白噪声激励下的 Duffing 振子方程为：

$$\ddot{x} + \eta\dot{x} + \alpha x + \beta x^3 = \sigma W(t) \tag{1}$$

令$\dot{x} = y$,将该方程扩展成为二维系统。令

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\eta y - \alpha x - \beta x^3 + \sigma W(t) \end{cases} \tag{2}$$

系统(2)在无外激励时的平衡点满足：

$$\begin{cases} y = 0 \\ -\eta y - \alpha x - \beta x^3 = 0 \end{cases} \tag{3}$$

解方程组(3)，可得:

(a) 当$\alpha < 0, \beta > 0$时，系统(2)在无外激励时的平衡点有三个，横坐标为

$x_1 = 0, x_2 = -\sqrt{-\alpha/\beta}, x_3 = \sqrt{-\alpha/\beta},$

此时，有两个稳定平衡点$\left(\pm\sqrt{-\alpha/\beta}\right)$;一个不稳定平衡点$(0,0)$。

(b) 当$\alpha > 0, \beta > 0$时，系统(2)在无外激励时的唯一平衡点，且为稳定平衡点

$(0,0)$。

系统(2)有精确的稳态概率密度为

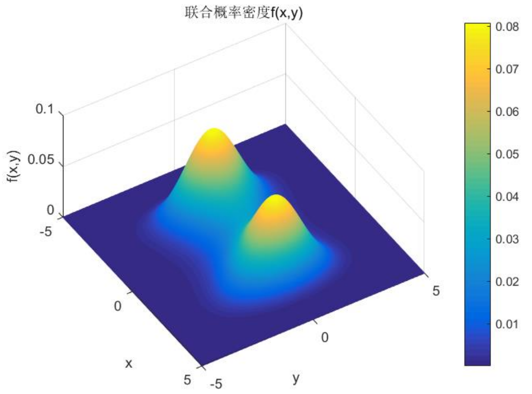
$$\mathrm{f}(x, y) = Cexp\left\{-\frac{\eta}{\sigma^2}\left[\alpha x^2 + \frac{\beta}{2}x^4 + y^2\right]\right\} \tag{4}$$

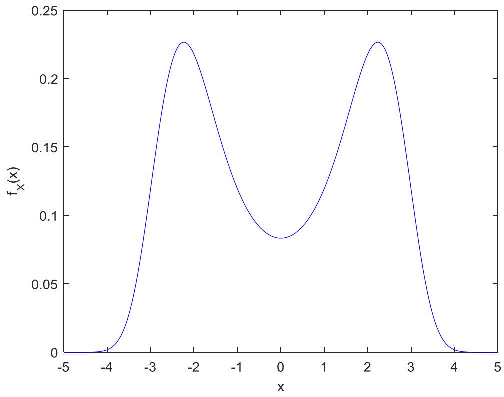
归一化参数C应满足：$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f}(x, y)\, dxdy = 1$。

参考计算参数：

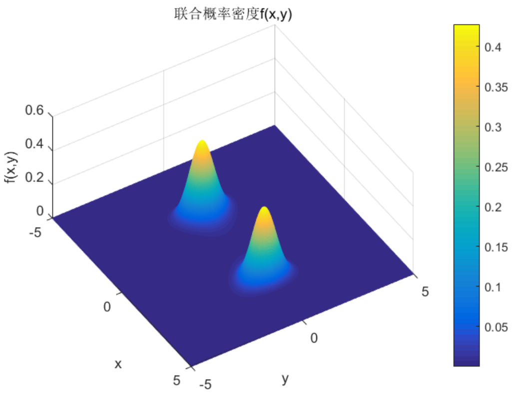
- 针对情形(a)，给出具有双峰的概率密度(4)两组参数取值：

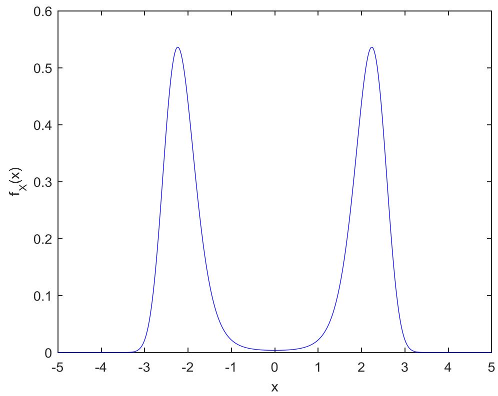
(1) $\alpha = -1.0$，$\beta = 0.2$，$\eta = 0.2$，$\sigma^2 = 0.5$，$C \approx 0.0297$





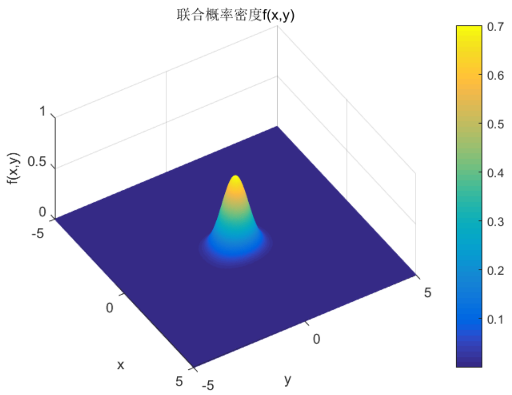
(2) $\alpha = -1.0$，$\beta = 0.2$，$\eta = 0.2$，$\sigma^2 = 0.1, C \approx 0.0029$

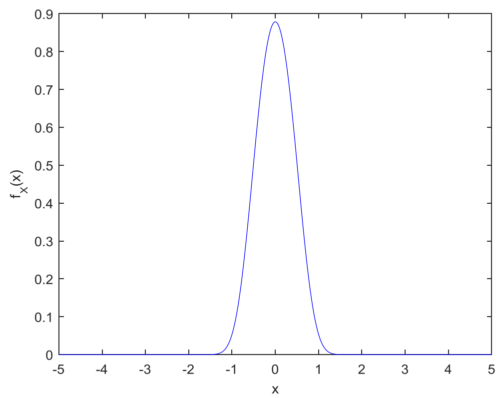




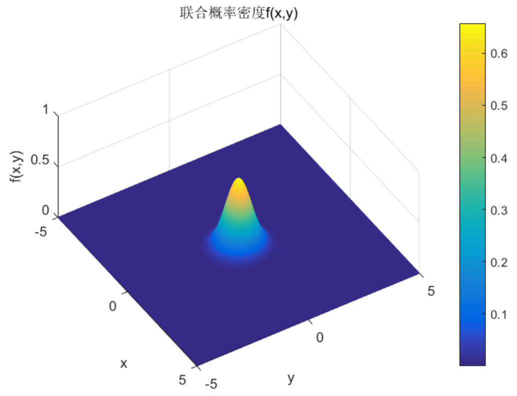
- 针对(b)情形，给出具有单峰的概率密度(4)两组参数取值：

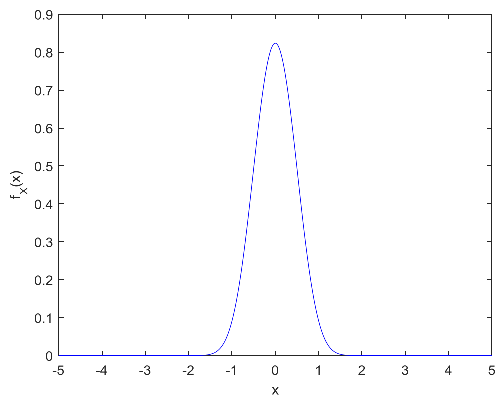
(1) $\alpha = 1.0$，$\beta = 0.8$，$\eta = 0.6$，$\sigma^2 = 0.3, C \approx 0.7011$





(2) $\alpha = 1.0$，$\beta = 0.2$，$\eta = 0.2$，$\sigma^2 = 0.1, C \approx 0.6573$





注意：

(1) 矩方程的计算时长是概率密度函数迭代的时间步长；

(2) σ^2 越大，计算区域越大；