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Immersed Interface/Boundary Method

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Introduction

The immersed interface method (IIM) is a numerical method for solving interface problems or problems on irregular domains. Interface problems are considered as partial differential equations (PDEs) with discontinuous coefficients, multi-physics, and/or singular sources along a co-dimensional space. The IIM was originally introduced by LeVeque and Li [7] and Li [8] and further developed in [1, 11]. A monograph of IIM has been published by SIAM in 2006 [12].

The original motivation of the immersed interface method is to improve accuracy of Peskin's immersed boundary (IB) method and to develop a higher-order method for PDEs with discontinuous coefficients. The IIM method is based on uniform or adaptive Cartesian/polar/spherical grids or triangulations. Standard finite difference or finite element methods are used away from interfaces or boundaries. A higher-order finite difference or finite element schemes are developed near or on the interfaces or boundaries according to the interface conditions, and it results in a higher accuracy in the entire domain. The method employs continuation of the solution from the one side to the other side of the domain separated by the interface. The continuation procedure uses the multivariable Taylor's expansion of the solution at selected interface points. The Taylor coefficients are then determined by incorporating the interface conditions and the equation. The necessary interface conditions are derived from the physical interface conditions.

Since interfaces or irregular boundaries are one dimensional lower than the solution domain, the extra costs in dealing with interfaces or irregular boundaries are generally insignificant. Furthermore, many available software packages based on uniform Cartesian/polar/spherical grids, such as FFT and fast Poisson solvers, can be applied easily with the immersed interface method. Therefore, the immersed interface method is simple enough to be implemented by researchers and graduate students who have reasonable background in finite difference or finite element methods, but it is powerful enough to solve complicated problems with a high-order accuracy.

Immersed Boundary Method and Interface Modeling

The immersed boundary (IB) method was originally introduced by Peskin [22, 23] for simulating flow patterns around heart valves and for studying blood flows in a heart [24]. First of all, the immersed boundary method is a *mathematical model* that describes elastic structures (or membranes) interacting with fluid flows. For instance, the blood flows in a heart can be considered as a Newtonian fluid governed by the Navier-Stokes equations

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)+\nabla p=\mu\Delta\mathbf{u}+\mathbf{F},$$

(1)

with the incompressibility condition $\nabla\cdot\mathbf{u}=0$, where ρ is fluid density, $\mathbf{u}$ fluid velocity, p pressure, and μ fluid viscosity. The geometry of the heart is complicated and is moving with time, so are the heart valves, which makes it difficult to simulate the flow patterns around the heart valves. In the immersed boundary model, the flow equations are extended to a rectangular box (domain) with a periodic boundary condition; the heart boundary and valves are modeled as elastic band that exerts force on the fluid. The immersed structure is typically represented by a collection of interacting particles X_k with a prescribed force law. Let $\delta(\mathbf{x})$ be the Dirac delta function. In Peskin's original immersed boundary model, the force is considered as source distribution along the boundary of the heart and thus can be written as

$$\mathbf{F}(\mathbf{x},t)=\int_{\Gamma(\mathbf{s},t)}\mathbf{f}(\mathbf{s},t)\delta(\mathbf{x}-\mathbf{X}(\mathbf{s},t))d\mathbf{s},$$

(2)

where $\Gamma(\mathbf{s},t)$ is the surface parameterized by $\mathbf{s}$ which is one dimensional in 2D and two dimensional in 3D, say a heart boundary, $\mathbf{f}(\mathbf{s},t)$ is the force density. Since the boundary now is immersed in the entire domain, it is called the *immersed boundary*. The system is closed by requiring that the elastic immersed boundary moves at the local fluid velocity:

$$\begin{aligned}\frac{d\mathbf{X}(\mathbf{s},t)}{dt}&=\mathbf{u}(\mathbf{X}(\mathbf{s},t),t)\\&=\int\mathbf{u}(\mathbf{x},t)\,\delta\,(\mathbf{x}-\mathbf{X}(\mathbf{s},t),t)\,d\mathbf{x},\end{aligned}$$

(3)

here the integration is over the entire domain.

For an elastic material, as first considered by Peskin, the force density is given by

$$\mathbf{f}(\mathbf{s},t) = \frac{\partial \mathbf{T}}{\partial \mathbf{s}}\,\tau, \qquad \mathbf{T}(\mathbf{s},t) = \sigma\left(\left|\frac{\partial \mathbf{X}}{\partial \mathbf{s}}\right| - 1\right),$$

(4)

the unit tangent vector $\tau(\mathbf{s},t)$ is given by $\tau(\mathbf{s},t) = \frac{\partial \mathbf{X}/\partial \mathbf{s}}{|\partial \mathbf{X}/\partial \mathbf{s}|}$. The tension $\mathbf{T}$ assumes that elastic fiber band obeys a linear Hooke's law with stiffness constant σ. For different applications, the key of the immersed boundary method is to derive the force density.

In Peskin's original IB method, the blood flow in a heart is embedded in a rectangular box with a periodic boundary condition. In numerical simulations, a uniform Cartesian grid (x_i, y_j, z_k) can be used. An important feature of the IB method is to use a discrete delta function $\delta_h(\mathbf{x})$ to approximate the Dirac delta function $\delta(\mathbf{x})$. There are quite a few discrete delta functions $\delta_h(\mathbf{x})$ that have been developed in the literature. In three dimensions, often a discrete delta function $\delta_h(\mathbf{x})$ is a product of one-dimensional ones,

$$\delta_h(\mathbf{x}) = \delta_h(x)\delta_h(y)\delta_h(z).$$

(5)

A traditional form for $\delta_h(x)$ was introduced in [24]:

$$\delta_h(x) = \begin{cases} \frac{1}{4h}\left(1+\cos(\pi x/2h)\right), & \text{if } |x| < 2h, \\ 0, & \text{if } |x| \geq 2h. \end{cases}$$

(6)

Another commonly used one is the hat function:

$$\delta_h(x) = \begin{cases} (h-|x|)/h^2, & \text{if } |x| < h, \\ 0, & \text{if } |x| \geq h. \end{cases}$$

(7)

With Peskin's discrete delta function approach, one can discretize a source distribution on a surface Γ as

$$\mathbf{F}_{ijk} = \sum_{l=1}^{N_b} \mathbf{f}(\mathbf{s}_l)\, \delta_h(x_i - X_\ell)\, \delta_h(y_j - Y_\ell)\, \delta_h(z_k - Z_\ell)\Delta \mathbf{s}_l,$$

(8)

where N_b is the number of discrete points $\{(X_\ell, Y_\ell, Z_\ell)\}$ on the surface $\Gamma(\mathbf{s},t)$. In this way, the singular source is distributed to the nearby grid points in a neighborhood of the immersed boundary $\Gamma(\mathbf{s},t)$. The discrete delta function approach cannot achieve second-order or higher accuracy except when the interface is aligned with a grid line.

In the immersed boundary method, we also need to interpolate the velocity at grid points to the immersed boundary corresponding to (3). This is done again through the discrete delta function

$$\begin{aligned} u(X,Y,Z) \quad &= \quad \textstyle\sum_{ijk} u(x_i,y_j,z_k)\delta_h(x_i - X) \\ &\qquad \delta_h(y_j - Y)\delta_h(z_k - Z)h_x h_y h_z \end{aligned}$$

(9)

assume (X, Y, Z) is a point on the immersed boundary $\Gamma(\mathbf{s},t)$, h_x, h_y, h_z are mesh sizes in each coordinate direction. Once the velocity is computed, the new location of the immersed boundary is updated through (3). Since the flow equation is defined on a rectangular domain, standard numerical methods can be applied. For many application problems in mathematical biology, the projection method is used for small to modest Reynolds numbers.

The immersed boundary method is simple and robust. It has been combined with and with adaptive mesh refinement [26, 27]. A few IB packages are available [3]. The IB method has been applied to many problems in mathematical biology and computational fluid mechanics. There are a few review articles on IB method given. Among them are the one given by Peskin in [25] and Mittal and Iaccarina [19] that highlighted the applications of IB method on computational fluid dynamics problems. The immersed boundary method is considered as a regularized method, and it is believed to be first-order accurate for the velocity, which has been confirmed by many numerical simulations and been partially proved [20].

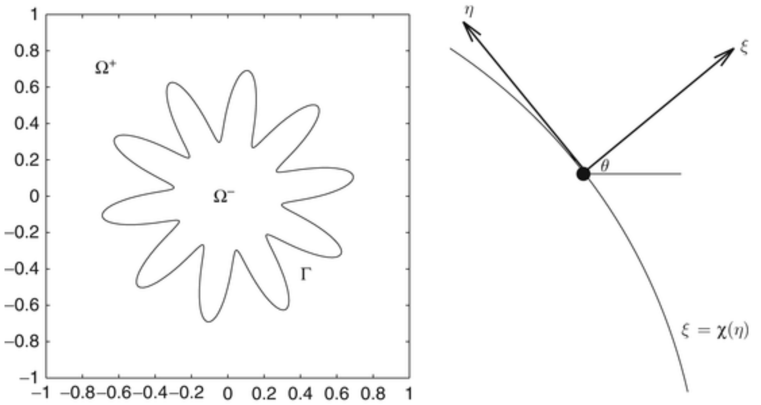
The Immersed Interface Method

We describe the second immersed interface method for a scalar elliptic equation in two-dimensional domain, and we refer to [17] and references therein for general equations, fourth-order method, and the three-dimensional case. A simplified Peskin's model can be rewritten as a Poisson equation of the form:

$$\begin{aligned} &\nabla \cdot (\beta(\mathbf{x})\nabla u) - \sigma(\mathbf{x})\,u = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega - \Gamma, \\ &[u]|_\Gamma = 0, \qquad [\beta u_n]|_\Gamma = v(s) \end{aligned}$$

(10)

where $v(\mathbf{s}) \in C^2(\Gamma)$, $f(\mathbf{x}) \in C(\Omega)$, Γ is a smooth interface, and β is a piecewise constant. Here $u_n = \frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$ is the normal derivative, and $\mathbf{n}$ is the unit normal direction, and $[u]$ is the difference of the limiting values from different side of the interface Γ, so is $[u_n]$; see Fig. 1 (Left diagram) for an illustration.



Immersed Interface/Boundary Method, Fig. 1

Left diagram: a *rectangular* domain $\Omega = \Omega^{+} \cup \Omega^{-}$ with an interface Γ. The coefficients such as $\beta(\mathbf{x})$ have a jump across the interface. *Right diagram*: the local coordinates in the normal and tangential directions, where θ is the angle between the x-axis and the normal direction

Given a Cartesian mesh $\{(x_i, y_j\}$; $x_i = i\,h_x,\ 0 \le i \le M\ \ y_j = j\,h_y,\ 0 \le j \le N$ with the mesh size $h_x,\ h_y$, the node (x_i, y_j) is irregular if the central five-point finite difference stencil at (x_i, y_j) has grid points from both side of the interface Γ, otherwise is regular. The IIM uses the standard five-point finite difference scheme at regular grid:

$$\frac{\beta_{i+\frac{1}{2},j} u_{i+1,j} \beta_{i-\frac{1}{2},j} u_{i-1,j} (\beta_{i+\frac{1}{2},j} \beta_{i-\frac{1}{2},j}) u_{ij}}{(h_x)^2} + \frac{\beta_{i,j+\frac{1}{2}} u_{i,j+1} \beta_{i,j-\frac{1}{2}} u_{i,j-1} (\beta_{i,j+\frac{1}{2}} \beta_{i,j-\frac{1}{2}}) u_{ij}}{(h_y)^2} - \sigma u_{ij} = f_{ij}.$$

(11)

The local truncation error at regular grid points is $O(h^2)$, where $h = \max\{h_x, h_y\}$.

If (x_i, y_j) is an irregular grid point, then the method of undetermined coefficients

$$\sum_{k=1}^{n_s} \gamma_k U_{i+i_k, j+j_k} - \sigma_{ij} U_{ij} = f_{ij} + C_{ij}$$

(12)

is used to determine γ_k's and C_{ij}, where n_s is the number of grid points in the finite difference stencil. We usually take $n_s = 9$. We determine the coefficients in such a way that the local truncation error

$$T_{ij} = \sum_{k=1}^{n_s} \gamma_k u\left(x_{i+i_k}, y_{j+j_k}\right) - \sigma_{ij} u(x_i, y_j) - f(x_i, y_j) - C_{ij},$$

(13)

is as small as possible in the magnitude.

We choose a projected point $\mathbf{x}_{ij}^{*} = (x_i^{*}, y_j^{*})$ on the interface Γ of irregular point (x_i, y_j). We use the Taylor expansion at $\mathbf{x}_{ij}^{*}$ in the local coordinates (ξ, η) so that (12) matches (10) up to second derivatives at $\mathbf{x}_{ij}^{*}$ from a particular side of the interface, say the − side. This will guarantee the consistency of the finite difference scheme. The local coordinates in the normal and tangential directions is

$$\xi = (x - x^{*})\ \cos\theta + (y - y^{*})\ \sin\theta,$$

$$\eta = -(x - x^{*})\ \sin\theta + (y - y^{*})\ \cos\theta,$$

(14)

where θ is the angle between the x-axis and the normal direction, pointing to the direction of a specified side. In the neighborhood of (x^{*}, y^{*}), the interface Γ can be parameterized as

$$\xi = \chi(\eta), \quad \text{with} \quad \chi(0) = 0, \quad \chi'(0) = 0.$$

(15)

The interface conditions are given

$$[u(\chi(\eta), \eta)] = 0, \ [\beta(u_{\xi}(\chi(\eta), \eta) - \chi'(\eta)\, u_{\eta}(\chi(\eta), \eta))] = \sqrt{1 + |\chi'(\eta)|^2}\, v(\eta)$$

and the curvature of the interface at (x^{*}, y^{*}) is $\chi''(0)$. The Taylor expansion of each $u(x_{i+i_k}, y_{j+j_k})$ at $\mathbf{x}_{ij}^{*}$ can be written as

$$u(x_{i+i_k}, y_{j+j_k}) = u(\xi_k, \eta_k) = u^{\pm} + \xi_k u_{\xi}^{\pm} + \eta_k u_{\eta}^{\pm} + \frac{1}{2}\xi_k^2 u_{\xi\xi}^{\pm} + \xi_k \eta_k u_{\xi\eta}^{\pm} + \frac{1}{2}\eta_k^2 u_{\eta\eta}^{\pm} + O(h^3),$$

(16)

where the + or − superscript depends on whether (ξ_k, η_k) lies on the + or − side of Ω. Therefore the local truncation error T_{ij} can be expressed as a linear combination of the values $u^{\pm}, u_{\xi}^{\pm}, u_{\eta}^{\pm}, u_{\xi\xi}^{\pm}, u_{\xi\eta}^{\pm}, u_{\eta\eta}^{\pm}$

$$
\begin{aligned}
\mathcal{I}_{ij} \;=\;& a_1 u^- + a_2 u^+ + a_3 u^-_\xi + a_4 u^+_\xi + a_5 u^-_\eta \\
& + a_6 u^+_\eta + a_7 u^-_{\xi\xi} + a_8 u^+_{\xi\xi} + a_9 u^-_{\eta\eta} \\
& + a_{10} u^+_{\eta\eta} + a_{11} u^-_{\xi\eta} + a_{12} u^+_{\xi\eta} \\
& - \sigma u^- - f^- - C_{ij} + O(\max_k |\gamma_k| \ldots
\end{aligned}
$$

(17)

where $h = \max\{h_x, h_y\}$. We drive addition… respect to η at $\eta = 0$, and then we can expres… η) as

$$
\begin{aligned}
u^+ &= u^-, \quad u^+_\xi = \rho\, u^-_\xi + \frac{v}{\beta^+}, \quad u^+_\eta = u^-_\eta, \\
u^+_{\xi\xi} &= -\chi'' u^-_\xi + \chi'' u^+_\xi + (\rho - 1)\, u^-_{\eta\eta} + \rho u^-_{\xi\xi}, \\
u^+_{\eta\eta} &= u^-_{\eta\eta} + (u^-_\xi - u^+_\xi)\, \chi'', \\
u^+_{\xi\eta} &= \left(u^+_\eta - \rho u^-_\eta\right) \chi'' + \rho\, u^-_{\xi\eta} + \frac{v'}{\beta^+},
\end{aligned}
$$

(18)

where $\rho = \dfrac{\beta^-}{\beta^+}$. An alternative is to use a collocation method, That is, we equate the interface conditions

$$
\begin{aligned}
u^+(\xi_k, \eta_k) \;=\;& u^-(\xi_k, \eta_k), \quad \beta^+ \tfrac{\partial u^+}{\partial \nu}(\xi_k, \eta_k) \\
& - \beta^- \tfrac{\partial u^-}{\partial \nu}(\xi_k, \eta_k) = v(\xi_k, \eta_k),
\end{aligned}
$$

where (ξ_k, η_k) is the local coordinates of the three closest projection points to (x_i, y_j) along the equation at (x_i^*, y_j^*); $[\beta(u_{\xi\xi} + u_{\eta\eta})] = 0$. In this way one can avoid the tangential derivative the data v, especially useful for the three-dimensional case.

If we define the index sets $K^\pm = \{ k : (\xi_k, \eta_k)$ is onthe ± side of $\Gamma\}$, then a_{2j-1} terms are defined by

$$
\begin{aligned}
a_1 &= \textstyle\sum_{k \in K^-} \gamma_k, & a_3 &= \textstyle\sum_{k \in K^-} \xi_k \gamma_k, \\
a_5 &= \textstyle\sum_{k \in K^-} \eta_k \gamma_k, & a_7 &= \tfrac{1}{2} \textstyle\sum_{k \in K^-} \xi_k^2 \gamma_k, \\
a_9 &= \tfrac{1}{2} \textstyle\sum_{k \in K^-} \eta_k^2 \gamma_k, & a_{11} &= \textstyle\sum_{k \in K^-} \xi_k \eta_k \gamma_k.
\end{aligned}
$$

(19)

The a_{2j} terms have the same expressions as a_{2j-1} except the summation is taken over K^+. From (18) equating the terms in (13) for $(u^-, u^-_\xi, u^-_\eta, u^-_{\xi\xi}, u^-_{\eta\eta}, u^-_{\xi\eta})$, we obtain the linear system of equations for γ_k's:

$$
\begin{aligned}
a_1 + a_2 &= 0 \\
a_3 + \rho a_4 - a_8 \frac{[\beta]\chi''}{\beta^+} + a_{10} \frac{[\beta]\chi''}{\beta^+} &= 0 \\
a_5 + a_6 + a_{12}(1 - \rho)\, \chi'' &= 0 \\
a_7 + a_8\, \rho &= \beta^- \\
a_9 + a_{10} + a_8\, (\rho - 1) &= \beta^- \\
a_{11} + a_{12}\, \rho &= 0.
\end{aligned}
$$

(20)

Once the γ_k's are obtained, we set $C_{ij} = a_{12} \frac{v'}{\beta^+} + \frac{1}{\beta^+}(a_4 + (a_8 - a_{10})\chi'')\, v$.

Remark 1

- If $[\beta] = 0$, then the finite difference scheme is the standard one. Only correction terms need to be added at irregular grid points. The correction terms can be regarded as second-order accurate discrete delta functions.
- If $v \equiv 0$, then the correction terms are zero.
- If we use a six-point stencil and (20) has a solution, then this leads to the original IIM [7].
- For more general cases, say both σ and f are discontinuous, we refer the reader to [7, 8, 12] for the derivation.

Enforcing the Maximum Principle Using an Optimization Approach

The stability of the finite difference equations is guaranteed by enforcing the sign constraint of the discrete maximum principle; see, for example, Morton and Mayers [21]. The sign restriction on the coefficients γ_k's in (12) are

$$
\begin{aligned}
&\gamma_k \ge 0 \quad \text{if} \quad (i_k, j_k) \ne (0, 0), \\
&\gamma_k < 0 \quad \text{if} \quad (i_k, j_k) = (0, 0).
\end{aligned}
$$

(21)
We form the following constrained quadratic optimization problem whose solution is the coefficients of the finite difference equation at the irregular grid point $\mathbf{x}_{ij}$:

$$\min_{\gamma}\left\{\frac{1}{2}\|, -g\|_2^2\right\}, \quad \text{subject to} \quad A\gamma = b,$$

$$\gamma_k \geq 0, \quad \text{if} \quad (i_k, j_k) \neq (0,0); \quad \gamma_k < 0, \text{ if } (i_k, j_k) = (0,0),$$

(22)
where $\gamma = [\gamma_1, \gamma_2, \cdots, \gamma_{n_s}]^T$ is the vector composed of the coefficients of the finite difference equation; $A\gamma = b$ is the system of linear equations (20); and $g \in R^{ns}$ has the following components: $g \in R^{ns}$,

$$g_k = \frac{\beta_{i+i_k, j+j_k}}{h^2}, \quad \text{if} \quad (i_k, j_k) \in \{(-1,0), (1,0), (0,-1), (0,1)\};$$

$$g_k = -\frac{4\beta_{i,j}}{h^2}, \quad \text{if} \quad (i_k, j_k) = (0,0);$$

$$g_k = 0, \quad \text{otherwise.}$$

(23)
With the maximum principle, the second-order convergence of the IIM has been proved in [11].

Augmented Immersed Interface Method

The original idea of the augmented strategy for interface problems was proposed in [9] for elliptic interface problems with a piecewise constant but discontinuous coefficient. With a few modifications, the augmented method developed in [9] was applied to generalized Helmholtz equations including Poisson equations on irregular domains in [14]. The augmented approach for the incompressible Stokes equations with a piecewise constant but discontinuous viscosity was proposed in [18], for slip boundary condition to deal with pressure boundary condition in [17], and for the Navier-Stokes equations on irregular domains in [6].

There are at least two motivations to use augmented strategies. The first one is to get a faster algorithm compared to a direct discretization, particularly to take advantages of existing fast solvers. The second reason is that, for some interface problems, an augmented approach may be the only way to derive an accurate algorithm. This is illustrated in the augmented immersed interface method [18] for the incompressible Stokes equations with discontinuous viscosity in which the jump conditions for the pressure and the velocity are coupled together. The augmented techniques enable us to decouple the jump conditions so that the idea of the immersed interface method can be applied.

While augmented methods have some similarities to boundary integral methods or the integral equation approach to find a source strength, the augmented methods have a few special features: (1) no Green function is needed, and therefore there is no need to evaluate singular integrals; (2) there is no need to set up the system of equations for the augmented variable explicitly; (3) they are applicable to general PDEs with or without source terms; and (4) the method can be applied to general boundary conditions. On the other hand, we may need estimate the condition number of the Schur complement system and develop preconditioning techniques.

Procedure of the Augmented IIM

We explain the procedure of the augmented IIM using the fast Poisson solver on an interior domain as an illustration.

Assume we have linear partial differential equations with a linear interface or boundary condition. The the Poisson equation on an irregular domain Ω, as an example,

$$\Delta u = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad q(u, u_n) = 0, \quad \mathbf{x} \in \partial\Omega,$$

(24)
where $q(u, u_n) = 0$ is either a Dirichlet or Neumann boundary condition along the boundary $\partial\Omega$. To use an augmented approach, the domain Ω is embedded into a rectangle $\Omega \subset R$; the PDE and the source term are extended to the entire rectangle R:

$$\Delta u = \begin{cases} f, & \text{if } \mathbf{x} \in \Omega, \\ 0, & \text{if } \mathbf{x} \in R \setminus \Omega, \end{cases} \qquad \begin{aligned} [u] &= g, \text{ on } \partial\Omega, \\ [u_n] &= 0, \text{ on } \partial\Omega, \\ u &= 0, \text{ on } \partial R. \end{aligned}$$

(25)
and

$$q(u, u_n) = 0 \quad \text{on } \partial\Omega.$$

The solution u to (25) is a functional $u(g)$ of g. We determine g such that the solution $u(g)$ satisfies the boundary condition $q(u, u_n) = 0$. Note that, given g, we can solve (25) using the immersed interface method with a single call to a fast Poisson solver.

On a Cartesian mesh (x_i, y_j), $i = 0, 1, \cdots, M$, $j = 0, 1, \cdots, N$, $M \sim N$, we use U and G to represent the discrete solution to (25). Note that the dimension of U is $O(N^2)$ while that of G is of $O(N)$. The augmented IIM can be written as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U \\ G \end{bmatrix} = \begin{bmatrix} F \\ Q \end{bmatrix},$$
(26)

where A is the matrix formed from the discrete five-point Laplacian; BG are correction terms due to the jump in u, and the boundary condition is discretized by an interpolation scheme $CU + DG = Q$, corresponding to the boundary condition $q(u, u_n) = 0$. The main reason to use an augmented approach is to take advantage of fast Poisson solvers. Eliminating U from (26) gives a linear system for G, the *Schur complement system*,

$$(D - CA^{-1}B)G = Q - CA^{-1}F \stackrel{\text{def}}{=} F_2.$$
(27)

This is an $N_b \times N_b$ system for $\mathbf{G}$, a much smaller linear system compared to the one for U, where N_b is the dimension of G. If we can solve the Schur complement system efficiently, then we obtain the solution of the original problem with one call to the fast Poisson solver. There are two approaches to solve the Schur complement system. One is the GMRES iterative method; the other one is a direct method such as the *LU* decomposition. In either of the cases, we need to know how to find the matrix vector multiplication without forming the sub-matrices A^{-1}, B, C, D explicitly. That is, first we set $G = 0$ and solve the first equation of the (26), to get $U(0) = A^{-1}F$. For a given G the residual vector of the boundary condition is then given by

$$R(G) = C(U(0) - U(G)) + DG - Q.$$

Remark 2

For different applications, the augmented variable(s) can be chosen differently but the above procedure is the same. For some problems, if we need to use the same Schur complement at every time step, it is then more efficient to use the *LU* decomposition just once. If the Schur complement is varying or only used a few times, then the GMRES iterative method may be a better option. One may need to develop efficient preconditioners for the Schur complement.

Immersed Finite Element Method (IFEM)

The IIM has also been developed using finite element formulation as well, which is preferred sometimes because there is rich theoretical foundation based on Sobolev space, and finite element approach may lead to a better conditioned system of equations. Finite element methods have less regularity requirements for the coefficients, the source term, and the solution than finite difference methods do. In fact, the weak form for one-dimensional elliptic interface problem $(\beta u')' - \sigma u = f(x) + v\delta(x - \alpha), \ 0 < x < 1$ with homogeneous Dirichlet boundary condition is

$$\begin{aligned} \int_0^1 (\beta u'\phi' - \sigma uv)\, dx &= -\int_0^1 f\phi dx + v\phi(\alpha), \\ & \forall \phi \in H_0^1(0,1). \end{aligned}$$
(28)

For two-dimensional elliptic interface problems (10), the weak form is

$$\begin{aligned} \int\int_\Omega (\beta \nabla u \nabla \phi - \sigma uv)\ d\mathbf{x} &= -\int\int_\Omega f\,\phi\, d\mathbf{x} \\ & - \int_\Gamma v\phi ds, \quad \forall \phi(\mathbf{x}) \in H_0^1(\Omega). \end{aligned}$$
(29)

Unless a body-fitted mesh is used, the solution obtained from the standard finite element method using the linear basis functions is only first-order accurate in the maximum norm. In [10], a new immersed finite element for the one-dimensional case is constructed using modified basis functions that satisfy homogeneous jump conditions. The modified basis functions satisfy

$$\phi_i(x_k) = \begin{cases} 1, & \text{if } k = i, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \ \ [\phi_i] = 0, \ \ [\beta \phi_i'] = 0.$$
(30)

Obviously, if $x_j < \alpha < x_{j+1}$, then only ϕ_j and ϕ_{j+1} need to be changed to satisfy the second jump condition. Using the method of undetermined coefficients, we can conclude that

$$\phi_j(x) = \begin{cases} 0, & 0 \le x < x_{j-1}, \\ \frac{x-x_{j-1}}{h}, & x_{j-1} \le x < x_j, \\ \frac{x_j-x}{D}+1, & x_j \le x < \alpha, \\ \frac{\rho(x_{j+1}-x)}{D}, & \alpha \le x < x_{j+1}, \\ 0, & x_{j+1} \le x \le 1, \end{cases}$$

$$\phi_{j+1}(x) = \begin{cases} 0, & 0 \le x < x_j, \\ \frac{x-x_j}{D}, & x_j \le x < \alpha, \\ \frac{\rho(x-x_{j+1})}{D}+1, & \alpha \le x < x_{j+1}, \\ \frac{x_{j+2}-x}{h}, & x_{j+1} \le x \le x_{j+2}, \\ 0, & x_{j+2} \le x \le 1. \end{cases}$$

where

$$\rho = \frac{\beta^-}{\beta^+}, \quad D = h - \frac{\beta^+ - \beta^-}{\beta^+}(x_{j+1} - \alpha).$$

Using the modified basis function, it has been shown in [10] that the Galerkin method is second-order accurate in the maximum norm. For 1D interface problems, the FD and FE methods discussed here are not very much different. The FE method likely perform better for self-adjoint problems, while the FD method is more flexible for general elliptic interface problems.

Modified Basis Functions for Two-Dimensional Problems

A similar idea above has been applied to two-dimensional problems with a uniform Cartesian triangulation [15]. The piecewise linear basis function centered at a node is defined as:

$$\phi_i(\mathbf{x}_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise,} \end{cases}$$

$$[u]|_\Gamma = 0, \quad \left[\beta \frac{\partial \phi_i}{\partial \mathbf{n}}\right]|_\Gamma = 0, \quad \phi_i|_{\partial\Omega} = 0.$$

(31)

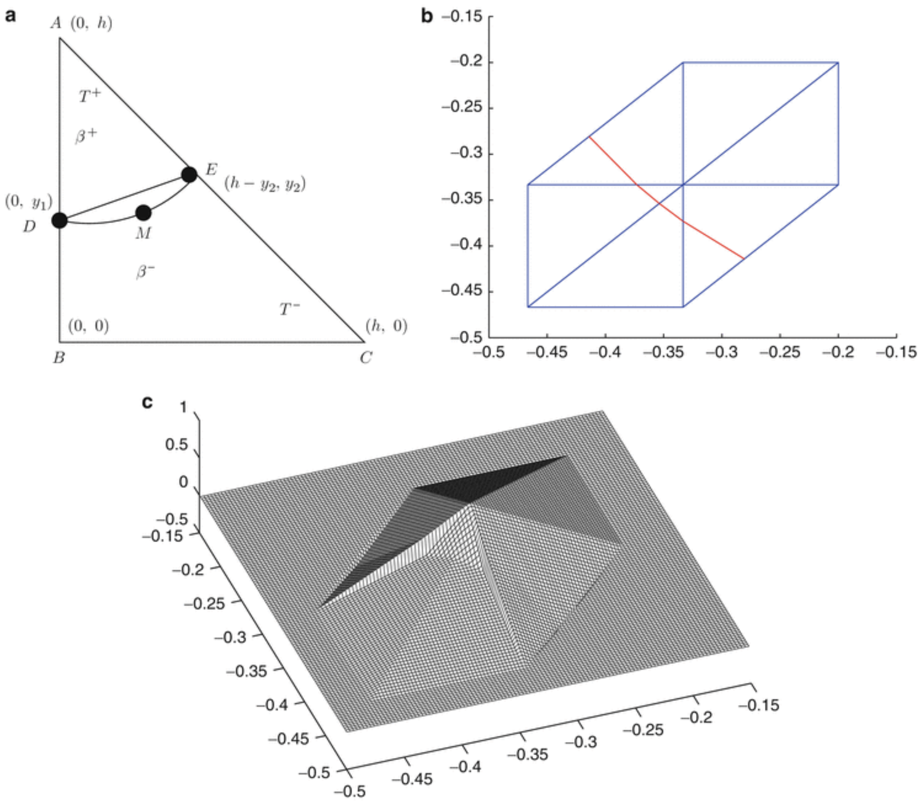
We call the space formed by all the basis function $\phi_i(\mathbf{x})$ as the immersed finite element space (IFE).

We consider a reference interface element T whose geometric configuration is given in Fig. 2a in which the curve between points D and E is a part of the interface. We assume that the coordinates at A, B, C, D, and E are

$$(0,h), \quad (0,0), \quad (h,0), \quad (0,y_1), \quad (h-y_2, y_2),$$

(32)

with the restriction $0 \le y_1 \le h, \quad 0 \le y_2 < h$.



Immersed Interface/Boundary Method, Fig. 2

(**a**) A typical *triangle* element with an interface cutting through. The curve between D and E is part of the interface curve Γ which is approximated by the *line* segment $\overline{DE}$. In this diagram, T is the *triangle* $\triangle ABC$, $T^+ = \triangle ADE$, $T^- = T - T^+$, and T_r is the region enclosed by the $\overline{DE}$ and the arc DME. (**b**) A standard domain of six *triangles* with an interface cutting through. (**c**) A global basis function on its support in the *nonconforming* immersed finite element space. The basis function has small jump across some edges

Once the values at vertices A, B, and C of the element T are specified, we construct the following piecewise linear function:

$$u(\mathbf{x}) = \begin{cases} u^+(\mathbf{x}) = a_0 + a_1 x + a_2(y - h), \\ \text{if } \mathbf{x} = (x, y) \in T^+, \\ u^-(\mathbf{x}) = b_0 + b_1 x + b_2 y, \\ \text{if } \mathbf{x} = (x, y) \in T^-, \end{cases} \tag{33a}$$

$$u^+(D) = u^-(D), \quad u^+(E) = u^-(E), \quad \beta^+ \frac{\partial u^+}{\partial n} = \beta^- \frac{\partial u^-}{\partial n}, \tag{33b}$$

where $\mathbf{n}$ is the unit normal direction of the line segment $\overline{DE}$. This is a piecewise linear function in T that satisfies the natural jump conditions along $\overline{DE}$. The existence and uniqueness of the basis functions and error estimates are given in [15].

It is easy to show that the linear basis function defined at a nodal point exists and it is unique. It has also been proved in [15] that for the solution of the interface problem (10), there is an interpolation function $u_I(\mathbf{x})$ in the IFE space that approximates $u(x)$ to second-order accuracy in the maximum norm.

However, as we can see from Fig. 2c, a linear basis function may be discontinuous along some edges. Therefore such IFE space is a nonconforming finite element space. Theoretically, it is easy to prove the corresponding Galerkin finite element method is at least first-order accurate; see [15]. In practice, its behaviors are much better than the standard finite element without any modifications. Numerically, the computed solution has super linear convergence. More theoretical analysis can be found in [2, 16].

The nonconforming immersed finite element space is also constructed for elasticity problems with interfaces in [4, 13, 29]. There are six coupled unknowns in one interface triangle for elasticity problems with interfaces.

A *conforming* IFE space is also proposed in [15]. The basis functions are still piecewise linear. The idea is to extend the support of the basis function along interface to one more triangle to keep the continuity. The conforming immersed finite element method is indeed second-order accurate. The trade-off is the increased complexity of the implementation. We refer the readers to [15] for the details. The conforming immersed finite element space is also constructed for elasticity problems with interfaces in [4].

Finally, one can construct the quadratic nonconforming element using the quadratic Taylor expansion (16) at the midpoint of the interface. The relation of coefficients of both sides is determined by the interface conditions (17). Then the quadratic element on the triangle is uniquely by the values of the basis six points of the triangle.

Hyperbolic Equations

We consider an advection equation as a model equation

$$u_t + (c(x)u)_x = 0,\ \ t > 0,\ x \in R,\ \ u(0,x) = u_0(x), \tag{34}$$

where $c = c(x) > 0$ is piecewise smooth. The second-order immersed interface method has been developed in [30]. We describe the higher-order method closely related to CIP methods [28]. CIP is one of the numerical methods that provides an accurate, less-dispersive and less-dissipative numerical solution. The method uses the exact integration in time by the characteristic method and uses the solution u and its derivative $v = u_x$ as unknowns. The piecewise cubic Hermite interpolation for each computational cell in each cell $[x_{j-1}, x_j]$ based on solution values and its derivatives at two endpoints $x_{j-1},\ x_j$. In this way the method allows us to take an arbitrary time step (no CFL limitation) without losing the stability and accuracy. That is, we use the exact simultaneous update formula for the solution u:

$$u(x_k, t+\Delta t) = \frac{c(y_k)}{c(x_k)} u(y_k, t) \tag{35}$$

and for its derivative v:

$$\begin{aligned} v(x_k, t+\Delta t) &= \left(\frac{c\prime(y_k)}{c(x_k)} - \frac{c\prime(x_k)}{c(x_k)} \right) \frac{c(y_k)}{c(x_k)} u(y_k, t) \\ &\quad + \left(\frac{c(y_k)}{c(x_k)} \right)^2 v(y_k, t). \end{aligned} \tag{36}$$

For the piecewise constant equation $u_t + c(x)\, u_x = 0$, we use the piecewise cubic interpolation: $F^-(x)$ in $[x_{j-1}, \alpha]$ and $F^+(x)$ in $[\alpha, x_j]$ of the form $F^{\pm}(x) = \sum_{k=0}^{3} a_k^{\pm}(x-\alpha)^k$. The eight unknowns are uniquely determined via the interface relations and the interpolation conditions at the interface $\alpha \in (x_{i-1}, x_i)$:

$$[u] = 0, \quad [cu_x] = 0, \quad [c^2 u_{xx}] = 0, \quad [c^3 u_{xxx}] = 0, \tag{37}$$

$$\begin{aligned} F^-(x_{j-1}) &= u_{j-1}^n, \quad F_x^-(x_{j-1}) = v_{j-1}^n, \\ F^+(x_j) &= u_j^n, \quad F_x^+(x_j) = v_j^n, \end{aligned} \tag{38}$$

Thus, we update solution (u^n, v^n) at node x_j by

$$u_j^{n+1} = F^+(x_j - c^+\Delta t), \quad v_j^{n+1} = F_x^+(x_j - c^+\Delta t). \tag{39}$$

Similarly, for (34) we have the method based on the interface conditions $[cu] = [c^2 u_x] = [c^3 u_{xx}] = [c^4 u_{xxx}] = 0$ and the updates (35)–(36).

The d'Alembert-based method for the Maxwell equation that extends our characteristic-based method to Maxwell system is developed for the piecewise constant media and then applied to Maxwell system with piecewise constant coefficients. Also, one can extend the exact time integration CIP method for equations in discontinuous media in R^2 and R^3 and the Hamilton Jacobi equation [5].

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