{n}_\kappa).$$
- Choose the upwind numerical flux:
$$\mathcal{H}(u_h^+, u_h^-, \mathbf{n}_\kappa)|_{\partial\kappa} = \mathbf{b}\cdot\mathbf{n}_\kappa \lim_{s\to 0^+} u_h(x - s\mathbf{b}) \quad \text{for } \kappa\in\mathcal{T}_h.$$
By convention, we set $u_h^-|_{\partial_-\kappa\cap\partial\Omega} = g$; thereby,
$$\mathcal{H}(u_h^+, u_h^-, \mathbf{n}_\kappa)|_{\partial_-\kappa\cap\partial\Omega} = \mathbf{b}\cdot\mathbf{n}_\kappa\, g \quad \text{for } \kappa\in\mathcal{T}_h.$$
- (Nonlinear Problems) Other Examples: Lax–Friedrichs flux, Roe's flux, Vijayasundaram flux, ...
- Note that both DGFEM formulations are equivalent (proof: exercise).

Given $\Omega \subset \mathbb{R}^d$, $d \geq 1$, and $f \in L^2(\Omega)$, find u such that

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

DG Discretization:

1. Rewrite as a first–order system.
2. Derive an elemental weak formulation.
3. Introduce numerical flux functions $\Rightarrow$ Flux Formulation.
4. Eliminate the auxiliary variables $\Rightarrow$ Primal Formulation.

- Rewrite as a first–order system.

$$\mathbf{s} - \nabla u = 0, \qquad -\nabla \cdot \mathbf{s} = f.$$

- Elemental weak formulation: find $(\mathbf{s}, u)$ such that

$$\begin{aligned} \int_\kappa \mathbf{s} \cdot \boldsymbol{\tau} \mathrm{d}x + \int_\kappa u \nabla \cdot \boldsymbol{\tau} \mathrm{d}x - \int_{\partial\kappa} u \boldsymbol{\tau} \cdot \mathbf{n}_\kappa \mathrm{d}s &= 0, \\ \int_\kappa \mathbf{s} \cdot \nabla v \mathrm{d}x - \int_{\partial\kappa} \mathbf{s} \cdot \mathbf{n}_\kappa v \mathrm{d}s &= \int_\kappa f v \mathrm{d}x. \end{aligned}$$

- Notation: ∇_h denotes the broken gradient operator, defined elementwise.
- Numerical flux functions:
  1. $\hat{u} = \hat{u}(u_h)$,
  2. $\hat{\mathbf{s}} = \hat{\mathbf{s}}(u_h, \nabla_h u_h)$

  are approximations to u_h and $\nabla_h u_h$, respectively.
- Flux Formulation: find $u_h \in V_h$ and $\mathbf{s}_h \in \Sigma_h = [V_h]^d$ such that

$$\int_\Omega \mathbf{s}_h \cdot \boldsymbol{\tau}_h \mathrm{d}x + \int_\Omega u_h \nabla_h \cdot \boldsymbol{\tau}_h \mathrm{d}x - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{u} \boldsymbol{\tau}_h \cdot \mathbf{n}_\kappa \mathrm{d}s = 0, \tag{1}$$

$$\int_\Omega \mathbf{s}_h \cdot \nabla v_h \mathrm{d}x - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{\mathbf{s}} \cdot \mathbf{n}_\kappa v_h \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x \tag{2}$$

for all $\boldsymbol{\tau}_h \in \Sigma_h$, $v_h \in V_h$.

- Setting $\boldsymbol{\tau}_h = \nabla_h v_h$ in (1) and integrating by parts gives

$$\int_\Omega \mathbf{s}_h \cdot \nabla_h v_h \mathrm{d}x - \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \mathrm{d}x + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} (u_h^+ - \hat{u}) \nabla_h v_h^+ \cdot \mathbf{n}_\kappa \mathrm{d}s = 0. \quad (3)$$

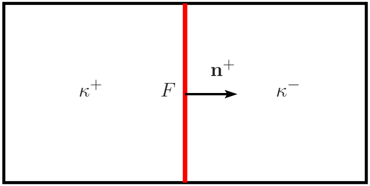
- Inserting (3) into (2) gives the primal formulation: find $u_h \in V_h$ such that

$$\int_\Omega \nabla_h u_h \cdot \nabla v_h \mathrm{d}x - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} (u_h^+ - \hat{u}) \nabla_h v_h^+ \cdot \mathbf{n}_\kappa \mathrm{d}s$$
$$- \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{\mathbf{s}} \cdot \mathbf{n}_\kappa v_h^+ \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x$$

for all $v_h \in V_h$.

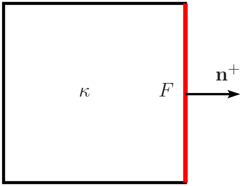
- Let $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}$ denote the set of all faces in the mesh $\mathcal{T}_h$.
- Notation:

$F \subset \mathcal{F}_h^{\mathcal{I}}$



$$\begin{aligned} [\![v]\!] &= v^+\mathbf{n}^+ + v^-\mathbf{n}^- \\ [\![\mathbf{q}]\!] &= \mathbf{q}^+\cdot\mathbf{n}^+ + \mathbf{q}^-\cdot\mathbf{n}^- \\ \{\!\{v\}\!\} &= (v^+ + v^-)/2 \end{aligned}$$

$F \subset \mathcal{F}_h^{\mathcal{B}}$



$$\begin{aligned} [\![v]\!] &= v\mathbf{n} \\ [\![\mathbf{q}]\!] &= \mathbf{q}\cdot\mathbf{n} \\ \{\!\{v\}\!\} &= v \end{aligned}$$

- The following identity holds:

$$\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}\mathbf{q}^+\cdot\mathbf{n}^+\,v^+\mathrm{d}s = \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\mathbf{q}\}\!\}\cdot[\![v]\!]\mathrm{d}s + \sum_{F\in\mathcal{F}_h^{\mathcal{I}}}\int_F [\![\mathbf{q}]\!]\{\!\{v\}\!\}\mathrm{d}s.$$

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DGFEM Primal Formulation

Find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = \int_\Omega f v_h \mathrm{d}x \qquad \forall v_h \in V_h,$$

where

$$\begin{aligned} A_h(u_h, v_h) &= \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \nabla u_h \cdot \nabla v_h \mathrm{d}x \\ &+ \sum_{F \in \mathcal{F}_h} \int_F (\llbracket \hat{u} - u_h \rrbracket \cdot \{\!\{ \nabla_h v_h \}\!\} - \{\!\{ \hat{\mathbf{s}} \}\!\} \cdot \llbracket v_h \rrbracket) \mathrm{d}s \\ &+ \sum_{F \in \mathcal{F}_h^{\mathcal{I}}} \int_F (\{\!\{ \hat{u} - u_h \}\!\} \llbracket \nabla_h v_h \rrbracket - \llbracket \hat{\mathbf{s}} \rrbracket \{\!\{ v_h \}\!\}) \mathrm{d}s, \end{aligned}$$

where $\hat{u} = \hat{u}(u_h)$ and $\hat{\mathbf{s}} = \hat{\mathbf{s}}(u_h, \nabla_h u_h)$.

Interior Penalty Methods

Given $\theta \in [-1, 1]$, find $u_h \in V_h$ such that

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h \mathrm{d}x - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h u_h\}\!\} \cdot [\![v_h]\!] \mathrm{d}s$$
$$+\theta \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot [\![u_h]\!] \mathrm{d}s + \sum_{F \in \mathcal{F}_h} \int_F \sigma [\![u_h]\!] \cdot [\![u_h]\!] \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x$$

for all $v_h \in V_h$.

- $\theta = -1$ – Symmetric Interior Penalty Method (SIP).
- $\theta = 1$ – Nonsymmetric Interior Penalty Method (NIP).
- $\theta = 0$ – Incomplete Interior Penalty Method (IIP).

Arnold, Brezzi, Cockburn, & Marini 2002

Guidelines for devising the methods.

- Use a characterization of the exact solution in terms of solutions of local problems and transmission conditions.
- Use <u>discontinuous</u> approximations for both the solution inside each element and its trace on the element boundary.
- Define the local solvers by using a <u>Galerkin</u> method to weakly enforce the equations on each element.
- Define a global problem by <u>weakly</u> imposing the transmission conditions.

We provide two different characterizations of the solution of the following second-order elliptic model problem:

$$
\begin{aligned}
\mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } \Omega,\\
\nabla \cdot \mathbf{q} &= f && \text{in } \Omega,\\
\widehat{u} &= u_D && \text{on } \partial\Omega.
\end{aligned}
$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

The main idea.

The general approach: Local problems and transmission conditions.

We have that the exact solution satisfies the local problems

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 \quad \text{in } K, \\ \nabla \cdot \mathbf{q} &= f \quad \text{in } K, \end{aligned}$$

the transmission conditions

$$\begin{aligned} [\![\widehat{u}]\!] &= 0 \quad \text{if } F \in \mathcal{E}_h^o, \\ [\![\widehat{\mathbf{q}}]\!] &= 0 \quad \text{if } F \in \mathcal{E}_h^o, \end{aligned}$$

and the Dirichlet boundary condition

$$\widehat{u} = u_D \quad \text{if } F \in \mathcal{E}_h^\partial.$$

The main idea.

A first approach: Rewriting the equations.

We can obtain $(\mathbf{q}, u)$ in K in terms of $\widehat{u}$ on ∂K and f by solving

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } K, \\ \nabla \cdot \mathbf{q} &= f && \text{in } K, \\ u &= \widehat{u} && \text{on } \partial K. \end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution, on each $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned} [\![\widehat{\mathbf{q}}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o, \\ \widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial, \end{aligned}$$

where $\widehat{\mathbf{q}}$ is the trace of $\mathbf{q} = \mathbf{q}(\widehat{u}, f)$ on ∂K.

The main idea.

A first approach: Characterization of the solution.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathsf{U}_{\widehat{u}}) + (\mathbf{Q}_f, \mathsf{U}_f)$, where

$$\begin{aligned}
\mathrm{c}\,\mathbf{Q}_{\widehat{u}} + \nabla \mathsf{U}_{\widehat{u}} &= 0 \quad \text{in } K, & \mathrm{c}\,\mathbf{Q}_f + \nabla \mathsf{U}_f &= 0 \quad \text{in } K,\\
\nabla\cdot\mathbf{Q}_{\widehat{u}} &= 0 \quad \text{in } K, & \nabla\cdot\mathbf{Q}_f &= f \quad \text{in } K,\\
\mathsf{U}_{\widehat{u}} &= \widehat{u} \quad \text{on } \partial K, & \mathsf{U}_f &= 0 \quad \text{on } \partial K.
\end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution, on each $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned}
-[\![\widehat{\mathbf{Q}}_{\widehat{u}}]\!] &= [\![\widehat{\mathbf{Q}}_f]\!] && \text{if } F \in \mathcal{E}_h^o,\\
\widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial.
\end{aligned}$$

A first approach. (B.C., J.Gopalakrishnan and R.Lazarov, SINUM, 2009.)

The local solvers: A weak formulation on each element.

On the element $K \in \Omega_h$, given $\widehat{u}$ on ∂K and f, we have that $(\mathbf{q}, u)$ satisfies the equations

$$\begin{aligned} (\mathrm{c}\,\mathbf{q}, \mathbf{v})_K - (u, \nabla \cdot \mathbf{v})_K + \langle \widehat{u}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= 0, \\ -(\mathbf{q}, \nabla w)_K + \langle \widehat{\mathbf{q}} \cdot \mathbf{n}, w \rangle_{\partial K} &= (f, w)_K, \end{aligned}$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, where

$$\widehat{\mathbf{q}} \cdot \mathbf{n} = \mathbf{q} \cdot \mathbf{n} \qquad \text{on } \partial K.$$

The first approach.

The local solvers: Definition.

On the element $K \in \Omega_h$, we define $(\mathbf{q}_h, u_h)$ terms of $(\widehat{u}_h, f)$ as the element of $\mathbf{V}(K) \times W(K)$ such that

$$
\begin{aligned}
(\mathrm{c}\, \mathbf{q}_h, \mathbf{v})_K - (u_h, \nabla \cdot \mathbf{v})_K + \langle \widehat{u}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= 0, \\
-(\mathbf{q}_h, \nabla w)_K + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial K} &= (f, w)_K,
\end{aligned}
$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, where

$$
\widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau (u_h - \widehat{u}_h) \qquad \text{on } \partial K.
$$

The first approach

The local solvers: The form of the numerical trace $\widehat{\mathbf{q}}_h$.

If we want that, at any given point x of ∂K at which the normal $\mathbf{n}$ is well defined,

- The numerical trace $\widehat{\mathbf{q}}_h(x) \cdot \mathbf{n}$ only depends on $\mathbf{q}_h(x) \cdot \mathbf{n}$, $u_h(x)$ and the numerical trace $\widehat{u}_h(\mathrm{x})$.
- The dependence is linear.
- The numerical trace $\widehat{\mathbf{q}}_h(x) \cdot \mathbf{n}$ is consistent, that is, $\widehat{\mathbf{q}}_h(x) \cdot \mathbf{n} = \mathbf{q}_h(x) \cdot \mathbf{n}$ whenever $u_h(x) = \widehat{u}_h(x)$,

we must have that $\widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h)$.

The first approach.

The local solvers are well defined.

Theorem

The local solver on K is well defined if

- $\tau > 0$ *on* ∂K,
- $\nabla W(K) \subset \mathbf{V}(K)$.

The first approach.

Proof.

The system is square. Set $\widehat{u}_h = 0$ and $f = 0$.
For $(\mathbf{v}, w) := (\mathbf{q}_h, u_h)$, the equations read

$$\begin{aligned} (\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K - (u_h, \nabla\cdot\mathbf{q}_h)_K &= 0, \\ -(\mathbf{q}_h, \nabla u_h)_K + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, u_h \rangle_{\partial K} &= 0. \end{aligned}$$

Hence

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K + \langle (\widehat{\mathbf{q}}_h - \mathbf{q}_h) \cdot \mathbf{n}, u_h \rangle_{\partial K} = 0,$$

and since $\widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h)$, we get

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K + \langle \tau\,(u_h), u_h \rangle_{\partial K} = 0.$$

This implies that $\mathbf{q}_h = 0$ on K, and that $u_h = 0$ on ∂K.

The first approach.

Proof.

Now, the first equation defining the local solvers reads

$$-(u_h, \nabla \cdot \mathbf{v})_K = 0,$$

for all $\mathbf{v} \in \mathbf{V}(K)$. Hence

$$(\nabla u_h, \mathbf{v})_K = 0,$$

and so $\nabla u_h = 0$. This proves the result.

The first approach.

The numerical trace $\widehat{u}_h$ is well defined.

Theorem

The numerical trace $\widehat{u}_h$ is well defined if, for each $K \in \partial\Omega_h$,

- $\tau > 0$ *on* ∂K,
- $\nabla W(K) \subset \mathbf{V}(K)$.

The first approach.

Proof.

The system is square. Set $u_D = 0$ and $f = 0$. For $\mu := \widehat{u}_h$, the equation reads

$$0 = \sum_{F \in \mathcal{E}_h^o} \langle \widehat{u}_h, [\![\widehat{\mathbf{q}}_h]\!] \rangle_F = \sum_{K \in \Omega_h} \langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial K} =: \langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h}.$$

Note that

$$\begin{aligned}
-\langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} &= - \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&= - \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} - \langle u_h, \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&\quad + \langle (u_h - \widehat{u}_h), \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&= - \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} - \langle u_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} + \langle u_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} \\
&\quad + \langle (u_h - \widehat{u}_h), \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h}
\end{aligned}$$

The first approach.

Proof.

For $(\mathbf{v}, w) := (\mathbf{q}_h, u_h)$, the equations of the local solvers read

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K - (u_h, \nabla\cdot\mathbf{q}_h)_K + \langle \widehat{u}_h, \mathbf{q}_h\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\ -(\mathbf{q}_h, \nabla u_h)_K + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, u_h\rangle_{\partial K} &= 0.\end{aligned}$$

Then

$$-\langle \widehat{u}_h, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h} = (\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_{\Omega_h} + \langle (u_h - \widehat{u}_h), \tau(u_h - \widehat{u}_h)\rangle_{\partial\Omega_h}.$$

As a consequence, $\langle \widehat{u}_h, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h} = 0$ implies $\mathbf{q}_h = 0$ on Ω_h and $u_h = \widehat{u}_h$ on $\partial\Omega_h$.

The first approach.
Proof.

Now, the first equation definign the local solvers reads

$$-(u_h, \nabla \cdot \mathbf{v})_K + \langle u_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} = 0,$$

for all $\mathbf{v} \in \mathbf{V}(K)$. Hence

$$(\nabla u_h, \mathbf{v})_K = 0,$$

and so $\nabla u_h = 0$.

This shows that u_h is a constant and, since $u_h = \widehat{u}_h = 0$ on $\partial\Omega$, we can conclude that $u_h = 0$ on Ω_h. We now have that $\widehat{u}_h = u_h = 0$ on $\partial\Omega_h$. This proves the result.

The first approach.

Characterization of the approximate solution.

We have that $(\mathbf{q}_h, u_h) = (\mathbf{Q}_{\widehat{u}_h}, \mathsf{U}_{\widehat{u}_h}) + (\mathbf{Q}_f, \mathsf{U}_f)$ where

$$(\mathbf{Q}_{\widehat{u}_h}, \mathsf{U}_{\widehat{u}_h}) := (\mathbf{Q}(\widehat{u}_h, 0), \mathsf{U}(\widehat{u}_h, 0)), \quad (\mathbf{Q}_f, \mathsf{U}_f) := (\mathbf{Q}(0, f), \mathsf{U}(0, f)).$$

where $(\mathbf{Q}(\widehat{u}_h, f), \mathsf{U}(\widehat{u}_h, f))$ is the linear mapping that associates $(\widehat{u}_h, f)$ to $(\mathbf{q}_h, u_h)$, and where the numerical trace $\widehat{u}_h$ is the element of the space

$$M_h := \{\mu \in L^2(\mathcal{E}_h) : \quad \mu|_F \in M(F) \ \ \forall\, F \in \mathcal{E}_h\},$$

satisfying the equations

$$\begin{aligned} a_h(\widehat{u}_h, \mu) &= \ell_h(\mu) && \forall\, \mu \in M_h : \mu|_{\partial\Omega} = 0, \\ \langle \mu, \widehat{u}_h \rangle_{\partial\Omega} &= \langle \mu, u_D \rangle_{\partial\Omega} && \forall\, \mu \in M_h, \end{aligned}$$

where $a_h(\mu, \lambda) := -\langle \mu, \widehat{\mathbf{Q}}_\lambda \cdot \mathbf{n} \rangle_{\partial\Omega_h}$, and $\ell_h(\mu) := \langle \mu, \widehat{\mathbf{Q}}_f \cdot \mathbf{n} \rangle_{\partial\Omega_h}$.

The first approach.

A rewriting of the method.

The approximate solution $(\mathbf{q}_h, u_h, \widehat{u}_h)$ is the element of the space $\mathbf{V}_h \times W_h \times M_h$ satisfying the equations

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{v})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \mu, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,\\
\langle \mu, \widehat{u}_h\rangle_{\partial\Omega} &= \langle \mu, u_D\rangle_{\partial\Omega},
\end{aligned}
$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where

$$
\widehat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial\Omega_h.
$$

The second approach.

A rewriting of the method.

The approximate solution $(\mathbf{q}_h, u_h, \widehat{\mathbf{q}}_h)$ is the element of the space $\mathbf{V}_h \times W_h \times \mathbf{N}_h$ satisfying the equations

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{v})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\ -(\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\ \langle \boldsymbol{\eta}\cdot\mathbf{n}, \widehat{u}_h\rangle_{\partial\Omega_h} &= \langle \boldsymbol{\eta}\cdot\mathbf{n}, u_D\rangle_{\partial\Omega},\end{aligned}$$

for all $(\mathbf{v}, w, \boldsymbol{\eta}) \in \mathbf{V}_h \times W_h \times \mathbf{N}_h$, where

$$\widehat{u}_h = u_h + s(\mathbf{q}_h - \widehat{\mathbf{q}}_h)\cdot\mathbf{n} \qquad \text{on } \partial\Omega_h.$$

Note that this method is strongly related to the method obtained with the first approach with $\tau = 1/s$.

The first approach.

The jumps $u_h - \widehat{u}_h$ control the four residuals.

The Galerkin formulation on the element K defining the local solver reads

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_K - (u_h, \nabla\cdot\mathbf{v})_K + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\
-(\mathbf{q}_h, \nabla w)_K + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial K} &= (f, w)_K,
\end{aligned}
$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K)\times W(K)$, or, equivalently,

$$
\begin{aligned}
(\mathbf{R}^u_K, \mathbf{v})_K &= \langle R^u_{\partial K}, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial K} \quad \forall\ \mathbf{v}\in\mathbf{V}(K),\\
(R^{\mathbf{q}}_K, w)_K &= \langle R^{\mathbf{q}}_{\partial K}, w\rangle_{\partial K} \quad \forall\ w\in W(K),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{R}^u_K &:= \mathrm{c}\mathbf{q}_h + \nabla u_h & R^u_{\partial K} &:= u_h - \widehat{u}_h\\
R^{\mathbf{q}}_K &:= \nabla\cdot\mathbf{q}_h - f & R^{\mathbf{q}}_{\partial K} &:= (\mathbf{q}_h - \widehat{\mathbf{q}}_h)\cdot\mathbf{n} = -\tau\,(u_h - \widehat{u}_h).
\end{aligned}
$$

Since

$$\tau = -\frac{R^{\mathbf{q}}_{\partial K}}{R^{u}_{\partial K}} \approx \frac{R^{\mathbf{q}}_{K}}{\mathbf{R}^{u}_{K}}.$$

where

$$\begin{aligned} \mathbf{R}^{u}_{K} &:= \mathrm{c}\mathbf{q}_h + \nabla u_h & \qquad R^{u}_{\partial K} &:= u_h - \widehat{u}_h \\ R^{\mathbf{q}}_{K} &:= \nabla \cdot \mathbf{q}_h - f & \qquad R^{\mathbf{q}}_{\partial K} &:= (\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}. \end{aligned}$$

we see that τ forces a ratio between the residuals.

Examples according to the local solver.

Numerical traces for simplexes K.

Method	$\widehat{\mathbf{q}}_h$
RT-H	$\mathbf{q}_h$
BDM-H	$\mathbf{q}_h$
LDG-H	$\mathbf{q}_h + \tau(u_h - \widehat{u}_h) \cdot \mathbf{n}$
IP-H	$-a\nabla u_h + \tau(u_h - \widehat{u}_h) \cdot \mathbf{n}$

Examples according to the local solver

Some remarks.

- The RT-H method is the hybridized version of the original RT method.
- The BDM-H method is the hybridized version of the original BDM method.
- The LDG-H method is not the hybridized version of the LDG method.
- The IP-H method is not the hybridized version of the IP method.
- The bilinear forms a_h of the RT-H, BDM-H and SF-H methods are the same on simplexes. (For these three methods, $\tau^* = 0$.)
- The LDG-H method is defined for any $\tau > 0$.
- The IP-H method is defined only for $\tau \approx h^{-1}$.
- The LDG-H and IP-H can be applied on any polyhedral element K.

Examples according to the local solver

Questions.

- HDG methods were devised so that they are efficiently implemented, but how about their accuracy?
- For $k = 0$, do they give (finite volume-like) convergence schemes?
- Can they have the same superconvergence properties than the mixed methods?
- Is the lack of commutative properties an essential barrier to achieving superconvergence?

We seek HDG methods for which a projection of the error, $\Pi_W u - u_h$, converges faster than the error $u - u_h$.

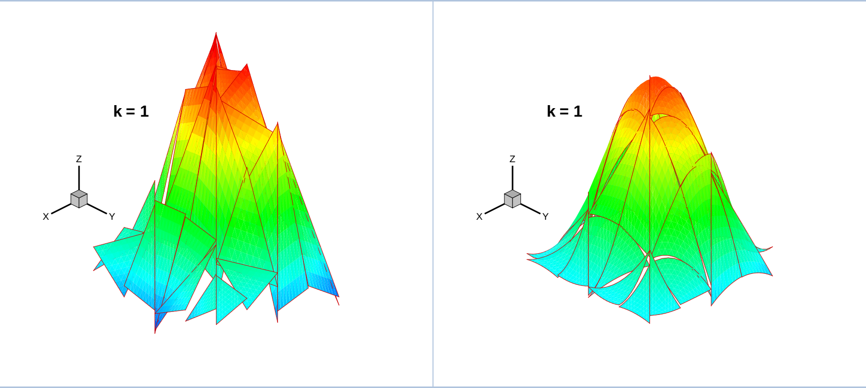
If this property holds, we introduce a new approximation $u_h^\star$. On each element K it lies in the space $W^*(K)$ and defined by

$$
\begin{aligned}
(\nabla u_h{}^\star, \nabla w)_K &= -(\mathrm{c}\mathbf{q}_h, \nabla w)_K \qquad \text{for all } w \in W^*(K),\\
(u_h{}^\star, 1)_K &= (u_h, 1)_K,
\end{aligned}
$$

If $\mathbf{q} - \mathbf{q}_h$ converges to zero fast enough, then $u - u_h^\star$ might converge as fast as $\Pi_W u - u_h$. This does happen for mixed methods!

Illustration of the postprocessing.

An HDG method for linear elasticity.



Comparison between the approximate solution (left) and the post-processed solution (right) for linear polynomial approximations.

(S.-C. Soon, B.C. and H. Stolarski, 2008.)

The EDG methods. (B.C., J.Guzmán, S.-C.Soon and H.Stolarski, SINUM, 2009 .)

Motivation.

- Is it possible to modify the HDG methods as to render them as efficiently implementable as the CG methods?
- If so, can we keep the superconvergence property of the original HDG method?

Given and HDG method, we define an associated EDG method as follows. The approximate solution is $(\mathbf{q}_h, u_h) = (\mathbf{Q}_{\widehat{u}_h}, \mathsf{U}_{\widehat{u}_h}) + (\mathbf{Q}_f, \mathsf{U}_f)$, where the numerical trace $\widehat{u}_h$ is the element of a subspace $\widetilde{M}_h$ of M_h satisfying the equations

$$\begin{aligned} a_h(\widehat{u}_h, \mu) &= \ell_h(\mu) && \forall\, \mu \in \widetilde{M}_h : \mu|_{\partial\Omega} = 0, \\ \langle \mu, \widehat{u}_h \rangle_{\partial\Omega} &= \langle \mu, u_D \rangle_{\partial\Omega} && \forall\, \mu \in \widetilde{M}_h. \end{aligned}$$