数学物理方法第一次作业
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§ 1.1

1. (2) z 在 a, b 的垂直平分线上

   (4) z 在 $y^2 \leq 1-2x$区域内

   (6) z 在左半平面，圆 $(x+1)^2+y^2=2$ 外

   (8)圆 $2x^2+2y^2-x=0$ 上（不包括 $(0,0)$ 点）

   (10)

2. (2)

$$z=\begin{cases}-1\\ e^{i\pi(2k+1/2)}\\ \cos\pi(2k+1/2)+i\sin\pi(2k+1/2)\end{cases}$$

   (4)

$$z=\begin{cases}1-\cos\alpha+i\sin\alpha\\ \rho e^{i(\phi+2k\pi)} \quad \rho=\sqrt{2-2\cos\alpha} \quad \phi=\arccos\dfrac{1-\cos\alpha}{\sqrt{2-2\cos\alpha}}\\ \rho(\cos\phi+i\sin\phi)\end{cases}$$

   (6)

$$z=\begin{cases}e\cos 1+\mathrm{i}e\sin 1\\ e\cdot e^{\mathrm{i}(1+2k\pi)}\\ e(\cos 1+\mathrm{i}\sin 1)\end{cases}$$

3. (2)

$$z=i, \frac{\sqrt{3}}{2}-\frac{1}{2}, -\frac{\sqrt{3}}{2}-\frac{1}{2}$$

   (4)

$$z=\sqrt[i]{2i}=\left(e^{i(\frac{\pi}{2}+2k\pi)+\ln 2}\right)^{-i}=2e^{\frac{\pi}{2}+2k\pi}e^{-i\ln 2}$$

   (6)

$$
\begin{aligned}
z &= \sin 5\varphi \\
&= \operatorname{Im}\left(e^{5i\varphi}\right) \\
&= \operatorname{Im}\left(\cos\varphi + i\sin\varphi\right)^5 \\
&= \operatorname{Im}\left(i\sum_{k=0}^{2}(-1)^k C_5^{2k+1}\cos^{4-2k}\varphi\sin^{2k+1}\varphi\right) \\
&= \sum_{k=0}^{2}(-1)^k C_5^{2k+1}\cos^{4-2k}\varphi\sin^{2k+1}\varphi
\end{aligned}
$$

§ 1.2

1. 令 $z = x + iy \quad x, y$ 均为实数

> (1.2.11)
>
> $$
> \begin{aligned}
> \sin(z+2\pi) &= \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} = \frac{e^{i(z)} - e^{-i(z)}}{2i} = \sin z \\
> \cos(z+2\pi) &= \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} = \frac{e^{i(z)} + e^{-i(z)}}{2} = \cos z
> \end{aligned}
> $$
>
> (1.2.12)
>
> $$
> \begin{aligned}
> &|\sin z| = \left|\frac{e^{i(z)} - e^{-i(z)}}{2i}\right| \\
> &= \frac{1}{2}\left|e^{i(x-\frac{\pi}{2})-y} - e^{-i(x+\frac{\pi}{2})+y}\right| \\
> &= \frac{1}{2}\sqrt{\left(e^{i(x-\frac{\pi}{2})-y} - e^{-i(x+\frac{\pi}{2})+y}\right)\left(e^{-i(x-\frac{\pi}{2})-y} - e^{i(x+\frac{\pi}{2})+y}\right)} \\
> &= \frac{1}{2}\sqrt{e^{2y} + e^{-2y} - e^{-2ix} - e^{2ix}} \\
> &= \frac{1}{2}\sqrt{e^{2y} + e^{-2y} + 2\left(\sin^2 x - \cos^2 x\right)}
> \end{aligned}
> $$

> $$
> (1.2.13)
> $$
>
> $$
> \begin{aligned}
> &|\cos z| = \left|\frac{e^{i(z)} + e^{-i(z)}}{2}\right| \\
> &= \frac{1}{2}\left|e^{i(x)-y} + e^{-i(x)+y}\right| \\
> &= \frac{1}{2}\sqrt{\left(e^{i(x)-y} + e^{-i(x)+y}\right)\left(e^{-i(x)-y} + e^{i(x)+y}\right)} \\
> &= \frac{1}{2}\sqrt{e^{2y} + e^{-2y} + e^{-2ix} + e^{2ix}} \\
> &= \frac{1}{2}\sqrt{e^{2y} + e^{-2y} + 2\left(\cos^2 x - \sin^2 x\right)}
> \end{aligned}
> $$
>
> $$
> (1.2.14)
> $$

> $$sh\left(z+2\pi i\right)=\frac{e^{z+2\pi i}-e^{-(z+2\pi i)}}{2}=\frac{e^{z}-e^{-(z)}}{2}=sh(z)$$
> $$ch\left(z+2\pi i\right)=\frac{e^{z+2\pi i}+e^{-(z+2\pi i)}}{2}=\frac{e^{z}+e^{-(z)}}{2}=ch(z)$$

2. (1)

$$\sin(a+\mathrm{i}b)=\frac{e^{\mathrm{i}(a+\mathrm{i}b)}-e^{-\mathrm{i}(a+\mathrm{i}b)}}{2i}=\frac{e^{-b}}{2}(\sin a-\mathrm{i}\cos a)+\frac{e^{b}}{2}(\sin a+\mathrm{i}\cos a)$$

(3)

$$\ln\left(-1\right)=\ln\left(e^{\mathrm{i}(2k\pi+\pi)}\right)=\mathrm{i}\left(2k\pi+\pi\right)$$

(5)

$$\cos\mathrm{i}x=\frac{e^{\mathrm{i}(\mathrm{i}x)}+e^{-\mathrm{i}(\mathrm{i}x)}}{2}=\frac{e^{x}+e^{-x}}{2}$$

(7)

$$\cosh\ \mathrm{i}x=\frac{e^{\mathrm{i}x}+e^{-\mathrm{i}x}}{2}=\cos\ \mathrm{x}$$

(9)

$$\begin{aligned}&\left|e^{\mathrm{i}az-\mathrm{i}b\sin z}\right|=\left|\exp\left(-ay+\mathrm{i}ax-b\frac{e^{\mathrm{i}z}-e^{-\mathrm{i}z}}{2}\right)\right|\\&=\left|\exp\left(-ay+\mathrm{i}ax-\frac{1}{2}b\left[e^{-y}\left(\cos x+\mathrm{i}\sin x\right)-e^{y}\left(\cos x-\mathrm{i}\sin x\right)\right]\right)\right|\\&=\left|\exp\left(\left[-ay+\frac{1}{2}b\cos x\left(e^{y}-e^{-y}\right)\right]+\mathrm{i}\left[ax+\frac{1}{2}b\sin x\left(e^{y}-e^{-y}\right)\right]\right)\right|\\&=e^{-ay+\frac{1}{2}b\cos x(e^{y}-e^{-y})}\end{aligned}$$

§ 1.4

1. 记区域为 R

由题

$$\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}=0\\\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}=0\end{cases}$$

考虑到函数的解析性，函数的实部与虚部必然有一阶连续偏导数

任取区域内点 P 记 $f(P)=K$，取定另一点 Q 在区域内做二者的通路，用 $\gamma(\tau)\quad\tau\in[0,1]$ 表示，其中 $\gamma(0)=P,\gamma(1)=Q$ 则

$$f\left(Q\right)=\int_{0}^{1}\left(\frac{\partial f}{\partial x}\frac{dx}{d\tau}+\frac{\partial f}{\partial y}\frac{dy}{d\tau}\right)d\tau+f(P)=\int_{0}^{1}0d\tau+f(P)=f(P)$$

> 因此，函数在区域内为常数
>
> 这么写到底严不严谨🤔

2. (2)

> $$
> \begin{aligned}
> u &= e^{x}\left(x\cos y - y\sin y\right) \\
> &= e^{\frac{z+z^*}{2}}\left(\frac{z+z^*}{2}\frac{\left(e^{z}+e^{z^*}\right)}{2e^{\frac{z+z^*}{2}}}+\left(\mathrm{i}z-\mathrm{i}\frac{z+z^*}{2}\right)\frac{\left(e^{z}-e^{z^*}\right)}{2\mathrm{i}e^{\frac{z+z^*}{2}}}\right) \\
> &= \frac{z+z^*}{2}\frac{\left(e^{z}+e^{z^*}\right)}{2}+\left(\frac{z-z^*}{2}\right)\frac{\left(e^{z}-e^{z^*}\right)}{2} \\
> &= \frac{ze^{z}+z^*e^{z^*}}{2} \\
> &= \frac{f\left(z\right)+f^*\left(z\right)}{2} \\
> &\Rightarrow f\left(z\right) = ze^{z}
> \end{aligned}
> $$

(4)

> 令 $x = r\cos\theta, y = x = r\sin\theta$，则原式可化为
>
> $$v = \frac{\sin\theta}{r}$$
>
> 进而得到
>
> $$
> \begin{aligned}
> \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\cos\theta}{r^2} \\
> \frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r} = \frac{\sin\theta}{r}
> \end{aligned}
> $$
>
> 容易求得
>
> $$u(r,\theta) = -\frac{\cos\theta}{r} + C$$
>
> 进而得到
>
> $$
> \begin{aligned}
> f\left(z\right) &= -\frac{\cos\theta}{r} + \mathrm{i}\frac{\sin\theta}{r} + C \\
> &= \frac{-z^*}{zz^*} + C = -\frac{1}{z} + C
> \end{aligned}
> $$
>
> 考虑 $f(2) = 0$，有
>
> $$f(z) = -\frac{1}{z} + \frac{1}{2}$$

(6)

> 容易求得
>
> $$
> \begin{aligned}
> &\begin{cases} v_x = -u_y = 2y - x \\ v_y = u_x = 2x + y \end{cases} \\
> &\Rightarrow v = \frac{1}{2}y^2 - \frac{1}{2}x^2 + 2xy + C \\
> &\Rightarrow f = \left(1 - \frac{1}{2}i\right)z^2 + C
> \end{aligned}
> $$

> 考虑 $f(0)=0$，得到
>
> $$f=\left(1-\frac{1}{2}i\right)z^2$$

(8)

> $$\begin{gathered}\begin{cases}v_x=-u_y=6y^2+6xy-6x^2\\ v_y=u_x=3x^2+12xy-3y^2\end{cases}\\ \Rightarrow v=-y^3-2x^3+6xy^2+3x^2y+C\\ \Rightarrow f=(1-2\mathrm{i})z^3+C\\ \because f\left(0\right)=0\\ \Rightarrow f=(1-2\mathrm{i})z^3\end{gathered}$$

(10)

> $$\begin{gathered}u=\ln\rho\\ =\ln\sqrt{zz^*}\\ =\frac{1}{2}(\ln z+\ln z^*)\\ =\frac{1}{2}(f\left(z\right)+f^*\left(z\right))\\ \Rightarrow f\left(z\right)=\ln z+C\\ \because f\left(1\right)=0\\ \Rightarrow f\left(z\right)=\ln z\end{gathered}$$

§ 1.5

2. 令电场线方程为

$$v(x,y)=F\left(y/x\right)=C$$

以构造调和函数，则

$$\begin{gathered}\frac{\partial^2F}{\partial x^2}+\frac{\partial^2F}{\partial y^2}=\frac{\partial}{\partial x}\left(-F'\frac{y}{x^2}\right)+\frac{\partial}{\partial y}\left(F'\frac{1}{x}\right)\\ =\left(F''\frac{y^2}{x^4}+2F'\frac{y}{x^3}\right)+\left(F''\frac{1}{x^2}\right)\\ =F''\frac{y^2+x^2}{x^4}+F'\frac{2y}{x^3}=0\end{gathered}$$

由此得到

$$\begin{gathered}F'=-\frac{y^2+x^2}{2xy}F''\\ =-\frac{t^2+1}{2t}F''\end{gathered}$$

解得

$$v(x,y)=F(\frac{y}{x})=A\arctan\frac{y}{x}+B$$

> 进而得到
>
> $$\begin{aligned}A\arctan\left(\frac{y}{x}\right)+B&=\frac{\mathrm{i}}{2}A\left[\ln\left(1-\mathrm{i}\frac{z-z^*}{\mathrm{i}\left(z+z^*\right)}\right)-\ln\left(1+\mathrm{i}\frac{z-z^*}{\mathrm{i}\left(z+z^*\right)}\right)\right]+B\\&=\frac{\mathrm{i}}{2}A\left[\ln\left(\frac{2z^*}{z+z^*}\right)-\ln\left(\frac{2z}{z+z^*}\right)\right]+B\\&=\frac{\mathrm{i}}{2}A\left[\ln\left(2z^*\right)-\ln\left(2z\right)\right]+B\\&=\frac{f\left(z\right)-f^*\left(z\right)}{2i}\\&\Rightarrow f\left(z\right)=A\ln\left(z\right)+Ci+D\end{aligned}$$
>
> A, B, C, D 均为实数
>
> 3. 圆族为
>
> $$\frac{x^2+y^2}{2x}=R$$
>
> 不妨令$x=\rho\cos\varphi, y=\rho\sin\varphi$,则圆族可表示为
>
> $$F\left(\frac{\rho}{2\cos\varphi}\right)=C$$
>
> 由题

$\S\,2.4$

对于解析域内解析函数$f(z)$和解析域内点 $x\in\mathbb{R}$

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}f\left(z\right)\right)\bigg|_{z=x\,0}=\left(\frac{\partial}{\partial x}f\left(z\right)\right)\bigg|_{z=x_0}=\left(\frac{\mathrm{d}}{\mathrm{d}x}f\left(x\right)\right)\bigg|_{x=x\,_0}$$

由于解析函数的任意阶导数解析，我们很容易递推得到

$$\left(\frac{\mathrm{d}^n}{\mathrm{d}^n z}f\left(z\right)\right)\bigg|_{z=x_0}=\left(\frac{\mathrm{d}^n}{\mathrm{d}^n x}f\left(x\right)\right)\bigg|_{x=x_0}$$

顺带一提，这堆玩意把 $z=0$ 的围道转成了 $z=x$ 的围道

1.

> $$\begin{aligned}\frac{\partial^n\psi}{\partial t^n}(0) &= \frac{n!}{2\pi \mathrm{i}}\oint\frac{\psi(\zeta)}{\zeta^{n+1}}\mathrm{d}\zeta \\ &= \frac{n!}{2\pi \mathrm{i}}\oint\frac{e^{2\zeta x-\zeta^2}}{\zeta^{n+1}}\mathrm{d}\zeta \\ &= \frac{n!}{2\pi \mathrm{i}}\oint\frac{e^{x^2-z^2}}{(x-z)^{n+1}}\mathrm{d}(x-z) \\ &= -\frac{n!}{2\pi \mathrm{i}}e^{x^2}\oint\frac{1}{e^{z^2}(x-z)^{n+1}}\mathrm{d}z \\ &= \begin{cases} n=2k \Rightarrow e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{1}{e^{z^2}}\Big|_{z=x} \\ n=2k+1 \Rightarrow -e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{1}{e^{z^2}}\Big|_{z=x}\end{cases} \quad k\in\mathbb{N} \\ \Rightarrow \frac{\partial^n\psi}{\partial t^n}(0) &= (-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2}\end{aligned}$$

2.

> $$\begin{aligned}\frac{\partial^n\psi}{\partial t^n}\Big|_{t=0} &= \frac{n!}{2\pi \mathrm{i}}\oint\frac{\psi(\zeta)}{\zeta^{n+1}}\mathrm{d}\zeta \\ &= \frac{n!}{2\pi \mathrm{i}}\oint\frac{z^{n+1}\frac{z}{x}e^{x-z}}{(z-x)^{n+1}}\mathrm{d}\left(1-\frac{x}{z}\right) \\ &= \frac{n!}{2\pi \mathrm{i}}e^x\oint\frac{z^n e^{-z}}{(z-x)^{n+1}}\mathrm{d}z \\ &= e^x\left(\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(z^n e^{-z}\right)\right)\Big|_{z=x} \\ &= e^x\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n e^{-x}\right)\end{aligned}$$

§ 3.2

3. (1)

> $$\frac{1}{r} = \lim_{k\to\infty}\sqrt[k]{\left(1+\frac{1}{k}\right)^{k^2}} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e \Rightarrow r = \frac{1}{e}$$
>
> 收敛圆为 $|z|^2 = \frac{1}{e^2}$

(3)

> $$\frac{1}{r} = \lim_{k\to\infty}\sqrt[k]{\left(\frac{1}{k}\right)^k} = \frac{1}{k} = 0 \Rightarrow r = \infty$$
>
> 在全平面收敛(不包含无穷远点)

(5)

> $$\frac{1}{r} = \lim_{k\to\infty}\sqrt[k]{k^k} = k = \infty \Rightarrow r = 0$$
>
> 仅在点 $z = 3$ 收敛

4. (2)

> $$\sum_{k=1}^{\infty}(a_k-b_k)z^k=\sum_{k=1}^{\infty}a_kz^k-\sum_{k=1}^{\infty}b_kz^k$$
> $$\Rightarrow R\leqslant\min\{R_1,R_2\}$$

(4)

> R_1,R_2 均为有限数，且均不为 0

§ 3.3

1. (2)

> 考虑邻域 $|z-\mathrm{i}|<1$，我们规定 $z_0=\mathrm{i}$ 的辐角为 $\frac{\pi}{2}$
>
> $$f^{(n)}(z_0)=\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{4-3n}{3}\right)}z_0^{\frac{1}{3}-n}=\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{4-3n}{3}\right)}e^{\mathrm{i}\frac{\pi}{2}\left(\frac{1}{3}-n\right)}$$
> $$\Rightarrow z^{\frac{1}{3}}=\sum_{n=0}^{\infty}\frac{\Gamma(1)}{\Gamma(n+1)}\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{4-3n}{3}\right)}(z-z_0)^ne^{\mathrm{i}\frac{\pi}{2}\left(\frac{1}{3}-n\right)}$$

(4)

> 考虑邻域 $|z-1|<1$ 我们规定 $z_0=1$ 的辐角为 0
>
> $$f^{(n)}(z_0)=\frac{\Gamma\left(\frac{1}{m}+1\right)}{\Gamma\left(\frac{1}{m}+1-n\right)}z_0^{\frac{1}{m}-n}=\frac{\Gamma\left(\frac{m+1}{m}\right)}{\Gamma\left(\frac{m+1-nm}{m}\right)}$$
> $$\Rightarrow z^{\frac{1}{3}}=\sum_{n=0}^{\infty}\frac{\Gamma(1)}{\Gamma(n+1)}\frac{\Gamma\left(\frac{m+1}{m}\right)}{\Gamma\left(\frac{m+1-nm}{m}\right)}(z-z_0)^n$$

(6)

> 考虑邻域 $|z|<1$，我们规定 $1+e^z$ 的辐角为 $()$