数值结果

2020 年 2 月 21 日

Impressible Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \mathrm{u}}{\partial t} + (\mathrm{u}\cdot\nabla)\mathrm{u} - \nu\Delta\mathrm{u} + \nabla p = \mathrm{f} & in \quad \Omega\times(0,T], \quad (1a)\\ \nabla\cdot\mathrm{u} = 0 & in \quad \Omega\times(0,T], \quad (1b)\\ \mathrm{u} = \mathrm{g_D} & on \quad \partial\Omega\times(0,T], \quad (1c)\\ \mathrm{u}(\mathrm{x},0) = \mathrm{u_0}(\mathrm{x}) & on \quad \Omega\times\{0\}, \quad (1d) \end{cases}$$

1 P2P1

1.1 Backward Euler Scheme

1.1.1 零 Dirichlet 边界条件

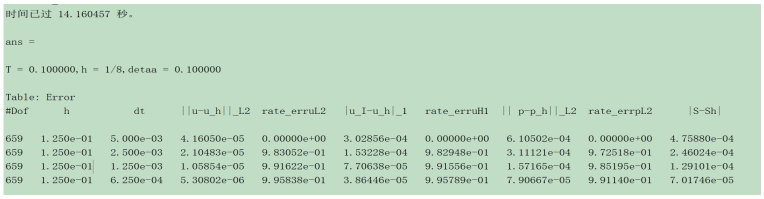
Example 1. $\Omega = [0,1]\times[0,1]$, T = 0.1s, $\delta = 0.1$

$$\begin{cases} u_1(x,y,t) = -\exp(t)x^2(x-1)^2y(y-1)(2y-1)/256\\ u_2(x,y,t) = \exp(t)x(x-1)(2x-1)y^2(y-1)^2/256\\ p(x,y,t) = \exp(t)(x^3-1/4) \end{cases}$$

Table: Error

#Dof	h	dt	||u-u_h||_L2	rate_erruL2	|u_I-u_h|_1	rate_erruH1	|| p-p_h||_L2	rate_errpL2	|S-Sh|
659	1.250e-01	3.906e-03	4.00145e-06	0.00000e+00	2.30398e-04	0.00000e+00	1.01308e-03	0.00000e+00	3.43967e-11
2467	6.250e-02	9.766e-04	2.53379e-07	3.98115e+00	3.02237e-05	2.93038e+00	2.50392e-04	2.01648e+00	1.54809e-12
9539	3.125e-02	2.441e-04	1.59349e-08	3.99103e+00	3.87371e-06	2.96389e+00	6.23831e-05	2.00496e+00	3.47278e-13
37507	1.563e-02	6.104e-05	1.00601e-09	3.98549e+00	4.95394e-07	2.96706e+00	1.55826e-05	2.00121e+00	7.12763e-14

Figure 1: space error

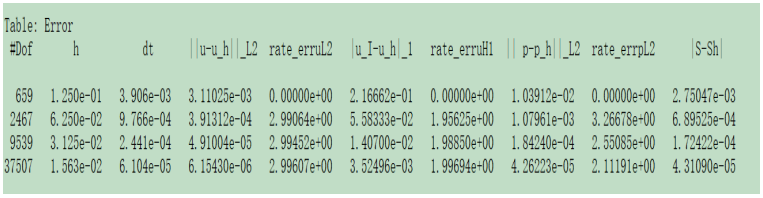


```
时间已过 14.160457 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error
#Dof      h           dt          ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2  |S-Sh|

659   1.250e-01   5.000e-03   4.16050e-05   0.00000e+00   3.02856e-04   0.00000e+00   6.10502e-04   0.00000e+00   4.75880e-04
659   1.250e-01   2.500e-03   2.10483e-05   9.83052e-01   1.53228e-04   9.82948e-01   3.11121e-04   9.72518e-01   2.46024e-04
659   1.250e-01   1.250e-03   1.05854e-05   9.91622e-01   7.70638e-05   9.91556e-01   1.57165e-04   9.85195e-01   1.29101e-04
659   1.250e-01   6.250e-04   5.30802e-06   9.95838e-01   3.86446e-05   9.95789e-01   7.90667e-05   9.91140e-01   7.01746e-05
```

Figure 2: time error

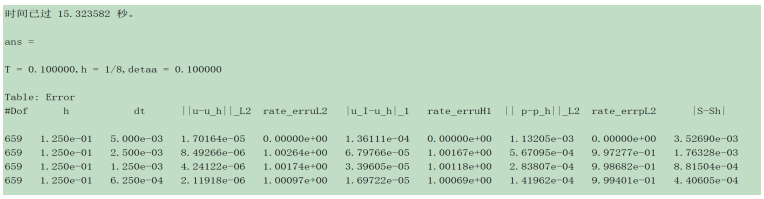
Example 2.$\Omega = [0,1]\times[0,1]$, T = 0.1s, $\delta = 0.1$

$$\begin{cases} u_1(x,y,t) = \exp(t)\sin^2(\pi x)\sin(2\pi y) \\ u_2(x,y,t) = -\exp(t)\sin(2\pi x)\sin^2(\pi y) \\ p(x,y,t) = \exp(t)(\sin(\pi y) - 2/\pi) \end{cases}$$



```
Table: Error
 #Dof      h           dt          ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2  |S-Sh|

  659   1.250e-01   3.906e-03   3.11025e-03   0.00000e+00   2.16662e-01   0.00000e+00   1.03912e-02   0.00000e+00   2.75047e-03
 2467   6.250e-02   9.766e-04   3.91312e-04   2.99064e+00   5.58333e-02   1.95625e+00   1.07961e-03   3.26678e+00   6.89525e-04
 9539   3.125e-02   2.441e-04   4.91004e-05   2.99452e+00   1.40700e-02   1.98850e+00   1.84240e-04   2.55085e+00   1.72422e-04
37507   1.563e-02   6.104e-05   6.15430e-06   2.99607e+00   3.52496e-03   1.99694e+00   4.26223e-05   2.11191e+00   4.31090e-05
```

Figure 3: space error ($\|p - p_h\|_{L^2}$ 空间收敛阶不稳定)

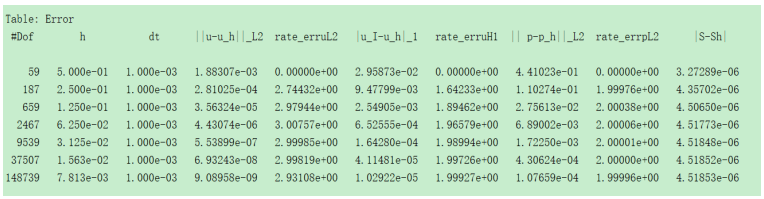


```
时间已过 15.323582 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error
#Dof      h           dt          ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2  |S-Sh|

659   1.250e-01   5.000e-03   1.70164e-05   0.00000e+00   1.36111e-04   0.00000e+00   1.13205e-03   0.00000e+00   3.52690e-03
659   1.250e-01   2.500e-03   8.49266e-06   1.00264e+00   6.79766e-05   1.00167e+00   5.67095e-04   9.97277e-01   1.76328e-03
659   1.250e-01   1.250e-03   4.24122e-06   1.00174e+00   3.39605e-05   1.00118e+00   2.83807e-04   9.98682e-01   8.81504e-04
659   1.250e-01   6.250e-04   2.11918e-06   1.00097e+00   1.69722e-05   1.00069e+00   1.41962e-04   9.99401e-01   4.40605e-04
```

Figure 4: time error

Example 3.$\Omega = [0,1]\times[0,1]$, T = 0.1s, $\delta = 0.1$

$$\begin{cases} u_1(x,y,t) = 20x^2(x-1)^2y(y-1)(2y-1)t \\ u_2(x,y,t) = -20x(x-1)(2x-1)y^2(y-1)^2t \\ p(x,y,t) = 10(2x-1)(2y-1) \end{cases}$$



```
Table: Error
 #Dof      h           dt          ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2  |S-Sh|

   59   5.000e-01   1.000e-03   1.88307e-03   0.00000e+00   2.95873e-02   0.00000e+00   4.41023e-01   0.00000e+00   3.27289e-06
  187   2.500e-01   1.000e-03   2.81025e-04   2.74432e+00   9.47799e-03   1.64233e+00   1.10274e-01   1.99976e+00   4.35702e-06
  659   1.250e-01   1.000e-03   3.56324e-05   2.97944e+00   2.54905e-03   1.89462e+00   2.75613e-02   2.00038e+00   4.50650e-06
 2467   6.250e-02   1.000e-03   4.43074e-06   3.00757e+00   6.52555e-04   1.96579e+00   6.89002e-03   2.00006e+00   4.51773e-06
 9539   3.125e-02   1.000e-03   5.53899e-07   2.99985e+00   1.64280e-04   1.98994e+00   1.72250e-03   2.00001e+00   4.51848e-06
37507   1.563e-02   1.000e-03   6.93243e-08   2.99819e+00   4.11481e-05   1.99726e+00   4.30624e-04   2.00000e+00   4.51852e-06
148739   7.813e-03   1.000e-03   9.08958e-09   2.93108e+00   1.02922e-05   1.99927e+00   1.07659e-04   1.99996e+00   4.51853e-06
```

Figure 5: space error

时间已过 14.738088 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error

#Dof	h	dt	||u-u_h||_L2	rate_erruL2	|u_I-u_h|_1	rate_erruH1	|| p-p_h||_L2	rate_errpL2	|S-Sh|
659	1.250e-01	5.000e-03	8.18055e-09	0.00000e+00	1.25709e-07	0.00000e+00	2.03921e-06	0.00000e+00	2.22315e-05
659	1.250e-01	2.500e-03	4.01139e-09	1.02810e+00	6.35742e-08	9.83572e-01	1.03945e-06	9.72189e-01	1.12098e-05
659	1.250e-01	1.250e-03	1.98698e-09	1.01352e+00	3.19734e-08	9.91569e-01	5.24685e-07	9.86293e-01	5.62843e-06
659	1.250e-01	6.250e-04	9.88989e-10	1.00655e+00	1.60342e-08	9.95720e-01	2.63583e-07	9.93195e-01	2.82009e-06

Figure 6: time error

Example 7.$\Omega = [0,1]\times[0,1]$, T = 0.1s, $\delta = 0.1$

$$\begin{cases} u_1(x,y,t) = 20x^2(x-1)^2y(y-1)(2y-1)t \\ u_2(x,y,t) = -20x(x-1)(2x-1)y^2(y-1)^2t \\ p(x,y,t) = 10(2x-1)(2y-1)t \end{cases}$$

T = 0.100000, dt = 1e-3, detaa = 0.100000

Table: Error

#Dof	h	dt	||u-u_h||_L2	rate_erruL2	|u_I-u_h|_1	rate_erruH1	|| p-p_h||_L2	rate_errpL2	|S-Sh|
187	2.500e-01	1.000e-03	2.81025e-04	0.00000e+00	9.47799e-03	0.00000e+00	1.13612e-02	0.00000e+00	4.35702e-06
659	1.250e-01	1.000e-03	3.56324e-05	2.97944e+00	2.54905e-03	1.89462e+00	2.76966e-03	2.03633e+00	4.50650e-06
2467	6.250e-02	1.000e-03	4.43074e-06	3.00757e+00	6.52555e-04	1.96579e+00	6.89420e-04	2.00625e+00	4.51773e-06
9539	3.125e-02	1.000e-03	5.53899e-07	2.99985e+00	1.64280e-04	1.98994e+00	1.72264e-04	2.00076e+00	4.51848e-06
37507	1.563e-02	1.000e-03	6.93243e-08	2.99819e+00	4.11481e-05	1.99726e+00	4.30710e-05	1.99983e+00	4.51852e-06

Figure 7: space error

时间已过 13.393792 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error

#Dof	h	dt	||u-u_h||_L2	rate_erruL2	|u_I-u_h|_1	rate_erruH1	|| p-p_h||_L2	rate_errpL2	|S-Sh|
659	1.250e-01	5.000e-03	8.18055e-09	0.00000e+00	1.25709e-07	0.00000e+00	2.03921e-06	0.00000e+00	2.22315e-05
659	1.250e-01	2.500e-03	4.01139e-09	1.02810e+00	6.35742e-08	9.83572e-01	1.03945e-06	9.72189e-01	1.12098e-05
659	1.250e-01	1.250e-03	1.98698e-09	1.01352e+00	3.19734e-08	9.91569e-01	5.24685e-07	9.86293e-01	5.62843e-06
659	1.250e-01	6.250e-04	9.88989e-10	1.00655e+00	1.60342e-08	9.95720e-01	2.63583e-07	9.93195e-01	2.82009e-06

Figure 8: time error

Example 4.$\Omega = [0,1]\times[0,1]$, T = 0.1s, $\delta = 0.1$

$$\begin{cases} u_1(x,y,t) = 2\pi sin^2(\pi x)sin(\pi y)cos(\pi y)cos(t) \\ u_2(x,y,t) = -2\pi sin(\pi x)cos(\pi x)sin^2(\pi y)cos(t) \\ p(x,y,t) = cos(\pi x)cos(\pi y) \end{cases}$$

Table: Error

#Dof	h	dt	||u-u_h||_L2	rate_erruL2	|u_I-u_h|_1	rate_erruH1	|| p-p_h||_L2	rate_errpL2	|S-Sh|
659	1.250e-01	1.563e-02	8.96676e-03	0.00000e+00	6.14689e-01	0.00000e+00	2.86317e-02	0.00000e+00	1.30541e-03
2467	6.250e-02	3.906e-03	1.12731e-03	2.99170e+00	1.58025e-01	1.95970e+00	2.29071e-03	3.64375e+00	3.55835e-04
9539	3.125e-02	9.766e-04	1.42117e-04	2.98774e+00	3.98039e-02	1.98917e+00	3.15726e-04	2.85905e+00	9.20853e-05
37507	1.563e-02	2.441e-04	1.81934e-05	2.96560e+00	9.97055e-03	1.99717e+00	7.45049e-05	2.08326e+00	2.31632e-05

Figure 9: space error ($\|p-p_h\|_{L^2}$ 空间收敛阶不稳定)

```
时间已过 18.710146 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error
#Dof      h            dt        ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

659   1.250e-01   5.000e-03   4.16050e-05   0.00000e+00   3.02856e-04   0.00000e+00   6.10502e-04   0.00000e+00   4.75880e-04
659   1.250e-01   2.500e-03   2.10483e-05   9.83052e-01   1.53228e-04   9.82948e-01   3.11121e-04   9.72518e-01   2.46024e-04
659   1.250e-01   1.250e-03   1.05854e-05   9.91622e-01   7.70638e-05   9.91556e-01   1.57165e-04   9.85195e-01   1.29101e-04
659   1.250e-01   6.250e-04   5.30802e-06   9.95838e-01   3.86446e-05   9.95789e-01   7.90667e-05   9.91140e-01   7.01746e-05
```

Figure 10: time error

1.1.2 非零 Dirichlet 边界条件

Example 5. $\Omega = [0,2]\times[-1,1]$, T = 0.1s, $\delta = 0.1$,$\mu = 0.01$.

$$\begin{cases} u = 2\cos(\pi y)\sin(\pi x)\sin t \\ v = -2\sin(\pi y)\cos(\pi x)\sin t \\ p = 2\sin(\pi y)\sin(\pi x)\cos t \end{cases}$$

```
T = 0.100000, dt = h^3*6, detaa = 0.100000


Table: Error
 #Dof      h            dt        ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

   659   2.500e-01   9.375e-02   8.99270e-02   0.00000e+00   2.36217e+00   0.00000e+00   8.35377e-02   0.00000e+00   2.99233e-01
  2467   1.250e-01   1.172e-02   1.25179e-02   2.84476e+00   7.36078e-01   1.68218e+00   1.34063e-02   2.63952e+00   4.65940e-02
  9539   6.250e-02   1.465e-03   8.20560e-04   3.93124e+00   1.00353e-01   2.87477e+00   2.99460e-03   2.16248e+00   6.01282e-03
 37507   3.125e-02   1.831e-04   5.09453e-05   4.00959e+00   1.26826e-02   2.98416e+00   7.23494e-04   2.04931e+00   7.52872e-04
148739   1.563e-02   2.289e-05   3.20904e-06   3.98874e+00   1.61251e-03   2.97548e+00   1.79250e-04   2.01301e+00   9.41205e-05
```

Figure 11: space error

```
时间已过 52.300063 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error
 #Dof      h            dt        ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

2467   1.250e-01   5.000e-03   7.28480e-05   0.00000e+00   3.98760e-03   0.00000e+00   7.82616e-04   0.00000e+00   2.05430e-02
2467   1.250e-01   2.500e-03   3.59604e-05   1.01848e+00   1.96328e-03   1.02225e+00   3.93141e-04   9.93256e-01   1.02815e-02
2467   1.250e-01   1.250e-03   1.78493e-05   1.01054e+00   9.73056e-04   1.01267e+00   1.96991e-04   9.96918e-01   5.14291e-03
2467   1.250e-01   6.250e-04   8.88992e-06   1.00562e+00   4.84254e-04   1.00676e+00   9.85959e-05   9.98530e-01   2.57201e-03
```

Figure 12: time error

Example 6. $\Omega = [0,1]\times[-0.25,0]$, T = 0.1s, $\delta = 0.1$

$$\begin{cases} u_1 = \left(x^2y^2 + e^{-y}\right)\cos(2\pi t) \\ u_2 = \left[-\frac{2}{3}xy^3 + 2 - \pi\sin(\pi x)\right]\cos(2\pi t) \\ p = -[2 - \pi\sin(\pi x)]\cos(2\pi y)\cos(2\pi t) \end{cases}$$

T = 0.100000, dt = h^3*8, detaa = 1.000000e-01

Table: Error

#Dof	h	dt	||u-u_h||_L2	rate_erruL2	|u_I-u_h|_1	rate_erruH1	|| p-p_h||_L2	rate_errpL2	|S-Sh|
197	1.250e-01	1.563e-02	3.10842e-04	0.00000e+00	1.76336e-02	0.00000e+00	5.07185e-02	0.00000e+00	3.79560e-02
679	6.250e-02	1.953e-03	3.69715e-05	3.07169e+00	4.15970e-03	2.08377e+00	6.58624e-03	2.94499e+00	5.53425e-03
2507	3.125e-02	2.441e-04	4.57165e-06	3.01563e+00	1.02325e-03	2.02332e+00	9.52862e-04	2.78912e+00	7.01307e-04
9619	1.563e-02	3.052e-05	5.69946e-07	3.00382e+00	2.54521e-04	2.00731e+00	1.68460e-04	2.49986e+00	8.78877e-05

Figure 13: space error ($\|p - p_h\|_{L^2}$ 空间收敛阶不稳定)

时间已过 6.696447 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error

#Dof	h	dt	||u-u_h||_L2	rate_erruL2	|u_I-u_h|_1	rate_erruH1	|| p-p_h||_L2	rate_errpL2	|S-Sh|
197	1.250e-01	5.000e-03	1.62076e-05	0.00000e+00	4.16475e-04	0.00000e+00	7.94964e-03	0.00000e+00	1.39298e-02
197	1.250e-01	2.500e-03	8.16098e-06	9.89859e-01	2.09641e-04	9.90308e-01	3.97084e-03	1.00145e+00	7.08091e-03
197	1.250e-01	1.250e-03	4.09471e-06	9.94981e-01	1.05170e-04	9.95202e-01	1.98439e-03	1.00075e+00	3.56860e-03
197	1.250e-01	6.250e-04	2.05090e-06	9.97504e-01	5.26719e-05	9.97613e-01	9.91934e-04	1.00038e+00	1.79027e-03

Figure 14: time error

1.2 Backward Euler Leap-Frog Method （$S^{n+1} = \frac{q^{n+1}}{\sqrt{\mathrm{E}(\mathrm{u}^{n-1})+\delta}}$）

1.2.1 零 Dirichlet 边界条件

Example 1.

T = 1.000000e-01; dt = h^2/2, detaa = 1.000000e-01

Table: Error

#Dof	h	dt	||u-u_h||_L2	rate_erruL2	|u_I-u_h|_1	rate_erruH1	|| p-p_h||_L2	rate_errpL2	|S-Sh|
187	2.500e-01	3.125e-02	5.85107e-05	0.00000e+00	1.57555e-03	0.00000e+00	1.03019e-02	0.00000e+00	1.71774e-08
659	1.250e-01	7.813e-03	3.92675e-06	3.89729e+00	2.25945e-04	2.80182e+00	2.59504e-03	1.98909e+00	5.51723e-11
2467	6.250e-02	1.953e-03	2.55126e-07	3.94405e+00	3.01658e-05	2.90499e+00	6.59765e-04	1.97573e+00	5.82578e-12
9539	3.125e-02	4.883e-04	1.78762e-08	3.83510e+00	3.86945e-06	2.96271e+00	1.64989e-04	1.99958e+00	1.43352e-12
37507	1.563e-02	1.221e-04	2.25432e-09	2.98727e+00	4.95556e-07	2.96501e+00	4.12904e-05	1.99849e+00	3.47500e-13

Figure 15: space error ($\|u - u_h\|_{L^2}$ 空间收敛阶不稳定)

时间已过 28.920554 秒。

ans =

T = 0.099687, h = 1/16, detaa = 0.100000

Table: Error

#Dof	h	dt	||u-u_h||_L2	rate_erruL2	|u_I-u_h|_1	rate_erruH1	|| p-p_h||_L2	rate_errpL2	|S-Sh|
2467	6.250e-02	5.000e-03	8.25482e-08	0.00000e+00	6.24918e-07	0.00000e+00	1.55473e-03	0.00000e+00	1.43749e-11
2467	6.250e-02	2.500e-03	4.14221e-08	9.94836e-01	3.13584e-07	9.94816e-01	7.80285e-04	9.94591e-01	7.36944e-12
2467	6.250e-02	1.250e-03	2.07486e-08	9.97388e-01	1.57077e-07	9.97380e-01	3.90874e-04	9.97295e-01	3.77609e-12
2467	6.250e-02	6.250e-04	1.03837e-08	9.98687e-01	7.86101e-08	9.98684e-01	1.95621e-04	9.98648e-01	1.94933e-12

Figure 16: time error

Example 2.

```
T = 0.100000;dt = h^3*8, detaa = 0.100000


Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

  659   1.250e-01   1.563e-02   1.14736e-02   0.00000e+00   2.24512e-01   0.00000e+00   1.55836e-02   0.00000e+00   2.13666e-02
 2467   6.250e-02   1.953e-03   1.42288e-03   3.01143e+00   5.66168e-02   1.98749e+00   1.84055e-03   3.08182e+00   2.75460e-03
 9539   3.125e-02   2.441e-04   1.73882e-04   3.03264e+00   1.41190e-02   2.00359e+00   2.39981e-04   2.93914e+00   3.44786e-04
37507   1.563e-02   3.052e-05   2.14649e-05   3.01806e+00   3.52791e-03   2.00075e+00   3.98253e-05   2.59117e+00   4.31071e-05
```

Figure 17: space error ($\|p - p_h\|_{L^2}$ 空间收敛阶不稳定)

```
时间已过 28.403521 秒。

ans =

T = 0.099687, h = 1/16, detaa = 0.100000

Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

2467   6.250e-02   5.000e-03   3.32776e-03   0.00000e+00   2.41457e-02   0.00000e+00   1.85205e-03   0.00000e+00   7.01420e-03
2467   6.250e-02   2.500e-03   1.66983e-03   9.94848e-01   1.21161e-02   9.94847e-01   9.38170e-04   9.81203e-01   3.51920e-03
2467   6.250e-02   1.250e-03   8.36425e-04   9.97393e-01   6.06899e-03   9.97393e-01   4.72189e-04   9.90486e-01   1.76261e-03
2467   6.250e-02   6.250e-04   4.18592e-04   9.98690e-01   3.03725e-03   9.98690e-01   2.36878e-04   9.95219e-01   8.82052e-04
```

Figure 18: time error

Example 3.

```
T = 0.100000;dt = h^3*8, detaa = 0.100000


Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

  659   1.250e-01   1.563e-02   1.22077e-03   0.00000e+00   9.12579e-03   0.00000e+00   2.75605e-02   0.00000e+00   7.36345e-05
 2467   6.250e-02   1.953e-03   1.52558e-04   3.00037e+00   1.29059e-03   2.82192e+00   6.89002e-03   2.00002e+00   1.70613e-05
 9539   3.125e-02   2.441e-04   1.90329e-05   3.00279e+00   2.15297e-04   2.58363e+00   1.72250e-03   2.00001e+00   2.20142e-06
37507   1.563e-02   3.052e-05   2.37664e-06   3.00151e+00   4.46684e-05   2.26900e+00   4.30624e-04   2.00000e+00   2.76418e-07
```

Figure 19: space error ($\|u - u_h\|_{H^1}$ 空间收敛阶不稳定)

```
时间已过 28.510349 秒。

ans =

T = 0.099687, h = 1/16, detaa = 0.100000

Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1  rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

2467   6.250e-02   5.000e-03   3.88770e-04   0.00000e+00   2.85697e-03   0.00000e+00   3.50247e-06   0.00000e+00   3.92882e-05
2467   6.250e-02   2.500e-03   1.94385e-04   1.00000e+00   1.42848e-03   1.00000e+00   1.97753e-06   8.24674e-01   2.11542e-05
2467   6.250e-02   1.250e-03   9.71924e-05   1.00000e+00   7.14241e-04   1.00000e+00   1.04586e-06   9.19003e-01   1.09546e-05
2467   6.250e-02   6.250e-04   4.85962e-05   1.00000e+00   3.57120e-04   1.00000e+00   5.37261e-07   9.61002e-01   5.57167e-06
```

Figure 20: time error

Example 7.

```
时间已过 13344.144350 秒。

ans =

T = 0.099945;dt = h^3*8, detaa = 0.100000


Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

  659   1.250e-01   1.563e-02   1.22077e-03   0.00000e+00   9.12579e-03   0.00000e+00   5.20836e-02   0.00000e+00   7.36345e-05
 2467   6.250e-02   1.953e-03   1.52558e-04   3.00037e+00   1.29059e-03   2.82192e+00   6.54255e-03   2.99291e+00   1.70613e-05
 9539   3.125e-02   2.441e-04   1.90329e-05   3.00279e+00   2.15297e-04   2.58363e+00   8.31534e-04   2.97601e+00   2.20142e-06
37507   1.563e-02   3.052e-05   2.37664e-06   3.00151e+00   4.46684e-05   2.26900e+00   1.10441e-04   2.91250e+00   2.76418e-07
```

Figure 21: space error ($\|u-u_h\|_{H^1}$ 空间收敛阶不稳定)

```
时间已过 30.644656 秒。

ans =

T = 0.099687, h = 1/16, detaa = 0.100000

Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

2467   6.250e-02   5.000e-03   3.88770e-04   0.00000e+00   2.85697e-03   0.00000e+00   1.66668e-02   0.00000e+00   3.92882e-05
2467   6.250e-02   2.500e-03   1.94385e-04   1.00000e+00   1.42848e-03   1.00000e+00   8.33340e-03   1.00000e+00   2.11542e-05
2467   6.250e-02   1.250e-03   9.71924e-05   1.00000e+00   7.14241e-04   1.00000e+00   4.16670e-03   1.00000e+00   1.09546e-05
2467   6.250e-02   6.250e-04   4.85962e-05   1.00000e+00   3.57120e-04   1.00000e+00   2.08335e-03   1.00000e+00   5.57167e-06
```

Figure 22: time error

Example 4.

```
T = 0.100000;dt = h^3*8, detaa = 0.100000


Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

  659   1.250e-01   1.563e-02   8.38097e-03   0.00000e+00   6.15665e-01   0.00000e+00   2.83414e-02   0.00000e+00   1.64317e-03
 2467   6.250e-02   1.953e-03   1.09372e-03   2.93787e+00   1.58009e-01   1.96214e+00   2.29071e-03   3.62904e+00   3.55835e-04
 9539   3.125e-02   2.441e-04   1.40705e-04   2.95850e+00   3.98037e-02   1.98903e+00   2.63932e-04   3.11756e+00   4.61581e-05
37507   1.563e-02   3.052e-05   1.78331e-05   2.98004e+00   9.97047e-03   1.99717e+00   4.89880e-05   2.42966e+00   5.79975e-06
```

Figure 23: space error ($\|p-p_h\|_{L^2}$ 空间收敛阶不稳定)

```
时间已过 29.664408 秒。

ans =

T = 0.099687, h = 1/16, detaa = 0.100000

Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

2467   6.250e-02   5.000e-03   8.01015e-04   0.00000e+00   5.81465e-03   0.00000e+00   9.29752e-04   0.00000e+00   8.09569e-04
2467   6.250e-02   2.500e-03   4.17536e-04   9.39929e-01   3.03092e-03   9.39935e-01   5.44630e-04   7.71570e-01   4.40190e-04
2467   6.250e-02   1.250e-03   2.13088e-04   9.70454e-01   1.54681e-03   9.70460e-01   2.92606e-04   8.96315e-01   2.29082e-04
2467   6.250e-02   6.250e-04   1.07631e-04   9.85355e-01   7.81295e-04   9.85358e-01   1.51412e-04   9.50480e-01   1.16856e-04
```

Figure 24: time error

1.2.2 非零 Dirichlet 边界条件

Example 5.

```
T = 0.100000;dt = h^3*8, detaa = 0.100000


Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2       |S-Sh|

  2467   1.250e-01   1.563e-02   4.53662e-02   0.00000e+00   6.40542e-01   0.00000e+00   1.42430e-02   0.00000e+00   8.71244e-02
  9539   6.250e-02   1.953e-03   5.58372e-03   3.02232e+00   1.03321e-01   2.63217e+00   3.21970e-03   2.14526e+00   1.58963e-02
 37507   3.125e-02   2.441e-04   6.92267e-04   3.01183e+00   1.30483e-02   2.98519e+00   7.39029e-04   2.12322e+00   2.00482e-03
148739   1.563e-02   3.052e-05   8.63576e-05   3.00293e+00   1.65743e-03   2.97684e+00   1.80243e-04   2.03569e+00   2.50910e-04
```

Figure 25: space error

```
时间已过 155.113664 秒。

ans =

T = 0.099687, h = 1/16, detaa = 0.100000

Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2       |S-Sh|

9539   6.250e-02   5.000e-03   1.41072e-02   0.00000e+00   6.26781e-02   0.00000e+00   1.65412e-03   0.00000e+00   3.90069e-02
9539   6.250e-02   2.500e-03   7.05309e-03   1.00011e+00   3.13368e-02   1.00010e+00   8.84592e-04   9.02982e-01   2.01018e-02
9539   6.250e-02   1.250e-03   3.52642e-03   1.00005e+00   1.56678e-02   1.00005e+00   4.55363e-04   9.57994e-01   1.01741e-02
9539   6.250e-02   6.250e-04   1.76318e-03   1.00003e+00   7.83376e-03   1.00003e+00   2.30788e-04   9.80448e-01   5.11472e-03
```

Figure 26: time error

Example 6.

```
T = 0.100000;dt = h^3*8, detaa = 0.100000


Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2       |S-Sh|

  659   1.250e-01   1.563e-02   6.00312e-02   0.00000e+00   3.00367e-01   0.00000e+00   1.99892e-01   0.00000e+00   5.30517e-02
 2467   6.250e-02   1.953e-03   1.02627e-02   2.54831e+00   5.14729e-02   2.54484e+00   2.54117e-02   2.97565e+00   1.45524e-02
 9539   3.125e-02   2.441e-04   1.29585e-03   2.98544e+00   6.73291e-03   2.93451e+00   3.31679e-03   2.93764e+00   1.91301e-03
37507   1.563e-02   3.052e-05   1.62288e-04   2.99727e+00   9.51366e-04   2.82316e+00   4.78032e-04   2.79461e+00   2.40803e-04
```

Figure 27: space error

```
时间已过 5.243066 秒。

ans =

T = 0.099687, h = 1/16, detaa = 0.100000

Table: Error
#Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2       |S-Sh|

679   6.250e-02   5.000e-03   1.29273e-02   0.00000e+00   6.11072e-02   0.00000e+00   1.78121e-02   0.00000e+00   2.42771e-02
679   6.250e-02   2.500e-03   6.69404e-03   9.49472e-01   3.16423e-02   9.49489e-01   8.83222e-03   1.01201e+00   1.32656e-02
679   6.250e-02   1.250e-03   3.40390e-03   9.75687e-01   1.60899e-02   9.75696e-01   4.39397e-03   1.00725e+00   6.91704e-03
679   6.250e-02   6.250e-04   1.71608e-03   9.88072e-01   8.11172e-03   9.88077e-01   2.19097e-03   1.00395e+00   3.52980e-03
```

Figure 28: time error

1.3 Crank-Nicolson Method($S^{n+1} = \frac{q^{n+1}}{\sqrt{\mathrm{E}\left(\tilde{\mathrm{u}}^{n+\frac{1}{2}}\right)+\delta}}$)

1.3.1 零 Dirichlet 边界条件

Example 1.

```
Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

   659   1.250e-01   1.000e-02   9.86499e-06   0.00000e+00   5.68008e-04   0.00000e+00   3.27474e-03   0.00000e+00   4.99600e-14
  2467   6.250e-02   1.000e-02   6.08922e-07   4.01799e+00   7.25183e-05   2.96949e+00   8.27102e-04   1.98524e+00   3.50830e-14
  9539   3.125e-02   1.000e-02   3.20389e-08   4.24836e+00   7.74886e-06   3.22629e+00   2.07862e-04   1.99244e+00   4.88498e-14
 37507   1.563e-02   1.000e-02   1.73370e-09   4.20790e+00   8.53325e-07   3.18282e+00   5.20384e-05   1.99798e+00   4.61853e-14
148739   7.813e-03   1.000e-02   1.09867e-10   3.98003e+00   1.11715e-07   2.93327e+00   1.30131e-05   1.99961e+00   4.88498e-14


ans =

dt = 0.01, detaa = 1
```

Figure 29: space error

```
时间已过 26.541212 秒。

ans =

T = 0.100000, h = 1/8, detaa = 1.000000

Table: Error
#Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

659   6.250e-02   5.000e-03   4.64240e-08   0.00000e+00   3.11023e-06   0.00000e+00   1.54426e-06   0.00000e+00   1.88738e-13
659   6.250e-02   2.500e-03   2.73685e-11   1.07281e+01   2.36228e-09   1.03626e+01   6.51720e-07   1.24459e+00   8.88178e-16
659   6.250e-02   1.250e-03   2.34051e-13   6.86955e+00   1.93046e-12   1.02570e+01   1.30345e-06   -1.00001e+00   3.55271e-15
659   6.250e-02   6.250e-04   1.14325e-13   1.03368e+00   8.59630e-13   1.16716e+00   2.60691e-06   -1.00000e+00   7.54952e-15
659   6.250e-02   3.125e-04   5.90641e-14   9.52781e-01   4.31577e-13   9.94098e-01   5.21381e-06   -1.00000e+00   1.77636e-14
```

Figure 30: time error ($\|p - p_h\|_{L^2}$ 时间收敛阶有问题)

Example 2.

```
Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

   659   1.250e-01   4.419e-02   7.52468e-03   0.00000e+00   5.19888e-01   0.00000e+00   1.78135e-02   0.00000e+00   1.29479e-04
  2467   6.250e-02   1.563e-02   9.67251e-04   2.95967e+00   1.37394e-01   1.91989e+00   2.02510e-03   3.13691e+00   1.77337e-05
  9539   3.125e-02   5.524e-03   1.21040e-04   2.99840e+00   3.46085e-02   1.98912e+00   5.40554e-04   1.90548e+00   2.24777e-06
 37507   1.563e-02   1.953e-03   1.51518e-05   2.99793e+00   8.67021e-03   1.99699e+00   8.68889e-05   2.63720e+00   2.82348e-07


ans =

dt = sqrt(h^3)*lam, lam=1.000000e+00, detaa = 10
```

Figure 31: space error($\|p - p_h\|_{L^2}$ 空间收敛阶不稳定)

```
时间已过 17.067617 秒。

ans =

T = 0.100000, h = 1/8, detaa = 1.000000

Table: Error
#Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

659   6.250e-02   5.000e-03   2.24207e-05   0.00000e+00   1.94214e-03   0.00000e+00   1.19549e-03   0.00000e+00   2.70677e-06
659   6.250e-02   2.500e-03   4.07411e-08   9.10413e+00   3.43473e-06   9.14323e+00   4.96852e-04   1.26672e+00   6.03423e-07
659   6.250e-02   1.250e-03   2.71102e-09   3.90958e+00   3.14656e-08   6.77028e+00   2.11973e-04   1.22893e+00   7.58721e-08
659   6.250e-02   6.250e-04   6.78024e-10   1.99943e+00   7.86811e-09   1.99969e+00   7.90401e-05   1.42323e+00   5.63668e-08
659   6.250e-02   3.125e-04   1.69528e-10   1.99981e+00   1.96716e-09   1.99990e+00   2.53779e-05   1.63901e+00   8.95514e-08
```

Figure 32: time error

Example 3.

```
Table: Error
 #Dof      h           dt       ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

   659   1.250e-01   1.000e-02   3.62687e-04   0.00000e+00   2.54899e-02   0.00000e+00   2.76919e-02   0.00000e+00   3.74025e-08
  2467   6.250e-02   1.000e-02   4.45302e-05   3.02587e+00   6.52554e-03   1.96576e+00   6.89413e-03   2.00602e+00   3.75593e-08
  9539   3.125e-02   1.000e-02   5.54611e-06   3.00523e+00   1.64280e-03   1.98994e+00   1.72262e-03   2.00077e+00   3.75672e-08
 37507   1.563e-02   1.000e-02   6.92920e-07   3.00071e+00   4.11481e-04   1.99726e+00   4.30627e-04   2.00009e+00   3.75676e-08
148739   7.813e-03   1.000e-02   8.66115e-08   3.00006e+00   1.02921e-04   1.99928e+00   1.07656e-04   2.00001e+00   3.75677e-08


ans =

dt = 0.01, detaa = 1
```

Figure 33: space error

```
时间已过 17.776227 秒。

ans =

T = 0.100000, h = 1/8, detaa = 1.000000

Table: Error
#Dof      h           dt       ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

659   6.250e-02   5.000e-03   5.30524e-10   0.00000e+00   3.50442e-08   0.00000e+00   1.07670e-08   0.00000e+00   9.41720e-09
659   6.250e-02   2.500e-03   6.67993e-11   2.98951e+00   5.59018e-10   5.97014e+00   2.86090e-10   5.23401e+00   2.35435e-09
659   6.250e-02   1.250e-03   1.67169e-11   1.99853e+00   1.39603e-10   2.00156e+00   7.20623e-11   1.98915e+00   5.88595e-10
659   6.250e-02   6.250e-04   4.18029e-12   1.99963e+00   3.49076e-11   1.99972e+00   1.80651e-11   1.99604e+00   1.47149e-10
659   6.250e-02   3.125e-04   1.04517e-12   1.99987e+00   8.72751e-12   1.99990e+00   4.50791e-12   2.00268e+00   3.67879e-11
```

Figure 34: time error

Example 7.

```
时间已过 7894.618601 秒。

ans =

T = 0.100000;dt = 1e-4, detaa = 1.000000

Table: Error
 #Dof      h           dt       ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

  187   2.500e-01   1.000e-04   2.81017e-04   0.00000e+00   9.47799e-03   0.00000e+00   1.13583e-02   0.00000e+00   3.65130e-12
  659   1.250e-01   1.000e-04   3.56320e-05   2.97941e+00   2.54905e-03   1.89462e+00   2.76957e-03   2.03601e+00   3.81206e-12
 2467   6.250e-02   1.000e-04   4.43072e-06   3.00756e+00   6.52555e-04   1.96579e+00   6.89419e-04   2.00621e+00   3.80496e-12
 9539   3.125e-02   1.000e-04   5.53892e-07   2.99986e+00   1.64280e-04   1.98994e+00   1.72262e-04   2.00078e+00   3.79963e-12
37507   1.563e-02   1.000e-04   6.92694e-08   2.99932e+00   4.11481e-05   1.99726e+00   4.30627e-05   2.00009e+00   3.81295e-12
```

Figure 35: space error

```
时间已过 29.528125 秒。

ans =

T = 0.100000, h = 1/16, detaa = 1.000000

Table: Error
#Dof      h           dt       ||u-u_h||_L2  rate_erruL2  |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

659   6.250e-02   5.000e-03   5.30524e-10   0.00000e+00   3.50442e-08   0.00000e+00   1.07670e-08   0.00000e+00   9.41720e-09
659   6.250e-02   2.500e-03   6.67993e-11   2.98951e+00   5.59018e-10   5.97014e+00   2.86094e-10   5.23399e+00   2.35435e-09
659   6.250e-02   1.250e-03   1.67169e-11   1.99853e+00   1.39603e-10   2.00156e+00   7.20612e-11   1.98919e+00   5.88595e-10
659   6.250e-02   6.250e-04   4.18029e-12   1.99963e+00   3.49076e-11   1.99972e+00   1.80629e-11   1.99619e+00   1.47149e-10
659   6.250e-02   3.125e-04   1.04517e-12   1.99987e+00   8.72750e-12   1.99990e+00   4.52498e-12   1.99705e+00   3.67879e-11
```

Figure 36: time error

Example 4.

```
T = 1;dt = sqrt(h^3),detaa = 1

Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2      |S-Sh|

  659   1.250e-01   4.419e-02   5.22276e-03   0.00000e+00   3.53434e-01   0.00000e+00   6.99431e-02   0.00000e+00   3.51482e-04
 2467   6.250e-02   1.563e-02   6.09517e-04   3.09907e+00   8.59874e-02   2.03924e+00   7.81204e-03   3.16241e+00   2.88294e-05
 9539   3.125e-02   5.524e-03   7.56006e-05   3.01120e+00   2.16206e-02   1.99172e+00   1.26629e-03   2.62509e+00   3.33856e-06
37507   1.563e-02   1.953e-03   9.46316e-06   2.99800e+00   5.41406e-03   1.99762e+00   2.34989e-04   2.42994e+00   4.09130e-07
```

Figure 37: space error ($\|p - p_h\|_{L^2}$ 空间收敛阶不稳定)

```
时间已过 21.397515 秒。

ans =

T = 0.100000,h = 1/8,detaa = 1.000000

Table: Error
#Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2      |S-Sh|

659   6.250e-02   5.000e-03   7.12106e-05   0.00000e+00   6.14595e-03   0.00000e+00   9.71345e-03   0.00000e+00   1.55233e-05
659   6.250e-02   2.500e-03   1.27381e-07   9.12679e+00   1.01310e-05   9.24472e+00   4.71752e-03   1.04195e+00   9.50899e-06
659   6.250e-02   1.250e-03   1.27862e-08   3.31649e+00   1.43473e-07   6.14185e+00   2.07603e-03   1.18420e+00   8.01551e-06
659   6.250e-02   6.250e-04   3.17517e-09   2.00968e+00   3.56714e-08   2.00793e+00   7.82106e-04   1.40839e+00   7.64890e-06
659   6.250e-02   3.125e-04   7.90135e-10   2.00666e+00   8.88914e-09   2.00465e+00   2.52257e-04   1.63247e+00   7.56224e-06
```

Figure 38: time error

1.3.2 非零 Dirichlet 边界条件

Example 5.

```
Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2      |S-Sh|

  2467   1.250e-01   4.419e-02   1.31284e-02   0.00000e+00   7.69882e-01   0.00000e+00   5.25659e-02   0.00000e+00   8.89879e-04
  9539   6.250e-02   1.563e-02   8.20773e-04   3.99957e+00   1.00370e-01   2.93931e+00   1.42109e-02   1.88713e+00   1.11144e-04
 37507   3.125e-02   5.524e-03   5.09354e-05   4.01024e+00   1.26827e-02   2.98439e+00   3.63502e-03   1.96696e+00   1.35171e-05
148739   1.563e-02   1.953e-03   3.20618e-06   3.98974e+00   1.61235e-03   2.97562e+00   9.14122e-04   1.99151e+00   1.67991e-06


ans =

dt = sqrt(h^3),detaa = 1
```

Figure 39: space error

```
时间已过 78.581570 秒。

ans =

T = 0.100000,h = 1/8,detaa = 1.000000

Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2      |S-Sh|

2467   6.250e-02   5.000e-03   2.78212e-06   0.00000e+00   1.58859e-04   0.00000e+00   1.71298e-06   0.00000e+00   1.10205e-05
2467   6.250e-02   2.500e-03   6.93990e-07   2.00320e+00   3.96338e-05   2.00294e+00   4.18630e-07   2.03276e+00   2.76434e-06
2467   6.250e-02   1.250e-03   1.73324e-07   2.00145e+00   9.89925e-06   2.00134e+00   1.03550e-07   2.01535e+00   7.08838e-07
2467   6.250e-02   6.250e-04   4.33103e-08   2.00069e+00   2.47372e-06   2.00064e+00   2.57585e-08   2.00720e+00   1.96165e-07
2467   6.250e-02   3.125e-04   1.08251e-08   2.00033e+00   6.18297e-07   2.00031e+00   6.42488e-09   2.00331e+00   6.82059e-08
```

Figure 40: time error

Example 6.

```
T = 0.1;dt = 0.001,detaa = 1

Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

  197   1.250e-01   1.000e-03   2.90452e-04   0.00000e+00   1.70246e-02   0.00000e+00   2.86835e-02   0.00000e+00   2.03694e-06
  679   6.250e-02   1.000e-03   3.47195e-05   3.06448e+00   4.14035e-03   2.03979e+00   7.51852e-03   1.93170e+00   2.71014e-06
 2507   3.125e-02   1.000e-03   4.27428e-06   3.02199e+00   1.02170e-03   2.01878e+00   1.93427e-03   1.95866e+00   2.73901e-06
 9619   1.563e-02   1.000e-03   5.32456e-07   3.00495e+00   2.54284e-04   2.00646e+00   4.87700e-04   1.98773e+00   2.74052e-06
37667   7.813e-03   1.000e-03   7.06073e-08   2.91477e+00   6.35224e-05   2.00110e+00   1.21095e-04   2.00985e+00   2.74061e-06
```

Figure 41: space error

```
时间已过 22.091005 秒。

ans =

T = 0.100000,h = 1/8,detaa = 1.000000

Table: Error
#Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

197   6.250e-02   5.000e-03   1.55481e-06   0.00000e+00   1.15230e-04   0.00000e+00   4.36673e-04   0.00000e+00   6.99651e-05
197   6.250e-02   2.500e-03   1.10607e-07   3.81323e+00   2.81128e-06   5.35715e+00   4.51924e-04   -4.95254e-02   1.66263e-05
197   6.250e-02   1.250e-03   2.75475e-08   2.00544e+00   6.99861e-07   2.00609e+00   9.38443e-04   -1.05419e+00   3.58626e-06
197   6.250e-02   6.250e-04   6.87456e-09   2.00258e+00   1.74653e-07   2.00258e+00   1.91389e-03   -1.02817e+00   3.64978e-07
197   6.250e-02   3.125e-04   1.71731e-09   2.00112e+00   4.36293e-08   2.00112e+00   3.84346e-03   -1.00590e+00   4.35159e-07
```

Figure 42: time error ($\|p - p_h\|_{L^2}$ 时间收敛阶有问题)

1.4 BDF2 Method($S^{n+1} = \frac{q^{n+1}}{\sqrt{\mathrm{E}(2\mathrm{u}^n - \mathrm{u}^{n-1}) + \delta}}$)

1.4.1 零 Dirichlet 边界条件

关于 BDF2 格式, 时间收敛阶用两种方法都算过了，结果都不好。算空间收敛阶的时候，只有几个算例的结果是好的，可能格式有问题。列出了 p2p1 元中几个好点的计算空间收敛阶的算例。

Example 1.

```
ans =

T =0.100000,dt = 0.001,detaa = 0.100000


Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

  187   2.500e-01   1.000e-03   6.21971e-05   0.00000e+00   1.66957e-03   0.00000e+00   4.24115e-03   0.00000e+00   1.21245e-09
  659   1.250e-01   1.000e-03   4.01084e-06   3.95487e+00   2.30938e-04   2.85390e+00   1.01545e-03   2.06233e+00   1.36998e-11
 2467   6.250e-02   1.000e-03   2.53479e-07   3.98397e+00   3.02355e-05   2.93319e+00   2.50490e-04   2.01930e+00   6.49480e-14
 9539   3.125e-02   1.000e-03   1.59375e-08   3.99137e+00   3.87427e-06   2.96425e+00   6.23922e-05   2.00532e+00   6.66134e-16
37507   1.563e-02   1.000e-03   1.00605e-09   3.98565e+00   4.95406e-07   2.96724e+00   1.55830e-05   2.00139e+00   1.55431e-15
```

Figure 43: space error

Example 3.

```
T =0.100000,dt = sqrt(h^3)/5,detaa = 0.100000


Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

  187   2.500e-01   2.500e-02   2.80784e-04   0.00000e+00   9.47808e-03   0.00000e+00   1.10274e-01   0.00000e+00   2.28177e-07
  659   1.250e-01   8.839e-03   3.46253e-05   3.01956e+00   2.47837e-03   1.93520e+00   2.75612e-02   2.00038e+00   2.82872e-09
 2467   6.250e-02   3.125e-03   4.43178e-06   2.96587e+00   6.52557e-04   1.92521e+00   6.89002e-03   2.00006e+00   6.84658e-11
 9539   3.125e-02   1.105e-03   5.74337e-07   2.94792e+00   1.63378e-04   1.99789e+00   1.72250e-03   2.00001e+00   5.64582e-12
37507   1.563e-02   3.906e-04   2.29518e-07   1.32329e+00   4.13139e-05   1.98351e+00   4.30636e-04   1.99996e+00   6.95111e-13
```

Figure 44: space error

1.4.2 非零 Dirichlet 边界条件

Example 5.

```
T =0.100000, dt = 0.001, detaa = 0.100000


Table: Error
 #Dof        h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

   659   2.500e-01   1.000e-03   1.13440e-01   0.00000e+00   3.00210e+00   0.00000e+00   7.73341e-02   0.00000e+00   1.21158e-06
  2467   1.250e-01   1.000e-03   1.29385e-02   3.13219e+00   7.58846e-01   1.98409e+00   1.29379e-02   2.57951e+00   4.24265e-06
  9539   6.250e-02   1.000e-03   8.20409e-04   3.97918e+00   1.00339e-01   2.91892e+00   2.95889e-03   2.12847e+00   4.43957e-06
 37507   3.125e-02   1.000e-03   5.09416e-05   4.00943e+00   1.26855e-02   2.98363e+00   7.21175e-04   2.03664e+00   4.42098e-06
148739   1.563e-02   1.000e-03   3.20664e-06   3.98971e+00   1.61258e-03   2.97574e+00   1.79103e-04   2.00956e+00   4.42038e-06
```

Figure 45: space error

2 P1P1

2.1 Backward Euler Scheme

2.1.1 零 Dirichlet 边界条件

Example 7.

```
时间已过 1775.056945 秒。

ans =

T = 0.100000, dt = 1e-4, detaa = 0.100000

Table: Error
 #Dof        h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

   75   2.500e-01   1.000e-04   5.22911e-03   0.00000e+00   5.21628e-02   0.00000e+00   1.36437e-01   0.00000e+00   3.75377e-07
  243   1.250e-01   1.000e-04   1.69346e-03   1.62660e+00   2.46525e-02   1.08128e+00   4.27212e-02   1.67521e+00   3.81195e-07
  867   6.250e-02   1.000e-04   4.56955e-04   1.88985e+00   1.11910e-02   1.13940e+00   1.27015e-02   1.74995e+00   4.30253e-07
 3267   3.125e-02   1.000e-04   1.17392e-04   1.96072e+00   5.26525e-03   1.08776e+00   3.77575e-03   1.75017e+00   4.47136e-07
12675   1.563e-02   1.000e-04   2.96637e-05   1.98456e+00   2.55463e-03   1.04339e+00   1.14501e-03   1.72140e+00   4.51672e-07
```

Figure 46: space error

```
时间已过 6.728721 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error
#Dof        h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

243   1.250e-01   5.000e-03   2.31031e-07   0.00000e+00   1.74324e-06   0.00000e+00   1.70052e-06   0.00000e+00   1.92514e-05
243   1.250e-01   2.500e-03   1.01606e-07   1.18510e+00   7.67031e-07   1.18442e+00   8.55804e-07   9.90627e-01   9.69994e-06
243   1.250e-01   1.250e-03   4.72348e-08   1.10506e+00   3.56697e-07   1.10459e+00   4.29510e-07   9.94588e-01   4.86473e-06
243   1.250e-01   6.250e-04   2.27165e-08   1.05611e+00   1.71579e-07   1.05583e+00   2.15188e-07   9.97097e-01   2.43221e-06
243   1.250e-01   3.125e-04   1.11321e-08   1.02901e+00   8.40905e-08   1.02886e+00   1.07706e-07   9.98497e-01   1.21222e-06
```

Figure 47: time error

2.1.2 非零 Dirichlet 边界条件

Example 6.

```
T = 0.100000, dt = 1e-5, detaa = 0.100000

Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

  81   1.250e-01   1.000e-05   1.24599e-02   0.00000e+00   3.21615e-01   0.00000e+00   1.01447e-01   0.00000e+00   1.33923e-03
 255   6.250e-02   1.000e-05   3.15099e-03   1.98341e+00   1.60682e-01   1.00113e+00   3.03889e-02   1.73911e+00   4.00191e-04
 891   3.125e-02   1.000e-05   7.91477e-04   1.99319e+00   8.02208e-02   1.00216e+00   9.26805e-03   1.71321e+00   1.26642e-04
3315   1.563e-02   1.000e-05   1.98218e-04   1.99746e+00   4.00858e-02   1.00089e+00   2.87159e-03   1.69042e+00   5.35819e-05
```

Figure 48: space error

```
时间已过 4.011139 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error
#Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

81   1.250e-01   5.000e-03   3.56173e-05   0.00000e+00   6.16566e-04   0.00000e+00   6.39397e-03   0.00000e+00   1.51646e-02
81   1.250e-01   2.500e-03   1.79842e-05   9.85848e-01   3.11183e-04   9.86493e-01   3.19193e-03   1.00228e+00   8.34395e-03
81   1.250e-01   1.250e-03   9.03618e-06   9.92946e-01   1.56320e-04   9.93260e-01   1.59468e-03   1.00116e+00   4.84684e-03
81   1.250e-01   6.250e-04   4.52921e-06   9.96455e-01   7.83439e-05   9.96610e-01   7.97021e-04   1.00058e+00   3.07655e-03
81   1.250e-01   3.125e-04   2.26741e-06   9.98212e-01   3.92184e-05   9.98289e-01   3.98431e-04   1.00029e+00   2.18601e-03
```

Figure 49: time error

2.2 Backward Euler Leap-Frog Method （$S^{n+1}=\frac{q^{n+1}}{\sqrt{\mathrm{E}(\mathrm{u}^{n-1})+\delta}}$）

2.2.1 零 Dirichlet 边界条件

Example 7.

```
时间已过 8145.203481 秒。

ans =

T = 0.099990, dt = 1e-5, detaa = 0.100000

Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

   75   2.500e-01   1.000e-05   5.22836e-03   0.00000e+00   5.21544e-02   0.00000e+00   1.36443e-01   0.00000e+00   7.65324e-08
  243   1.250e-01   1.000e-05   1.69341e-03   1.62643e+00   2.46490e-02   1.08126e+00   4.27403e-02   1.67463e+00   6.80320e-08
  867   6.250e-02   1.000e-05   4.57152e-04   1.88918e+00   1.11895e-02   1.13938e+00   1.27222e-02   1.74825e+00   8.35523e-08
 3267   3.125e-02   1.000e-05   1.17654e-04   1.95813e+00   5.26460e-03   1.08775e+00   3.79451e-03   1.74536e+00   8.87846e-08
12675   1.563e-02   1.000e-05   2.99475e-05   1.97404e+00   2.55433e-03   1.04338e+00   1.16115e-03   1.70836e+00   9.01770e-08
```

Figure 50: space error

```
时间已过 3.589477 秒。

ans =

T = 0.099844, h = 1/8, detaa = 0.100000

Table: Error
#Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

243   1.250e-01   5.000e-03   3.58202e-04   0.00000e+00   2.78670e-03   0.00000e+00   1.51367e-02   0.00000e+00   3.40653e-05
243   1.250e-01   2.500e-03   1.78924e-04   1.00143e+00   1.39205e-03   1.00135e+00   7.56838e-03   9.99999e-01   1.82987e-05
243   1.250e-01   1.250e-03   8.94232e-05   1.00063e+00   6.95740e-04   1.00059e+00   3.78419e-03   1.00000e+00   9.46223e-06
243   1.250e-01   6.250e-04   4.47026e-05   1.00029e+00   3.47804e-04   1.00027e+00   1.89209e-03   1.00000e+00   4.80555e-06
243   1.250e-01   3.125e-04   2.23492e-05   1.00014e+00   1.73886e-04   1.00013e+00   9.46047e-04   1.00000e+00   2.41755e-06
```

Figure 51: time error

2.2.2 非零 Dirichlet 边界条件

Example 6.

```
时间已过 1618.748723 秒。

ans =

T = 0.099990,dt = 1e-5,detaa = 0.100000


Table: Error
 #Dof        h             dt        ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

   81   1.250e-01   1.000e-05   1.24544e-02   0.00000e+00   3.21630e-01   0.00000e+00   1.01447e-01   0.00000e+00   1.36755e-03
  255   6.250e-02   1.000e-05   3.14386e-03   1.98605e+00   1.60689e-01   1.00113e+00   3.03815e-02   1.73946e+00   4.28350e-04
  891   3.125e-02   1.000e-05   7.84213e-04   2.00322e+00   8.02344e-02   1.00216e+00   9.25955e-03   1.71418e+00   1.55401e-04
 3315   1.563e-02   1.000e-05   1.92187e-04   2.02873e+00   4.00877e-02   1.00088e+00   2.86424e-03   1.69279e+00   8.23681e-05
```

Figure 52: space error

```
时间已过 1.992889 秒。

ans =

T = 0.099844,h = 1/8,detaa = 0.100000

Table: Error
#Dof        h             dt        ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

81   1.250e-01   5.000e-03   1.28400e-02   0.00000e+00   6.07501e-02   0.00000e+00   1.38583e-02   0.00000e+00   2.53208e-02
81   1.250e-01   2.500e-03   6.64982e-03   9.49255e-01   3.14568e-02   9.49514e-01   6.82545e-03   1.02176e+00   1.44236e-02
81   1.250e-01   1.250e-03   3.38166e-03   9.75582e-01   1.59955e-02   9.75709e-01   3.38347e-03   1.01242e+00   8.13879e-03
81   1.250e-01   6.250e-04   1.70493e-03   9.88020e-01   8.06407e-03   9.88083e-01   1.68401e-03   1.00660e+00   4.78561e-03
81   1.250e-01   3.125e-04   8.55978e-04   9.94066e-01   4.04857e-03   9.94098e-01   8.40024e-04   1.00340e+00   3.05626e-03
```

Figure 53: time error

2.3 Crank-Nicolson Method($S^{n+1} = \frac{q^{n+1}}{\sqrt{\mathrm{E}\left(\widetilde{\mathrm{u}}^{n+\frac{1}{2}}\right)+\delta}}$)

2.3.1 零 Dirichlet 边界条件

Example 7.

```
时间已过 18047.086868 秒。

ans =

T = 0.100000;dt = 1e-5,detaa = 0.100000

Table: Error
 #Dof        h             dt        ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

   75   2.500e-01   1.000e-05   5.22911e-03   0.00000e+00   5.21628e-02   0.00000e+00   1.36437e-01   0.00000e+00   1.81371e-09
  243   1.250e-01   1.000e-05   1.69345e-03   1.62660e+00   2.46525e-02   1.08128e+00   4.27212e-02   1.67521e+00   1.02730e-08
  867   6.250e-02   1.000e-05   4.56954e-04   1.88985e+00   1.11910e-02   1.13940e+00   1.27015e-02   1.74995e+00   3.13218e-09
 3267   3.125e-02   1.000e-05   1.17391e-04   1.96072e+00   5.26325e-03   1.08776e+00   3.77574e-03   1.75017e+00   8.11049e-10
12675   1.563e-02   1.000e-05   2.96637e-05   1.98456e+00   2.55463e-03   1.04339e+00   1.14501e-03   1.72140e+00   2.04020e-10
```

Figure 54: space error

```
时间已过 8.035923 秒。

ans =

T = 0.100000,h = 1/8,detaa = 1.000000

Table: Error
#Dof        h             dt        ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

243   1.250e-01   5.000e-03   1.25581e-08   0.00000e+00   9.47252e-08   0.00000e+00   3.18672e-07   0.00000e+00   6.96408e-09
243   1.250e-01   2.500e-03   3.15478e-09   1.99301e+00   2.37693e-08   1.99465e+00   8.19077e-09   5.28193e+00   9.97771e-10
243   1.250e-01   1.250e-03   7.90056e-10   1.99751e+00   5.95258e-09   1.99751e+00   2.05059e-09   1.99796e+00   5.08348e-10
243   1.250e-01   6.250e-04   1.97599e-10   1.99938e+00   1.48878e-09   1.99938e+00   5.12840e-10   1.99946e+00   8.91501e-10
243   1.250e-01   3.125e-04   4.94050e-11   1.99985e+00   3.72236e-10   1.99985e+00   1.28223e-10   1.99985e+00   9.90502e-10
```

Figure 55: time error

2.3.2 非零 Dirichlet 边界条件

Example 6.

```
时间已过 4127.210148 秒。

ans =

T = 0.100000;dt = 1e-5,detaa = 0.100000

Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2       |S-Sh|

  81   1.250e-01   1.000e-05   1.24599e-02   0.00000e+00   3.21615e-01   0.00000e+00   1.01446e-01   0.00000e+00   1.31048e-03
 255   6.250e-02   1.000e-05   3.15098e-03   1.98342e+00   1.60682e-01   1.00113e+00   3.03911e-02   1.73900e+00   3.71398e-04
 891   3.125e-02   1.000e-05   7.91470e-04   1.99320e+00   8.02208e-02   1.00216e+00   9.27109e-03   1.71284e+00   9.78380e-05
3315   1.563e-02   1.000e-05   1.98211e-04   1.99750e+00   4.00858e-02   1.00089e+00   2.87445e-03   1.68945e+00   2.47745e-05
```

Figure 56: space error

```
时间已过 7.807992 秒。

ans =

T = 0.100000, h = 1/8, detaa = 1.000000

Table: Error
#Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2       |S-Sh|

81   1.250e-01   5.000e-03   1.20321e-06   0.00000e+00   2.08269e-05   0.00000e+00   7.40849e-05   0.00000e+00   3.71212e-04
81   1.250e-01   2.500e-03   3.01366e-07   1.99730e+00   5.21948e-06   1.99647e+00   1.91959e-05   1.94838e+00   3.21933e-04
81   1.250e-01   1.250e-03   7.51928e-08   2.00285e+00   1.30231e-06   2.00284e+00   4.78970e-06   2.00279e+00   3.10771e-04
81   1.250e-01   6.250e-04   1.88523e-08   1.99585e+00   3.26510e-07   1.99587e+00   1.20076e-06   1.99599e+00   3.08461e-04
81   1.250e-01   3.125e-04   4.76171e-09   1.98519e+00   8.24662e-08   1.98525e+00   3.03197e-07   1.98562e+00   3.08122e-04
```

Figure 57: time error

3 P1P0

3.1 Backward Euler Scheme

3.1.1 零 Dirichlet 边界条件

Example 7.

```
时间已过 5958.261742 秒。

ans =

T = 0.100000, dt = 1e-4, detaa = 0.100000

Table: Error
 #Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2       |S-Sh|

  82   2.500e-01   1.000e-04   5.85996e-03   0.00000e+00   5.54005e-02   0.00000e+00   1.71294e-01   0.00000e+00   3.92849e-07
 290   1.250e-01   1.000e-04   1.94838e-03   1.58862e+00   2.53853e-02   1.12590e+00   6.66054e-02   1.36276e+00   3.66045e-07
1090   6.250e-02   1.000e-04   5.11074e-04   1.93067e+00   1.13301e-02   1.16384e+00   2.79202e-02   1.25433e+00   4.23458e-07
4226   3.125e-02   1.000e-04   1.28179e-04   1.99537e+00   5.29122e-03   1.09849e+00   1.29086e-02   1.11297e+00   4.45332e-07
16642  1.563e-02   1.000e-04   3.19232e-05   2.00549e+00   2.55980e-03   1.04757e+00   6.27044e-03   1.04170e+00   4.51218e-07
```

Figure 58: space error

```
时间已过 12.963741 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error
#Dof        h            dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2       |S-Sh|

290   1.250e-01   5.000e-03   2.59796e-07   0.00000e+00   1.95285e-06   0.00000e+00   1.62533e-06   0.00000e+00   1.85577e-05
290   1.250e-01   2.500e-03   1.13107e-07   1.19969e+00   8.50557e-07   1.19910e+00   8.06986e-07   1.01012e+00   9.34936e-06
290   1.250e-01   1.250e-03   5.22713e-08   1.11359e+00   3.93188e-07   1.11319e+00   4.02552e-07   1.00337e+00   4.68814e-06
290   1.250e-01   6.250e-04   2.50581e-08   1.06074e+00   1.88520e-07   1.06050e+00   2.01107e-07   1.00121e+00   2.34320e-06
290   1.250e-01   3.125e-04   1.22590e-08   1.03143e+00   9.22370e-08   1.03130e+00   1.00520e-07   1.00048e+00   1.16714e-06
```

Figure 59: time error

3.1.2 非零 Dirichlet 边界条件

Example 6.

```
时间已过 815.690578 秒。

ans =

T = 0.100000, dt = 1e-4, detaa = 0.100000

Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

  86   1.250e-01   1.000e-04   1.25572e-02   0.00000e+00   3.21754e-01   0.00000e+00   1.24400e-01   0.00000e+00   1.54625e-03
 298   6.250e-02   1.000e-04   3.18900e-03   1.97734e+00   1.60775e-01   1.00092e+00   4.99642e-02   1.31603e+00   6.55596e-04
1106   3.125e-02   1.000e-04   7.95676e-04   2.00285e+00   8.02398e-02   1.00265e+00   2.15576e-02   1.21270e+00   3.83855e-04
4258   1.563e-02   1.000e-04   1.98456e-04   2.00336e+00   4.00874e-02   1.00117e+00   1.01574e-02   1.08567e+00   3.12080e-04
```

Figure 60: space error

```
时间已过 6.715954 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error
#Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

86   1.250e-01   5.000e-03   3.97560e-05   0.00000e+00   6.84173e-04   0.00000e+00   6.22680e-03   0.00000e+00   1.51614e-02
86   1.250e-01   2.500e-03   2.00881e-05   9.84832e-01   3.45535e-04   9.85529e-01   3.10713e-03   1.00291e+00   8.34026e-03
86   1.250e-01   1.250e-03   1.00967e-05   9.92449e-01   1.73633e-04   9.92788e-01   1.55197e-03   1.00148e+00   4.84291e-03
86   1.250e-01   6.250e-04   5.06165e-06   9.96211e-01   8.70348e-05   9.96379e-01   7.75587e-04   1.00074e+00   3.07250e-03
86   1.250e-01   3.125e-04   2.53417e-06   9.98091e-01   4.35725e-05   9.98175e-01   3.87694e-04   1.00037e+00   2.18190e-03
```

Figure 61: time error

3.2 Backward Euler Leap-Frog Method （$S^{n+1}=\frac{q^{n+1}}{\sqrt{\mathrm{E}(\mathrm{u}^{n-1})+\delta}}$）

3.2.1 零 Dirichlet 边界条件

Example 7.

```
时间已过 22398.742079 秒。

ans =

T = 0.099990, dt = 1e-5, detaa = 0.100000


Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

   82   2.500e-01   1.000e-05   5.85908e-03   0.00000e+00   5.53914e-02   0.00000e+00   1.71292e-01   0.00000e+00   8.10238e-08
  290   1.250e-01   1.000e-05   1.94830e-03   1.58845e+00   2.53819e-02   1.12586e+00   6.66149e-02   1.36254e+00   6.41618e-08
 1090   6.250e-02   1.000e-05   5.11309e-04   1.92995e+00   1.13288e-02   1.16381e+00   2.79280e-02   1.25413e+00   8.17909e-08
 4226   3.125e-02   1.000e-05   1.28501e-04   1.99242e+00   5.29067e-03   1.09847e+00   1.29131e-02   1.11287e+00   8.83836e-08
16642   1.563e-02   1.000e-05   3.22729e-05   1.99338e+00   2.55956e-03   1.04756e+00   6.27285e-03   1.04165e+00   9.00832e-08
```

Figure 62: space error

```
时间已过 8.646034 秒。

ans =

T = 0.099844, h = 1/8, detaa = 0.100000

Table: Error
#Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2       |S-Sh|

290   1.250e-01   5.000e-03   3.50919e-04   0.00000e+00   2.72672e-03   0.00000e+00   1.48077e-02   0.00000e+00   3.28639e-05
290   1.250e-01   2.500e-03   1.75250e-04   1.00172e+00   1.36183e-03   1.00162e+00   7.40386e-03   1.00000e+00   1.76446e-05
290   1.250e-01   1.250e-03   8.75795e-05   1.00075e+00   6.80584e-04   1.00070e+00   3.70193e-03   1.00000e+00   9.12162e-06
290   1.250e-01   6.250e-04   4.37793e-05   1.00034e+00   3.40216e-04   1.00032e+00   1.85096e-03   1.00000e+00   4.63148e-06
290   1.250e-01   3.125e-04   2.18872e-05   1.00016e+00   1.70090e-04   1.00015e+00   9.25481e-04   1.00000e+00   2.32918e-06
```

Figure 63: time error

3.2.2 非零 Dirichlet 边界条件

Example 6.

```
时间已过 3322.245791 秒。

ans =

T = 0.099990, dt = 1e-5, detaa = 0.100000


Table: Error
 #Dof      h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2      |S-Sh|

   86   1.250e-01   1.000e-05   1.25518e-02   0.00000e+00   3.21766e-01   0.00000e+00   1.24374e-01   0.00000e+00   1.31670e-03
  298   6.250e-02   1.000e-05   3.18174e-03   1.98001e+00   1.60781e-01   1.00091e+00   4.99659e-02   1.31567e+00   4.25508e-04
 1106   3.125e-02   1.000e-05   7.88269e-04   2.01306e+00   8.02433e-02   1.00265e+00   2.15621e-02   1.21244e+00   1.53591e-04
 4258   1.563e-02   1.000e-05   1.92328e-04   2.03512e+00   4.00893e-02   1.00116e+00   1.01576e-02   1.08594e+00   8.17729e-05
```

Figure 64: space error

```
时间已过 3.037632 秒。

ans =

T = 0.099844, h = 1/8, detaa = 0.100000

Table: Error
#Dof      h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2      |S-Sh|

86   1.250e-01   5.000e-03   1.28389e-02   0.00000e+00   6.07476e-02   0.00000e+00   1.35960e-02   0.00000e+00   2.53191e-02
86   1.250e-01   2.500e-03   6.64932e-03   9.49239e-01   3.14556e-02   9.49511e-01   6.68163e-03   1.02491e+00   1.44206e-02
86   1.250e-01   1.250e-03   3.38143e-03   9.75574e-01   1.59949e-02   9.75708e-01   3.30825e-03   1.01413e+00   8.13520e-03
86   1.250e-01   6.250e-04   1.70482e-03   9.88017e-01   8.06376e-03   9.88083e-01   1.64556e-03   1.00749e+00   4.78173e-03
86   1.250e-01   3.125e-04   8.55922e-04   9.94065e-01   4.04841e-03   9.94098e-01   8.20585e-04   1.00385e+00   3.05224e-03
```

Figure 65: time error

3.3 Crank-Nicolson Method($S^{n+1} = \frac{q^{n+1}}{\sqrt{\mathrm{E}\left(\tilde{\mathrm{u}}^{n+\frac{1}{2}}\right)+\delta}}$)

3.3.1 零 Dirichlet 边界条件

Example 7.

```
时间已过 4807.966926 秒。

ans =

T = 0.100000;dt = 1e-4, detaa = 0.100000

Table: Error
 #Dof      h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2      |S-Sh|

   82   2.500e-01   1.000e-04   5.85996e-03   0.00000e+00   5.54005e-02   0.00000e+00   1.71294e-01   0.00000e+00   3.08534e-09
  290   1.250e-01   1.000e-04   1.94837e-03   1.58862e+00   2.53853e-02   1.12590e+00   6.66054e-02   1.36276e+00   1.12740e-08
 1090   6.250e-02   1.000e-04   5.11072e-04   1.93067e+00   1.13301e-02   1.16384e+00   2.79202e-02   1.25433e+00   3.59415e-09
 4226   3.125e-02   1.000e-04   1.28179e-04   1.99537e+00   5.29122e-03   1.09849e+00   1.29086e-02   1.11297e+00   8.22954e-10
16642   1.563e-02   1.000e-04   3.19231e-05   2.00549e+00   2.55980e-03   1.04757e+00   6.27044e-03   1.04170e+00   1.70274e-10
```

Figure 66: space error

```
时间已过 19.945570 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error
#Dof      h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1   || p-p_h||_L2  rate_errpL2      |S-Sh|

290   1.250e-01   5.000e-03   1.42506e-08   0.00000e+00   1.07085e-07   0.00000e+00   1.82959e-07   0.00000e+00   6.52977e-08
290   1.250e-01   2.500e-03   3.58829e-09   1.98966e+00   2.69360e-08   1.99116e+00   1.05788e-08   4.11228e+00   8.13325e-09
290   1.250e-01   1.250e-03   8.98811e-10   1.99721e+00   6.74703e-09   1.99721e+00   2.64934e-09   1.99746e+00   6.31822e-09
290   1.250e-01   6.250e-04   2.24811e-10   1.99930e+00   1.68757e-09   1.99930e+00   6.62635e-10   1.99935e+00   1.00039e-08
290   1.250e-01   3.125e-04   5.62096e-11   1.99983e+00   4.21944e-10   1.99983e+00   1.65678e-10   1.99983e+00   1.09608e-08
```

Figure 67: time error

3.3.2 非零 Dirichlet 边界条件

Example 6.

```
时间已过 664.505087 秒。

ans =

T = 0.100000;dt = 1e-4,detaa = 0.100000

Table: Error
 #Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

   86   1.250e-01   1.000e-04   1.25572e-02   0.00000e+00   3.21752e-01   0.00000e+00   1.24366e-01   0.00000e+00   1.25887e-03
  298   6.250e-02   1.000e-04   3.18890e-03   1.97738e+00   1.60774e-01   1.00091e+00   4.99686e-02   1.31550e+00   3.67906e-04
 1106   3.125e-02   1.000e-04   7.95595e-04   2.00295e+00   8.02397e-02   1.00265e+00   2.15656e-02   1.21229e+00   9.60778e-05
 4258   1.563e-02   1.000e-04   1.98380e-04   2.00377e+00   4.00874e-02   1.00117e+00   1.01597e-02   1.08587e+00   2.42818e-05
```

Figure 68: space error

```
时间已过 5.137013 秒。

ans =

T = 0.100000, h = 1/8, detaa = 0.100000

Table: Error
#Dof       h           dt       ||u-u_h||_L2  rate_erruL2   |u_I-u_h|_1   rate_erruH1  || p-p_h||_L2  rate_errpL2      |S-Sh|

86   1.250e-01   5.000e-03   9.75126e-07   0.00000e+00   1.67688e-05   0.00000e+00   4.57223e-05   0.00000e+00   1.55364e-03
86   1.250e-01   2.500e-03   2.44348e-07   1.99665e+00   4.20171e-06   1.99673e+00   1.19924e-05   1.93078e+00   1.34212e-03
86   1.250e-01   1.250e-03   6.16052e-08   1.98781e+00   1.05929e-06   1.98788e+00   3.01911e-06   1.98993e+00   1.29545e-03
86   1.250e-01   6.250e-04   1.58694e-08   1.95681e+00   2.72827e-07   1.95704e+00   7.73962e-07   1.96379e+00   1.28669e-03
86   1.250e-01   3.125e-04   4.25183e-09   1.90009e+00   7.30721e-08   1.90059e+00   2.05392e-07   1.91388e+00   1.28598e-03
```

Figure 69: time error

4 稳态问题

向后欧拉格式可以计算，用 CN 格式算的时候，数值解会出现负值（可能是和需要压力的初值有关，初值赋零，初值取稳态解，改变时间间隔，改变 δ 都试过了，还是会出现负值）。所以在论文中用的向后欧拉格式。

5 方腔流

雷诺数为 400 和 1000 的时候，速度分量 u 和文献中的数据都比较接近，但是速度分量 v 在雷诺数为 400 时有一个值和文献中差的有点大（-0.23827（文献），-0.3705 （数值算出来的）），雷诺数为 1000 时，用向后欧拉和 CN 算出的速度的值基本一样，但是压力的值不一样，CN 格式可能受到压力的初值的影响，所以在论文中用的向后欧拉格式。