The HDG methods

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The DG methods are attracting the interest of many scientists because:

- They enforce the equations in an element-by-element fashion through a Galerkin formulation which can give rise to locally conservative methods.
- They can handle any type of mesh, element shape and basis functions: They are ideally suited for *hp*-adaptivity.
- They have a built-in stabilization mechanism which does not degrade their (high-order) accuracy.
- They can be applied to a wide variety of partial differential equations.

However, the DG methods (for second-order elliptic equations) have been criticized because:

- For the same mesh and the same polynomial degree, the number of globally coupled degrees of freedom of the DG methods is much bigger than those of the CG method. Moreover, the orders of convergence of both the vector and scalar variables are also the same.
- For the same mesh and the same index, the number of globally coupled degrees if freedom of the DG methods are much bigger than those of the hybridized version of the RT and BDM methods. Moreover, the orders of convergence of both the vector and the local average of the scalar variables are smaller by one.

The HDG methods.

The main features of the HDG methods.

- The HDG methods are obtained by discretizing characterizations of the exact solution written in terms of many local problems, one for each element of the mesh Ω_h, with suitably chosen data, and in terms of a single global problem that actually determines them.
- This permits an efficiently implementation since they inherit the above-mentioned structure of the exact solution. This is what renders them efficiently implementable, especially within the framework of $hp-$adaptive methods, as is typical of DG methods.

- The way in which they are defined allows them to be, in some instances, more accurate than already existing DG methods. In fact, in some cases when standard DG methods do not converge, HDG methods do.
- The HDG methods are ideally suited for steady-state problems and for time-dependent problems when implicit time-marching methods are used.

The HDG methods.

Guidelines for devising the methods.

- Use a characterization of the exact solution in terms of solutions of local problems and transmission conditions.
- Use discontinuous approximations for both the solution inside each element and its trace on the element boundary.
- Define the local solvers by using a Galerkin method to weakly enforce the equations on each element.
- Define a global problem by weakly imposing the transmission conditions.

The HDG methods.

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The HDG methods.

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We provide two different characterizations of the solution of the following second-order elliptic model problem:

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } \Omega, \\ \nabla \cdot \mathbf{q} &= f && \text{in } \Omega, \\ \widehat{u} &= u_D && \text{on } \partial\Omega. \end{aligned}$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

The main idea.

The general approach: Local problems and transmission conditions.

We have that the exact solution satisfies the local problems

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 \quad \text{in } K, \\ \nabla \cdot \mathbf{q} &= f \quad \text{in } K, \end{aligned}$$

the transmission conditions

$$\begin{aligned} [\![\widehat{u}]\!] &= 0 \quad \text{if } F \in \mathcal{E}_h^o, \\ [\![\widehat{\mathbf{q}}]\!] &= 0 \quad \text{if } F \in \mathcal{E}_h^o, \end{aligned}$$

and the Dirichlet boundary condition

$$\widehat{u} = u_D \quad \text{if } F \in \mathcal{E}_h^\partial.$$

The main idea.

A first approach: Rewriting the equations.

We can obtain $(\mathbf{q}, u)$ in K in terms of $\widehat{u}$ on ∂K and f by solving

$$
\begin{aligned}
\mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } K,\\
\nabla \cdot \mathbf{q} &= f && \text{in } K,\\
u &= \widehat{u} && \text{on } \partial K.
\end{aligned}
$$

The function $\widehat{u}$ can now be determined as the solution, on each $F \in \mathcal{E}_h$, of the equations

$$
\begin{aligned}
[\![\widehat{\mathbf{q}}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o,\\
\widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial,
\end{aligned}
$$

where $\widehat{\mathbf{q}}$ is the trace of $\mathbf{q} = \mathbf{q}(\widehat{u}, f)$ on ∂K.

The main idea.

A first approach: Characterization of the solution.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathsf{U}_{\widehat{u}}) + (\mathbf{Q}_f, \mathsf{U}_f)$, where

$$\begin{aligned} \mathrm{c}\,\mathbf{Q}_{\widehat{u}} + \nabla \mathsf{U}_{\widehat{u}} &= 0 && \text{in } K, & \mathrm{c}\,\mathbf{Q}_f + \nabla \mathsf{U}_f &= 0 && \text{in } K,\\ \nabla\cdot\mathbf{Q}_{\widehat{u}} &= 0 && \text{in } K, & \nabla\cdot\mathbf{Q}_f &= f && \text{in } K,\\ \mathsf{U}_{\widehat{u}} &= \widehat{u} && \text{on } \partial K, & \mathsf{U}_f &= 0 && \text{on } \partial K. \end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution, on each $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned} -[\![\widehat{\mathbf{Q}}_{\widehat{u}}]\!] &= [\![\widehat{\mathbf{Q}}_f]\!] && \text{if } F \in \mathcal{E}_h^o,\\ \widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial. \end{aligned}$$

The main idea.

A first approach: The one-dimensional case $K = (x_{i-1}, x_i)$ for $i = 1, \ldots, I$, with $\mathrm{c} = 1$.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathrm{U}_{\widehat{u}}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where

$$
\begin{aligned}
\mathbf{Q}_{\widehat{u}} + \frac{d}{dx}\mathrm{U}_{\widehat{u}} &= 0 && \text{in } (x_{i-1}, x_i), & \qquad \mathbf{Q}_f + \frac{d}{dx}\mathrm{U}_f &= 0 && \text{in } (x_{i-1}, x_i),\\
\frac{d}{dx}\mathbf{Q}_{\widehat{u}} &= 0 && \text{in } (x_{i-1}, x_i), & \frac{d}{dx}\mathbf{Q}_f &= f && \text{in } (x_{i-1}, x_i),\\
\mathrm{U}_{\widehat{u}} &= \widehat{u} && \text{on } \{x_{i-1}, x_i\}, & \mathrm{U}_f &= 0 && \text{on } \{x_{i-1}, x_i\}.
\end{aligned}
$$

The function $\widehat{u}$ is the solution of

$$
\begin{aligned}
\widehat{\mathbf{Q}}_{\widehat{u}}(x_i^+) - \widehat{\mathbf{Q}}_{\widehat{u}}(x_i^-) &= -\widehat{\mathbf{Q}}_f(x_i^+) + \widehat{\mathbf{Q}}_f(x_i^-) && \text{for } i = 1, \ldots, I-1,\\
\widehat{u}(x_i) &= u_D(x_i) && \text{for } i = 0, I.
\end{aligned}
$$

The main idea.

A first approach: The one-dimensional case $K = (x_{i-1}, x_i)$ for $i = 1, \dots, I$, with $\mathrm{c} = 1$.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathrm{U}_{\widehat{u}}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where, for $x \in (x_{i-1}, x_i)$,

$$\mathbf{Q}_{\widehat{u}}(x) = -\frac{1}{h}(\widehat{u}_i - \widehat{u}_{i-1}), \qquad \mathbf{Q}_f(x) = -\int_{x_{i-1}}^{x_i} G_x(x,s) f(s)\, ds,$$

$$\mathrm{U}_{\widehat{u}}(x) = \frac{1}{h}(x - x_{i-1})\widehat{u}_i + \frac{1}{h}(x_i - x)\widehat{u}_{i-1} \qquad \mathrm{U}_f(x) = \int_{x_{i-1}}^{x_i} G(x,s) f(s)\, ds.$$

The function $\widehat{u}$ is the solution of

$$\begin{aligned} \frac{1}{h}(-\widehat{u}_{i-1} + 2\widehat{u}_i - \widehat{u}_{i+1}) &= -\widehat{\mathbf{Q}}_f(x_i^+) + \widehat{\mathbf{Q}}_f(x_i^-) && \text{for } i = 1, \dots, I-1, \\ \widehat{u}(x_i) &= u_D(x_i) && \text{for } i = 0, I. \end{aligned}$$

The main idea.

A second approach: Rewriting the equations. We use $\overline{\zeta} := (\zeta, 1)_K / |K|$ and $\overline{\widehat{\mathbf{q}} \cdot \mathbf{n}} := \langle \widehat{\mathbf{q}} \cdot \mathbf{n}, 1 \rangle_{\partial K} / |K|$.

We can obtain $(\mathbf{q}, u)$ in K in terms of $\widehat{\mathbf{q}} \cdot \mathbf{n}$ on ∂K, $\overline{u}$ and f by solving

$$
\begin{aligned}
\mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } K,\\
\nabla \cdot \mathbf{q} &= f - \overline{f} + \overline{\widehat{\mathbf{q}} \cdot \mathbf{n}} && \text{in } K,\\
\mathbf{q} \cdot \mathbf{n} &= \widehat{\mathbf{q}} \cdot \mathbf{n} && \text{on } \partial K.
\end{aligned}
$$

The functions $\widehat{\mathbf{q}} \cdot \mathbf{n}$ and $\overline{u}$ can now be determined as the solution of the equations

$$
\begin{aligned}
[\![\widehat{u}]\!] &= 0 && \text{for } F \in \mathcal{E}_h^o,\\
\overline{\widehat{\mathbf{q}} \cdot \mathbf{n}} &= \overline{f} && \text{for } K \in \mathcal{T}_h,\\
\widehat{u} &= u_D && \text{for } F \in \mathcal{E}_h^\partial,
\end{aligned}
$$

where $\widehat{u}$ is the trace of $u = u(\widehat{\mathbf{q}} \cdot \mathbf{n}, \overline{u}, f)$ on ∂K.

The main idea.

A second approach: Characterization of the solution.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{q}}, \mathrm{U}_{\widehat{q}}) + (\mathbf{0}, \overline{u}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where

$$\begin{aligned}
\mathrm{c}\,\mathbf{Q}_{\widehat{\mathbf{q}}} + \nabla \mathrm{U}_{\widehat{\mathbf{q}}} &= 0 && \text{in } K, & \mathrm{c}\,\mathbf{Q}_f + \nabla \mathrm{U}_f &= 0 && \text{in } K,\\
\nabla \cdot \mathbf{Q}_{\widehat{\mathbf{q}}} &= \overline{\widehat{\mathbf{q}} \cdot \mathbf{n}} && \text{in } K, & \nabla \cdot \mathbf{Q}_f &= f - \overline{f} && \text{in } K,\\
\mathbf{Q}_{\widehat{\mathbf{q}}} \cdot \mathbf{n} &= \widehat{\mathbf{q}} \cdot \mathbf{n} && \text{on } \partial K, & \mathbf{Q}_f \cdot \mathbf{n} &= 0 && \text{on } \partial K,\\
\overline{\mathrm{U}}_{\widehat{\mathbf{q}}} &= 0, & && \overline{\mathrm{U}}_f &= 0.
\end{aligned}$$

The functions $\widehat{\mathbf{q}} \cdot \mathbf{n}$ and $\overline{u}$ can now be determined as the solution of the equations

$$\begin{aligned}
-[\![\widehat{\mathbf{U}}_{\widehat{\mathbf{q}}}]\!] - [\![\overline{u}]\!] &= [\![\widehat{\mathbf{U}}_f]\!] && \text{for } F \in \mathcal{E}_h^o,\\
\overline{\widehat{\mathbf{q}} \cdot \mathbf{n}} &= \overline{f} && \text{for } K \in \mathcal{T}_h,\\
\widehat{\mathbf{U}}_{\widehat{\mathbf{q}}} + \overline{u} + \widehat{\mathbf{U}}_f &= u_D && \text{for } F \in \mathcal{E}_h^\partial.
\end{aligned}$$

The main idea.

A second approach: The one-dimensional case $K = (x_{i-1}, x_i)$ for $i = 1, \ldots, I$, with $\mathrm{c} = 1$.

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathrm{U}_{\widehat{u}}) + (\mathbf{0}, \overline{u}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where

$$\begin{aligned}
\mathbf{Q}_{\widehat{\mathbf{q}}} + \frac{d}{dx}\mathrm{U}_{\widehat{\mathbf{q}}} &= 0 && \text{in } (x_{i-1}, x_i), & \mathbf{Q}_f + \frac{d}{dx}\mathrm{U}_f &= 0 && \text{in } (x_{i-1}, x_i),\\
\frac{d}{dx}\mathbf{Q}_{\widehat{\mathbf{q}}} &= \overline{\widehat{\mathbf{q}} \cdot \mathbf{n}} && \text{in } (x_{i-1}, x_i), & \frac{d}{dx}\mathbf{Q}_f &= f - \overline{f} && \text{in } (x_{i-1}, x_i),\\
\mathbf{Q}_{\widehat{\mathbf{q}}} \cdot \mathbf{n} &= \widehat{\mathbf{q}} \cdot \mathbf{n} && \text{on } \{x_{i-1}, x_i\}, & \mathbf{Q}_f \cdot \mathbf{n} &= 0 && \text{on } \{x_{i-1}, x_i\},\\
\overline{\mathrm{U}}_{\widehat{\mathbf{q}}} &= 0 && \text{on } \{x_{i-1}, x_i\}, & \overline{\mathrm{U}}_f &= 0 && \text{on } \{x_{i-1}, x_i\}.
\end{aligned}$$

The functions $\widehat{\mathbf{q}}$ and $\overline{u}$ are the solution of

$$\begin{aligned}
\widehat{\mathbf{U}}_{\widehat{\mathbf{q}}}(x_i^+) - \widehat{\mathbf{U}}_{\widehat{\mathbf{q}}}(x_i^-) + \overline{u}_{i+1/2} - \overline{u}_{i-1/2} &= -\widehat{\mathbf{U}}_f(x_i^+) + \widehat{\mathbf{U}}_f(x_i^-) && \text{for } i = 1, \ldots, I-1,\\
\widehat{\mathbf{q}}_i - \widehat{\mathbf{q}}_{i-1} &= h\,\overline{f}_{i-1/2} && \text{for } i = 1, \ldots, I-1,\\
\widehat{\mathbf{U}}_{\widehat{\mathbf{q}}}(x_0^+) + \overline{u}_{1/2} + \widehat{\mathbf{U}}_f(x_0^+) &= u_D(x_0),\\
\widehat{\mathbf{U}}_{\widehat{\mathbf{q}}}(x_I^-) + \overline{u}_{I-1/2} + \widehat{\mathbf{U}}_f(x_I^-) &= u_D(x_I).
\end{aligned}$$

We have that $(\mathbf{q}, u) = (\mathbf{Q}_{\widehat{u}}, \mathrm{U}_{\widehat{u}}) + (\mathbf{0}, \overline{u}) + (\mathbf{Q}_f, \mathrm{U}_f)$, where, for $x \in (x_{i-1}, x_i)$,

$$\mathbf{Q}_{\widehat{\mathbf{q}}}(x) = \frac{1}{h}(x - x_{i-1})\widehat{\mathbf{q}}_i + \frac{1}{h}(x_i - x)\widehat{\mathbf{q}}_{i-1}, \qquad \mathbf{Q}_f(x) = -\int_{x_{i-1}}^{x_i} \mathrm{G}_x(x,s)(f - \overline{f})(s)\, ds,$$

$$\mathrm{U}_{\widehat{u}}(x) = \frac{1}{6h}(h^2 - 3(x - x_{i-1})^2)\widehat{\mathbf{q}}_i - \frac{1}{6h}(h^2 - 3(x_i - x)^2)\widehat{\mathbf{q}}_{i-1}, \qquad \mathrm{U}_f(x) = \int_{x_{i-1}}^{x_i} \mathrm{G}(x,s)(f - \overline{f})(s)\, ds.$$

The functions $\widehat{\mathbf{q}}$ and $\overline{u}$ are the solution of

$$\frac{h}{6}(\widehat{\mathbf{q}}_{i-1} + 4\,\widehat{\mathbf{q}}_i + \widehat{\mathbf{q}}_{i+1}) + \overline{u}_{i+1/2} - \overline{u}_{i-1/2} = -\widehat{\mathbf{U}}_f(x_i^+) + \widehat{\mathbf{U}}_f(x_i^-) \quad \text{for } i = 1, \ldots, I-1,$$

$$\widehat{\mathbf{q}}_i - \widehat{\mathbf{q}}_{i-1} = h\,\overline{f}_{i-1/2} \quad \text{for } i = 1, \ldots, I-1,$$

$$\frac{h}{6}(2\widehat{\mathbf{q}}_0 + \widehat{\mathbf{q}}_1) + \overline{u}_{1/2} + \widehat{\mathbf{U}}_f(x_0^+) = u_D(x_0),$$

$$-\frac{h}{6}(\widehat{\mathbf{q}}_{I-1} + 2\widehat{\mathbf{q}}_I) + \overline{u}_{I-1/2} + \widehat{\mathbf{U}}_f(x_I^-) = u_D(x_I).$$

The main idea.

Summary.

- The HDG methods are obtained by constructing discrete versions of the above characterizations of the exact solution.
- In this way, the globally coupled degrees of freedom will be those of the corresponding global formulations.

A first approach. (B.C., J.Gopalakrishnan and R.Lazarov, SINUM, 2009.)

The local solvers: A weak formulation on each element.

On the element $K \in \Omega_h$, given $\widehat{u}$ on ∂K and f, we have that $(\mathbf{q}, u)$ satisfies the equations

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}, \mathbf{v})_K - (u, \nabla\cdot\mathbf{v})_K + \langle \widehat{u}, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\ -(\mathbf{q}, \nabla w)_K + \langle \widehat{\mathbf{q}}\cdot\mathbf{n}, w\rangle_{\partial K} &= (f, w)_K,\end{aligned}$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, where

$$\widehat{\mathbf{q}}\cdot\mathbf{n} = \mathbf{q}\cdot\mathbf{n} \qquad \text{on } \partial K.$$

On the element $K \in \Omega_h$, we define $(\mathbf{q}_h, u_h)$ terms of $(\widehat{u}_h, f)$ as the element of $\mathbf{V}(K) \times W(K)$ such that

$$\begin{aligned} (\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_K - (u_h, \nabla \cdot \mathbf{v})_K + \langle \widehat{u}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= 0, \\ -(\mathbf{q}_h, \nabla w)_K + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial K} &= (f, w)_K, \end{aligned}$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, where

$$\widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial K.$$

If we want that, at any given point x of ∂K at which the normal $\mathbf{n}$ is well defined,

- The numerical trace $\widehat{\mathbf{q}}_h(x) \cdot \mathbf{n}$ only depends on $\mathbf{q}_h(x) \cdot \mathbf{n}$, $u_h(x)$ and the numerical trace $\widehat{u}_h(\mathrm{x})$.
- The dependence is linear.
- The numerical trace $\widehat{\mathbf{q}}_h(x) \cdot \mathbf{n}$ is consistent, that is, $\widehat{\mathbf{q}}_h(x) \cdot \mathbf{n} = \mathbf{q}_h(x) \cdot \mathbf{n}$ whenever $u_h(x) = \widehat{u}_h(x)$,

we must have that $\widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h)$.

The first approach.

The local solvers are well defined.

Theorem

The local solver on K is well defined if

- $\tau > 0$ *on* ∂K,
- $\nabla W(K) \subset \mathbf{V}(K)$.

The first approach.

Proof.

The system is square. Set $\widehat{u}_h = 0$ and $f = 0$.
For $(\mathbf{v}, w) := (\mathbf{q}_h, u_h)$, the equations read

$$\begin{aligned} (\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K - (u_h, \nabla \cdot \mathbf{q}_h)_K &= 0, \\ -(\mathbf{q}_h, \nabla u_h)_K + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, u_h \rangle_{\partial K} &= 0. \end{aligned}$$

Hence

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K + \langle (\widehat{\mathbf{q}}_h - \mathbf{q}_h) \cdot \mathbf{n}, u_h \rangle_{\partial K} = 0,$$

and since $\widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h)$, we get

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K + \langle \tau\,(u_h), u_h \rangle_{\partial K} = 0.$$

This implies that $\mathbf{q}_h = 0$ on K, and that $u_h = 0$ on ∂K.

The first approach.

Proof.

Now, the first equation defining the local solvers reads

$$-(u_h, \nabla \cdot \mathbf{v})_K = 0,$$

for all $\mathbf{v} \in \mathbf{V}(K)$. Hence

$$(\nabla u_h, \mathbf{v})_K = 0,$$

and so $\nabla u_h = 0$. This proves the result.

The first approach

The local solvers: Examples of the stabilization function τ.

- If K simplex, $\mathbf{V}(K) = \boldsymbol{\mathcal{P}}_k(K)$, $W(K) = \mathcal{P}_k(K)$, $\mathbf{N}(F) \cdot \mathbf{n} = \mathcal{P}_k(K)$: we can take $\tau = 0$ on $\partial K \setminus F^*$, $\tau > 0$ on F^* (SF-H).

- The Bassi-Rebay stabilization function:

$$\tau(\phi)|_F := \eta_F \, \mathbf{r}_F(\phi) \cdot \mathbf{n}, \quad \mathbf{r}_F \in \mathbf{V}(K): \quad (\mathbf{r}_F(\phi), \mathbf{v})_K = \langle \phi, \mathbf{v} \cdot \mathbf{n} \rangle_F$$

Note that, with $\phi := u_h - \widehat{u}_h$,

$$\langle \tau(\phi), u_h - \widehat{u}_h \rangle_F \; = \; \eta_F \, \langle \mathbf{n} \cdot \mathbf{r}_F(\phi), \phi \rangle_F \; = \; \eta_F \, (\mathbf{r}_F(\phi), \mathbf{r}_F(\phi))_K.$$

The first approach

The local solvers: The lifting operator $\mathbf{r}$.

Set

$$\mathbf{r} := \sum_{F \in \mathcal{F}(K)} \mathbf{r}_F.$$

Then the first equation defining the local solver,

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_K - (u_h, \nabla \cdot \mathbf{v})_K + \langle \widehat{u}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} = 0,$$

for all $\mathbf{v} \in \mathbf{V}(K)$, or,

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_K = -(\nabla u_h, \mathbf{v})_K + \langle u_h - \widehat{u}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K},$$

can be rewritten as

$$\mathrm{c}\,\mathbf{q}_h = -\nabla u_h + \mathbf{r}(u_h - \widehat{u}_h),$$

provided c is constant on K and $W(K) \subset \mathbf{V}(K)$.

The first approach.

The global problem: The weak formulation for $\widehat{u}_h$.

For each face $F \in \mathcal{E}_h^o$, we take $\widehat{u}_h|_F$ in the space $M(F)$. We determine $\widehat{u}_h$ by requiring that,

$$\begin{aligned} \langle \mu, [\![\widehat{\mathbf{q}}_h]\!] \rangle_F = 0 \quad \forall\, \mu \in M(F) \quad & \text{if } F \in \mathcal{E}_h^o, \\ \widehat{u}_h = u_D \quad & \text{if } F \in \mathcal{E}_h^\partial. \end{aligned}$$

The first approach.

The transmission condition.

Suppose that the transmission condition implies that $[\![\widehat{\mathbf{q}}_h]\!] = 0$ on a face $F \in \mathcal{E}_h^o$. Then, on that face, we have that

$$[\![\mathbf{q}_h]\!] + \tau^+({u_h}^+ - \widehat{u}_h) + \tau^-({u_h}^- - \widehat{u}_h) = 0,$$

which holds if

$$\widehat{u}_h = \frac{\tau^+ {u_h}^+ + \tau^- {u_h}^-}{\tau^+ + \tau^-} + \frac{1}{\tau^+ + \tau^-} [\![\mathbf{q}_h]\!],$$
$$\widehat{\mathbf{q}}_h = \frac{\tau^- {\mathbf{q}_h}^+ + \tau^+ {\mathbf{q}_h}^-}{\tau^+ + \tau^-} + \frac{\tau^+ \tau^-}{\tau^+ + \tau^-} [\![u_h]\!]$$

provided $\tau^+ + \tau^- > 0$.

The first approach.

The numerical trace $\widehat{u}_h$ is well defined.

Theorem

The numerical trace $\widehat{u}_h$ is well defined if, for each $K \in \partial\Omega_h$,

- $\tau > 0$ *on* ∂K,
- $\nabla W(K) \subset \mathbf{V}(K)$.

The system is square. Set $u_D = 0$ and $f = 0$. For $\mu := \widehat{u}_h$, the equation reads

$$0 = \sum_{F \in \mathcal{E}_h^o} \langle \widehat{u}_h, [\![\widehat{\mathbf{q}}_h]\!] \rangle_F = \sum_{K \in \Omega_h} \langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial K} =: \langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h}.$$

Note that

$$\begin{aligned}
-\langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} &= - \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&= - \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} - \langle u_h, \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&\quad + \langle (u_h - \widehat{u}_h), \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \\
&= - \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} - \langle u_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} + \langle u_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} \\
&\quad + \langle (u_h - \widehat{u}_h), \tau(u_h - \widehat{u}_h) \rangle_{\partial \Omega_h}
\end{aligned}$$

The first approach.

Proof.

For $(\mathbf{v}, w) := (\mathbf{q}_h, u_h)$, the equations of the local solvers read

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K - (u_h, \nabla \cdot \mathbf{q}_h)_K + \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial K} &= 0,\\ -(\mathbf{q}_h, \nabla u_h)_K + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, u_h \rangle_{\partial K} &= 0.\end{aligned}$$

Then

$$-\langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial\Omega_h} = (\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_{\Omega_h} + \langle (u_h - \widehat{u}_h), \tau (u_h - \widehat{u}_h) \rangle_{\partial\Omega_h}.$$

As a consequence, $\langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial\Omega_h} = 0$ implies $\mathbf{q}_h = 0$ on Ω_h and $u_h = \widehat{u}_h$ on $\partial\Omega_h$.

The first approach.

Proof.

Now, the first equation definign the local solvers reads

$$-(u_h, \nabla \cdot \mathbf{v})_K + \langle u_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} = 0,$$

for all $\mathbf{v} \in \mathbf{V}(K)$. Hence

$$(\nabla u_h, \mathbf{v})_K = 0,$$

and so $\nabla u_h = 0$.

This shows that u_h is a constant and, since $u_h = \widehat{u}_h = 0$ on $\partial\Omega$, we can conclude that $u_h = 0$ on Ω_h. We now have that $\widehat{u}_h = u_h = 0$ on $\partial\Omega_h$. This proves the result.

The first approach.

Characterization of the approximate solution.

We have that $(\mathbf{q}_h, u_h) = (\mathbf{Q}_{\widehat{u}_h}, \mathrm{U}_{\widehat{u}_h}) + (\mathbf{Q}_f, \mathrm{U}_f)$ where

$$(\mathbf{Q}_{\widehat{u}_h}, \mathrm{U}_{\widehat{u}_h}) := (\mathbf{Q}(\widehat{u}_h, 0), \mathrm{U}(\widehat{u}_h, 0)), \quad (\mathbf{Q}_f, \mathrm{U}_f) := (\mathbf{Q}(0, f), \mathrm{U}(0, f)).$$

where $(\mathbf{Q}(\widehat{u}_h, f), \mathrm{U}(\widehat{u}_h, f))$ is the linear mapping that associates $(\widehat{u}_h, f)$ to $(\mathbf{q}_h, u_h)$, and where the numerical trace $\widehat{u}_h$ is the element of the space

$$M_h := \{\mu \in L^2(\mathcal{E}_h) : \quad \mu|_F \in M(F) \;\; \forall\, F \in \mathcal{E}_h\},$$

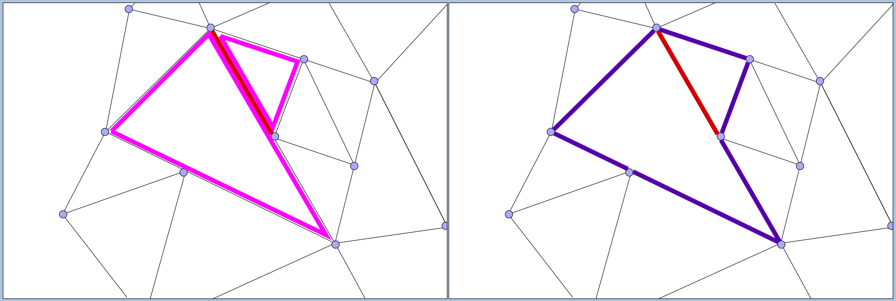
satisfying the equations

$$\begin{aligned} a_h(\widehat{u}_h, \mu) &= \ell_h(\mu) && \forall\, \mu \in M_h : \mu|_{\partial\Omega} = 0, \\ \langle \mu, \widehat{u}_h \rangle_{\partial\Omega} &= \langle \mu, u_D \rangle_{\partial\Omega} && \forall\, \mu \in M_h, \end{aligned}$$

where $a_h(\mu, \lambda) := -\langle \mu, \widehat{\mathbf{Q}}_\lambda \cdot \mathbf{n} \rangle_{\partial\Omega_h}$, and $\ell_h(\mu) := \langle \mu, \widehat{\mathbf{Q}}_f \cdot \mathbf{n} \rangle_{\partial\Omega_h}$.

The stiffness matrix is sparse by blocks:

$$a_h(\mu, \eta) = -\langle \mu, \widehat{\mathbf{Q}}_\eta \cdot \mathbf{n} \rangle_{\partial\Omega_h} \neq 0.$$



The first approach.

Symmetry and condition number of the stiffness matrix.

Theorem

We have that

$$a_h(\mu,\lambda) = (\mathrm{c}\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_{\partial\Omega_h} + \langle \tau(\mathsf{U}_\mu - \mu), (\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h}.$$

Moreover, if $\mathbf{V}(K) = \boldsymbol{\mathcal{P}}_k(K)$, $W(K) = \mathcal{P}_k(K)$ and $M(F) = \mathcal{P}_k(K)$, $k \geq 0$, the condition number of $a_h(\cdot,\cdot)$ (on $M_{h,0} \times M_{h,0}$) is of order $(1 + (\tau^*\, h)^2) h^{-2}$.

Here $\tau^* := \max_{K\in\Omega_h} \tau|_{\partial K \setminus F_K^*}$, where F_K^* is an arbitrary face of the simplex K.

Note that the matrix is invertible even if $\tau \equiv 0$!

The first approach.

Proof.

$$
\begin{aligned}
a_h(\mu,\lambda) &= -\langle \mu, \widehat{\mathbf{Q}}_\lambda \cdot \mathbf{n}\rangle_{\partial\Omega_h} \\
&= -\langle \mu, \mathbf{Q}_\lambda \cdot \mathbf{n} + \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h} \\
&= -\langle \mu, \mathbf{Q}_\lambda \cdot \mathbf{n}\rangle_{\partial\Omega_h} - \langle \mathsf{U}_\mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h} \\
&\quad + \langle \mathsf{U}_\mu - \mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h} \\
&= -\langle \mu, \mathbf{Q}_\lambda \cdot \mathbf{n}\rangle_{\partial\Omega_h} - \langle \mathsf{U}_\mu, \widehat{\mathbf{Q}}_\lambda \cdot \mathbf{n}\rangle_{\partial\Omega_h} + \langle \mathsf{U}_\mu, \mathbf{Q}_\lambda \cdot \mathbf{n}\rangle_{\partial\Omega_h} \\
&\quad + \langle \mathsf{U}_\mu - \mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h}
\end{aligned}
$$

For $(\mathbf{v}, w) := (\mathbf{Q}_\lambda, \mathsf{U}_\mu)$, the equations of the local solvers read

$$\begin{aligned}(\mathrm{c}\,\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_K - (\mathsf{U}_\mu, \nabla\cdot\mathbf{Q}_\lambda)_K + \langle \mu, \mathbf{Q}_\lambda\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\ -(\mathbf{Q}_\lambda, \nabla\mathsf{U}_\mu)_K + \langle \widehat{\mathbf{Q}}_\lambda\cdot\mathbf{n}, \mathsf{U}_\mu\rangle_{\partial K} &= 0.\end{aligned}$$

Then

$$a_h(\mu,\lambda) = (\mathrm{c}\,\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_K + \langle \mathsf{U}_\mu - \mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h}.$$

This completes the proof.

The first approach.

A rewriting of the method.

The approximate solution $(\mathbf{q}_h, u_h, \widehat{u}_h)$ is the element of the space $\mathbf{V}_h \times W_h \times M_h$ satisfying the equations

$$\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{v})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \mu, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,\\
\langle \mu, \widehat{u}_h\rangle_{\partial\Omega} &= \langle \mu, u_D\rangle_{\partial\Omega},
\end{aligned}$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where

$$\widehat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial\Omega_h.$$

For any $(w, \mu) \in W_h \times M_h$, define $\mathbf{q}_{w,\mu} \in \mathbf{V}_h$ as the solution of

$$(\mathrm{c}\, \mathbf{q}_{w,\mu}, \mathbf{v})_{\Omega_h} - (w, \nabla \cdot \mathbf{v})_{\Omega_h} + \langle \mu, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial \Omega_h} = 0,$$

for all $\mathbf{v} \in V_h$. Set also,

$$J_h(w, \mu) := \frac{1}{2}(\mathrm{c}\, \mathbf{q}_{w,\mu}, \mathbf{q}_{w,\mu})_{\Omega_h} + \frac{1}{2}\langle \tau(w - \mu), (w - \mu) \rangle_{\partial \Omega_h} - (f, w)_{\Omega_h},$$

for all $(w, \mu) \in W_h \times M_h$.

Theorem

$J_h(u_h, \widehat{u}_h) \leq J_h(w, \mu) \quad \forall (w, \mu) \in W_h \times M_h : \mu = u_D$ *on* $\partial \Omega$.

The first approach.

Proof.

Note that $\mathbf{q}_{w,\mu} = \mathbf{q}_h$ when $(w, \mu) = (u_h, \widehat{u}_h)$.
Then, we have,

$$\begin{aligned}
\delta_w J_h(u_h, \widehat{u}_h) &= (\mathrm{c}\, \mathbf{q}_h, \mathbf{q}_{\delta w, 0})_{\Omega_h} + \langle \tau(u_h - \widehat{u}_h), \delta w\rangle_{\partial\Omega_h} - (f, \delta w)_{\Omega_h},\\
\delta_\mu J_h(u_h, \widehat{u}_h) &= (\mathrm{c}\, \mathbf{q}_h, \mathbf{q}_{0, \delta\mu})_{\Omega_h} - \langle \tau(u_h - \widehat{u}_h), \delta\mu\rangle_{\partial\Omega_h}.
\end{aligned}$$

By definition of $\mathbf{q}_{w,\mu}$, we get

$$\begin{aligned}
\delta_w J_h(u_h, \widehat{u}_h) &= \quad (\delta w, \nabla \cdot \mathbf{q}_h)_{\Omega_h} + \langle \tau(u_h - \widehat{u}_h), \delta w\rangle_{\partial\Omega_h} - (f, \delta w)_{\Omega_h},\\
\delta_\mu J_h(u_h, \widehat{u}_h) &= -\langle \delta\mu, \mathbf{q}_h \cdot \mathbf{n}\rangle_{\partial\Omega_h} - \langle \tau(u_h - \widehat{u}_h), \delta\mu\rangle_{\partial\Omega_h},
\end{aligned}$$

and so

$$\begin{aligned}
\delta_w J_h(u_h, \widehat{u}_h) &= -(\mathbf{q}_h, \nabla \delta w)_{\Omega_h} + \langle \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h), \delta w\rangle_{\partial\Omega_h} - (f, \delta w)_{\Omega_h},\\
\delta_\mu J_h(u_h, \widehat{u}_h) &= -\langle \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h), \delta\mu\rangle_{\partial\Omega_h}.
\end{aligned}$$

The first approach.

The jumps $u_h - \widehat{u}_h$ stabilize the method.

The energy identity for the exact solution is

$$(\mathrm{c}\,\mathbf{q}, \mathbf{q})_\Omega = (f, u)_\Omega - \langle u_D, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial\Omega},$$

and for the approximate solution,

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_\Omega + \Theta_\tau(u_h - \widehat{u}_h) = (f, u_h)_\Omega - \langle u_D, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial\Omega}.$$

where $\Theta_\tau(u_h - \widehat{u}_h) := \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial\Omega_h}$.

$\Theta_\tau(u_h - \widehat{u}_h)$ is a dissipative term of the same form of that of the original DG method, when the stabilization function τ is positive.

The first approach.

The jumps $u_h - \widehat{u}_h$ control the four residuals.

The Galerkin formulation on the element K defining the local solver reads

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_K - (u_h, \nabla\cdot\mathbf{v})_K + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\
-(\mathbf{q}_h, \nabla w)_K + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial K} &= (f, w)_K,
\end{aligned}
$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, or, equivalently,

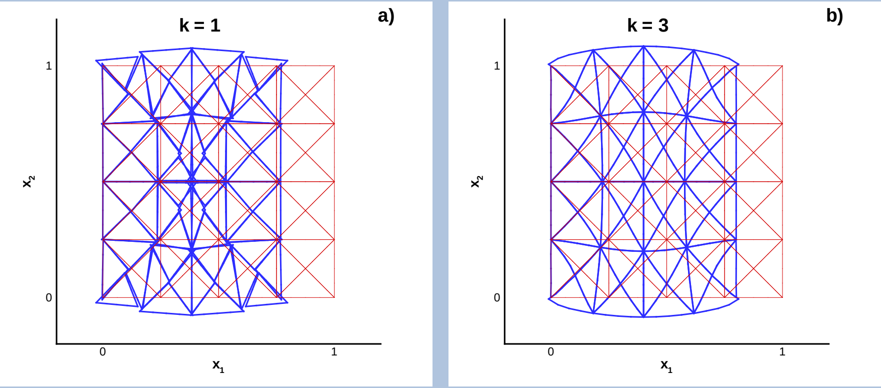
$$
\begin{aligned}
(\mathbf{R}^u_K, \mathbf{v})_K &= \langle R^u_{\partial K}, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial K} \quad \forall\ \mathbf{v}\in\mathbf{V}(K),\\
(R^{\mathbf{q}}_K, w)_K &= \langle R^{\mathbf{q}}_{\partial K}, w\rangle_{\partial K} \quad \forall\ w\in W(K),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{R}^u_K &:= \mathrm{c}\mathbf{q}_h + \nabla u_h & R^u_{\partial K} &:= u_h - \widehat{u}_h\\
R^{\mathbf{q}}_K &:= \nabla\cdot\mathbf{q}_h - f & R^{\mathbf{q}}_{\partial K} &:= (\mathbf{q}_h - \widehat{\mathbf{q}}_h)\cdot\mathbf{n} = -\tau\,(u_h - \widehat{u}_h).
\end{aligned}
$$

The first approach.

An illustration: An HDG method for nonlinear elasticity.

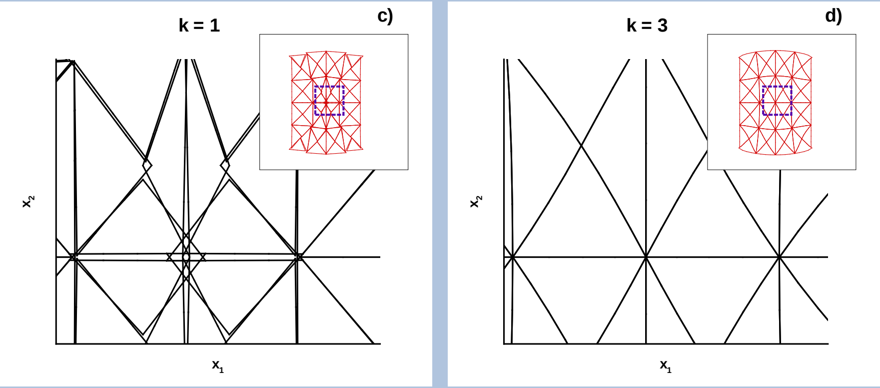


a) deformed shape using $\mathcal{P}^1$, b) deformed shape using $\mathcal{P}^3$.

(S.-C. Soon, B.C. and H. Stolarski, 2008.)

The first approach.

An Illustration: An HDG method for nonlinear elasticity.



c) closeup view of Figure a), d) closeup view of Figure b).

(S.-C. Soon, B.C. and H. Stolarski, 2008.)

The first approach.

An interpretation of the role of τ.

Since

$$\tau = -\frac{R^{\mathbf{q}}_{\partial K}}{R^{u}_{\partial K}} \approx \frac{R^{\mathbf{q}}_{K}}{\mathbf{R}^{u}_{K}}.$$

where

$$\begin{aligned} \mathbf{R}^{u}_{K} &:= \mathrm{c}\mathbf{q}_h + \nabla u_h \qquad & R^{u}_{\partial K} &:= u_h - \widehat{u}_h \\ R^{\mathbf{q}}_{K} &:= \nabla \cdot \mathbf{q}_h - f \qquad & R^{\mathbf{q}}_{\partial K} &:= (\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}. \end{aligned}$$

we see that τ forces a ratio between the residuals.

The first approach.

Questions.

- How to impose Neumann or Robin boundary conditions?
- What methods are associated with $\tau = 0$?
- What methods are associated with $\tau = \infty$?
- How to pick τ as to render the method as accurate as possible?
- How to pick the local spaces as to render the method as accurate as possible?
- Should the a posteriori error estimates depend only on the jumps $u_h - \widehat{u}_h$ and the stabilization function τ?

The first approach.

The effect of the local paces and τ on the accuracy of the method on simplexes.

Method	$\mathbf{V}(K)$	$W(K)$	$M(F)$	k
RT	$\boldsymbol{\mathcal{P}}_k(K) + \mathbf{x}\,\mathcal{P}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$	≥ 0
BDM	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(F)$	≥ 1
HDG	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$	≥ 0
CG	$\boldsymbol{\mathcal{P}}_{k-1}(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$	≥ 1

The first approach.

The effect of the locals paces and τ on the accuracy of the method on simplexes.

Method	$R^u_{\partial K}$	$R^{\mathbf{q}}_{\partial K}$	$\tau = -R^{\mathbf{q}}_{\partial K}/R^u_{\partial K}$	$\mathbf{q}_h$	u_h	$\overline{u}_h$	k
RT	–	0	0	$k+1$	$k+1$	$k+2$	≥ 0
BDM	–	0	0	$k+1$	k	$k+2$	≥ 2
HDG	–	–	$\mathcal{O}(h)$	$k+1$	k	$k+2$	≥ 1
HDG	–	–	$\mathcal{O}(1)$	$k+1$	$k+1$	$k+2$	≥ 1
HDG	–	–	$\mathcal{O}(1)$	1	1	1	$= 0$
HDG	–	–	$\mathcal{O}(1/h)$	k	$k+1$	$k+1$	≥ 1
CG	0	–	∞	k	$k+1$	$k+1$	≥ 1

On the element $K \in \Omega_h$, given $\widehat{\mathbf{q}} \cdot \mathbf{n}$ on ∂K, $\overline{u}$ and f, we have that $(\mathbf{q}, u)$ satisfies

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}, \mathbf{v})_K - (u, \nabla \cdot \mathbf{v})_K + \langle \widehat{u}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= 0, \\
-(\mathbf{q}, \nabla w)_K + \langle \widehat{\mathbf{q}} \cdot \mathbf{n}, w \rangle_{\partial K} &= (f - \overline{f} + \overline{\widehat{\mathbf{q}} \cdot \mathbf{n}}, w)_K, \\
(u, 1)_K &= (\overline{u}, 1)_K
\end{aligned}
$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, where

$$
\widehat{u} = u \qquad \text{on } \partial K.
$$

On the element $K \in \Omega_h$, we define $(\mathbf{q}_h, u_h)$ in terms of $(\widehat{\mathbf{q}}_h, \overline{u}_h, f)$ as the element of $\mathbf{V}(K) \times W(K)$ such that

$$\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_K - (u_h, \nabla \cdot \mathbf{v})_K + \langle \widehat{u}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= 0, \\
-(\mathbf{q}_h, \nabla w)_K + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial K} &= (f - \overline{f} + \overline{\widehat{\mathbf{q}}_h \cdot \mathbf{n}}, w)_K, \\
(u_h, 1)_K &= (\overline{u}_h, 1)_K,
\end{aligned}$$

for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$, where

$$\widehat{u}_h = u_h + s(\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n} \qquad \text{on } \partial K.$$

The second approach.

The local solvers are well defined.

Theorem

The local solver on K is well defined if

- $s \geq 0$ *on* ∂K,
- $\nabla W(K) \subset \mathbf{V}(K)$.

The system is square. Set $\widehat{\mathbf{q}}_h \cdot \mathbf{n} = 0$, $\overline{u}_h = 0$ and $f = 0$.
For $(\mathbf{v}, w) := (\mathbf{q}_h, u_h)$, the equations read

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K - (u_h, \nabla \cdot \mathbf{q}_h)_K + \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial K} &= 0,\\ -(\mathbf{q}_h, \nabla u_h)_K &= 0,\\ (u_h, 1)_K &= 0.\end{aligned}$$

Hence

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K + \langle \widehat{u}_h - u_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial K} = 0,$$

and since $\widehat{u}_h = u_h + s\mathbf{q}_h \cdot \mathbf{n}$, we get

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K + \langle s\,\mathbf{q}_h \cdot \mathbf{n}, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial K} = 0.$$

This implies that $\mathbf{q}_h = 0$ on K and that $\widehat{u}_h - u_h = s\,\mathbf{q}_h \cdot \mathbf{n} = 0$ on ∂K.

The second approach.

Proof.

Since $\mathbf{q}_h = 0$ and $\widehat{u}_h = u_h$, the first equation reads

$$-(u_h, \nabla \cdot \mathbf{v})_K + \langle u_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} = 0 \qquad \forall \, \mathbf{v} \in \mathbf{V}(K).$$

Hence

$$(\nabla u_h, \mathbf{v})_K = 0 \qquad \forall \, \mathbf{v} \in \mathbf{V}(K),$$

and so $\nabla u_h = 0$. Since $(u_h, 1)_K = 0$, $u_h = 0$. This proves the result.

The second approach.

The global problem: The weak formulation for $\widehat{\mathbf{q}}_h$ and $\overline{u}_h$.

For each face $F \in \mathcal{E}_h$, we take $\widehat{\mathbf{q}}_h|_F$ in the space $\mathbf{N}(F)$. Of course, if $F \in \mathcal{E}_h^o$ we impose the condition that $[\![\widehat{\mathbf{q}}_h]\!] = 0$.

We determine the numerical trace $\widehat{\mathbf{q}}_h$ and the local average $\overline{u}_h$ by requiring that, for each face $F \in \mathcal{E}_h$,

$$\begin{aligned}
\langle \boldsymbol{\eta}, [\![\widehat{u}_h]\!]\rangle_F &= 0 && \forall\, \boldsymbol{\eta} \in \mathbf{N}(F) \quad \text{if } F \in \mathcal{E}_h^o,\\
\langle \widehat{u}_h, \boldsymbol{\eta}\cdot\mathbf{n}\rangle_F &= \langle u_D, \boldsymbol{\eta}\cdot\mathbf{n}\rangle_F && \forall\, \boldsymbol{\eta} \in \mathbf{N}(F) \quad \text{if } F \in \mathcal{E}_h^\partial,
\end{aligned}$$

and by requiring that, for each element $K \in \Omega_h$,

$$\langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, 1\rangle_{\partial K} = (f, 1)_K.$$

Suppose that the transmission condition implies that $[\![\widehat{u}_h]\!] = 0$ on a face $F \in \mathcal{E}_h^o$. Then, on that face, we have that

$$[\![u_h]\!] + s^+(\mathbf{q}_h{}^+ - \widehat{\mathbf{q}}_h) + s^-(\mathbf{q}_h{}^- - \widehat{\mathbf{q}}_h) = 0,$$

which holds if

$$\widehat{u}_h = \frac{s^- u_h{}^+ + s^+ u_h{}^-}{s^+ + s^-} + \frac{s^+ s^-}{s^+ + s^-} [\![\mathbf{q}_h]\!],$$
$$\widehat{\mathbf{q}}_h = \frac{s^+ \mathbf{q}_h{}^+ + s^- \mathbf{q}_h{}^-}{s^+ + s^-} + \frac{1}{s^+ + s^-} [\![u_h]\!]$$

provided $s^+ + s^- > 0$.

The second approach.

The numerical trace $\widehat{\mathbf{q}}_h$ and the local average $\overline{u}$ are well defined.

Theorem

The numerical trace $\widehat{\mathbf{q}}_h$ and the local average $\overline{u}$ are well defined if, for each $K \in \partial\Omega_h$,

- $s \geq 0$ *on* ∂K,
- $\nabla W(K) \subset \mathbf{V}(K)$.

The second approach.

Proof.

The system is square. Set $u_D = 0$ and $f = 0$. For $\boldsymbol{\eta} := \widehat{\mathbf{q}}_h$, the first equation reads

$$0 = \sum_{F \in \mathcal{E}_h^o} \langle \widehat{\mathbf{q}}_h, [\![\widehat{u}_h]\!] \rangle_F = \sum_{K \in \Omega_h} \langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial K} =: \langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h}.$$

Note that

$$\begin{aligned} -\langle \widehat{u}_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} &= -\langle u_h + s(\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} \\ &= -\langle u_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} - \langle s(\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} \\ &\quad + \langle s(\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}, (\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n} \rangle_{\partial \Omega_h} \\ &= -\langle u_h, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} - \langle \widehat{u}_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} + \langle u_h, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\partial \Omega_h} \\ &\quad + \langle s(\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}, (\mathbf{q}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n} \rangle_{\partial \Omega_h}. \end{aligned}$$

The second approach.

Proof.

For $(\mathbf{v}, w) := (\mathbf{q}_h, u_h)$, the equations of the local solvers read

$$\begin{aligned}(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_K - (u_h, \nabla\cdot\mathbf{q}_h)_K + \langle \widehat{u}_h, \mathbf{q}_h\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\ -(\mathbf{q}_h, \nabla u_h)_K + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, u_h\rangle_{\partial K} &= 0.\end{aligned}$$

Then

$$-\langle \widehat{u}_h, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h} = (\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_{\Omega_h} + \langle s(\mathbf{q}_h - \widehat{\mathbf{q}}_h)\cdot\mathbf{n}, (\mathbf{q}_h - \widehat{\mathbf{q}}_h)\cdot\mathbf{n}\rangle_{\partial\Omega_h}.$$

As a consequence, $\langle \widehat{u}_h, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h} = 0$ implies $\mathbf{q}_h = 0$ on Ω_h and $\widehat{u}_h - u_h = s(\mathbf{q}_h - \widehat{\mathbf{q}}_h)\cdot\mathbf{n} = 0$ on $\partial\Omega_h$,

The second approach.

Proof.

Since, $\widehat{u}_h = u_h$ on $\partial\Omega_h$, the first equation defining the local solvers read

$$-(u_h, \nabla \cdot \mathbf{v})_K + \langle u_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} = 0,$$

for all $\mathbf{v} \in \mathbf{V}(K)$. This implies that u_h is a constant on Ω. Since $u_h = \widehat{u}_h = 0$ on $\partial\Omega$, we have that $u_h = 0$. As a consequence, $\overline{u}_h = 0$. This proves the result.

The second approach.

Characterization of the approximate solution.

We have that $(\mathbf{q}_h, u_h) = (\mathbf{Q}_{\widehat{\mathbf{q}}_h}, \mathsf{U}_{\widehat{\mathbf{q}}_h}) + (0, \overline{u}_h) + (\mathbf{Q}_f, \mathsf{U}_f)$ where

$$(\mathbf{Q}_{\widehat{\mathbf{q}}_h}, \mathsf{U}_{\widehat{\mathbf{q}}_h}) := (\mathbf{Q}(\widehat{u}_h, 0), \mathsf{U}(\widehat{\mathbf{q}}_h, 0)), \quad (\mathbf{Q}_f, \mathsf{U}_f) := (\mathbf{Q}(0, f), \mathsf{U}(0, f)).$$

where $(\mathbf{Q}(\widehat{\mathbf{q}}_h, f), \mathsf{U}(\widehat{\mathbf{q}}_h, f))$ is the linear mapping that associates $(\widehat{\mathbf{q}}_h, f)$ to $(\mathbf{q}_h, u_h)$.
Here, we take $(\widehat{\mathbf{q}}_h, \overline{u}_h) \in \mathbf{N}_h \times W_h^0$, where

$$\mathbf{N}_h := \{\boldsymbol{\eta} \in \mathbf{L}^2(\mathcal{E}_h) : \quad \boldsymbol{\eta}|_F \in \mathbf{N}(F) \ \ \forall\, F \in \mathcal{E}_h \ \ [\![\boldsymbol{\eta}]\!] = 0 \text{ on } \mathcal{E}_h^o\},$$
$$W_h^0 := \{\bar{w} \in L^2(\Omega) : \bar{w}|_K \text{ is a constant } \forall K \in \Omega_h\}.$$

The function $(\widehat{\mathbf{q}}_h, \overline{u}_h)$ satisfies the equations

$$\begin{aligned}
a_h(\widehat{\mathbf{q}}_h, \boldsymbol{\eta}) + b_h(\overline{u}_h, \boldsymbol{\eta}) &= \ell_{1,h}(\boldsymbol{\eta}) && \forall\, \boldsymbol{\eta} \in \mathbf{N}_h,\\
b_h(\bar{w}, \widehat{\mathbf{q}}_h) &= \ell_{2,h}(\bar{w}) &&\\
\langle \boldsymbol{\eta}\cdot\mathbf{n}, \widehat{u}_h\rangle_{\partial\Omega} &= \langle \boldsymbol{\eta}\cdot\mathbf{n}, u_D\rangle_{\partial\Omega} && \forall\, \boldsymbol{\eta} \in \mathbf{N}_h,
\end{aligned}$$

where

$$\begin{aligned}
a_h(\boldsymbol{\eta}, \boldsymbol{\zeta}) &:= -\,\langle \boldsymbol{\eta}\cdot\mathbf{n}, \widehat{\mathbf{U}}_{\boldsymbol{\zeta}}\rangle_{\partial\Omega_h},\\
b_h(\bar{w}, \boldsymbol{\eta}) &:= -\,\langle \bar{w}, \boldsymbol{\eta}\cdot\mathbf{n}\rangle_{\partial\Omega},\\
\ell_{1,h}(\boldsymbol{\eta}) &:= \langle \boldsymbol{\eta}\cdot\mathbf{n}, \widehat{\mathbf{U}}_f\rangle_{\partial\Omega_h},\\
\ell_{2,h}(\bar{w}) &:= (f, \bar{w})_{\Omega_h}.
\end{aligned}$$

Theorem

We have that

$$a_h(\boldsymbol{\eta}, \zeta) = (\mathrm{c}\mathbf{Q}_{\boldsymbol{\eta}}, \mathbf{Q}_{\zeta})_{\partial\Omega_h} + \langle s(\mathbf{Q}_{\boldsymbol{\eta}} - \boldsymbol{\eta}) \cdot \mathbf{n}, (\mathbf{Q}_{\zeta} - \zeta) \cdot \mathbf{n}) \rangle_{\partial\Omega_h}.$$

Estimate of condition number still open!

The second approach.

A rewriting of the method.

The approximate solution $(\mathbf{q}_h, u_h, \widehat{\mathbf{q}}_h)$ is the element of the space $\mathbf{V}_h \times W_h \times \mathbf{N}_h$ satisfying the equations

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{v})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \boldsymbol{\eta}\cdot\mathbf{n}, \widehat{u}_h\rangle_{\partial\Omega_h} &= \langle \boldsymbol{\eta}\cdot\mathbf{n}, u_D\rangle_{\partial\Omega},
\end{aligned}
$$

for all $(\mathbf{v}, w, \boldsymbol{\eta}) \in \mathbf{V}_h \times W_h \times \mathbf{N}_h$, where

$$\widehat{u}_h = u_h + s(\mathbf{q}_h - \widehat{\mathbf{q}}_h)\cdot\mathbf{n} \qquad \text{on } \partial\Omega_h.$$

Note that this method is strongly related to the method obtained with the first approach with $\tau = 1/s$.

The limit case $s = 0$. (B.C., J.Gopalakrishnan and H.Wang, Math. Comp., 2007.)

Local conservativity of the CG method.

We take K simplex, $\mathbf{V}(K) = \mathcal{P}_k(K)$, $W(K) = \mathcal{P}_k(K)$ and $\mathbf{N}(F)\cdot\mathbf{n} = \mathcal{P}_k(K)$.

We decompose $\mathbf{V}(K)$ as $\tilde{\mathbf{V}}(K) \oplus \mathbf{V}^\perp(K)$, where $\tilde{\mathbf{V}}(K) := \mathcal{P}_{k-1}(K)$ and $\mathbf{V}^\perp(K)$ consists of the elements of $\mathbf{V}(K)$ which are $L^2(K)-$orthogonal to the elements of $\tilde{\mathbf{V}}(K)$.

We then rewrite the method as follows:

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h^\perp, \mathbf{v}^\perp)_{\Omega_h} + \langle \widehat{u}_h - u_h, \mathbf{v}^\perp \cdot \mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
(\mathrm{c}\,\tilde{\mathbf{q}}_h + \nabla u_h, \tilde{\mathbf{v}})_{\Omega_h} + \langle \widehat{u}_h - u_h, \tilde{\mathbf{v}} \cdot \mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\tilde{\mathbf{q}}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \boldsymbol{\eta}\cdot\mathbf{n}, \widehat{u}_h\rangle_{\partial\Omega_h} &= \langle \boldsymbol{\eta}\cdot\mathbf{n}, u_D\rangle_{\partial\Omega},
\end{aligned}
$$

where $\widehat{u}_h = u_h + s(\mathbf{q}_h^\perp + \tilde{\mathbf{q}}_h - \widehat{\mathbf{q}}_h)\cdot\mathbf{n}$ on $\partial\Omega_h$.

The limit case $s = 0$

K simplex, $\mathbf{V}(K) = \mathcal{P}_k(K)$, $W(K) = \mathcal{P}_k(K)$ and $\mathbf{N}(F) \cdot \mathbf{n} = \mathcal{P}_k(K)$.

Setting $\mathbb{Q}_h^\perp := \mathbf{q}_h^\perp / s$ and $\mathbb{U} := u_h/s$, we eliminate $\widehat{u}_h$ from the equations

$$
\begin{aligned}
(\mathrm{c}\,\mathbb{Q}_h^\perp, \mathbf{v}^\perp)_{\Omega_h} + \langle (s\,\mathbb{Q}_h^\perp + \tilde{\mathbf{q}}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}, \mathbf{v}^\perp \cdot \mathbf{n} \rangle_{\partial\Omega_h} &= 0,\\
(\mathrm{c}\,\tilde{\mathbf{q}}_h + \nabla u_h, \tilde{\mathbf{v}})_{\Omega_h} + s\langle (s\mathbb{Q}_h^\perp + \tilde{\mathbf{q}}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}, \tilde{\mathbf{v}} \cdot \mathbf{n} \rangle_{\partial\Omega_h} &= 0,\\
-(\tilde{\mathbf{q}}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \boldsymbol{\eta} \cdot \mathbf{n}, \mathbb{U}_h + (s\mathbb{Q}_h^\perp + \tilde{\mathbf{q}}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n} \rangle_{\partial\Omega_h \setminus \partial\Omega} &= 0,\\
\langle \boldsymbol{\eta} \cdot \mathbf{n}, u_h + s\,(s\mathbb{Q}_h^\perp + \tilde{\mathbf{q}}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n} \rangle_{\partial\Omega_h} &= \langle \boldsymbol{\eta} \cdot \mathbf{n}, u_D \rangle_{\partial\Omega}.
\end{aligned}
$$

The limit case $s = 0$

K simplex, $\mathbf{V}(K) = \boldsymbol{\mathcal{P}}_k(K)$, $W(K) = \mathcal{P}_k(K)$ and $\mathbf{N}(F) \cdot \mathbf{n} = \mathcal{P}_k(K)$.

We (formally) pass to the limit and let s go to zero:

$$
\begin{aligned}
(\mathrm{c}\,\mathbb{Q}_h^{\perp}, \mathbf{v}^{\perp})_{\Omega_h} + \langle (\tilde{\mathbf{q}}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}, \mathbf{v}^{\perp} \cdot \mathbf{n} \rangle_{\partial\Omega_h} &= 0, \\
(\mathrm{c}\,\tilde{\mathbf{q}}_h + \nabla u_h, \tilde{\mathbf{v}})_{\Omega_h} &= 0, \\
-(\tilde{\mathbf{q}}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h}, \\
\langle \boldsymbol{\eta} \cdot \mathbf{n}, \mathbb{U}_h + (\tilde{\mathbf{q}}_h - \widehat{\mathbf{q}}_h) \cdot \mathbf{n} \rangle_{\partial\Omega_h \setminus \partial\Omega} &= 0, \\
\langle \boldsymbol{\eta} \cdot \mathbf{n}, u_h \rangle_{\partial\Omega_h} &= \langle \boldsymbol{\eta} \cdot \mathbf{n}, I_h u_D \rangle_{\partial\Omega}.
\end{aligned}
$$

The limit case $s = 0$

K simplex, $\mathbf{V}(K) = \boldsymbol{\mathcal{P}}_k(K)$, $W(K) = \mathcal{P}_k(K)$ and $\mathbf{N}(F) \cdot \mathbf{n} = \mathcal{P}_k(K)$.

Since $\tilde{\mathbf{q}}_h = -\mathrm{a}\,\nabla u_h$ on Ω_h, we have that

$$(\mathrm{a}\,\nabla u_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial\Omega_h} = (f, w)_{\Omega_h} \quad \forall w \in W_h,$$

where $\widehat{\mathbf{q}}_h = -\{\mathrm{a}\nabla u_h\} + \frac{1}{2}\,[\![\mathbb{U}_h]\!]$ on $\partial\Omega_h$, and $u_h \in W_h \cap \mathcal{C}^0(\Omega)$ with $u_h = I_h u_D$ on $\partial\Omega$.

The limit case $s = 0$

K simplex, $\mathbf{V}(K) = \boldsymbol{\mathcal{P}}_k(K)$, $W(K) = \mathcal{P}_k(K)$ and $\mathbf{N}(F) \cdot \mathbf{n} = \mathcal{P}_k(K)$.

- We first compute u_h, which is nothing but the approximation given by the CG method:

$$(\mathrm{a}\,\nabla u_h, \nabla w)_{\Omega_h} = (f, w)_{\Omega_h} \quad \forall w \in W_h \cap \mathcal{C}^0_0(\Omega).$$

- We then compute $[\![\mathbb{U}_h]\!]$ by solving a global problem whose matrix has condition of order one:

$$\frac{1}{2}\langle [\![\mathbb{U}_h]\!], [\![w]\!]\rangle_{\partial\Omega_h} = (f, w)_{\Omega_h} - (\mathrm{a}\,\nabla u_h, \nabla w)_{\Omega_h} + \langle\{\mathrm{a}\nabla u_h\} \cdot \mathbf{n}, w\rangle_{\partial\Omega_h},$$

for all $w \in W_h$.

- We finally compute the numerical trace $\widehat{\mathbf{q}}_h$ which renders the CG method locally conservative. Moreover, by using $\tilde{\mathbf{q}}_h$ and $\widehat{\mathbf{q}}_h$, we can compute, in an element-by-element fashion, an $H(div)$-conforming approximations to the flux by using suitable modifications of the RT or BDM projections.

Examples according to the local solver. (B.C., J.Gopalakrishnan and R.Lazarov, SINUM, 2009.)

Local spaces for simplexes K.

Method	$\mathbf{V}(K)$	$W(K)$	$M(F)$
RT-H	$\boldsymbol{\mathcal{P}}_k(K) + \mathbf{x}\,\mathcal{P}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
BDM-H	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_{k-1}(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
IP-H	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$

Method	$\widehat{\mathbf{q}}_h$
RT-H	$\mathbf{q}_h$
BDM-H	$\mathbf{q}_h$
LDG-H	$\mathbf{q}_h + \tau(u_h - \widehat{u}_h) \cdot \mathbf{n}$
IP-H	$-a\nabla u_h + \tau(u_h - \widehat{u}_h) \cdot \mathbf{n}$

Examples according to the local solver.

The bilinear form a_h.

Method	$a_h(\eta,\mu)$
RT-H	$(\mathrm{c}\,\mathbf{Q}\eta,\mathbf{Q}\mu)_{\Omega_h}$
BDM-H	$(\mathrm{c}\,\mathbf{Q}\eta,\mathbf{Q}\mu)_{\Omega_h}$
LDG-H	$(\mathrm{c}\,\mathbf{Q}\eta,\mathbf{Q}\mu)_{\Omega_h}+\langle\tau(\mathsf{U}\mu-\mu),\mathsf{U}\eta-\eta\rangle_{\partial\Omega_h}$
IP-H[†]	$(\mathrm{c}\nabla\mathsf{U}\mu,\nabla\mathsf{U}\eta)_{\Omega_h}+\langle\tau(\mathsf{U}\mu-\mu),\mathsf{U}\eta-\eta\rangle_{\partial\Omega_h}$
	$\langle(\eta-\mathsf{U}\eta),\mathrm{c}\nabla\mathsf{U}\mu\rangle_{\partial\Omega_h}+\langle\mu-\mathsf{U}\mu,\mathrm{c}\nabla\mathsf{U}\eta\rangle_{\partial\Omega_h}.$

[†] We assume that c is a constant on each element.

Examples according to the local solver

Some remarks.

- The RT-H method is the hybridized version of the original RT method.
- The BDM-H method is the hybridized version of the original BDM method.
- The LDG-H method is not the hybridized version of the LDG method.
- The IP-H method is not the hybridized version of the IP method.
- The bilinear forms a_h of the RT-H, BDM-H and SF-H methods are the same on simplexes. (For these three methods, $\tau^* = 0$.)
- The LDG-H method is defined for any $\tau > 0$.
- The IP-H method is defined only for $\tau \approx h^{-1}$.
- The LDG-H and IP-H can be applied on any polyhedral element K.

Examples according to the local solver

Questions.

- HDG methods were devised so that they are efficiently implemented, but how about their accuracy?
- For $k = 0$, do they give (finite volume-like) convergence schemes?
- Can they have the same superconvergence properties than the mixed methods?
- Is the lack of commutative properties an essential barrier to achieving superconvergence?

We seek HDG methods for which a projection of the error, $\Pi_W u - u_h$, converges faster than the error $u - u_h$.

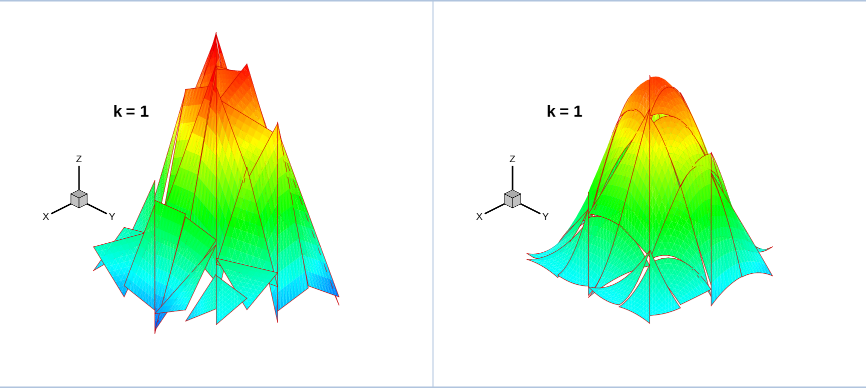
If this property holds, we introduce a new approximation $u_h^\star$. On each element K it lies in the space $W^*(K)$ and defined by

$$\begin{aligned}(\nabla {u_h}^\star, \nabla w)_K =& -(\mathrm{c}\mathbf{q}_h, \nabla w)_K \qquad \text{for all } w \in W^*(K),\\ ({u_h}^\star, 1)_K =& (u_h, 1)_K,\end{aligned}$$

If $\mathbf{q} - \mathbf{q}_h$ converges to zero fast enough, then $u - u_h^\star$ might converge as fast as $\Pi_W u - u_h$. This does happen for mixed methods!

Illustration of the postprocessing.

An HDG method for linear elasticity.

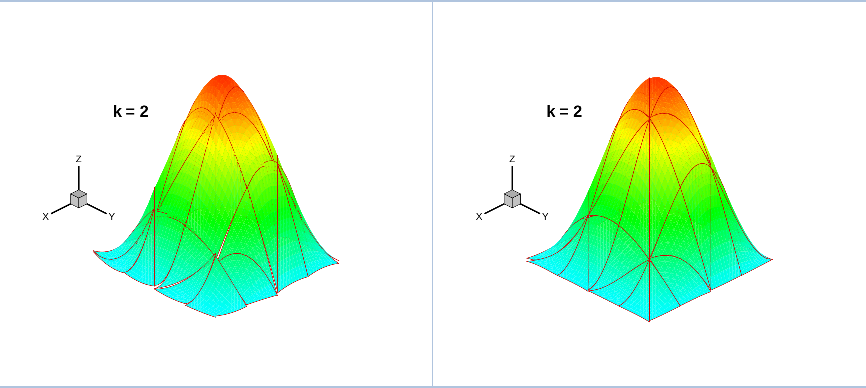


Comparison between the approximate solution (left) and the post-processed solution (right) for linear polynomial approximations.

(S.-C. Soon, B.C. and H. Stolarski, 2008.)

Illustration of the postprocessing.

An HDG method for linear elasticity.

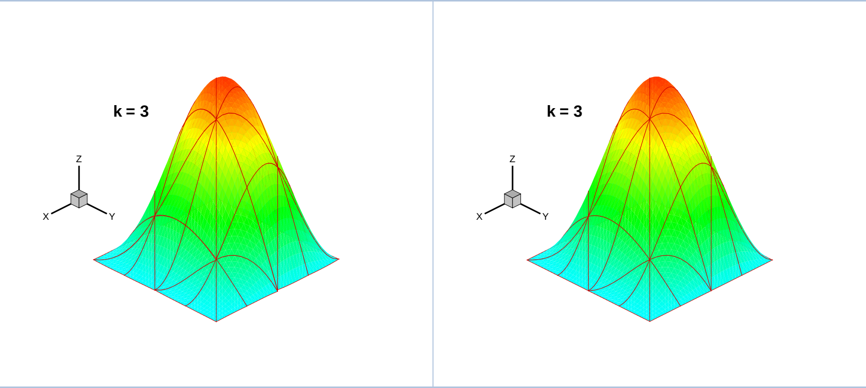


Comparison between the approximate solution (left) and the post-processed solution (right) for quadratic polynomial approximations.

(S.-C. Soon, B.C. and H. Stolarski, 2008.)

Illustration of the postprocessing.

An HDG method for linear elasticity.



Comparison between the approximate solution (left) and the post-processed solution (right) for cubic polynomial approximations.

(S.-C. Soon, B.C. and H. Stolarski, 2008.)

Some examples of superconvergent methods.

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a simplex.

method	$\mathbf{V}(K)$	$W(K)$
$\mathbf{BDFM}_{k+1}$	$\{\mathbf{q} \in \mathcal{P}_{k+1}(K) :$ $\mathbf{q} \cdot \mathbf{n}\vert_{\partial K} \in \mathcal{R}^k(\partial K)\}$	$\mathcal{P}_k(K)$
$\mathbf{RT}_k$	$\mathcal{P}_k(K) \oplus \mathbf{x}\mathcal{P}_k(K)$	$\mathcal{P}_k(K)$
$^*\mathbf{HDG}_k$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(K)$
$\mathbf{BDM}_k$ $k \geq 2$	$\mathcal{P}_k(K)$	$\mathcal{P}_{k-1}(K)$

* (B.C., B. Dong and J.Guzmán, Math. Comp., 2008.)

* (B.C., J.Gopalakrishnan and F.-J.Sayas, Math. Comp., 2010 .)

The numerical traces of the LDG method are:

$$\begin{aligned}\widehat{u}_h &= \{\!\{u_h\}\!\} + \mathbf{C}_{21}\cdot [\![u_h]\!] + C_{22}\,[\![\mathbf{q}_h]\!],\\ \widehat{\mathbf{q}}_h &= \{\!\{\mathbf{q}_h\}\!\} + \mathbf{C}_{12}\,[\![\mathbf{q}_h]\!] + C_{11}\,[\![u_h]\!],\end{aligned}$$

where $\mathbf{C}_{21} + \mathbf{C}_{12} = 0$ and $C_{22} = 0$.

The numerical traces of the LDG-H method are:

$$\begin{aligned}\widehat{u}_h &= \frac{\tau^+ {u_h}^+ + \tau^- {u_h}^-}{\tau^+ + \tau^-} + \frac{1}{\tau^+ + \tau^-}\,[\![\mathbf{q}_h]\!],\\ \widehat{\mathbf{q}}_h &= \frac{\tau^- {\mathbf{q}_h}^+ + \tau^+ {\mathbf{q}_h}^-}{\tau^+ + \tau^-} + \frac{\tau^+\tau^-}{\tau^+ + \tau^-}\,[\![u_h]\!]\end{aligned}$$

Consider DG methods on conforming meshes $\partial\Omega_h$ of simplexes K. Assume they use the local spaces $\mathbf{V}(K) := \mathcal{P}_k(K)$ and $W(K) := \mathcal{P}_k(K)$.

Theorem

For a very smooth solutions, we have, for $k \geq 1$,

$$\|\mathbf{q} - \mathbf{q}_h\|_\Omega \leq C(h^{k+1} + \|\widehat{\mathbf{q}}_h - \mathbf{q}_h\|_{\partial\Omega_h,h}),$$
$$\|u - u_h^*\|_\Omega \leq C\,h\,(h^{k+1} + \|\widehat{\mathbf{q}}_h - \mathbf{q}_h\|_{\partial\Omega_h,h}),$$

where $\|\widehat{\mathbf{q}}_h - \mathbf{q}_h\|^2_{\partial\Omega_h,h} := \sum_{K\in\Omega_h} h_K\,\|(\widehat{\mathbf{q}}_h - \mathbf{q}_h)\cdot\mathbf{n}\|^2_{\partial K}$*.Moreover,*

$$\|\widehat{\mathbf{q}}_h - \mathbf{q}_h\|_{\partial\Omega_h,h} \leq C \max_{K\in\Omega_h}\{C_{22}, 1/C_{22}, C_{11}, 1/C_{11}\}\, h^{k+1}.$$

Hence, for C_{11} and C_{22} of order one, the DG method superconverges.

Superconvergent DG methods

The effect of τ on the accuracy.

- If $\tau^{\pm}$, C_{11} are of order h^{-1} and $C_{22} = 0$, the LDG and HDG methods have the same convergence properties. The scalar variable converges with order $k+1$ but the vector variable only with order k. They do not converge for $k = 0$.
- If $\tau^{\pm}$, C_{11} and C_{22} are of order one, the DG and HDG methods have the same convergence properties. Both variables converge with order $k+1$ for $k \geq 0$. For $k \geq 1$, the local average of the scalar variable superconverges with order $k+2$.

The energy identity is

$$(\mathrm{c}\,\mathbf{q}_h, \mathbf{q}_h)_\Omega + \Theta_\tau(u_h - \widehat{u}_h) = (f, u_h)_\Omega - \langle u_D, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial\Omega}.$$

where, for the HDG,

$$\begin{aligned}
\Theta_\tau(u_h - \widehat{u}_h) &= \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial\Omega_h} \\
&= \langle \tau(u_h - P_M u_D), u_h - P_M u_D \rangle_{\partial\Omega} + \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial\Omega_h \setminus \partial\Omega} \\
&= \langle \tau(u_h - P_M u_D), u_h - P_M u_D \rangle_{\partial\Omega} \\
&\quad + \langle \frac{\tau^+ \tau^-}{\tau^+ + \tau^-} [\![u_h]\!], [\![u_h]\!] \rangle_{\mathcal{E}_h^\circ} + \langle \frac{1}{\tau^+ + \tau^-} [\![\mathbf{q}_h]\!], [\![\mathbf{q}_h]\!] \rangle_{\mathcal{E}_h^\circ}.
\end{aligned}$$

For the LDG,

$$\begin{aligned}
\Theta_\tau(u_h - \widehat{u}_h) = &\langle \tau(u_h - P_M u_D), u_h - P_M u_D \rangle_{\partial\Omega} \\
&+ \langle C_{11} [\![u_h]\!], [\![u_h]\!] \rangle_{\mathcal{E}_h^\circ} + \langle C_{22} [\![\mathbf{q}_h]\!], [\![\mathbf{q}_h]\!] \rangle_{\mathcal{E}_h^\circ}.
\end{aligned}$$

Devising superconvergent DG methods.(B.C., W.Qiu and K.Shi, Math. Comp. + SINUM, to appear.)

How to systematically devise superconvergent HDG methods?

We proceed as follows:

- Assume there is an auxiliary projection for which the error equations are in a form which guarantees superconvergence properties of the method. (The projections of the mixed methods are our role model!)
- Find out all the sufficient properties of the projection for that to happen.
- Reduce the satisfaction of those properties to a suitable choice of the local spaces.

We will then construct superconvergent HDG methods (and some new mixed methods as well).

We want to be able to write

- for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$,

$$(\mathrm{c}\,\varepsilon_{\mathbf{q}}, \mathbf{v})_K - (\varepsilon_u, \nabla \cdot \mathbf{v})_K + \langle \varepsilon_{\widehat{u}}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} = (\mathrm{c}\,(\mathbf{\Pi_V q} - \mathbf{q}), \mathbf{v})_K,$$
$$-(\varepsilon_{\mathbf{q}}, \nabla w)_K + \langle \widehat{\varepsilon_{\mathbf{q}}} \cdot \mathbf{n}, w \rangle_{\partial K} = 0,$$

- $\boldsymbol{\varepsilon}_{\widehat{\mathbf{q}}} \cdot \mathbf{n} = \widehat{\boldsymbol{\varepsilon}_{\mathbf{q}}} \cdot \mathbf{n} := \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \mathbf{n} + \tau(\varepsilon_u - \varepsilon_{\widehat{u}})$,
- for all $F \in \mathcal{E}_h$,

$$\langle \mu, [\![\widehat{\boldsymbol{\varepsilon}_{\mathbf{q}}}]\!] \rangle_F = 0, \qquad \forall \ \mu \in M(F)$$
$$\boldsymbol{\varepsilon}_{\widehat{u}} = 0 \qquad \text{if } F \in \mathcal{E}_h^{\partial}.$$

where

- $(\varepsilon_{\mathbf{q}}, \varepsilon_u) := (\mathbf{\Pi_V}(\mathbf{q} - \mathbf{q}_h), \Pi_W(u - u_h))$,
- $(\varepsilon_{\widehat{\mathbf{q}}} \cdot \mathbf{n}, \varepsilon_{\widehat{u}}) := (P_M(\mathbf{q} \cdot \mathbf{n} - \widehat{\mathbf{q}}_h \cdot \mathbf{n}), P_M(u - \widehat{u}_h))$,

Devising superconvergent methods

Energy and duality arguments.

By an energy argument, we get

$$(\mathrm{c}\varepsilon_{\mathbf{q}}, \varepsilon_{\mathbf{q}})_\Omega + \langle \tau(\varepsilon_u - \varepsilon_{\widehat{u}}), (\varepsilon_u - \varepsilon_{\widehat{u}})\rangle_{\partial\mathcal{T}_h} = (\mathrm{c}(\mathbf{\Pi_V q} - \mathbf{q}), \varepsilon_{\mathbf{q}})_\Omega,$$

By an duality argument, we get

$$\|\varepsilon_u\|_\Omega \leq C\, h\, \|\mathbf{\Pi_V q} - \mathbf{q}\,\|_\Omega.$$

Devising superconvergent methods

Assumptions A. Estimates of the error in flux and of the jumps.

- *Orthogonality properties.*

(*A.1*) $(\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}, \mathbf{v})_K = (\mathbf{q}, \mathbf{v})_K$ for all $\mathbf{v} \in \nabla W(K)$,

(*A.2*) $(\Pi_W u, w)_K = (u, w)_K$ for all $w \in \nabla \cdot \mathbf{V}(K)$,

(*A.3*) For all faces F of the element K,

$$\langle \boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q} \cdot \mathbf{n} + \tau(\Pi_W u), \mu \rangle_F = \langle \mathbf{q} \cdot \mathbf{n} + \tau(P_M u), \mu \rangle_F \quad \text{for all } \mu \in M(F).$$

- *Properties of the traces of the local spaces.*

(*A.4*) $\mathbf{V}(K) \cdot \mathbf{n}|_F \subset M(F)$,

(*A.5*) $W(K)|_F \subset M(F)$.

- *The semi-positivity property of* τ.

(A.6) $\langle \tau(\mu), \mu \rangle_F \geq 0$ for all $\mu \in M(F)$.

Theorem

Suppose that the Assumptions A are satisfied. Then we have

$$\|\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q} - \mathbf{q}_h\|_{c,\Omega} \leq \|\mathbf{q} - \boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}\|_{c,\Omega}.$$

We have

- for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$,

$$
\begin{aligned}
(\mathrm{c}\, \epsilon_{\mathbf{q}}, \mathbf{v})_K - (\epsilon_u, \nabla \cdot \mathbf{v})_K + \langle \epsilon_{\widehat{u}}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= 0, \\
-(\epsilon_{\mathbf{q}}, \nabla w)_K + \langle \boldsymbol{\epsilon}_{\widehat{\mathbf{q}}} \cdot \mathbf{n}, w \rangle_{\partial K} &= 0,
\end{aligned}
$$

- for all $F \in \mathcal{E}_h$,

$$
\begin{aligned}
\langle \mu, [\![\boldsymbol{\epsilon}_{\widehat{\mathbf{q}}}]\!] \rangle_F &= 0, \qquad && \forall \;\; \mu \in M(F) \\
\boldsymbol{\epsilon}_{\widehat{u}} &= 0 && \text{if } F \in \mathcal{E}_h^{\partial}.
\end{aligned}
$$

where

- $(\epsilon_{\mathbf{q}}, \epsilon_u) := (\mathbf{q} - \mathbf{q}_h, u - u_h)$,
- $(\epsilon_{\widehat{\mathbf{q}}} \cdot \mathbf{n}, \epsilon_{\widehat{u}}) := ((\mathbf{q} - \widehat{\mathbf{q}}_h) \cdot \mathbf{n}, u - \widehat{u}_h)$.

By (*A.1*),

- for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$,

$$
\begin{aligned}
(\mathrm{c}\, \mathbf{\Pi_V} \epsilon_{\mathbf{q}}, \mathbf{v})_K - (\epsilon_u, \nabla \cdot \mathbf{v})_K + \langle \epsilon_{\widehat{u}}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= (\mathrm{c}\, (\mathbf{\Pi_V q} - \mathbf{q}), \mathbf{v})_K, \\
-(\mathbf{\Pi_V} \epsilon_{\mathbf{q}}, \nabla w)_K + \langle \boldsymbol{\epsilon}_{\widehat{\mathbf{q}}} \cdot \mathbf{n}, w \rangle_{\partial K} &= 0,
\end{aligned}
$$

By (*A.2*),

- for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$,

$$
\begin{aligned}
(\mathrm{c}\, \mathbf{\Pi_V} \epsilon_{\mathbf{q}}, \mathbf{v})_K - (\Pi_W \epsilon_u, \nabla \cdot \mathbf{v})_K + \langle \epsilon_{\widehat{u}}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= (\mathrm{c}\, (\mathbf{\Pi_V q} - \mathbf{q}), \mathbf{v})_K, \\
-(\mathbf{\Pi_V} \epsilon_{\mathbf{q}}, \nabla w)_K + \langle \boldsymbol{\epsilon}_{\widehat{\mathbf{q}}} \cdot \mathbf{n}, w \rangle_{\partial K} &= 0,
\end{aligned}
$$

By *(A.4)*,

- for all $(\mathbf{v}, w) \in \mathbf{V}(K) \times W(K)$,

$$(\mathrm{c}\,\boldsymbol{\Pi}_{\mathbf{V}} \epsilon_{\mathbf{q}}, \mathbf{v})_K - (\Pi_W \epsilon_u, \nabla \cdot \mathbf{v})_K + \langle P_M \epsilon_{\widehat{u}}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} = (\mathrm{c}\,(\boldsymbol{\Pi}_{\mathbf{V}} \mathbf{q} - \mathbf{q}), \mathbf{v})_K,$$
$$-(\boldsymbol{\Pi}_{\mathbf{V}} \epsilon_{\mathbf{q}}, \nabla w)_K + \langle P_M \boldsymbol{\epsilon}_{\widehat{\mathbf{q}}} \cdot \mathbf{n}, w \rangle_{\partial K} = 0,$$

- for all $F \in \mathcal{E}_h$,

$$\langle \mu, [\![P_M \boldsymbol{\epsilon}_{\widehat{\mathbf{q}}}]\!] \rangle_F = 0, \qquad \forall \;\mu \in M(F)$$

But

- $\boldsymbol{\Pi}_{\mathbf{V}} \boldsymbol{\epsilon}_{\mathbf{q}} = \boldsymbol{\varepsilon}_{\mathbf{q}}$,
- $\Pi_W \epsilon_u = \varepsilon_u$,
- $P_M \boldsymbol{\epsilon}_{\widehat{\mathbf{q}}} = \boldsymbol{\varepsilon}_{\widehat{\mathbf{q}}}$,
- $P_M \epsilon_{\widehat{u}} = \varepsilon_{\widehat{u}}$,
- By (it A.3),

$$\begin{aligned} \boldsymbol{\varepsilon}_{\widehat{\mathbf{q}}} &= P_M(\boldsymbol{\epsilon}_{\mathbf{q}}) + \tau\, P_M(\epsilon_u - \epsilon_{\widehat{u}}) \\ &= \boldsymbol{\varepsilon}_{\mathbf{q}} + \tau\,(\varepsilon_u - \varepsilon_{\widehat{u}}) \\ &= \widehat{\boldsymbol{\varepsilon}_{\mathbf{q}}}. \end{aligned}$$

For $(\mathbf{v}, w) := (\varepsilon_{\mathbf{q}}, \varepsilon_u)$, we get

$$(\mathrm{c}\,\varepsilon_{\mathbf{q}}, \varepsilon_{\mathbf{q}})_K - (\varepsilon_u, \nabla \cdot \varepsilon_{\mathbf{q}})_K + \langle \varepsilon_{\widehat{u}}, \varepsilon_{\mathbf{q}} \cdot \mathbf{n} \rangle_{\partial K} = (\mathrm{c}\,(\mathbf{\Pi_V q} - \mathbf{q}), \varepsilon_{\mathbf{q}})_K,$$
$$-(\varepsilon_{\mathbf{q}}, \nabla \varepsilon_u)_K + \langle \widehat{\varepsilon}_{\mathbf{q}} \cdot \mathbf{n}, \varepsilon_u \rangle_{\partial K} = 0,$$

Adding the equations, and adding on the elements, we get

$$(\mathrm{c}\,\varepsilon_{\mathbf{q}}, \varepsilon_{\mathbf{q}})_{\Omega_h} + \langle (\widehat{\varepsilon}_{\mathbf{q}} - \varepsilon_{\mathbf{q}}) \cdot \mathbf{n}, \varepsilon_u - \varepsilon_{\widehat{u}} \rangle_{\partial\Omega_h} =$$
$$(\mathrm{c}\,\varepsilon_{\mathbf{q}}, \varepsilon_{\mathbf{q}})_{\Omega_h} + \langle \tau\,(\widehat{\varepsilon}_u - \varepsilon_u), \varepsilon_u - \varepsilon_{\widehat{u}} \rangle_{\partial\Omega_h} = (\mathrm{c}\,(\mathbf{\Pi_V q} - \mathbf{q}), \varepsilon_{\mathbf{q}})_{\Omega_h}.$$

Finally, by *(A.6)*,

$$(\mathrm{c}\,\varepsilon_{\mathbf{q}}, \varepsilon_{\mathbf{q}})_{\Omega_h} \le (\mathrm{c}\,(\mathbf{\Pi_V q} - \mathbf{q}), \varepsilon_{\mathbf{q}})_{\Omega_h}.$$

Devising superconvergent methods

Assumptions B. Estimate of the projection of the error in the scalar variable.

- *The approximation property*

(B.1) $\|\mathbf{\Pi_V q} - \mathbf{q}\|_K \leq C_{app}\, h_K(|u|_{1,K} + |\mathbf{q}|_{1,K})$.

- *The local space $W(K)$ is not small.*

(B.2) $\mathbf{P}^0(K) \subset \nabla W(K)$.

Theorem

Suppose that the Assumptions A and B are satisfied. Also, suppose that the classic elliptic regularity property holds. Then we have

$$\|\Pi_W u - u_h\|_\Omega \leq C\, h \|\mathbf{q} - \mathbf{\Pi_V q}\|_\Omega,$$

for some constant C depending on C_{app} but independent of h and the exact solution.

We begin by introducing the dual problem for any given θ in $L^2(\Omega)$:

$$
\begin{aligned}
\mathrm{c}\,\boldsymbol{\phi} - \nabla\psi &= 0 && \text{on } \Omega,\\
\nabla\cdot\boldsymbol{\phi} &= \theta && \text{on } \Omega,\\
\psi &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

We assume that this boundary value problem admits the regularity estimate

$$\|\boldsymbol{\phi}\|_{H^1(\Omega)} + \|\psi\|_{H^2(\Omega)} \leq C\,\|\theta\|_\Omega$$

for all θ in $L^2(\Omega)$. This is well known to hold if Ω is a convex polygon.

Lemma

Suppose Assumptions A hold. Then, for any $\psi_h \in W_h$, we have

$$(\varepsilon_u, \theta)_{\Omega_h} = (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \boldsymbol{\Pi}_{\mathbf{V}}\phi - \phi)_{\Omega_h} + (\mathbf{q} - \boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}, \nabla\psi - \nabla\psi_h)_{\Omega_h}.$$

Consequently,

$$\|\,\varepsilon_u\,\|_{\Omega_h} \leq C\,H(\theta)\,\|\,\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q} - \mathbf{q}\,\|_{\Omega_h},$$

where

$$H(\theta) := \sup_{\theta \in L^2(\Omega)\setminus\{0\}} \frac{\|\boldsymbol{\Pi}_{\mathbf{V}}\phi - \phi\|_{\Omega_h}}{\|\theta\|_{\Omega_h}} + \sup_{\theta \in L^2(\Omega)\setminus\{0\}} \inf_{\psi_h \in W_h} \frac{\|\nabla\psi - \nabla\psi_h\,\|_{\Omega_h}}{\|\theta\|_{\Omega_h}}.$$

By (B.1) and (B.2), we have that $H(\theta) \leq C\,h$. Without Assumptions B, we can still have $H(\theta) \leq C$.

Proof.

We have

$$\begin{aligned}
(\varepsilon_u, \theta)_{\Omega_h} &= (\varepsilon_u, \nabla \cdot \boldsymbol{\phi})_{\Omega_h} \\
&= -(\nabla \varepsilon_u, \boldsymbol{\phi})_{\Omega_h} + \langle \varepsilon_u, \mathbf{n} \cdot \boldsymbol{\phi} \rangle_{\partial\Omega_h} \\
&= -(\nabla \varepsilon_u, \boldsymbol{\Pi}_{\mathbf{V}} \boldsymbol{\phi})_{\Omega_h} + \langle \varepsilon_u, \mathbf{n} \cdot \boldsymbol{\phi} \rangle_{\partial\Omega_h} && \text{by (A.1)}, \\
&= (\varepsilon_u, \nabla \cdot \boldsymbol{\Pi}_{\mathbf{V}} \boldsymbol{\phi})_{\Omega_h} + \langle \varepsilon_u, \mathbf{n} \cdot (\boldsymbol{\phi} - \boldsymbol{\Pi}_{\mathbf{V}} \boldsymbol{\phi}) \rangle_{\partial\Omega_h} \\
&= (\varepsilon_u, \nabla \cdot \boldsymbol{\Pi}_{\mathbf{V}} \boldsymbol{\phi})_{\Omega_h} + \langle \varepsilon_u, \tau(\Pi_W \psi - \psi) \rangle_{\partial\Omega_h},
\end{aligned}$$

by the orthogonality property (A.3), and the inclusions (A.4) and(A.5).

By the first equation defining the local solver with $\mathbf{v} := \mathbf{\Pi_V \Phi}$,

$$
\begin{aligned}
(\varepsilon_u, \theta)_{\Omega_h} = &\, (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi_V}\phi)_{\Omega_h} + \langle \varepsilon_{\widehat{u}}, \mathbf{\Pi_V}\phi \cdot \mathbf{n}\rangle_{\partial\Omega_h} \\
&+ \langle \varepsilon_u, \tau(\Pi_W \psi - \psi)\rangle_{\partial\Omega_h}, \\
= &\, (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi_V}\phi)_{\Omega_h} + \langle \varepsilon_{\widehat{u}}, (\mathbf{\Pi_V}\phi - \phi) \cdot \mathbf{n}\rangle_{\partial\Omega_h} \\
&+ \langle \varepsilon_u, \tau(\Pi_W \psi - \psi)\rangle_{\partial\Omega_h},
\end{aligned}
$$

by the continuity of $\phi \cdot \mathbf{n}$ and the fact that $\varepsilon_{\widehat{u}} = 0$ on $\partial\Omega$. Then, by $(A.3)$, we get that

$$
(\varepsilon_u, \theta)_{\Omega_h} = (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi_V}\phi)_{\Omega_h} + \langle \varepsilon_u - \varepsilon_{\widehat{u}}, \tau(\Pi_W \psi - \psi)\rangle_{\partial\Omega_h}.
$$

Then

$$\begin{aligned}(\varepsilon_u, \theta)_{\Omega_h} &= (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \boldsymbol{\Pi}_{\mathbf{V}}\phi)_{\Omega_h} + \langle(\boldsymbol{\varepsilon}_{\widehat{\mathbf{q}}} - \varepsilon_{\mathbf{q}}) \cdot \mathbf{n}, \Pi_W\psi - \psi\rangle_{\partial\Omega_h} \\ &= (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \boldsymbol{\Pi}_{\mathbf{V}}\phi)_{\Omega_h} + \langle(\boldsymbol{\varepsilon}_{\widehat{\mathbf{q}}} - \varepsilon_{\mathbf{q}}) \cdot \mathbf{n}, \Pi_W\psi\rangle_{\partial\Omega_h} \\ &\quad - \langle\varepsilon_{\mathbf{q}} \cdot \mathbf{n}, \psi\rangle_{\partial\Omega_h}\end{aligned}$$

and, by the second equation defining the local solvers with $w := \Pi_W\psi$, we get that

$$(\varepsilon_u, \theta)_{\Omega_h} = (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \boldsymbol{\Pi}_{\mathbf{V}}\phi)_{\Omega_h} + (\nabla \cdot \varepsilon_{\mathbf{q}}, \Pi_W\psi)_{\Omega_h} - \langle\varepsilon_{\mathbf{q}} \cdot \mathbf{n}, \psi\rangle_{\partial\Omega_h}.$$

Moreover, by $(A.2)$,

$$
\begin{aligned}
(\varepsilon_u, \theta)_{\Omega_h} &= (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi_V}\phi)_{\Omega_h} + (\nabla \cdot \varepsilon_{\mathbf{q}}, \psi)_{\Omega_h} - \langle \varepsilon_{\mathbf{q}} \cdot \mathbf{n}, \psi \rangle_{\partial\Omega_h} \\
&= (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi_V}\phi)_{\Omega_h} - (\varepsilon_{\mathbf{q}}, \nabla\psi)_{\Omega_h} \\
&= (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi_V}\phi - \phi)_{\Omega_h} + (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \phi)_{\Omega_h} \\
&\quad - (\mathbf{\Pi_V}\mathbf{q} - \mathbf{q}_h, \nabla\psi)_{\Omega_h} \\
&= (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi_V}\phi - \phi)_{\Omega_h} + (\mathbf{q} - \mathbf{\Pi_V}\mathbf{q}, \nabla\psi)_{\Omega_h} \\
&= (\mathrm{c}\,(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi_V}\phi - \phi)_{\Omega_h} + (\mathbf{q} - \mathbf{\Pi_V}\mathbf{q}, \nabla\psi - \nabla\psi_h)_{\Omega_h},
\end{aligned}
$$

by $(A.1)$.

Assumption C:

- *The local space* $\mathbf{V}(K)$ *is not small.*

$(C.1)$ $P^0(K) \subset \nabla \cdot \mathbf{V}(K)$.

Theorem

Suppose that the Assumption C is satisfied. Then, we have

$$\|u - u_h^*\|_\Omega \le \|\Pi_W(u - u_h)\|_\Omega + C\,h\,(\|\mathbf{q}_h - \mathbf{q}\|_\Omega + \inf_{w \in W_h^*} \|\nabla(u - w)\|_\Omega).$$

Devising superconvergent methods.

Proof.

We have

$$\begin{aligned}\|u - u_h^*\|_K &\le \|\overline{u} - \overline{u_h^*}\|_K + C\,h\,\|\nabla(u - u_h^*)\|_K \\ &= \|\overline{u} - \overline{u_h}\|_K + C\,h\,\|\nabla(u - u_h^*)\|_K \\ &\le \|\Pi_W(u - u_h)\|_K + C\,h\,\|\nabla(u - u_h^*)\|_K,\end{aligned}$$

by $(C.1)$. Moreover

$$\begin{aligned}\|\nabla(u - u_h^*)\|_K^2 &= (\nabla(u - w), \nabla(u - u_h^*))_K - (\nabla(u_h^* - w), \nabla(u - u_h^*))_K \\ &= (\nabla(u - w), \nabla(u - u_h^*))_K - (\nabla(u_h^* - w), \nabla u + \mathrm{c}\mathbf{q}_h)_K \\ &= (\nabla(u - w), \nabla(u - u_h^*))_K - (\nabla(u_h^* - w), \mathrm{c}(\mathbf{q}_h - \mathbf{q}))_K\end{aligned}$$

The estimate follows after applying the Cauchy-Schwarz inequality.

Suppose that the local spaces satisfy

$$\nabla W(K) \subset \widetilde{\mathbf{V}}(K),$$
$$\nabla \cdot \mathbf{V}(K) \subset \widetilde{W}(K),$$
$$\mathbf{V}(K) \cdot \mathbf{n} + W(K) \subset \mathcal{R}(\partial K),$$
$$\mathbf{V}^{\perp}(K) \cdot \mathbf{n} \oplus W^{\perp}(K) = \mathcal{R}(\partial K),$$

where $\mathcal{R}(\partial K) = \{\mu \in L^2(\partial K) : \mu|_F \in M(F) \ \ \forall \mathcal{F}(K)\}$ and

$$\mathbf{V}(K) = \widetilde{\mathbf{V}}(K) \oplus \mathbf{V}^{\perp}(K),$$
$$W(K) = \widetilde{W}(K) \oplus W^{\perp}(K).$$

Then, the function $\Pi_h(\mathbf{q}, u) := (\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}, \Pi_W u)$ is the element of $\mathbf{V}(K) \times W(K)$ satisfying the equations

$$\begin{aligned}
(\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}, \widetilde{\mathbf{v}})_K &= (\mathbf{q}, \widetilde{\mathbf{v}})_K && \forall\, \widetilde{\mathbf{v}} \in \widetilde{\mathbf{V}}(K),\\
(\Pi_W u, \widetilde{w})_K &= (u, \widetilde{w})_K && \forall\, \widetilde{w} \in \widetilde{W}(K),\\
\langle \boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q} \cdot \mathbf{n} + \tau(\Pi_W u), \mu \rangle_F &= \langle \mathbf{q} \cdot \mathbf{n} + \tau(P_M u), \mu \rangle_F && \forall\, \mu \in M(F),
\end{aligned}$$

for all faces F of the element K, is well defined and satisfies the assumptions.

Construction of a superconvergent HDG methods

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a simplex.

method	$\mathbf{V}(K)$	$W(K)$	$\widetilde{\mathbf{V}}(K)$	$\widetilde{W}(K)$
$\mathbf{BDFM}_{k+1}$	$\{\mathbf{q} \in \mathbf{P}^{k+1}(K) :$ $\mathbf{q} \cdot \mathbf{n}\|_{\partial K} \in \mathcal{R}^k(\partial K)\}$	$P^k(K)$	$\nabla P^k(K) \oplus \mathbf{\Phi}_{k+1}(K)$	$P^k(K)$
$\mathbf{RT}_k$	$\mathbf{P}^k(K) \oplus \mathbf{x}\widetilde{P}^k(K)$	$P^k(K)$	$\mathbf{P}^{k-1}(K)$	$P^k(K)$
$\mathbf{HDG}_k$	$\mathbf{P}^k(K)$	$P^k(K)$	$\mathbf{P}^{k-1}(K)$	$P^{k-1}(K)$
$\underset{k\geq 2}{\mathbf{BDM}_k}$	$\mathbf{P}^k(K)$	$P^{k-1}(K)$	$\nabla P^{k-1}(K) \oplus \mathbf{\Phi}_k(K)$	$P^{k-1}(K)$

The auxiliary projection.

The HDG method for which $M(F) = P^k(F), k \geq 1$, and K is a simplex.

The function $\Pi_h(\mathbf{q}, u) := (\mathbf{\Pi_V q}, \Pi_W u)$ is the element of $\boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ satisfying the equations

$$
\begin{aligned}
(\mathbf{\Pi_V q}, \mathbf{v})_K &= (\mathbf{q}, \mathbf{v})_K && \forall\, \mathbf{v} \in \boldsymbol{\mathcal{P}}_{k-1}(K),\\
(\Pi_W u, w)_K &= (u, w)_K && \forall\, w \in \mathcal{P}_{k-1}(K),\\
\langle \mathbf{\Pi_V q} \cdot \mathbf{n} + \tau(\Pi_W u), \mu \rangle_F &= \langle \mathbf{q} \cdot \mathbf{n} + \tau(u), \mu \rangle_F && \forall\, \mu \in \mathcal{P}_k(F),
\end{aligned}
$$

for all faces F of the element K.

The associated projection.

The HDG method for which $M(F) = P^k(F), k \geq 1$, and K is a simplex.

Theorem

Suppose $k \geq 0$, $\tau|_{\partial K}$ *is nonnegative and* $\tau_K^{\max} := \max \tau|_{\partial K} > 0$. *Then* $\mathbf{\Pi_V q}$ *and* $\Pi_W u$ *are well defined. Furthermore, there is a constant* C *independent of* K *and* τ *such that*

$$\|\mathbf{\Pi_V q} - \mathbf{q}\|_K \leq C\, h_K^{\ell_{\mathbf{q}}+1}\, |\mathbf{q}|_{\mathbf{H}^{\ell_{\mathbf{q}}+1}(K)} + C\, h_K^{\ell_u+1}\, \tau_K^*\, |u|_{H^{\ell_u+1}(K)},$$

$$\|\Pi_W u - u\|_K \leq C\, h_K^{\ell_u+1}\, |u|_{H^{\ell_u+1}(K)} + C\, \frac{h_K^{\ell_{\mathbf{q}}+1}}{\tau_K^{\max}}\, |\nabla \cdot \mathbf{q}|_{H^{\ell_{\mathbf{q}}}(K)},$$

for $\ell_u, \ell_{\mathbf{q}}$ *in* $[0, k]$. *Here* $\tau_K^* := \max \tau|_{\partial K \setminus F^*}$, *where* F^* *is a face of* K *at which* $\tau|_{\partial K}$ *is maximum.*

The associated projection.

Sketch of the proof.

The function $\Pi_W u$ is the element of $\mathcal{P}_k(K)$ satisfying the equations

$$\begin{aligned}
(\Pi_W u, w)_K &= (u, w)_K && \forall\, w \in \mathcal{P}_{k-1}(K),\\
\langle \tau(\Pi_W u), w\rangle_{\partial K} &= \langle \tau(u) + (\mathbf{q} - \boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}) \cdot \mathbf{n}, w\rangle_{\partial K} && \forall\, w \in \mathcal{P}_k(K)^{\perp}.
\end{aligned}$$

Note that

$$\begin{aligned}
\langle (\mathbf{q} - \boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}) \cdot \mathbf{n}, w\rangle_{\partial K} &= (\mathbf{q} - \boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}, \nabla w)_K + (\nabla \cdot (\mathbf{q} - \boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}), w)_K\\
&= (\nabla \cdot \mathbf{q}, w)_K\\
&= ((\mathrm{I} - \mathrm{P}_{k-1})(\nabla \cdot \mathbf{q}), w)_K.
\end{aligned}$$

The function $\mathbf{\Pi_V q}$ is the element of $\mathcal{P}_k(K)$ satisfying the equations

$$\begin{aligned} (\mathbf{\Pi_V q}, \mathbf{v})_K &= (\mathbf{q}, \mathbf{v})_K && \forall\, \mathbf{v} \in \mathcal{P}_{k-1}(K), \\ \langle \mathbf{\Pi_V q} \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K \setminus F^*} &= \langle \mathbf{q} \cdot \mathbf{n} + \tau(u - \Pi_W u), \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K \setminus F^*} && \forall\, \mathbf{v} \in \mathcal{P}_k(K)^{\perp}, \end{aligned}$$

Note that, when τ is constant on ∂K, we can write

$$\begin{aligned} \langle \tau(u - \Pi_W u), \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} &= \tau\, (u - \Pi_W u, \nabla \cdot \mathbf{v})_K + \tau\, (\nabla(u - \Pi_W u), \mathbf{v})_K \\ &= \tau\, ((\mathrm{I} - \mathrm{P}_{k-1})(\nabla u), \mathbf{v})_K. \end{aligned}$$

Examples of superconvergent methods. (B.C., W.Qiu and K.Shi, Math. Comp. + SINUM, to appear.)

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a simplex.

method	τ	$\|\mathbf{q}-\mathbf{q}_h\|_\Omega$	$\|\Pi_W u - u_h\|_\Omega$	$\|u-u_h^\star\|_\Omega$
$\mathbf{BDFM}_{k+1}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{RT}_k$	0	$k+1$	$k+2$	$k+2$
$\mathbf{HDG}_k$	$\mathcal{O}(1), > 0$	$k+1$	$k+2$	$k+2$
$\mathbf{BDM}_k$ $k\geq 2$	0	$k+1$	$k+2$	$k+2$

Examples of superconvergent methods

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a square.

method	$\mathbf{V}(K)$	$W(K)$
$\mathbf{BDFM}_{[k+1]}$	$P^{k+1}(K)\backslash\{y^{k+1}\}$ $\times(P^{k+1}(K)\backslash\{x^{k+1}\})$	$P^k(K)$
$\mathbf{HDG}^P_{[k]}$	$\mathbf{P}^k(K)$ $\oplus\nabla\times(xy\widetilde{P}^k(K))$	$P^k(K)$
$\mathbf{BDM}_{[k]}$ $k \geq 2$	$\mathbf{P}^k(K)$ $\oplus\nabla\times(xy\,x^k)$ $\oplus\nabla\times(xy\,y^k)$	$P^{k-1}(K)$

Examples of superconvergent methods

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a cube.

method	$\mathbf{V}(K)$	$W(K)$
$\mathbf{BDFM}_{[k+1]}$	$P^{k+1}(K)\backslash\widetilde{P}^{k+1}(y,z)$ $\times P^{k+1}(K)\backslash\widetilde{P}^{k+1}(x,z)$ $\times P^{k+1}(K)\backslash\widetilde{P}^{k+1}(x,y)$	$P^k(K)$
$\mathbf{HDG}^{P}_{[k]}$	$\mathbf{P}^k(K)$ $\oplus\nabla\times(yz\widetilde{P}^k(K),0,0)$ $\oplus\nabla\times(0,zx\widetilde{P}^k(K),0)$	$P^k(K)$
$\mathbf{BDM}_{[k]}$ $k \geq 2$	$\mathbf{P}^k(K)$ $\oplus\nabla\times(0,0,xy\widetilde{P}^k(y,z))$ $\oplus\nabla\times(0,xz\widetilde{P}^k(x,y),0)$ $\oplus\nabla\times(yz\widetilde{P}^k(x,z),0,0)$	$P^{k-1}(K)$

Examples of superconvergent methods

Methods for which $M(F) = P^k(F), k \geq 1$, and K is a square or a cube.

method	τ	$\lVert \mathbf{q} - \mathbf{q}_h \rVert_\Omega$	$\lVert \Pi_W u - u_h \rVert_\Omega$	$\lVert u - u_h^\star \rVert_\Omega$
$\mathbf{BDFM}_{[k+1]}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{HDG}^P_{[k]}$	$\mathcal{O}(1), > 0$	$k+1$	$k+2$	$k+2$
$\mathbf{BDM}_{[k]}$ $k \geq 2$	0	$k+1$	$k+2$	$k+2$

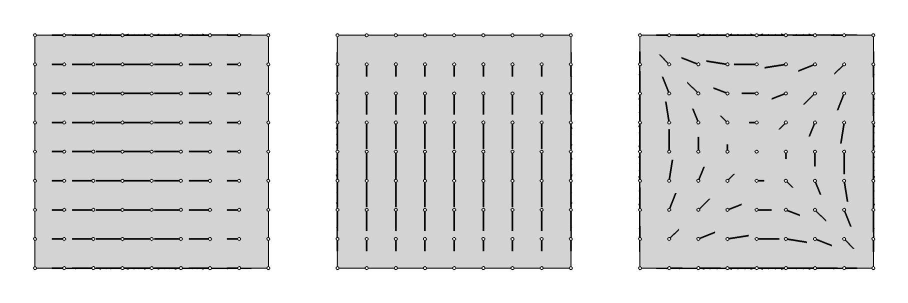
Examples of superconvergent methods

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a square.

method	$\mathbf{V}(K)$	$W(K)$
$\mathbf{RT}_{[k]}$	$P^{k+1,k}(K) \times P^{k,k+1}(K)$	$Q^k(K)$
$\mathbf{TNT}_{[k]}$	$\mathbf{Q}^k(K) \oplus \mathbf{H}_3^k(K)$	$Q^k(K)$
$\mathbf{HDG}_{[k]}^{Q}$	$\mathbf{Q}^k(K) \oplus \mathbf{H}_2^k(K)$	$Q^k(K)$

Examples of superconvergent methods

The space $\mathbf{H}_3^k(K)$.



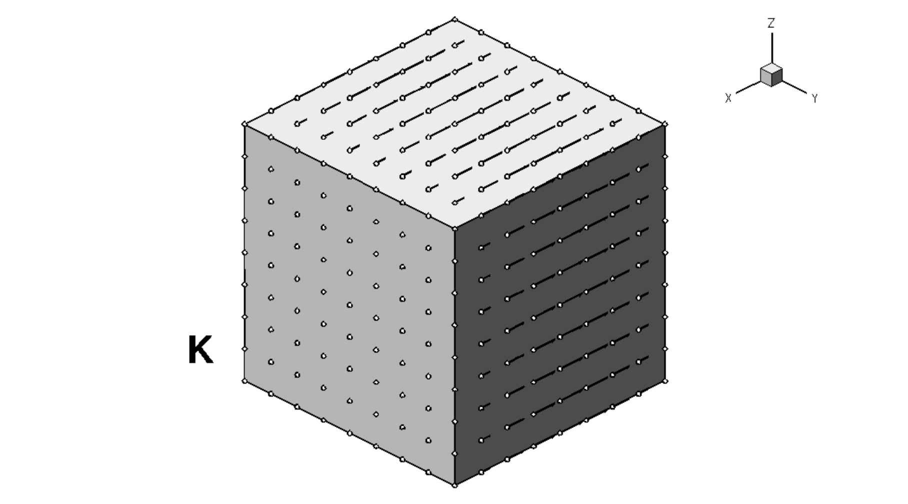
Examples of superconvergent methods

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a cube.

method	$\mathbf{V}(K)$	$W(K)$
$\mathbf{RT}_{[k]}$	$P^{k+1,k,k}(K) \times P^{k,k+1,k}(K) \times P^{k,k,k+1}(K)$	$Q^k(K)$
$\mathbf{TNT}_{[k]}$	$\mathbf{Q}^k(K) \oplus \mathbf{H}_7^k(K)$	$Q^k(K)$
$\mathbf{HDG}_{[k]}^{Q}$	$\mathbf{Q}^k(K) \oplus \mathbf{H}_6^k(K)$	$Q^k(K)$

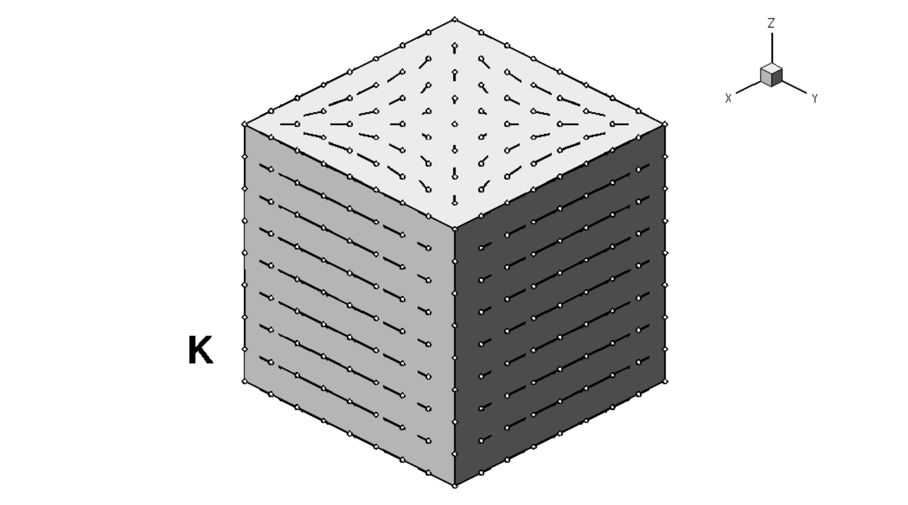
Examples of superconvergent methods

The space $\mathbf{H}_7^k(K)$.



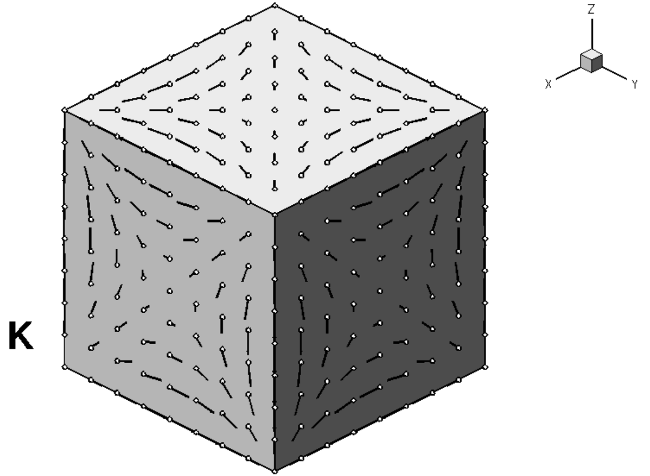
Examples of superconvergent methods

The space $\mathbf{H}_7^k(K)$.



Examples of superconvergent methods

The space $\mathbf{H}_7^k(K)$.



Examples of superconvergent methods

Methods for which $M(F) = Q^k(F), k \geq 1$, and K is a square or a cube.

method	τ	$\|\mathbf{q} - \mathbf{q}_h\|_\Omega$	$\|\Pi_W u - u_h\|_\Omega$	$\|u - u_h^\star\|_\Omega$
$\mathbf{RT}_{[k+1]}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{TNT}_{[k]}$	0	$k+1$	$k+2$	$k+2$
$\mathbf{HDG}^{Q}_{[k]}$	$\mathcal{O}(1) > 0$	$k+1$	$k+2$	$k+2$

Examples of superconvergent methods

Questions.

- Do we retain superconverence for general non-confoming meshes?
- If not, are there nonconforming meshes for which superconvergence holds?
- Do we still have superconvergence for variable-degree HDG methods?

Variable-degree HDG methods on nonconforming meshes. (Y.Chen and B.C., IMA + Math. Comp., to appear.)

Definition.

$$\begin{aligned}
\mathbf{V}_h &= \{\mathbf{r} \in \mathbf{L}^2(\mathcal{T}_h) : & \mathbf{r}|_K \in \mathbf{P}_{k(K)}(K) & \quad \forall\, K \in \mathcal{T}_h\},\\
W_h &= \{w \in L^2(\mathcal{T}_h) : & w|_K \in P_{k(K)}(K) & \quad \forall\, K \in \mathcal{T}_h\},\\
M_h &= \{\mu \in L^2(\mathcal{E}_h) : & \mu|_F \in P_{k(F)}(F) & \quad \forall\, F \in \mathcal{E}_h\}.
\end{aligned}$$

and

$$\begin{aligned}
k(F) &= k(K) & \text{if } F = \partial K \cap \partial\Omega,\\
k(F) &= \max\{k(K^+), k(K^-)\} & \text{if } F = \partial K^+ \cap \partial K^-.
\end{aligned}$$

Variable-degree HDG methods on nonconforming meshes

Overview of convergence properties

method	conformity of the meshes $\mathcal{T}_h$	order (flux)	order (scalar)
DG pure diffusion	conforming	k	$k+1$
LDG pure diffusion	conforming Cartesian meshes	$k+1/2$	$k+1$
LDG pure diffusion	nonconforming	k	$k+1$
HDG pure diffusion	conforming	$k+1$	$k+1+\min\{k,1\}$ projection of the scalar variable
HDG	nonconforming	$k+1/2$	$k+1$
HDG	nonconforming semimatching	$k+1$	$k+1+\min\{k,1\}$ projection of the scalar variable

Variable-degree HDG methods on nonconforming meshes

General meshes

Theorem

For any mesh of shape-regular simplexes, we have

$$\|\varepsilon_{\mathbf{q}}\|_c \leq \|\mathbf{\Pi_V q} - \mathbf{q}\|_c + C\,\|(P_M - P_{\mathcal{M}})(\mathbf{q}\cdot\mathbf{n} + \tau\, u)\|_{\partial\Omega_h},$$

Moreover,

$$\|\varepsilon_u\| \leq C\, h^{1/2}\left(\|\mathbf{\Pi_V q} - \mathbf{q}\| + \|(P_M - P_{\mathcal{M}})(\mathbf{q}\cdot\mathbf{n} + \tau\, u)\|_{\partial\Omega_h}\right).$$

$$\|(P_M - P_{\mathcal{M}})(\mathbf{q}\cdot\mathbf{n} + \tau\, u)\|_{\partial\Omega_h} \leq C\,|S_{P,h}|^{1/2}\, h_P^{k+1}\, D(\mathbf{q}, u),$$
$$D(\mathbf{q}, u) := |\mathbf{q}\cdot\mathbf{n} + \tau\, u|_{W^{k+1,\infty}(S_{P,h})},$$
$$S_{P,h} := \{F :\ P_M \neq P_{\mathcal{M}} \text{ on } F\}.$$

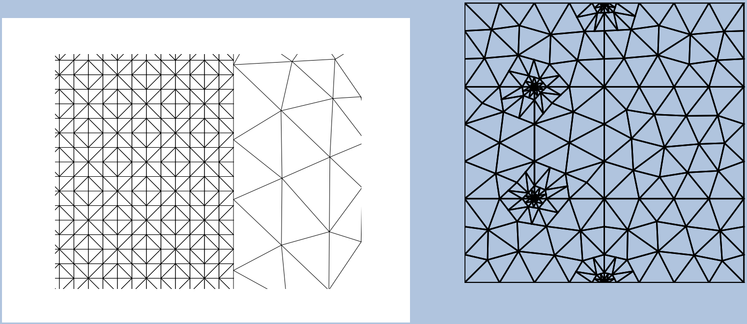


Figure: Examples of sets $S_{P,h}$ of size of order one.

Variable-degree HDG methods on nonconforming meshes

General meshes

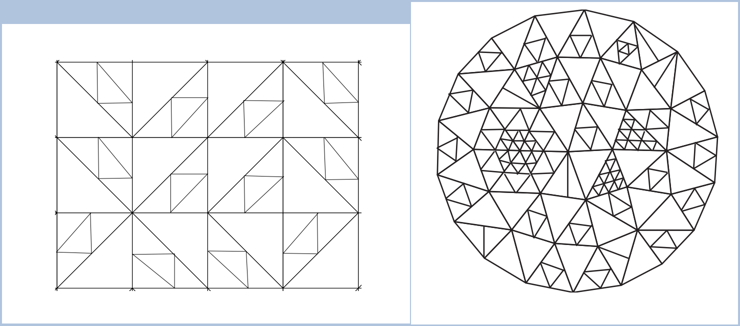


Figure: Examples of sets $S_{P,h}$ of size of order h^{-1}.

Variable-degree HDG methods on nonconforming meshes

The semimatching nonconforming meshes.

For every level index $\ell \geq 1$,

- Shape regularity:

$$\mathsf{T}_h^\ell \text{ is made of simplexes } \mathsf{K} \text{ such that } \frac{h_\mathsf{K}}{\rho_\mathsf{K}} \leq \sigma.$$

- Mandatory refinement:

$$\mathsf{T}_h^{\ell+1} \text{ is a refinement of } \mathsf{T}_h^\ell : \text{ no element of } \mathsf{T}_h^\ell \text{ is unrefined.}$$

- Local Uniformity:

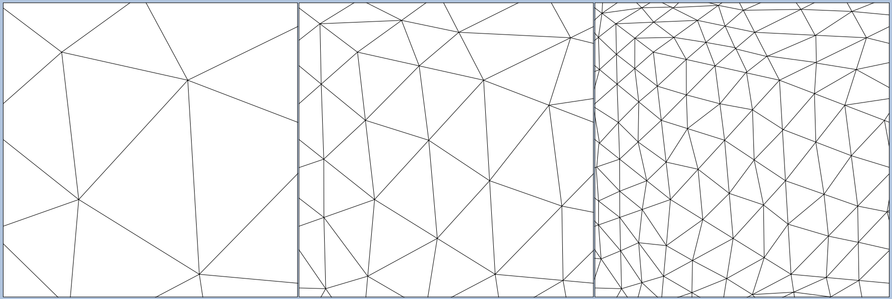
$$\forall\, \mathsf{K} \in \mathsf{T}_h^\ell : \max_{\mathsf{K}' \in \mathsf{T}_h^{\ell+1} : \mathsf{K}' \subset \mathsf{K}} h_{\mathsf{K}'} \leq \kappa \min_{\mathsf{K}' \in \mathsf{T}_h^{\ell+1} : \mathsf{K}' \subset \mathsf{K}} h_{\mathsf{K}'}.$$

- Uniform refinement:

$$\forall\, \mathsf{K} \in \mathsf{T}_h^\ell : \max_{\mathsf{K}' \in \mathsf{T}_h^{\ell+n} : \mathsf{K}' \subset \mathsf{K}} h_{\mathsf{K}'} \leq \mathsf{c}\, \eta^n\, h_\mathsf{K}.$$

Variable-degree HDG methods on nonconforming meshes

The semimatching nonconforming meshes



T_h^1 T_h^2 T_h^3

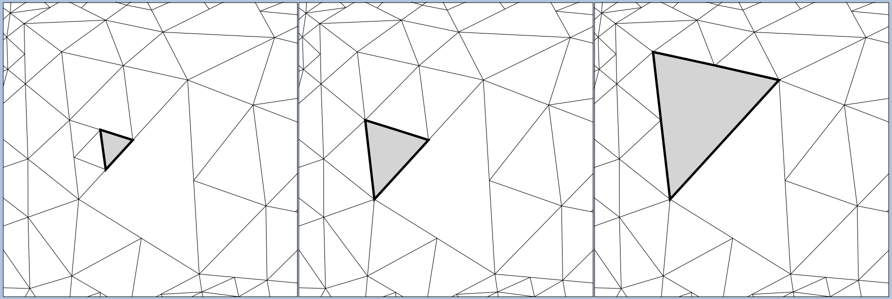
Figure: An example of a family of triangulations $\{\mathrm{T}_h^\ell\}_{\ell\geq 1}$ for which $\eta = 1/2$.

$\mathcal{T}_h = \{K\}$ is a semimatching nonconforming mesh if, for each element $K \in \mathcal{T}_h$ there is a set $\{\mathsf{K}_K^\ell\}_{\ell=1}^{\ell_K}$ such that:

- $\mathsf{K}_K^\ell \in \mathsf{T}_h^\ell,$ for $\ell = 1, \cdots, \ell_K$.
- $\mathsf{K}_K^\ell \supset K,$ for $\ell = 1, \cdots, \ell_K$.
- $\mathsf{K}_K^{\ell_K} = K$.

Variable-degree HDG methods on nonconforming meshes

The semimatching meshes.



$K = \mathsf{K}_K^3 \in \mathsf{T}_h^3, \ell_K = 3$ $\qquad$ $\mathsf{K}_K^2 \in \mathsf{T}_h^2$ $\qquad$ $\mathsf{K}_K^1 \in \mathcal{T}_h^1$

Figure: A nonconforming mesh $\mathcal{T}_h$ (left) and of the set $\{\mathsf{K}_K^\ell\}_{\ell=1}^{\ell_K}$ (in gray).

Variable-degree HDG methods on nonconforming meshes

The condition on the degree.

We *further* require that

$$k(K^+) \geq k(K^-) \text{ whenever } \ell_{K^+} \geq \ell_{K^-}.$$

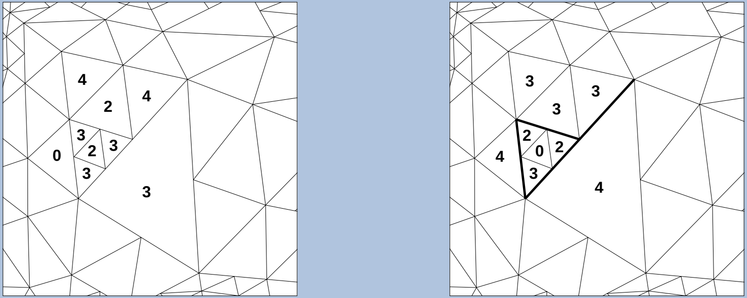


Figure: Illustration of the last condition: Yes (left), no (right).

Theorem

For any semimatching mesh, we have

$$\|\varepsilon_{\mathbf{q}}\| \leq C\,(\|\mathbf{q} - \mathbf{\Pi_V q}\| + \|(P_M - P_{\mathcal{M}})(\mathbf{q}\cdot\mathbf{n} + \tau\, u)\|_{\partial\Omega_h,h}),$$

Moreover, if the standard elliptic regularity holds,

$$\|\epsilon_u\| \leq C\, h^{\min_{K\in\mathcal{T}_h}\{1,k(K)\}}\,(\|\mathbf{q} - \mathbf{\Pi_V q}\| + \|(P_M - P_{\mathcal{M}})(\mathbf{q}\cdot\mathbf{n} + \tau\, u)\|_{\partial\Omega_h,h}).$$

$$\|\mathbf{q}\cdot\mathbf{n} + \tau\, u\|_{\partial K,h} \leq C h_K^{k(K)+1}\,\mathcal{D}_K(\mathbf{q},u)$$
$$\mathcal{D}_K(\mathbf{q},u) := |\mathbf{q}|_{H^{k+1}(K)} + \|\tau\|_{L^\infty(\partial K)}\,|u|_{H^{k+1}(K)}.$$

Variable-degree HDG methods on nonconforming meshes

Conclusions.

For uniform-degree methods on simplexes,

- HDG as well as DG methods always converge with order $k+1$ in the scalar variable.
- HDG methods can converge in the flux with order $k+1$ on some general nonconforming meshes. In this case, they superconverge with order $k+3/2$ for $k \geq 1$ in the scalar variable.
- For general meshes, they might lose $1/2$ an order of convergence in the flux and might not exhibit superconvergence of the scalar variable.
- HDG methods superconverge with order $k+2$ on semimatching meshes for $k \geq 1$.

The EDG methods. (B.C., J.Guzmán, S.-C.Soon and H.Stolarski, SINUM, 2009 .)

Motivation.

- Is it possible to modify the HDG methods as to render them as efficiently implementable as the CG methods?
- If so, can we keep the superconvergence property of the original HDG method?

Given and HDG method, we define an associated EDG method as follows. The approximate solution is $(\mathbf{q}_h, u_h) = (\mathbf{Q}_{\widehat{u}_h}, \mathsf{U}_{\widehat{u}_h}) + (\mathbf{Q}_f, \mathsf{U}_f)$, where the numerical trace $\widehat{u}_h$ is the element of a subspace $\widetilde{M}_h$ of M_h satisfying the equations

$$
\begin{aligned}
a_h(\widehat{u}_h, \mu) &= \ell_h(\mu) && \forall\, \mu \in \widetilde{M}_h : \mu|_{\partial\Omega} = 0, \\
\langle \mu, \widehat{u}_h \rangle_{\partial\Omega} &= \langle \mu, u_D \rangle_{\partial\Omega} && \forall\, \mu \in \widetilde{M}_h.
\end{aligned}
$$

Main example: $\widetilde{M}_h := M_h \cap \mathcal{C}_0(\Omega)$. (MDG method by Hughes et al.)

- The local solvers are the same for the HDG and the associated EDG.
- The sparsity of the stiffness matrix is identical to that of the (statically condensed) CG methods.
- Its condition number is smaller.
- For linear elements on simplexes, $\widehat{u}_h = u_h$ given by the CG method.
- Loss of the local conservativity of the numerical trace of the flux.
- Loss of the optimality of the order of convergence of the flux.
- Loss of the superconvergence of the scalar variable.

The EDG methods

Numerical experiments.

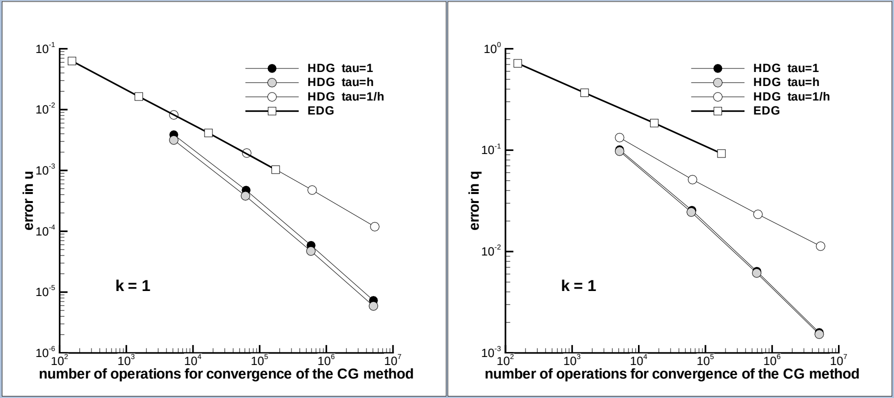


Figure: History of convergence of the HDG methods (for different values of τ) and the corresponding EDG method (for $\tau = 1$) for the same fixed polynomial degree of the numerical trace $\widehat{u}_h$.

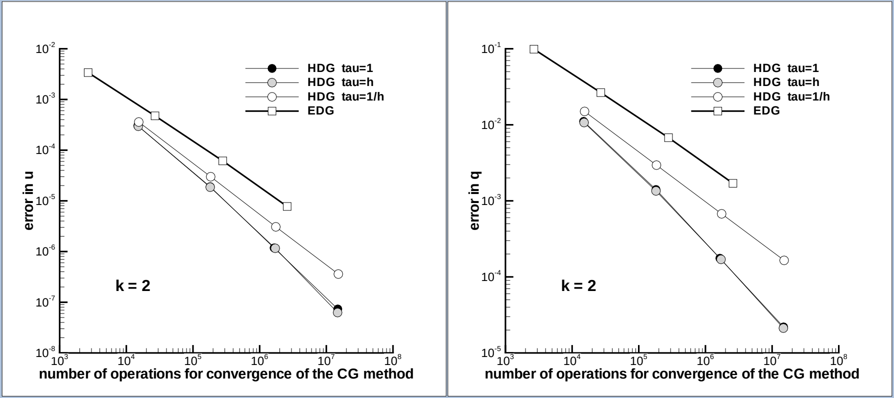


Figure: History of convergence of the HDG methods (for different values of τ) and the corresponding EDG method (for $\tau = 1$) for the same fixed polynomial degree of the numerical trace $\widehat{u}_h$.

The EDG methods

Numerical experiments.

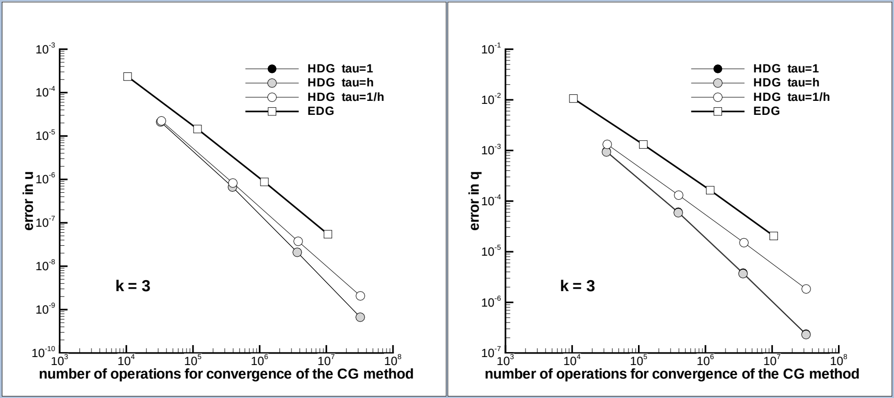
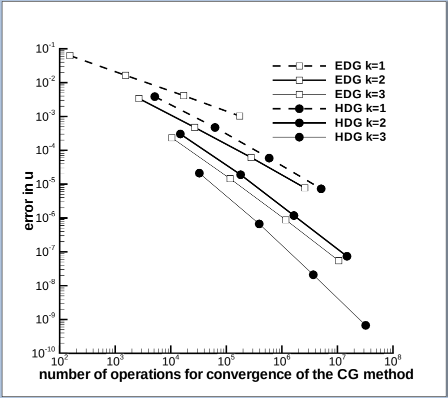


Figure: History of convergence of the HDG methods (for different values of τ) and the corresponding EDG method (for $\tau = 1$) for the same fixed polynomial degree of the numerical trace $\widehat{u}_h$.

Numerical experiments.



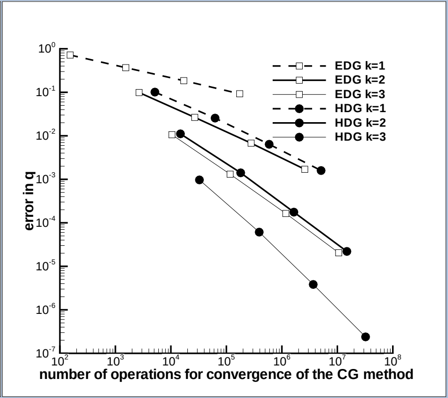


Figure: History of convergence of the HDG and the corresponding EDG method when $\tau = 1$.

Consider the model problem:

$$
\begin{aligned}
\mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } \Omega \times (0, T),\\
u_t + \nabla \cdot \mathbf{q} &= f && \text{in } \Omega \times (0, T),\\
\widehat{u} &= u_D && \text{on } \partial\Omega \times (0, T),\\
u &= u_0 && \text{on } \Omega \times \{0\}.
\end{aligned}
$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

We can obtain $(\mathbf{q}, u)$ in $K \times (0, T)$ in terms of $\widehat{u}$ on $\partial K \times (0, T)$, f and u_0 by solving

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } K \times (0, T), \\ u_t + \nabla \cdot \mathbf{q} &= f && \text{in } K \times (0, T), \\ u &= \widehat{u} && \text{on } \partial K \times (0, T), \\ u &= u_0 && \text{on } K \times \{0\}. \end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution on each $F \times (0, T)$, $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned} [\![\widehat{\mathbf{q}}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o, \\ \widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial, \end{aligned}$$

where $\widehat{\mathbf{q}}$ is the trace of $\mathbf{q} = \mathbf{q}(\widehat{u}, f, u_0)$ on ∂K.

At any time, the approximate solution $(\mathbf{q}_h, u_h, \widehat{u}_h)$ is an element of the space $\mathbf{V}_h \times W_h \times M_h$. It satisfies the equations

$$\begin{aligned}
(\mathrm{c}\, \mathbf{q}_h, \mathbf{v})_{\Omega_h} - (u_h, \nabla \cdot \mathbf{v})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial\Omega_h} &= 0, \\
((u_h)_t, \nabla w)_{\Omega_h} - (\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h}, \\
\langle \mu, \widehat{\mathbf{q}}_h \cdot \mathbf{n} \rangle_{\partial\Omega_h \setminus \partial\Omega} &= 0, \\
\langle \mu, \widehat{u}_h \rangle_{\partial\Omega} &= \langle \mu, u_D \rangle_{\partial\Omega},
\end{aligned}$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where

$$\widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau (u_h - \widehat{u}_h) \qquad \text{on } \partial\Omega_h.$$

The HDG method retains all the convergence and superconvergence, uniformly in time, of the HDG method for the steady-state case provided the initial condition is properly defined.

To approximate the time derivative at time $t^n := n\Delta t$, we could use the BDF approximation

$$(u_h)_t^n \approx (\sum_{j=0}^{\ell} \gamma_j {u_h}^{n-j})/\Delta t,$$

and set

$$\tilde{f}^n = f^n - (\sum_{j=1}^{\ell} \gamma_j {u_h}^{n-j})/\Delta t,$$

HDG methods for the heat equation.

A fully discrete method.

Then, at any time $t^n = n\,\Delta t$, the approximate solution $(\mathbf{q}_h, u_h, \widehat{u}_h)$ is an element of the space $\mathbf{V}_h \times W_h \times M_h$. It satisfies the equations

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{v})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{v})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
\frac{\gamma_0}{\Delta t}(u_h, \nabla w)_{\Omega_h} - (\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (\tilde{f}, w)_{\Omega_h},\\
\langle \mu, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,\\
\langle \mu, \widehat{u}_h\rangle_{\partial\Omega} &= \langle \mu, u_D\rangle_{\partial\Omega},
\end{aligned}
$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where

$$
\widehat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial\Omega_h.
$$

HDG methods for the wave equation. (N.C.Nguyen, J. Peraire and B.C., Math. Comp., JCP, 2011.) (B.C. and V.Queneville-Bélair, Math. Comp., 2nd. revision.)

The model problem.

Consider the model problem:

$$\begin{aligned} u_{tt} + \nabla \cdot (\mathrm{c}\nabla u) &= f && \text{in } \Omega \times (0, T), \\ \widehat{u} &= (u_D) && \text{on } \partial\Omega \times (0, T), \\ u &= u_0 && \text{on } \Omega \times \{0\}, \\ u_t &= u_1 && \text{on } \Omega \times \{0\}. \end{aligned}$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

We rewrite it in terms of $(\mathbf{q}, v) := (-\mathrm{c}\nabla u, u_t)$ as follows:

$$
\begin{aligned}
\mathrm{c}\,\mathbf{q}_t + \nabla v &= 0 && \text{in } \Omega \times (0, T),\\
v_t + \nabla \cdot \mathbf{q} &= f && \text{in } \Omega \times (0, T),\\
v &= (u_D)_t && \text{on } \partial\Omega \times (0, T),\\
\mathrm{c}\,\mathbf{q} &= -\nabla u_0 && \text{on } \Omega \times \{0\},\\
v &= u_1 && \text{on } \Omega \times \{0\}.
\end{aligned}
$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

We can obtain $(\mathbf{q}, v)$ in $K \times (0, T)$ in terms of $\widehat{v}$ on $\partial K \times (0, T)$, f, u_0 and u_1 by solving

$$
\begin{aligned}
\mathrm{c}\,\mathbf{q}_t + \nabla u &= 0 && \text{in } K \times (0, T),\\
v_t + \nabla \cdot \mathbf{q} &= f && \text{in } K \times (0, T),\\
\mathrm{c}\,\mathbf{q} &= -\nabla u_0 && \text{on } \Omega \times \{0\},\\
v &= u_1 && \text{on } \Omega \times \{0\}.
\end{aligned}
$$

The function $\widehat{v}$ can now be determined as the solution on each $F \times (0, T)$, $F \in \mathcal{E}_h$, of the equations

$$
\begin{aligned}
[\![\widehat{\mathbf{q}}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o,\\
\widehat{v} &= (u_D)_t && \text{if } F \in \mathcal{E}_h^\partial,
\end{aligned}
$$

where $\widehat{\mathbf{q}}$ is the trace of $\mathbf{q} = \mathbf{q}(\widehat{u}, f, u_0, u_1)$ on ∂K.

At any time, the approximate solution $(\mathbf{q}_h, v_h, \widehat{v}_h)$ is an element of the space $\mathbf{V}_h \times W_h \times M_h$. It satisfies the equations

$$
\begin{aligned}
(\mathrm{c}\,(\mathbf{q}_h)_t, \mathbf{r})_{\Omega_h} - (v_h, \nabla\cdot\mathbf{r})_{\Omega_h} + \langle \widehat{v}_h, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
((v_h)_t, \nabla w)_{\Omega_h} - (\mathbf{q}_h, \nabla w)_{\Omega_h} + \langle \widehat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \mu, \widehat{\mathbf{q}}_h\cdot\mathbf{n}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,\\
\langle \mu, \widehat{v}_h\rangle_{\partial\Omega} &= \langle \mu, (u_D)_t\rangle_{\partial\Omega},
\end{aligned}
$$

for all $(\mathbf{r}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where

$$
\widehat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(v_h - \widehat{v}_h) \qquad \text{on } \partial\Omega_h.
$$

HDG methods for the wave equation.

The semidiscrete method.

For simplexes, $\mathbf{V}(K) := \boldsymbol{\mathcal{P}}_k(K)$ and $W(K) := \mathcal{P}_k(K)$:

- The HDG method converges in $\mathbf{q}_h$ and v_h with the optimal order of $k+1$, for $k \geq 0$, in the $L^\infty(0,T;L^2(\Omega))$-norm.
- The variable $\int_0^t v_h$ superconverges with order $k+2$, for $k \geq 1$, in the $L^\infty(0,T;L^2(\Omega))$-norm provided the initial conditions are suitably defined.
- In this case, the postprocessed solution u_h^* superconverges with order $k+2$, for $k \geq 1$, in the $L^\infty(0,T;L^2(\Omega))$-norm.

Recall that, on each element K, u_h^* lies in the space $\mathcal{P}_{k+1}(K)$ and is defined by

$$(\nabla {u_h}^\star, \nabla w)_K = -\,(\mathrm{c}\mathbf{q}_h, \nabla w)_K \qquad \text{for all } w \in \mathcal{P}_{k+1}(K),$$

$$({u_h}^\star, 1)_K = (u_h, 1)_K = (\int_0^t v_h + u_h(0), 1)_K.$$

HDG methods for convection-diffusion equations. (N.C.Nguyen, J. Peraire and B.C., JCP, 2009).(Y.Chen and B.C., IMA + Math. Comp., to appear.)

The model problem.

Consider the model problem:

$$\begin{aligned} \mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } \Omega \times (0, T), \\ \nabla \cdot (\mathbf{q} + \mathsf{v}\, u) &= f && \text{in } \Omega \times (0, T), \\ \widehat{u} &= u_D && \text{on } \partial\Omega \times (0, T). \end{aligned}$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

HDG methods for convection-diffusion equations.

The approach.

We can obtain $(\mathbf{q}, u)$ in $K \times (0, T)$ in terms of $\widehat{u}$ on $\partial K \times (0, T)$, f and u_0 by solving

$$\begin{aligned}
\mathrm{c}\,\mathbf{q} + \nabla u &= 0 && \text{in } K \times (0, T),\\
\nabla \cdot (\mathbf{q} + \mathrm{v}\, u) &= f && \text{in } K \times (0, T),\\
u &= \widehat{u} && \text{on } \partial K \times (0, T).
\end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution on each $F \times (0, T)$, $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned}
[\![\widehat{\mathbf{q}} + \mathrm{v}\,\widehat{u}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o,\\
\widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial,
\end{aligned}$$

where $\widehat{\mathbf{q}}$ is the trace of $\mathbf{q} = \mathbf{q}(\widehat{u}, f, u_0)$ on ∂K.

The HDG method defines the approximation $(\mathbf{q}_h, u_h, \widehat{u}_h)$ in $\mathbf{V}_h \times W_h \times M_h$ by requiring that

$$
\begin{aligned}
(\mathrm{c}\,\mathbf{q}_h, \mathbf{r})_{\Omega_h} - (u_h, \nabla\cdot\mathbf{r})_{\Omega_h} + \langle \widehat{u}_h, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\mathbf{q}_h + u_h\mathbf{v}, \nabla w)_{\Omega_h} + \langle (\widehat{\mathbf{q}}_h + \widehat{u}_h\,\mathbf{v})\cdot\mathbf{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h},\\
\langle \mu, \widehat{u}_h\rangle_{\partial\Omega} &= \langle \mu, g\rangle_{\partial\Omega},\\
\langle \mu, (\widehat{\mathbf{q}}_h + \widehat{u}_h\,\mathbf{v})\cdot\mathbf{n}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0,
\end{aligned}
$$

hold for all $(\mathbf{r}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where

$$
\widehat{\mathbf{q}}_h + \widehat{u}_h\,\mathbf{v} = \mathbf{q}_h + \widehat{u}_h\,\mathbf{v} + \tau(u_h - \widehat{u}_h)\mathbf{n} \quad \text{on } \partial\Omega_h.
$$

Theorem

The method is well defined if

A1 *There is a constant* $\gamma_0 > 0$*:* $\min(\tau - \frac{1}{2}\mathbf{v}\cdot\mathbf{n})|_{\partial K} \geq \gamma_0 \ \forall\, K \in \mathcal{T}_h$.

A2 *On any face* $F \in \mathcal{E}_h$, τ *is a constant.*

The following practical choices of stabilization functions τ do satisfy these two conditions:

$$\tau^+ = \tau^- = |\mathbf{v}\cdot\mathbf{n}| + \frac{\kappa}{\ell},$$

$$(\tau^+, \tau^-) = \begin{cases} (|\mathbf{v}\cdot\mathbf{n}| + \frac{\kappa}{\ell}, 0) & \text{when } \mathbf{v}\cdot\mathbf{n}^- \leq 0, \\ (0, |\mathbf{v}\cdot\mathbf{n}| + \frac{\kappa}{\ell}) & \text{when } \mathbf{v}\cdot\mathbf{n}^- > 0. \end{cases}$$

Here κ is a scalar proportional to some norm of the diffusivity matrix c^{-1} and ℓ denotes a representative length scale.

For the first choice of τ, we have

$$\begin{aligned}\widehat{u}_h &= \{\!\{u_h\}\!\} + \frac{1}{2\,\tau}\,[\![\mathbf{q}_h\cdot\mathbf{n}]\!],\\ \widehat{u}_h\,\mathbf{v} + \widehat{\mathbf{q}_h} &= \{\!\{u_h\}\!\}\mathbf{v} + \{\!\{\mathbf{q}_h\}\!\} + \frac{1}{2\,\tau}\,[\![\mathbf{q}_h\cdot\mathbf{n}]\!]\mathbf{v} + \frac{\tau}{2}\,[\![u_h\mathbf{n}]\!],\end{aligned}$$

whereas for the second choice for τ,

$$\begin{cases}\widehat{u}_h &= {u_h}^{+} + \frac{1}{\tau^{+}}\,[\![\mathbf{q}_h\cdot\mathbf{n}]\!],\\ \widehat{u}_h\,\mathbf{v} + \widehat{\mathbf{q}_h} &= u_h^{+}\mathbf{v} + {\mathbf{q}_h}^{-} + \frac{1}{\tau^{+}}\,[\![\mathbf{q}_h\cdot\mathbf{n}]\!]\,\mathbf{v}\end{cases} \qquad \text{if } \mathbf{v}\cdot\mathbf{n}^{-} \leq 0,$$

and

$$\begin{cases}\widehat{u}_h &= {u_h}^{-} + \frac{1}{\tau^{-}}\,[\![\mathbf{q}_h\cdot\mathbf{n}]\!],\\ \widehat{u}_h\,\mathbf{v} + \widehat{\mathbf{q}_h} &= u_h^{-}\mathbf{v} + {\mathbf{q}_h}^{+} + \frac{1}{\tau^{-}}\,[\![\mathbf{q}_h\cdot\mathbf{n}]\!]\mathbf{v},\end{cases} \qquad \text{if } \mathbf{v}\cdot\mathbf{n}^{-} > 0.$$

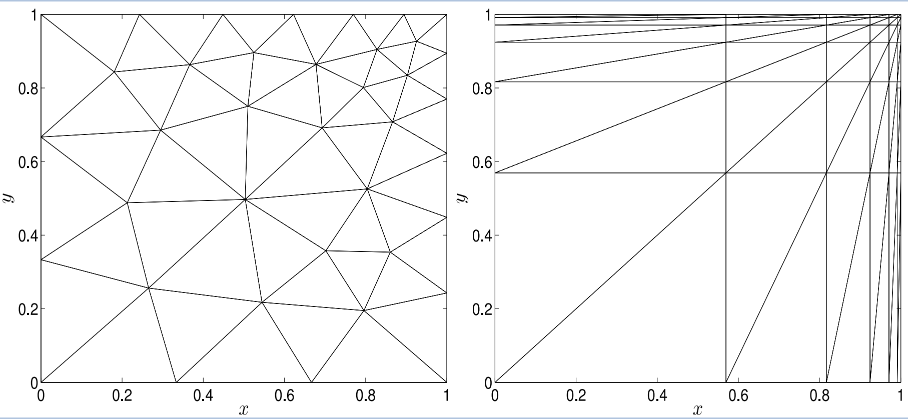
On any simplex K, the projection $(\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q}, \Pi_W u)$ is the element of $\mathcal{P}_k(K) \times \mathcal{P}_k(K)$ which solves the equations

$$\begin{aligned} ((\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q} - \mathbf{q}) + \mathbf{v}\,(\Pi_W u - u), \mathbf{r})_K &= 0 \;\; \forall\, \mathbf{r} \in \boldsymbol{\mathcal{P}}_{k-1}(K), \\ (\Pi_W u - u, w)_K &= 0 \;\; \forall\, w \in \mathcal{P}_{k-1}(K), \\ \langle((\boldsymbol{\Pi}_{\mathbf{V}}\mathbf{q} - \mathbf{q}) + \mathbf{v}\,(P_M u - u)) \cdot \mathbf{n} + \tau(\Pi_W u - u), \mu\rangle_F &= 0 \;\; \forall\, \mu \in \mathcal{P}_k(F), \end{aligned}$$

for all faces F of the simplex K.

HDG methods for convection-diffusion.

Numerical examples.

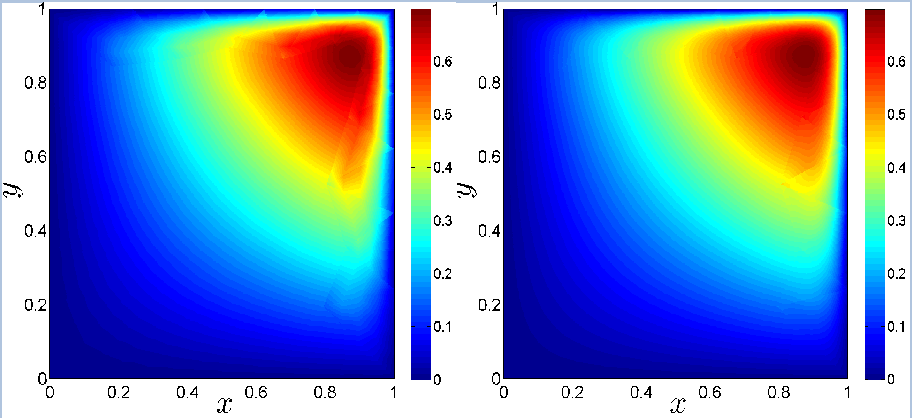


Unstructured and anisotropic meshes.

(N.C.Nguyen, J. Peraire and B.C., JCP, 2009.

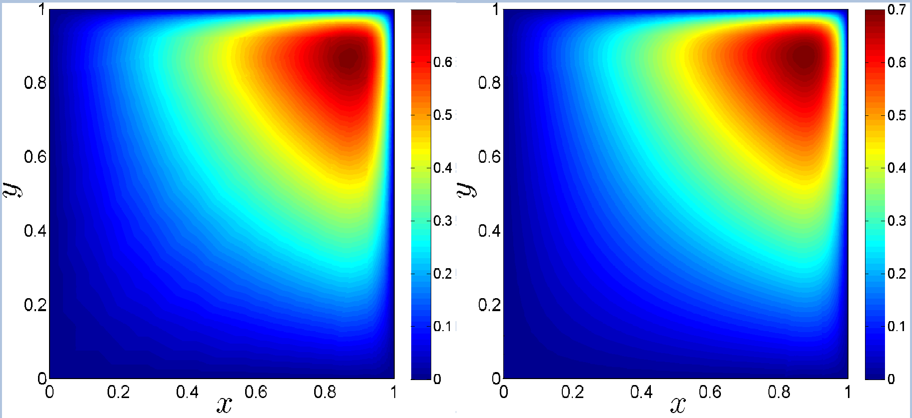
HDG methods for convection-diffusion.

Numerical examples.



HDG approximation with quadratic polynomials on the unstructured triangulation.

(N.C.Nguyen, J. Peraire and B.C., JCP, 2009.



HDG approximation with quadratic polynomials on refined triangulations.

(N.C.Nguyen, J. Peraire and B.C., JCP, 2009.

Consider the following problem:

$$\begin{aligned}
\sigma_{ij,j} + b_i &= 0 && \text{in } \Omega,\\
\epsilon_{ij} - \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) &= 0 && \text{in } \Omega,\\
\sigma_{ij} - D_{ijkl}\,\epsilon_{kl} &= 0 && \text{in } \Omega,\\
\widehat{u}_i &= u_i && \text{on } \partial\Omega_D,\\
\widehat{\sigma}_{ij}\, n_j &= t_i && \text{on } \partial\Omega_N.
\end{aligned}$$

We can obtain (σ, u) in K in terms of $\widehat{u}$ by solving

$$\begin{aligned}
\sigma_{ij,j} + b_i &= 0 && \text{in } K,\\
\epsilon_{ij} - \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) &= 0 && \text{in } K,\\
\sigma_{ij} - D_{ijkl}\,\epsilon_{kl} &= 0 && \text{in } K,\\
\widehat{u}_i &= \widehat{u}_i && \text{on } \partial K.
\end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution of the transmission condition

$$\begin{aligned}
[\![\widehat{\sigma}_{ij}\, n_j]\!] &= 0 && \text{on } \mathcal{E}_h^o,\\
\widehat{u}_i &= u_i && \text{on } \partial\Omega_D,\\
\widehat{\sigma}_{ij}\, n_j &= t_i && \text{on } \partial\Omega_N.
\end{aligned}$$

The approximation $(\mathbf{u}^h, \underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\epsilon}}^h, \widehat{\mathbf{u}}^h)$ is taken in the finite dimensional space $\mathbf{V}^h \times \underline{\mathbf{W}}^h \times \underline{\mathbf{Z}}^h \times \mathbf{M}^h$ where

$$\begin{aligned}
\mathbf{V}^h &= \{\mathbf{v} \in \mathbf{L}^2(\Omega) \;:\; v_i\,|_K \in \mathcal{P}_k(K) \quad \forall K \in \Omega_h, \quad i = 1,2,3\},\\
\underline{\mathbf{W}}^h &= \{\underline{\mathbf{w}} \in \underline{\mathbf{L}}^2(\Omega) \;:\; w_{ij}\,|_K \in \mathcal{P}_k(K) \quad \forall K \in \Omega_h, \quad i,j = 1,2,3\},\\
\underline{\mathbf{Z}}^h &= \{\underline{\mathbf{z}} \in \underline{\mathbf{L}}^2(\Omega) \;:\; z_{ij}\,|_K \in \mathcal{P}_k(K) \quad \forall K \in \Omega_h, \quad i,j = 1,2,3\},\\
\mathbf{M}^h &= \{\boldsymbol{\mu} \in \mathbf{L}^2(\mathcal{E}_h) \;:\; \mu_i\,|_F \in \mathcal{P}_k(F) \quad \forall F \in \mathcal{E}_h, \quad i = 1,2,3\}.
\end{aligned}$$

On the element K, $(\mathbf{u}^h, \underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\epsilon}}^h)$ is obtained in terms of $\widehat{u}^h$ by solving

$$
\begin{aligned}
&\left(v_{i,j}, \sigma_{ij}^h\right)_K - \langle v_i, \widehat{\sigma}_{ij}^h n_j\rangle_{\partial K} - \left(v_i, b_i\right)_K = 0,\\
&\left(w_{ij}, \epsilon_{ij}^h\right)_K - \frac{1}{2}\langle w_{ij}, (\widehat{u}_i^h n_j + \widehat{u}_j^h n_i)\rangle_{\partial K} + \frac{1}{2}\left(w_{ij,j}, u_i^h\right)_K + \frac{1}{2}\left(w_{ij,i}, u_j^h\right)_K = 0,\\
&\left(z_{ij}, \sigma_{ij}^h\right)_K - \left(z_{ij}, D_{ijkl}\epsilon_{kl}^h\right)_K = 0,
\end{aligned}
$$

for all $(\mathbf{v}, \underline{\mathbf{w}}, \underline{\mathbf{z}}, \boldsymbol{\mu}) \in \mathcal{P}_k(K) \times \underline{\mathcal{P}_k(K)} \times \underline{\mathcal{P}_k(K)} \times \mathcal{P}_k(K)$, where

$$
\widehat{\sigma}_{ij}^h = \sigma_{ij}^h - \tau_{ijkl}\left(u_k^h - \widehat{u}_k^h\right) n_l \qquad \text{on } \partial\Omega_h.
$$

The function $\widehat{u}^h$ is now determined as the element of $\mathbf{M}_h$ satisfying

$$
\begin{aligned}
\left\langle \mu_i, \widehat{\sigma}_{ij}^h n_j \right\rangle_{\partial\Omega_h \setminus \partial\Omega_D} &= \left\langle \mu_i, t_i \right\rangle_{\partial\Omega_N},\\
\left\langle \mu_i, \widehat{u}_i^h \right\rangle_{\partial\Omega_D} &= \left\langle \mu_i, u_i \right\rangle_{\partial\Omega_D}.
\end{aligned}
$$

for all $\boldsymbol{\mu} \in \mathbf{M}_h$.

Linear elasticity.
An HDG method

In compact form:

$$\left(v_{i,j},\sigma_{ij}^h\right)_{\Omega_h}-\langle v_i,\widehat{\sigma}_{ij}^h n_j\rangle_{\partial\Omega_h}-\left(v_i,b_i\right)_{\Omega_h}=0,$$

$$\left(w_{ij},\epsilon_{ij}^h\right)_{\Omega_h}-\frac{1}{2}\langle w_{ij},\left(\widehat{u}_i^h n_j+\widehat{u}_j^h n_i\right)\rangle_{\partial\Omega_h}+\frac{1}{2}\left(w_{ij,j},u_i^h\right)_{\Omega_h}+\frac{1}{2}\left(w_{ij,i},u_j^h\right)_{\Omega_h}=0,$$

$$\left(z_{ij},\sigma_{ij}^h\right)_{\Omega_h}-\left(z_{ij},D_{ijkl}\epsilon_{kl}^h\right)_{\Omega_h}=0,$$

$$\left\langle\mu_i,\widehat{\sigma}_{ij}^h n_j\right\rangle_{\partial\Omega_h\setminus\partial\Omega_D}=\langle\mu_i,t_i\rangle_{\partial\Omega_N},$$

$$\left\langle\mu_i,\widehat{u}_i^h\right\rangle_{\partial\Omega_D}=\langle\mu_i,u_i\rangle_{\partial\Omega_D},$$

for all $(\mathbf{v},\underline{\mathbf{w}},\underline{\mathbf{z}},\boldsymbol{\mu})\in\mathbf{V}^h\times\underline{\mathbf{W}}^h\times\underline{\mathbf{Z}}^h\times\mathbf{M}^h$, where

$$\widehat{\sigma}_{ij}^h=\sigma_{ij}^h-\tau_{ijkl}\left(u_k^h-\widehat{u}_k^h\right)n_l\qquad\text{on }\partial\Omega_h.$$

Theorem

The approximate solution

$$(\mathbf{u}^h, \underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\epsilon}}^h) = (\mathtt{U}^{(\widehat{\mathbf{u}}^h)}, \mathtt{S}^{(\widehat{\mathbf{u}}^h)}, \mathtt{E}^{(\widehat{\mathbf{u}}^h)}) + (\mathtt{U}^{(\mathbf{u})}, \mathtt{S}^{(\mathbf{u})}, \mathtt{E}^{(\mathbf{u})}),$$

is well defined if we take $\tau_{ijkl}\, n_j\, n_l$ positive definite on $\partial\Omega_h$. Moreover, the function $\boldsymbol{\lambda}^h := \widehat{\mathbf{u}}^h - \mathbf{u}$, is the only element of $\mathbf{M}^h$ satisfying

$$a^h\left(\boldsymbol{\mu}, \boldsymbol{\lambda}^h\right) = b^h\left(\boldsymbol{\mu}\right) \qquad \forall \boldsymbol{\mu} \in \mathbf{M}^h(\mathbf{0}),$$

where

$$\begin{aligned} a^h\left(\boldsymbol{\zeta}, \boldsymbol{\eta}\right) &= \left(D_{ijkl}\, \mathtt{E}_{ij}^{(\zeta)}, \mathtt{E}^{(\eta)}{}_{kl}\right)_{\Omega_h} + \left\langle \left(\mathtt{U}_i^{(\eta)} - \eta_i\right), \tau_{ijkl}\, n_j\, n_l \left(\mathtt{U}_k^{(\zeta)} - \zeta_k\right)\right\rangle_{\partial\Omega_h}, \\ b^h\left(\boldsymbol{\zeta}\right) &= \left\langle \zeta_i, t_i \right\rangle_{\partial\Omega_N} - \left\langle \widehat{\mathtt{S}}_{ij}^{(\zeta)} n_j, u_i \right\rangle_{\partial\Omega_D} + \left(\mathtt{U}_i^{(\zeta)}, b_i\right)_{\Omega_h}, \end{aligned}$$

for all $\boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbf{L}^2(\mathscr{E}^h)$.

- For $k \geq 0$ all unknowns converge with order $k+1$.
- For $k \geq 2$ the local average of the displacement superconverges with order $k+2$. A local postprocessing can be devised that provides another approximate displacement converging with order $k+2$.
- Analysis: Still open!

HDG methods for the Stokes flow. (N.C Nguyen, J. Peraire and B.C., JCP+CMAME, 2010.) (B.C., J. Gopalakrishnan, N.C.Nguyen, J. Peraire and F.-J. Sayas, Math. Comp., 2011.) (B.C. and K. Shi, Math. Comp. + SINUM, to appear.)

The model problem.

Consider the model problem:

$$
\begin{aligned}
-\nu\Delta u + \nabla p &= f && \text{in } \Omega,\\
\nabla \cdot u &= 0 && \text{on } \Omega,\\
\widehat{u} &= u_D && \text{on } \partial\Omega,
\end{aligned}
$$

where $\langle u_D \cdot \mathbf{n}, 1\rangle_{\partial\Omega} = 0$ and $(p, 1)_\Omega = 0$.

We begin by rewriting it as follows:

$$
\begin{aligned}
\boldsymbol{\omega} - \nabla \times \mathbf{u} &= 0 && \text{in } \Omega,\\
\nu \nabla \times \boldsymbol{\omega} + \nabla p &= f && \text{in } \Omega,\\
\nabla \cdot u &= 0 && \text{on } \Omega,\\
\widehat{u} &= u_D && \text{on } \partial\Omega,
\end{aligned}
$$

where $\langle u_D \cdot \mathbf{n}, 1 \rangle_{\partial\Omega} = 0$ and $(p, 1)_\Omega = 0$.

HDG methods for the Stokes flow

Using the vorticity.

We can express $(\boldsymbol{\omega}, \mathbf{u}, p)$ in K in terms of $\widehat{\mathbf{u}}$ on ∂K and $\overline{p} := (p,1)_K/|K|$ by solving

$$\begin{aligned}
\boldsymbol{\omega} - \nabla \times \mathbf{u} &= 0, \qquad \nu \nabla \times \boldsymbol{\omega} + \nabla p = \mathbf{f} && \text{in } K,\\
\nabla\cdot \mathbf{u} &= \frac{1}{|K|}\langle \widehat{\mathbf{u}} \cdot \mathbf{n}, 1\rangle_{\partial K} && \text{in } K,\\
\mathbf{u} &= \widehat{\mathbf{u}} && \text{on } \partial K.
\end{aligned}$$

The functions $\widehat{\mathbf{u}}$ and $\overline{p}$ are the solution of

$$\begin{aligned}
[\![-\nu \widehat{\boldsymbol{\omega}} \times \mathbf{n} + \widehat{p}\,\mathbf{n}]\!] &= 0 && \text{for all } F \in \mathcal{E}_h^o,\\
\langle \widehat{\mathbf{u}} \cdot \mathbf{n}, 1\rangle_{\partial K} &= 0 && \text{for all } K \in \Omega_h,\\
\widehat{\mathbf{u}} &= \mathbf{u}_D && \text{on } \partial\Omega,\\
(\overline{p}, 1)_\Omega &= 0.
\end{aligned}$$

HDG methods for the Stokes flow.

Using the velocity gradient.

We begin by rewriting it as follows:

$$\begin{aligned} \mathrm{L} - \nabla \mathbf{u} &= 0 && \text{in } \Omega, \\ -\nu \nabla \cdot \mathrm{L} + \nabla p &= f && \text{in } \Omega, \\ \nabla \cdot u &= 0 && \text{on } \Omega, \\ \widehat{u} &= u_D && \text{on } \partial\Omega, \end{aligned}$$

where $\langle u_D \cdot \mathbf{n}, 1 \rangle_{\partial\Omega} = 0$ and $(p, 1)_\Omega = 0$.

HDG methods for the Stokes flow

Using the velocity gradient.

We can express $(\mathrm{L}, \mathbf{u}, p)$ in K in terms of $\widehat{\mathbf{u}}$ on ∂K and $\overline{p} := (p, 1)_K / |K|$ by solving

$$\begin{aligned} \mathrm{L} - \nabla \mathbf{u} &= 0, \qquad -\nu \nabla \cdot \mathrm{L} + \nabla p = \mathbf{f} && \text{in } K, \\ \nabla \cdot \mathbf{u} &= \frac{1}{|K|} \langle \widehat{\mathbf{u}} \cdot \mathbf{n}, 1 \rangle_{\partial K} && \text{in } K, \\ \mathbf{u} &= \widehat{\mathbf{u}} && \text{on } \partial K. \end{aligned}$$

The functions $\widehat{\mathbf{u}}$ and $\overline{p}$ are the solution of

$$\begin{aligned} [\![-\nu \widehat{\mathrm{L}} \mathbf{n} + \widehat{p}\, \mathbf{n}]\!] &= 0 && \text{for all } F \in \mathcal{E}_h^o, \\ \langle \widehat{\mathbf{u}} \cdot \mathbf{n}, 1 \rangle_{\partial K} &= 0 && \text{for all } K \in \Omega_h, \\ \widehat{\mathbf{u}} &= \mathbf{u}_D && \text{on } \partial \Omega, \\ (\overline{p}, 1)_\Omega &= 0. \end{aligned}$$

The HDG methods for the Stokes flow

Which approach should we use?

- Both approaches give rise to saddle-point problems of the same sparsiy structure.
- In both approaches, the only globally coupled degrees of freedom are those of the velocity trace $\widehat{u}$ and the average of the pressure on each element $\overline{p}$.
- The local solvers for the vorticity formulation have less degrees of freedom. However, there is no superconvergence of the velocity.
- The local solvers for the velocity gradient formulation have more degrees of freedom. However, there is superconvergence of the velocity.

On the element $K \in \Omega_h$, we define $(\mathrm{L}_h, \mathbf{u}_h, p_h)$ in terms of $(\widehat{\mathbf{u}}_h, \overline{p}_h, f)$ as the element of $\mathrm{G}(K) \times \mathbf{V}(K) \times Q(K)$ solving

$$\begin{aligned}
(\mathrm{L}_h, \mathrm{G})_K + (\mathbf{u}_h, \nabla \cdot \mathrm{G})_K - \langle \widehat{\mathbf{u}}_h, \mathrm{G}\mathbf{n} \rangle_{\partial K} &= 0, \\
(\nu \mathrm{L}_h, \nabla \mathbf{v})_K - (p_h, \nabla \cdot \mathbf{v})_K - \langle \nu \widehat{\mathrm{L}}_h \mathbf{n} - \widehat{p}_h \mathbf{n}, \mathbf{v} \rangle_{\partial K} &= (\mathbf{f}, \mathbf{v})_K, \\
-(\mathbf{u}_h, \nabla q)_{\Omega_h} + \langle \widehat{\mathbf{u}}_h \cdot \mathbf{n}, q - \overline{q} \rangle_{\partial K} &= 0,
\end{aligned}$$

for all $(\mathrm{G}, \mathbf{v}, q) \in \mathrm{G}(K) \times \mathbf{V}(K) \times Q(K)$, where

$$-\nu \widehat{\mathrm{L}}_h \mathbf{n} + \widehat{p}_h \mathbf{n} = -\nu \mathrm{L}_h \mathbf{n} + p_h \mathbf{n} + \nu\, \tau \,(\mathbf{u}_h - \widehat{\mathbf{u}}_h) \quad \text{on } \partial K,$$

and $(p_h, 1)_K / |K| = \overline{p}_h$.

We take $\widehat{\mathbf{u}}_h|_F$ in $\mathbf{M}(F)$ and $\overline{p}_h|_K$ in $\mathcal{P}_0(K)$ and determine them by requiring

$$\begin{aligned}
\langle [\![-\nu\widehat{\mathrm{L}}_h\mathbf{n} + \widehat{p}_h\,\mathbf{n}]\!], \boldsymbol{\mu}\rangle_F &= 0 \qquad \forall \boldsymbol{\mu} \in \mathbf{M}(F) \;\; \forall\, F \in \mathcal{E}_h^o,\\
\langle \widehat{\mathbf{u}}_h \cdot \mathbf{n}, 1\rangle_{\partial K} &= 0 \qquad \forall\, K \in \Omega_h,\\
\widehat{\mathbf{u}}_h &= \mathbf{u}_D \quad \text{ on } \partial\Omega,\\
(\overline{p}_h, 1)_\Omega &= 0.
\end{aligned}$$

Theorem

The HDG methods are well defined if

- $\tau > 0$ *on* $\partial\Omega_h$,
- $\nabla \mathbf{V}(K) \in \mathrm{G(K)} \;\; \forall K \in \Omega_h$,
- $\nabla Q(K) \in \mathbf{V}(K) \;\; \forall K \in \Omega_h$.

We denote by $(\mathsf{L}, \mathbf{U}, \mathsf{P})$ the linear mapping that associates $(\widehat{\mathbf{u}}_h, \overline{p}_h, f)$ to $(\mathrm{L}_h, \mathbf{u}_h, p_h)$, and set

$$\begin{aligned}
(\mathsf{L}^{\widehat{\mathbf{u}}_h}, \mathbf{U}^{\widehat{\mathbf{u}}_h}, \mathsf{P}^{\widehat{\mathbf{u}}_h}) &:= (\mathsf{L}, \mathbf{U}, \mathsf{P})(\widehat{\mathbf{u}}_h, 0, 0),\\
(\mathsf{L}^{\overline{p}_h}, \mathbf{U}^{\overline{p}_h}, \mathsf{P}^{\overline{p}_h}) &:= (\mathsf{L}, \mathbf{U}, \mathsf{P})(0, \overline{p}_h, 0),\\
(\mathsf{L}^{f}, \mathbf{U}^{f}, \mathsf{P}^{f}) &:= (\mathsf{L}, \mathbf{U}, \mathsf{P})(0, 0, f).
\end{aligned}$$

Then we have that

$$(\mathrm{L}_h, \mathbf{u}_h, p_h) = (\mathsf{L}^{\widehat{\mathbf{u}}_h}, \mathbf{U}^{\widehat{\mathbf{u}}_h}, \mathsf{P}^{\widehat{\mathbf{u}}_h}) + (\mathsf{L}^{\overline{p}_h}, \mathbf{U}^{\overline{p}_h}, \mathsf{P}^{\overline{p}_h}) + (\mathsf{L}^{f}, \mathbf{U}^{f}, \mathsf{P}^{f}).$$

The function $(\widehat{\mathbf{u}}_h, \overline{p}_h)$ is the only element in $\mathbf{M}_h \times \overline{P}_h$ such that

$$
\begin{aligned}
a_h(\widehat{\mathbf{u}}_h, \boldsymbol{\mu}) + b_h(\overline{p}_h, \boldsymbol{\mu}) &= \ell_h(\boldsymbol{\mu}), && \forall\, \boldsymbol{\mu} \in \mathbf{M}_h : \boldsymbol{\mu}|_{\partial\Omega} = \mathbf{0}),\\
b_h(\overline{q}, \widehat{\mathbf{u}}_h) &= 0, && \forall\, \overline{q} \in \overline{P}_h,\\
\widehat{\mathbf{u}}_h &= \mathbf{u}_D, &&\\
(\overline{p}_h, 1)_\Omega &= 0. &&
\end{aligned}
$$

where $\mathbf{M}_h := \{\boldsymbol{\mu} \in \mathbf{L}^2(\mathcal{E}_h) : \quad \boldsymbol{\mu}|_F \in \mathbf{M}(F) \;\; \forall\, F \in \mathcal{E}_h^o\}$.

- The bilinear form $a_h(\cdot, \cdot)$ is symmetric and positive definite on $\mathbf{M}_{h,0} \times \mathbf{M}_{h,0}$.

The energy identity for the exact solution is

$$(\mathrm{L},\mathrm{L})_\Omega = (\mathbf{f},\mathbf{u})_\Omega + \langle -\nu \mathrm{L}\mathbf{n} + p\,\mathbf{n}, \mathbf{u}_D\rangle_{\partial\Omega},$$

and for the approximate solution we have,

$$(\mathrm{L}_h,\mathrm{L}_h)_\Omega + \Theta_\tau(\mathbf{u}_h - \widehat{\mathbf{u}}_h) = (\mathbf{f},\mathbf{u}_h)_\Omega + \langle(-\nu\widehat{\mathrm{L}}_h + \widehat{p}_h\,\mathrm{I})\mathbf{n}, \mathbf{u}_D\rangle_{\partial\Omega},$$

where $\Theta_\tau(\mathbf{u}_h - \widehat{\mathbf{u}}_h) := \sum_{K\in\Omega_h}\langle \tau(\mathbf{u}_h - \widehat{\mathbf{u}}_h), \mathbf{u}_h - \widehat{\mathbf{u}}_h\rangle_{\partial K}$. We see that the jumps $\mathbf{u}_h - \widehat{\mathbf{u}}_h$ stabilize the method if we require the function τ to be positive on $\partial\Omega_h$.

The Galerkin formulation on the element K reads

$$
\begin{aligned}
(\mathrm{R}_K^{\mathbf{u}}, \mathrm{G})_K &= \langle \mathrm{R}_{\partial K}^{\mathbf{u}}, \mathrm{G}\rangle_{\partial K} \\
(\mathbf{R}_K^{\mathrm{L},p}, \mathbf{v})_K &= \langle R_{\partial K}^{\mathrm{L},p}, \mathbf{v}\rangle_{\partial K}, \\
(R_K^{\nabla\cdot\mathbf{u}}, q)_K &= \langle tr\mathrm{R}_{\partial K}^{\mathbf{u}}, q\rangle_{\partial K},
\end{aligned}
$$

for all $(\mathrm{G}, \mathbf{v}, q) \in \mathrm{G}(K) \times \mathbf{V}(K) \times P(K)$ where

$$
\begin{aligned}
\mathrm{R}_K^{\mathbf{u}} &:= \mathrm{L}_h - \nabla \mathbf{u}_h, \\
\mathbf{R}_K^{\mathrm{L},p} &:= \nabla \cdot (-\nu \mathrm{L}_h + p_h\, \mathrm{I}) - \mathbf{f}, \\
R_K^{\nabla\cdot\mathbf{u}} &:= \nabla \cdot \mathbf{u}_h, \\
\mathrm{R}_{\partial K}^{\mathbf{u}} &:= (\widehat{\mathbf{u}}_h - \mathbf{u}_h) \otimes \mathbf{n}, \\
\mathbf{R}_{\partial K}^{\mathrm{L},p} &:= (-\nu \mathrm{L}_h \mathbf{n} + p_h\, \mathbf{n}) - (-\nu \widehat{\mathrm{L}}_h \mathbf{n} + \widehat{p}_h\, \mathbf{n}) = -\nu\, \tau\, (\mathbf{u}_h - \widehat{\mathbf{u}}_h)
\end{aligned}
$$

For simplexes K and

$$\mathrm{G}(K) \times \mathbf{V}(K) \times W(K) \times \mathbf{M}(K) := \mathrm{P}_k(K) \times \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K) \times \boldsymbol{\mathcal{P}}_k(F),$$

we have:

$$\begin{aligned} (\mathrm{E}^{\mathrm{L}}, \mathrm{G})_K + (\varepsilon_{\boldsymbol{u}}, \boldsymbol{\nabla}\cdot \mathrm{G})_K - \langle \varepsilon_{\widehat{\boldsymbol{u}}}, \mathrm{G}\mathbf{n}\rangle_{\partial K} &= (\Pi\ \mathrm{L} - \mathrm{L}, \mathrm{G})_K, \\ -(\nabla\cdot(\nu \mathrm{E}^{\mathrm{L}}), \mathbf{v})_K + (\nabla \varepsilon^p, \mathbf{v})_K + \langle \nu\tau\,(\varepsilon_{\boldsymbol{u}} - \varepsilon_{\widehat{\boldsymbol{u}}}), \mathbf{v}\rangle_{\partial K} &= 0, \\ -(\varepsilon_{\boldsymbol{u}}, \nabla q)_K + \langle \varepsilon_{\widehat{\boldsymbol{u}}}, q\,\mathbf{n}\rangle_{\partial K} &= 0, \end{aligned}$$

for all $(\mathrm{G}, \mathbf{v}, q)$ in $\mathrm{G}(K) \times \mathbf{V}(K) \times Q(K)$. Moreover,

$$\begin{aligned} \langle -\nu \mathrm{E}^{\mathrm{L}}\mathbf{n} + \varepsilon^p\mathbf{n} + \nu\tau\,(\varepsilon_{\boldsymbol{u}} - \varepsilon_{\widehat{\boldsymbol{u}}}), \boldsymbol{\mu}\rangle_F &= 0 \quad \forall\ \boldsymbol{\mu} \in \mathbf{M}(F)\ ; \forall F \in \mathcal{E}_h^o, \\ \varepsilon_{\widehat{\boldsymbol{u}}} &= 0 \quad \text{on } \partial\Omega, \\ (\varepsilon^p, 1)_\Omega &= 0. \end{aligned}$$

Theorem

We have

$$\|\,\mathrm{E}^{\mathrm{L}}\,\|_\Omega \le C\,\|\,\Pi\,\mathrm{L}-\mathrm{L}\,\|_\Omega,$$
$$\|\varepsilon^p\|_\Omega \le C\,\sqrt{C_\tau}\,\nu\,\|\Pi\ \mathrm{L}-\mathrm{L}\|_\Omega,$$

where $C_\tau := \max_{K\in\Omega_h}\{1,\tau_K\,h_K\}$. *Moreover,*

$$\|\,\varepsilon_{\boldsymbol{u}}\,\|_\Omega \le C\,C_\tau\,h^{\min\{k,1\}}\,\|\,\Pi\,\mathrm{L}-\mathrm{L}\,\|_\Omega,$$

provided a standard elliptic regularity result holds.

Note that, by an energy argument, we get

$$(\mathrm{E}^{\mathrm{L}},\mathrm{E}^{\mathrm{L}})_\Omega + \Theta_\tau(\varepsilon_{\boldsymbol{u}}-\varepsilon_{\widehat{\boldsymbol{u}}}) = (\Pi\,\mathrm{L}-\mathrm{L},\mathrm{E}^{\mathrm{L}})_\Omega.$$

A new approximate velocity $\mathbf{u}_h^\star$ can be obtained which has the following properties:

- It is computed in an element-by-element fashion.
- $\mathbf{u}_h^\star \in \mathbf{H}(div, \Omega)$.
- $\nabla\cdot \mathbf{u}_h^\star = 0$ on Ω.
- $\|\mathbf{u}_h^\star - \mathbf{u}\|_\Omega \le C\, C_\tau\, h^{\min\{k,1\}}\, \|\Pi\,\mathrm{L} - \mathrm{L}\|_\Omega + C\, h^{k+2} |\,\mathbf{u}\,|_{\mathbf{H}^{k+2}(\Omega)}$.

- Let $\mathbf{V}^D(K)$, $W^D(K)$ and $M^D(F)$ be the local spaces of a superconvergent HDG method for diffusion.

- Set $\mathrm{G}_i(K) := \mathbf{V}^D(K)$, $\mathbf{V}_i(K) := W^D(K)$ and $\mathbf{M}_i(F) := M^D(F)$.

- Take a local space $Q(K)$ such that

$$\nabla \cdot \mathbf{V}(K) \subset Q(K), \quad Q(K)\,\mathrm{I} \subset \mathrm{G}(K).$$

Theorem

The previous theorem holds for the resulting HDG method.

The incompressible Navier-Stokes equations. (N.C. Nguyen, J.Peraire and B.C., Math. Comp., JCP, 2011.)

The model problem.

Consider the model problem:

$$\begin{aligned} -\nu\Delta u + \nabla\cdot(u\otimes u) + \nabla p &= f && \text{in } \Omega,\\ \nabla\cdot u &= 0 && \text{on } \Omega,\\ \widehat{u} &= u_D && \text{on } \partial\Omega, \end{aligned}$$

where $\langle u_D\cdot\mathbf{n}, 1\rangle_{\partial\Omega} = 0$ and $(p,1)_\Omega = 0$.

The incompressible Navier-Stokes equations.

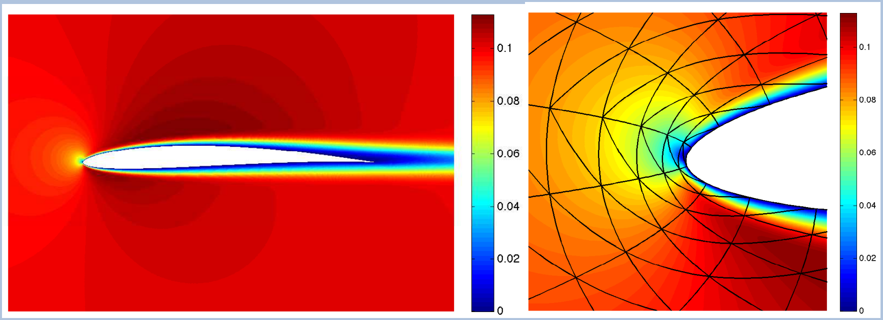
Compact form of the HDG methods.

$(\mathrm{L}_h, \mathbf{u}_h, p_h, \widehat{\mathbf{u}}_h)$ is the element of $\mathrm{G}_h \times \mathbf{V}_h \times Q_h \times \mathbf{M}_h$ solving

$$
\begin{aligned}
(\mathrm{L}_h, \mathrm{G})_{\Omega_h} + (\mathbf{u}_h, \nabla\cdot\mathrm{G})_{\Omega_h} - \langle \widehat{\mathbf{u}}_h, \mathrm{G}\mathbf{n}\rangle_{\partial\Omega_h} &= 0,\\
(\nu\mathrm{L}_h, \nabla\mathbf{v})_{\Omega_h} - (\mathbf{u}_h\otimes\mathbf{u}_h, \nabla\mathbf{v})_{\Omega_h} \qquad&\\
-(p_h, \nabla\cdot\mathbf{v})_{\Omega_h} - \langle \nu\widehat{\mathrm{L}}_h\mathbf{n} + \widehat{\mathbf{u}}_h\,\widehat{\mathbf{u}}_h\cdot\mathbf{n} - \widehat{p}_h\mathbf{n}, \mathbf{v}\rangle_{\partial\Omega_h} &= (\mathbf{f}, \mathbf{v})_{\Omega_h},\\
-(\mathbf{u}_h, \nabla q)_{\Omega_h} + \langle \widehat{\mathbf{u}}_h\cdot\mathbf{n}, q\rangle_{\partial\Omega_h} &= 0,\\
\langle -\nu\widehat{\mathrm{L}}_h\mathbf{n} + \widehat{\mathbf{u}}_h\,\widehat{\mathbf{u}}_h\cdot\mathbf{n} + \widehat{p}_h\,\mathbf{n}, \boldsymbol{\mu}\rangle_{\partial\Omega_h\setminus\partial\Omega} &= 0\\
\langle \widehat{\mathbf{u}}_h, \boldsymbol{\mu}\rangle_{\partial\Omega} &= \langle \mathbf{u}_D, \boldsymbol{\mu}\rangle_{\partial\Omega}\\
(p_h, 1)_{\Omega} &= 0,
\end{aligned}
$$

for all $(\mathrm{G}, \mathbf{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \mathbf{V}_h \times Q_h \times \mathbf{M}_h$, where

$$
-\nu\widehat{\mathrm{L}}_h\mathbf{n} + \widehat{p}_h\mathbf{n} = -\nu\mathrm{L}_h\mathbf{n} + p_h\mathbf{n} + \nu\,\tau\,(\mathbf{u}_h - \widehat{\mathbf{u}}_h) \quad \text{on } \partial\Omega_h.
$$

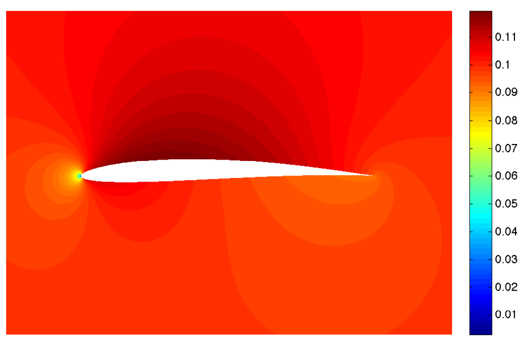


Viscous flow over a Kármán-Trefftz airfoil: $M_\infty = 0.1$, $Re = 4000$ and $\alpha = 0$. Mach number distribution (left) and detail of the mesh and Mach number solution near the leading edge region (right) using fourth order polynomial approximations.

(N.-C. Nguyen, J. Peraire and B.C., 2011.)

The Euler equations of gas dynamics.

A numerical example.



Inviscid flow over a Kármán-Trefftz airfoil: $M_\infty = 0.1$, $\alpha = 0$. Detail of the mesh employed (left) and Mach number contours of the solution using fourth order polynomial approximations (right).

(N.-C. Nguyen, J. Peraire and B.C., 2011.)

Ongoing work and open problems

- Other stabilization functions? Other choices of local spaces?
- Superconvergence for pyramidal, hexahedral elements?
- A posteriori error estimates: Only in terms of $u_h - \widehat{u}_h$ and τ?
- Efficient solvers: Domain decomposition methods? Efficient preconditionners?
- Stokes flow: Superconvergence with other formulations?
- Solid mechanics: Optimal convergence for all variables?
- Linear transport: Which unknowns superconverge? EDG methods?
- HDG methods for KdV equations: Superconvergence?
- Nonlinear hyperbolic conservation laws: How to deal with shocks?