Noname manuscript No.
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The scalar auxiliary variable(SAV) and FEM for the time-dependent Navier-Stokes equations

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the date of receipt and acceptance should be inserted later

1 Problem Description

We consider in this work the following unsteady incompressible Navier-Stokes system:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & in \quad \Omega\times(0,T], & (1a)\\ \nabla\cdot\mathbf{u} = 0 & in \quad \Omega\times(0,T], & (1b)\\ \mathbf{u} = \mathbf{g_D} & on \quad \partial\Omega\times(0,T], & (1c)\\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & on \quad \Omega\times\{0\}. & (1d)\end{cases}$$

We firstly describe how to reformulate the original system been composed of the equations (1a)-(1d) inspired by the work [1].

Define the total energy of the original system by

$$E(t) = E[\mathbf{u}] = \frac{1}{2}\int_{\Omega} |\mathbf{u}|^2 d\mathbf{x}. \tag{2}$$

Introduce the SAV $r(t)$ by

$$r(t) = \sqrt{E[\mathbf{u}] + \delta}, \tag{3}$$

where δ is an arbitrarily constant such that $E[\mathbf{u}] + \delta > 0$. Then

$$2r\frac{dr}{dt} = \int_{\Omega} \frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{u}d\mathbf{x} = \int_{\Omega}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right)\cdot\mathbf{u}d\mathbf{x} - \frac{1}{2}\int_{\partial\Omega}(\mathbf{n}\cdot\mathbf{u})|\mathbf{u}|^2 ds. \tag{4}$$

Note that both $r(t)$ and $E[\mathbf{u}]$ are scalar variables about space. At the same time, inspired by the work in [1] and [2], the original system can be reformulated into the following equivalent system:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \dfrac{r(t)}{\sqrt{E[\mathbf{u}]+\delta}}(\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, & (5)\\ \nabla\cdot\mathbf{u} = 0, & (6)\\ \dfrac{dr}{dt}\cdot 2r = \displaystyle\int_{\Omega}\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{r(t)}{\sqrt{E[\mathbf{u}]+\delta}}(\mathbf{u}\cdot\nabla)\mathbf{u}\right)\cdot\mathbf{u}d\mathbf{x} - \frac{1}{2}\int_{\partial\Omega}(\mathbf{n}\cdot\mathbf{u})\,|\mathbf{u}|^2 ds, & (7)\\ \mathbf{u} = \mathbf{g_D} \qquad on \quad \partial\Omega, & (8)\\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), & (9)\\ r(0) = \sqrt{E[\mathbf{u}_0(\mathbf{x})]+\delta]} = \sqrt{\dfrac{1}{2}\displaystyle\int_{\Omega}|\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x} + \delta}. & (10)\end{cases}$$

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2 Algorithm Constructions

2.1 The Semi-discrete Backward Euler Scheme

$$
\begin{cases}
\dfrac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t}-\nu\Delta\mathbf{u}^{n+1}+\dfrac{r^{n+1}}{\sqrt{\mathrm{E}[\mathbf{u}^n]+\delta}}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n+\nabla p^{n+1}=\mathbf{f}^{n+1}, & (11)\\
2r^{n+1}\dfrac{r^{n+1}-r^{n}}{\Delta t}=\displaystyle\int_{\Omega}\Big(\dfrac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t}+\dfrac{r^{n+1}}{\sqrt{\mathrm{E}[\mathbf{u}^n]+\delta}}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n\Big)\cdot\mathbf{u}^{n+1}\,\mathrm{d}\mathbf{x}, & (12)\\
\nabla\cdot\mathbf{u}^{n+1}=\mathbf{0}. & (13)
\end{cases}
$$

where $\mathbf{u}^0=\mathbf{u}_0, q^0=q(0)$, $\mathrm{E}[\mathbf{u}^n]=\int_\Omega \frac{1}{2}\left|\mathbf{u}^n\right|^2\mathrm{d}\mathbf{x}$.

We now describe how to solve the semi-discrete-in-time scheme (11)-(13) efficiently. Denote

$$
S^{n+1}=\frac{r^{n+1}}{\sqrt{\mathrm{E}(\mathbf{u}^n)+\delta}},\quad \mathbf{u}^{n+1}=\mathbf{u}_1^{n+1}+S^{n+1}\mathbf{u}_2^{n+1},\quad p^{n+1}=p_1^{n+1}+S^{n+1}p_2^{n+1}. \tag{14}
$$

Then we find that (11)–(13) are split into two generalized Stokes equations:

$$
\begin{cases}
\dfrac{\mathbf{u}_1^{n+1}-\mathbf{u}^{n}}{\Delta t}-\nu\Delta\mathbf{u}_1^{n+1}+\nabla p_1^{n+1}=\mathbf{f}^{n+1}, & (15a)\\
\nabla\cdot\mathbf{u}_1^{n+1}=\mathbf{0}, & (15b)
\end{cases}
$$

$$
\begin{cases}
\dfrac{\mathbf{u}_2^{n+1}}{\Delta t}-\nu\Delta\mathbf{u}_2^{n+1}+\nabla p_2^{n+1}=-(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n, & (16a)\\
\nabla\cdot\mathbf{u}_2^{n+1}=\mathbf{0}. & (16b)
\end{cases}
$$

From (11) and (12), we can get a quadratic equation for S^{n+1}:

$$
\alpha_2^{n+1}(S^{n+1})^2+\alpha_1^{n+1}S^{n+1}+\alpha_0^{n+1}=0, \tag{17}
$$

where

$$
\begin{aligned}
\alpha_2^{n+1}&=\frac{\triangle t}{2}(\mathrm{E}[\mathbf{u}^n]+\delta)+\nu\|\nabla\mathbf{u}_2^{n+1}\|_{L^2}^2,\\
\alpha_1^{n+1}&=2\nu(\nabla\mathbf{u}_1^{n+1},\nabla\mathbf{u}_2^{n+1})_\Omega-(\mathbf{f}^{n+1},\mathbf{u}_2^{n+1})_\Omega-\frac{2q^n}{\triangle t}\sqrt{E[\mathbf{u}^n]+\delta},\\
\alpha_0^{n+1}&=\nu\|\nabla\mathbf{u}_1^{n+1}\|_{L^2}^2-(\mathbf{f}^{n+1},\mathbf{u}_1^{n+1}).
\end{aligned}
$$

2.2 Backward Euler Leap-Frog Scheme

$$
\begin{cases}
\dfrac{\mathbf{u}^{n+1}-\mathbf{u}^{n-1}}{2\Delta t}+\dfrac{r^{n+1}}{\sqrt{\mathrm{E}[\mathbf{u}^{n-1}]+\delta}}(\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n-1}-\nu\Delta\mathbf{u}^{n+1}+\nabla p^{n+1}=\mathbf{f}^{n+1}, & (18)\\
r^{n+1}\dfrac{r^{n+1}-r^{n-1}}{\Delta t}=\displaystyle\int_{\Omega}\Big(\dfrac{\mathbf{u}^{n+1}-\mathbf{u}^{n-1}}{2\Delta t}+\dfrac{r^{n+1}}{\sqrt{\mathrm{E}[\mathbf{u}^{n-1}]+\delta}}(\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n-1}\Big)\cdot\mathbf{u}^{n+1}\,\mathrm{d}\mathbf{x}, & (19)\\
\nabla\cdot\mathbf{u}^{n+1}=\mathbf{0}, & (20)
\end{cases}
$$

the first iterative value $\mathbf{u}^1$ is computed by

$$
\frac{\mathbf{u}^1-\mathbf{u}^0}{\triangle t}+(\mathbf{u}^0\cdot\nabla)\mathbf{u}^0-\nu\Delta\mathbf{u}^1+\nabla p^1=\mathbf{f}^1.
$$

[1] X.L. Li and J. Shen. Error analysis of the SAV-MAC scheme for the Navier-Stokes equations. arXiv:1909.05131v1.

[2] L.L Lin, Z.G. Yang, and S.C. Dong. Numerical approximation of incompressible Navier-Stokes equations based on an auxiliary energy variable. J. Comput. Phys., 388:1C22, 2019

Define

$$S^{n+1}=\frac{r^{n+1}}{\sqrt{\mathrm{E}[\mathbf{u}^n]+\delta}}, \quad \text{or} \quad S^{n+1}=\frac{r^{n+1}}{\sqrt{\mathrm{E}[\mathbf{u}^{n-1}]+\delta}}, \quad \mathbf{u}^{n+1}=\mathbf{u}_1^{n+1}+S^{n+1}\mathbf{u}_2^{n+1}, \quad p^{n+1}=p_1^{n+1}+S^{n+1}p_2^{n+1}. \tag{21}$$

Now we split (18)-(20) into two generalized Stokes equations:

$$\begin{cases} \dfrac{\mathbf{u}_1^{n+1}-\mathbf{u}^{n-1}}{2\triangle t}-\nu\Delta\mathbf{u}_1^{n+1}+\nabla p_1^{n+1}=\mathbf{f}^{n+1} & \text{(22a)}\\ \nabla\cdot\mathbf{u}_1^{n+1}=\mathbf{0}, & \text{(22b)}\end{cases}$$

$$\begin{cases} \dfrac{\mathbf{u}_2^{n+1}}{2\triangle t}-\nu\Delta\mathbf{u}_2^{n+1}+\nabla p_2^{n+1}=-(\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n-1} & \text{(23a)}\\ \nabla\cdot\mathbf{u}_2^{n+1}=\mathbf{0}. & \text{(23b)}\end{cases}$$

From (18) and (19), we can get a quadratic equation for S^{n+1}:

$$\alpha_2^{n+1}(S^{n+1})^2+\alpha_1^{n+1}S^{n+1}+\alpha_0^{n+1}=0, \tag{24}$$

where

$$\begin{aligned}
\alpha_2^{n+1}&=\frac{\triangle t}{2}(\mathrm{E}[\mathbf{u}^{n-1}]+\delta)+\nu\|\nabla\mathbf{u}_2^{n+1}\|_{L^2}^2,\\
\alpha_1^{n+1}&=2\nu(\nabla\mathbf{u}_1^{n+1},\nabla\mathbf{u}_2^{n+1})_\Omega-(\mathbf{f}^{n+1},\mathbf{u}_2^{n+1})_\Omega-\frac{2r^{n-1}}{\triangle t}\sqrt{E[\mathbf{u}^{n-1}]+\delta},\\
\alpha_0^{n+1}&=\nu\|\nabla\mathbf{u}_1^{n+1}\|_{L^2}^2-(\mathbf{f}^{n+1},\mathbf{u}_1^{n+1}).
\end{aligned}$$

2.3 Crank-Nicolson Scheme

Set

$$\frac{r^{n+\frac12}}{\sqrt{E[\overline{\mathbf{u}}^{n+\frac12}]+\delta}}=S^{n+\frac12}, \tag{25}$$

$$\sqrt{E[\overline{\mathbf{u}}^{n+\frac12}]+\delta}=s_1, \tag{26}$$

$$\mathbf{u}^{n+1}=\mathbf{u}_1^{n+1}+S^{n+\frac12}\mathbf{u}_2^{n+1}, \tag{27}$$

$$p^{n+1}=p_1^{n+1}+S^{n+\frac12}p_2^{n+1}, \tag{28}$$

We can get blew two Stokes equations:

$$\begin{cases} \dfrac{\mathbf{u}_1^{n+1}}{\Delta t}-\dfrac{\nu}{2}\Delta\mathbf{u}_1^{n+1}+\dfrac12\nabla p_1^{n+1}=\mathbf{f}^{n+\frac12}+\dfrac{\nu}{2}\Delta\mathbf{u}^n-\dfrac12\nabla p^n+\dfrac{\mathbf{u}^n}{\Delta t}, & (29)\\ \nabla\cdot\mathbf{u}_1^{n+1}=0, & (30)\\ \mathbf{u}_1^{n+1}=\mathbf{g_D}^{n+1} \qquad on \quad \partial\Omega, & (31)\\ \mathbf{u}^0=\mathbf{u}_0(\mathbf{x}). & (32)\end{cases}$$

$$\begin{cases} \dfrac{\mathbf{u}_2^{n+1}}{\Delta t}-\dfrac{\nu}{2}\Delta\mathbf{u}_2^{n+1}+\dfrac12\nabla p_2^{n+1}=-(\overline{\mathbf{u}}^{n+\frac12}\cdot\nabla)\overline{\mathbf{u}}^{n+\frac12}, & (33)\\ \nabla\cdot\mathbf{u}_2^{n+1}=0, & (34)\\ \mathbf{u}_2^{n+1}=\mathbf{0} \qquad on \quad \partial\Omega, & (35)\\ \mathbf{u}^0=\mathbf{u}_0(\mathbf{x}), & (36)\end{cases}$$

where $\overline{\mathbf{u}}^{n+\frac12}=\frac32\mathbf{u}^n-\frac12\mathbf{u}^{n-1}$ for $n\geq 1$. And we get $\overline{\mathbf{u}}^{\frac12}$ to start the solving of equations (33)-(36) by the below first-order-scheme:

$$
\begin{cases}
\dfrac{\overline{\mathbf{u}}^{\frac{1}{2}}}{\Delta t/2} - \nu\Delta\overline{\mathbf{u}}^{\frac{1}{2}} + \dfrac{1}{2}\nabla p^1 = \mathbf{f}^{\frac{1}{2}} - \dfrac{1}{2}\nabla p^0 + \dfrac{\mathbf{u}^0}{\Delta t/2} - (\mathbf{u}^0\cdot\nabla)\mathbf{u}^0, & (37)\\
\nabla\cdot\overline{\mathbf{u}}^{\frac{1}{2}} = 0, & (38)\\
\overline{\mathbf{u}}^{\frac{1}{2}} = \mathbf{g_D}^{\frac{1}{2}} \qquad on \quad \partial\Omega, & (39)\\
\mathbf{u}^0 = \mathbf{u}_0(\mathbf{x}), & (40)
\end{cases}
$$

which has a local truncation error of $O(\Delta t^2)$.

It remains to determine the coefficients of an quadratic equation about S^{n+1},

$$
\begin{aligned}
\alpha_2^{n+1} &= \frac{4}{\triangle t}(\mathrm{E}[\widetilde{\mathbf{u}}^{n+1/2}]+\delta) + \frac{\nu}{4}\|\nabla\mathbf{u}_2^{n+1}\|_{L^2}^2,\\
\alpha_1^{n+1} &= \frac{\nu}{2}(\nabla(\mathbf{u}_1^{n+1}+\mathbf{u}^n),\nabla\mathbf{u}_2^{n+1})_\Omega - \frac{4r^n}{\triangle t}\sqrt{E[\widetilde{\mathbf{u}}^{n+1/2}]+\delta} - \frac{1}{2}(\mathbf{f}^{n+1/2},\mathbf{u}_2^{n+1})_\Omega,\\
\alpha_0^{n+1} &= \frac{\nu}{4}\|\nabla(\mathbf{u}_1^{n+1}+\mathbf{u}^n)\|_{L^2}^2 - \frac{1}{2}(\mathbf{f}^{n+1/2},\mathbf{u}^n+\mathbf{u}_1^{n+1}).
\end{aligned}
$$

2.4 BDF2 Scheme

We set

$$
\begin{aligned}
\frac{r^{n+1}}{\sqrt{E[\overline{\mathbf{u}}]+\delta}} &= S^{n+1}, && (41)\\
\sqrt{E[\overline{\mathbf{u}}]+\delta} &= s_2, && (42)\\
\mathbf{u}^{n+1} &= \mathbf{u}_1^{n+1} + S^{n+1}\mathbf{u}_2^{n+1}, && (43)\\
p^{n+1} &= p_1^{n+1} + S^{n+1}p_2^{n+1}, && (44)
\end{aligned}
$$

The semi-discrete BDF2 scheme is defined as

$$
\begin{cases}
\dfrac{3\mathbf{u}_1^{n+1}}{2\Delta t} - \nu\Delta\mathbf{u}_1^{n+1} + \nabla p_1^{n+1} = \mathbf{f}^{n+1} + \dfrac{\tilde{\mathbf{u}}}{2\Delta t}, & (45)\\
\nabla\cdot\mathbf{u}_1^{n+1} = 0, & (46)\\
\mathbf{u}_1^{n+1} = \mathbf{g_D}^{n+1} \qquad on \quad \partial\Omega, & (47)\\
\mathbf{u}^0 = \mathbf{u}_0(\mathbf{x}). & (48)
\end{cases}
$$

$$
\begin{cases}
\dfrac{\mathbf{u}_2^{n+1}}{2\Delta t} - \nu\Delta\mathbf{u}_2^{n+1} + \nabla p_2^{n+1} = -(\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}}, & (49)\\
\nabla\cdot\mathbf{u}_2^{n+1} = 0, & (50)\\
\mathbf{u}_2^{n+1} = \mathbf{0} \qquad on \quad \partial\Omega, & (51)\\
\mathbf{u}^0 = \mathbf{u}_0(\mathbf{x}). & (52)
\end{cases}
$$

we need to get $\mathbf{u}^1$ by solving the blew Stokes equations.

$$
\begin{cases}
\dfrac{\mathbf{u}^1}{\Delta t} - \nu\Delta\mathbf{u}^1 + \nabla p^1 = \mathbf{f}^1 + \dfrac{\mathbf{u}^0}{\Delta t} - (\mathbf{u}^0\cdot\nabla)\mathbf{u}^0, & (53)\\
\nabla\cdot\mathbf{u}^1 = 0, & (54)\\
\mathbf{u}^1 = g_D^1 \qquad on \quad \partial\Omega, & (55)\\
\mathbf{u}^0 = \mathbf{u}_0(\mathbf{x}). & (56)
\end{cases}
$$

which equals to

$$
\begin{cases}
\dfrac{3\mathbf{u}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1}}{2\Delta t} + \dfrac{r^{n+1}}{\sqrt{\mathrm{E}[\mathbf{u}^n]+\delta}}((2\mathbf{u}^n-\mathbf{u}^{n-1})\cdot\nabla)\mathbf{u}^n - \nu\Delta\mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1}, & (57)\\
\dfrac{3r^{n+1}-4r^n+r^{n-1}}{2\Delta t} = \dfrac{1}{2r^{n+1}}\displaystyle\int_\Omega \Big(\dfrac{3\mathbf{u}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1}}{2\Delta t} & \\
\qquad\qquad + \dfrac{1}{2\sqrt{\mathrm{E}[\mathbf{u}^n]+\delta}}((2\mathbf{u}^n-\mathbf{u}^{n-1})\cdot\nabla)\mathbf{u}^n\Big)\cdot\mathbf{u}^{n+1}\,\mathrm{d}\mathbf{x}, & (58)\\
\nabla\cdot\mathbf{u}^{n+1} = \mathbf{0}, & (59)
\end{cases}
$$

An quadratic equation about S_h^{n+1} is in the following form:

$$\alpha^{n+1}(S_h^{n+1})^2 + \beta^{n+1}S_h^{n+1} + \gamma^{n+1} = 0, \tag{60}$$

where

$$\begin{cases} \alpha^{n+1} = \displaystyle\int_\Omega 3(\mathbf{u}_{2h}^{n+1})^2 + 2\Delta t\mathbf{u}_{2h}^{n+1}(\overline{\mathbf{u}}_h \cdot \nabla\overline{\mathbf{u}}_h)d\mathbf{x} - 6(s_2)_h^2, & (61) \\ \beta^{n+1} = \displaystyle\int_\Omega 6\mathbf{u}_{1h}^{n+1} \cdot \mathbf{u}_{2h}^{n+1} - \mathbf{u}_{2h}^{n+1} \cdot \widetilde{\mathbf{u}}_h + 2\Delta t\mathbf{u}_{1h}^{n+1}(\overline{\mathbf{u}}_h \cdot \nabla\overline{\mathbf{u}}_h)d\mathbf{x} + 2\tilde{r}_h(s_2)_h, & (62) \\ \gamma^{n+1} = \displaystyle\int_\Omega 3(\mathbf{u}_{1h}^{n+1})^2 - \mathbf{u}_{1h}^{n+1} \cdot \widetilde{\mathbf{u}}_h d\mathbf{x} - \Delta t\int_{\partial\Omega}(\mathbf{n} \cdot \mathbf{u}_h^{n+1})|\overline{\mathbf{u}}_h|^2 d\mathbf{s}. & (63) \end{cases}$$