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An immersed interface method for the Navier-Stokes equations on irregular domains

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An augmented method based on a Cartesian grid is proposed for the incompressible Navier Stokes equations on an irregular domain. The irregular domain is embedded into a rectangular one so that a fast Poisson solver can be used in the projection method. Unlike several methods suggested in the literature that set the force strengths as unknowns, which often results an ill-conditioned linear system, we set the jump in the normal derivative of the velocity as the augmented variable. The new approach improve the condition number of the system for the augmented variable significantly. Using the immersed interface method, we achieve second order accuracy for the velocity. Numerical results and comparisons are given to validate the new method. Some interesting new numerical experiments results are also presented.

We consider the incompressible Navier-Stokes equations in a bounded domain $R \setminus \Omega$:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) + \nabla p = \mu \Delta \mathbf{u} + \mathbf{G}, \quad \mathbf{x} \in R \setminus \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{x} \in R \setminus \Omega, \tag{2}$$

$$\mathbf{u}|_{\partial R} = \mathbf{u}_{\partial R}, \qquad \mathbf{u}|_{\partial \Omega} = \mathbf{u}_{\partial \Omega}, \quad \text{BC}, \tag{3}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, \quad \text{IC}, \tag{4}$$

where $\mathbf{G}$ is an external force. We assume that R is a rectangular domain and Ω is a set of exclusions.

While there are a number of numerical methods for solving Navier Stokes equations on irregular domains, we are interested in Cartesian grid based finite difference methods. One of main motivations using Cartesian grid methods is to avoid mesh generation for moving boundary problems.

For simplicity, we assume that the domain Ω is a rectangle $[a,\ b] \times [c,\ d]$ with holes, and the spatial spacing is $h_x = (b-a)/M$, $h_y = (d-c)/N$, where M and N are the number of grid points in the x and y directions, respectively. We use a standard uniform Cartesian grid for simplicity. Our method from time t^k to t^{k+1} can be written as:

$$\frac{\mathbf{u}^* - \mathbf{u}^k}{\Delta t} + (\mathbf{u} \cdot \nabla \mathbf{u})^{k+\frac{1}{2}} = \begin{cases} \nabla p^{k-\frac{1}{2}} + \dfrac{\mu}{2} \left(\Delta \mathbf{u}^* + \Delta \mathbf{u}^k \right) + \mathbf{G}^{k+\frac{1}{2}}, & \text{if } \mathbf{x} \in R \setminus \Omega \\ \dfrac{\mu}{2} \left(\Delta \mathbf{u}^* + \Delta \mathbf{u}^k \right), & \text{if } \mathbf{x} \in \Omega \end{cases} \tag{5}$$

$$\mathbf{u}^*|_{\partial R} = \mathbf{u}_R(\mathbf{x}_{\partial R}, t^{k+1}) \tag{6}$$

$$[\mathbf{u}^*]_{\partial \Omega} = 0, \qquad \left[\frac{\partial \mathbf{u}^*}{\partial n} \right]_{\partial \Omega} = \mathbf{q}^{k+1}, \tag{7}$$

$$\begin{cases} \Delta \phi^{k+1} = \dfrac{\nabla \cdot \mathbf{u}^*}{\Delta t}, & \mathbf{x} \in R, \\ \left. \dfrac{\partial \phi^{k+1}}{\partial \mathbf{n}} \right|_{\partial R} = 0, \qquad [\phi^{k+1}]_{\partial \Omega} = 0, & \left[\dfrac{\partial \phi^{k+1}}{\partial n} \right]_{\partial \Omega} = 0, \end{cases} \tag{8}$$

$$\mathbf{u}^{k+1} = \mathbf{u}^* - \Delta t \, \nabla \phi^{k+1}, \qquad \mathbf{x} \in R, \tag{9}$$

where $(\mathbf{u} \cdot \nabla \mathbf{u})^{k+\frac{1}{2}}$ is approximated by

$$(\mathbf{u} \cdot \nabla \mathbf{u})^{k+\frac{1}{2}} = \frac{3}{2} \left(\mathbf{u}^k \cdot \nabla \right) \mathbf{u}^k - \frac{1}{2} \left(\mathbf{u}^{k-1} \cdot \nabla \right) \mathbf{u}^{k-1}. \tag{10}$$

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The solution above is a functional of the augmented variable $\mathbf{q}^{k+1}$ which should be chosen to satisfy the boundary condition

$$\mathbf{u}^{k+1}|_{\partial\Omega} = \mathbf{u}_{\partial\Omega}(\mathbf{x}_{\partial\Omega}, t^{k+1}) \tag{11}$$

The equations (5)-(11) now is a complete system for $(\mathbf{u}^*, \phi^{k+1}, \mathbf{q}^{k+1})$. After we have solved this system, then the pressure is determined from

$$\nabla p^{k+1/2} = \nabla p^{k-1/2} + \nabla\phi^{k+1}, \qquad \mathbf{x} \in R \setminus \Omega. \tag{12}$$

There are two new and novel ideas that are significant different from other methods in the literature: (1) we set the jump in the normal derivative of the velocity as augmented variables; (2),we apply the augmented method for the coupled system for $(\mathbf{u}^*, \phi^{k+1})$ and a fast Poisson solver can be applied to evaluate the Schur-complement system for the augmented variable $\mathbf{q}^{k+1}$.

Note that the equation (8) is also solved on the entire rectangular domain R so that the same fast Poisson solver can be used.

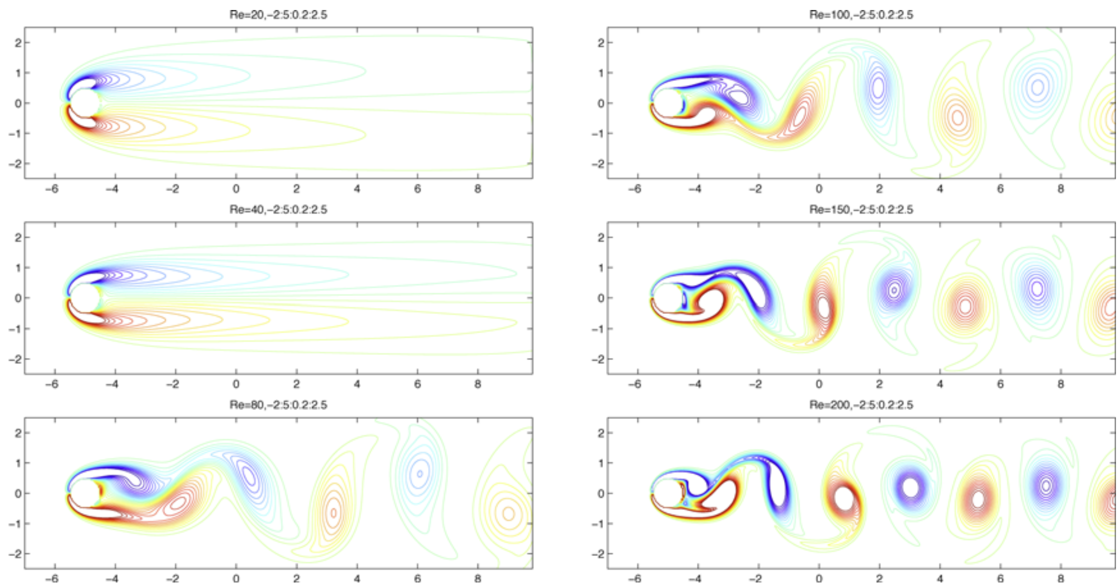

Fig. 1 Vorticity plots for $Re = 20, 40, 80, 100, 150, 200$ at time $T = 60$.

In Fig.1, we show the vorticity plots of the vorticity contours for $Re = 20, 40, 80, 100, 150, 200$. The characteristic vortex shedding has been fully developed for all the cases when $Re \geq 80$. More complete test results can be found in the complete paper.

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