Ten Lectures on the Convergence Analysis of

Discontinuous Galerkin Finite Element Methods

for Nonlinear PDEs

Lectures 1 and 2

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Introduction

Continuum mechanics is a fertile source of partial differential equations. This lecture course focuses on the numerical solution of elliptic, parabolic and hyperbolic PDEs by DGFEMs.

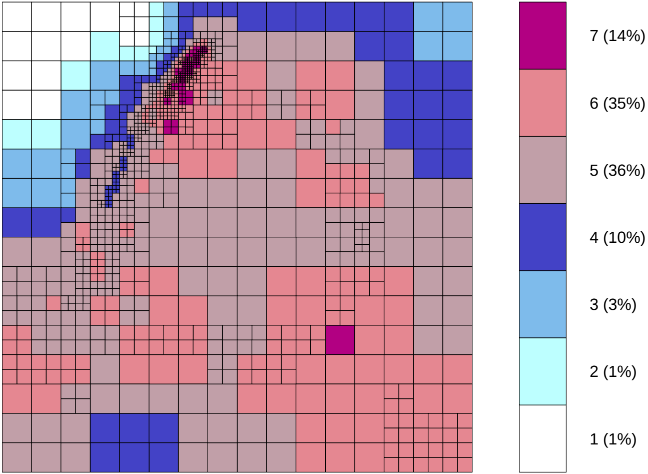
- Why DGFEM?

Physical reasons: transmission problems, fracture, shocks

Computational reasons:

- transport-dominated problems: built-in stabilisation
- local-conservation properties
- mesh adaptation on irregular grids
- *hp*-adaptivity with locally varying polynomial degrees

hp-DGFEM mesh



An essential tool: integration-by-parts

$$\int_a^b w'(x)v(x)\,\mathrm{d}x = w(x)v(x)\big|_{x=a}^{x=b} - \int_a^b w(x)v'(x)\,\mathrm{d}x, \quad w, v \in \mathrm{H}^1(a,b);$$

$$\int_\Omega (\nabla\cdot\mathbf{w})v\,\mathrm{d}x = \int_{\partial\Omega}(\mathbf{w}\cdot\mathbf{n})\,v\,\mathrm{d}s - \int_\Omega \mathbf{w}\cdot\nabla v\,\mathrm{d}x, \quad \mathbf{w}\in\mathrm{H}(\mathrm{div},\Omega),\ v\in\mathrm{H}^1(\Omega);$$

$$\int_\Omega (\nabla\cdot(A\nabla u))\,v\,\mathrm{d}x = \int_{\partial\Omega}(A\nabla u\cdot\mathbf{n})\,v\,\mathrm{d}s - \int_\Omega A\nabla u\cdot\nabla v\,\mathrm{d}x,$$

$$A\nabla u\in\mathrm{H}(\mathrm{div},\Omega),\ v\in\mathrm{H}^1(\Omega).$$

Historical roots (in the elliptic/parabolic community)

Enforcing a Dirichlet boundary condition via penalty:

- 1968: Lions considered the problem of solving $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$, with $f \in \mathrm{L}^2(\Omega)$ and $g \in \mathrm{H}^{-1/2}(\partial\Omega)$.

  $-\Delta w = f$ & regularized boundary condition: $w + \frac{1}{\mu}\frac{\partial w}{\partial n} = g$, with $\mu \gg 1$ a penalty parameter.

  Find $w \in \mathrm{H}^1(\Omega)$ such that

  $$\int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \mu(w - g)\, v \,\mathrm{d}s = \int_\Omega f\, v \,\mathrm{d}x \qquad \forall v \in \mathrm{H}^1(\Omega).$$

- 1970 Aubin (FD): $\mu \asymp h^{-1+\varepsilon}$ implies convergence.
  1973 Babuška (FE): $\mu \asymp h^{-(2p+1)/3}$, H^1-error$= \mathcal{O}(h^{(2p+1)/3})$.

Lack of optimality $\Leftarrow$ Lack of consistency

Note that the exact solution u does not satisfy the weak formulation. In fact,

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \,\mathrm{d}s + \int_{\partial\Omega} \mu(u-g) v \,\mathrm{d}s = \int_\Omega f\, v \,\mathrm{d}x \quad \forall v \in \mathrm{H}^2(\Omega).$$

- 1971 Nitsche

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \,\mathrm{d}s - \int_{\partial\Omega} u \frac{\partial v}{\partial n} \,\mathrm{d}s + \int_{\partial\Omega} \mu(u-g) v \,\mathrm{d}s$$
$$= \int_\Omega f\, v \,\mathrm{d}x - \int_{\partial\Omega} g \frac{\partial v}{\partial n} \,\mathrm{d}s \quad \forall v \in \mathrm{H}^2(\Omega).$$

$\mu = \alpha h^{-1}$ and $\alpha \geq 1$ sufficiently large implies:

$$\mathrm{L}^2\text{-error} = \mathcal{O}(h^{p+1}),\ \mathrm{H}^1\text{-error} = \mathcal{O}(h^p).$$

Interior penalty methods: elliptic & parabolic PDEs

- 1973 Babuška & Zlámal: weak imposition of C^1 continuity for 4th-order PDEs [à la Lions–Aubin–Babuška].
- 1978 Wheeler: interior penalty FEM [á la Nitsche] motivated by earier work of Douglas & Dupont (1976) and Baker (1977).
- 1979 Arnold's Ph.D. thesis ($\longrightarrow$ SINUM (1982)).
- 1980–late 1990's very little work.

DGFEM methods: hyperbolic PDEs

- Reed & Hill (Los Alamos Technical Report, 1973)
- Lesaint & Raviart (1974, 1978); Lesaint (1975, MAFELAP)
- Johnson, Nävert & Pitkäranta (CMAME, 1984)
- Johnson & Pitkäranta (Math. Comp., 1986)
- Cockburn & Shu (1989 $\longrightarrow$)
- Biswas, Devine & Flaherty (1994)   *hp* DGFEM
- Baumann (Ph.D. Thesis, 1997); Baumann & Oden (1999)
- Houston & Süli (1999 $\longrightarrow$)

Literature

B. Cockburn, G.E. Karniadakis, C.-W. Shu,
The development of discontinuous Galerkin methods.
In: B. Cockburn, G.E. Karniadakis, and C.-W. Shu (Eds.), Discontinuous Galerkin Finite Element Methods, Lecture Notes in Computational Science and Engineering 11, Springer-Verlag 2000.

D.N. Arnold, F. Brezzi, B. Cockburn, L.D. Marini,
Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Num. Anal. **39** (2002), 1749–1779.

Books

- J. Hesthaven & T. Warburton, *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*. Springer; Series: Texts in Applied Mathematics, Vol. 54. 2008; XIV + 501 p., Hardcover. ISBN: 978-0-387-72065-4.
- B. M. Rivière, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*. SIAM; Series: Frontiers in Applied Mathematics). 212 p., Paperback. ISBN-10: 089871656X, ISBN-13: 978-0898716566.
- H. Fahs, *High-Order Discontinuous Galerkin Methods for the Maxwell Equations*. Editions universitaires europeennes, 15 April 2010. 208 p., Paperback. ISBN-10: 6131500207, ISBN-13: 978-6131500206.
- A. Di Pietro & A. Ern *Mathematical Aspects of Discontinuous Galerkin Methods*. Springer; Series: Computational Science & Engineering, Vol. 69, 2012, 384 p., Paperback. ISBN 978-3-642-22979-4.

Overview

Lecture 1 Introduction

Lecture 2 *hp*-DGFEM in 1-D for $-u'' = f$;

Lecture 3 Broken Sobolev spaces: broken Poincaré–Sobolev inequality

Lecture 4 *hp*-DGFEM for semilinear elliptic equations;

Lecture 5 *hp*-DGFEM for quasilinear elliptic PDEs (scalar case);

Lecture 6 *hp*-DGFEM for quasilinear elliptic PDEs (systems);

Lecture 7 DGFEM for nonlinear elliptic and hyperbolic systems (I);

Lecture 8 DGFEM for nonlinear elliptic and hyperbolic systems (II);

Lecture 9 DGFEM for 4th-order nonlinear PDEs: Cahn-Hilliard eq. (I);

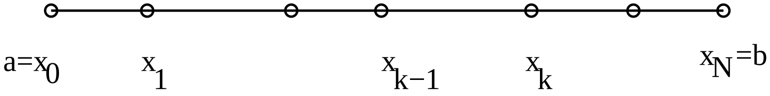
Lecture10 DGFEM for 4th-order nonlinear PDEs: Cahn-Hilliard eq. (II);

The lectures are based on the following journal papers

- Lectures 3 & 4: A Lasis and E Süli. hp-Version discontinuous Galerkin finite element method for semilinear parabolic problems. SIAM Journal on Numerical Analysis, 45 (4), 1544–1569 (2007).
- Lecture 5: P Houston, J A Robson, and E Süli. Discontinuous Galerkin finite element approximation of quasilinear elliptic boundary value problems I. IMA Journal of Numerical Analysis, 25(4), 726–749 (2005).
- Lecture 6: S Congreve, P Houston, E Süli, and T Wihler. Discontinuous Galerkin finite element approximation of quasilinear elliptic boundary value problems II: Strongly monotone quasi-Newtonian flows. IMA Journal of Numerical Analysis. Submitted for publication, 2012.
- Lectures 7 & 8: C Ortner and E Süli. Discontinuous Galerkin finite element approximation of nonlinear second-order elliptic and hyperbolic systems. SIAM J. Numer. Anal., 45(4), 1370–1397 (2007).
- Lectures 9 & 10: D Kay, V Styles and E Süli. Discontinuous Galerkin finite element approximation of the Cahn-Hilliard equation with convection. SIAM J. Numer. Anal., 47(4), 2660–2685 (2009).

One-dimensional model problem

$$-u'' = f \in \mathrm{L}^2(a,b), \quad x \in \Omega = (a,b),$$
$$u(a) = A, \qquad u(b) = B.$$



Broken Sobolev space

Partition

$$\mathcal{T}_h = \{\kappa_k = (x_{k-1}, x_k),\ k = 1, \ldots, N\}$$

Broken Sobolev space

$$\mathrm{H}^s(\Omega, \mathcal{T}_h) = \{v \in \mathrm{L}^2(a,b) \,:\, v|_\kappa \in \mathrm{H}^s(\kappa) \quad \forall \kappa \in \mathcal{T}_h\}$$

DG finite element space

$$V_{DG} = V^p(\Omega, \mathcal{T}_h) := \{v \in \mathrm{L}^2(a,b) \,:\, v|_\kappa \in \mathcal{P}^p(\kappa) \quad \forall \kappa \in \mathcal{T}_h\}$$

A technical remark: Sobolev embedding

Recall that $\mathrm{H}^1(a,b) \hookrightarrow \mathrm{C}^{1/2}[a,b]$. This is easily seen by noting:

$$|u(x) - u(y)|^2 = \left| \int_x^y u'(t)\,\mathrm{d}t \right|^2 \le |x-y| \left| \int_x^y |u'(t)|^2\,\mathrm{d}t \right|.$$

Hence, for any $x, y \in [a,b]$:

$$|u(x) - u(y)| \le |x-y|^{1/2} \left[\int_a^b |u'(t)|^2\,\mathrm{d}t \right]^{\frac{1}{2}}.$$

Replacing u by u' above, we deduce that $H^2(a,b) \hookrightarrow \mathrm{C}^{1,\,1/2}[a,b]$.

Broken weak formulation

$$-u''(x) = f(x) \qquad \forall x \in \Omega = (a, b).$$

$$-u''(x)v(x) = f(x)v(x) \qquad \forall v \in \mathrm{H}^2(\Omega, \mathcal{T}_h), \quad x \in \Omega.$$

$$-\int_{x_{k-1}}^{x_k} u''(x)v(x)\,\mathrm{d}x = \int_{x_{k-1}}^{x_k} f(x)v(x)\,\mathrm{d}x \qquad \forall v \in \mathrm{H}^2(\Omega, \mathcal{T}_h).$$

$$\int_{x_{k-1}}^{x_k} u'(x)v'(x)\,\mathrm{d}x - u'(x)v(x)\Big|_{x_{k-1}+}^{x_k-} = \int_{x_{k-1}}^{x_k} f(x)v(x)\,\mathrm{d}x \quad \forall v \in \mathrm{H}^2(\Omega, \mathcal{T}_h).$$

Broken weak formulation

$$\sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} u'(x)v'(x)\,\mathrm{d}x - \sum_{k=1}^{N} u'(x)v(x)\Big|_{x_{k-1}+}^{x_k-}$$
$$= \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} f(x)v(x)\,\mathrm{d}x \qquad \forall v \in \mathrm{H}^2(\Omega, \mathcal{T}_h).$$

$$\begin{aligned}
\sum_{k=1}^{N} u'(x)v(x)\Big|_{x_{k-1}+}^{x_k-} &= u'(x_1)v(x_1-) - u'(x_0)v(x_0+) \\
&+ u'(x_2)v(x_2-) - u'(x_1)v(x_1+) \\
&+ u'(x_3)v(x_3-) - u'(x_2)v(x_2+) \\
&+ \dots \\
&+ u'(x_{N-1})v(x_{N-1}-) - u'(x_{N-2})v(x_{N-2}+) \\
&+ u'(x_N)v(x_N-) - u'(x_{N-1})v(x_{N-1}+),
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{N} u'(x)v(x)\Big|_{x_{k-1}+}^{x_k-} &= (-1)u'(x_0)v(x_0+) + u'(x_N)v(x_N-) \\
&+ \sum_{k=1}^{N-1} u'(x_k)(v(x_k-) - v(x_k+)).
\end{aligned}$$

Jump:

$$
\begin{aligned}
[\![v(x_k)]\!] &= v(x_k-) - v(x_k+) = v(x_k-)(+1) + v(x_k+)(-1) \\
[\![v(x_0)]\!] &= \qquad\qquad - v(x_0+) \\
[\![v(x_N)]\!] &= v(x_N-).
\end{aligned}
$$

Average:
$$
\begin{aligned}
\{\!\{v(x_k)\}\!\} &= \tfrac{1}{2}(v(x_k-) + v(x_k+)) \\
\{\!\{v(x_0)\}\!\} &= \qquad\qquad\qquad v(x_0+) \\
\{\!\{v(x_N)\}\!\} &= v(x_N-).
\end{aligned}
$$

$$
\sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} u'(x)v'(x)\,\mathrm{d}x - \sum_{k=0}^{N} \{\!\{u'(x_k)\}\!\}[\![v(x_k)]\!] \quad \longleftarrow
$$

$$
= \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} f(x)v(x)\,\mathrm{d}x \qquad \forall v \in \mathrm{H}^2(\Omega, \mathcal{T}_h).
$$

$$
\begin{aligned}
&\sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} u'(x)v'(x)\,\mathrm{d}x - \sum_{k=0}^{N} \{\!\{u'(x_k)\}\!\} [\![v(x_k)]\!] \pm \sum_{k=0}^{N} [\![u(x_k)]\!] \{\!\{v'(x_k)\}\!\} \\
&\quad = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} f(x)v(x)\,\mathrm{d}x \pm \left([\![u(x_N)]\!] \{\!\{v'(x_N)\}\!\} + [\![u(x_0)]\!] \{\!\{v'(x_0)\}\!\} \right)
\end{aligned}
$$

for all $v \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$. However,

$$
[\![u(x_N)]\!] = B, \qquad [\![u(x_0)]\!] = -A.
$$

$$\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} u'(x)v'(x)\,\mathrm{d}x - \sum_{k=0}^{N}\{\!\{u'(x_k)\}\!\}[\![v(x_k)]\!] \pm \sum_{k=0}^{N}[\![u(x_k)]\!]\{\!\{v'(x_k)\}\!\}$$

$$+\sum_{k=0}^{N}\sigma_k[\![u(x_k)]\!]\,[\![v(x_k)]\!] \quad \longleftarrow \text{ penalty}$$

$$= \sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} f(x)v(x)\,\mathrm{d}x \pm \left(B\{\!\{v'(x_N)\}\!\} - A\{\!\{v'(x_0)\}\!\}\right)$$

$$+\left(\sigma_N B v(x_N-) + \sigma_0 A v(x_0+)\right) \quad \longleftarrow \text{ penalty}$$

for all $v \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$. Penalty parameter: $\sigma_k > 0$, $k = 0, \ldots, N$.

Definition

Broken weak formulation: Find $\hat{u} \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$ such that

$$B_\pm(\hat{u}, v) = \ell_\pm(v) \qquad \forall v \in \mathrm{H}^2(\Omega, \mathcal{T}_h).$$

Discontinuous Galerkin finite element method

$$V_{DG} := \{v \in \mathrm{L}^2(a,b) \, : \, v|_\kappa \in \mathcal{P}^p(\kappa) \quad \forall \kappa \in \mathcal{T}_h\}.$$

Definition

Find $U \in V_{DG}$ such that

$$B_\pm(U,V) = \ell_\pm(V) \qquad \forall V \in V_{DG}.$$

$+$ nonsymmetric interior penalty method

$-$ symmetric interior penalty method

Galerkin orthogonality (=consistency)

Note that $\hat{u} := u \in \mathrm{H}^2(a, b)(\subset \mathrm{C}^{1,1/2}[a, b])$ satisfies the broken weak formulation of the problem. Thus,

$$B_\pm(u, V) = \ell_\pm(V) = B_\pm(U, V) \qquad \forall V \in V_{DG}.$$

Therefore, we have Galerkin orthogonality:

$$B_\pm(u - U, V) = 0 \qquad \forall V \in V_{DG}.$$

The nonsymmetric interior penalty method

Find $U \in V_{DG}$ such that

$$B_+(U, V) = \ell_+(v) \qquad \forall V \in V_{DG}.$$

$$
\begin{aligned}
B_{+}(u,v) &= \sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} u'(x)v'(x)\,\mathrm{d}x - \sum_{k=0}^{N}\{\!\{u'(x_k)\}\!\}[\![v(x_k)]\!] \\
&\quad + \sum_{k=0}^{N}[\![u(x_k)]\!]\{\!\{v'(x_k)\}\!\} + \sum_{k=0}^{N}\sigma_k[\![u(x_k)]\!]\,[\![v(x_k)]\!],
\end{aligned}
$$

$$
\begin{aligned}
\ell_{+}(v) &= \sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} f(x)v(x)\,\mathrm{d}x + \big(B\{\!\{v'(x_N)\}\!\} - A\{\!\{v'(x_0)\}\!\}\big) \\
&\qquad + (\sigma_N B v(x_N-) + \sigma_0 A v(x_0+)).
\end{aligned}
$$

Existence and uniqueness of solution

Observe that, for all $V \neq 0$ and any $\sigma_k > 0$,

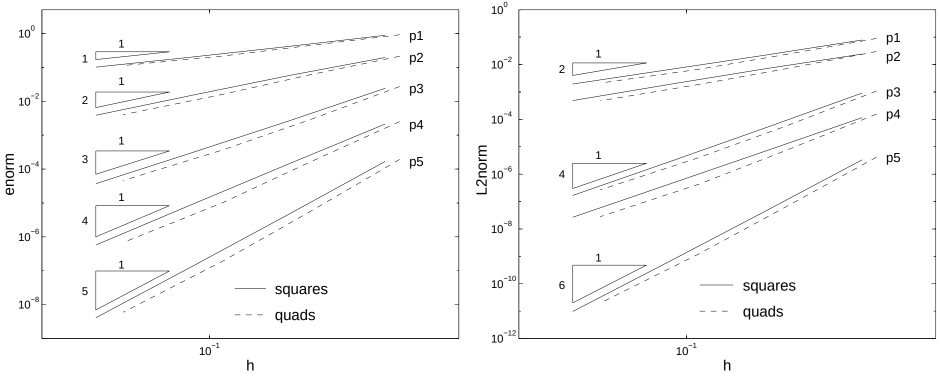
$$B_+(V, V) = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} |V'(x)|^2 \,\mathrm{d}x + \sum_{k=0}^{N} \sigma_k [\![V(x_k)]\!]^2 =: \|V\|_{DG}^2 > 0.$$

Hence the homogeneous problem (with $f \equiv 0$ and $A = B = 0$) has $U \equiv 0$ as its unique solution.

Since V_{DG} is a finite-dimensional linear space, uniqueness of solution implies existence of solution.

Error curves in 2D

$-\Delta u = f$ in Ω, with $u|_{\partial\Omega} = 0$:



Convergence analysis in 1D

Let $\pi_h^p u \in V_{DG}$ denote a (suitable) continuous piecewise polynomial interpolant of u, and write

$$u - U = (u - \pi_h^p u) - (U - \pi_h^p u) =: \eta - \xi.$$

Now

$$\|\xi\|_{DG}^2 = B_+(\xi, \xi) = B_+(\eta, \xi) - B_+(u - U, \xi) = B_+(\eta, \xi).$$

Hence,

$$\begin{aligned}
\|\xi\|_{DG}^2 \quad = \quad & \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \eta'(x)\xi'(x)\,\mathrm{d}x - \sum_{k=0}^{N} \{\!\{\eta'(x_k)\}\!\} [\![\xi(x_k)]\!] \\
& + \sum_{k=0}^{N} [\![\eta(x_k)]\!] \{\!\{\xi'(x_k)\}\!\} + \sum_{k=0}^{N} \sigma_k [\![\eta(x_k)]\!] \, [\![\xi(x_k)]\!].
\end{aligned}$$

Note that $\eta(x_k) = 0$, $k = 0, \ldots, N$; therefore

$$\|\xi\|_{DG}^2 \leq \|\xi\|_{DG} \left(\sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} |\eta'(x)|^2 \, \mathrm{d}x + \sum_{k=0}^{N} \frac{1}{\sigma_k} |\{\!\{\eta'(x_k)\}\!\}|^2 \right)^{1/2} .$$

Therefore,

$$\|\xi\|_{DG} \leq \left(\sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} |\eta'(x)|^2 \, \mathrm{d}x + \sum_{k=0}^{N} \frac{1}{\sigma_k} |\{\!\{\eta'(x_k)\}\!\}|^2 \right)^{1/2} .$$

$$
\begin{aligned}
\|u-U\|_{DG} &\leq \|\eta\|_{DG} + \|\xi\|_{DG} \\
&\leq 2\left(\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} |\eta'(x)|^2 \,\mathrm{d}x\right)^{1/2} + \left(\sum_{k=0}^{N}\frac{1}{\sigma_k}|\{\!\{\eta'(x_k)\}\!\}|^2\right)^{1/2} \\
&=: \mathrm{T}_1 + \mathrm{T}_2.
\end{aligned}
$$

Approximation theory:[1] for $0 < s_k \leq p_k$, $1 \leq p_k \leq p$,

$$
\mathrm{T}_1 \leq C\left(\sum_{k=1}^{N}\frac{h_k^{2s_k}}{p_k^{2s_k}}|u|^2_{\mathrm{H}^{s_k+1}(x_{k-1},x_k)}\right)^{1/2},
$$

$$
\mathrm{T}_2 \leq C\left(\sum_{k=1}^{N}\frac{h_k^{2s_k}}{p_k^{2s_k}}|u|^2_{\mathrm{H}^{s_k+1}(x_{k-1},x_k)}\right)^{1/2}, \qquad \sigma_k = \alpha\left\{\!\!\left\{\frac{p_k^{3/2}}{h_k}\right\}\!\!\right\}, \quad \alpha > 0.
$$

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[1] See later for proofs!

Final convergence estimate

For $0 < s_k \leq p_k$, $1 \leq p_k \leq p$:

$$\|u - U\|_{DG} \leq C \left(\sum_{k=1}^{N} \frac{h_k^{2s_k}}{p_k^{2s_k}} |u|^2_{\mathrm{H}^{s_k+1}(x_{k-1},x_k)} \right)^{1/2}$$

with

$$\sigma_k = \alpha \left\{\!\!\left\{ \frac{p_k^{3/2}}{h_k} \right\}\!\!\right\} \quad \text{and any} \quad \alpha > 0.$$

Open question: If $p_k^{3/2} \longrightarrow p_k$, is the error bound still optimal?

Comments

- In the multi-dimensional case, use of *continuous* projection π_h^p will not work in general because of hanging nodes.
- Multi-dimensional analogue of the argument presented, with

$$\sigma_k = \alpha \left\{\!\!\left\{ \frac{p_k^{3/2}}{h_k} \right\}\!\!\right\} \quad \text{with arbitrary} \quad \alpha > 0,$$

  leads to loss of optimality with respect to p by $p^{1/2}$.
- Georgoulis & Süli (IMA Journal of Numerical Analysis, 2005): Use of augmented Sobolev spaces restores full *hp*-optimality.

Approximation theory: Legendre polynomials

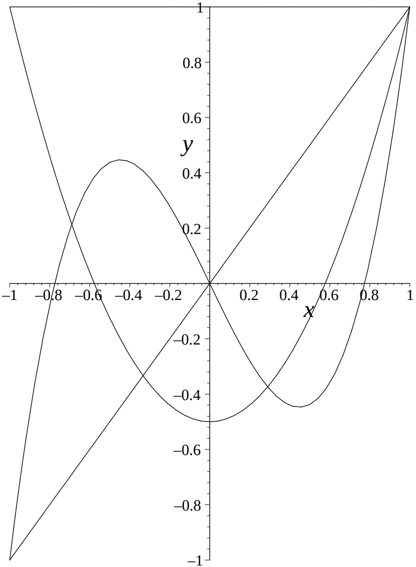
System of orthogonal polynomials on the interval $[-1,1]$ with respect to the weight-function $w(x) \equiv 1$:

$$L_n \quad \text{is a polynomial of degree } n,\ n \geq 0,$$

$$\int_{-1}^{1} L_n(x) L_m(x)\, \mathrm{d}x = 0 \quad \text{when } m \neq n.$$

$L_0(x) \equiv 1$, $L_1(x) = x$, $L_2(x) = \frac{3}{2}x^2 - \frac{1}{2}x$, $L_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$, etc.

Legendre polynomials



Properties of Legendre polynomials

$$L_n(1) = 1, \qquad L_n(-1) = (-1)^n,$$

$$\int_{-1}^{1} L_n^2(x)\,\mathrm{d}x = \frac{2}{2n+1}, \qquad n \geq 0,$$

$$\int_{-1}^{1} (1-x^2)^s L_n^{(s)}(x) L_m^{(s)}(x)\,\mathrm{d}x = \begin{cases} \frac{2}{2n+1} \cdot \frac{(n+s)!}{(n-s)!} & m = n \geq s \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$((1-x^2)L_n'(x))' + n(n+1)L_n(x) = 0, \qquad x \in [-1,1], \quad n \geq 1.$$

W. Gautschi: Orthogonal Polynomials (Oxford Univ. Press, 2004).

Let $u \in \mathrm{L}^2(-1,1)$; then,

$$u(x) = \sum_{n=0}^{\infty} u_n L_n(x)$$

where

$$u_n = \frac{2n+1}{2} \int_{-1}^{1} u(x) L_n(x) \,\mathrm{d}x.$$

Legendre projector:

$$\Pi_p u(x) = \sum_{n=0}^{p} u_n L_n(x) \quad \in \mathcal{P}_p.$$

Legendre projector

Note that

$$\|u - \Pi_p u\|^2_{\mathrm{L}^2(-1,1)} = \sum_{n=p+1}^{\infty} \frac{2}{2n+1} |u_n|^2,$$

with

$$u_n = \frac{2n+1}{2} \int_{-1}^{1} u(x) L_n(x) \,\mathrm{d}x,$$

$$\|u - \Pi_p u\|_{\mathrm{L}^2(-1,1)} \leq \left[\frac{\Gamma(p+2-s)}{\Gamma(p+2+s)}\right]^{1/2} |u|_{\mathrm{H}^s(-1,1)}, \qquad 0 \leq s \leq p+1.$$

Proof: Theorem 3.11 (p.72), Schwab (Oxford Univ. Press, 1998).

Modified Legendre projector

Let $u \in \mathrm{H}^1(-1,1)$. Consider the projector

$$P_p u(x) = u(-1) + \int_{-1}^{x} (\Pi_{p-1} u')(s)\,\mathrm{d}s, \qquad p \geq 1.$$

Note that $P_p u(-1) = u(-1)$ and $P_p(1) = u(1)$, since

$$\begin{aligned} u(1) - P_p(1) &= u(1) - u(-1) - \int_{-1}^{1} (\Pi_{p-1} u')(s)\,\mathrm{d}s \\ &= \int_{-1}^{1} (u' - \Pi_{p-1} u')(s)\,\mathrm{d}s = 0. \end{aligned}$$

Approximation error for modified Legendre projector: H^1

$$
\begin{aligned}
\|u' - (P_p u)'\|_{\mathrm{L}^2(-1,1)} &= \|u' - \Pi_{p-1} u'\|_{\mathrm{L}^2(0,1)} \\
&\le \left[\frac{\Gamma(p+1-s)}{\Gamma(p+1+s)}\right]^{1/2} |u'|_{\mathrm{H}^s(-1,1)} \\
&= \left[\frac{\Gamma(p+1-s)}{\Gamma(p+1+s)}\right]^{1/2} |u|_{\mathrm{H}^{s+1}(-1,1)}, \quad 0 \le s \le p.
\end{aligned}
$$

Approximation error for modified Legendre projector: L^2

$$\|u - P_p u\|^2_{\mathrm{L}^2(-1,1)} \leq \int_{-1}^{1} \frac{1}{1-x^2} |u(x) - P_p u(x)|^2 \, \mathrm{d}x.$$

Now,

$$\begin{aligned}
u(x) - P_p u(x) &= \int_{-1}^{x} [u'(s) - (\Pi_{p-1} u')(s)] \, \mathrm{d}s \\
&= \int_{-1}^{x} \left(\sum_{n=p}^{\infty} b_n L_n(s) \right) \, \mathrm{d}s \\
&= \sum_{n=p}^{\infty} b_n \int_{-1}^{x} L_n(s) \, \mathrm{d}s \equiv \sum_{n=p}^{\infty} b_n \psi_n(x),
\end{aligned}$$

where

$$b_n = \frac{2n+1}{2} \int_{-1}^{1} u'(x) L_n(x) \, \mathrm{d}x.$$

Approximation error for modified Legendre projector: L^2

From Legendre differential equation, by integration from -1 to x,

$$-\psi_n(x) = \frac{1}{n(n+1)}(1-x^2)L_n'(x).$$

Therefore,

$$\begin{aligned}\int_{-1}^{1} \frac{1}{1-x^2}\psi_n(x)\psi_m(x) &= \frac{1}{n(n+1)m(m+1)}\int_{-1}^{1}(1-x^2)L_n'(x)L_m'(x)\,\mathrm{d}x \\ &= \frac{2\delta_{nm}}{n(n+1)(2n+1)}.\end{aligned}$$

Approximation error for modified Legendre projector: L^2

$$
\begin{aligned}
\|u - P_p u\|^2_{\mathrm{L}^2(-1,1)} &\leq \int_{-1}^{1} \frac{1}{1-x^2} |u(x) - P_p u(x)|^2 \,\mathrm{d}x \\
&= \int_{-1}^{1} \frac{1}{1-x^2} \left(\sum_{n=p}^{\infty} b_n \psi_n(x) \right)^2 \,\mathrm{d}x \\
&= \sum_{n=p}^{\infty} \frac{2}{n(n+1)(2n+1)} |b_n|^2 \\
&\leq \frac{1}{p(p+1)} \sum_{n=p}^{\infty} \frac{2}{2n+1} |b_n|^2 \\
&= \frac{1}{p(p+1)} \|u' - \Pi_{p-1} u'\|^2_{\mathrm{L}^2(-1,1)} \\
&\leq \frac{1}{p(p+1)} \frac{\Gamma(p+1-s)}{\Gamma(p+1+s)} |u|^2_{\mathrm{H}^{s+1}(-1,1)}.
\end{aligned}
$$

Summary

$$\begin{aligned}
\|u - P_p u\|_{\mathrm{L}^2(-1,1)} &\leq \left[\frac{1}{p(p+1)}\frac{\Gamma(p+1-s)}{\Gamma(p+1+s)}\right]^{1/2} |u|_{\mathrm{H}^{s+1}(-1,1)} \\
&\leq C\, p^{-s-1}|u|_{\mathrm{H}^{s+1}(-1,1)},
\end{aligned}$$

$$\begin{aligned}
\|u' - (P_p u)'\|_{\mathrm{L}^2(-1,1)} &\leq \left[\frac{\Gamma(p+1-s)}{\Gamma(p+1+s)}\right]^{1/2} |u|_{\mathrm{H}^{s+1}(-1,1)} \\
&\leq C\, p^{-s}|u|_{\mathrm{H}^{s+1}(-1,1)}
\end{aligned}$$

for $0 \leq s \leq p$.

Maximum norm estimate

$$
\begin{aligned}
\|u' - (P_p u)'\|_{\mathrm{C}[-1,1]} &= \|u' - \Pi_{p-1} u'\|_{\mathrm{C}[-1,1]} \\
&\leq C \, \|u' - \Pi_{p-1} u'\|_{\mathrm{L}^2(-1,1)}^{1/2} \|u' - \Pi_{p-1} u'\|_{\mathrm{H}^1(-1,1)}^{1/2}.
\end{aligned}
$$

Canuto & Quarteroni (1982):

$$
\|u' - \Pi_{p-1} u'\|_{\mathrm{H}^1(-1,1)} \leq C \, p^{\frac{3}{2} - s} |u|_{\mathrm{H}^{s+1}(-1,1)}, \qquad 0 < s \leq p.
$$

Therefore,

$$
\|u' - \Pi_{p-1} u'\|_{\mathrm{C}[-1,1]} \leq C \, p^{\frac{3}{4} - s} |u|_{\mathrm{H}^{s+1}(-1,1)}, \qquad 0 < s \leq p,
$$

and

$$
\|u' - (P_p u)'\|_{\mathrm{C}[-1,1]} \leq C \, p^{\frac{3}{4} - s} |u|_{\mathrm{H}^{s+1}(-1,1)}, \qquad 0 < s \leq p.
$$

Definition of π_h^p

$$(\pi_h^p u)(x) = P_p \hat{u}(\hat{x}), \qquad x \in (x_{k-1}, x_k),$$

where

$$\hat{u}(\hat{x}) = u(x) \qquad \text{with} \qquad x = x_{k-1} + \frac{h_k}{2}(\hat{x} + 1), \quad \hat{x} \in (-1, 1).$$

Approximation results

Using a scaling argument on each element $[x_{k-1}, x_k]$, we obtain:

$$\|u' - (\pi_h^p u)'\|^2_{\mathrm{L}^2(x_{k-1},x_k)} \leq C\,\left(\frac{h_k}{p_k}\right)^{2s_k} |u|^2_{\mathrm{H}^{s_k}(x_{k-1},x_k)},$$

$$\begin{aligned}\frac{1}{\sigma_k}|\{\!\{u'(x_k) - (\pi_h^p u)'(x_k)\}\!\}|^2 \quad &\leq \quad C\,\left(\frac{h_k}{p_k}\right)^{2s_k} |u|^2_{\mathrm{H}^{s_k}(x_{k-1},x_k)} \\ &\qquad + C\,\left(\frac{h_{k+1}}{p_{k+1}}\right)^{2s_{k+1}} |u|^2_{\mathrm{H}^{s_{k+1}}(x_k,x_{k+1})}.\end{aligned}$$

These yield the bounds on the terms T_1 and T_2, stated earlier.

Inverse inequality

Let $h_k = x_{k+1} - x_k$ and let $\mathcal{P}_{p_k}$ denote the set of all polynomials of degree $\leq p_k$ on the interval (x_k, x_{k+1}). Then, there exists a positive constant C_{inv}, independent of h_k and p_k, such that

$$\|v'\|_{\mathrm{L}^2(x_k,x_{k+1})} \leq C_{\text{inv}} \frac{p_k^2}{h_k} \|v\|_{\mathrm{L}^2(x_k,x_{k+1})} \qquad \forall v \in \mathcal{P}_{p_k}.$$

Proof: Ch. Schwab (Oxford Univ. Press, 1998).

Below we show that the factor p_k^2 on the right-hand side is sharp. It suffices to do so for the reference interval $(-1, 1)$. For the sake of simplicity we shall write N instead of p_k.

A note on the inverse inequality

Suppose that $v \in \mathrm{L}^2(-1,1)$ and write:

$$v(x) = \sum_{k=0}^{\infty} \alpha_k L_k(x).$$

Then,

$$\begin{aligned} v'(x) &= \sum_{k=1}^{\infty} \alpha_k L_k'(x) = \sum_{k=0}^{\infty} \alpha_{k+1} L_{k+1}'(x) \\ &= \sum_{k=0}^{\infty} \alpha_{k+1} \left\{(2k+1)L_k(x) + [2(k-2)+1]L_{k-2}(x) \right. \\ &\qquad \left. + [2(k-4)+1]L_{k-4}(x) + \ldots .\right\} \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{\infty} \alpha_{2i+k+1} \right) (2k+1)\, L_k(x). \end{aligned}$$

Suppose that $N \geq 1$ and let

$$\alpha_k = \begin{cases} k & \text{for } 0 \leq k \leq N, \\ 0 & \text{for } k > N. \end{cases}$$

We shall denote the corresponding function v by v_N. Clearly, $v_N \in \mathcal{P}_N$. Then,

$$\|v_N\|^2_{\mathrm{L}^2(-1,1)} = \sum_{k=0}^{N} \alpha_k^2 \|L_k\|^2_{\mathrm{L}^2(-1,1)} = \sum_{k=1}^{N} k^2 \frac{2}{2k+1} \asymp \frac{1}{2} N^2.$$

On the other hand,

$$v_N'(x) = \sum_{k=0}^{N-1} \left(\sum_{\substack{i \geq 0 \\ 2i+k+1 \leq N}} (2i+k+1) \right) (2k+1) L_k(x).$$

Hence,

$$
\begin{aligned}
\|v_N'\|^2_{\mathrm{L}^2(-1,1)} &= \sum_{k=0}^{N-1} \left(\sum_{0 \le i \le \frac{1}{2}(N-k-1)} (2i+k+1) \right)^2 (2k+1)^2 \frac{2}{2k+1} \\
&= 2 \sum_{k=0}^{N-1} \left(\sum_{0 \le i \le \frac{1}{2}(N-k-1)} (2i+k+1) \right)^2 (2k+1) \\
&\asymp C \sum_{k=0}^{N-1} \left((N-k)^2\right)^2 (2k+1) \\
&\asymp CN^6.
\end{aligned}
$$

Hence,

$$\|v_N'\|^2_{\mathrm{L}^2(-1,1)} \asymp CN^4 \|v_N\|^2_{\mathrm{L}^2(-1,1)}.$$

Outlook

We shall extend this analysis to:

- Semilinear PDEs of the form
$$u_t - \Delta u = f(u);$$
- Quasilinear PDE of the form
$$-\nabla \cdot (\mu(x, |\nabla u|)\, \nabla u) = f, \qquad \Omega \subset \mathbb{R}^d;$$
  with monotone and globally Lipschitz-continuous nonlinearity for both the scalar case and for systems;
- More general second-order elliptic and hyperbolic systems; and
- Fourth-order nonlinear PDEs (e.g. the Cahn–Hilliard eqn.).

Ten Lectures on the Convergence Analysis of

Discontinuous Galerkin Finite Element Methods

for Nonlinear PDEs

Lectures 3 and 4

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Introduction

Let Ω be a bounded open polyhedral domain in $\mathbb{R}^d$, $d \geq 2$, with Lipschitz-continuous boundary.

Consider the semilinear parabolic PDE

$$u' - \Delta u = f(x, t, u) \quad \text{in} \quad \Omega \times (0, T], \tag{1.1}$$

where $u' := \partial u / \partial t$ and $T > 0$.

We will assume that f is a real-valued function defined on $\Omega \times (0, T] \times \mathbb{R}$, which satisfies the following assumption:

Hypothesis A

$f(\cdot, \cdot, v) : (x, t) \in \Omega \times (0, T] \mapsto f(x, t, v) \in \mathbb{R}$ is measurable for all $v \in \mathbb{R}$, with $f(x, t, 0) = 0$ for all $(x, t) \in \Omega \times (0, T]$; and

$f(x, t, \cdot) : v \in \mathbb{R} \mapsto f(x, t, v) \in \mathbb{R}$ is locally Lipschitz continuous for a.e. $(x, t) \in \Omega \times (0, T]$; i.e., there exist real numbers $G_f > 0$ and $\gamma \geq 0$ such that

$$|f(x, t, w) - f(x, t, v)| \leq G_f(1+|w|+|v|)^{\gamma}\, |w - v| \quad \begin{cases} \forall w,\ v \in \mathbb{R}, \\ \text{a.e. } (x, t) \\ \in \Omega \times (0, T]. \end{cases} \tag{1.2}$$

We shall suppose that:

$0 \leq \gamma < \infty$ when $d = 2$, and $0 \leq \gamma \leq 2/(d - 2)$ when $d \geq 3$.

Let $\partial\Omega$ be the union of all $(d-1)$-dimensional open faces of Ω. We decompose $\partial\Omega$ into Γ_{D} and Γ_{N}, s.t. $\bar{\Gamma}_{\mathrm{D}} \cup \bar{\Gamma}_{\mathrm{N}} = \overline{\partial\Omega}$, $|\Gamma_{\mathrm{D}}| > 0$; and let $\nu = (\nu_1, \ldots, \nu_d)^{\mathrm{T}}$ be the unit outward normal to $\partial\Omega$.

We impose Dirichlet and Neumann boundary conditions on Γ_{D} and Γ_{N}, respectively:

$$
\begin{aligned}
u &= g_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}} \times (0, T], \\
\nabla u \cdot \nu &= g_{\mathrm{N}} \quad \text{on} \quad \Gamma_{\mathrm{N}} \times (0, T],
\end{aligned}
\tag{1.3}
$$

with $g_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$ and $g_{\mathrm{N}} \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$.

Given a $u_0 \in \mathrm{L}^2(\Omega)$, we add to (1.1) and (1.3) the initial condition

$$
u = u_0 \quad \text{on} \quad \Omega \times \{0\}\,. \tag{1.4}
$$

We shall be concerned with the error analysis of the spatially semidiscrete *hp*-version interior penalty discontinuous Galerkin finite element method (*hp*-DGFEM) on shape-regular meshes.

Remark:
We do not assume that f satisfies a global Lipschitz condition.

Broken weak formulation

For a Banach space X equipped with a norm $\|\cdot\|$, the space $\mathrm{L}^q(0,T;X)$ consists of all strongly measurable functions $u:(0,T)\to X$ with the norm

$$\|u\|_{\mathrm{L}^q(0,T;X)} := \left(\int_0^T \|u(t)\|^q \,\mathrm{d}t\right)^{1/q} < \infty \quad \text{for} \quad 1 \le q < \infty,$$

and

$$\|u\|_{\mathrm{L}^\infty(0,T;X)} := \operatorname*{ess.\,sup}_{0\le t\le T} \|u(t)\| < \infty \quad \text{for} \quad q = \infty.$$

The Sobolev space $\mathrm{W}^{1,q}(0, T; X)$ consists of all functions $u \in \mathrm{L}^q(0, T; X)$ such that u' exists in the weak sense and belongs to $\mathrm{L}^q(0, T; X)$, with the associated norm, for $1 \le q < \infty$:

$$\|u\|_{\mathrm{W}^{1,q}(0,T;X)} := \left(\int_0^T \left\{ \|u(t)\|^q + \|u'(t)\|^q \right\} \mathrm{d}t \right)^{1/q} < \infty,$$

and

$$\|u\|_{\mathrm{W}^{1,\infty}(0,T;X)} := \operatorname*{ess.\,sup}_{0<t<T} (\|u(t)\| + \|u'(t)\|).$$

We shall write $\mathrm{H}^1(0, T; X) := \mathrm{W}^{1,2}(0, T; X)$.

Let $\mathcal{T}_h$ be a subdivision of Ω into disjoint open elements κ s.t.:

$$\bar{\Omega} = \cup_{\kappa \in \mathcal{T}_h} \bar{\kappa},$$

where $\mathcal{T}_h$ is regular or 1-irregular, i.e. each face of κ has at most one hanging node.

Let

$$h_\kappa := \mathrm{diam}(\bar{\kappa}) \qquad \text{and} \qquad h := \max_{\kappa \in \mathcal{T}_h} h_\kappa \in (0, 1].$$

Let the family $\{\mathcal{T}_h\}_{h \in (0,1]}$ be shape-regular, and let each $\kappa \in \mathcal{T}_h$ be an affine image $F_\kappa(\hat{\kappa})$ of a fixed master element $\hat{\kappa}$ for all $\kappa \in \mathcal{T}_h$, where $\hat{\kappa}$ is the open unit simplex or the open unit hypercube in $\mathbb{R}^d$.

Let $\mathcal{P}_p(\hat{\kappa})$ be the set of all polynomials of degree p or less on $\hat{\kappa}$. If $\hat{\kappa}$ is the open unit hypercube in $\mathbb{R}^d$, let $\mathcal{Q}_p(\hat{\kappa})$ be the set of all tensor-product polynomials on $\hat{\kappa}$ of degree p or less in each coordinate direction.

Finite element space

To each $\kappa \in \mathcal{T}_h$ we assign $p_\kappa \in \mathbb{N}$ (the local polynomial degree) and $s_\kappa \in \mathbb{N}$ (the local Sobolev index), collect p_κ, s_κ and F_κ into $\mathbf{p} := \{p_\kappa : \kappa \in \mathcal{T}_h\}$, $\mathbf{s} := \{s_\kappa : \kappa \in \mathcal{T}_h\}$ and $\mathbf{F} := \{F_\kappa : \kappa \in \mathcal{T}_h\}$, and consider the finite element space

$$S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F}) := \left\{ v \in \mathrm{L}^2(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{R}_{p_\kappa}(\hat{\kappa}),\ \kappa \in \mathcal{T}_h \right\},$$

where $\mathcal{R}$ is either $\mathcal{P}$ or $\mathcal{Q}$ if $\hat{\kappa}$ is the open unit hypercube in $\mathbb{R}^d$, and $\mathcal{R}$ is $\mathcal{P}$ if $\hat{\kappa}$ is the open unit simplex in $\mathbb{R}^d$.

We shall assume that the polynomial degree vector $\mathbf{p}$, with $p_\kappa \geq 1$ for each $\kappa \in \mathcal{T}_h$, has bounded local variation, i.e. there exists a constant $\rho \geq 1$, independent of h, such that, for any pair of elements κ and κ' in $\mathcal{T}_h$ which share a $(d-1)$-dimensional face,

$$\rho^{-1} \leq p_\kappa / p_{\kappa'} \leq \rho.$$

For $q \in [1,\infty)$, we consider the broken Sobolev space of composite order $\mathbf{s}$ on the subdivision $\mathcal{T}_h$:

$$\mathrm{W}^{\mathbf{s},q}(\Omega,\mathcal{T}_h) := \left\{u \in \mathrm{L}^q(\Omega) : u|_\kappa \in \mathrm{W}^{s_\kappa,q}(\kappa) \text{ for all } \kappa \in \mathcal{T}_h\right\},$$

equipped with the broken Sobolev norm/seminorm, respectively,

$$\|u\|_{\mathbf{s},q,\mathcal{T}_h} := \left(\sum_{\kappa\in\mathcal{T}_h} \|u\|^q_{s_\kappa,q,\kappa}\right)^{1/q}, \quad |u|_{\mathbf{s},q,\mathcal{T}_h} := \left(\sum_{\kappa\in\mathcal{T}_h} |u|^q_{s_\kappa,q,\kappa}\right)^{1/q}.$$

Remark:
When $s_\kappa = s$ for all $\kappa \in \mathcal{T}_h$, we write $\mathrm{W}^{s,q}(\Omega,\mathcal{T}_h)$, $\|u\|_{s,q,\mathcal{T}_h}$, $|u|_{s,q,\mathcal{T}_h}$, and for $q = 2$ we let $\mathrm{H}^{\mathbf{s}} := \mathrm{W}^{\mathbf{s},2}$ and omit the index q in the notations of the norm and seminorm.

Let $\mathcal{E}$ be the set of all open $(d-1)$-dimensional faces in $\mathcal{T}_h$, containing the smallest common $(d-1)$-dimensional interfaces e of neighbouring elements.

Let $\mathcal{E}_{\mathrm{int}}$ be the set of all faces in $\mathcal{E}$ contained in Ω, and let

$$\Gamma_{\mathrm{int}} := \{x \in \Omega : x \in e \text{ for some } e \in \mathcal{E}_{\mathrm{int}}\}.$$

We denote by $\mathcal{E}_{\partial}$ the set of all $(d-1)$-dimensional boundary faces.

Let each $e \in \mathcal{E}_{\partial}$ be a subset of the interior of exactly one of Γ_{D} and Γ_{N}, and label the corresponding sets of faces by $\mathcal{E}_{\mathrm{D}}$ and $\mathcal{E}_{\mathrm{N}}$.

For each $e \in \mathcal{E}_{\mathrm{int}}$ there exist integers i, j such that $i > j > 0$ and κ_i and κ_j share the face e; for $v \in \mathrm{W}^{\mathbf{s},q}(\Omega, \mathcal{T}_h)$, $s_\kappa > 1/q$, $\kappa \in \mathcal{T}_h$,

$$[\![v]\!]_e := v|_{\partial\kappa_i \cap e} - v|_{\partial\kappa_j \cap e}$$

and

$$\{\!\{ v \}\!\}_e := \frac{1}{2}\left(v|_{\partial\kappa_i \cap e} + v|_{\partial\kappa_j \cap e} \right),$$

with $\partial\kappa$ denoting the union of all open faces of the element κ.

With each face e we associate the unit normal vector ν pointing from the element κ_i to κ_j when $i > j$; when the face belongs to $\mathcal{E}_\partial$, we choose ν to be the unit outward normal vector.

Consider the bilinear form

$$B(w,v) := \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \nabla w\cdot\nabla v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{D}}}\{\theta(\nabla v\cdot\nu)w - (\nabla w\cdot\nu)v\}\,\mathrm{d}s$$
$$+ \int_{\Gamma_{\mathrm{D}}}\sigma wv\,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}}\{\theta\{\!\{\nabla v\cdot\nu\}\!\}[\![w]\!] - \{\!\{\nabla w\cdot\nu\}\!\}[\![v]\!]\}\,\mathrm{d}s$$
$$+ \int_{\Gamma_{\mathrm{int}}}\sigma[\![w]\!][\![v]\!]\,\mathrm{d}s \quad (2.1)$$

and the linear functional

$$\ell(v) := \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}} v\,\mathrm{d}s + \theta\int_{\Gamma_{\mathrm{D}}}(\nabla v\cdot\nu)g_{\mathrm{D}}\,\mathrm{d}s + \int_{\Gamma_{\mathrm{D}}}\sigma g_{\mathrm{D}} v\,\mathrm{d}s. \quad (2.2)$$

Here σ is called the penalty parameter:

$$\sigma|_e = \sigma_e \quad \text{for} \quad e \in \mathcal{E}_{\rm int} \cup \mathcal{E}_\partial,$$

where σ_e is a positive constant on the face e, to be chosen.

The subscript e will be suppressed when no confusion can occur.

It is assumed that $\theta \in [-1, 1]$. In particular:

$\theta = -1$: Symmetric Interior Penalty, or SIP, method.
$\theta = 1$: Nonsymmetric Interior Penalty, or NSIP, method.

We shall label the bilinear form (2.1) and the linear functional (2.2) with indices S and NS in the symmetric and nonsymmetric cases.

The broken weak formulation of the problem (1.1)–(1.4) is:

$$\text{find } u \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathfrak{A}) \text{ such that}$$
$$\int_\Omega u' v \,\mathrm{d}x + B(u,v) - \int_\Omega f(x,t,u) v \,\mathrm{d}x = \ell(v) \quad \forall v \in \mathrm{H}^2(\Omega,\mathcal{T}_h),$$
$$u(0) = u_0, \tag{2.3}$$

where $\mathfrak{A}$ denotes the function space

$$\mathfrak{A} = \left\{ w \in \mathrm{H}^2(\Omega,\mathcal{T}_h) : w,\ \nabla w \cdot \nu \text{ are continuous across each } e \in \mathcal{E}_{\mathrm{int}} \right\}.$$

Remark:

Note that if $u \in \mathrm{H}^1(0, T; \mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0, T; \mathrm{H}^1(\Omega))$ is a weak solution of (1.1)–(1.4) and $u \in \mathrm{L}^2(0, T; \mathfrak{A})$, then u also solves (2.3). In the sequel we shall always assume that such a u exists.

The *hp*-DGFEM approximation of the problem (1.1)–(1.4) is:

$$
\begin{aligned}
&\text{find } u_{\mathrm{DG}} \in \mathrm{H}^1(0, T; S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})) \text{ such that} \\
\int_\Omega u'_{\mathrm{DG}} v \,\mathrm{d}x + B(u_{\mathrm{DG}}, v) - \int_\Omega f(x, t, u_{\mathrm{DG}}) v \,\mathrm{d}x &= \ell(v) \quad \forall v \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F}), \\
u_{\mathrm{DG}}(0) &= u_0^{\mathrm{DG}},
\end{aligned}
\tag{2.4}
$$

where u_0^{DG} is an approximation to u_0 from $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$.

The equation (2.4) is a system of ODEs for the coefficients in the expansion of $u_{\mathrm{DG}}(\,\cdot\,, t)$ in terms of the basis functions of the finite-dimensional space $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$.

Thus, (2.4) is a nonautonomous system of ODEs with locally Lipschitz right-hand side, since $f(x, t, \,\cdot\,)$ is locally Lipschitz, uniformly in $(x, t) \in \Omega \times (0, T]$, and the other terms are linear.

By Carathéodory's theorem this implies the existence of a unique local solution to (2.4) on a certain maximal subinterval $[0, t_{\star\star})$ of $[0, T]$. We shall show that u_{DG} exists on the whole of $[0, T]$.

Coddington & Levinson:
Theory of Ordinary Differential Equations (McGraw-Hill, 1955).

Error analysis

We state and prove some preliminary results.

Lemma

Suppose that f satisfies Hypothesis A. Then, there exists a positive constant $C_f = C_f(\gamma, G_f, |\Omega|)$ such that

$$\|f(\,\cdot\,,t,w) - f(\,\cdot\,,t,v)\|_{0,\Omega} \le C_f \|w - v\|_{0,2(\gamma+1),\Omega} \times \Big(1 + \|w\|^{\gamma}_{0,2(\gamma+1),\Omega} + \|v\|^{\gamma}_{0,2(\gamma+1),\Omega}\Big) \quad (3.1)$$

for all $w,\ v \in \mathrm{L}^{2(\gamma+1)}(\Omega)$ and a.e. $t \in (0,\,T]$.

Proof.

Let $q = 2(\gamma + 1)$.

If $d = 2$, $0 \le \gamma < \infty$ then $2 \le q < \infty$; and
if $d \ge 3$, $0 \le \gamma \le 2/(d-2)$ then $2 \le q \le 2d/(d-2)$.

Let $\gamma > 0$; so $q > 2$. If $\gamma = 0$, (3.1) trivially holds with $C_f = G_f/3$.

Let $w,\ v \in \mathrm{L}^{2(\gamma+1)}(\Omega)$; from (1.2), by Hölder, for a.e. $t \in (0, T]$,

$$\|f(\,\cdot\,, t, w) - f(\,\cdot\,, t, v)\|_{0,\Omega}^2 \le G_f^2 \int_\Omega (w-v)^2 (1 + |w| + |v|)^{2\gamma}\,\mathrm{d}x$$

$$\le G_f^2 \left(\int_\Omega |w - v|^{2 \cdot q/2}\ \mathrm{d}x \right)^{2/q}$$

$$\times \left(\int_\Omega (1 + |w| + |v|)^{2\gamma \cdot (1-2/q)^{-1}}\,\mathrm{d}x \right)^{1-2/q}.$$

As $1 - 2/q = (q-2)/q = 2\gamma/q$ and $q > 2$, we have

$$\begin{aligned}
&\|f(\,\cdot\,,t,w) - f(\,\cdot\,,t,v)\|_{0,\Omega}^2 \\
&\qquad \le G_f^2 \|w - v\|_{0,q,\Omega}^2 \left(\int_\Omega (1 + |w| + |v|)^q \,\mathrm{d}x \right)^{2\gamma/q} \\
&\qquad \le G_f^2 \|w - v\|_{0,q,\Omega}^2 \left(|\Omega|^{1/q} + \|w\|_{0,q,\Omega} + \|v\|_{0,q,\Omega} \right)^{2\gamma} \\
&\qquad \le C_f^2 \|w - v\|_{0,q,\Omega}^2 \left(1 + \|w\|_{0,q,\Omega}^\gamma + \|v\|_{0,q,\Omega}^\gamma \right)^2 .
\end{aligned}$$

Hence (3.1) for a.e. $t \in (0,T]$ and all $w, v \in \mathrm{L}^q(\Omega)$, $q = 2(\gamma + 1)$.

$\square$

We equip $\mathrm{H}^1(\Omega, \mathcal{T}_h)$ with the norm $\|\cdot\|_{1,h}$ defined by

$$\|w\|_{1,h} := \left(\sum_{\kappa \in \mathcal{T}_h} \|\nabla w\|_{0,\kappa}^2 + \int_{\Gamma_{\mathrm{D}}} \sigma w^2 \,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \sigma \, [\![w]\!]^2 \,\mathrm{d}s \right)^{1/2},$$

where σ is a positive penalty parameter.

We define the norm $|||\cdot|||_{1,h}$ by

$$|||w|||_{1,h} := \left(\sum_{\kappa \in \mathcal{T}_h} \|\nabla w\|_{0,\kappa}^2 + |\Gamma_{\mathrm{D}}|^{-1} \int_{\Gamma_{\mathrm{D}}} w^2 \,\mathrm{d}s + \sum_{e \in \mathcal{E}_{\mathrm{int}}} h_e^{-1} \int_e [\![w]\!]^2 \,\mathrm{d}s \right)^{1/2}.$$

The parameter σ will be chosen so that $\sigma|_e = \sigma_e$ on each face $e \in \mathcal{E}$ and $\sigma_e \geq C_\sigma / h_e$ where C_σ is a positive constant whose value will be fixed later on; here h_e denotes the diameter of the face e.

With such σ, by noting that $1 \leq h_e^{-1} \leq \sigma_e / C_\sigma$ (as $h \leq 1$) and

$$|\Gamma_{\mathrm{D}}|^{-1} \int_{\Gamma_{\mathrm{D}}} w^2 \,\mathrm{d}s = |\Gamma_{\mathrm{D}}|^{-1} \sum_{e \in \mathcal{E}_{\mathrm{D}}} 1 \cdot \int_e w^2 \,\mathrm{d}s,$$

we have

$$|||w|||_{1,h} \leq C \|w\|_{1,h} \qquad \forall w \in \mathrm{H}^1(\Omega, \mathcal{T}_h), \tag{3.2}$$

where C is a constant independent of h and w.

Lemma (Broken Sobolev–Poincaré inequality)

There exists a positive constant C, independent of h, such that for any $q \in [1,\infty)$ if $d = 2$ and any $q \in [1, 2d/(d-2)]$ if $d \geq 3$,

$$\|w\|_{0,q,\Omega} \leq C|||w|||_{1,h} \qquad \forall w \in \mathrm{H}^1(\Omega, \mathcal{T}_h). \tag{3.3}$$

Proof.

`http://eprints.maths.ox.ac.uk/1198/` $\square$

Lemma

Suppose that f satisfies Hypothesis A. Then, there exists a positive constant $C_f = C_f(\gamma, G_f, d, |\Omega|)$ such that

$$|(f(\,\cdot\,,t,u) - f(\,\cdot\,,t,v), w)| \leq C_f \|u - v\|_{0,\Omega} \times \left(1 + |||u|||_{1,h}^{\gamma} + |||v|||_{1,h}^{\gamma}\right) \|w\|_{1,h}, \quad (3.4)$$

for all u, v, $w \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$ and a.e. $t \in (0, T]$.

Proof.

Let $u, v, w \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$. Let $p > 1$ and $1/p + 1/q = 1$. By Hölder's inequality, for a.e. $t \in (0, T]$,

$$
\begin{aligned}
|(f(\cdot, t, u) - f(\cdot, t, v), w)| &\leq G_f \int_\Omega |u - v| \, (1 + |u| + |v|)^\gamma \, |w| \, \mathrm{d}x \\
&\leq G_f \left(\int_\Omega |u - v|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega (1 + |u| + |v|)^{2\gamma} \, |w|^2 \, \mathrm{d}x \right)^{1/2} \\
&\leq G_f \left(\int_\Omega |u - v|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega (1 + |u| + |v|)^{2\gamma p} \, \mathrm{d}x \right)^{\frac{1}{2p}} \\
&\quad \times \left(\int_\Omega |w|^{2q} \, \mathrm{d}x \right)^{\frac{1}{2q}} .
\end{aligned}
$$

When $d \geq 3$ we take $p = d/2$, $q = d/(d-2)$; while if $d = 2$ we take any $p > 1$ and put $q = p/(p-1)$. The result follows by the broken Sobolev–Poincaré inequality (3.3) and (3.2). $\square$

Lemma

Let $\kappa \in \mathcal{T}_h$ *with* $h_\kappa = \mathrm{diam}(\bar{\kappa})$ *and* $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$ *for some* $k_\kappa \geq 0$*; then, there exists a* $z_{p_\kappa}^{h_\kappa}(u) \in \mathcal{R}_{p_\kappa}(\kappa)$, $p_\kappa \geq 1$, *s.t. for any* l, *with* $0 \leq l \leq s_\kappa$,

$$\|u - z_{p_\kappa}^{h_\kappa}(u)\|_{l,\kappa} \leq C \, \frac{h_\kappa^{s_\kappa - l}}{p_\kappa^{k_\kappa - l}} \, \|u\|_{k_\kappa,\kappa}, \tag{3.5}$$

and when $k_\kappa > 1/2$, *then*

$$\|u - z_{p_\kappa}^{h_\kappa}(u)\|_{0,e_\kappa} \leq C \, \frac{h_\kappa^{s_\kappa - 1/2}}{p_\kappa^{k_\kappa - 1/2}} \, \|u\|_{k_\kappa,\kappa}; \tag{3.6}$$

further, if $k_\kappa > 3/2$, *then*

$$\|\nabla(u - z_{p_\kappa}^{h_\kappa}(u))\|_{0,e_\kappa} \leq C \, \frac{h_\kappa^{s_\kappa - 3/2}}{p_\kappa^{k_\kappa - 3/2}} \, \|u\|_{k_\kappa,\kappa}, \tag{3.7}$$

where e_κ *is any face* $e_\kappa \subset \partial\kappa$, $s_\kappa = \min\{p_\kappa + 1, k_\kappa\}$.

Proof.

For the proof of (3.5), see Lemma 4.5 in

I. Babuška & M. Suri:
The h-p version of the finite element method with quasi-uniform meshes, RAIRO Modél. Math. Anal. Numér., 21 (1987), 199–238.

for $d = 2$ (the argument being analogous when $d > 2$).

Using the multiplicative trace inequality

$$\|u\|_{0,\partial\kappa} \leq C(d) \left(h_\kappa^{-1/2} \|u\|_{0,\kappa} + \|u\|_{0,\kappa}^{1/2} \|\nabla u\|_{0,\kappa}^{1/2} \right),$$

we obtain (3.6) and (3.7) from (3.5).

If the reference element $\hat{\kappa}$ is the d-dimensional hypercube, instead of the Babuška–Suri projector $z_{p_\kappa}^{h_\kappa}$ one can use a Jackson-type quasi-interpolation operator $J_{p_\kappa}^{k_\kappa}$; cf. Theorem 5.1 in

M. Melenk:
hp-interpolation of nonsmooth functions and an application to *hp*-a posteriori error estimation. SIAM J. Numer. Anal. 43 (2005), 127–155.

We need the following bound on the bilinear form $B(\cdot, \cdot)$; see

P. Houston, Ch. Schwab, & E. Süli:
Discontinuous *hp*-finite element methods for advection-diffusion-reaction problems. SIAM J. Numer. Anal. 39 (2002), no. 6, 2133–2163.

a key ingredient being the inverse inequality

$$\|\nabla w\|^2_{0,\partial\kappa\cap\Gamma_{\mathrm{D}}} \le C_{\mathrm{inv}} \frac{p_\kappa^2}{h_\kappa} \|\nabla w\|^2_{0,\kappa}, \tag{3.8}$$

where the constant C_{inv} depends only on the shape-regularity constant of the family $\{\mathcal{T}_h\}$; see Theorem 4.76 in:

Ch. Schwab:
p- and *hp*-finite element methods. Theory and applications in solid and fluid mechanics. Oxford University Press, 1998.

Lemma (Continuity)

There exists a positive constant C, independent of the discretisation parameters, s.t. for all $v \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$ and all $w \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$:

$$
\begin{aligned}
|B(v,w)| \leq C \|w\|_{1,h} \Bigg\{ & \int_{\Gamma_{\mathrm{D}}} \sigma |v|^2 \, \mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \sigma [\![v]\!]^2 \, \mathrm{d}s + \sum_{\kappa \in \mathcal{T}_h} \|\nabla v\|_{0,\kappa}^2 \\
& + \sum_{\kappa \in \mathcal{T}_h} \left(\|\sqrt{\tau} v\|_{0,\partial\kappa \cap \Gamma_{\mathrm{D}}}^2 + \|\sigma^{-1/2} \nabla v\|_{0,\partial\kappa\cap\Gamma_{\mathrm{D}}}^2 \right) \\
& + \sum_{\kappa \in \mathcal{T}_h} \left(\|\sqrt{\tau} [\![v]\!] \|_{0,\partial\kappa \cap \Gamma_{\mathrm{int}}}^2 + \|\sigma^{-1/2} \nabla v\|_{0,\partial\kappa\cap\Gamma_{\mathrm{int}}}^2 \right) \Bigg\}^{1/2},
\end{aligned}
\tag{3.9}
$$

where $\tau_e = \{\!\{ p^2 \}\!\}_e / h_e$ and h_e is the diameter of a face $e \subset \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$; if $e \in \mathcal{E}_{\mathrm{D}}$ the contribution from outside Ω in the definition of τ_e is $= 0$.

Consider now the symmetric bilinear form (2.1) (with $\theta = -1$).

We have, for any $w \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$,

$$\begin{aligned} B_{\mathrm{S}}(w, w) &= \sum_{\kappa \in \mathcal{T}_h} \|\nabla w\|_{0,\kappa}^2 + \int_{\Gamma_{\mathrm{D}}} \big(\sigma w^2 - 2w(\nabla w \cdot \nu)\big) \mathrm{d}s \\ &\quad + \int_{\Gamma_{\mathrm{int}}} \Big(\sigma \, [\![w]\!]^2 - 2 \, [\![w]\!] \, \{\!\{\nabla w \cdot \nu\}\!\}\Big) \mathrm{d}s. \end{aligned}$$

Clearly, the integrands in the last two terms need not be positive for $w \neq 0$ unless σ is chosen sufficiently large: the purpose of the analysis that now follows is to assess just how large σ needs to be to ensure coercivity of $B_{\mathrm{S}}(\,\cdot\,,\,\cdot\,)$ over $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F}) \times S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$.

For any $\tau_e > 0$ we have

$$-2\int_{\Gamma_{\mathrm{D}}} w(\nabla w\cdot\nu)\,\mathrm{d}s \geq -\sum_{e\in\mathcal{E}_{\mathrm{D}}}\left(\int_e \tau_e w^2\,\mathrm{d}s + \int_e \tau_e^{-1}(\nabla w\cdot\nu)^2\,\mathrm{d}s\right).$$

Using the inverse inequality (3.8), the shape-regularity condition (to relate h_κ to h_e, where κ is the element whose face $e\in\mathcal{E}_{\mathrm{D}}$ is), taking $p_e = p_\kappa$ for $e\subset\partial\kappa\cap\Gamma_{\mathrm{D}}$, and absorbing all constants in C_τ:

$$-\int_e \tau_e^{-1}(\nabla w\cdot\nu)^2\,\mathrm{d}s \geq -\int_e \tau_e^{-1}\,|\nabla w|^2\,|\nu|^2\,\mathrm{d}s \geq -\tau_e^{-1}C_\tau\frac{p_e^2}{h_e}\,\|\nabla w\|_{0,\kappa}^2,$$

and hence

$$-2\int_{\Gamma_{\mathrm{D}}} w(\nabla w\cdot\nu)\,\mathrm{d}s \geq -\sum_{e\in\mathcal{E}_{\mathrm{D}}}\left(\int_e \tau_e w^2\,\mathrm{d}s + \tau_e^{-1}C_\tau\frac{p_e^2}{h_e}\|\nabla w\|_{0,\kappa}^2\right).$$

Similarly, for the term involving faces $e \in \mathcal{E}_{\mathrm{int}}$, we have, using the bounded local variation condition (to relate p_κ^2 to $\{\!\{p^2\}\!\}_e$),

$$
\begin{aligned}
&-2\int_{\Gamma_{\mathrm{int}}} [\![w]\!] \{\!\{\nabla w \cdot \nu\}\!\} \,\mathrm{d}s \\
&\quad \geq -\sum_{e\in\mathcal{E}_{\mathrm{int}}} \left(\int_e \tau_e [\![w]\!]^2 \,\mathrm{d}s + \tau_e^{-1} C_\tau \frac{\{\!\{p^2\}\!\}_e}{2h_e} \big(\|\nabla w\|_{0,\kappa'}^2 + \|\nabla w\|_{0,\kappa''}^2 \big) \right).
\end{aligned}
$$

Here κ' and κ'' are the two elements that have e as common face.

As no face e of any element $\kappa \in \mathcal{T}_h$ contains more than one hanging node, no element κ can have more than $2d \cdot 2^{d-1} = 2^d d$ faces if $\hat{\kappa}$ is the d-dimensional hypercube, or more than $(d+1)d$ faces if $\hat{\kappa}$ is the d-dimensional simplex.

Letting

$$c_d := \max\left\{2^d d, (d+1)d\right\} = 2^d d, \quad \tau_e := c_d C_\tau \{\!\{p^2\}\!\}_e / h_e$$

for $e \in \mathcal{E}_{\mathrm{D}} \cup \mathcal{E}_{\mathrm{int}}$ (with the convention that, for any face $e \in \mathcal{E}_{\mathrm{D}}$, $\{\!\{p^2\}\!\}_e = p_e^2/2 = p_\kappa^2/2$ where κ is the element in $\mathcal{T}_h$ with face e):

$$B_{\mathrm{S}}(w,w) \geq \frac{1}{2} \sum_{\kappa \in \mathcal{T}_h} \|\nabla w\|_{0,\kappa}^2 + \int_{\Gamma_{\mathrm{D}}} (\sigma - \tau) w^2 \,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} (\sigma - \tau) [\![w]\!]^2 \,\mathrm{d}s.$$

Choosing $\sigma_e = C_\sigma \{\!\{p^2\}\!\}_e / h_e$ with $C_\sigma > c_d C_\tau$ will ensure that $\sigma_e > \tau_e$ for all $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$, and hence that $B_{\mathrm{S}}(\cdot, \cdot)$ is coercive.

When $e \in \mathcal{E}_{\mathrm{D}}$ the contribution from outside Ω in the definition of σ_e is set to 0. Thus, we adopt the following hypothesis concerning the choice of the penalty constant C_σ.

Hypothesis B

In the nonsymmetric case (when $\theta = 1$) we take any $C_\sigma > 0$.
In the symmetric case (when $\theta = -1$) we take $C_\sigma > c_d C_\tau$.

Lemma

The nonsymmetric bilinear form $B_{\mathrm{NS}}(\,\cdot\,,\,\cdot\,)$ *is coercive in the norm* $\|\cdot\|_{1,h}$ *over the space* $\mathrm{H}^1(\Omega,\mathcal{T}_h)\times\mathrm{H}^1(\Omega,\mathcal{T}_h)$*; more precisely,*

$$B_{\mathrm{NS}}(w,w)=\|w\|_{1,h}^2 \qquad \forall w\in\mathrm{H}^1(\Omega,\mathcal{T}_h).$$

With C_σ *as in* Hypothesis B*, the symmetric bilinear form* $B_{\mathrm{S}}(\,\cdot\,,\,\cdot\,)$ *induces a norm* $\|\cdot\|_B$ *on the finite element space* $S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathbf{F})$*; moreover, there exists a positive constant* c_0 *such that*

$$B_{\mathrm{S}}(w,w)=\|w\|_B^2\geq c_0\|w\|_{1,h}^2 \qquad \forall w\in S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathbf{F}),$$

i.e. $B_{\mathrm{S}}(\,\cdot\,,\,\cdot\,)$ *is coercive in* $\|\cdot\|_{1,h}$ *over* $S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathbf{F})\times S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathbf{F})$*.*

Let us consider the projection operator Π from $\mathrm{H}^2(\Omega, \mathcal{T}_h)$ onto the finite element space $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ defined (for $u \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$) by

$$B(u - \Pi u, v) = 0 \qquad \forall v \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F}). \tag{3.10}$$

As $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ is a finite-dimensional linear space, the existence of a unique Πu in $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ for each $u \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$ follows from the coercivity of $B(\,\cdot\,,\,\cdot\,)$ on $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F}) \times S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$.

Πu is called the broken elliptic projection of u, and the operator

$$\Pi\,:\, \mathrm{H}^2(\Omega, \mathcal{T}_h) \to S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$$

the broken elliptic projector.

Lemma

Let $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$ *for some* $k_\kappa \geq 2$ *and each* $\kappa \in \mathcal{T}_h$*, let* $p_\kappa \geq 1$ *for each* $\kappa \in \mathcal{T}_h$*, and define*

$$\sigma_e := C_\sigma \{\!\{p^2\}\!\}|_e / h_e,$$

where C_σ *is as in* Hypothesis B*, and* h_e *is the diameter of a face* $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$*; when* $e \in \mathcal{E}_{\mathrm{D}}$*, the contribution from outside* Ω *in the definition of* σ_e *is set to* 0*. Then,*

$$\|u - \Pi u\|_{1,h}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{k_\kappa,\kappa}^2, \tag{3.11}$$

where $s_\kappa = \min\{p_\kappa + 1, k_\kappa\}$*. Moreover, if* Ω *is convex,* $\theta = -1$*, and* Γ_{N} *is empty (i.e.* $\partial\Omega = \Gamma_{\mathrm{D}}$*), then, letting* $s_\kappa := \min\{p_\kappa + 1, k_\kappa\}$*,*

$$\|u - \Pi u\|_{0,\Omega}^2 \leq C \left(\max_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^2}{p_\kappa} \right) \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{k_\kappa,\kappa}^2. \tag{3.12}$$

Proof.

Note that $B(w, w) \geq c_0 \|w\|_{1,h}^2$ for all $w \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$.

On writing

$$u - \Pi u = (u - z_{p_\kappa}^{h_\kappa}(u)) + (z_{p_\kappa}^{h_\kappa}(u) - \Pi u) =: \eta + \xi,$$

and taking $v = \xi$ in the definition of the broken elliptic projector (3.10), we deduce that

$$c_0 \|\xi\|_{1,h}^2 \leq B(\xi, \xi) = B((u - \Pi u) - \eta, \xi) = -B(\eta, \xi) \leq |B(\eta, \xi)| \, .$$

By the lemma asserting the continuity of B, with $v = \eta$, $w = \xi$, and the above inequality, noting that $\sigma_e = C_\sigma \{\!\{p^2\}\!\}|_e / h_e > \tau_e$, with h_e the diameter of a face $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$ and with the contribution from outside Ω in $\{\!\{p^2\}\!\}|_e$ set to 0 for $e \in \mathcal{E}_{\mathrm{D}}$:

$$\|\xi\|_{1,h}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \Big(\|\sqrt{\sigma}\eta\|_{0,\partial\kappa\cap\Gamma_{\mathrm{D}}}^2 + \|\sqrt{\sigma}\,[\![\eta]\!]\,\|_{0,\partial\kappa\cap\Gamma_{\mathrm{int}}}^2 + \|\nabla\eta\|_{0,\kappa}^2$$

$$+\|\sigma^{-1/2}\nabla\eta\|_{0,\partial\kappa\cap\Gamma_{\mathrm{D}}}^2 + \|\sigma^{-1/2}\nabla\eta\|_{0,\partial\kappa\cap\Gamma_{\mathrm{int}}}^2 \Big).$$

Using the triangle inequality $\|u - \Pi u\|_{1,h} \leq \|\eta\|_{1,h} + \|\xi\|_{1,h}$ and the definition of the norm $\|\cdot\|_{1,h}$ gives the bound:

$$\|u - \Pi u\|_{1,h}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \Big(\|\sqrt{\sigma}\eta\|_{0,\partial\kappa\cap\Gamma_{\mathrm{D}}}^2 + \|\sqrt{\sigma}\,[\![\eta]\!]\,\|_{0,\partial\kappa\cap\Gamma_{\mathrm{int}}}^2 + \|\nabla\eta\|_{0,\kappa}^2 + \|\sigma^{-1/2}\nabla\eta\|_{0,\partial\kappa\cap\Gamma_{\mathrm{D}}}^2 + \|\sigma^{-1/2}\nabla\eta\|_{0,\partial\kappa\cap\Gamma_{\mathrm{int}}}^2 \Big). \quad (3.13)$$

From inequalities (3.5)–(3.7), we deduce that

$$\|\eta\|_{0,\partial\kappa}^2 \leq C\frac{h_\kappa^{2s_\kappa-1}}{p_\kappa^{2k_\kappa-1}}\|u\|_{k_\kappa,\kappa}^2, \quad \|\nabla\eta\|_{0,\partial\kappa}^2 \leq C\frac{h_\kappa^{2s_\kappa-3}}{p_\kappa^{2k_\kappa-3}}\|u\|_{k_\kappa,\kappa}^2, \quad \|\eta\|_{1,\kappa}^2 \leq C\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-2}}\|u\|_{k_\kappa,\kappa}^2.$$

Applying these to the right-hand side of (3.13), choosing σ_e as in the statement of the Lemma, noting the bounded local variation condition and the shape-regularity of $\mathcal{T}_h$ to relate h_e to h_κ:

$$\|u - \Pi u\|_{1,h}^2 \leq C \sum_{\kappa\in\mathcal{T}_h} \left(\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-2}} + \frac{p_\kappa^2}{h_\kappa}\,\frac{h_\kappa^{2s_\kappa-1}}{p_\kappa^{2k_\kappa-1}}\right)\|u\|_{k_\kappa,\kappa}^2,$$

and hence (3.11).

To estimate $\|u - \Pi u\|_{0,\Omega}$ in the case of $\theta = -1$ and $\Gamma_{\mathrm{D}} = \partial\Omega$, we shall use the Aubin–Nitsche duality argument.

Let $(\,\cdot\,,\,\cdot\,)$ be the L^2 inner product over Ω. Then,

$$\|u - \Pi u\|_{0,\Omega} = \sup_{\substack{g \in \mathrm{L}^2(\Omega) \\ g \neq 0}} \frac{(u - \Pi u, g)}{\|g\|_{0,\Omega}}. \tag{3.14}$$

For $g \in \mathrm{L}^2(\Omega)$ fixed, $g \neq 0$, let $w = w_g$ be the weak solution of

$$\begin{aligned} -\Delta w &= g \quad \text{in} \quad \Omega, \\ w &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.15}$$

As Ω is a convex polyhedron, $w \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ by elliptic regularity theory and $\exists C > 0$, independent of g and w, s.t.

$$\|w\|_{2,\Omega} \leq C\|g\|_{0,\Omega}. \tag{3.16}$$

Further, $w \in \mathrm{C}^1(\Omega)$. The SIP DGFEM approximation of (3.15) is:

find $w_{\mathrm{DG}} \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ s.t. $B_{\mathrm{S}}(w_{\mathrm{DG}}, v) = \ell_g(v) \quad \forall v \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$,

where $B_{\mathrm{S}}(w, v)$ is defined by (2.1) with $\theta = -1$, and

$$\ell_g(v) = (g, v) + \ell_{\mathrm{S}}(v),$$

where $\ell_{\mathrm{S}}(v)$ is defined by (2.2) with $\theta = -1$ and $g_{\mathrm{D}} = 0$ on $\Gamma_{\mathrm{D}} = \partial\Omega$ (also, as $\Gamma_{\mathrm{N}} = \emptyset$, the integral over Γ_{N} in (2.2) vanishes); clearly, then, $\ell_{\mathrm{S}}(v) = 0$ for all v in $\mathrm{H}^2(\Omega, \mathcal{T}_h)$.

Using the fact that $w \in \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega) \cap \mathrm{C}^1(\Omega)$, we deduce that $B_{\mathrm{S}}(w, v) = \ell_g(v)$ for all $v \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$.

Moreover, by the definition of the broken elliptic projector (3.10), $B_{\mathrm{S}}(u - \Pi u, v) = 0$ for all $v \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$, and hence

$$\begin{aligned}(u - \Pi u, g) &= (g, u - \Pi u) = \ell_g(u - \Pi u) = B_{\mathrm{S}}(w, u - \Pi u) \\ &= B_{\mathrm{S}}(u - \Pi u, w) = B_{\mathrm{S}}(u - \Pi u, w - z_{p_\kappa}^{h_\kappa}(w)).\end{aligned}$$

Letting $\eta_w := w - z_{p_\kappa}^{h_\kappa}(w)$, by (3.9), noting that

$$\sigma_e = C_\sigma \{\!\{p^2\}\!\}|_e / h_e > \tau_e,$$

with $h_e = \mathrm{diam}(e)$, $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$ and with the contribution from outside Ω in $\{\!\{p^2\}\!\}|_e$ set to 0 for $e \in \mathcal{E}_{\mathrm{D}}$, we have:

$$\begin{aligned}
&(u - \Pi u, g) = B_{\mathrm{S}}(u - \Pi u, \eta_w) \\
&\leq C \, \|u - \Pi u\|_{1,h} \Bigg\{ \int_{\Gamma_{\mathrm{D}}} \sigma \, |\eta_w|^2 \, \mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \sigma \, [\![\eta_w]\!]^2 \, \mathrm{d}s \\
&+ \sum_{\kappa \in \mathcal{T}_h} \left(\|\nabla \eta_w\|_{0,\kappa}^2 + \|\sigma^{-1/2} \nabla \eta_w\|_{0,\partial\kappa \cap \Gamma_{\mathrm{D}}}^2 + \|\sigma^{-1/2} \nabla \eta_w\|_{0,\partial\kappa \cap \Gamma_{\mathrm{int}}}^2 \right) \Bigg\}^{\frac{1}{2}}.
\end{aligned}$$

Using (3.11) and inequalities (3.5)–(3.7) on the right-hand side of (3.17), choosing σ_e as described in the statement above, noting the bounded local variation condition and the shape-regularity of $\mathcal{T}_h$ to relate h_e to h_κ, we obtain

$$(u - \Pi u, g) \leq C \left(\sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{k_\kappa,\kappa}^2 \times \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^2}{p_\kappa} \|w\|_{2,\kappa}^2 \right)^{1/2} .$$

By observing that

$$\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^2}{p_\kappa}\|w\|_{2,\kappa}^2 \le \left(\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^2}{p_\kappa}\right)\sum_{\kappa\in\mathcal{T}_h}\|w\|_{2,\kappa}^2 = \left(\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^2}{p_\kappa}\right)\|w\|_{2,\Omega}^2$$

and noting (3.16), we obtain

$$(u-\Pi u, g) \le C\left(\left(\max_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^2}{p_\kappa}\right)\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}}\|u\|_{k_\kappa,\kappa}^2\right)^{1/2}\|g\|_{0,\Omega}.$$

Therefore, for any $g \in \mathrm{L}^2(\Omega)$, $g \neq 0$,

$$\frac{(u - \Pi u, g)}{\|g\|_{0,\Omega}} \leq C \left(\left(\max_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^2}{p_\kappa} \right) \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{k_\kappa,\kappa}^2 \right)^{1/2} .$$

Recalling (3.14), taking the supremum over $g \in \mathrm{L}^2(\Omega)$, $g \neq 0$, and squaring the resulting expression yields (3.12). $\square$

Error analysis of the nonsymmetric DGFEM

We take $\theta = 1$ in (2.1), and write $B_{\mathrm{NS}}(\,\cdot\,,\,\cdot\,)$ in place of $B(\,\cdot\,,\,\cdot\,)$.

We shall derive a bound on the H^1 norm of the error $u - u_{\mathrm{DG}}$. Here u_{DG} is the NSIP DGFEM approximation of u. Decompose:

$$u - u_{\mathrm{DG}} = \eta + \xi = (u - \Pi u) + (\Pi u - u_{\mathrm{DG}}),$$

with Π denoting the broken elliptic projector defined above, with $\theta = 1$. We assume for simplicity that $u_0^{\mathrm{DG}} = \Pi u_0$; thus $\xi(0) = 0$.

As in Hypothesis B, we assume that $\sigma_e = C_\sigma \{\!\{p^2\}\!\}_e / h_e$, where $C_\sigma > 0$ is any positive constant, and h_e is the diameter of a face $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$; when $e \in \mathcal{E}_{\mathrm{D}}$, the contribution from outside Ω in the definition of σ_e is set to 0.

Lemma

Let f satisfy Hypothesis A *and let* $u \in \mathrm{L}^\infty(0, T; \mathrm{H}^1(\Omega))$. *Suppose:*

a) $p_\kappa \geq 2$ *and* $u|_\kappa \in \mathrm{H}^1(0, T; \mathrm{H}^{k_\kappa}(\kappa))$ *with* $k_\kappa \geq 3\frac{1}{2}$, $\forall \kappa \in \mathcal{T}_h$;

b) *the hp-mesh is quasi-uniform in the sense that there exists a positive constant* C_0 *such that*

$$\max_{\kappa \in \mathcal{T}_h} \frac{h_\kappa}{p_\kappa^2} \leq C_0 \min_{\kappa \in \mathcal{T}_h} \frac{h_\kappa}{p_\kappa^2}. \tag{3.17}$$

Then, $\exists h_0 \in (0, 1]$ *and a constant* $C > 0$ *independent of* h *and* $\mathbf{p}$, *s.t. for all* $h \in (0, h_0]$, $h = \max_{\kappa \in \mathcal{T}_h} h_\kappa$, *and for all* $t \in [0, T]$:

$$\int_0^t \|(u - u_{\mathrm{DG}})(s)\|_{1,h}^2 \, \mathrm{d}s \leq C \int_0^t \left\{ \|\eta(s)\|_{1,h}^2 + \|\eta'(s)\|_{0,\Omega}^2 \right\} \mathrm{d}s. \tag{3.18}$$

Proof. Let $t_{\star\star} \in (0, T]$ be such that $u_{\mathrm{DG}}(t) \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ exists for all $t \in [0, t_{\star\star})$. The existence of such a $t_{\star\star}$ is ensured by Carathéodory's theorem.

Thus:

- either $t_{\star\star} = T$, or
- $t_{\star\star} < T$ and $\limsup_{t \to t_{\star\star}} \|u_{\mathrm{DG}}(t)\|_{1,h} = +\infty$.

We shall show that, for h sufficiently small, $t_{\star\star} = T$.

From the hp-DGFEM (2.4), for all $v \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$, we have

$$\int_\Omega u_{\mathrm{DG}}' v \,\mathrm{d}x + B_{\mathrm{NS}}(u_{\mathrm{DG}}, v) = \int_\Omega f(x, t, u_{\mathrm{DG}}) v \,\mathrm{d}x + \ell_{\mathrm{NS}}(v) \quad (3.19)$$

for all $t \in (0, t_{\star\star})$. Also, the broken weak formulation (2.3) gives:

$$\begin{aligned}\int_\Omega (\Pi u') v \,\mathrm{d}x + B_{\mathrm{NS}}(\Pi u, v) = &\int_\Omega f(x, t, u) v \,\mathrm{d}x + \ell_{\mathrm{NS}}(v) \\ &+ \int_\Omega (\Pi u' - u') v \,\mathrm{d}x + B_{\mathrm{NS}}(\Pi u - u, v)\end{aligned} \quad (3.20)$$

for all $v \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$ and all $t \in (0, T]$.

By subtracting (3.19) from (3.20) and taking $v = \xi = \Pi u - u_{\mathrm{DG}}$:

$$\int_\Omega \xi'\xi \,\mathrm{d}x + B_{\mathrm{NS}}(\xi,\xi) = \int_\Omega \{f(x,t,u) - f(x,t,u_{\mathrm{DG}})\}\,\xi \,\mathrm{d}x$$
$$- \int_\Omega \eta'\xi \,\mathrm{d}x - B_{\mathrm{NS}}(\eta,\xi)$$

for all $t \in (0, t_{\star\star})$. By (3.10) we have $B_{\mathrm{NS}}(\eta,\xi) = 0$.

Hence, by noting that $\|\xi\|_{1,h}^2 = B_{\mathrm{NS}}(\xi,\xi)$, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_{0,\Omega}^2 + \|\xi\|_{1,h}^2 \leq \left|\int_\Omega \{f(x,t,u) - f(x,t,\Pi u)\}\,\xi\,\mathrm{d}x\right| + \left|\int_\Omega \{f(x,t,\Pi u) - f(x,t,u_{\mathrm{DG}})\}\,\xi\,\mathrm{d}x\right| + \left|\int_\Omega \eta'\xi\,\mathrm{d}x\right| \quad (3.21)$$

for all $t \in (0, t_{\star\star})$. By the Cauchy–Schwarz inequality and Cauchy's inequality, with $\varepsilon_1 > 0$, we have

$$\left|\int_\Omega \eta'\xi\,\mathrm{d}x\right| \leq \|\eta'\|_{0,\Omega}\|\xi\|_{0,\Omega} \leq \frac{\varepsilon_1}{2}\|\eta'\|_{0,\Omega}^2 + \frac{1}{2\varepsilon_1}\|\xi\|_{0,\Omega}^2.$$

By the same argument, with ε_2, $\varepsilon_3 > 0$,

$$\left|\int_\Omega \{f(x,t,u) - f(x,t,\Pi u)\}\,\xi\,\mathrm{d}x\right| \\ \leq \frac{\varepsilon_2}{2}\|f(\,\cdot\,,t,u) - f(\,\cdot\,,t,\Pi u)\|_{0,\Omega}^2 + \frac{1}{2\varepsilon_2}\|\xi\|_{0,\Omega}^2,$$

$$\left|\int_\Omega \{f(x,t,\Pi u) - f(x,t,u_{\mathrm{DG}})\}\,\xi\,\mathrm{d}x\right| \\ \leq \frac{\varepsilon_3}{2}\|f(\,\cdot\,,t,\Pi u) - f(\,\cdot\,,t,u_{\mathrm{DG}})\|_{0,\Omega}^2 + \frac{1}{2\varepsilon_3}\|\xi\|_{0,\Omega}^2.$$

Using the broken Sobolev–Poincaré inequality and (3.2), for a.e. $t \in (0, T]$ we have that

$$
\begin{aligned}
&\|f(\,\cdot\,, t, u) - f(\,\cdot\,, t, \Pi u)\|_{0,\Omega}^2 \\
&\leq C_f^2 \|\eta\|_{0,2(\gamma+1),\Omega}^2 \left(1 + \|u\|_{0,2(\gamma+1),\Omega}^{\gamma} + \|\Pi u\|_{0,2(\gamma+1),\Omega}^{\gamma}\right)^2 \\
&\leq C \|\eta\|_{0,2(\gamma+1),\Omega}^2 \left(1 + \|u\|_{0,2(\gamma+1),\Omega}^{2\gamma} + \|\Pi u - u\|_{0,2(\gamma+1),\Omega}^{2\gamma}\right) \\
&= C \|\eta\|_{0,2(\gamma+1),\Omega}^2 \left(1 + \|u\|_{0,2(\gamma+1),\Omega}^{2\gamma} + \|\eta\|_{0,2(\gamma+1),\Omega}^{2\gamma}\right) \\
&\leq C \|\eta\|_{0,2(\gamma+1),\Omega} \left(1 + \|u\|_{0,2(\gamma+1),\Omega}^{2\gamma} + \|\eta\|_{1,h}^{2\gamma}\right) \\
&\leq C \|\eta\|_{0,2(\gamma+1),\Omega}^2,
\end{aligned}
$$

where $C = C(\Omega, \gamma, \text{norms of } u \text{ over } (0, T))$.

Applying these bounds on the right-hand side of (3.21) with $\varepsilon_1 = \varepsilon_2 = 1$ (the value of ε_3 will be fixed below) and absorbing all constants into C_1 and $C_2 = C_2(\varepsilon_3)$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|_{0,\Omega}^2 + 2\|\xi(t)\|_{1,h}^2 \\
&\quad \leq C_1(\|\eta(t)\|_{0,2(\gamma+1),\Omega}^2 + \|\eta'(t)\|_{0,\Omega}^2) + C_2\|\xi(t)\|_{0,\Omega}^2 \\
&\qquad +\varepsilon_3\|f(\,\cdot\,,t,\Pi u) - f(\,\cdot\,,t,u_{\mathrm{DG}})\|_{0,\Omega}^2,
\end{aligned} \tag{3.22}$$

for a.e. $t \in (0, t_{\star\star})$.

To bound $\|f(\,\cdot\,,t,\Pi u)-f(\,\cdot\,,t,u_{\mathrm{DG}})\|_{0,\Omega}^2$, we first note that, by a similar argument to the one above, we have, for a.e. $t\in(0,t_{\star\star})$,

$$\begin{aligned}&\|f(\,\cdot\,,t,\Pi u)-f(\,\cdot\,,t,u_{\mathrm{DG}})\|_{0,\Omega}^2\\&\qquad\leq C\|\xi(t)\|_{0,2(\gamma+1),\Omega}^2\left(1+\|\xi(t)\|_{0,2(\gamma+1),\Omega}^{2\gamma}\right),\end{aligned}\tag{3.23}$$

where the constant $C>0$ depends only on the domain Ω, the exponent γ in the growth condition for the function f, and on Lebesgue and Sobolev norms of u over the time interval $(0,T)$.

For $\mathcal{T}_h$ and $\mathbf{p}$ fixed, let $t_\star = t_\star(\mathcal{T}_h, \mathbf{p}) \in (0, t_{\star\star}]$ be the largest t s.t. u_{DG} exists for all $t \in [0, t_\star]$ and $\|\xi(t)\|_{1,h}^2 \leq 1$ for all $t \in [0, t_\star]$.

The existence of such a $t_\star$ follows from the definition of $t_{\star\star}$, together with the fact that $t \mapsto \|\xi(t)\|_{1,h}^2$ is continuous in the neighbourhood of $t = 0$ and $\|\xi(0)\|_{1,h}^2 = 0$.

We shall show that $t_\star = T$ for all h, sufficiently small; thereby, we will have also shown that $t_{\star\star} = T$. We have that

$$\|\xi(t)\|_{0,2(\gamma+1),\Omega}^2 \leq \mathit{Const.}\|\xi(t)\|_{1,h}^2 \qquad \forall t \in [0, t_\star]$$

by the broken Sobolev–Poincaré inequality and (3.2); here *Const.* is a constant that is independent of h, $\mathbf{p}$ and t.

This and (3.23), together with the fact that $\|\xi(t)\|_{1,h}^2 \leq 1$ for all $t \in [0, t_\star]$, imply that, for a.e. $t \in (0, t_\star]$,

$$\|f(\,\cdot\,, t, \Pi u) - f(\,\cdot\,, t, u_{\mathrm{DG}})\|_{0,\Omega}^2 \leq \tilde{C}\|\xi(t)\|_{1,h}^2,$$

where the constant $\tilde{C} > 0$ depends only on the domain Ω, the exponent γ in the growth condition for the function f, and on Lebesgue and Sobolev norms of u over the time interval $(0, t_\star)$.

On choosing $\varepsilon_3 \tilde{C} \leq 1$, after integration from 0 to $t \leq t_\star$ and noting that $\xi(0) = 0$, the inequality (3.22) yields that

$$\begin{aligned}\|\xi(t)\|_{0,\Omega}^2 + \int_0^t \|\xi(s)\|_{1,h}^2 \,\mathrm{d}s \leq & C_1 \int_0^t \Big\{ \|\eta(s)\|_{0,2(\gamma+1),\Omega}^2 + \|\eta'(s)\|_{0,\Omega}^2 \Big\} \mathrm{d}s \\ & + C_2 \int_0^t \|\xi(s)\|_{0,\Omega}^2 \,\mathrm{d}s \qquad \forall t \in [0, t_\star],\end{aligned} \tag{3.24}$$

with the constant $C_1 > 0$ depending only on the domain Ω, the exponent γ in the growth condition for the function f, and on Lebesgue and Sobolev norms of u over the time interval $(0, T)$.

The first integral on the right-hand side of (3.24) can be made as small as we like (for example, by fixing the local polynomial degree p_κ on each element $\kappa \in \mathcal{T}_h$ and reducing $h = \max_{\kappa\in\mathcal{T}_h} h_\kappa$). In particular, let us take $h_0 \in (0,1]$ so small that, for all $h \le h_0$ and $t \in [0, T]$, the following inequality holds:

$$C_1 \int_0^t \left\{ \|\eta(s)\|^2_{0,2(\gamma+1),\Omega} + \|\eta'(s)\|^2_{0,\Omega} \right\} \mathrm{d}s$$
$$< \frac{1}{1+T}\,\mathrm{e}^{-C_2 T} \times C_{\mathrm{inv}}^{-1} C_0^{-2} \left(\max_{\kappa\in\mathcal{T}_h} \frac{h_\kappa}{p_\kappa^2} \right)^2,$$

where C_{inv} is the constant from the inverse inequality

$$\|\xi(t)\|^2_{1,h} \le C_{\mathrm{inv}} \left(\max_{\kappa\in\mathcal{T}_h} \frac{p_\kappa^2}{h_\kappa} \right)^2 \|\xi(t)\|^2_{0,\Omega} \qquad \forall t \in [0, t_\star]. \quad (3.25)$$

We note in passing that in order to be able to extract the factor $(\max_{\kappa\in\mathcal{T}_h}(h_\kappa/p_\kappa^2))^2$ above from $\|\eta(s)\|^2_{0,2(\gamma+1),\Omega} + \|\eta'(s)\|^2_{0,\Omega}$ with strict inequality (by using (3.3), (3.2) and (3.11)), we require assumption a) of the Lemma. Hence (3.24) yields

$$\begin{aligned}
&\|\xi(t)\|^2_{0,\Omega} + \int_0^t \|\xi(s)\|^2_{1,h}\,\mathrm{d}s \\
&\qquad < \frac{\mathrm{e}^{-C_2 T}}{1+T} \times C_{\mathrm{inv}}^{-1} C_0^{-2} \left(\max_{\kappa\in\mathcal{T}_h} \frac{h_\kappa}{p_\kappa^2}\right)^2 + C_2 \int_0^t \|\xi(s)\|^2_{0,\Omega}\,\mathrm{d}s
\end{aligned}$$

for all $t \in [0, t_\star]$.

Gronwall's inequality then implies that

$$\|\xi(t)\|_{0,\Omega}^2 < C_{\mathrm{inv}}^{-1} C_0^{-2} \left(\max_{\kappa \in \mathcal{T}_h} \frac{h_\kappa}{p_\kappa^2} \right)^2 \qquad \forall t \in [0, t_\star].$$

By the inverse inequality (3.25) we have that, for all $t \in [0, t_\star]$,

$$\begin{aligned} \|\xi(t)\|_{1,h}^2 &< C_0^{-2} \left(\max_{\kappa \in \mathcal{T}_h} \frac{h_\kappa}{p_\kappa^2} \right)^2 \left(\max_{\kappa \in \mathcal{T}_h} \frac{p_\kappa^2}{h_\kappa} \right)^2 \\ &= C_0^{-2} \left(\max_{\kappa \in \mathcal{T}_h} \frac{h_\kappa}{p_\kappa^2} \right)^2 \left(\min_{\kappa \in \mathcal{T}_h} \frac{h_\kappa}{p_\kappa^2} \right)^{-2}, \end{aligned}$$

which, by the quasi-uniformity hypothesis b) above, is ≤ 1.

Thus, for $h \leq h_0$, we have that $\|\xi(t)\|_{1,h}^2 < 1$ for all $t \in [0, t_\star]$.

By continuity of $t \mapsto \|\xi(t)\|_{1,h}^2$ on $[0, t_\star]$ it follows that $t_\star = t_{\star\star}$, provided that $h \in (0, h_0]$ (for, else $t_\star$ would *not* be the largest real number in $(0, t_{\star\star}]$ such that $\|\xi(t)\|_{1,h}^2 \leq 1$ for all $t \in [0, t_\star]$).

Now, since $\|\xi(t)\|_{1,h}^2 < 1$ for all $t \in [0, t_{\star\star}]$, and hence

$$\limsup_{t \to t_{\star\star}} \|\xi(t)\|_{1,h} \leq 1,$$

it follows by the definition of ξ and the triangle inequality that

$$\limsup_{t \to t_{\star\star}} \|u_{\mathrm{DG}}(t)\|_{1,h} \leq 1 + \limsup_{t \to t_{\star\star}} \|\Pi u(t)\|_{1,h} \leq Const. \, .$$

Therefore $t_{\star\star}$ cannot be strictly smaller than T (for, if it were, then we would have that $\limsup_{t\to t_{\star\star}} \|u_{\mathrm{DG}}\|_{1,h} = +\infty$).

To summarise, we have shown that, for $h \leq h_0$, u_{DG} exists on the whole of the interval $[0, T]$ and $\|\xi(t)\|_{1,h} \leq 1$ for all $t \in [0, T]$.

From (3.24), by Gronwall's inequality, we then obtain

$$\|\xi(t)\|_{0,\Omega}^2+\int_0^t \|\xi(s)\|_{1,h}^2 \,\mathrm{d}s \le C \int_0^t \left\{\|\eta(s)\|_{0,2(\gamma+1),\Omega}^2 + \|\eta'(s)\|_{0,\Omega}^2\right\} \mathrm{d}s$$

and hence, for all $t \in [0, T]$,

$$\int_0^t \|\xi(s)\|_{1,h}^2 \,\mathrm{d}s \le C \int_0^t \left\{\|\eta(s)\|_{0,2(\gamma+1),\Omega}^2 + \|\eta'(s)\|_{0,\Omega}^2\right\} \mathrm{d}s,$$

with the constant $C > 0$ depending only on the domain Ω, the quasi-uniformity constant C_0, the final time T, the exponent γ in the growth condition for the function f, and on Lebesgue and Sobolev norms of u over the time interval $(0, T)$.

By the triangle inequality and the broken Sobolev–Poincaré inequality and (3.2), for all $t \in [0, T]$, we have

$$\int_0^t \|(u - u_{\mathrm{DG}})(s)\|_{1,h}^2 \,\mathrm{d}s \leq C \int_0^t \left\{ \|\eta(s)\|_{1,h}^2 + \|\eta'(s)\|_{0,\Omega}^2 \right\} \mathrm{d}s.$$

That completes the proof. $\square$

Using this yields the following error bound for the NSIP DGFEM.

Theorem

Suppose that $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded polyhedral domain with Lipschitz-continuous boundary, let $\{\mathcal{T}_h\}$ be a family of shape-regular and hp-quasi-uniform subdivisions of Ω, and let $\mathbf{p}$ be a polynomial degree vector of bounded local variation. Let each face $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$ be assigned a positive real number

$$\sigma_e = \{\!\{p^2\}\!\}_e / h_e,$$

where h_e is the diameter of e, with the convention that for $e \in \mathcal{E}_{\mathrm{D}}$ the contributions from outside Ω in the definition of σ_e are set to 0. Suppose that f satisfies Hypothesis A and $u \in \mathrm{L}^\infty(0, T; \mathrm{H}^1(\Omega))$.

Theorem (... continued)

Then, if $p_\kappa \geq 2$ *and* $u|_\kappa \in \mathrm{H}^1(0,T;\mathrm{H}^{k_\kappa}(\kappa))$ *with* $k_\kappa \geq 3\frac{1}{2}$ *on each* $\kappa \in \mathcal{T}_h$*, then* $\exists h_0 \in (0,1]$ *s.t.* $\forall h \in (0,h_0]$, $h = \max_{\kappa\in\mathcal{T}_h} h_\kappa$*, and* $\forall t \in [0,T]$*, the solution* $u_{\mathrm{DG}}(\,\cdot\,,t) \in S^{\mathbf{p}}(\Omega,\mathcal{T}_h,\mathbf{F})$ *of the NSIP DGFEM* (2.4) *satisfies the following error bound:*

$$\|u - u_{\mathrm{DG}}\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega,\mathcal{T}_h))} \leq C \sum_{\kappa\in\mathcal{T}_h} \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}} \|u\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^{k_\kappa}(\kappa))},$$

with $1 \leq s_\kappa \leq \min\{p_\kappa+1, k_\kappa\}$, $p_\kappa \geq 2$ *on each* $\kappa \in \mathcal{T}_h$*, where* C *is a positive constant depending only on the domain* Ω*, the shape-regularity and quasi-uniformity constants of* $\mathcal{T}_h$*, the final time* T*, the exponent* γ *in the growth condition for the function* f*, the parameter* ρ*, on* $k = \max_{\kappa\in\mathcal{T}_h} k_\kappa$*, and norms of* u *on* $(0,T)$.

Proof. As before, let us choose the projector Π to be the broken elliptic projector defined by (3.10), with $\theta = 1$. We then have that

$$\|\eta(s)\|_{1,h}^2 \le C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u(s)\|_{k_\kappa,\kappa}^2 \qquad \forall s \in [0, T],$$

with $1 \le s_\kappa \le \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \ge 1$ on each $\kappa \in \mathcal{T}_h$.

By differentiating (3.10) with respect to t we deduce that $B(u' - \Pi u', v) = 0$ for all $v \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$. Hence,

$$\|\eta'(s)\|_{1,h}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u'(s)\|_{k_\kappa,\kappa}^2 \qquad \forall s \in [0, T],$$

and with $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$ on each $\kappa \in \mathcal{T}_h$.

Thus, by the broken Sobolev–Poincaré inequality (3.3) and (3.2), an identical bound holds for the norm $\|\eta'(s)\|_{0,\Omega}$, for all $s \in [0, T]$.

Using these bounds in the r.h.s. of (3.18) for $t \in [0, T]$, we obtain the desired bound, with $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$ and $p_\kappa \geq 2$ on each $\kappa \in \mathcal{T}_h$, where C is a positive constant depending only on the domain Ω, the shape-regularity and quasi-uniformity constants of $\mathcal{T}_h$, the final time T, the exponent γ in the growth condition for the function f, the parameter ρ, on $k = \max_{\kappa \in \mathcal{T}_h} k_\kappa$, and on Lebesgue and Sobolev norms of u over the time interval $(0, T)$. $\square$

Remark.

When f is globally Lipschitz-continuous the assumptions a) and b) stated earlier are redundant as (3.24) holds automatically for all $t \in [0, T]$, and it is not necessary to separately prove that $\|\xi(t)\|_{1,h} \leq 1$ holds for all $t \in [0, T]$ and all h sufficiently small.

As we shall now see, a) and b) are also redundant for SIP DGFEM.

Error analysis of the symmetric version of DGFEM

The symmetric version of the interior penalty discontinuous Galerkin finite element method appeared in the literature much earlier than the nonsymmetric formulation; see:

M.F. Wheeler:
An elliptic collocation-finite element method with interior penalties, SIAM J. Numer. Anal., 15 (1978), 152–161.

It was not widely accepted as an effective numerical method until very recently, due to the additional condition on the minimum size of the penalty parameter, which is required in order to ensure the coercivity of the bilinear form of the method.

The renewed interest in SIP DGFEM for second-order elliptic problems can be attributed to the optimality of its convergence rate in the L^2 norm. Henceforth $\theta = -1$.

Theorem

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded polyhedral domain with Lipschitz boundary. Suppose that $\{\mathcal{T}_h\}$ is a family of shape-regular subdivisions of Ω, and $\mathbf{p}$ has bounded local variation. Let each face $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$ be assigned the positive real number

$$\sigma_e = C_\sigma \{\!\{p^2\}\!\}_e / h_e, \tag{3.26}$$

where $h_e = diam(e)$, with the convention that for $e \in \mathcal{E}_{\mathrm{D}}$ the contributions from outside Ω in the definition of σ_e are set to 0, and let C_σ be as in Hypothesis B. Let f satisfy Hypothesis A.

Then, if $u|_\kappa \in \mathrm{H}^1(0, T; \mathrm{H}^{k_\kappa}(\kappa))$, $k_\kappa \geq 2$, $\kappa \in \mathcal{T}_h$ and $u \in \mathrm{L}^\infty(0, T; \mathrm{H}^1(\Omega))$, there exists $h_0 \in (0, 1]$ such that, for all $h \in (0, h_0]$, $h = \max_{\kappa \in \mathcal{T}_h} h_\kappa$, and all $t \in [0, T]$, the solution $u_{\mathrm{DG}}(\cdot, t) \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ of the SIP DGFEM (2.4) satisfies:

Theorem (... continued)

$$\operatorname*{ess.sup}_{0\leq t\leq T}\|u(t)-u_{\mathrm{DG}}(t)\|_{1,h}^2\leq C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}}\|u\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^{k_\kappa}(\kappa))}, \tag{3.27}$$

with $1\leq s_\kappa\leq\min\{p_\kappa+1,k_\kappa\}$*,* $p_\kappa\geq 1$*, for* $\kappa\in\mathcal{T}_h$*, where* C *is a positive constant depending only on the domain* Ω*, the shape-regularity constant of* $\mathcal{T}_h$*, the final time* T*, the exponent* γ *in the growth condition for the function* f*, the parameter* ρ*, Lebesgue and Sobolev norms of* u *over the interval* $(0,T)$*, and* $k=\max_{\kappa\in\mathcal{T}_h}k_\kappa$*.*

Proof.

Let $t_{\star\star} \in (0, T]$ be such that $u_{\mathrm{DG}}(t) \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ exists for all $t \in [0, t_{\star\star})$. Again, the existence of such a $t_{\star\star}$ is ensured by Carathéodory's theorem. Thus, either:

- $t_{\star\star} = T$, or
- $t_{\star\star} < T$ and $\limsup_{t \to t_{\star\star}} \|u_{\mathrm{DG}}(t)\|_{1,h} = +\infty$.

We shall show that, for h sufficiently small, $t_{\star\star} = T$. Let us write

$$u - u_{\mathrm{DG}} = (u - \Pi u) + (\Pi u - u_{\mathrm{DG}}) =: \eta + \xi.$$

Subtract (3.19) from (3.20) and take $v = \xi'$; for a.e. $t \in (0, t_{\star\star})$:

$$\begin{aligned}\|\xi'\|_{0,\Omega}^2 + B_{\mathrm{S}}(\xi, \xi') &= \int_\Omega \{f(x, t, u) - f(x, t, u_{\mathrm{DG}})\}\, \xi' \,\mathrm{d}x \\ &\quad - \int_\Omega \eta' \xi' \,\mathrm{d}x - B_{\mathrm{S}}(\eta, \xi').\end{aligned}$$

By (3.10), $B_{\mathrm{S}}(\eta, \xi') = 0$. With the constant C_σ in (3.26) chosen large enough, $B_{\mathrm{S}}(\,\cdot\,,\,\cdot\,)$ is coercive, and therefore defines an inner product on $\mathrm{H}^1(\Omega, \mathcal{T}_h)$, which induces the norm $\|\cdot\|_B$ on this space. Hence $B_{\mathrm{S}}(\xi, \xi') = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_B^2$. We thus infer that

$$
\begin{aligned}
\|\xi'\|_{0,\Omega}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_B^2 \leq \left|\int_\Omega \eta'\xi' \,\mathrm{d}x\right| + \left|\int_\Omega \{f(x,t,u) - f(x,t,\Pi u)\}\,\xi' \,\mathrm{d}x\right| \\
+ \left|\int_\Omega \{f(x,t,\Pi u) - f(x,t,u_{\mathrm{DG}})\}\,\xi' \,\mathrm{d}x\right| \quad (3.28)
\end{aligned}
$$

for a.e. $t \in (0, t_{\star\star})$.

By the Cauchy–Schwarz inequality and Cauchy's inequality:

$$\left|\int_\Omega \eta'\xi'\,\mathrm{d}x\right| \le \|\eta'\|_{0,\Omega}\|\xi'\|_{0,\Omega} \le \frac{\varepsilon_1}{2}\|\eta'\|_{0,\Omega}^2 + \frac{1}{2\varepsilon_1}\|\xi'\|_{0,\Omega}^2,$$

and, similarly,

$$\begin{aligned}\left|\int_\Omega \{f(x,t,u) - f(x,t,\Pi u)\}\,\xi'\,\mathrm{d}x\right| \\ \le \frac{\varepsilon_2}{2}\|f(\,\cdot\,,t,u) - f(\,\cdot\,,t,\Pi u)\|_{0,\Omega}^2 + \frac{1}{2\varepsilon_2}\|\xi'\|_{0,\Omega}^2,\end{aligned}$$

and

$$
\begin{aligned}
\left| \int_\Omega \{f(x,t,\Pi u) - f(x,t,u_{\mathrm{DG}})\}\, \xi' \,\mathrm{d}x \right| \\
&\le \frac{\varepsilon_3}{2} \| f(\,\cdot\,,t,\Pi u) - f(\,\cdot\,,t,u_{\mathrm{DG}}) \|_{0,\Omega}^2 + \frac{1}{2\varepsilon_3} \|\xi'\|_{0,\Omega}^2,
\end{aligned}
$$

with $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$. Also, as before, for a.e. $t \in [0,T]$:

$$
\| f(\,\cdot\,,t,u) - f(\,\cdot\,,t,\Pi u) \|_{0,\Omega}^2 \le C \|\eta(t)\|_{0,2(\gamma+1),\Omega}^2,
$$

where the constant $C > 0$ depends only on the domain Ω, the exponent γ in the growth condition for the function f, and on Lebesgue and Sobolev norms of u over the time interval $(0,T)$.

Choosing $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that $\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_3^{-1} \le 2$, and inserting the above bounds into (3.28), we obtain for all $t \in (0, t_{\star\star})$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_B^2 \le C_1 \left(\|\eta\|^2_{0,2(\gamma+1),\Omega} + \|\eta'\|^2_{0,\Omega}\right) \\ + \tilde{C}_2\|f(\,\cdot, t, \Pi u) - f(\,\cdot\,, t, u_{\mathrm{DG}})\|^2_{0,\Omega}. \tag{3.29}$$

To bound $\|f(\,\cdot\,, t, \Pi u) - f(\,\cdot\,, t, u_{\mathrm{DG}})\|^2_{0,\Omega}$ we note that, by the same argument as above, for a.e. $t \in (0, t_{\star\star})$, we have

$$
\begin{aligned}
&\|f(\,\cdot\,,t,\Pi u) - f(\,\cdot\,,t,u_{\mathrm{DG}})\|_{0,\Omega}^2 \\
&\qquad \le C\|\xi(t)\|_{0,2(\gamma+1),\Omega}^2 \left(1 + \|\xi(t)\|_{0,2(\gamma+1),\Omega}^{2\gamma}\right),
\end{aligned}
$$

where the constant $C > 0$ depends only on the domain Ω, the exponent γ in the growth condition for the function f, and on Lebesgue and Sobolev norms of u over the time interval $(0, T)$.

For $\mathcal{T}_h$ and the polynomial degree vector $\mathbf{p}$ fixed, let $t_\star = t_\star(\mathcal{T}_h, \mathbf{p})$ denote the largest time $t \in (0, t_{\star\star}]$ such that $u_{\mathrm{DG}}(t)$ exists for all $t \in [0, t_\star]$ and $\|\xi(t)\|_{1,h} \leq 1$ for all $t \in [0, t_\star]$.

The existence of such a $t_\star$ is guaranteed by the definition of $t_{\star\star}$, together with the fact that $t \mapsto \|\xi(t)\|_{1,h}$ is continuous in the neighbourhood of $t = 0$ and $\|\xi(0)\|_{1,h} = 0$.

By the broken Sobolev–Poincaré inequality (3.3) and (3.2), for a.e. $t \in (0, t_\star]$, we have that

$$\|f(\,\cdot\,, t, \Pi u) - f(\,\cdot\,, t, u_{\mathrm{DG}})\|_{0,\Omega}^2 \leq C\|\xi(t)\|_{1,h}^2.$$

Inserting this into (3.29), integrating from 0 to $t \leq t_\star$, noting that $c_0\|\xi(t)\|_{1,h}^2 \leq \|\xi(t)\|_B^2$ and $\xi(0) = 0$, we have for all $t \in [0, t_\star]$:

$$c_0\|\xi(t)\|_{1,h}^2 \leq C_1 \int_0^t \left\{\|\eta(s)\|_{0,2(\gamma+1),\Omega}^2 \|\eta'(s)\|_{0,\Omega}^2\right\} \mathrm{d}s + C_2 \int_0^t \|\xi(s)\|_{1,h}^2 \, \mathrm{d}s. \tag{3.30}$$

The first integral on the r.h.s. can be bounded in terms of h_κ and p_κ. Define $h = \max_{\kappa \in \mathcal{T}_h} h_\kappa$, $C_3 = C_2/c_0$, and let $h_0 \in (0, 1]$ be small enough so that for all $h \leq h_0$ and all $t \in [0, t_\star]$ we have

$$C_1 \int_0^t \left\{\|\eta(s)\|_{0,2(\gamma+1),\Omega}^2 + \|\eta'(s)\|_{0,\Omega}^2\right\} \mathrm{d}s < \frac{c_0}{1+T} \, \mathrm{e}^{-C_3 T}.$$

Thus, for $h \leq h_0$ and all $t \in [0, t_\star]$, from (3.30) we have that

$$\|\xi(t)\|_{1,h}^2 < \frac{1}{1+T}\,\mathrm{e}^{-C_3 T} + C_3 \int_0^t \|\xi(s)\|_{1,h}^2 \,\mathrm{d}s;$$

using Gronwall's inequality, we deduce that $\|\xi(t)\|_{1,h}^2 < 1$ for all $t \in [0, t_\star]$ with $h \leq h_0$.

By a continuity argument, as before, applied to the mapping $t \mapsto \|\xi(t)\|_{1,h}^2$, we deduce that $t_\star = t_{\star\star} = T$ for all $h \in (0, h_0]$.

For $h \leq h_0$ by Gronwall's inequality and (3.30):

$$\|\xi(t)\|_{1,h}^2 \leq C \int_0^t \left\{ \|\eta(s)\|_{0,2(\gamma+1),\Omega}^2 + \|\eta'(s)\|_{0,\Omega}^2 \right\} \mathrm{d}s \qquad \forall t \in [0, T],$$

where $C > 0$ depends only on Ω, the exponent γ, the final time T, and on Lebesgue and Sobolev norms of u over $(0, T)$.

Further, by the broken Sobolev–Poincaré inequality (3.3) and (3.2), we have that $\|\eta\|^2_{0,2(\gamma+1),\Omega} \leq C\|\eta\|^2_{1,h}$; by the triangle inequality,

$$\|(u - u_{\mathrm{DG}})(t)\|^2_{1,h} \leq C\left(\|\eta(t)\|^2_{1,h} + \int_0^t \left\{\|\eta(s)\|^2_{1,h} + \|\eta'(s)\|^2_{0,\Omega}\right\} \mathrm{d}s\right),$$

for all $t \in [0, T]$.

As in the proof of the previous Theorem, to bound $\|\eta'(s)\|_{0,\Omega}^2$, and noting the embedding $\mathrm{H}^1(0,T;\mathrm{H}^{k_\kappa}(\kappa)) \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{H}^{k_\kappa}(\kappa))$ yields (3.27), with $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}_h$, where $C > 0$ depends only on Ω, the shape-regularity constant of $\mathcal{T}_h$, the final time T, the parameter ρ, the exponent γ, on $k = \max_{\kappa \in \mathcal{T}_h} k_\kappa$, and Lebesgue and Sobolev norms of u on $(0,T)$.

$\square$

Theorem

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded convex polyhedral domain with Lipschitz boundary. Suppose that Γ_{N} is empty, $\{\mathcal{T}_h\}$ is a family of shape-regular subdivisions of Ω, and $\mathbf{p}$ has bounded local variation. Let each face $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}}$ be assigned the positive real number

$$\sigma_e = C_\sigma \{\!\{p^2\}\!\}_e / h_e,$$

where $h_e = diam(e)$, with the convention that for $e \in \mathcal{E}_{\mathrm{D}}$ the contributions from outside Ω in the definition of σ_e are set to 0, and C_σ is as in Hypothesis B. *Let f satisfy* Hypothesis A.

Then, if $u|_\kappa \in \mathrm{H}^1(0, T; \mathrm{H}^{k_\kappa}(\kappa))$, $k_\kappa \geq 2$, $\kappa \in \mathcal{T}_h$ and $u \in \mathrm{L}^\infty(0, T; \mathrm{H}^1(\Omega) \cap \mathrm{C}(\bar{\Omega}))$, there exists $h_0 \in (0, 1]$ such that for all $h \in (0, h_0]$, $h = \max_{\kappa \in \mathcal{T}_h} h_\kappa$, and all $t \in (0, T]$, the solution $u_{\mathrm{DG}}(\,\cdot\,, t) \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ of the SIP DGFEM (2.4) satisfies:

Theorem (... continued)

$$\|u - u_{\mathrm{DG}}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} \le C \left(\max_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^2}{p_\kappa} \right) \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^{k_\kappa}(\kappa))}$$

with $1 \le s_\kappa \le \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \ge 1$, *for* $\kappa \in \mathcal{T}_h$, *where* $C > 0$ *depends only on* Ω, *the shape-regularity constant of* $\mathcal{T}_h$, *the final time* T, *the exponent* γ, *the parameter* ρ, *on Lebesgue and Sobolev norms of* u *over* $(0, T)$, *and on* $k = \max_{\kappa \in \mathcal{T}_h} k_\kappa$.

Proof.

By the same argument as before, and with ξ and η as before:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_{0,\Omega}^2 + \|\xi\|_B^2 \leq \left|\int_\Omega \{f(x,t,u) - f(x,t,\Pi u)\}\,\xi\,\mathrm{d}x\right| + \left|\int_\Omega \{f(x,t,\Pi u) - f(x,t,u_{\mathrm{DG}})\}\,\xi\,\mathrm{d}x\right| + \left|\int_\Omega \eta'\xi\,\mathrm{d}x\right|,$$

for a.e. $t \in [0,T]$.

Applying (3.4) and the Cauchy–Schwarz inequality to the right-hand side of the above inequality gives, for a.e. $t \in [0, T]$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_{0,\Omega}^2 + \|\xi\|_B^2 \le C\|\eta\|_{0,\Omega}(1 + |||u|||_{1,h}^{\gamma} + |||\Pi u|||_{1,h}^{\gamma})\|\xi\|_{1,h}$$
$$+ C\|\xi\|_{0,\Omega}(1 + |||\Pi u|||_{1,h}^{\gamma} + |||u_{\mathrm{DG}}|||_{1,h}^{\gamma})\|\xi\|_{1,h} + \|\eta'\|_{0,\Omega}\|\xi\|_{0,\Omega}.$$

Let us show that

$$\operatorname*{ess.\,sup}_{0\le t\le T} |||\Pi u(t)|||_{1,h} \quad \text{and} \quad \operatorname*{ess.\,sup}_{0\le t\le T} |||u_{\mathrm{DG}}(t)|||_{1,h}$$

are bounded uniformly with respect to $h \in (0, h_0]$. We have that

$$\begin{aligned} |||\Pi u(t)|||_{1,h} &\le |||(\Pi u - u)(t)|||_{1,h} + |||u(t)|||_{1,h} \\ &= |||\eta(t)|||_{1,h} + |||u(t)|||_{1,h} \\ &\le \|\eta(t)\|_{1,h} + |||u(t)|||_{1,h} \le Const. \qquad \forall t \in [0, T], \end{aligned}$$

where $Const. > 0$ is independent of h, $\mathbf{p}$ and $t \in [0, T]$.

The last inequality follows from (3.11), the continuous embedding $\mathrm{H}^1(0, T; \mathrm{H}^{k_\kappa}(\kappa)) \hookrightarrow \mathrm{C}([0, T]; \mathrm{H}^{k_\kappa}(\kappa))$, recalling the definition of the norm $|||\cdot|||_{1,h}$, and noting that $u \in \mathrm{L}^\infty(0, T; \mathrm{H}^1(\Omega) \cap \mathrm{C}(\bar{\Omega}))$.

By the above and the fact that $\operatorname{ess.sup}_{0\le t\le T} \|\xi(t)\|^2_{1,h} \le 1$ uniformly in $h \le h_0$, we have that

$$\begin{aligned}
|||u_{\mathrm{DG}}(t)|||_{1,h} &\le |||(u_{\mathrm{DG}} - \Pi u)(t)|||_{1,h} + |||\Pi u(t)|||_{1,h} \\
&= |||\xi(t)|||_{1,h} + |||\Pi u(t)|||_{1,h} \\
&\le \|\xi(t)\|_{1,h} + |||\Pi u(t)|||_{1,h} \\
&\le \|\xi(t)\|_{1,h} + \|\eta(t)\|_{1,h} + |||u(t)|||_{1,h} \le Const.,
\end{aligned}$$

for all $t \in [0, T]$ and uniformly in $h \in (0, h_0]$, — again by (3.11) on observing that $\mathrm{H}^1(0, T; \mathrm{H}^{k_\kappa}(\kappa)) \hookrightarrow \mathrm{C}([0, T]; \mathrm{H}^{k_\kappa}(\kappa))$, the definition of $|||\cdot|||_{1,h}$ and that $u \in \mathrm{L}^\infty(0, T; \mathrm{H}^1(\Omega) \cap \mathrm{C}(\bar{\Omega}))$.

Once again, *Const.* denotes a positive constant, independent of the discretisation parameters and of $t \in [0, T]$. Hence we deduce that

$$\begin{aligned}
\operatorname*{ess.\,sup}_{0 \leq t \leq T} \left(1 + |||u(t)|||_{1,h}^{\gamma} + |||\Pi u(t)|||_{1,h}^{\gamma}\right) &\leq \textit{Const.}, \\
\operatorname*{ess.\,sup}_{0 \leq t \leq T} \left(1 + |||\Pi u(t)|||_{1,h}^{\gamma} + |||u_{\mathrm{DG}}(t)|||_{1,h}^{\gamma}\right) &\leq \textit{Const.},
\end{aligned}$$

uniformly in $h \in (0, h_0]$.

Therefore, by Cauchy's inequality,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_{0,\Omega}^2 + \|\xi\|_{1,h}^2 &\leq C\left(\|\eta\|_{0,\Omega}\|\xi\|_{1,h} + \|\xi\|_{0,\Omega}\|\xi\|_{1,h} + \|\eta'\|_{0,\Omega}\|\xi\|_{0,\Omega}\right)\\ &\leq \frac{1}{2}\|\xi\|_{1,h}^2 + \frac{1}{2}C^2\left(\|\eta\|_{0,\Omega}^2 + \|\eta'\|_{0,\Omega}^2 + \|\xi\|_{0,\Omega}^2\right),\end{aligned}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_{0,\Omega}^2 + \|\xi\|_{1,h}^2 \leq C^2\left(\|\eta\|_{0,\Omega}^2 + \|\eta'\|_{0,\Omega}^2 + \|\xi\|_{0,\Omega}^2\right).$$

By integrating from 0 to $t \in (0, T]$ and applying Gronwall's ineq.,

$$\|\xi(t)\|_{0,\Omega}^2 + \int_0^t \|\xi(s)\|_{1,h}^2 \,\mathrm{d}s \leq C \int_0^T \left\{ \|\eta(s)\|_{0,\Omega}^2 + \|\eta'(s)\|_{0,\Omega}^2 \right\} \mathrm{d}s. \tag{3.31}$$

Applying (3.12) to the right-hand side of (3.31), we deduce that

$$\|\xi(t)\|_{0,\Omega}^2 \leq C \left(\max_{\kappa \in \mathcal{T}_h} \frac{h_k^2}{p_\kappa} \right) \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{\mathrm{H}^1(0,T;\mathrm{H}^{k_\kappa}(\kappa))}^2 \qquad \forall t \in [0, T].$$

Finally, by the triangle inequality

$$\|u(t) - u_{\mathrm{DG}}(t)\|_{0,\Omega} \leq \|\eta(t)\|_{0,\Omega} + \|\xi(t)\|_{0,\Omega}$$

and applying (3.12) to $\|\eta(t)\|_{0,\Omega}$, we get the desired bound. $\square$

Remark

Suppose, for example, that $u \in \mathrm{H}^1(0, T; \mathrm{H}^k(\Omega))$, $k \geq 2$, *and* $p_\kappa = p$ *for all* $\kappa \in \mathcal{T}_h$; *let* $h = \max_{\kappa \in \mathcal{T}_h} h_\kappa$. *Then, we have:*

$$\frac{h}{p^{3/2}} \|u - u_{\mathrm{DG}}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega,\mathcal{T}_h))} + \frac{1}{p} \|u - u_{\mathrm{DG}}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} \leq C \frac{h^s}{p^k} \|u\|_{\mathrm{H}^1(0,T;\mathrm{H}^k(\Omega))},$$

where $C > 0$ *is as above,* $1 \leq s \leq \min\{p+1, k\}$, *and* $p \geq 1$.

Hence our error bounds are fully optimal in h; *the error bound* (3.27) *in the broken* H^1 *norm is suboptimal in* p *by half-a-power of* p, *while the error bound in the* L^2 *norm is suboptimal in* p *by a single power of the polynomial degree* p.

Ten Lectures on the Convergence Analysis of

Discontinuous Galerkin Finite Element Methods

for Nonlinear PDEs

Lecture 5

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Outline

- Problem Formulation
- *hp*-Discontinuous Galerkin (DG) FEM
- *hp*-A Priori Error Analysis
- Numerical Experiments

Problem Formulation

- Given: Polyhedron $\Omega \subset \mathbb{R}^d$, r.h.s. $f \in \mathrm{L}^2(\Omega)$, nonlinearity μ.
- PDE: Find a solution u of

$$-\nabla \cdot (\mu(x, |\nabla u|)\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma := \partial\Omega.$$

- Nonlinearity μ: Assume that
  - (A1) $\mu \in C(\overline{\Omega} \times [0, \infty))$;
  - (A2) there exist positive constants λ and Λ such that

$$\lambda\, |t - s|^2 \leq (\mu(x, |t|)t - \mu(x, |s|)s, t - s),$$

$$|\mu(x, |t|)t - \mu(x, |s|)s| \leq \Lambda\, |t - s|$$

  for all $t, s \in \mathbb{R}^d$ and $x \in \overline{\Omega}$.

Weak formulation

- $\mathrm{L}^2(\Omega)$ inner product: $(w, v) := \int_\Omega w(x)v(x)\,\mathrm{d}x$,
- $\mathrm{L}^2(\Omega)$ norm: $\|v\| := (v, v)^{1/2}$.

Find: $u \in \mathrm{H}^1_0(\Omega)$ such that

$$(\mu(\cdot, |\nabla u|)\nabla u, \nabla v) = (f, v) \quad \forall v \in \mathrm{H}^1_0(\Omega).$$

- Thanks to (A2), we have:

  1. Lipschitz continuity:

  $$\|\mu(\cdot, |\nabla w_1|)\nabla w_1 - \mu(\cdot, |\nabla w_2|)\nabla w_2\| \leq \Lambda \|\nabla(w_1 - w_2)\|$$

  for all $w_1, w_2 \in \mathrm{H}_0^1(\Omega)$;

  2. Strong monotonicity:

  $$\lambda \|\nabla(w_1 - w_2)\|^2 \leq (\mu(\cdot, |\nabla w_1|)\nabla w_1 - \mu(\cdot, |\nabla w_2|)\nabla w_2, \nabla(w_1 - w_2))$$

  for all $w_1, w_2 \in \mathrm{H}_0^1(\Omega)$.

Consequences:

1. Boundedness:

$$|(\mu(\cdot, |\nabla w|)\nabla w, \nabla v)| \leq \Lambda \|\nabla w\| \, \|\nabla v\|$$

for all $w, v \in \mathrm{H}_0^1(\Omega)$;

2. By Riesz Representation Theorem:
$\forall u \in \mathrm{H}_0^1(\Omega)$ there exists $Tu \in \mathrm{H}_0^1(\Omega)$ such that

$$(\mu(\cdot, |\nabla u|)\nabla u, \nabla v) = (\nabla Tu, \nabla v) \qquad \forall v \in \mathrm{H}_0^1(\Omega).$$

Existence and uniqueness

Proposition

Let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$, and let T be an operator from H into itself. Suppose that T is Lipschitz continuous, i.e. there exists $\Lambda > 0$ such that

$$\|T(\mathrm{w}_1) - T(\mathrm{w}_2)\|_H \leq \Lambda \|\mathrm{w}_1 - \mathrm{w}_2\|_H \qquad \forall \mathrm{w}_1, \mathrm{w}_2 \in H,$$

and strongly monotone, i.e. there exists $\lambda > 0$ such that

$$(T(\mathrm{w}_1) - T(\mathrm{w}_2), \mathrm{w}_1 - \mathrm{w}_2)_H \geq \lambda \|\mathrm{w}_1 - \mathrm{w}_2\|_H^2 \qquad \forall \mathrm{w}_1, \mathrm{w}_2 \in H.$$

Then, T is a bijection of H onto itself, and the inverse T^{-1} of T is Lipschitz continuous on H:

$$\|T^{-1}\mathrm{f} - T^{-1}\mathrm{g}\|_H \leq (\Lambda/\lambda)\|\mathrm{f} - \mathrm{g}\|_H \qquad \forall \mathrm{f}, \mathrm{g} \in H.$$

J. Nečas, Introd. to the Theory of Nonlin. Elliptic Eq., 1986.

Taking $H = \mathrm{H}^1_0(\Omega)$ and $\| \cdot \|_H := \|\nabla \cdot\|$ yields the following result.

Corollary

For $f \in \mathrm{L}^2(\Omega)$ and assuming (A1)–(A2) for the nonlinearity μ, there exists a unique solution u of the quasilinear PDE in $\mathrm{H}^1_0(\Omega)$.

Continuum mechanics

- More generally: Non-Newtonian fluids: $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$
- Example: Generalized power-law model

$$\begin{aligned} -\nabla \cdot (\mu(x, |\varepsilon(x)|)\varepsilon(\mathbf{u})) + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned}$$

S. Berrone and E. Süli
Two-sided a posteriori error bounds for incompressible quasi-Newtonian flows.
IMA J. Numer. Anal., 28(2):382–421, 2008.

Non-Newtonian fluids

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Jonathan Rothstein's Non-Newtonian Fluids Dynamics Lab, University of Massachusetts

We shall consider here the more general problem:

$$\begin{aligned} -\nabla \cdot (\mu(x, |\nabla u|)\nabla u) &= f && \text{in } \Omega, \\ u &= g_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}, \\ \mu(x, |\nabla u|)\frac{\partial u}{\partial \nu} &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}, \end{aligned}$$

where $f \in \mathrm{L}^2(\Omega)$, $g_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$ and $g_{\mathrm{N}} \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$.

Integration-by-parts formula

Consider a partition of Ω into a finite set $\mathcal{T} = \mathcal{T}_h$ of disjoint open Lipschitz domains $\kappa \subset \Omega$, the union of whose closures covers $\overline{\Omega}$.

$$\int_\kappa (\mu(x, |\nabla u|)\nabla u) \cdot \nabla v \,\mathrm{d}x - \int_{\partial\kappa} \mu(x, |\nabla u|) \frac{\partial u}{\partial \nu} \, v \,\mathrm{d}s = \int_\kappa f \, v \,\mathrm{d}x.$$

$$\begin{aligned}\sum_{\kappa \in \mathcal{T}} \int_\kappa (\mu(x, |\nabla u|)\nabla u) \cdot \nabla v \,\mathrm{d}x \quad &- \quad \sum_{\kappa \in \mathcal{T}} \int_{\partial\kappa} \mu(x, |\nabla u|) \frac{\partial u}{\partial \nu} \, v \,\mathrm{d}s \\ &= \quad \sum_{\kappa \in \mathcal{T}} \int_\kappa f \, v \,\mathrm{d}x.\end{aligned}$$

Transformation of the element-edge term

Observe that

$$
\begin{aligned}
-\sum_{\kappa\in\mathcal{T}}\int_{\partial\kappa}\mu(x,|\nabla u|)\frac{\partial u}{\partial\nu}\,v\,\mathrm{d}s \quad = \quad & -\sum_{\kappa\in\mathcal{T}}\int_{\partial\kappa\setminus\Gamma}\mu(x,|\nabla u|)\frac{\partial u}{\partial\nu}\,v\,\mathrm{d}s \\
& -\sum_{\kappa\in\mathcal{T}}\int_{\partial\kappa\cap\Gamma_{\mathrm{D}}}\mu(x,|\nabla u|)\frac{\partial u}{\partial\nu}\,v\,\mathrm{d}s \\
& -\sum_{\kappa\in\mathcal{T}}\int_{\partial\kappa\cap\Gamma_{\mathrm{N}}}\mu(x,|\nabla u|)\frac{\partial u}{\partial\nu}\,v\,\mathrm{d}s.
\end{aligned}
$$

Transformation of the element-edge term

Observe that

$$
\begin{aligned}
-\sum_{\kappa\in\mathcal{T}}\int_{\partial\kappa}\mu(x,|\nabla u|)\frac{\partial u}{\partial\nu}\,v\,\mathrm{d}s \quad = \quad & -\sum_{\kappa\in\mathcal{T}}\int_{\partial\kappa\setminus\Gamma}\mu(x,|\nabla u|)\frac{\partial u}{\partial\nu}\,v\,\mathrm{d}s \\
& -\sum_{\kappa\in\mathcal{T}}\int_{\partial\kappa\cap\Gamma_{\mathrm{D}}}\mu(x,|\nabla u|)\frac{\partial u}{\partial\nu}\,v\,\mathrm{d}s \\
& -\int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}}\,v\,\mathrm{d}s.
\end{aligned}
$$

hp-DGFEM

- *hp*-DG finite element space:

$$V_{DG} = \left\{ v \in \mathrm{L}^2(\Omega) : v|_\kappa \in \mathcal{R}_{p_\kappa}(\kappa), \kappa \in \mathcal{T} \right\},$$

  with $\mathcal{R} = \mathcal{P}$, or $\mathcal{R} = \mathcal{Q}$.

- Trace operators:

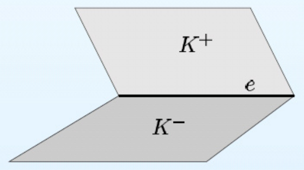
  Jump:

$$[\![v]\!] = v^+ \boldsymbol{\nu}^+ + v^- \boldsymbol{\nu}^-.$$

  Average:

$$\{\!\{v\}\!\} = \frac{1}{2}(v^+ + v^-).$$

jumps/averages



- DG-norm:

$$\|u\|_{DG}^2 = \|\nabla u\|_{\mathrm{L}^2(\Omega)}^2 + \alpha \int_{\Gamma_{\mathrm{int,D}}} \mathtt{h}^{-1} \mathtt{p}^2 |[\![u]\!]|^2 \, \mathrm{d}s.$$

Transformation of the element-edge term

Observe that [assuming that $u \in \mathrm{C}^1(\Omega)$]:

$$
\begin{aligned}
\sum_{\kappa \in \mathcal{T}} \int_{\partial\kappa \setminus \Gamma} \mu(x, |\nabla u|) \frac{\partial u}{\partial \nu} \, v \, \mathrm{d}s &= \sum_{\kappa \in \mathcal{T}} \int_{\partial\kappa \setminus \Gamma} \mu(x, |\nabla u|) \nabla u \cdot v \, \boldsymbol{\nu} \, \mathrm{d}s \\
&= \sum_{\kappa \in \mathcal{T}} \int_{\partial\kappa \setminus \Gamma} \{\!\{ \mu(x, |\nabla u|) \nabla u \}\!\} \cdot v \, \boldsymbol{\nu} \, \mathrm{d}s \\
&= \int_{\Gamma_{\mathrm{int}}} \{\!\{ \mu(x, |\nabla u|) \nabla u \}\!\} \cdot [\![v]\!] \, \mathrm{d}s,
\end{aligned}
$$

where $\Gamma_{\mathrm{int}} := \mathcal{E} \setminus \Gamma$. On the other hand,

$$
\sum_{\kappa \in \mathcal{T}} \int_{\partial\kappa \cap \Gamma_{\mathrm{D}}} \mu(x, |\nabla u|) \frac{\partial u}{\partial \nu} \, v \, \mathrm{d}s = \int_{\Gamma_{\mathrm{D}}} \mu(x, |\nabla u|) \nabla u \cdot v \, \boldsymbol{\nu} \, \mathrm{d}s.
$$

Forms

Semilinear form:

$$
\begin{aligned}
B(u, v) = \sum_{\kappa \in \mathcal{T}} \int_{\kappa} \mu(|\nabla u|) \nabla u \cdot \nabla v \,\mathrm{d}x \\
- \int_{\Gamma_{\mathrm{int}}} \{\!\{ \mu(|\nabla u|) \nabla u \}\!\} \cdot [\![v]\!] \,\mathrm{d}s - \int_{\Gamma_{\mathrm{D}}} \mu(|\nabla u|) \nabla u \cdot v \boldsymbol{\nu} \,\mathrm{d}s \\
+ \theta \int_{\Gamma_{\mathrm{int}}} \{\!\{ \mu(h^{-1} |[\![u]\!]|) \nabla v \}\!\} \cdot [\![u]\!] \,\mathrm{d}s + \theta \int_{\Gamma_{\mathrm{D}}} \mu(h^{-1} |u - g_{\mathrm{D}}|) \nabla v \cdot (u - g_{\mathrm{D}}) \boldsymbol{\nu} \,\mathrm{d}s \\
+ \int_{\Gamma_{\mathrm{int}}} \sigma \, [\![u]\!] \cdot [\![v]\!] \,\mathrm{d}s + \int_{\Gamma_{\mathrm{D}}} \sigma \, u \, v \,\mathrm{d}s.
\end{aligned}
$$

Method parameter: $\theta \in [-1, 1]$. *Penalty parameter:* $\sigma = \alpha \frac{\mathtt{p}^2}{\mathtt{h}}$.
Stability parameter: $\alpha > 0$ sufficiently large.

$$
\ell(v) := \int_{\Omega} f \, v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}} \, v \,\mathrm{d}s + \int_{\Gamma_{\mathrm{D}}} \sigma g_{\mathrm{D}} \, v \,\mathrm{d}s.
$$

hp-DGFEM

- Variational formulation (IP form): Find $u_{DG} \in V_{DG}$ such that

$$B(u_{DG}, v) = \ell(v) \qquad \forall v \in V_{DG}.$$

A priori results for the *hp*-DGFEM

Proposition (Existence and Uniqueness)

Let $f \in \mathrm{L}^2(\Omega)$ and assume that (A1)–(A2) hold. Then, for a sufficiently large stability parameter $\alpha > 0$, there exists a unique solution u_{DG} to the hp-DGFEM in V_{DG}.

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Preliminary results

Lemma

The semilinear form $B(\cdot,\cdot)$ is Lipschitz–continuous in its first argument in the sense that: $\exists C > 0$ such that

$$|B(w_1, v) - B(w_2, v)| \leq C\|w_1 - w_2\|_{DG}\|v\|_{DG}$$

for all $w_1, w_2, v \in V_{DG}$, and all $\theta \in [-1, 1]$.

Proof:

$$
\begin{aligned}
|B(w_1,v)-B(w_2,v)| \le &\sum_{\kappa\in\mathcal{T}}\int_\kappa |\mu(|\nabla w_1|)\nabla w_1-\mu(|\nabla w_2|)\nabla w_2|\;|\nabla v|\,\mathrm{d}x\\
&+\int_{\Gamma_{\mathrm{D}}} |\mu(|\nabla w_1|)\nabla w_1-\mu(|\nabla w_2|)\nabla w_2|\,|v|\,\mathrm{d}s\\
&+\int_{\Gamma_{\mathrm{int}}} \{\!\{|\mu(|\nabla w_1|)\nabla w_1-\mu(|\nabla w_2|)\nabla w_2|\}\!\}\,|[\![v]\!]|\,\mathrm{d}s\\
&+|\theta|\int_{\Gamma_{\mathrm{D}}} \left|\mu(h^{-1}|w_1-g_{\mathrm{D}}|)(w_1-g_{\mathrm{D}})-\mu(h^{-1}|w_2-g_{\mathrm{D}}|)(w_2-g_{\mathrm{D}})\right|\;|\nabla v|\;\mathrm{d}s\\
&+|\theta|\int_{\Gamma_{\mathrm{int}}} \left|\mu(h^{-1}|[\![w_1]\!]|)[\![w_1]\!]-\mu(h^{-1}|[\![w_2]\!]|)[\![w_2]\!]\right|\;\{\!\{|\nabla v|\}\!\}\,\mathrm{d}s\\
&+\int_{\Gamma_{\mathrm{D}}} \sigma|w_1-w_2|\,|v|\,\mathrm{d}s+\int_{\Gamma_{\mathrm{int}}} \sigma|[\![w_1-w_2]\!]|\,|[\![v]\!]|\,\mathrm{d}s\\
&\equiv \mathrm{T}_1+\mathrm{T}_2+\mathrm{T}_3+\mathrm{T}_4+\mathrm{T}_5+\mathrm{T}_6+\mathrm{T}_7.
\end{aligned}
$$

Writing $w = w_1 - w_2$, we now proceed in the following steps.

Step 1: Using the Lipschitz continuity of the nonlinearity and the Cauchy–Schwarz inequality,

$$\mathrm{T}_1 \leq \Lambda \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \, \mathrm{d}x \right)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla v|^2 \, \mathrm{d}x \right)^{1/2}.$$

For T_2, analogously, we have that

$$\mathrm{T}_2 \leq \Lambda \int_{\Gamma_{\mathrm{D}}} |\nabla w|\,|v| \,\mathrm{d}s \leq \Lambda \left(\int_{\Gamma_{\mathrm{D}}} \sigma^{-1} |\nabla w|^2 \,\mathrm{d}s \right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma |v|^2 \,\mathrm{d}s \right)^{1/2}.$$

Hence, using inverse inequalities and recalling the definition of the penalty parameter σ_e on $e \subset \Gamma_{\mathrm{D}}$, we have that

$$\mathrm{T}_2 \leq C \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla w|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma |v|^2 \,\mathrm{d}s \right)^{1/2}.$$

Analogously,

$$\begin{aligned} \mathrm{T}_3 &\leq \Lambda \int_{\Gamma_{\mathrm{int}}} \{\!\{|\nabla w|\}\!\} \, |[\![v]\!]| \, \mathrm{d}s \\ &\leq \Lambda \left(\int_{\Gamma_{\mathrm{int}}} \sigma^{-1} \{\!\{|\nabla w|\}\!\}^2 \, \mathrm{d}s \right)^{1/2} \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![v]\!]^2 \, \mathrm{d}s \right)^{1/2}. \end{aligned}$$

Let us write

$$\int_{\Gamma_{\mathrm{int}}} \sigma^{-1} \{\!\{|\nabla w|\}\!\}^2 \, \mathrm{d}s = \sum_{e \in \mathcal{E}_{\mathrm{int}}} \sigma_e^{-1} \int_e \{\!\{|\nabla w|\}\!\}^2 \, \mathrm{d}s.$$

For $e \in \mathcal{E}_{\mathrm{int}}$, let κ and κ' be the two elements that share e. Then,

$$\begin{aligned} \int_e \{\!\{|\nabla w|\}\!\}^2 \, \mathrm{d}s &\leq \frac{1}{2} \int_e |\nabla w|_\kappa|^2 \, \mathrm{d}s + \frac{1}{2} \int_e |\nabla w|_{\kappa'}|^2 \, \mathrm{d}s \\ &\leq C \frac{p_\kappa^2}{2h_e} \int_\kappa |\nabla w|^2 \, \mathrm{d}x + C \frac{p_{\kappa'}^2}{2h_e} \int_{\kappa'} |\nabla w|^2 \, \mathrm{d}x \\ &\leq C \frac{\{\!\{p^2\}\!\}|_e}{h_e} \max \left\{ \int_\kappa |\nabla w|^2 \, \mathrm{d}x, \int_{\kappa'} |\nabla w|^2 \, \mathrm{d}x \right\}. \end{aligned}$$

On recalling from the definition of σ that

$$\sigma_e = \alpha \frac{\{\!\{ p^2 \}\!\}|_e}{h_e} \qquad \text{for } e \in \mathcal{E}_{\mathrm{int}},$$

we have that

$$\sum_{e \in \mathcal{E}_{\mathrm{int}}} \sigma_e^{-1} \int_e \{\!\{ |\nabla w| \}\!\}^2 \, \mathrm{d}s \le C\alpha^{-1} \sum_{e \in \mathcal{E}_{\mathrm{int}}} \max_{\{\kappa \,:\, e \subset \partial\kappa\}} \int_\kappa |\nabla w|^2 \, \mathrm{d}x.$$

Thus,

$$\sum_{e \in \mathcal{E}_{\mathrm{int}}} \sigma_e^{-1} \int_e \{\!\{ |\nabla w| \}\!\}^2 \, \mathrm{d}s \le C \sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \, \mathrm{d}x,$$

and hence

$$\mathrm{T}_3 \quad \le \quad C \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![v]\!]^2 \, \mathrm{d}s \right)^{1/2}.$$

For T_4, analogously as for T_2 (only, exchanging v and w),

$$\mathrm{T}_4 \leq C\,|\theta| \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla v|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma |w|^2 \,\mathrm{d}s \right)^{1/2} .$$

For T_5, in the same way as for T_3 (only, exchanging v and w),

$$\mathrm{T}_5 \quad \leq \quad C\,|\theta| \left(\sum_{\kappa \in \mathcal{T}} \int_{\kappa} |\nabla v|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![w]\!]^2 \,\mathrm{d}s \right)^{1/2} .$$

Step 2: The terms T_6 and T_7 may be bounded by a straightforward application of the Cauchy–Schwarz inequality as follows:

$$\mathrm{T}_6 \leq \left(\int_{\Gamma_{\mathrm{D}}} \sigma |w|^2 \, \mathrm{d}s \right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma |v|^2 \, \mathrm{d}s \right)^{1/2},$$

$$\mathrm{T}_7 \leq \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![w]\!]^2 \, \mathrm{d}s \right)^{1/2} \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![v]\!]^2 \, \mathrm{d}s \right)^{1/2}.$$

Step 3: Collecting the bounds on $\mathrm{T}_1, \ldots, \mathrm{T}_7$ and recalling that $w = w_1 - w_2$ we deduce the statement of the lemma. $\square$

Lemma

Let $\theta \in [-1, 1]$ *and* $\alpha \geq C_0\,(1 + |\theta|)^2$ *where* $C_0 = C_0(\Lambda, \lambda)$ *is a certain fixed positive constant; then, the semilinear form* $B(\cdot, \cdot)$ *is strongly monotone in the sense that*

$$B(w_1, w_1 - w_2) - B(w_2, w_1 - w_2) \geq \frac{1}{2}\min\{\lambda, 1\}\|w_1 - w_2\|_{DG}^2,$$

for all $w_1, w_2 \in V_{DG}$.

Proof:

By writing $w = w_1 - w_2$, we have that

$$
\begin{aligned}
&B(w_1, w_1 - w_2) - B(w_2, w_1 - w_2) \\
&\qquad \geq \lambda \sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \,\mathrm{d}x \\
&\qquad\qquad - \Lambda(1 + |\theta|) \left(\int_{\Gamma_{\mathrm{D}}} \sigma^{-1} |\nabla w|^2 \,\mathrm{d}s \right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma |w|^2 \,\mathrm{d}s \right)^{1/2} \\
&\qquad\qquad + \int_{\Gamma_{\mathrm{D}}} \sigma |w|^2 \,\mathrm{d}s \\
&\qquad\qquad - \Lambda(1 + |\theta|) \left(\int_{\Gamma_{\mathrm{int}}} \sigma^{-1} \{\!\{ |\nabla w| \}\!\}^2 \,\mathrm{d}s \right)^{1/2} \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![w]\!]^2 \,\mathrm{d}s \right)^{1/2} \\
&\qquad\qquad + \int_{\Gamma_{\mathrm{int}}} \sigma [\![w]\!]^2 \,\mathrm{d}s.
\end{aligned}
$$

In exactly the same way as in the case of terms T_2 and T_3 in the proof of Lemma 1, we have

$$\left(\int_{\Gamma_{\mathrm{D}}} \sigma^{-1}|\nabla w|^2 \, \mathrm{d}s\right)^{1/2} \leq C \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \, \mathrm{d}x\right)^{1/2}$$

and

$$\left(\int_{\Gamma_{\mathrm{int}}} \sigma^{-1}\{\!\{|\nabla w|\}\!\}^2 \, \mathrm{d}s\right)^{1/2} \leq C \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \, \mathrm{d}x\right)^{1/2}.$$

Thus, we have that:

$$
\begin{aligned}
&B(w_1, w_1 - w_2) - B(w_2, w_1 - w_2) \geq \lambda \sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \,\mathrm{d}x \\
&- \frac{C\,(1 + |\theta|)}{\sqrt{\alpha}} \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma |w|^2 \,\mathrm{d}s \right)^{1/2} + \int_{\Gamma_{\mathrm{D}}} \sigma |w|^2 \,\mathrm{d}s \\
&- \frac{C\,(1 + |\theta|)}{\sqrt{\alpha}} \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![w]\!]^2 \,\mathrm{d}s \right)^{1/2} + \int_{\Gamma_{\mathrm{int}}} \sigma [\![w]\!]^2 \,\mathrm{d}s.
\end{aligned}
$$

Applying Cauchy's inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ to the second and fourth terms on the right–hand side, and writing $w = w_1 - w_2$:

$$B(w_1, w) - B(w_2, w) \geq \lambda \left(1 - \left(C\,(1 + |\theta|)^2/\alpha\right)\right) \sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \,\mathrm{d}x + \frac{1}{2} \int_{\Gamma_{\mathrm{D}}} \sigma |w|^2 \,\mathrm{d}s + \frac{1}{2} \int_{\Gamma_{\mathrm{int}}} \sigma [\![w]\!]^2 \,\mathrm{d}s.$$

The result follows by taking $\alpha \geq C_0(1 + |\theta|)^2$, where $C_0 = 2C$. $\square$

A priori results for the *hp*-DGFEM

Theorem (A Priori Error Bound)

Assume that the exact solution $u \in \mathrm{C}^1(\Omega)$*, and* $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$*,* $\kappa \in \mathcal{T}$*. Then, the following hp-a priori error estimate holds:*

$$\|u - u_{DG}\|_{DG}^2 \leq C \sum_{\kappa \in \mathcal{T}} \frac{h^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{\mathrm{H}^{k_\kappa}(\kappa)}^2,$$

with $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$*,* $p_\kappa \geq 1$*;* $C > 0$ *is a constant independent of the element sizes and the polynomial degrees.*

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Proof:

Lemma (hp-Approximation Result – Babuška & Suri (1987))

Let $\kappa \in \mathcal{T}$ *be a* d*–simplex or* d*–parallelepiped of diameter* h_κ*. Let* $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$*,* $k_\kappa \geq 0$*, for* $\kappa \in \mathcal{T}$*. Then, there exists a sequence* $z_{p_\kappa}^{h_\kappa}$ *in* $\mathcal{R}_{p_\kappa}(\kappa)$*,* $p_\kappa = 1, 2, \ldots,$ *such that for* $0 \leq q \leq k_\kappa$*,*

$$\|u - z_{p_\kappa}^{h_\kappa}\|_{\mathrm{H}^q(\kappa)} \leq C \frac{h_\kappa^{s_\kappa - q}}{p_\kappa^{k_\kappa - q}} \|u\|_{\mathrm{H}^{k_\kappa}(\kappa)},$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$*,* $p_\kappa \geq 1$*, for* $\kappa \in \mathcal{T}$*, and* C *is a positive constant, independent of* u*,* $\mathtt{h}$ *and* $\mathtt{p}$*.*

We now apply this Lemma.

Given $u \in \mathrm{H}^2(\Omega, \mathcal{T})$, we now define $\Pi_p^h u \in V_{DG}$ by

$$(\Pi_p^h u)|_\kappa = z_{p_\kappa}^{h_\kappa}(u|_\kappa).$$

Then, assuming that $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, for $\kappa \in \mathcal{T}$, and writing

$$\eta = u - \Pi_p^h u,$$

the above Lemma gives:

$$\|\eta\|_{\mathrm{L}^2(\kappa)}^2 \leq C \frac{h_\kappa^{2s_\kappa}}{p_\kappa^{2k_\kappa}} \|u\|_{\mathrm{H}^{k_\kappa}(\kappa)}^2 \quad \text{and} \quad \|\nabla\eta\|_{\mathrm{L}^2(\kappa)}^2 \leq C \frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-2}} \|u\|_{\mathrm{H}^{k_\kappa}(\kappa)}^2,$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u, $\mathtt{h}$ and $\mathtt{p}$.

The multiplicative trace inequality asserts the existence of a positive constant $C = C(d)$ such that

$$\|\eta\|^2_{\mathrm{L}^2(\partial\kappa)} \leq C(d)\left(\|\eta\|_{\mathrm{L}^2(\kappa)}\|\nabla\eta\|_{\mathrm{L}^2(\kappa)} + h_\kappa^{-1}\|\eta\|^2_{\mathrm{L}^2(\kappa)}\right).$$

Hence,

$$\|\eta\|^2_{\mathrm{L}^2(\partial\kappa)} \leq C\frac{h_\kappa^{2s_\kappa-1}}{p_\kappa^{2k_\kappa-1}}\|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)} \quad \text{and} \quad \|\nabla\eta\|^2_{\mathrm{L}^2(\partial\kappa)} \leq C\frac{h_\kappa^{2s_\kappa-3}}{p_\kappa^{2k_\kappa-3}}\|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)}$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u, $\mathtt{h}$ and $\mathtt{p}$.

We assume that $\mathcal{T}$ is shape-regular and that the polynomial degree vector $\mathtt{p}$ has bounded local variation. Hence,

$$\sum_{\kappa\in\mathcal{T}}\int_\kappa |\nabla\eta|^2\,\mathrm{d}x + \int_{\Gamma_{\mathrm{D}}}\sigma^{-1}|\nabla\eta|^2\,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}}\sigma^{-1}\{\!\{|\nabla\eta|\}\!\}^2\,\mathrm{d}s$$
$$+\int_{\Gamma_{\mathrm{D}}}\sigma\eta^2\,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}}\sigma[\![\eta]\!]^2\,\mathrm{d}s \le C\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}}\|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)}.$$

Thus,

$$\|\eta\|_{DG} = \left(\sum_{\kappa\in\mathcal{T}}\int_\kappa |\nabla\eta|^2\,\mathrm{d}x + \int_{\Gamma_{\mathrm{D}}}\sigma\eta^2\,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}}\sigma[\![\eta]\!]^2\,\mathrm{d}s\right)^{1/2}$$
$$\le C\left(\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa^{2s_\kappa-2}}{p_\kappa^{2k_\kappa-3}}\|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)}\right)^{1/2},\ 1\le s_\kappa\le\min\{p_\kappa+1,k_\kappa\},$$

$p_\kappa\ge 1$, for $\kappa\in\mathcal{T}$, and $C>0$ is independent of u, $\mathtt{h}$ and $\mathtt{p}$.

Consistency of the semilinear form $B(\cdot, \cdot)$

Lemma

Let $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, for $\kappa \in \mathcal{T}$, $\theta \in [-1, 1]$ and $\alpha \geq C_0 (1 + |\theta|)^2$. Then, for all $v \in V_{DG}$,

$$|B(u, v) - B(\Pi_p^h u, v)| \leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)} \right)^{1/2} \|v\|_{DG},$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u, v, $\mathtt{h}$ and $\mathtt{p}$.

Proof:

We proceed in the same way as in the proof of the previous lemma:

$$
\begin{aligned}
|B(u,v) - B(\Pi_p^h u, v)| \leq & \sum_{\kappa \in \mathcal{T}} \int_\kappa \left| \mu(|\nabla u|)\nabla u - \mu(|\nabla \Pi_p^h u|)\nabla \Pi_p^h u \right| \, |\nabla v| \,\mathrm{d}x \\
& + \int_{\Gamma_{\mathrm{D}}} |\mu(|\nabla u|)\nabla u - \mu(|\nabla \Pi_p^h u|)\nabla \Pi_p^h u| \, |v| \,\mathrm{d}s \\
& + \int_{\Gamma_{\mathrm{int}}} \{\!\{ |\mu(|\nabla u|)\nabla u - \mu(|\nabla \Pi_p^h u|)\nabla \Pi_p^h u| \}\!\} \, |[\![v]\!]| \,\mathrm{d}s \\
& + |\theta| \int_{\Gamma_{\mathrm{D}}} \left| \mu(h^{-1}|u - g_{\mathrm{D}}|)(u - g_{\mathrm{D}}) - \mu(h^{-1}|\Pi_p^h u - g_{\mathrm{D}}|)(\Pi_p^h u - g_{\mathrm{D}}) \right| \, |\nabla v| \,\mathrm{d}s \\
& + |\theta| \int_{\Gamma_{\mathrm{int}}} \left| \mu(h^{-1}|[\![u]\!]|)[\![u]\!] - \mu(h^{-1}|[\![\Pi_p^h u]\!]|)[\![\Pi_p^h u]\!] \right| \, \{\!\{ |\nabla v| \}\!\} \,\mathrm{d}s \\
& + \int_{\Gamma_{\mathrm{D}}} \sigma |u - \Pi_p^h u| \, |v| \,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \sigma |[\![u - \Pi_p^h u]\!]| \, |[\![v]\!]| \,\mathrm{d}s \\
& \equiv \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3 + \mathrm{T}_4 + \mathrm{T}_5 + \mathrm{T}_6 + \mathrm{T}_7.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{T}_1 &\leq \Lambda \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla (u - \Pi_p^h u)|^2 \, \mathrm{d}x \right)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla v|^2 \, \mathrm{d}x \right)^{1/2} \\
&\leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 2}} \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)} \right)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla v|^2 \, \mathrm{d}x \right)^{1/2} \qquad \forall v \in V_{DG},
\end{aligned}
$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and $C > 0$ is independent of u, v, $\mathtt{h}$ and $\mathtt{p}$.

$$
\begin{aligned}
\mathrm{T}_2 &\leq \Lambda \int_{\Gamma_{\mathrm{D}}} |\nabla(u - \Pi_p^h u)| \, |v| \, \mathrm{d}s \\
&\leq \Lambda \left(\int_{\Gamma_{\mathrm{D}}} \sigma^{-1} |\nabla(u - \Pi_p^h u)|^2 \, \mathrm{d}s \right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma |v|^2 \, \mathrm{d}s \right)^{1/2} .
\end{aligned}
$$

Hence, we have that

$$
\mathrm{T}_2 \leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 1}} \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)} \right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma |v|^2 \, \mathrm{d}s \right)^{1/2} \quad \forall v \in V_{DG},
$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u, v, $\mathtt{h}$ and $\mathtt{p}$.

Analogously,

$$\begin{aligned}
\mathrm{T}_3 &\leq \Lambda \int_{\Gamma_{\mathrm{int}}} \{\!\{|\nabla(u - \Pi_p^h u)|\}\!\}\, |[\![v]\!]| \,\mathrm{d}s \\
&\leq \Lambda \left(\int_{\Gamma_{\mathrm{int}}} \sigma^{-1} \{\!\{|\nabla(u - \Pi_p^h u)|\}\!\}^2 \,\mathrm{d}s \right)^{1/2} \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![v]\!]^2 \,\mathrm{d}s \right)^{1/2} \\
&\leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 1}} \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)} \right)^{1/2} \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![v]\!]^2 \,\mathrm{d}s \right)^{1/2} \quad \forall v \in V_{DG},
\end{aligned}$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and $C > 0$ is independent of u, v, $\mathtt{h}$ and $\mathtt{p}$.

For T_4,

$$\begin{aligned} \mathrm{T}_4 &\leq |\theta|\,\Lambda \int_{\Gamma_{\mathrm{D}}} |u - \Pi_p^h u|\,|\nabla v|\,\mathrm{d}s \\ &\leq |\theta|\,\Lambda \left(\int_{\Gamma_{\mathrm{D}}} \sigma |u - \Pi_p^h u|^2\,\mathrm{d}s\right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma^{-1} |\nabla v|^2\,\mathrm{d}s\right)^{1/2}. \end{aligned}$$

By an inverse inequality in the second factor on the right-hand side and the definition of the penalty parameter σ_e on $e \subset \Gamma_{\mathrm{D}}$,

$$\mathrm{T}_4 \leq |\theta| C \left(\sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)}\right)^{1/2} \left(\sum_{\kappa\in\mathcal{T}} \int_\kappa |\nabla v|^2\,\mathrm{d}x\right)^{1/2} \quad \forall v \in V_{DG},$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u, v, $\mathtt{h}$ and $\mathtt{p}$.

Analogously,

$$\mathrm{T}_5 \leq |\theta| C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)} \right)^{1/2} \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla v|^2 \,\mathrm{d}x \right)^{1/2} \quad \forall v \in V_{DG},$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and $C > 0$ is independent of u, v, $\mathtt{h}$, $\mathtt{p}$.

Similarly,

$$
\begin{aligned}
\mathrm{T}_6 &\leq \left(\int_{\Gamma_{\mathrm{D}}} \sigma |u - \Pi_p^h u|^2 \,\mathrm{d}s\right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma |v|^2 \,\mathrm{d}s\right)^{1/2} \\
&\leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)}\right)^{1/2} \left(\int_{\Gamma_{\mathrm{D}}} \sigma |v|^2 \,\mathrm{d}s\right)^{1/2} \quad \forall v \in V_{DG},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{T}_7 &\leq \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![u - \Pi_p^h u]\!]^2 \,\mathrm{d}s\right)^{1/2} \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![v]\!]^2 \,\mathrm{d}s\right)^{1/2} \\
&\leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)}\right)^{1/2} \left(\int_{\Gamma_{\mathrm{int}}} \sigma [\![v]\!]^2 \,\mathrm{d}s\right)^{1/2} \quad \forall v \in V_{DG},
\end{aligned}
$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and $C > 0$ is independent of u, v, $\mathtt{h}$ and $\mathtt{p}$. Collecting the bounds on $\mathrm{T}_1, \ldots, \mathrm{T}_7$ completes the proof. $\square$

Main theorem

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain, $\mathcal{T} = \{\kappa\}$ a shape-regular subdivision of Ω into d–simplexes or d–parallelepipeds, and $\mathtt{p}$ a polynomial degree vector of bounded local variation.

Let $\theta \in [-1, 1]$, $\alpha \geq C_0(1 + |\theta|)^2$. Assuming that $u \in \mathrm{C}^1(\Omega)$ and $u|_\kappa \in \mathrm{H}^{k_\kappa}(\kappa)$, $k_\kappa \geq 2$, for $\kappa \in \mathcal{T}$, $u_{DG} \in V_{DG}$ satisfies:

$$\|u - u_{DG}\|_{DG}^2 \leq C \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{\mathrm{H}^{k_\kappa}(\kappa)}^2,$$

with $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant independent of u, $\mathtt{h}$ and $\mathtt{p}$.

Proof:

Let us write

$$u - u_{DG} = (u - \Pi_p^h u) + (\Pi_p^h u - u_{DG}) \equiv \eta + \xi.$$

As $u \in \mathrm{C}^1(\Omega) \cap \mathrm{H}^2(\Omega, \mathcal{T})$, we have that $B(u, v) = \ell(v)$ for all v in V_{DG}; in particular, $B(u, \xi) = \ell(\xi)$. We begin by estimating ξ:

$$\begin{aligned}
\frac{1}{2}\min\{\lambda, 1\}\|\xi\|_{DG}^2 &= \frac{1}{2}\min\{\lambda, 1\}\|\Pi_p^h u - u_{DG}\|_{DG}^2 \\
&\leq B(\Pi_p^h u, \Pi_p^h u - u_{DG}) - B(u_{DG}, \Pi_p^h u - u_{DG}) \\
&= B(\Pi_p^h u, \xi) - \ell(\xi) = B(\Pi_p^h u, \xi) - B(u, \xi) \\
&\leq C\left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|_{\mathrm{H}^{k_\kappa}(\kappa)}^2\right)^{1/2} \|\xi\|_{DG}.
\end{aligned}$$

Therefore,

$$\|\xi\|_{DG} \leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)} \right)^{1/2},$$

which, by the triangle inequality and the bound on $\|\eta\|_{DG}$, gives

$$\|u - u_{DG}\|_{DG} \leq \|\xi\|_{DG} + \|\eta\|_{DG} \leq C \left(\sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2k_\kappa - 3}} \|u\|^2_{\mathrm{H}^{k_\kappa}(\kappa)} \right)^{1/2},$$

where $1 \leq s_\kappa \leq \min\{p_\kappa + 1, k_\kappa\}$, $p_\kappa \geq 1$, for $\kappa \in \mathcal{T}$, and C is a positive constant, independent of u, $\mathtt{h}$, $\mathtt{p}$. $\square$

Numerical experiments

We take $d = 2$ and $\alpha = 10$.

A (damped) Newton method is used to solve the resulting set of nonlinear equations. In each inner (linear) iteration, we use a (left) preconditioned GMRES algorithm with a block symmetric Gauss–Seidel preconditioner.

High-order quadrature was required to integrate the term involving the forcing function f to a sufficient accuracy.

Example 1: Problem with a smooth solution

$\Omega := (-1,1)^2$ with $\Gamma_{\mathrm{D}} := [-1,1] \times \{-1\} \cup \{1\} \times [-1,1]$,
$\Gamma_{\mathrm{N}} := [-1,1] \times \{1\} \cup \{-1\} \times [-1,1]$.

Nonlinear diffusion coefficient:

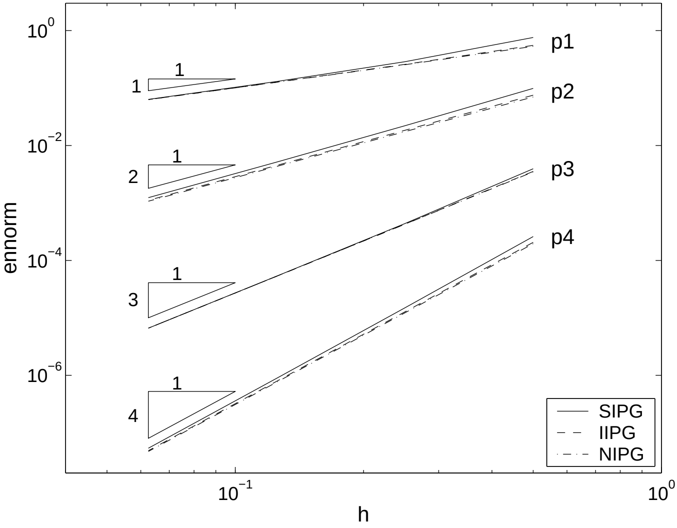
$$\mu(x, |\nabla u|) = 2 + \frac{1}{1 + |\nabla u|}.$$

The functions g_{D}, g_{N} and f are then chosen so that the solution is:

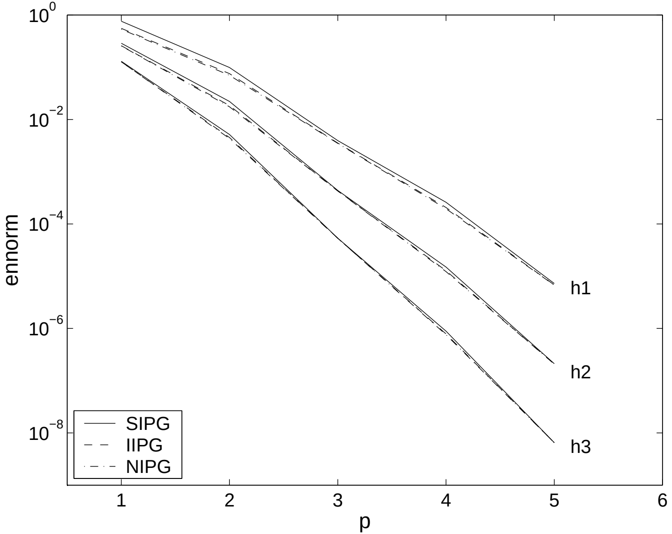
$$u(x,y) = \cos(\pi x/2)\cos(\pi y/2).$$

A simple calculation verifies that $\lambda = 2$ and $\Lambda = 3$.

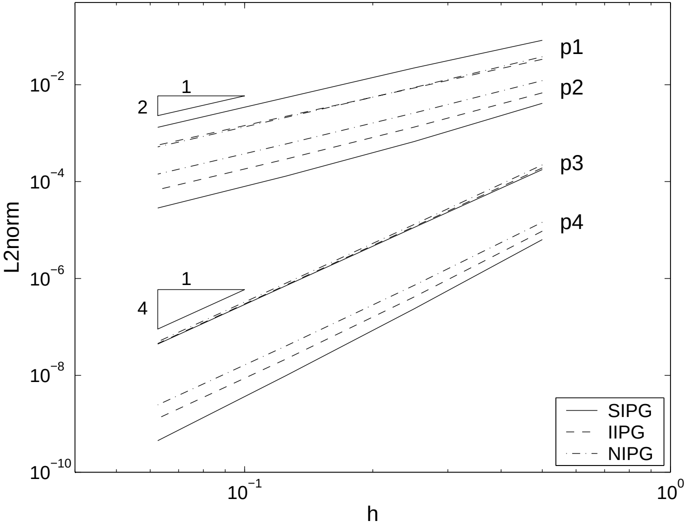
Convergence of the DGFEM with *h*-refinement: p=1,2,3,4



Convergence of the DGFEM with p-refinement: $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$



Convergence of DGFEM with h-refinement: $p = 1, 2, 3, 4$



Example 2: Problem with a nonsmooth solution

Let Ω be the L-shaped domain $(-1,1)^2 \setminus [0,1) \times (-1,0]$; $\Gamma_{\mathrm{D}} = \partial\Omega$,

$$\mu(x, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2},$$

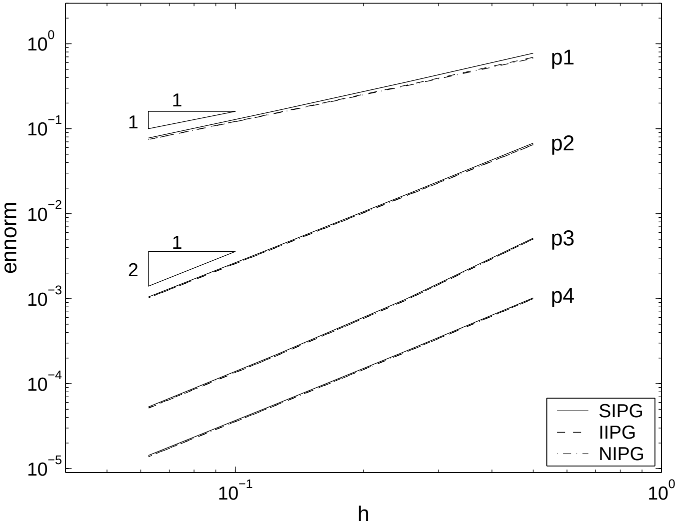
and select g_{D} and f so that the solution is

$$u(x,y) = \cos(\pi y/2)\, \chi_{(0,1)}(x)\, x^{2.5},$$

where $\chi_{(0,1)} : \mathbb{R} \to \mathbb{R}$ denotes the characteristic function of the interval $(0,1) \subset \mathbb{R}$; $\lambda = 1 - \sqrt{2/\mathrm{e}}$ and $\Lambda = 1$.

u has a singularity along the line $x = 0$; $u \in \mathrm{H}^{3-\varepsilon}(\Omega)$, $\forall \varepsilon > 0$.

Convergence of DGFEM with h-refinement: $p = 1, 2, 3, 4$



p	SIPG ($\theta = -1$) $\|u - u_{\mathrm{DG}}\|_{DG}$	k	IIPG ($\theta = 0$) $\|u - u_{\mathrm{DG}}\|_{DG}$	k	NIPG ($\theta = 1$) $\|u - u_{\mathrm{DG}}\|_{DG}$	k
1	7.745e-1	-	6.927e-1	-	6.737e-1	-
2	6.749e-2	3.52	6.505e-2	3.41	6.463e-2	3.38
3	5.163e-3	6.34	5.033e-3	6.31	5.017e-3	6.30
4	1.021e-3	5.63	9.994e-4	5.62	9.949e-4	5.62
5	3.813e-4	4.41	3.731e-4	4.41	3.715e-4	4.41
6	1.759e-4	4.24	1.722e-4	4.24	1.715e-4	4.24
7	9.242e-5	4.18	9.044e-5	4.18	9.005e-5	4.18
8	5.327e-5	4.13	5.218e-5	4.12	5.198e-5	4.11
9	3.304e-5	4.06	3.237e-5	4.05	3.225e-5	4.05
10	2.170e-5	3.99	2.130e-5	3.97	2.125e-5	3.96

Table: Example 2. Convergence of the DGFEM with p–refinement.

Example 3: Structural hypothesis on μ violated

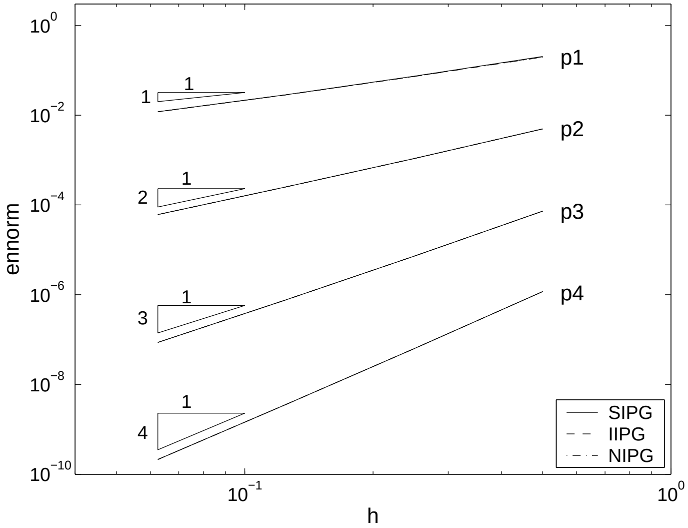
We take $\Omega = (0, 1)^2$, $\Gamma_{\mathrm{D}} = \partial\Omega$ and

$$\mu(x, |\nabla u|) = |\nabla u|^{r-2}, \qquad 1 < r < \infty.$$

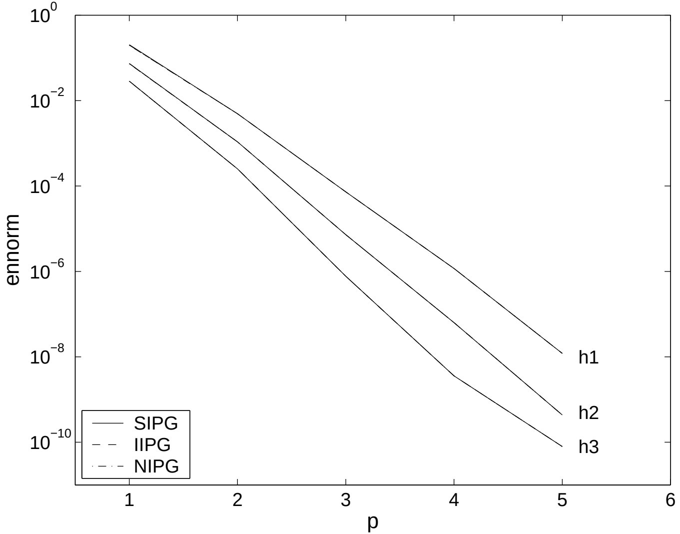
We choose $r = 3$, g_{D} and f so that the analytical solution is

$$u(x, y) = \mathrm{e}^{xy}.$$

Convergence of DGFEM with h-refinement: $p = 1, 2, 3, 4$



Convergence of the DGFEM with p-refinement: $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$



Ten Lectures on the Convergence Analysis of

Discontinuous Galerkin Finite Element Methods

for Nonlinear PDEs

Lecture 6

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Introduction

Quasi-Newtonian fluid flow problem:

$$
\begin{aligned}
-\nabla \cdot \{\mu(\mathbf{x}, |\mathbf{e}(\mathbf{u})|)\, \mathbf{e}(\mathbf{u})\} + \nabla p &= \mathbf{f} && \text{in } \Omega, && (1)\\
\nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, && (2)\\
\mathbf{u} &= 0 && \text{on } \Gamma. && (3)
\end{aligned}
$$

Here, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded polygonal Lipschitz domain with boundary $\Gamma = \partial\Omega$, $\mathbf{f} \in \mathrm{L}^2(\Omega)^d$ is a given source term;

$\mathbf{u} = (u_1, \ldots, u_d)^{\mathrm{T}}$ is the velocity vector, p is the pressure;
$\mathbf{e}(\mathbf{u})$ is the symmetric $d \times d$ strain tensor defined by

$$
e_{ij}(\mathbf{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, \ldots, d.
$$

Furthermore, $|\mathbf{e}(\mathbf{u})|$ is the Frobenius norm of $\mathbf{e}(\mathbf{u})$.

We formulate a class of *hp*-version interior penalty DGFEMs for the numerical approximation of the quasi-Newtonian problem (1)–(3).

We establish the existence and uniqueness of both the analytical solution to (1)–(3) and of its DGFEM counterpart.

The *a priori* error analysis of the underlying class of DGFEMs is then undertaken in the associated natural energy norm.

Weak Formulation

For a bounded Lipschitz domain $D \subset \mathbb{R}^d, d \geq 1$, $\mathrm{H}^t(D)$ denotes Sobolev space of real-valued functions with norm $\|\cdot\|_{t,D}$, $t \geq 0$.

$\mathrm{H}^1_0(D) :=$ subspace of functions in $\mathrm{H}^1(D)$ with zero trace on ∂D.

$\mathrm{L}^2_0(D) := \{q \in \mathrm{L}^2(D) : \int_D q \, \mathrm{d}x = 0\}$.

For a function space $X(D)$, let $X(D)^d$ and $X(D)^{d \times d}$ denote the spaces of vector and tensor fields, respectively, whose components belong to $X(D)$. These spaces are equipped with the usual product norms which we denote in the same way as the norm in $X(D)$.

For the d-component vector-valued functions $\mathbf{v}, \mathbf{w}$ and $d \times d$ matrix-valued functions $\sigma, \tau \in \mathbb{R}^{d \times d}$, we define the operators

$$(\nabla \mathbf{v})_{ij} := \frac{\partial v_i}{\partial x_j}, \qquad (\nabla \cdot \sigma)_i := \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j},$$

$$(\mathbf{v} \otimes \mathbf{w})_{ij} := v_i w_j, \qquad \sigma : \tau := \sum_{i=1}^{d} \sigma_{ij} \tau_{ij}.$$

For matrix-valued functions the Frobenius norm is defined by

$$|\tau|^2 = \tau : \tau.$$

Variational Form

By introducing the forms

$$A(\mathbf{u},\mathbf{v}) := \int_\Omega \mu(|\mathbf{e}(\mathbf{u})|)\mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v})\,\mathrm{d}x, \quad B(\mathbf{v},q) := -\int_\Omega q\nabla\cdot\mathbf{v}\,\mathrm{d}x,$$

a natural weak formulation of the quasi-Newtonian problem (1)–(3) is to find $(\mathbf{u}, p) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega)$ such that

$$A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x, \tag{4}$$

$$-B(\mathbf{u},q) = 0 \tag{5}$$

for all $(\mathbf{v}, q) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega)$.

B satisfies the following inf-sup condition (Brezzi & Fortin, 1991): there exists a constant $\kappa > 0$ such that

$$\inf_{0\neq q\in\mathrm{L}^2_0(\Omega)} \sup_{0\neq\mathbf{v}\in\mathrm{H}^1_0(\Omega)^d} \frac{B(\mathbf{v},q)}{\|q\|_{0,\Omega}\,\|\mathbf{e}(\mathbf{v})\|_{0,\Omega}} \geq \kappa. \tag{6}$$

Assumption

We assume that the nonlinearity μ satisfies the following conditions:

(A1) $\mu \in \mathrm{C}(\overline{\Omega} \times [0, \infty))$.

(A2) *There exist constants $m_\mu, M_\mu > 0$ such that*

$$m_\mu(t - s) \leq \mu(\mathbf{x}, t)t - \mu(\mathbf{x}, s)s \leq M_\mu(t - s), \qquad (7)$$

for all $t \geq s \geq 0$ and $\mathbf{x} \in \overline{\Omega}$.

Lemma 2.1 in Barrett & Liu (1994) implies that as μ satisfies (7), there exist $C_1 > 0$ and $C_2 > 0$, such that for all symmetric τ, ω in $\mathbb{R}^{d\times d}$ and all $\mathbf{x} \in \overline{\Omega}$:

$$|\mu(\mathbf{x}, |\tau|)\tau - \mu(\mathbf{x}, |\omega|)\omega| \leq C_1|\tau - \omega|, \tag{8}$$

$$C_2|\tau - \omega|^2 \leq (\mu(\mathbf{x}, |\tau|)\tau - \mu(\mathbf{x}, |\omega|)\omega) : (\tau - \omega). \tag{9}$$

For ease of notation we shall suppress the dependence of μ on $\mathbf{x}$ and write $\mu(t)$ instead of $\mu(\mathbf{x}, t)$.

Well-Posedness

Theorem

Let X be a real Hilbert space. Consider forms $a : X \times X \to \mathbb{R}$ and $l : X \to \mathbb{R}$ such that

(a) *l is linear and continuous on X,*

(b) *the functional $v \mapsto a(w, v)$ is linear and continuous on X for any fixed $w \in X$,*

(c) *there exists a constant $L > 0$ such that*

$$|a(u, w) - a(v, w)| \leq L\|u - v\|_X \|w\|_X$$

for any $u, v, w \in X$,

(d) *there exists a constant $c > 0$ such that for any $u, w \in X$, there exists $v \in X$ with*

$$a(u, v) - a(w, v) \geq c\|u - w\|_X, \qquad \|v\|_X \leq 1.$$

Theorem (continued)

(e) *for any* $0 \neq v \in X$,

$$\sup_{u \in X} a(u, v) > 0.$$

Then, there exists a unique solution $u \in X$ *of the variational equation*

$$a(u, v) = l(v) \qquad \forall v \in X. \tag{10}$$

Proof.

Step 1: Denoting by $(\cdot,\cdot)_X$ the inner product in X, by applying Riesz' theorem and (b), we deduce that, for any $w \in X$, there is an element denoted by $\mathrm{A}w \in X$ such that

$$a(w,u) = (\mathrm{A}w,u)_X \qquad \forall u \in X.$$

This defines an operator $\mathrm{A} : X \to X$. Using (d), we note that for any $u,w \in X$, there is $v \in X$ with $\|v\|_X \leq 1$ and

$$c\|u-w\|_X \leq a(u,v) - a(w,v) = (\mathrm{A}u - \mathrm{A}w, v)_X. \tag{11}$$

Furthermore, by (c), we observe that

$$|(\mathrm{A}u - \mathrm{A}v, w)_X| = |a(u, w) - a(v, w)| \leq L\|u - v\|_X \|w\|_X,$$

for any $u, v, w \in X$. Therefore,

$$\|\mathrm{A}u - \mathrm{A}v\|_X = \sup_{\|w\|_X \leq 1} |(\mathrm{A}u - \mathrm{A}v, w)_X| \leq L\|u - v\|_X. \tag{12}$$

In addition, again by Riesz' theorem and by recalling (a), there exists $\ell \in X$ such that

$$l(u) = (\ell, u)_X \qquad \forall u \in X.$$

Hence, the variational problem (10) corresponds to the operator equation

$$\mathsf{A}u = \ell, \qquad u \in X. \tag{13}$$

Step 2: We shall prove that the image $\mathrm{Im}(\mathsf{A})$ of A is the entire space X. This implies that (13) has a solution for any $\ell \in X$.

Step 2a: We first show that $\mathrm{Im}(\mathsf{A})$ is closed in X. To this end, let us consider a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathrm{Im}(\mathsf{A})$ that converges to $\bar{z} \in X$. Evidently, this sequence is a Cauchy sequence. Furthermore, there is a sequence $\{w_n\}_{n=1}^{\infty} \subset X$ such that $z_n = \mathsf{A}w_n$ for any $n \in \mathbb{N}$. Hence, for any $m, n \in \mathbb{N}$, using (11), we have that

$$\begin{aligned}\|z_m - z_n\|_X &= \|\mathsf{A}w_m - \mathsf{A}w_n\|_X \\ &= \sup_{\|v\|_X \leq 1} (\mathsf{A}w_m - \mathsf{A}w_n, v)_X \geq c\|w_m - w_n\|_X.\end{aligned}$$

Thus, $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence, with a limit $\bar{w} \in X$.

Furthermore,

$$\bar{z} = \lim_{n\to\infty} \mathsf{A}w_n = \lim_{n\to\infty} (\mathsf{A}w_n - \mathsf{A}\bar{w}) + \mathsf{A}\bar{w},$$

and by (12), we have

$$\|\mathsf{A}w_n - \mathsf{A}\bar{w}\|_X \leq L\|w_n - \bar{w}\|_X \to 0 \qquad \text{as } n \to \infty.$$

Thus, $\lim_{n\to\infty} \mathsf{A}w_n = \mathsf{A}\bar{w}$, and hence, $\bar{z} = \mathsf{A}\bar{w} \in \mathrm{Im}(\mathsf{A})$.

Step 2b: Suppose now that $\mathrm{Im}(\mathsf{A}) \neq X$. Then, since $\mathrm{Im}(\mathsf{A})$ is closed, we apply the Hahn–Banach theorem to deduce that there exists $0 \neq \widetilde{w} \in X$ with $(v, \widetilde{w})_X = 0$ for all $v \in \mathrm{Im}(\mathsf{A})$. In particular,

$$0 = (\mathsf{A}u, \widetilde{w}) = a(u, \widetilde{w}) \qquad \forall u \in X,$$

which contradicts (e).

Consequently, $\mathrm{Im}(\mathsf{A}) = X$ and (13) has a solution $u \in X$.

Step 3: We complete the proof by demonstrating that the solution of (13) is unique.

Suppose that there are two solutions $u_1, u_2 \in X$ of (13). Then, recalling (11) there exists $w \in X$ such that

$$0 = (\mathrm{A}u_1 - \mathrm{A}u_2, w)_X \geq c\|u_1 - u_2\|_X,$$

and therefore, $u_1 = u_2$. $\square$

We will now apply the above result to (4)–(5).

To this end, we define the form

$$\mathcal{A}((\mathbf{u},p);(\mathbf{v},q)) := A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) - B(\mathbf{u},q)$$

on the space $(\mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)) \times (\mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega))$, and the norm

$$|||(\mathbf{u},p)|||^2 := \|\mathbf{e}(\mathbf{u})\|_{0,\Omega}^2 + \|p\|_{0,\Omega}^2.$$

Proposition

There exist two constants $L, c > 0$ such that the following hold:

(a) *Continuity: For any $(\mathbf{u}, p)$, $(\mathbf{v}, q)$, $(\mathbf{w}, r) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$:*

$$|\mathcal{A}((\mathbf{u}, p); (\mathbf{v}, q)) - \mathcal{A}((\mathbf{w}, r); (\mathbf{v}, q))| \leq L \|(\mathbf{u} - \mathbf{w}, p - r)\| \|(\mathbf{v}, q)\|.$$

(b) *Inf-sup stability: For any $(\mathbf{u}, p), (\mathbf{w}, r) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ there exists $(\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ such that*

$$\mathcal{A}((\mathbf{u}, p); (\mathbf{v}, q)) - \mathcal{A}((\mathbf{w}, r); (\mathbf{v}, q)) \geq c \|(\mathbf{u} - \mathbf{w}, p - r)\|,$$

for all $\|(\mathbf{v}, q)\| \leq 1$.

(c) *For any $0 \neq (\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$,*

$$\sup_{(\mathbf{u}, p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)} \mathcal{A}((\mathbf{u}, p); (\mathbf{v}, q)) > 0.$$

Proof.

(a) Applying the triangle inequality, we have that

$$
\begin{aligned}
&|\mathcal{A}((\mathbf{u},p);(\mathbf{v},q)) - \mathcal{A}((\mathbf{w},r);(\mathbf{v},q))| \\
&\qquad\qquad \leq |A(\mathbf{u},\mathbf{v}) - A(\mathbf{w},\mathbf{v})| + |B(\mathbf{v},p-r)| + |B(\mathbf{u}-\mathbf{w},q)|.
\end{aligned}
$$

Then, recalling (8) leads to

$$
\begin{aligned}
|A(\mathbf{u},\mathbf{v}) - A(\mathbf{w},\mathbf{v})| &\leq \int_\Omega |\mu(|\mathbf{e}(\mathbf{u})|)\mathbf{e}(\mathbf{u}) - \mu(|\mathbf{e}(\mathbf{w})|)\mathbf{e}(\mathbf{w})||\mathbf{e}(\mathbf{v})|\,\mathrm{d}x \\
&\leq C_1 \int_\Omega |\mathbf{e}(\mathbf{u}) - \mathbf{e}(\mathbf{w})||\mathbf{e}(\mathbf{v})|\,\mathrm{d}x \\
&\leq C_1 \|\mathbf{e}(\mathbf{u}) - \mathbf{e}(\mathbf{w})\|_{0,\Omega} \|\mathbf{e}(\mathbf{v})\|_{0,\Omega}.
\end{aligned}
$$

Furthermore,

$$|B(\mathbf{v}, p - r)| \leq \int_\Omega |p - r||\nabla \cdot \mathbf{v}| \,\mathrm{d}x \leq \|p - r\|_{0,\Omega} \|\nabla \mathbf{v}\|_{0,\Omega}.$$

According to Korn's inequality, there exist a positive constant C_* such that $\|\mathbf{v}\|_{1,\Omega} \leq C_* \|\mathbf{e}(\mathbf{v})\|_{0,\Omega}$ for all $\mathbf{v} \in \mathrm{H}_0^1(\Omega)^d$; thus,

$$|B(\mathbf{v}, p - r)| \leq C_* \|p - r\|_{0,\Omega} \|\mathbf{e}(\mathbf{v})\|_{0,\Omega}.$$

Similarly,

$$|B(\mathbf{u}-\mathbf{w},q)| \leq C_* \|q\|_{0,\Omega} \|\mathbf{e}(\mathbf{u})-\mathbf{e}(\mathbf{w})\|_{0,\Omega}.$$

Combining these estimates we obtain

$$\begin{aligned} |\mathcal{A}((\mathbf{u},p);(\mathbf{v},q)) &- \mathcal{A}((\mathbf{w},r);(\mathbf{v},q))| \\ &\leq C_1 \|\mathbf{e}(\mathbf{u})-\mathbf{e}(\mathbf{w}))\|_{0,\Omega} \|\mathbf{e}(\mathbf{v})\|_{0,\Omega} \\ &\quad + C_* \|p-r\|_{0,\Omega} \|\mathbf{e}(\mathbf{v})\|_{0,\Omega} \\ &\quad + C_* \|q\|_{0,\Omega} \|\mathbf{e}(\mathbf{u})-\mathbf{e}(\mathbf{w})\|_{0,\Omega}. \end{aligned}$$

Thence, using the Cauchy–Schwarz inequality, we deduce (a).

(b) Let $p - r \in \mathrm{L}_0^2(\Omega)$, then, from the inf-sup condition (6) there exists $\boldsymbol{\xi} \in \mathrm{H}_0^1(\Omega)^d$ such that

$$
\begin{aligned}
-\int_\Omega (p - r)\nabla \cdot \boldsymbol{\xi}\, \mathrm{d}x \geq \kappa \left\| p - r \right\|_{0,\Omega}^2 , \\
\left\| \mathbf{e}(\boldsymbol{\xi}) \right\|_{0,\Omega} \leq \left\| p - r \right\|_{0,\Omega} .
\end{aligned}
\tag{14}
$$

Now, we choose

$$\hat{\mathbf{v}} := \alpha(\mathbf{u} - \mathbf{w}) + \beta\boldsymbol{\xi}, \qquad \hat{q} := \alpha(p - r),$$

with

$$\alpha := C_2^{-1}(1 + C_1^2\kappa^{-2}), \qquad \beta := 2\kappa^{-1},$$

where C_1, C_2 are the constants from (8)–(9).

Using (8), (9), (14) and the arithmetic-geometric mean inequality:

$$
\begin{aligned}
&\mathcal{A}((\mathbf{u}, p); (\hat{\mathbf{v}}, \hat{q})) - \mathcal{A}((\mathbf{w}, r); (\hat{\mathbf{v}}, \hat{q})) \\
&\quad = \int_\Omega \{\mu(|\mathbf{e}(\mathbf{u})|)\mathbf{e}(\mathbf{u}) - \mu(|\mathbf{e}(\mathbf{w})|)\mathbf{e}(\mathbf{w})\} : \mathbf{e}(\hat{\mathbf{v}})\,\mathrm{d}x \\
&\qquad - \int_\Omega (p - r)\nabla\cdot\hat{\mathbf{v}}\,\mathrm{d}x + \int_\Omega \hat{q}\nabla\cdot(\mathbf{u} - \mathbf{w})\,\mathrm{d}x \\
&\quad = \alpha\int_\Omega \{\mu(|\mathbf{e}(\mathbf{u})|)\mathbf{e}(\mathbf{u}) - \mu(|\mathbf{e}(\mathbf{w})|)\mathbf{e}(\mathbf{w})\} : \mathbf{e}(\mathbf{u} - \mathbf{w})\,\mathrm{d}x \\
&\qquad + \beta\int_\Omega \{\mu(|\mathbf{e}(\mathbf{u})|)\mathbf{e}(\mathbf{u}) - \mu(|\mathbf{e}(\mathbf{w})|)\mathbf{e}(\mathbf{w})\} : \mathbf{e}(\boldsymbol{\xi})\,\mathrm{d}x \\
&\qquad - \beta\int_\Omega (p - r)\nabla\cdot\boldsymbol{\xi}\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\geq \alpha C_2 \int_\Omega |\mathbf{e}(\mathbf{u}-\mathbf{w})|^2 \,\mathrm{d}x \\
&\quad - \frac{1}{2}\kappa\beta \int_\Omega |\mathbf{e}(\boldsymbol{\xi})|^2 \,\mathrm{d}x + \beta\kappa \,\|p-r\|_{0,\Omega}^2 \\
&\quad - \frac{1}{2}\kappa^{-1}\beta \int_\Omega |\mu(|\mathbf{e}(\mathbf{u})|)\mathbf{e}(\mathbf{u}) - \mu(|\mathbf{e}(\mathbf{w})|)\mathbf{e}(\mathbf{w})|^2 \,\mathrm{d}x \\
&\geq (\alpha C_2 - \frac{1}{2}\kappa^{-1}\beta C_1^2) \,\|\mathbf{e}(\mathbf{u}-\mathbf{w})\|_{0,\Omega}^2 + \frac{1}{2}\beta\kappa \,\|p-r\|_{0,\Omega}^2 \\
&= |\!|\!|(\mathbf{u}-\mathbf{w}, p-r)|\!|\!|^2.
\end{aligned}
$$

Using the triangle inequality, we deduce that

$$\begin{aligned}
\|\!|(\hat{\mathbf{v}}, \hat{q})|\!\|^2 &= \|\mathbf{e}(\hat{\mathbf{v}})\|^2_{0,\Omega} + \|q\|^2_{0,\Omega} \\
&\leq 2\alpha^2\|\mathbf{e}(\mathbf{u}-\mathbf{w})\|^2_{0,\Omega} + 2\beta^2\|\mathbf{e}(\boldsymbol{\xi})\|^2_{0,\Omega} + \alpha^2\,\|p-r\|^2_{0,\Omega} \\
&\leq 2\alpha^2\|\mathbf{e}(\mathbf{u}-\mathbf{w})\|^2_{0,\Omega} + (\alpha^2+2\beta^2)\,\|p-r\|^2_{0,\Omega} \\
&\leq \max(2\alpha^2, \alpha^2+2\beta^2)\|\!|(\mathbf{u}-\mathbf{w}, p-r)|\!\|^2.
\end{aligned}$$

Setting

$$(\mathbf{v}, q) = \max(2\alpha^2, \alpha^2+2\beta^2)^{-\frac{1}{2}}\|\!|(\mathbf{u}-\mathbf{w}, p-r)|\!\|^{-1}(\hat{\mathbf{v}}, \hat{q})$$

completes the proof of (b).

(c) Let $(\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega) \setminus \{(\mathbf{0}, 0)\}$. For $\mathbf{v} \neq 0$, we have:

$$\sup_{(\mathbf{u},p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)} \mathcal{A}((\mathbf{u}, p); (\mathbf{v}, q)) \geq \mathcal{A}((\mathbf{v}, q); (\mathbf{v}, q)) = A(\mathbf{v}, \mathbf{v}).$$

Noting (9), yields

$$A(\mathbf{v}, \mathbf{v}) = \int_\Omega \mu(|\mathbf{e}(\mathbf{v})|)\mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{v}) \, \mathrm{d}x \geq C_2 \|\mathbf{e}(\mathbf{v})\|_{0,\Omega}^2 > 0.$$

If $\mathbf{v} = \mathbf{0}, q \neq 0$, we use the inf-sup condition (6) to find a function $\mathbf{v}_q \in \mathrm{H}_0^1(\Omega)^d$ such that

$$\begin{aligned} \sup_{(\mathbf{u},p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)} \mathcal{A}((\mathbf{u}, p); (\mathbf{0}, q)) &\geq \mathcal{A}(-(\mathbf{v}_q, 0); (\mathbf{0}, q)) \\ &= B(\mathbf{v}_q, q) \geq \kappa \|q\|_{0,\Omega} > 0. \end{aligned}$$

This completes the proof. $\square$

Theorem

There exists exactly one solution $(\mathbf{u}, p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ *to the weak formulation* (4)–(5).

Proof.

Notice that (4)–(5) is equivalent to finding a pair of functions $(\mathbf{u}, p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ such that

$$\mathcal{A}((\mathbf{u}, p); (\mathbf{v}, q)) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}x \qquad \forall (\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega).$$

Thence, noticing that by Korn's inequality the linear form

$$\mathbf{v} \mapsto \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}x$$

is continuous, and applying Theorem 1, in combination with the above proposition, implies the well-posedness of (4)–(5). $\square$

Meshes and finite element spaces

Let $\mathcal{T}_h$ be a subdivision of Ω into disjoint open element-domains K such that $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$.

We assume that the family of subdivisions $\{\mathcal{T}_h\}_{h>0}$ is shape regular and each $K \in \mathcal{T}_h$ is an affine image of a fixed master element $\hat{K}$; i.e., for each $K \in \mathcal{T}_h$, there exists an affine mapping $T_K : \hat{K} \to K$ such that $K = T_K(\hat{K})$, where $\hat{K}$ is the open cube $(-1,1)^3$ in $\mathbb{R}^3$ or the open square $(-1,1)^2$ in $\mathbb{R}^2$.

By h_K we denote the element diameter of $K \in \mathcal{T}_h$,

$$h = \max_{K \in \mathcal{T}_h} h_K,$$

and $\mathbf{n}_K$ signifies the unit outward normal vector to K.

We allow the meshes $\mathcal{T}_h$ to be 1-irregular, i.e., each face of any one element $K \in \mathcal{T}_h$ contains at most one hanging node (which, for simplicity, we assume to be at the centre of the corresponding face) and each edge of each face contains at most one hanging node (yet again assumed to be at the centre of the edge).

We suppose that $\mathcal{T}_h$ is regularly reducible (Ortner & Süli, 2006), i.e., there exists a shape-regular conforming mesh $\widetilde{\mathcal{T}_h}$ such that the closure of each element in $\mathcal{T}_h$ is a union of closures in $\widetilde{\mathcal{T}_h}$, and that there exists a constant $C > 0$, independent of mesh sizes, such that for any two elements $K \in \mathcal{T}_h$ and $\widetilde{K} \in \widetilde{\mathcal{T}_h}$ with $\widetilde{K} \subseteq K$, we have that $h_K / h_{\widetilde{K}} \leq C$.

Note that these assumptions imply that the family $\{\mathcal{T}_h\}_{h>0}$ is of *bounded local variation*, i.e., there exists a constant $\rho_1 \geq 1$, independent of element sizes, such that

$$\rho_1^{-1} \leq h_K/h_{K'} \leq \rho_1 \tag{15}$$

for any pair of elements $K, K' \in \mathcal{T}_h$ which share a common face $F = \partial K \cap \partial K'$.

We store the element sizes in the vector $\mathbf{h} := \{h_K : K \in \mathcal{T}_h\}$.

For a non-negative integer k, we denote by $\mathcal{Q}_k(\hat{K})$ the set of all tensor-product polynomials on $\hat{K}$ of degree k in each coordinate direction.

To each $K \in \mathcal{T}_h$, we assign a local polynomial degree $k_K \geq 1$ and store these in a vector $\mathbf{k} = \{k_K : K \in \mathcal{T}_h\}$.

We suppose that $\mathbf{k}$ is also of bounded local variation, i.e., there exists a constant $\rho_2 \geq 1$, independent of the element sizes and $\mathbf{k}$, such that, for any pair of neighbouring elements $K, K' \in \mathcal{T}_h$,

$$\rho_2^{-1} \leq k_K / k_{K'} \leq \rho_2. \tag{16}$$

With this notation we introduce the finite element spaces

$$\mathbf{V}_h := \left\{ \mathbf{v} \in \mathrm{L}^2(\Omega)^d : \mathbf{v}|_K \circ T_K \in \mathcal{Q}_{k_K}(\hat{K})^d, K \in \mathcal{T}_h \right\},$$
$$Q_h := \left\{ q \in \mathrm{L}_0^2(\Omega) : q|_K \circ T_K \in \mathcal{Q}_{k_K - 1}(\hat{K}), K \in \mathcal{T}_h \right\}.$$

We define an interior face F of $\mathcal{T}_h$ as the intersection of two neighbouring elements $K, K' \in \mathcal{T}_h$, i.e., $F = \partial K \cap \partial K'$.

Similarly, we define a boundary face $F \subset \Gamma$ as the entire face of an element K on the boundary.

We denote by $\mathcal{F}_{\mathcal{I}}$ the set of all interior faces, $\mathcal{F}_{\mathcal{B}}$ the set of all boundary faces and $\mathcal{F} = \mathcal{F}_{\mathcal{I}} \cup \mathcal{F}_{\mathcal{B}}$ the set of all faces.

Face operators

Let q, $\mathbf{v}$, and τ be scalar-, vector- and matrix-valued functions, respectively, which are smooth inside each element $K \in \mathcal{T}_h$.

Given two adjacent elements, $K^+, K^- \in \mathcal{T}_h$, which share a common face $F \in \mathcal{F}_\mathcal{I}$, i.e., $F = \partial K^+ \cap \partial K^-$, we write $q^\pm$, $\mathbf{v}^\pm$, and $\tau^\pm$ to denote the traces of the functions q, $\mathbf{v}$, and τ, respectively, on the face F, taken from the interior of $K^\pm$, respectively.

The averages of q, $\mathbf{v}$, and τ at $\mathbf{x} \in F$ are given by

$$\{\!\{q\}\!\} := \frac{1}{2}(q^+ + q^-), \qquad \{\!\{\mathbf{v}\}\!\} := \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-), \qquad \{\!\{\tau\}\!\} := \frac{1}{2}(\tau^+ + \tau^-),$$

respectively.

Similarly, the jumps of q, $\mathbf{v}$, and τ at $\mathbf{x} \in F$ are given by

$$
\begin{aligned}
[\![q]\!] &:= q^+\mathbf{n}_{K^+} + q^-\mathbf{n}_{K^-}, \qquad & [\![\mathbf{v}]\!] &:= \mathbf{v}^+ \cdot \mathbf{n}_{K^+} + \mathbf{v}^- \cdot \mathbf{n}_{K^-}, \\
[\![\mathbf{v}]\!] &:= \mathbf{v}^+ \otimes \mathbf{n}_{K^+} + \mathbf{v}^- \otimes \mathbf{n}_{K^-}, \qquad & [\![\tau]\!] &:= \tau^+\mathbf{n}_{K^+} + \tau^-\mathbf{n}_{K^-}.
\end{aligned}
$$

On a boundary face $F \in \mathcal{F}_{\mathcal{B}}$, we set

$$
\{\!\{q\}\!\} := q, \quad \{\!\{\mathbf{v}\}\!\} := \mathbf{v}, \quad \{\!\{\tau\}\!\} := \tau
$$

and

$$
[\![q]\!] := q\mathbf{n}, \quad [\![\mathbf{v}]\!] := \mathbf{v} \cdot \mathbf{n}, \quad [\![\mathbf{v}]\!] := \mathbf{v} \otimes \mathbf{n}, \quad [\![\tau]\!] := \tau\mathbf{n},
$$

with $\mathbf{n}$ denoting the unit outward normal vector on the boundary Γ.

We note the following elementary identities for any scalar–, vector–, and matrix–valued functions $q, \mathbf{v}$, and τ, respectively:

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} q\mathbf{v}\cdot\mathbf{n}_K\,\mathrm{d}s = \sum_{F\in\mathcal{F}}\int_F [\![q]\!]\cdot\{\!\{\mathbf{v}\}\!\}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_\mathcal{I}}\int_F \{\!\{q\}\!\}[\![\mathbf{v}]\!]\,\mathrm{d}s,$$

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} \tau : (\mathbf{v}\otimes\mathbf{n}_K)\,\mathrm{d}s = \sum_{F\in\mathcal{F}}\int_F [\![\mathbf{v}]\!] : \{\!\{\tau\}\!\}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_\mathcal{I}}\int_F \{\!\{\mathbf{v}\}\!\}\cdot[\![\tau]\!]\,\mathrm{d}s. \tag{17}$$

$\mathbf{n}_K$ denotes the unit outward normal vector to the element $K\in\mathcal{T}_h$.

DGFEM Discretization

Given a partition $\mathcal{T}_h$ of Ω, together with the corresponding polynomial degree vector $\mathbf{k}$, the IP DGFEM formulation is defined as follows: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$A_h(\mathbf{u}_h, \mathbf{v}) + B_h(\mathbf{v}, p_h) = F_h(\mathbf{v}), \tag{18}$$

$$-B_h(\mathbf{u}_h, q) = 0 \tag{19}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$, where

$$
\begin{aligned}
A_h(\mathbf{u},\mathbf{v}) :=& \int_\Omega \mu\left(|\mathbf{e}_h(\mathbf{u})|\right)\mathbf{e}_h(\mathbf{u}) : \mathbf{e}_h(\mathbf{v})\,\mathrm{d}x \\
&- \sum_{F\in\mathcal{F}} \int_F \{\!\{\mu\left(|\mathbf{e}_h(\mathbf{u})|\right)\mathbf{e}_h(\mathbf{u})\}\!\} : [\![\mathbf{v}]\!]\,\mathrm{d}s \\
&+ \theta \sum_{F\in\mathcal{F}} \int_F \{\!\{\mu(h_F^{-1}|[\![\mathbf{u}]\!]|)\mathbf{e}_h(\mathbf{v})\}\!\} : [\![\mathbf{u}]\!]\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}} \int_F \sigma[\![\mathbf{u}]\!] : [\![\mathbf{v}]\!]\,\mathrm{d}s, \\
B_h(\mathbf{v},q) :=& - \int_\Omega q\nabla_h \cdot \mathbf{v}\,\mathrm{d}x + \sum_{F\in\mathcal{F}} \int_F \{\!\{q\}\!\}[\![\mathbf{v}]\!]\,\mathrm{d}s
\end{aligned}
$$

and

$$
F_h(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x.
$$

Here, $\mathbf{e}_h(\cdot)$ and ∇_h denote the element-wise strain tensor and gradient operator, respectively, and $\theta \in [-1, 1]$.

The interior penalty parameter σ is defined as follows:

$$\sigma := \gamma \frac{k_F^2}{h_F}, \tag{20}$$

where $\gamma \geq 1$ is a constant, which must be chosen sufficiently large (independent of the local element sizes and polynomial degree).

For a face $F \in \mathcal{F}$, we define h_F as the diameter of the face and the face polynomial degree k_F as

$$k_F := \begin{cases} \max(k_K, k_{K'}), & \text{if } F = \partial K \cap \partial K' \in \mathcal{F}_{\mathcal{I}}, \\ k_K, & \text{if } F = \partial K \cap \Gamma \in \mathcal{F}_{\mathcal{B}}. \end{cases}$$

Remark

We note that the formulation (18)–(19) corresponds to the symmetric interior penalty (SIP) method when $\theta = -1$, the non-symmetric interior penalty (NIP) method when $\theta = 1$ and the incomplete interior penalty method (IIP) when $\theta = 0$.

We introduce the energy norms $\|\cdot\|_{1,h}$ and $\|(\cdot,\cdot)\|_{\mathrm{DG}}$ by

$$\|\mathbf{v}\|_{1,h}^2 := \|\mathbf{e}_h(\mathbf{v})\|_{0,\Omega}^2 + \sum_{F\in\mathcal{F}} \int_F \sigma |[\![\mathbf{v}]\!]|^2 \,\mathrm{d}s$$

and

$$\|(\mathbf{v},q)\|_{\mathrm{DG}}^2 := \|\mathbf{v}\|_{1,h}^2 + \|q\|_{0,\Omega}^2 \,. \tag{21}$$

Lemma

There exists a constant $C_{\mathcal{K}} > 0$*, independent of* $\mathbf{h}$ *and* $\mathbf{k}$*, such that*

$$\|\mathbf{e}_h(\mathbf{v})\|_{0,\Omega}^2 \leq \|\nabla_h \mathbf{v}\|_{0,\Omega}^2 \leq C_{\mathcal{K}} \left(\|\mathbf{e}_h(\mathbf{v})\|_{0,\Omega}^2 + \sum_{F \in \mathcal{F}} \int_F h_F^{-1} |[\![\mathbf{v}]\!]|^2 \,\mathrm{d}s \right)$$

for all $\mathbf{v} \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$*, where*

$$\mathrm{H}^1(\Omega, \mathcal{T}_h) = \left\{ \mathbf{v} \in \mathrm{L}^2(\Omega)^d : \mathbf{v}|_K \in \mathrm{H}^1(K)^d, K \in \mathcal{T}_h \right\}.$$

Proof.

The proof of the first bound follows from elementary manipulations and application of the Cauchy–Schwarz inequality.

The second estimate is a discrete Korn inequality for piecewise H^1 vector fields; see, Brenner (2004). $\square$

Well-Posedness of the DGFEM Formulation

Next, we prove that the DGFEM (18)–(19) admits a unique solution. To this end, let us assume that the bilinear form B_h satisfies the following discrete inf-sup condition:

$$\inf_{0 \neq q \in Q_h} \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}_h} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,h} \|q\|_{0,\Omega}} \geq c \left(\max_{K \in \mathcal{T}_h} k_K \right)^{-1}. \tag{22}$$

This inf-sup condition holds

- for $k_K \geq 2$, $K \in \mathcal{T}_h$, or
- for $k \geq 1$ if $\mathcal{T}_h$ is conforming and $k_K = k$ for all $K \in \mathcal{T}_h$;

see Theorems 6.2 and 6.12 in Schötzau, Schwab & Toselli (2002).

Theorem

Provided that the penalty parameter γ *arising in* (20) *is chosen sufficiently large, there is exactly one solution* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *of the hp-DGFEM* (18)–(19).

Proof.

We set

$$\mathcal{A}_h((\mathbf{u}, p); (\mathbf{v}, q)) := A_h(\mathbf{u}, \mathbf{v}) + B_h(\mathbf{v}, p) - B_h(\mathbf{u}, q),$$

which allows the DGFEM defined in (18)–(19) to be written in the following compact form: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\mathcal{A}_h((\mathbf{u}_h, p_h); (\mathbf{v}, q)) = F_h(\mathbf{v}) \tag{23}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$.

We will now check conditions (a)–(e) of Theorem 1 separately.

(a) The continuity of the linear form F_h follows from applying the Cauchy–Schwarz inequality together with Lemma 4.

(b) The linearity of $(\mathbf{v}, q) \mapsto \mathcal{A}_h((\mathbf{u}, p); (\mathbf{v}, q))$, for fixed $(\mathbf{u}, p) \in \mathbf{V}_h \times Q_h$, follows directly from the definition of $\mathcal{A}_h$. Furthermore, the continuity is shown by using the Cauchy–Schwarz inequality and invoking (8).

(c) The argument is as in the proof of Proposition 1 (a), by (8), the Cauchy–Schwarz inequality, and the discrete Korn ineq. from Lemma 4.

The flux terms are treated as in Houston, Robson & Süli (2005).

(d) The inf-sup stability of $\mathcal{A}_h$ is proved as Theorem 5.4 in Wihler & Wirtz (2011), provided that $\gamma > 0$ is large enough, using the discrete inf-sup condition (22), where the nonlinear terms are estimated as in Proposition 1 (b) using (8)–(9).

More precisely, it can be seen that, for any $(\mathbf{u}, p)$ and $(\mathbf{w}, r)$ in $\mathbf{V}_h \times Q_h$, there exists $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} &\mathcal{A}_h((\mathbf{u}, p); (\mathbf{v}, q)) - \mathcal{A}_h((\mathbf{w}, r); (\mathbf{v}, q)) \\ &\qquad \geq c \left(\max_{K \in \mathcal{T}_h} k_K \right)^{-2} \|(\mathbf{u} - \mathbf{w}, p - r)\|_{\mathrm{DG}}, \end{aligned} \tag{24}$$

for all $\|(\mathbf{v}, q)\|_{\mathrm{DG}} \leq 1$.

(e) We proceed as in the proof of Proposition 1 (c), and consider $(\mathbf{0}, 0) \neq (\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$. Firstly, if $\mathbf{v} \neq \mathbf{0}$, then

$$\sup_{(\mathbf{u},p)\in\mathbf{V}_h\times Q_h} \mathcal{A}_h((\mathbf{u}, p); (\mathbf{v}, q)) \geq \mathcal{A}_h((\mathbf{v}, q); (\mathbf{v}, q)) = A_h(\mathbf{v}, \mathbf{v}).$$

By Lemma 2.3 in Houston, Robson & Süli (2005) (with $w_1 = v$, $w_2 = 0$ and $\gamma > 0$ sufficiently large), there exists a constant $C > 0$ independent of the mesh size and the local polynomial degrees s.t.:

$$A_h(\mathbf{v}, \mathbf{v}) \geq C \|\mathbf{v}\|_{1,h}^2 \qquad \forall \mathbf{v} \in \mathbf{V}_h. \tag{25}$$

Hence,

$$\sup_{(\mathbf{u},p) \in \mathbf{V}_h \times Q_h} \mathcal{A}_h((\mathbf{u}, p); (\mathbf{v}, q)) > 0.$$

Secondly, if $\mathbf{v} = \mathbf{0}$, $q \neq 0$, then we apply the discrete inf-sup condition (22), as in the proof of Proposition 1 (c), to obtain:

$$\sup_{(\mathbf{u},p)\in \mathbf{V}_h \times Q_h} \mathcal{A}_h((\mathbf{u},p);(\mathbf{0},q)) > 0.$$

This completes the proof. $\square$

A Priori Error Analysis

Theorem

Let the penalty parameter γ be sufficiently large, and let

$$(\mathbf{u}, p) \in (\mathrm{C}^1(\Omega) \cap \mathrm{H}^2(\Omega))^d \times (\mathrm{C}^0(\Omega) \cap \mathrm{H}^1(\Omega)),$$

and $\mathbf{u}|_K \in \mathrm{H}^{s_K+1}(K)^d$, $p|_K \in \mathrm{H}^{s_K}(K)$, *with* $s_K \geq 1$, $K \in \mathcal{T}_h$. *Then, provided that the discrete inf-sup condition* (22) *is valid:*

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathrm{DG}}^2 \leq C \max_{K \in \mathcal{T}_h} k_K^4 \sum_{K \in \mathcal{T}_h} \left(\frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K - 1}} \|\mathbf{u}\|_{s_K+1,K}^2 + \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K}} \|p\|_{s_K,K}^2 \right),$$

where $(\mathbf{u}_h, p_h)$ *is the DGFEM solution defined in* (18)–(19), *and* $C > 0$ *is independent of the mesh size and polynomial degrees.*

Proof.

Let us consider two interpolants $\Pi_{\mathbf{u}}$ and Π_p satisfying

$$\|\mathbf{u} - \Pi_{\mathbf{u}}\mathbf{u}\|_{1,h}^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K - 1}} \|\mathbf{u}\|_{s_K+1,K}^2,$$

$$\sum_{K \in \mathcal{T}_h} \left(\|p - \Pi_p p\|_{0,K}^2 + h_K k_K^{-1} \|p - \Pi_p p\|_{0,\partial K}^2 \right) \qquad (26)$$

$$\leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K}} \|p\|_{s_K,K}^2;$$

see equation (3.2) in Houston, Robson & Süli (2005) and Houston, Schwab & Süli (2002), respectively.

Thus, defining

$$
\begin{aligned}
\mathbf{u} - \mathbf{u}_h &= (\mathbf{u} - \Pi_{\mathbf{u}}\mathbf{u}) + (\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}_h) =: \boldsymbol{\eta}_{\mathbf{u}} + \boldsymbol{\xi}_{\mathbf{u}}, \\
p - p_h &= (p - \Pi_p p) + (\Pi_p p - p_h) =: \eta_p + \xi_p,
\end{aligned}
$$

we have $(\boldsymbol{\xi}_{\mathbf{u}}, \xi_p) \in \mathbf{V}_h \times Q_h$. Next, by the inf-sup stability (24) we find $(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p) \in \mathbf{V}_h \times Q_h$ with $\|(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)\|_{\mathrm{DG}} \leq 1$ and

$$
\begin{aligned}
c \left(\max_{K \in \mathcal{T}_h} k_K \right)^{-2} & \|(\boldsymbol{\xi}_{\mathbf{u}}, \xi_p)\|_{\mathrm{DG}} \\
& \leq \mathcal{A}_h((\Pi_{\mathbf{u}}\mathbf{u}, \Pi_p p); (\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)) - \mathcal{A}_h((\mathbf{u}_h, p_h); (\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)).
\end{aligned}
$$

Then, thanks to our regularity assumptions, the DGFEM (18)–(19) is consistent, and thus,

$$
\begin{aligned}
& c\left(\max_{K\in\mathcal{T}_h} k_K\right)^{-2} \|(\boldsymbol{\xi}_{\mathbf{u}},\xi_p)\|_{\mathrm{DG}} \\
& \le \mathcal{A}_h((\Pi_{\mathbf{u}}\mathbf{u},\Pi_p p);(\hat{\boldsymbol{\xi}}_{\mathbf{u}},\hat{\xi}_p)) - \mathcal{A}_h((\mathbf{u},p);(\hat{\boldsymbol{\xi}}_{\mathbf{u}},\hat{\xi}_p)) \\
& \le |A_h(\Pi_{\mathbf{u}}\mathbf{u},\hat{\boldsymbol{\xi}}_{\mathbf{u}}) - A_h(\mathbf{u},\hat{\boldsymbol{\xi}}_{\mathbf{u}})| \\
& \qquad + |B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}},\Pi_p p - p)| + |B_h(\Pi_{\mathbf{u}}\mathbf{u}-\mathbf{u},\hat{\xi}_p)| \\
& =: T_1 + T_2 + T_3.
\end{aligned}
$$

For term T_1, we apply Lemma 3.2 in Houston, Robson & Süli (2005) to obtain

$$
\begin{aligned}
T_1 &= |A_h(\Pi_{\mathbf{u}}\mathbf{u}, \hat{\boldsymbol{\xi}}_{\mathbf{u}}) - A_h(\mathbf{u}, \hat{\boldsymbol{\xi}}_{\mathbf{u}})| \\
&\leq C \left(\sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K - 1}} \|\mathbf{u}\|^2_{s_K+1,K} \right)^{\frac{1}{2}} \|\hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{1,h}.
\end{aligned}
$$

For T_2, by the Cauchy–Schwarz inequality we arrive at

$$\begin{aligned} T_2 &= |B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \Pi_p p - p)| \\ &\leq \|\nabla \cdot \hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{0,\Omega} \|\Pi_p p - p\|_{0,\Omega} \\ &\quad + \left(\sum_{F \in \mathcal{F}} \int_F \sigma^{-1} |\{\!\{ \Pi_p p - p \}\!\}|^2 \,\mathrm{d}s \right)^{\frac{1}{2}} \left(\sum_{F \in \mathcal{F}} \int_F \sigma |[\![\hat{\boldsymbol{\xi}}_{\mathbf{u}})]\!]|^2 \,\mathrm{d}s \right)^{\frac{1}{2}} . \end{aligned}$$

By noting Korn's inequality and recalling (20) we have that

$$\begin{aligned} &|B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \Pi_p p - p)| \\ &\leq C \|\hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{1,h} \left(\sum_{K \in \mathcal{T}_h} \left(\|p - \Pi_p p\|_{0,K}^2 + h_K k_K^{-2} \|p - \Pi_p p\|_{0,\partial K}^2 \right) \right)^{\frac{1}{2}} . \end{aligned}$$

Invoking (26) results in

$$|B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \Pi_p p - p)| \leq C \|\hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{1,h} \left(\sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K}} \|p\|_{s_K,K}^2 \right)^{\frac{1}{2}} .$$

Similarly,

$$\begin{aligned}
T_3 &= |B_h(\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}, \hat{\xi}_p)| \\
&\leq C\|\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}\|_{1,h} \left(\sum_{K \in \mathcal{T}_h} \left(\|\hat{\xi}_p\|_{0,K}^2 + h_K k_K^{-2} \|\hat{\xi}_p\|_{0,\partial K}^2 \right) \right)^{\frac{1}{2}}.
\end{aligned}$$

By applying an inverse estimate to the boundary term, see, e.g. Theorem 4.76 in Schwab (1998) and scaling, and using (26), yields

$$
\begin{aligned}
|B_h(\Pi_{\mathbf{u}}\mathbf{u}-\mathbf{u},\hat{\xi}_p)| &\leq C\|\Pi_{\mathbf{u}}\mathbf{u}-\mathbf{u}\|_{1,h}\|\hat{\xi}_p\|_{0,\Omega} \\
&\leq C\|\hat{\xi}_p\|_{0,\Omega}\left(\sum_{K\in\mathcal{T}_h}\frac{h_K^{2\min\{s_K,k_K\}}}{k_K^{2s_K-1}}\|\mathbf{u}\|_{s_K+1,K}^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Finally, recalling that $\|(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)\|_{\mathrm{DG}} \leq 1$, using that

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathrm{DG}} \leq \|(\boldsymbol{\eta}_{\mathbf{u}}, \eta_p)\|_{\mathrm{DG}} + \|(\boldsymbol{\xi}_{\mathbf{u}}, \xi_p)\|_{\mathrm{DG}},$$

and noting the bounds on T_1, T_2, T_3 completes the proof. $\square$

Example 1: Smooth solution

Let Ω be the L-shaped domain $(-1,1)^2 \setminus [0,1) \times (-1,0]$, and consider the nonlinearity

$$\mu(|\mathbf{e}(\mathbf{u})|) = 2 + \frac{1}{1 + |e(\mathbf{u})|^2}.$$

In addition, we select $\mathbf{f}$ so that the analytical solution to (1)–(3) is:

$$\mathbf{u}(x,y) = \begin{pmatrix} -\mathrm{e}^x(y\cos(y) + \sin(y)) \\ \mathrm{e}^x y \sin(y) \end{pmatrix},$$

$$p(x,y) = 2\mathrm{e}^x \sin(y) - (2(1-e)(\cos(1) - 1))/3.$$

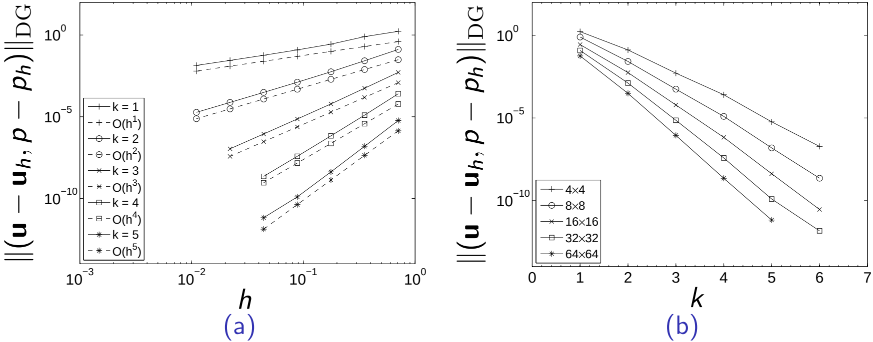

Figure: Convergence of the DGFEM with h-refinement; p-refinement.

Example 2: Cavity problem

We consider the cavity-like problem from Berrone & Süli (2008) using the Carreau law nonlinearity

$$\mu(|\mathbf{e}(\mathbf{u})|) = k_\infty + (k_0 - k_\infty)(1 + \lambda|\mathbf{e}(\mathbf{u})|^2)^{(\theta-2)/2},$$

with $k_\infty = 1, k_0 = 2, \lambda = 1$ and $\theta = 1.2$. We let Ω be the unit square $(0,1)^2 \subset \mathbb{R}^2$ and select the forcing function $\mathbf{f}$ so that the analytical solution to (1)–(3) is given by

$$\mathbf{u}(x,y) = \begin{pmatrix} \left(1 - \cos\left(2\frac{\pi\left(\mathrm{e}^{\theta x}-1\right)}{\mathrm{e}^{\theta}-1}\right)\right)\sin(2\pi y) \\ -\theta\mathrm{e}^{\theta x}\sin\left(2\frac{\pi\left(\mathrm{e}^{\theta x}-1\right)}{\mathrm{e}^{\theta}-1}\right)\frac{1-\cos(2\pi y)}{\mathrm{e}^{\theta}-1} \end{pmatrix},$$

$$p(x,y) = 2\pi\theta\mathrm{e}^{\theta x}\sin\left(2\frac{\pi\left(\mathrm{e}^{\theta x}-1\right)}{\mathrm{e}^{\theta}-1}\right)\frac{\sin(2\pi y)}{\mathrm{e}^{\theta}-1}.$$

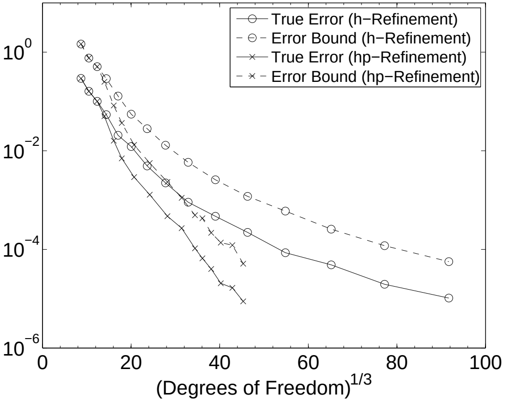

Figure: Example 2. Comparison of the error in the DGFEM norm with both *h*- and *hp*-refinement, w.r.t. the number of degrees of freedom.

Ten Lectures on the Convergence Analysis of

Discontinuous Galerkin Finite Element Methods

for Nonlinear PDEs

Lectures 7 and 8

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May 20, 2012

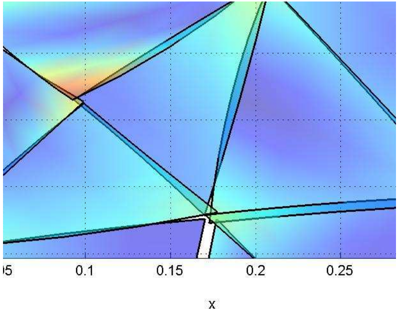
Background

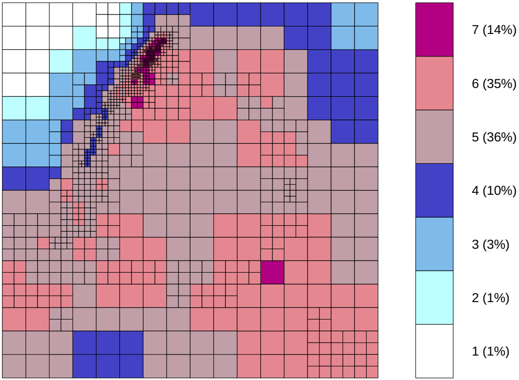
Over the past decade there has been growing interest in a class of finite element methods for PDEs, commonly referred to as Discontinuous Galerkin Finite Element Methods (DGFEMs).

Why DGFEM?

- Local conservation properties
- High-order extension of finite volume methods
- Linear hyperbolic PDEs: no numerical stabilisation needed
- Ease of implementing h-refinement with hanging nodes
- Ease of implementing p-refinement with locally varying p

Why DGFEM?





Süli & Houston (2001)

hp-adaptive DGFEM for hyperbolic & mixed elliptic-hyperbolic PDEs

Survey article

Cockburn, Karniadakis, Shu (Springer, 2000)

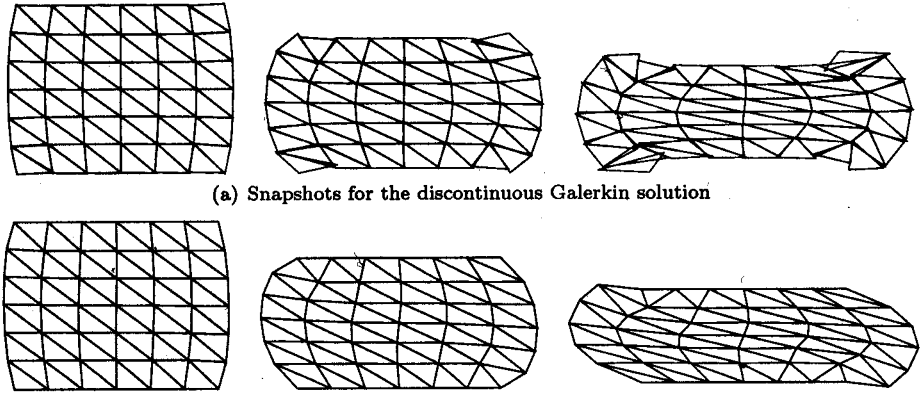
- ✓ Fluid dynamics
- ✓ Electromagnetism

The development and analysis of DGFEMs for nonlinear PDEs in the area of solid mechanics have lagged behind.

DGFEM in solid mechanics?

DGFEM $\Longrightarrow$ No locking:

T.P. Wihler, Locking-free DGFEM for elasticity problems in polygons, IMA J. Numer. Anal., 24 (2004), 45-75.



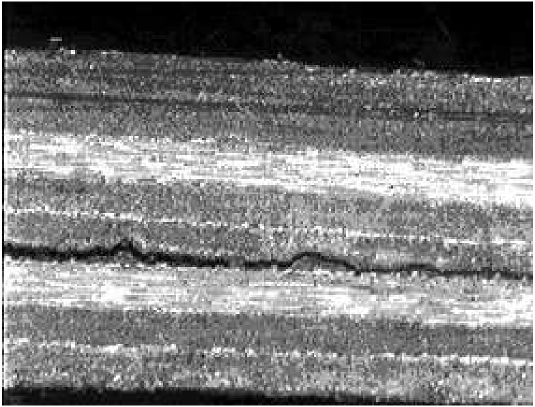
(a) Snapshots for the discontinuous Galerkin solution

(b) Snapshots for the conforming approximation solution

Ten Eyck & Lew (IJNME 2006)

DGFEM in solid mechanics?

Delamination



DGFEM in solid mechanics?

Fracture mechanics

Overview

DG finite element approximation of nonlinear elliptic and hyperbolic PDEs that arise in elastostatics and elastodynamics.

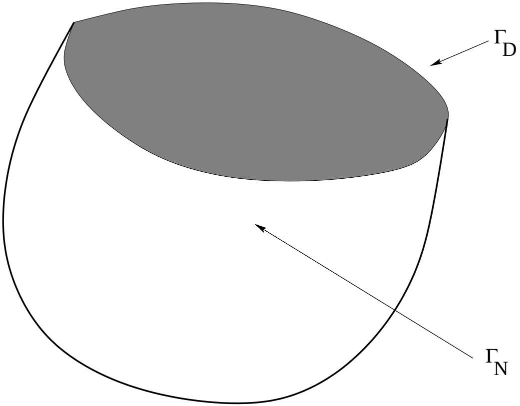
- Statement of the problem: static & dynamic problems
- Analytical difficulties:
  – Legendre–Hadamard condition and Gårding's inequality;
  – Added difficulties in the case of DGFEMs
- DGFEM: convergence proof for static and dynamic problems
- Conclusions and open problems

Ortner & Süli:

`SIAM Journal of Numerical Analysis` **45**`(4) 1370-1397, July 2007.`

1. Statement of the problem

$\Omega \subset \mathbb{R}^d$, $d = 2, 3$, bounded open with Lipschitz boundary $\Gamma = \partial\Omega$



$\Gamma_{\mathrm{D}} \subset \Gamma$, $\mathrm{meas}_{d-1}(\Gamma_{\mathrm{D}}) > 0$ and $\Gamma_{\mathrm{N}} = \Gamma \setminus \Gamma_{\mathrm{D}}$.

$x \in \Omega \subset \mathbb{R}^d$	position of a point in undeformed state
$y \in \mathbb{R}^d$	position of a point in deformed state, $y = y(x)$

A configuration is stable if it (locally) minimizes the total energy:

$$E(y) = \int_\Omega W(\nabla y(x))\,\mathrm{d}x - \int_\Omega f(x) \cdot y(x)\,\mathrm{d}x, \quad \nabla y(x) \in \mathbb{R}^{d\times d}_+, \quad x \in \Omega.$$

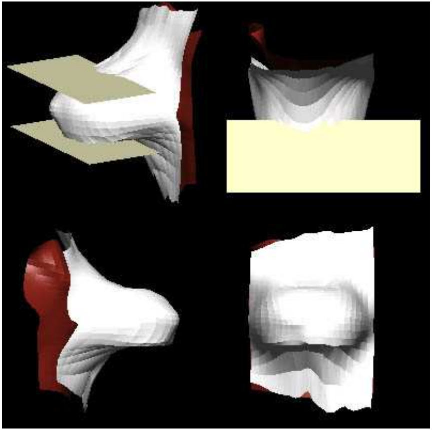
$W = W(F)$ is the isothermal stored energy density function.

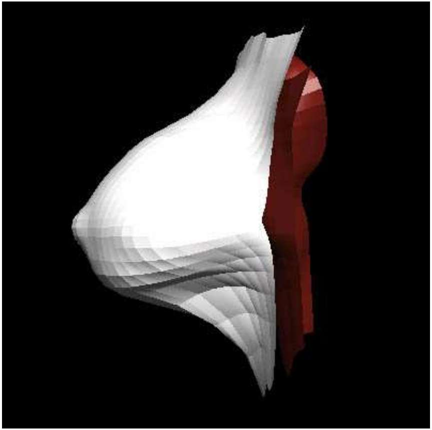
Examples

$$\begin{aligned} F &:= \nabla y && \text{Deformation gradient} \\ C &:= F^{\mathrm{T}} F && \text{Cauchy–Green tensor} \\ E &:= \tfrac{1}{2}(F^{\mathrm{T}} F - I) && \text{Green strain tensor} \end{aligned}$$

- Neohookean material:
  $$W(F) = \frac{\mu}{2}|F|^2 + g(\det F), \qquad g(z) = \frac{\lambda}{4} z^2 - \frac{\lambda+2\mu}{2} \log(z).$$
- Mooney–Rivlin material: $\longrightarrow$ rubber & foam-like materials
  $$W(F) = a|F|^2 + b|\mathrm{cof}\, F|^2 + g(\det F).$$
- Fung material: $\longrightarrow$ soft tissue
  $$W(F) = a(\mathrm{e}^{b\,\mathrm{tr}(E)} - 1).$$

Fung material in soft-tissue modelling





Pras Pathmanathan (Oxford)

Seek a displacement field $u : \overline{\Omega} \to \mathbb{R}^d$ such that u is a stationary point of the energy functional

$$E(v) := \int_{\Omega} W(\nabla v(x))\,\mathrm{d}x - \int_{\Omega} f(x)\cdot v(x)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}}(s)\cdot v(s)\,\mathrm{d}s$$

over all $v = (v_1, \ldots, v_d)^{\mathrm{T}}$ such that $v = g_{\mathrm{D}}$ on Γ_{D}.

$$\begin{array}{rcll} W & \in & \mathrm{C}^2(\mathbb{R}^{d\times d};\mathbb{R}) & \quad \text{– stored energy density function} \\ f & \in & \mathrm{L}^2(\Omega)^d & \quad \text{– given body force} \\ g_{\mathrm{N}} & \in & \mathrm{L}^2(\Gamma_{\mathrm{N}})^d & \quad \text{– normal component of stress on } \Gamma_{\mathrm{N}} \end{array}$$

Euler–Lagrange equation

Define $S = \nabla W$ — the Piola–Kirchhoff stress tensor, with entries

$$S_{i\alpha}(\eta) := \frac{\partial}{\partial \eta_{i\alpha}} W(\eta), \qquad \eta \in \mathbb{R}^{d\times d}.$$

$\longrightarrow$ [Ball, 2002]

Sufficiently smooth stationary points $u = u(x)$ of E satisfy

$$-\sum_{\alpha=1}^{d} \partial_{x_\alpha} S_{i\alpha}(\nabla u(x)) = f_i(x), \qquad i = 1, \dots, d, \qquad x \in \Omega,$$

$$u = g_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}} \qquad \text{and} \qquad S(\nabla u)\nu = g_{\mathrm{N}} \quad \text{on } \Gamma_{\mathrm{N}},$$

ν is the unit outward normal vector to Γ, $\partial_{x_\alpha} = \partial/\partial x_\alpha$.

Static and dynamic problems

Static problem

$$-\sum_{\alpha=1}^{d} \partial_{x_\alpha} S_{i\alpha}(\nabla u(x)) = f_i(x), \qquad i = 1, \dots, d, \qquad x \in \Omega,$$

$$u = g_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}} \qquad \text{and} \qquad S(\nabla u)\nu = g_{\mathrm{N}} \quad \text{on } \Gamma_{\mathrm{N}}.$$

Dynamic problem

$$\partial_t^2 u_i - \sum_{\alpha=1}^{d} \partial_{x_\alpha}(S_{i\alpha}(\nabla u)) = f_i(x,t), \quad i = 1, \dots, d, \quad x \in \Omega,\ t \in (0,T],$$

$$u(\cdot,t) = g_{\mathrm{D}}(\cdot,t) \quad \text{on } \Gamma_{\mathrm{D}} \qquad \text{and} \qquad S(\nabla u(\cdot,t))\nu = g_{\mathrm{N}}(\cdot,t) \quad \text{on } \Gamma_{\mathrm{N}},$$

$$u(\cdot,0) = u_0, \qquad \partial_t u(\cdot,0) = u_1.$$

Existence of solutions to dynamic problem

Hughes, Kato, Marsden (Arch. Rational Mech. Anal., 1977)
Chen & von Wahl (J. Reine Angew. Math., 1982)
Dafermos & Hrusa (Arch. Rational Mech. Anal., 1985)

Related problem: nonlinear viscoelasticity:
$u_{tt} = \nabla \cdot (S(\nabla u) + \nabla u_t)$

Friesecke & Dolzmann (SIMA, 1997)
Demoulini (Arch. Rational Mech. Anal., 2000)

FE approximation of dynamic problem

C^0 finite element methods:
Makridakis (Math. Comput., 1993): $u_{tt} = \nabla \cdot (S(\nabla u))$

Related problem: nonlinear viscoelasticity
Carstensen & Dolzmann (SINUM, 2004): $u_{tt} = \nabla \cdot (S(\nabla u) + \nabla u_t)$

Discontinuous Galerkin Finite Element Methods?
The analysis of DGFEMs for static and dynamic *nonlinear* elasticity had not been considered prior to Ortner & Süli (2007).

Discontinuous Galerkin method for elasticity

- ⋆ Hansbo & Larson (CMAME 2002):
  incompressible & nearly incompressible elasticity
- ⋆ Arnold, Brezzi, & Marini (J. Sci. Comp. 2005):
  Reissner–Mindlin plate
- ⋆ Wihler (IMAJNA 2004, MathComp 2006):
  Adaptive volume-locking-free DGFEMs

- ⋆ Negri (2004 SISSA Report; Adv. Math. Sci. Appl. **15**, 283–306):
  Discontinuous FE approximation of free-discontinuity problems
- ⋆ Ten Eyck & Lew (IJNME 2006):
  Linearized stability of DGFEM for nonlinear elastostatics

► However, the analysis of DGFEMs for static and dynamic *nonlinear* elasticity is un(der)developed. ◄

2. Analytical difficulties

The function W is required to satisfy

$$W(F) = W(QF), \quad Q \in SO(d) \qquad \& \qquad \lim_{\det(F)\to 0_+} W(F) = +\infty.$$

$\Rightarrow$ W can't be convex [Antman (2005); Ball (1977); Morrey (1952)]:

convexity $\Rightarrow$ polyconvexity $\Rightarrow$ quasiconvexity $\Rightarrow$ rank-1 convexity

Rank-1 convexity of W, viz.

$$t \mapsto W(F + t\, a \otimes b) \text{ convex for all } F \in \mathbb{R}^{d\times d}, \ \ a, b \in \mathbb{R}^d$$

corresponds to requiring that $A = \nabla S$, where $S = \nabla W$, satisfies the Legendre–Hadamard condition.

Legendre–Hadamard condition

$$A_{i\alpha j\beta}(\chi) := \frac{\partial}{\partial \chi_{j\beta}} S_{i\alpha}(\chi) = \frac{\partial^2}{\partial \chi_{i\alpha} \partial \chi_{j\beta}} W(\chi), \qquad \chi \in \mathbb{R}^{d\times d}.$$

Legendre–Hadamard condition:

$$\sum_{i,\alpha,j,\beta=1}^{d} A_{i\alpha j\beta}(\chi) a_i a_j b_\alpha b_\beta \geq 0 \qquad \forall a \in \mathbb{R}^d,\ \ \forall b \in \mathbb{R}^d.$$

Strong ellipticity (SE) (Nirenberg)

$\exists M_1 > 0$ such that $\forall \chi \in \mathbb{R}^{d\times d}$:

$$\mathrm{Re} \sum_{i,\alpha,j,\beta=1}^{d} A_{i\alpha j\beta}(\chi) a_i \overline{a_j} b_\alpha b_\beta \geq M_1 |a|^2 |b|^2 \quad \forall a \in \mathbb{C}^d,\ \ \forall b \in \mathbb{R}^d.$$

Remark: A stronger hypothesis

Uniform ellipticity:

$\exists M_1 > 0$ such that, for all $\chi \in \mathbb{R}^{d\times d}$,

$$\boxed{\operatorname{Re} \sum_{i,\alpha,j,\beta=1}^{d} A_{i\alpha j\beta}(\chi)\xi_{i\alpha}\overline{\xi_{j\beta}} \geq M_1|\xi|^2 \quad \forall \xi \in \mathbb{C}^{d\times d}} \qquad \text{(UE)}$$

$\xi = a \otimes b$, $a \in \mathbb{C}^d$, $b \in \mathbb{R}^d$: (UE) $\Rightarrow$ (SE)

W is convex $\quad\Rightarrow\quad$ W is rank-1 convex

Gårding inequality

If (SE) condition holds, then the elliptic differential operator

$$\mathcal{L} : u \mapsto -\sum_{\alpha,j,\beta=1}^{d} \frac{\partial}{\partial x_\alpha}\left(A_{i\alpha j\beta}(\chi(\cdot))\frac{\partial u_j}{\partial x_\beta}\right), \qquad i = 1, \ldots, d,$$

with $A_{i\alpha j\beta} \circ \chi \in \mathrm{C}(\overline{\Omega})$, satisfies the following

Gårding inequality (Morrey (1966))

$\exists M_1 > 0,\ M_0 \geq 0$ such that

$$\operatorname{Re}(\mathcal{L}u, u) \geq \frac{1}{2}M_1\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} - M_0\|u\|^2_{\mathrm{L}^2(\Omega)}, \qquad u \in \mathrm{H}^1_0(\Omega)^d.$$

Special case: $\begin{cases} A_{i\alpha j\beta} \text{ constant for } i,\alpha,j,\beta = 1,\ldots,d, \\ \Omega = \mathbb{R}^d. \end{cases}$

By Plancherel's Theorem:

$$\begin{aligned} \operatorname{Re}(\mathcal{L}u, u) &= (2\pi)^d \operatorname{Re}(F(\mathcal{L}u), Fu) \\ &= (2\pi)^d \operatorname{Re} \int_{\mathbb{R}^d} \sum_{i,\alpha,j,\beta=1}^{d} A_{i\alpha,j\beta} \xi_\alpha \xi_\beta F(u_i) \overline{F(u_j)} \, \mathrm{d}\xi \\ &\geq (2\pi)^d \int_{\mathbb{R}^d} M_1 |\xi|^2 |F(u)|^2 \, \mathrm{d}\xi \\ &= M_1 \|\nabla u\|^2_{\mathrm{L}^2(\Omega)} \\ &= M_1 \|u\|^2_{\mathrm{H}^1(\mathbb{R}^d)} - M_1 \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}. \end{aligned}$$

General case: partition of unity on Ω together with $A_{i\alpha j\beta} \in \mathrm{C}(\overline{\Omega})$.

Analysis of non

Choose finite-dimensional subspace

$$V_{DG} := V^p(\Omega, \mathcal{T}_h) \not\subseteq \mathrm{H}^1_0(\Omega)^d$$

consisting of discontinuous piecewise polynomial functions of degree p over a subdivision $\mathcal{T}_h$ of Ω.

$\longrightarrow$ Gårding inequality holds over V_{DG}. ?

This leads to optimal-order convergence in $\| \cdot \|_{\mathrm{H}^1(\Omega)}$. ?

Tool no. 1: Broken Gårding inequality

$$\mathcal{Z}_\delta := \{\Phi \in \mathrm{L}^\infty(\Omega)^{d\times d} \,:\, \|\Phi - \nabla u\|_{\mathrm{L}^\infty(\Omega)} \le \delta\},$$

$$a(\Phi; v, w) := \sum_{i,\alpha,j,\beta=1}^{d} \int_\Omega A_{i\alpha j\beta}(\Phi)\partial_{x_\alpha} v_i \partial_{x_\beta} w_j \,\mathrm{d}x, \qquad \Phi \in \mathcal{Z}_\delta.$$

Theorem (Ortner & Süli (2007))

Let $u \in \mathrm{C}^1(\overline{\Omega})^d$ be such that the following Gårding inequality holds:

$$a(\nabla u; v, v) \ge \tfrac{1}{2} M_1 \|\nabla v\|^2_{\mathrm{L}^2(\Omega)} - M_0 \|v\|^2_{\mathrm{L}^2(\Omega)} \qquad \forall v \in \mathrm{H}^1_{\mathrm{D},0}(\Omega)^d,$$

where $M_1 > 0$ and $M_0 \ge 0$. Assume furthermore that $\delta \le M_1/(4L_\delta)$, where L_δ is the local Lipschitz constant of $A = \nabla S$ over $\mathcal{Z}_\delta$. Then, for all $\Phi \in \mathcal{Z}_\delta$ and $h \le 1$, the following broken Gårding inequality holds:

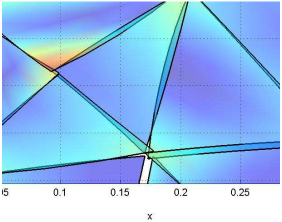
$$a(\Phi; v, v) \ge \tfrac{1}{4} M_1 \|\nabla v\|^2_{\mathrm{L}^2(\Omega)} - 2M_0 \|v\|^2_{\mathrm{L}^2(\Omega)} - C_1 \int_{\Gamma_{\mathrm{int}} \cup \Gamma_{\mathrm{D}}} h^{-1} |[\![v]\!]|^2 \,\mathrm{d}s,$$

for all $v \in V_{DG}$, where $C_1 = C_1(M_0, M_1) > 0$ is independent of h.

3. DGFEM: static problem

$$
\begin{aligned}
\mathcal{E}_{\mathrm{D}} &= \text{set of all faces } e \text{ contained in } \Gamma_{\mathrm{D}} \\
\mathcal{E}_{\mathrm{N}} &= \text{set of all faces } e \text{ contained in } \Gamma_{\mathrm{N}} \\
\mathcal{E}_{\mathrm{int}} &= \text{set of all faces } e \text{ internal to } \Omega \\
\Gamma_{\mathrm{int}} &= \{x \in e \,:\, e \in \mathcal{E}_{\mathrm{int}}\}.
\end{aligned}
$$

$$
[\![v]\!]_e = v^+ \otimes \nu^+ + v^- \otimes \nu^-, \qquad \{\!\{v\}\!\}_e = \frac{1}{2}\left(v^+ + v^-\right).
$$



Construction

$$-\int_\kappa (\nabla \cdot S(\nabla u)) \cdot v \,\mathrm{d}x = \int_\kappa f \cdot v \,\mathrm{d}x,$$

$$\int_\kappa S(\nabla u) : \nabla v \,\mathrm{d}x - \int_{\partial\kappa} S(\nabla u)\nu \cdot v \,\mathrm{d}s = \int_\kappa f \cdot v \,\mathrm{d}x,$$

$$\int_\kappa S(\nabla u) : \nabla v \,\mathrm{d}x - \int_{\partial\kappa} \{\!\{S(\nabla u)\}\!\} : v \otimes \boldsymbol{\nu} \,\mathrm{d}s + \int_{\partial\kappa} \sigma [\![u]\!] : [\![v]\!] \,\mathrm{d}s = \int_\kappa f \cdot v \,\mathrm{d}x,$$

where $\sigma > 0$ is the discontinuity penalization parameter.

Sum over all $\kappa \in \mathcal{T}_h$, and reorganize sums over element boundaries $\partial\kappa$ into sums over element edges $e \in \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{D}} \cup \mathcal{E}_{\mathrm{N}}$.

Semilinear form $B_{DG}(\cdot,\cdot)$ and linear functional $\ell_{DG}(\cdot)$

$$
\begin{aligned}
B_{DG}(w,v) \;&=\; \int_\Omega S(\nabla w) : \nabla v \,\mathrm{d}x - \int_{\Gamma_{\mathrm{int,D}}} \{\!\{ S(\nabla w) \}\!\} : [\![v]\!] \,\mathrm{d}s \\
&\quad + \int_{\Gamma_{\mathrm{int,D}}} \sigma [\![w]\!] : [\![v]\!] \,\mathrm{d}s,
\end{aligned}
$$

$$
\ell_{DG}(v) = \int_\Omega f \cdot v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{D}}} \sigma g_{\mathrm{D}} \cdot v \,\mathrm{d}s + \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}} \cdot v \,\mathrm{d}s.
$$

Find $u_{\mathrm{DG}} \in V_{DG}$ such that

$$
B_{DG}(u_{\mathrm{DG}}, v) = \ell_{DG}(v) \qquad \forall v \in V_{DG}.
$$

Convergence analysis

Broken Sobolev space

$$\mathrm{H}^1(\Omega, \mathcal{T}_h)^d = \{v \in \mathrm{L}^2(\Omega)^d : \ v|_\kappa \in \mathrm{H}^1(\kappa)^d \quad \forall \kappa \in \mathcal{T}_h\}$$

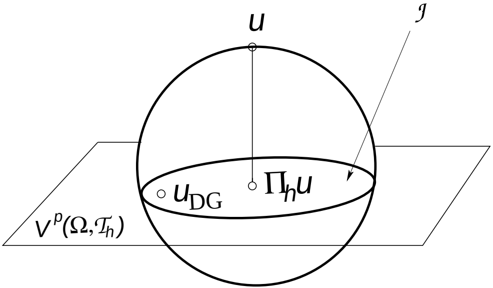
equipped with the norm

$$\|v\|_{DG} = \left(\int_\Omega |\nabla v|^2 \, \mathrm{d}x + \int_{\Gamma_{\mathrm{int,D}}} \sigma |[\![v]\!]|^2 \, \mathrm{d}s \right)^{1/2}.$$

Discontinuity penalisation parameter:

$$\sigma|_e = \frac{\alpha}{h_e}, \qquad \alpha > 0, \qquad e \subset \Gamma_{\mathrm{int,D}} := \Gamma_{\mathrm{int}} \cup \Gamma_{\mathrm{DG}}.$$

$$u - u_{\mathrm{DG}} = (u - \Pi_h u) - (u_{\mathrm{DG}} - \Pi_h u).$$



$$\mathcal{J} := \left\{ v \in V_{DG} \, : \, \|v - \Pi_h u\|_{DG} \leq C_* h^r \|u\|_{\mathrm{H}^{r+1}(\Omega)} \right\}.$$

Construct nonlinear mapping $\mathcal{N} : \mathcal{J} \to \mathcal{J}$ with fixed point u_{DG}.

$$B_{DG}(u_{\mathrm{DG}}, v) = \ell_{DG}(v) = B_{DG}(u, v) \qquad \forall v \in V_{DG}.$$

$$B_{DG}(u_{\mathrm{DG}}, v) - B_{DG}(\Pi_h u, v) = B_{DG}(u, v) - B_{DG}(\Pi_h u, v) \qquad \forall v \in V_{DG}.$$

Consider the integral mean-value linearization:

$$B_{DG}(w_1, v) - B_{DG}(w_2, v) = \int_0^1 \tilde{b}(w_2 + \tau(w_1 - w_2); w_1 - w_2, v)\,\mathrm{d}\tau.$$

$$\int_0^1 \tilde{b}(\Pi_h u + \tau(u_{\mathrm{DG}} - \Pi_h u); u_{\mathrm{DG}} - \Pi_h u, v)\,\mathrm{d}\tau$$
$$= \int_0^1 \tilde{b}(\Pi_h u + \tau(u - \Pi_h u); u - \Pi_h u, v)\,\mathrm{d}\tau \quad \forall v \in V_{DG}.$$

Given $\varphi \in \mathcal{J}$, we denote by $u_\varphi = \mathcal{N}(\varphi) \in V_{DG}$ the solution to the following linear variational problem: find $u_\varphi \in V_{DG}$ s.t.

$$\int_0^1 \tilde{b}(\Pi_h u + \tau(\varphi - \Pi_h u); u_\varphi - \Pi_h u, v)\,\mathrm{d}\tau$$
$$= \int_0^1 \tilde{b}(\Pi_h u + \tau(u - \Pi_h u); u - \Pi_h u, v)\,\mathrm{d}\tau \quad \forall v \in V_{DG}.$$

What is $\tilde{b}$?

$\tilde{b}(w;\cdot,\cdot)$ is a bilinear form defined by

$$\begin{aligned}
\tilde{b}(w;z,v) \;&=\; \int_\Omega \sum_{i,\alpha,j,\beta=1}^{d} A_{i\alpha j\beta}(\nabla w)\frac{\partial v_i}{\partial x_\alpha}\frac{\partial z_j}{\partial x_\beta}\,\mathrm{d}x \\
&- \int_{\Gamma_{\mathrm{int,D}}} \sum_{i,\alpha,j,\beta=1}^{d} \left\{\!\!\left\{ A_{i\alpha j\beta}(\nabla w)\frac{\partial z_j}{\partial x_\beta} \right\}\!\!\right\} [\![v]\!]_{i\alpha}\,\mathrm{d}s \\
&+ \int_{\Gamma_{\mathrm{int,D}}} \sigma [\![z]\!] : [\![v]\!]\,\mathrm{d}s.
\end{aligned}$$

Assuming Gårding with $M_0 = 0$ ... [to be weakened to $M_0 \geq 0$]:

- $\mathcal{N}(\mathcal{J}) \subset \mathcal{J}$
- $\mathcal{N}$ is a contraction in the norm $\|\cdot\|_{DG}$.

Banach's fixed point theorem $\Rightarrow \exists_1 u_{\mathrm{DG}} \in \mathcal{J}$, fixed point of $\mathcal{N}$:

$$\|u_{\mathrm{DG}} - \Pi_h u\|_{DG} \leq C_* h^r \|u\|_{\mathrm{H}^{r+1}(\Omega)}, \quad d/2 < r \leq \min(m, p).$$

Hence,

$$\begin{aligned} \|u - u_{\mathrm{DG}}\|_{DG} &\leq \|u - \Pi_h u\|_{DG} + \|u_{\mathrm{DG}} - \Pi_h u\|_{DG} \\ &\lesssim h^r \|u\|_{\mathrm{H}^{r+1}(\Omega)}, \quad d/2 < r \leq \min(m, p). \end{aligned}$$

4. DGFEM: dynamic problem

Let $t \in [0, T]$. Find $u_{\mathrm{DG}}(\cdot, t) \in V_{DG}$ such that

$$(\ddot{u}_{\mathrm{DG}}, v) + B_{DG}(u_{\mathrm{DG}}, v) + \int_{\Gamma_{\mathrm{int,D}}} \sigma [\![\dot{u}_{\mathrm{DG}}]\!] : [\![v]\!] \,\mathrm{d}s$$
$$= \ell_{DG}(v) + \int_{\Gamma_{\mathrm{D}}} \sigma \dot{g}_{\mathrm{DG}} \cdot v \,\mathrm{d}s \quad \forall v \in V_{DG},$$

for all $t \in (0, T]$ and

$$u_{\mathrm{DG}}(x, 0) = u^0_{\mathrm{DG}}(x), \qquad \dot{u}_{\mathrm{DG}}(x, 0) = u^1_{\mathrm{DG}}(x), \qquad x \in \Omega,$$

with u^0_{DG} and u^1_{DG} in V_{DG}.

We are now assuming that Gårding's inequality holds with $M_0 \geq 0$.

Tool no. 2: Nonlinear elliptic projector

Define $Z(t) \in V_{DG}$ by

$$B_{DG}(Z(t), v) + 2M_0(Z(t), v) = B_{DG}(u(t), v) + 2M_0(u(t), v), \quad t \in [0, T],$$

where M_0 is as in the broken Gårding inequality. Then,

$$\|u(t) - Z(t)\|_{DG} \lesssim h^r \|u(t)\|_{\mathrm{H}^{r+1}(\Omega)},$$

$$\|\dot{u}(t) - \dot{Z}(t)\|_{DG} \lesssim h^r \left(\|u(t)\|_{\mathrm{H}^{r+1}(\Omega)} + \|\dot{u}(t)\|_{\mathrm{H}^{r+1}(\Omega)} \right),$$

$$\|\ddot{u}(t) - \ddot{Z}(t)\|_{DG} \lesssim h^r \left(\|u(t)\|_{\mathrm{H}^{r+1}(\Omega)} + \|\dot{u}(t)\|_{\mathrm{H}^{r+1}(\Omega)} + \|\ddot{u}(t)\|_{\mathrm{H}^{r+1}(\Omega)} \right).$$

Idea of convergence analysis

$$u(t) - u_{\mathrm{DG}}(t) = (u(t) - Z(t)) - (u_{\mathrm{DG}} - Z(t)) =: \eta(t) - \xi(t).$$

Using a fixed point argument, similarly as in the elliptic case,

$$\begin{aligned} &\|\dot{\xi}(t)\|^2_{\mathrm{L}^2(\Omega)} + \|\xi(t)\|^2_{DG} + \int_0^t \int_{\Gamma_{\mathrm{int,D}}} \sigma |[\![\dot{\xi}]\!]|^2 \,\mathrm{d}s\,\mathrm{d}\tau \\ &\lesssim \left(\int_0^t \|\ddot{\eta}(\tau)\|^2_{\mathrm{L}^2(\Omega)} \,\mathrm{d}\tau + \int_0^t \int_{\Gamma_{\mathrm{int,D}}} \sigma |[\![\dot{\eta}]\!]|^2 \,\mathrm{d}s\,\mathrm{d}\tau \right). \end{aligned}$$

Final bound

Assuming that $(d/2) + 1 < r \leq \min(m, p)$, we have proved the following optimal-order error bound:

$$\max_{t \in [0,T]} (\|\dot{u}(t) - \dot{u}_{\mathrm{DG}}(t)\|_{\mathrm{L}^2(\Omega)} + \|u(t) - u_{\mathrm{DG}}(t)\|_{DG})$$
$$\lesssim h^r \max_{t \in [0,T]} (\|u(t)\|_{\mathrm{H}^{r+1}(\Omega)} + \|\dot{u}(t)\|_{\mathrm{H}^{r+1}(\Omega)} + \|\ddot{u}(t)\|_{\mathrm{H}^{r+1}(\Omega)}).$$

5. Conclusions

1. We developed discontinuous Galerkin finite element methods for static and dynamic PDEs in nonlinear elasticity.
2. We showed that the method exhibits optimal rates of convergence in the broken energy norm. Key new result: broken Gårding inequality.
   Related work: Brenner (SINUM 2003, Math. Comput. 2004)
3. Extension of the analysis to the case of nonlinear viscoelasticity $u_{tt} = \nabla \cdot (S(\nabla u) + \nabla u_t)$ is trivial.
4. Open problem: analysis of space-time discrete schemes.
5. Implementation of the method: cf. Ten Eyck & Lew (2006).

Ten Lectures on the Convergence Analysis of

Discontinuous Galerkin Finite Element Methods

for Nonlinear PDEs

Lectures 9 and 10

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Cahn–Hilliard equation with a convection term

Find functions c and w defined on $\Omega \times [0, T]$, $T > 0$, s.t.:

$$
\begin{aligned}
\partial_t c - \frac{1}{\mathrm{Pe}}\Delta w + \nabla \cdot (\mathbf{u} c) = 0 \qquad & \text{in } \Omega_T := \Omega \times (0, T], \\
w = \Phi'(c) - \gamma^2 \Delta c \qquad & \text{in } \Omega_T, \\
c(\cdot, 0) = c_0(\cdot) \qquad & \text{in } \Omega, \\
\partial_{\mathbf{n}} c = \partial_{\mathbf{n}} w = 0 \qquad & \text{on } \partial\Omega_T := \partial\Omega \times (0, T].
\end{aligned}
$$

Ω: bounded convex polygonal domain in $\mathbb{R}^2$, with boundary $\partial\Omega$;
$\mathbf{n}$: outward-pointing unit normal to $\partial\Omega$;
c: order-parameter, such that $c(x,t) \approx 1$ (resp. $c(x,t) \approx -1$) if, at time $t \in [0, T]$ fluid 1 (resp. fluid 2) is present at $x \in \Omega$;
$\mathbf{u} \in \mathrm{H}(\mathrm{div}, \Omega) \cap [\mathrm{C}(\overline{\Omega})]^2$ is a given function such that $\nabla \cdot \mathbf{u} = 0$ in Ω and $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$; where, $\mathrm{H}(\mathrm{div}, \Omega) := \{\mathbf{v} \in [\mathrm{L}^2(\Omega)]^2 \,:\, \nabla \cdot \mathbf{v} \in \mathrm{L}^2(\Omega)\}$.

$\gamma > 0$: interface parameter; typically in the range $10^{-3} - 10^{-2}$.

$\Phi(\cdot)$: free-energy; taken to be

$$\Phi(c) := \frac{1}{4}(1 - c^2)^2.$$

Pe is the Péclet number, which, for ease of writing, is taken $= 1$.

$\partial_t \eta := \frac{\partial \eta}{\partial t}$ and $\partial_{\mathbf{n}} \eta := \nabla \eta \cdot \mathbf{n}$.

The Cahn–Hilliard equation was originally introduced as a phenomenological model of phase separation in a binary alloys [Cahn–Hilliard (1958), Cahn (1961)].

More recently: in studies of phase transitions and interface dynamics, related free-boundary problems, multiphase fluids and polymer solutions [Boyer (2002), Kim (2005), Modica (1987)].

Derivation and analysis of the equation: cf. Elliott (1989).

Finite element approximation

Results for continuous finite element approximations of the Cahn–Hilliard equation include:

- optimal order error estimates for a semidiscrete splitting method by Elliott, French and Milner (1989),
- optimal order error estimates for a fully-discrete splitting method in one space dimension with weaker regularity assumptions by Du and Nicolaides (1991), and
- convergence of a fully-discrete splitting method with a nonsmooth logarithmic free-energy proved by Copetti and Elliott in (1992).

- Continuous FE approximations of Cahn–Hilliard systems, modelling phase separation of multi-component alloys, have been studied by Blowey, Copetti, Elliott (1996), Barrett and Blowey (1996, 2001).
- Feng and Prohl (2001) proved near-optimal error estimates for a fully discrete mixed finite element approximations of the Cahn–Hilliard equation and studied the dependence on the interface parameter γ.

Computational difficulties

Explicit discretizations require severe time-step restrictions of the form $\tau \sim h^4$, and therefore implicit methods should be used.

To accurately capture the interface dynamics, high spatial resolution is needed: typically at least 8–10 elements are required across the interfacial region (cf. Elliott and Stuart (1993)); otherwise, spurious numerical solutions can be introduced.

Near the interface, the leading-order term in the asymptotic expansion of c for $0 < \gamma \ll 1$ is (Elliott & French (1987)):

$$\tanh\left(\frac{1}{\gamma\sqrt{2}}\,\mathrm{d}\right),$$

where d is the signed distance to the centre of the interface.

Thus if the interfacial region is taken to be located where the order-parameter c varies between -0.99 and 0.99 a simple calculation yields that the width of the interface is $\approx 7.5\gamma$.

Cahn–Hilliard system with convection

Arises in Cahn–Hilliard–Navier–Stokes systems (cf. Badalassi et al. (2003), Boyer (1999), Boyer (2004), Jacqmin (1999), Kim (2005), Kim, Kang, Lowengrub (2004)).

If the Péclet number is large (convection-dominated problem), C^0 finite elements are inadequate without stabilization.

Due to their built-in numerical dissipation, no extra stabilization is needed with DGFEMs.

DGFEM for Cahn–Hilliard equation

- Computations:
  Choo and Lee (2005), Wells, Kuhl, Garikipati (2006), Xia, Xu, Shu (2007), Feng (2007).

- Analysis:
  Feng and Karakashian (2007), for equation written as a 4th-order PDE. Optimal-order error bound is derived for c in the broken $\mathrm{L}^2(\mathrm{H}^2)$ norm, for $p \geq 2$.

  Optimal error bounds in $\mathrm{L}^\infty(\mathrm{L}^2)$ and broken $\mathrm{L}^2(\mathrm{H}^1)$ norms for c, for $p \geq 3$. For $p = 2$ suboptimal by one full order w.r.t. h.

We construct a method that is of optimal order for all $p \geq 1$.

Notation and auxiliary results

$$\begin{aligned}\mathrm{V} &:= \{v \in \mathrm{H}^1(\Omega) : (v,1) = 0\},\\ \mathcal{F} &:= \{v \in (\mathrm{H}^1(\Omega))' : \langle v, 1\rangle = 0\},\\ \mathrm{H}^2_{\mathrm{N}}(\Omega) &:= \{v \in \mathrm{H}^2(\Omega) : \partial_{\mathbf{n}} v = 0 \ \text{ on } \partial\Omega\}.\end{aligned}$$

Green's operator: $\mathcal{G} : \mathcal{F} \to \mathrm{V}$, defined by

$$(\nabla(\mathcal{G}z), \nabla\eta) = \langle z, \eta\rangle \qquad \forall \eta \in \mathrm{H}^1(\Omega).$$

The existence of a unique element $\mathcal{G}z \in \mathrm{V}$ for any $z \in \mathcal{F}$ follows by the Lax–Milgram theorem on V, by noting that

$$\mathrm{H}^1(\Omega) = \mathrm{V} \oplus \operatorname{span}\{1\} \quad \text{and} \quad \langle z, 1\rangle = 0 \qquad \forall z \in \mathcal{F}.$$

Weak formulation of the problem

(P) For $\mathbf{u} \in \mathrm{H}(\mathrm{div};\Omega) \cap [\mathrm{C}(\overline{\Omega})]^2$, with $\nabla \cdot \mathbf{u} = 0$ in Ω and $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, find $\{c(\cdot,t), w(\cdot,t)\}$ in $\mathrm{V} \times \mathrm{H}^1(\Omega)$, $t \in [0,T]$:

$$\begin{aligned} (\partial_t c, \eta) + (\nabla w, \nabla \eta) &= b(\mathbf{u}; c, \eta) & \forall \eta \in \mathrm{H}^1(\Omega), \\ (w, \eta) &= (\Phi'(c), \eta) + \gamma^2 (\nabla c, \nabla \eta) & \forall \eta \in \mathrm{H}^1(\Omega), \\ c(\cdot, 0) &= c_0(\cdot) \in \mathrm{H}^2_{\mathsf{N}}(\Omega) \cap \mathrm{V}, & \end{aligned}$$

where

$$b(\mathbf{u}; c, \eta) := \int_\Omega c\, \mathbf{u} \cdot \nabla \eta \,\mathrm{d}x.$$

DGFEM approximation

$\{\mathcal{T}_h\}_{h>0}$ = shape-regular family of partitions of Ω into disjoint open triangles or quadrilaterals κ, such that $\overline{\Omega} = \cup_{\kappa \in \mathcal{T}_h} \overline{\kappa}$;

$h := \max_{\kappa \in \mathcal{T}_h} h_\kappa$ is the spatial discretization parameter, $h_\kappa := \mathrm{diam}(\kappa)$.

As we need to use inverse inequalities, we assume that the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is quasiuniform.

For $p \geq 1$ we associate with $\mathcal{T}_h$ the finite element space

$$\begin{aligned} \mathrm{S}(\Omega, \mathcal{T}_h) \quad := \quad & \{v \in \mathrm{L}^2(\Omega) : v|_\kappa \text{ is a polynomial of degree} \leq p \\ & \qquad \text{on each } \kappa \in \mathcal{T}_h\}. \end{aligned}$$

Broken spaces

Broken Sobolev space:

$$\mathrm{H}^s(\Omega, \mathcal{T}_h) := \{v \in \mathrm{L}^2(\Omega) : v|_\kappa \in \mathrm{H}^s(\kappa) \quad \forall \kappa \in \mathcal{T}_h\}$$

and

$$\mathrm{V}(\Omega, \mathcal{T}_h) := \{v \in \mathrm{H}^1(\Omega, \mathcal{T}_h) : (v, 1) = 0\},$$

$$\mathrm{V}_h := \{v \in \mathrm{S}(\Omega, \mathcal{T}_h) : (v, 1) = 0\}.$$

These spaces are equipped with the norms

$$\|u\|_{s,\mathcal{T}_h} := \left(\sum_{\kappa \in \mathcal{T}_h} \|u\|_{s,\kappa}^2\right)^{\frac{1}{2}} \qquad \text{and} \qquad \|\cdot\|_{1,\mathcal{T}_h},$$

respectively.

Notation

For interior edge e shared by elements κ^+, κ^- and $v \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$:

$$[\![v]\!] := v^+ \boldsymbol{\nu}^+ + v^- \boldsymbol{\nu}^- \qquad \text{and} \qquad \{\!\{v\}\!\} := \frac{1}{2}(v^+ + v^-),$$

where $\boldsymbol{\nu}^i$ is the unit normal vector on e, outward of κ^i, $i = +, -$.

Notation

$\mathcal{E}$ is the set of all interior edges of all elements $\kappa \in \mathcal{T}_h$,

$\sigma|_e := \sigma_e = \frac{\alpha}{h_e}$, where α is a sufficiently large positive constant,

$c_e \geq \theta_0 |\mathbf{u} \cdot \mathbf{n}|$, with θ_0 a positive constant, independent of e and h_e,

h_e is the length of edge e,

$(u, v)_\kappa := \int_\kappa u\, v \,\mathrm{d}x$, with a similar definition of $(u, v)_e$,

∇v, for $v \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$, denotes the element-wise gradient of v.

Bilinear forms

$$\begin{aligned} B_{DG}(v,w) \quad &:= \quad \sum_{\kappa\in\mathcal{T}_h} (\nabla v, \nabla w)_\kappa \\ &\quad - \sum_{e\in\mathcal{E}} \left[([\![v]\!], \{\!\{\nabla w\}\!\})_e + ([\![w]\!], \{\!\{\nabla v\}\!\})_e - (\sigma [\![v]\!], [\![w]\!])_e \right]. \end{aligned}$$

$$\begin{aligned} b_{DG}(\mathbf{u}; v, w) \quad &:= \quad \sum_{\kappa\in\mathcal{T}_h} \int_\kappa v\mathbf{u}\cdot\nabla w \,\mathrm{d}x \\ &\quad - \sum_{e\in\mathcal{E}} \int_e \{\!\{\mathbf{u}v\}\!\}\cdot[\![w]\!] \;\mathrm{d}s - \sum_{e\in\mathcal{E}} c_e \int_e [\![v]\!]\cdot[\![w]\!] \;\mathrm{d}s. \end{aligned}$$

DG-norm(s)

Let us define $\|\cdot\|_{DG}$ by

$$\|w\|_{DG}^2 := \|\nabla w\|^2 + \sum_{e\in\mathcal{E}} \left(2\sigma_e \, \|[\![w]\!]\|_e^2 + \frac{1}{\sigma_e} \, \|\{\!\{\nabla w\}\!\}\|_e^2 \right).$$

$\|\cdot\|_{DG}$ is a seminorm on $\mathrm{S}(\Omega, \mathcal{T}_h)$ and a norm on V_h.

When $B_{DG}(w,w) \geq 0$, we also define the broken (semi)norm $\|\cdot\|_B$:

$$\|w\|_B^2 := B_{DG}(w,w).$$

$\|\cdot\|_B$ is equivalent to $\|\cdot\|_{DG}$, uniformly in h:

- $\star$ on $\mathrm{S}(\Omega, \mathcal{T}_h)$ as a seminorm, and
- $\star$ on V_h as a norm.

Properties of B_{DG} and b_{DG}

$B_{DG}(\cdot,\cdot)$ and $\|\cdot\|_{DG}$ have the following properties:

1. Consistency: let $v \in \mathrm{H}^2_{\mathrm{N}}(\Omega) \cap \mathrm{V}$; then,

$$B_{DG}(v,w) = (-\Delta v, w) \qquad \forall w \in \mathrm{S}(\Omega, \mathcal{T}_h).$$

   See (2.1) and line 2 on p.746 in Arnold (1982).

2. Continuity: There exists $C > 0$, independent of h, such that

$$|B_{DG}(v,w)| \leq C\|v\|_{DG}\,\|w\|_{DG} \qquad \forall v,w \in \mathrm{H}^2(\Omega,\mathcal{T}_h). \quad (2.1)$$

3. Coercivity: There exists $\alpha_0 > 0$, and for each $\alpha \geq \alpha_0$ there exists $C_0 = C_0(\alpha) > 0$, independent of h, such that

$$C_0 \|w\|_{DG}^2 \leq B_{DG}(w, w) \qquad \forall w \in \mathrm{S}(\Omega, \mathcal{T}_h). \tag{2.2}$$

We take $\alpha = \alpha_0$ in the definition of the penalty parameter σ.

4. Broken Friedrichs' inequality: Let $r \in [2, \infty)$; there exists $C = C(r) > 0$, independent of h, s.t. for all $w \in \mathrm{V}(\Omega, \mathcal{T}_h)$:

$$\|w\|_{0,r} \leq C \left(\|\nabla w\|^2 + \sum_{e \in \mathcal{E}} 2\sigma_e \, \|[\![w]\!]\|_e^2 \right)^{\frac{1}{2}} . \qquad (2.3)$$

Trivially,

$$\|w\|_{0,r} \leq C\|w\|_{DG} \qquad \forall w \in \mathrm{H}^2(\Omega, \mathcal{T}_h) \cap \mathrm{V}(\Omega, \mathcal{T}_h).$$

See, Lasis & Süli (2003); Buffa & Ortner (IMAJNA, 2009).

5. Consistency: Suppose that $(v, w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$; then,

$$b_{DG}(\mathbf{u}; v, w) = b(\mathbf{u}; v, w).$$

6. Continuity: For α_0 as in Item 3 above there exists $C > 0$ such that, for any $C_1 > 0$ and $(v, w) \in \mathrm{H}^1(\Omega, \mathcal{T}_h) \times \mathrm{H}^1(\Omega, \mathcal{T}_h)$:

$$\begin{aligned} b_{DG}(\mathbf{u}; v, w) \leq C \|\mathbf{u}\|_{0,\infty} \Bigg[\sum_{e \in \mathcal{E}} \Bigg(\frac{C_1}{2\sigma_e} \|\{\!\{v\}\!\}\|_e^2 \\ + \frac{C_1}{2\sigma_e} c_e^2 \|[\![v]\!]\|_e^2 + \frac{1}{C_1} \sigma_e \|[\![w]\!]\|_e^2 \Bigg) \\ + \sum_{\kappa \in \mathcal{T}_h} \left(\frac{C_1}{2} \|v\|_\kappa^2 + \frac{1}{2C_1} \|\nabla w\|_\kappa^2 \right) \Bigg]. \end{aligned} \tag{2.4}$$

Thus, there exists a constant $C > 0$ such that, for any $C_1 > 0$:

$$b_{DG}(\mathbf{u}; v, w) \leq C \|\mathbf{u}\|_{0,\infty} \left(C_1 \|v\|^2 + \frac{1}{C_1} \|w\|_{DG}^2 \right)$$

$$\forall (v, w) \in \mathrm{S}(\Omega, \mathcal{T}_h) \times \mathrm{S}(\Omega, \mathcal{T}_h). \quad (2.5)$$

We shall assume henceforth that:

$$(c(\cdot,t),w(\cdot,t)) \in (\mathrm{H}^2(\Omega)\cap \mathrm{V}) \times \mathrm{H}^2(\Omega), \qquad t \in (0,T].$$

Then:

$$
\begin{aligned}
(\partial_t c,\eta) + B_{DG}(w,\eta) &= b_{DG}(\mathbf{u};c,\eta) && \forall \eta \in \mathrm{H}^2(\Omega,\mathcal{T}_h), \quad (2.6a)\\
(w,\eta) &= (\Phi'(c),\eta) + \gamma^2 B_{DG}(c,\eta) && \forall \eta \in \mathrm{H}^2(\Omega,\mathcal{T}_h), \quad (2.6b)\\
c(\cdot,0) &= c_0(\cdot) \in \mathrm{H}^2_{\mathrm{N}}(\Omega)\cap \mathrm{V}. && \quad (2.6c)
\end{aligned}
$$

The discrete Laplacian Δ_h

Given $w \in \mathrm{S}(\Omega, \mathcal{T}_h)$, the Riesz Representation Theorem implies the existence of a unique $\Delta_h w \in \mathrm{V}_h$ such that

$$(-\Delta_h w, v) = B_{DG}(w, v) \qquad \forall v \in \mathrm{V}_h.$$

We equip V_h with the $\mathrm{L}^2(\Omega)$ inner product to obtain a Hilbert space, and we then note that, for $w \in \mathrm{S}(\Omega, \mathcal{T}_h)$ fixed,

$$v \in \mathrm{V}_h \mapsto B_{DG}(w, v) \in \mathbb{R}$$

is a bounded linear functional over this Hilbert space by (2.1) and the property of norm-equivalence in finite-dimensional vector spaces.

The discrete Green's function $\mathcal{G}_h : \mathcal{F} \cap \mathrm{L}^2(\Omega) \to \mathrm{V}_h$ is defined by

$$B_{DG}(\mathcal{G}_h z, v) = (z, v) \qquad \forall v \in \mathrm{S}(\Omega, \mathcal{T}_h). \tag{2.7}$$

For $z \in \mathcal{F} \cap \mathrm{L}^2(\Omega)$ we have $(z, 1) = 0$ and $B_{DG}(\mathcal{G}_h z, 1) = 0$.

Hence (2.7) holds trivially for any constant function v. Thus, we can define, for $z \in \mathcal{F} \cap \mathrm{L}^2(\Omega)$, the function $\mathcal{G}_h z \in \mathrm{V}_h$ by

$$B_{DG}(\mathcal{G}_h z, v) = (z, v) \qquad \forall v \in \mathrm{V}_h. \tag{2.8}$$

The existence of a unique $\mathcal{G}_h z$, for any $z \in \mathcal{F} \cap \mathrm{L}^2(\Omega)$, follows from the Lax–Milgram theorem.

Broken elliptic projector $P_h : \mathrm{H}^2(\Omega, \mathcal{T}_h) \to \mathrm{S}(\Omega, \mathcal{T}_h)$

For $v \in \mathrm{H}^2(\Omega, \mathcal{T}_h)$, we define P_h by

$$B_{DG}(P_h v, \chi) = B_{DG}(v, \chi) \qquad \forall \chi \in \mathrm{S}(\Omega, \mathcal{T}_h), \tag{2.9}$$

$$(P_h v, 1) = (v, 1). \tag{2.10}$$

We note that P_h satisfies the bounds:

$$\|P_h v - v\| \leq Ch \, \|P_h v - v\|_{DG} \quad \text{and} \quad \|P_h v - v\|_{DG} \leq Ch^s \, \|v\|_{s+1}\,, \\ \forall v \in \mathrm{H}^{s+1}(\Omega), \quad 1 \leq s \leq p. \tag{2.11}$$

Observe, in particular, that $P_h : \mathrm{H}^2(\Omega, \mathcal{T}_h) \cap \mathrm{V} \to \mathrm{V}_h$.

The orthogonal projector $\Pi_h : \mathrm{L}^2(\Omega) \to \mathrm{S}(\Omega, \mathcal{T}_h)$ defined by

$$(v - \Pi_h v, \chi) = 0 \qquad \forall \chi \in \mathrm{S}(\Omega, \mathcal{T}_h)$$

satisfies the error bound

$$\|\Pi_h v - v\| \leq C h^s \|v\|_s \qquad \forall v \in \mathrm{H}^s(\Omega), \quad 0 \leq s \leq p + 1. \quad (2.12)$$

Note

$(v, 1) = 0 \Longrightarrow (\Pi_h v, 1) = 0$; thus, $v \in \mathrm{V}(\Omega, \mathcal{T}_h) \Longrightarrow \Pi_h v \in \mathrm{V}_h$.

Broken Agmon and Gagliardo–Nirenberg inequalities

Lemma (Broken Agmon inequality)

$$\|z\|_{0,\infty} \leq C\|z\|^{\frac{1}{2}}\|\Delta_h z\|^{\frac{1}{2}} \qquad \forall z \in \mathrm{V}_h.$$

Lemma (Broken Gagliardo–Nirenberg inequality)

Let $\|\cdot\|_{0,3}$ denote the $\mathrm{L}^3(\Omega)$ norm on Ω. There exists a positive constant C, independent of h, such that

$$\|\nabla z\|_{0,3} \leq C\,\|z\|^{\frac{1}{3}}\,\|\Delta_h z\|^{\frac{2}{3}} \qquad \forall z \in \mathrm{V}_h. \tag{2.13}$$

DGFEM approximation of (**P**)

Let $0 = t_0 < t_1 < \cdots < t_N = T$, with $\tau := t_n - t_{n-1}$, $n = 1 \to N$.

($\mathbf{P}_{h,\tau}$) Given $\mathbf{u} \in \mathrm{H}(\mathrm{div};\Omega) \cap [\mathrm{C}(\overline{\Omega})]^2$, $\nabla \cdot \mathbf{u} = 0$ in Ω and $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, for $n = 1 \to N$ find $\{c_h^n, w_h^n\} \in \mathrm{V}_h \times \mathrm{S}(\Omega, \mathcal{T}_h)$ s.t.:

$$(\delta_\tau c_h^n, \chi) + B_{DG}(w_h^n, \chi) = b_{DG}(\mathbf{u}; c_h^n, \chi) \qquad \forall \chi \in \mathrm{S}(\Omega, \mathcal{T}_h), \tag{3.1a}$$

$$(w_h^n, \chi) = \gamma^2 B_{DG}(c_h^n, \chi) + (\Phi'(c_h^n), \chi) \qquad \forall \chi \in \mathrm{S}(\Omega, \mathcal{T}_h), \tag{3.1b}$$

$$c_h^0 := \Pi_h c_0 \in \mathrm{V}_h, \tag{3.1c}$$

where

$$\delta_\tau c_h^n := \frac{c_h^n - c_h^{n-1}}{\tau}, \qquad n = 1 \to N.$$

Uniform bounds on the sequence of numerical solutions

Lemma

For any $h > 0$ and $\tau \leq C_\star \gamma^2$, where $C_\star$ is a sufficiently small but fixed positive constant, $\exists_1$ solution $\{c_h^n, w_h^n\} \in \mathrm{V}_h \times \mathrm{S}(\Omega, \mathcal{T}_h)$ to the n-th step of $(\mathbf{P}_{h,\tau})$, $n \in \{1, 2, \ldots, N\}$. Also, there exists $C > 0$, independent of h and τ, such that

$$\max_{n=1\to N} \left[\gamma^2 \|c_h^n\|_{DG}^2 + \|c_h^n\|_{0,\infty} + (\Phi(c_h^n), 1) + \|w_h^n\|^2\right] + \sum_{n=1}^{N} \tau \|w_h^n\|_{DG}^2 + \gamma^2 \sum_{n=1}^{N} \tau \left\|\delta_\tau c_h^n\right\|^2 \leq C, \tag{3.2}$$

$$\max_{n=1\to N} \|\Delta_h c_h^n\| \leq C. \tag{3.3}$$

Error estimates

We define the continuous-in-time functions for $n = 1, \dots, N$:

$$c_{h,\tau}(\cdot, t) := \frac{(t - t_{n-1})}{\tau} c_h^n(\cdot) + \frac{(t_n - t)}{\tau} c_h^{n-1}(\cdot), \qquad t \in (t_{n-1}, t_n],$$

and

$$w_{h,\tau}(\cdot, t) := \frac{(t - t_{n-1})}{\tau} w_h^n(\cdot) + \frac{(t_n - t)}{\tau} w_h^{n-1}(\cdot), \qquad t \in (t_{n-1}, t_n].$$

We also define the piecewise-constant-in-time functions

$$\widehat{c}_{h,\tau}(\cdot, t) := c_h^n \quad \text{and} \quad \widehat{w}_{h,\tau}(\cdot, t) := w_h^n, \qquad t \in (t_{n-1}, t_n].$$

Now, problem $(\mathbf{P}_{\mathbf{h},\tau})$ can be restated as follows.

Given $\mathbf{u} \in \mathrm{H}(\mathrm{div};\Omega) \cap [\mathrm{C}(\overline{\Omega})]^2$, $\mathrm{div}\ \mathbf{u} = 0$ in Ω and $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, find $\{c_{h,\tau}, w_{h,\tau}\} \in \mathrm{H}^1(0,T;\mathrm{V}_h) \times \mathrm{L}^2(0,T;\mathrm{S}(\Omega,\mathcal{T}_h))$ such that

$$(\delta_\tau \widehat{c}_{h,\tau}, \chi) + B_{DG}(\widehat{w}_{h,\tau}, \chi) = b_{DG}(\mathbf{u}; \widehat{c}_{h,\tau}, \chi) \qquad \forall \chi \in \mathrm{S}(\Omega,\mathcal{T}_h), \tag{4.1a}$$

$$(\widehat{w}_{h,\tau}, \chi) = \gamma^2 B_{DG}(\widehat{c}_{h,\tau}, \chi) + (\Phi'(\widehat{c}_{h,\tau}), \chi) \qquad \forall \chi \in \mathrm{S}(\Omega,\mathcal{T}_h). \tag{4.1b}$$

We define

$$\widehat{S}^c(\cdot,t) \quad := \quad \delta_\tau P_h c(\cdot,t) - \partial_t P_h c(\cdot,t), \qquad t \in (t_{n-1},t_n].$$

For $c \in \mathrm{W}^{2,\infty}((0,T);\mathrm{H}^2(\Omega)\cap\mathrm{V})$ using (2.3) with $r=2$, the equivalence of the norms $\|\cdot\|_{DG}$ and $\|\cdot\|_B$ on V_h, and the stability of P_h in the norm $\|\cdot\|_B$, we have

$$\|\widehat{S}^c(\cdot,t)\| \quad \le \quad C\tau \tag{4.2}$$

for $t \in [0,T]$. Finally, for $t \in (t_{n-1},t_n]$, $n = 1 \to N$, we define

$$\begin{aligned}
&E^c(\cdot,t) := c(\cdot,t) - \widehat{c}_{h,\tau}(\cdot,t),\\
&E^c_A(\cdot,t) := c(\cdot,t) - P_h c(\cdot,t), \quad E^c_h(\cdot,t) := P_h c(\cdot,t) - \widehat{c}_{h,\tau}(\cdot,t),
\end{aligned}$$

and $\widehat{c}_{h,\tau}(\cdot,0) = c^0_h = P_h c_0 = P_h c(\cdot,0)$ (whereby $E^c_h(\cdot,0)=0$), with analogous error functions for w and $\partial_t c$.

For $1 \le s \le p$, assuming that $c \in \mathrm{L}^\infty((0,T);\mathrm{H}^{s+1}(\Omega)\cap \mathrm{V})$, $w \in \mathrm{L}^\infty((0,T);\mathrm{H}^{s+1}(\Omega))$, (2.11) yields, for $t \in [0,T]$:

$$\|E_A^c(\cdot,t)\| + h\|E_A^c(\cdot,t)\|_{DG} \le Ch^{s+1}, \tag{4.3}$$
$$\|E_A^w(\cdot,t)\| + h\|E_A^w(\cdot,t)\|_{DG} \le Ch^{s+1}. \tag{4.4}$$

Using the definition of $\|\cdot\|_B$, (2.3) with $r=2$, (2.8), the equivalence of this norm with $\|\cdot\|_{DG}$ on V_h and (2.12), for $c \in \mathrm{W}^{1,\infty}((0,T);\mathrm{H}^{s+1}(\Omega)\cap\mathrm{V})$, $1 \le s \le p$, for $t \in [0,T]$:

$$\|\mathcal{G}_h(\partial_t E_A^c)(\cdot,t)\|_{DG} \le C\,\|\partial_t E_A^c(\cdot,t)\| \le Ch^{s+1}. \tag{4.5}$$

From the definition of E^c, (2.3) with $r=2$, and (4.3):

$$\|E^c(\cdot,t)\| \le \|E_h^c(\cdot,t)\| + \|E_A^c(\cdot,t)\| \le C\|E_h^c(\cdot,t)\|_{DG} + Ch^{s+1}$$

for $t \in [0,T]$ and all s with $1 \le s \le p$.

Using (2.9) we deduce from (2.6a,b) that, for all $\chi \in \mathrm{S}(\Omega, \mathcal{T}_h)$,

$$(\delta_\tau P_h c, \chi) + B_{DG}(P_h w, \chi) = b_{DG}(\mathbf{u}; c, \chi) + (\widehat{S}^c, \chi) - (\partial_t E_A^c, \chi)\,, \quad (4.6a)$$

$$(P_h w, \chi) = \gamma^2 B_{DG}(P_h c, \chi) + \ (\Phi'(c), \chi) - (E_A^w, \chi). \quad (4.6b)$$

We shall also need the following lemma.

Lemma

For $c \in \mathrm{L}^\infty((0,T); \mathrm{H}^2(\Omega))$ *and* $t \in [0,T]$:

$$\|\Phi'(c(\cdot,t)) - \Phi'(\widehat{c}_{h,\tau}(\cdot,t))\|_{DG} \quad \leq \quad C\|E^c(\cdot,t)\|_{DG}. \quad (4.7)$$

Proof.

For ease of writing we shall suppress the dependence of $c, \widehat{c}_{h,\tau}$ and E^c on t. Let us define

$$Q^c := c^2 + c\ \widehat{c}_{h,\tau} + \widehat{c}^2_{h,\tau}.$$

Clearly,

$$\|\Phi'(c) - \Phi'(\widehat{c}_{h,\tau})\|_{DG} \quad \leq \quad \|E^c Q^c\|_{DG} + \|E^c\|_{DG}. \qquad (4.8)$$

We bound the first term in (4.8) using the definition of $\|\cdot\|_{DG}$:

$$\begin{aligned}
\|E^c Q^c\|^2_{DG} &= \|\nabla(E^c Q^c)\|^2 \\
&\quad + \sum_{e\in\mathcal{E}} \left(2\sigma_e \, \|[\![E^c Q^c]\!]\|^2_e + \frac{1}{\sigma_e} \, \|\{\!\{E^c Q^c\}\!\}\|^2_e \right) \\
&:= \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3. \qquad (4.9)
\end{aligned}$$

We begin by bounding the term T_1. For any element $\kappa \in \mathcal{T}_h$:

$$
\begin{aligned}
\left\|\nabla\left(E^c Q^c\right)\right\|_\kappa &\leq \left\|Q^c \nabla E^c\right\|_\kappa + \left\|E^c \nabla Q^c\right\|_\kappa \\
&\leq \left\|Q^c\right\|_{0,\infty} \left\|\nabla E^c\right\|_\kappa + \left\|E^c \nabla Q^c\right\|_\kappa \\
&\leq \frac{3}{2}\left(\|c\|_{0,\infty}^2 + \|\widehat{c}_{h,\tau}\|_{0,\infty}^2\right) \left\|\nabla E^c\right\|_\kappa \\
&\quad + \left\|E^c\left(2c\nabla c + c\nabla\widehat{c}_{h,\tau} + \widehat{c}_{h,\tau}\nabla c + 2\widehat{c}_{h,\tau}\nabla\widehat{c}_{h,\tau}\right)\right\|_\kappa \\
&\leq \frac{3}{2}\left(\|c\|_{0,\infty}^2 + \|\widehat{c}_{h,\tau}\|_{0,\infty}^2\right) \left\|\nabla E^c\right\|_\kappa \\
&\quad + 2\left\|E^c\right\|_{0,6,\kappa}\left(\|c\|_{0,\infty} + \|\widehat{c}_{h,\tau}\|_{0,\infty}\right) \\
&\quad \times\left(\|\nabla c\|_{0,3,\kappa} + \|\nabla\widehat{c}_{h,\tau}\|_{0,3,\kappa}\right).
\end{aligned}
$$

After squaring, summing over $\kappa \in \mathcal{T}_h$, using inequality (2.3) with $r = 6$, taking square roots, applying Hölder's inequality for finite sums and the interpolation inequality (2.13) with $z = \widehat{c}_{h,\tau}$:

$$\begin{aligned} \|\nabla(E^c Q^c)\| &\leq C(c, \|\widehat{c}_{h,\tau}\|_{0,\infty}, \|\Delta_h \widehat{c}_{h,\tau}\|) \|E^c\|_{DG} \\ &\leq C\|E^c\|_{DG}. \end{aligned}$$

The last bound follows from the second inequality in (3.2) and from (3.3). Hence our bound on term T_1.

Next we bound the term T_2. For any $e \in \mathcal{E}$ we have

$$
\begin{aligned}
\sigma_e \left\| [\![E^c Q^c]\!] \right\|_e^2 &= \sigma_e \left\| [\![E^c]\!] (Q^c)^+ + [\![Q^c]\!] (E^c)^- \right\|_e^2 \\
&\leq 2\sigma_e \int_e \left(\left| [\![E^c]\!] \right|^2 \left((Q^c)^+ \right)^2 + \left| [\![Q^c]\!] \right|^2 \left((E^c)^- \right)^2 \right) \mathrm{d}s. \quad (4.10)
\end{aligned}
$$

Since c is continuous, we have

$$
[\![Q^c]\!] = c [\![\widehat{c}_{h,\tau}]\!] + [\![\widehat{c}_{h,\tau}^2]\!].
$$

Squaring yields

$$
\begin{aligned}
\left| [\![Q^c]\!] \right|^2 &\leq 2 \left(c^2 \left| [\![\widehat{c}_{h,\tau}]\!] \right|^2 + \left| [\![\widehat{c}_{h,\tau}^2]\!] \right|^2 \right) \\
&= 2 \left(c^2 \left| [\![\widehat{c}_{h,\tau}]\!] \right|^2 + \{\!\{ \widehat{c}_{h,\tau} \}\!\}^2 \left| [\![\widehat{c}_{h,\tau}]\!] \right|^2 \right) \\
&= 2 \left(c^2 + \{\!\{ \widehat{c}_{h,\tau} \}\!\}^2 \right) \left| [\![E^c]\!] \right|^2 . \quad (4.11)
\end{aligned}
$$

Combining (4.10) and (4.11) we obtain

$$
\begin{aligned}
\sigma_e \left\| [\![E^c Q^c]\!] \right\|_e^2 &\leq 2\sigma_e \int_e \Big(\left| [\![E^c]\!] \right|^2 \left((Q^c)^+ \right)^2 \\
&\qquad + 2 \left(c^2 + \{\!\{ \widehat{c}_{h,\tau} \}\!\}^2 \right) \left| [\![E^c]\!] \right|^2 \left((E^c)^- \right)^2 \Big) \, \mathrm{d}s \\
&\leq C \left(\|c\|_{0,\infty}^4 + \|\widehat{c}_{h,\tau}\|_{0,\infty}^4 \right) \sigma_e \int_e \left| [\![E^c]\!] \right|^2 \mathrm{d}s \\
&\leq C \sigma_e \int_e \left| [\![E^c]\!] \right|^2 ,
\end{aligned}
$$

where the last line follows by the second inequality in (3.2). Summation over element edges $e \in \mathcal{E}$ yields the bound on T_2.

Finally, we bound the term T_3. For any $\kappa \in \mathcal{T}_h$ and $e \subset \partial\kappa$:

$$
\begin{aligned}
\frac{1}{\sigma_e}\|\{\!\{E^c Q^c\}\!\}\|_e^2 &= \frac{1}{4\sigma_e}\left\|(E^c)^+(Q^c)^+ + (Q^c)^-(E^c)^-\right\|_e^2 \\
&= \frac{1}{4\sigma_e}\int_e \left[\left((E^c)^+ + (E^c)^-\right)(Q^c)^+ - \left((Q^c)^+ - (Q^c)^-\right)(E^c)^-\right]^2 \mathrm{d}s \\
&\le \frac{2}{\sigma_e}\int_e \{\!\{E^c\}\!\}^2 \left((Q^c)^+\right)^2 \mathrm{d}s + \frac{1}{2\sigma_e}\int_e |[\![Q^c]\!]|^2 \left((E^c)^-\right)^2 \mathrm{d}s \\
&\le C\left(\|c\|_{0,\infty}^4 + \|\widehat{c}_{h,\tau}\|_{0,\infty}^4\right)\left(\frac{1}{\sigma_e}\int_e \{\!\{E^c\}\!\}^2 \mathrm{d}s + \frac{1}{\sigma_e}\int_e \left((E^c)^-\right)^2 \mathrm{d}s\right) \\
&\le C\left(\|c\|_{0,\infty}^4 + \|\widehat{c}_{h,\tau}\|_{0,\infty}^4\right)\left(\frac{1}{\sigma_e}\int_e \{\!\{E^c\}\!\}^2 \mathrm{d}s + \int_\kappa |\nabla E^c|^2 \mathrm{d}x\right).
\end{aligned}
$$

We sum over all element edges $e \in \mathcal{E}$ to obtain our bound on T_3.

We substitute our bounds on T_1, T_2 and T_3 into (4.9), insert the resulting bound on $\|E^c Q^c\|_{DG}$ into (4.8), and note that

$$
c(\cdot, t) \in \mathrm{H}^2(\Omega) \cap \mathrm{V} \subset \mathrm{L}^\infty(\Omega) \cap \mathrm{W}^{1,3}(\Omega).
$$

We require two further preparatory lemmas.

Lemma

Suppose that:

$c_0 \in \mathrm{H}^{s+1}(\Omega) \cap \mathrm{H}^2_{\mathrm{N}}(\Omega) \cap \mathrm{V}$,

$c \in \mathrm{L}^\infty((0, T); \mathrm{H}^{s+1}(\Omega) \cap \mathrm{V}) \cap \mathrm{W}^{2,\infty}((0, T); \mathrm{L}^2(\Omega))$ *and*

$w \in \mathrm{L}^\infty(0, T; \mathrm{H}^{s+1}(\Omega))$, *with* $1 \leq s \leq p$, $p \geq 1$.

Then, for every $t \in [0, T]$, *we have*

$$|(\Phi'(c(\cdot, t)) - \Phi'(\widehat{c}_{h,\tau}(\cdot, t)), \delta_\tau E_h^c(\cdot, t))| \leq C(h^{2s+2} + \tau^2 + \|E^c(\cdot, t)\|_{DG}^2 + \|E_h^c(\cdot, t)\|_{DG}^2) + \frac{1}{8}\|E_h^w(\cdot, t)\|_B^2. \tag{4.12}$$

Proof.

Noting (2.8), (2.1), (2.11) and using standard inverse estimates:

$$\begin{aligned}
&|(\Phi'(c) - \Phi'(\widehat{c}_{h,\tau}), \delta_\tau E_h^c)| \\
&\leq \|P_h\left[\Phi'(c) - \Phi'(\widehat{c}_{h,\tau})\right] - \left(\Phi'(c) - \Phi'(\widehat{c}_{h,\tau})\right)\| \, \|\delta_\tau E_h^c\| \\
&\qquad + C\|P_h\left[\Phi'(c) - \Phi'(\widehat{c}_{h,\tau})\right]\|_{DG} \, \|\mathcal{G}_h(\delta_\tau E_h^c)\|_{DG} \\
&\leq Ch\|\Phi'(c) - \Phi'(\widehat{c}_{h,\tau})\|_{DG} \, \|\delta_\tau E_h^c\| \\
&\qquad + C\|P_h\left[\Phi'(c) - \Phi'(\widehat{c}_{h,\tau})\right]\|_{DG} \, \|\mathcal{G}_h(\delta_\tau E_h^c)\|_{DG} \\
&\leq C\|\Phi'(c) - \Phi'(\widehat{c}_{h,\tau})\|_{DG}^2 + C_1\|\mathcal{G}_h(\delta_\tau E_h^c)\|_{DG}^2.
\end{aligned} \tag{4.13}$$

We shall use (4.7) to bound the first term on the right-hand side of (4.13). In order to handle the second term on the right-hand side of (4.13), we now bound $\|\mathcal{G}_h(\delta_\tau E_h^c)\|_{DG}$.

Subtracting (4.1a) from (4.6a) and noting (2.8), (2.1), (2.5) and (2.3) with $r = 2$, for any $\chi \in \mathrm{V}_h$ and $t \in [0, T]$:

$$
\begin{aligned}
B_{DG}(\mathcal{G}_h(\delta_\tau E_h^c), \chi) \quad &= \quad (\delta_\tau E_h^c, \chi) \\
= -B_{DG}(E_h^w, \chi) &+ b_{DG}(\mathbf{u}; E^c, \chi) + (\widehat{S}^c, \chi) - (\partial_t E_A^c, \chi) \qquad (4.14) \\
&\le C\|E_h^w\|_{DG}\,\|\chi\|_{DG} + |b_{DG}(\mathbf{u}; E_h^c, \chi)| + |b_{DG}(\mathbf{u}; E_A^c, \chi)| \\
&\qquad + \|\widehat{S}^c\|\,\|\chi\| + C\|\mathcal{G}_h(\partial_t E_A^c)\|_{DG}\,\|\chi\|_{DG} \\
&\le C\|E_h^w\|_{DG}\,\|\chi\|_{DG} + C\|E_h^c\|\,\|\chi\|_{DG} + |b_{DG}(\mathbf{u}; E_A^c, \chi)| \\
&\qquad + C\|\widehat{S}^c\|\,\|\chi\|_{DG} + C\|\mathcal{G}_h(\partial_t E_A^c)\|_{DG}\,\|\chi\|_{DG}. \qquad (4.15)
\end{aligned}
$$

Now, by (2.4), the multiplicative trace inequality, and (2.11):

$$
|b_{DG}(\mathbf{u}; E_A^c, \chi)| \le C h^{s+1} \|\chi\|_{DG} \qquad \forall \chi \in \mathrm{S}(\Omega, \mathcal{T}_h). \qquad (4.16)
$$

We substitute (4.16), (4.2) and (4.5) into (4.15), recalling the equivalence of the seminorms $\|\cdot\|_{DG}$ and $\|\cdot\|_{B}$ on $\mathrm{S}(\Omega, \mathcal{T}_h)$:

$$B_{DG}(\mathcal{G}_h(\delta_\tau E_h^c), \chi) \leq C\left(h^{s+1} + \tau + \|E_h^w\|_B + \|E_h^c\|\right)\|\chi\|_{DG}, \tag{4.17}$$

where $1 \leq s \leq p$.

Take $\chi = \mathcal{G}_h(\delta_\tau E_h^c)$ $(\in \mathrm{V}_h)$ in (4.17), apply (2.2) to the left-hand side of (4.17) and (2.3) with $r = 2$ to the last term in the brackets on the right-hand side of (4.17):

$$\|\mathcal{G}_h(\delta_\tau E_h^c)(\cdot, t)\|_{DG}^2 \leq C_2(h^{2s+2} + \tau^2 + \|E_h^w(\cdot, t)\|_B^2 + \|E_h^c(\cdot, t)\|_{DG}^2) \tag{4.18}$$

where $1 \leq s \leq p$.

Choosing C_1 such that $8C_1C_2 < 1$ and using (4.13), (4.7) and (4.18) we obtain the desired result.

Lemma

Suppose that

$c_0 \in \mathrm{H}^{s+1}(\Omega) \cap \mathrm{H}^2_{\mathrm{N}}(\Omega) \cap \mathrm{V}$,
$c \in \mathrm{L}^\infty((0,T);\mathrm{H}^{s+1}(\Omega) \cap \mathrm{V}) \cap \mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Omega))$ *and*
$w \in \mathrm{L}^\infty(0,T;\mathrm{H}^{s+1}(\Omega))$, *where* $1 \leq s \leq p$, $p \geq 1$.

Then, for every $t \in [0,T]$, *we have that*

$$\begin{aligned} &\gamma^2 B_{DG}(E_h^c(\cdot,t), \delta_\tau E_h^c(\cdot,t)) + B_{DG}(E_h^w(\cdot,t), E_h^w(\cdot,t)) \\ &\quad \leq C(h^{2s} + \tau^2) + C\|E_h^c(\cdot,t)\|_{DG}^2 + \frac{1}{2}\|E_h^w(\cdot,t)\|_B^2. \end{aligned} \tag{4.19}$$

Proof.

Subtracting (4.1b) from (4.6b) and choosing $\chi = \delta_\tau E_h^c$:

$$\begin{aligned}(E_h^w, \delta_\tau E_h^c) &= \gamma^2 B_{DG}(E_h^c, \delta_\tau E_h^c) \\ &\quad + (\Phi'(c) - \Phi'(\widehat{c}_{h,\tau}), \delta_\tau E_h^c) - (E_A^w, \delta_\tau E_h^c). \quad (4.20)\end{aligned}$$

Next, set $\chi = E_h^w$ in (4.14), combine the resulting equation with (4.20). Hence, for $t \in [0, T]$ and any real number β:

$$\begin{aligned}&\gamma^2 B_{DG}(E_h^c, \delta_\tau E_h^c) + B_{DG}(E_h^w, E_h^w) \\ &\quad = b_{DG}(\mathbf{u}; E^c, E_h^w) + (\widehat{S}^c - \partial_t E_A^c, E_h^w) \\ &\qquad\quad - (\Phi'(c) - \Phi'(\widehat{c}_{h,\tau}), \delta_\tau E_h^c) + (E_A^w, \delta_\tau E_h^c) \\ &\quad = b_{DG}(\mathbf{u}; E_h^c, E_h^w) + b_{DG}(\mathbf{u}; E_A^c, E_h^w) + (\widehat{S}^c - \partial_t E_A^c, E_h^w - \beta) \\ &\qquad\quad - (\Phi'(c) - \Phi'(\widehat{c}_{h,\tau}), \delta_\tau E_h^c) + (E_A^w, \delta_\tau E_h^c),\end{aligned}$$

since $(\widehat{S}^c, 1) = 0$ and $(\partial_t E_A^c, 1) = 0$.

From (2.5), (4.16), (4.5), (4.3) and the broken Poincaré–Friedrichs inequality:

$$\inf_{\beta \in \mathbb{R}} \|E_h^w - \beta\| \leq C\|E_h^w\|_{DG},$$

we deduce, for $1 \leq s \leq p$ and $t \in [0, T]$, that

$$\begin{aligned}
&\gamma^2 B_{DG}(E_h^c, \delta_\tau E_h^c) + B_{DG}(E_h^w, E_h^w) \\
&\leq C\|E_h^c\| \, \|E_h^w\|_{DG} + Ch^{s+1}\|E_h^w\|_{DG} \\
&\qquad +(\|\widehat{S}^c\| + Ch^{s+1}) \inf_{\beta \in \mathbb{R}} \|E_h^w - \beta\| \\
&\qquad +|(\Phi'(c) - \Phi'(\widehat{c}_{h,\tau}), \delta_\tau E_h^c)| + Ch^{s+1} \, \|\delta_\tau E_h^c\| \\
&\leq C\|\widehat{S}^c\|^2 + C_1\|\mathcal{G}_h(\delta_\tau E_h^c)\|_{DG}^2 + C\|E_h^c\|_{DG}^2 \\
&\qquad +\frac{1}{4}\|E_h^w\|_B^2 + |(\Phi'(c) - \Phi'(\widehat{c}_{h,\tau}), \delta_\tau E_h^c)| + Ch^{2s}. \quad (4.21)
\end{aligned}$$

In the transition to the first line of the second inequality in (4.21) we used the inverse inequality

$$h\|\delta_\tau E_h^c\| \leq C\|\mathcal{G}_h(\delta_\tau E_h^c)\|_{DG}.$$

Noting (4.2), (4.18), (4.12), the triangle inequality

$$\|E^c\|_{DG} \leq \|E_h^c\|_{DG} + \|E_A^c\|_{DG}$$

in conjunction with (4.3), and choosing C_1 as in the proof of Lemma 3.3, the inequality (4.21) yields the required bound.

The main result

We shall write $\|\cdot\|_{\mathrm{H}^1(\Omega,\mathcal{T}_h)} := \|\cdot\|_{DG}$.

Theorem

Suppose that $p \geq 1$ and $1 \leq s \leq p$. Assume further that:

$c_0 \in \mathrm{H}^{s+1}(\Omega) \cap \mathrm{H}^2_{\mathrm{N}}(\Omega) \cap \mathrm{V}$,
$c \in \mathrm{L}^\infty((0,T);\mathrm{H}^{s+1}(\Omega) \cap \mathrm{V}) \cap \mathrm{W}^{1,\infty}(0,T;\mathrm{H}^2(\Omega)) \cap$
$\quad \mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Omega))$, *and*
$w \in \mathrm{L}^\infty(0,T;\mathrm{H}^{s+1}(\Omega)) \cap \mathrm{W}^{1,\infty}(0,T;\mathrm{H}^2(\Omega))$.

Then,

$$\|c_0 - c_h^0\|_{\mathrm{H}^1(\Omega,\mathcal{T}_h)} \leq Ch^s$$

and

$$\|c - c_{h,\tau}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega,\mathcal{T}_h))} + \|w - w_{h,\tau}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega,\mathcal{T}_h))} \leq C(h^s + \tau).$$

Proof.

(2.12) implies the bound on the error between c_0 and $c_h^0 := \Pi_h c_0$.

Setting $t = t_n$ in (4.19) and applying a discrete Gronwall inequality, we deduce that for all $\tau \in (0, \tau_0)$, where $\tau_0 = \tau_0(\gamma)$ is a sufficiently small (depending on γ) but fixed real number and $1 \le s \le p$,

$$\begin{aligned}\|E_h^c(\cdot, t_n)\|_B^2 + \tau \sum_{m=1}^{n} \|E_h^w(\cdot, t_m)\|_B^2 &\le C(\|E_h^c(\cdot, 0)\|_B^2 + h^{2s} + \tau^2) \\ &\le C(h^{2s} + \tau^2), \quad n = 1 \to N.\end{aligned}$$

For $t \in (t_{n-1}, t_n]$, $n = 1 \to N$, we have that

$$\begin{aligned}\|E_h^c(\cdot, t)\|_B &= \|P_h c(\cdot, t) - c_h^n\|_B \\ &\le \|P_h(c(\cdot, t) - c(\cdot, t_n))\|_B + \|P_h c(\cdot, t_n) - c_h^n\|_B \\ &\le \|c(\cdot, t) - c(\cdot, t_n)\|_B + \|P_h c(\cdot, t_n) - c_h^n\|_B \\ &\le \tau \cdot \text{ess.sup}_{t \in [0,T]} \|\partial_t c(\cdot, t)\|_B + \|E_h^c(\cdot, t_n)\|_B \\ &\le C\tau + \|E_h^c(\cdot, t_n)\|_B.\end{aligned}$$

Therefore,

$$\|E_h^c(\cdot,t)\|_B^2 \leq C(h^{2s}+\tau^2), \qquad \text{for } t \in [0,T], \quad 1 \leq s \leq p.$$

Similarly, for $1 \leq s \leq p$,

$$\int_0^T \|E_h^w(\cdot,t)\|_B^2 \,\mathrm{d}t = \sum_{m=1}^{N} \int_{t_{m-1}}^{t_m} \|E_h^w(\cdot,t)\|_B^2 \,\mathrm{d}t \leq C(h^{2s}+\tau^2).$$

Noting the definitions of E_h^c and E_h^w, we get, for $1 \leq s \leq p$,

$$\begin{aligned} &\text{ess.sup}_{t\in[0,T]}\|P_h c(\cdot,t) - \widehat{c}_{h\tau}(\cdot,t)\|_B^2 \\ &\qquad + \int_0^T \|P_h w(\cdot,t) - \widehat{w}_{h\tau}(\cdot,t)\|_B^2 \,\mathrm{d}t \leq C(h^{2s}+\tau^2). \end{aligned}$$

By recalling the equivalence of $\|\cdot\|_B$ and $\|\cdot\|_{DG}$ as norms on V_h and seminorms on $\mathrm{S}(\Omega,\mathcal{T}_h)$, the desired bounds follow on applying the triangle inequality and (4.3).

Evolution of ellipse without convection: Q1 elements

We apply our DGFEM using Q1 elements on a 32 by 32 uniform square mesh. We consider the Cahn–Hilliard equation without convection (i.e. $\mathbf{u} = \mathbf{0}$), with $\gamma = 1/100$. The initial datum c_0 is a piecewise constant function whose jump-set is an ellipse:

$$c_0(x, y) := \begin{cases} 0.95 & \text{if } 9(x-0.5)^2 + (y-0.5)^2 < 1/9, \\ -0.95 & \text{otherwise.} \end{cases}$$

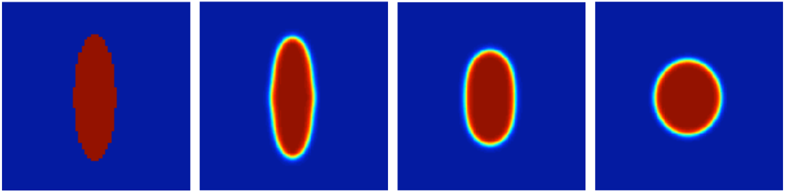

Figure: The evolution of an ellipse without convection.

Evolution of cross without convection: Q2 elements

The initial datum c_0 is a piecewise constant function whose jump-set has the shape of a cross; see Figure 2. We use a 32 by 32 uniform square mesh and quadratic elements.

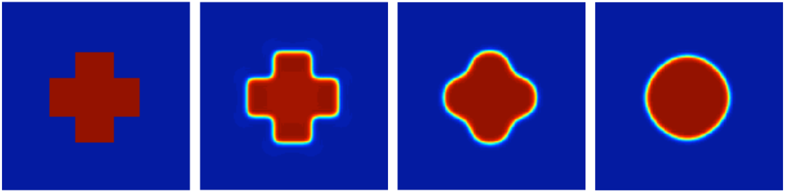

Figure: The evolution of a cross.

As in the first example, the initial datum c_0 with a cross-shaped jump-set evolves to a steady state exhibiting a circular interface.

Spinodal decomposition: Q2 elements

Spinodal decomposition is the separation of a mixture of two, or more, components to bulk regions of each. Such a phenomenon occurs when a high-temperature mixture of two, or more, alloys is rapidly cooled. The initial datum c_0 is chosen to be a small uniformly distributed random perturbation about zero. We take $\gamma = 1/100$ using Q2 elements on a 32 by 32 uniform square mesh.

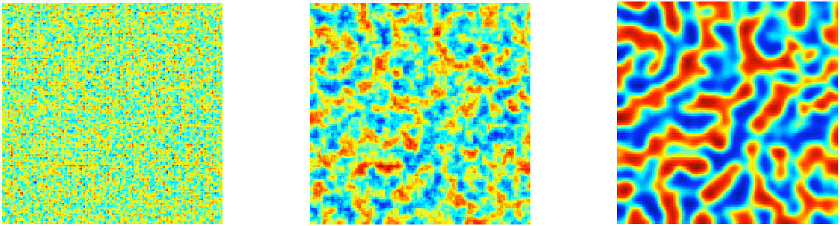

Figure: Early stages of spinodal decomposition.

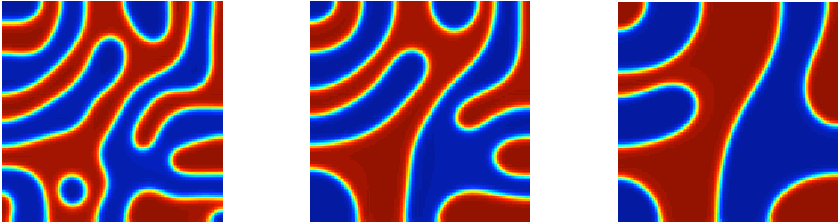

Figure: Later stages of spinodal decomposition.

The separation of the two components into bulk regions is clearly seen. The initial separation happens over a very short time-scale relative to the motion thereafter. Later, the bulk regions begin to move more slowly and separation continues until the interfaces develop a constant curvature.

Convection-dominated problems: Q2 elements

In all of the following examples we will take $\mathrm{Pe} = 200$,

$$\mathbf{u}(x, y) := f(r)(2y - 1, 1 - 2x)^{\mathrm{T}}, \qquad (x, y) \in \Omega := (0, 1)^2,$$

where

$$f(r) := \tfrac{1}{2}\left(1 + \tanh\left(\beta\left(\tfrac{1}{2} - \varepsilon - r\right)\right)\right), \qquad r^2 := (x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2,$$

with $\beta = 200$, $\varepsilon = 0.1$.

Clearly $\nabla \cdot \mathbf{u} = 0$, and $\mathbf{u} \cdot \mathbf{n} = 0$ to machine precision on $\partial\Omega$.

Evolution of a cross: Q2 elements

We start from the same cross-shaped initial datum as before. We use a quadratic discontinuous Galerkin method on a 32 by 32 square mesh and apply the above velocity field, taking $\gamma = 1/100$.

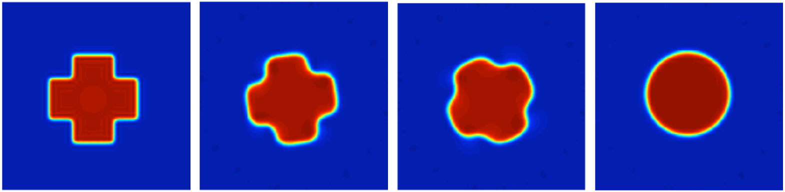

Figure: Evolution of a cross in circular convection: $t = 0.02, 0.04, 0.3, 1.0$.

The convection term is rotating the two components anti-clockwise while the interface is reducing to a circle. In the final frame of figure both bulk regions are still rotating under the velocity field.

Spinodal decomposition under convection: Q3 elements

We show the effects of convection on spinodal decomposition. We use a cubic DGFEM take $\gamma = 1/200$. To model the resulting thinner interface we apply a cubic polynomial approximation on a 64 by 64 square mesh. The initial datum c_0 in the circular domain is a small uniformly distributed random perturbation of zero.

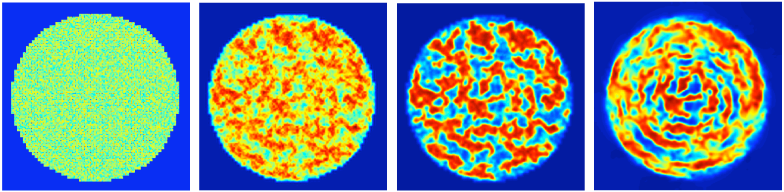

Figure: Formation of bulk regions: spinodal decomposition under circular convection: $\gamma = 1/200$, $\mathrm{Pe} = 200$, $t = 0, 0.05, 0.1, 0.15$.

Initially the two components are driven into bulk regions. As before this initial motion occurs over a relatively short time-scale.

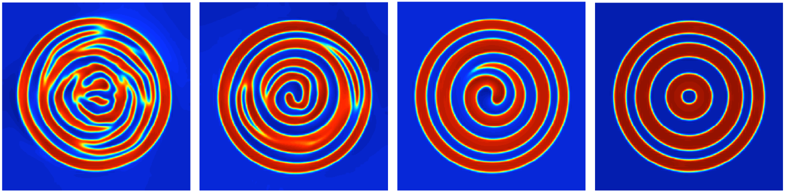

Figure: Convection of bulk regions into circular regions: spinodal decomposition under circular convection: $\gamma = 1/200$, Pe $= 200$, $t = 0.3, 0.35, 0.4, 10$.

Due to the convection term, these bulk regions form concentric circles with filament structure. This convection-dominated motion occurs on a relatively short time-scale.

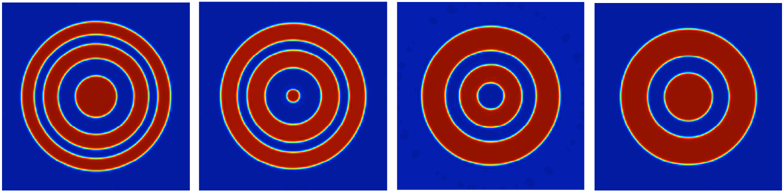

Figure: Spinodal decomposition under circular convection: $\gamma = 1/200$, $\mathrm{Pe} = 200$, $t = 80, 140, 200, 300$.

Finally, motion of the phases continues due to the fact that we have considered a constant mobility function of $B(c) = 1$ in our model. This motion, due to the diffusion coefficient, occurs over a very large time-scale. In general, such a function restricts diffusion away from interfaces by degenerating to zero when $c = \pm 1$ and is introduced into the model as follows:

$$\partial_t c - \frac{1}{\mathrm{Pe}} \nabla \cdot (B(c) \nabla w) + \nabla \cdot (\mathbf{u} c) = 0.$$