Midterm Exam I

Estimation and Detection (Fall 2016)

*** There are 4 problems in 2 pages.**

*** Total points: 105**

1. (60 points) Let

$$x[n] = A + \omega[n], \; n = 0, 1, ..., N-1,$$

where $\omega[n] \sim \mathcal{N}(0, \sigma^2)$

(a) (5 points) Assume σ^2 is known. Obtain the CRLB and MVUE for A.

Sol:

see p.11-p.12 in Ch3.
$\frac{\sigma^2}{N}$ is CRLB and $\bar{x}$ is the MVUE.

(b) (6 points) Followed (a), obtain the PDF for the MVUE.

Sol:

$$\hat{A} \sim \mathcal{N}(A, \frac{\sigma^2}{N})$$

(c) (6 points) Followed (a), what is the CRLB for A^2?

Sol:

see p.21 in Ch3.

$$\frac{4A^2\sigma^2}{N}$$

(d) (15 points) Followed (a), let the MVUE be $g(x)$. Obtain the mean and variance for $(g(x))^2$. (Hint: If $x \sim \mathcal{N}(\mu, \sigma^2)$, $\mathbb{E}\{x^2\} = \mu^2 + \sigma^2$ $\mathbb{E}\{x^4\} = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$)

Sol:

see p.36 in Ch5.

$$
\begin{aligned}
\mathbb{E}\{\bar{x}^2\} &= A^2 + \frac{\sigma^2}{N} \\
\mathrm{var}\{\bar{x}^2\} &= \mathbb{E}\{(\bar{x}^2)^2\} - (\mathbb{E}\{\bar{x}^2\})^2 \\
&= A^4 + 6A^2\frac{\sigma^2}{N} + 3(\frac{\sigma^2}{N})^2 - (A^4 + 2A^2\frac{\sigma^2}{N} + \frac{\sigma^4}{N^2}) \\
&= 4A^2\frac{\sigma^2}{N} + 2\frac{\sigma^4}{N^2} \\
&= \frac{\sigma^2}{N}(4A^2 + 2\frac{\sigma^2}{N})
\end{aligned}
$$

(e) (5 points) Followed (d), is $(g(x))^2$ the MVUE for A^2?

Sol:

No.

$$\mathbb{E}\{\bar{x}^2\} = A^2 + \frac{\sigma^2}{N} \neq A^2$$

is biased.

(f) (8 points) Followed (d), in what condition the estimator $(g(x))^2$ can asymptotically achieve the CRLB and tend to be the MVUE?

Sol:

When $N \to \infty$ or $\sigma^2 \to 0$

$\mathbb{E}\{\bar{x}^2\} = A^2 + \frac{\sigma^2}{N} \to A^2$ asymptotically unbiased

$$\text{var}\{\bar{x}^2\} = \frac{\sigma^2}{N}(4A^2 + \frac{\sigma^2}{N}) \rightarrow \frac{4A^2\sigma^2}{N} \text{ asymptotically achieve CRLB}$$

(g) (15 points) Assume now both A and σ^2 are unknown.Obtain the sufficient statistics for (A, σ^2).Assuming completeness, using these sufficient statistics to obtain the MVUE.

Sol:

see p.34-p.36 in Ch5.

2. (15 points) Given the following system model:

$$x[n] = A + Bn + Cn^2 + \omega[n], \; n = 0, 1, 2$$

Answer the following questions with "explanations" or proof.

(a) (8 points) Obtain the MVUEs for A,B and C.

Sol:

Let

$$\theta = \begin{bmatrix} A \\ B \\ C \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

and

$$x = \mathbf{H}\theta + \omega$$

$$\hat{\theta} = \begin{bmatrix} \hat{A} \\ \hat{B} \\ \hat{C} \end{bmatrix} = \mathbf{H}^{-1}x = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 2 & -0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix} x$$

(b) (7 points) Indicate the minimum variance of these estimators.

Sol:

$$\begin{aligned} \text{minimum variance} &= \mathcal{I}^{-1}(\theta) = \sigma^2(\mathbf{H}^T\mathbf{H})^{-1} \\ \text{var}\{\hat{A}\} &\geq \sigma^2 \\ \text{var}\{\hat{B}\} &\geq 6.5\sigma^2 \\ \text{var}\{\hat{C}\} &\geq 1.5\sigma^2 \end{aligned} \tag{1}$$

3. (15 points) Given i.i.d observations $x[n]$ for $n = 0, 1, \ldots, N-1$, where $x[n]$ has the following PDF

$$p(x;\sigma^2) = \begin{cases} \frac{x}{\sigma^2}\exp\left\{-\frac{1}{2}\frac{x^2}{\sigma^2}\right\} & , x > 0 \\ 0 & , x < 0 \end{cases},$$

(a) (10 points) Find a sufficient statistic for σ^2.

Sol:

$$\begin{aligned} p(\mathbf{x};\sigma^2) &= \prod_{n=0}^{N-1} u\left[x[n]\right]\frac{x[n]}{\sigma^2}\exp\left[\frac{-x^2[n]}{2\sigma^2}\right] \\ &= \frac{1}{\sigma^{2N}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right]u\left[min(x[n])\right]\prod_{n=0}^{N-1}x[n] \end{aligned}$$

From Neyman-fisher factorization ,the sufficient statistic for σ^2 is

$$\mathbf{T}(\mathbf{x}) = \sum_{n=0}^{N-1}x^2[n]$$

(b) (5 points) Find a MVU estimator for σ^2. (assuming the sufficient statistic found in (a) is complete)

Sol:

Find some function g so that $\hat{\sigma}^2 = g(\mathbf{T})$ is an unbiased estimator of σ^2.

$$\because \mathrm{E}\left[\mathbf{T}(\mathbf{x})\right] = \sum_{n=0}^{N-1} \mathrm{E}\left\{x^2[n]\right\} = 2N\sigma^2$$

$$\therefore \textit{Let} \quad g(\mathbf{T}(\mathbf{x})) = \frac{1}{2N}\mathbf{T}(\mathbf{x}) \quad \textit{such} \quad \textit{that} \quad \mathrm{E}\{g(\mathbf{T}(\mathbf{x}))\} = \sigma^2$$

4. (15 points) Given the following system model

$$x[n] = A\cos(2\pi f_0 n + \phi) + \omega[n],\ n = 0, 1, \ldots, N-1,$$

where $\omega[n]$ is WGN with zero mean and variance σ^2. A and f_0 is assumed known.

(a) (7 points) Find the CRLB for var$\{\phi\}$.

Sol: The PDF is

$$\begin{aligned} p(\mathbf{x};\phi) &= \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x[n] - A\cos(2\pi f_0 n + \phi))^2}{2\sigma^2}\right] \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - A\cos(2\pi f_0 n + \phi))^2\right] \end{aligned}$$

The log-likelihood function

$$\ln p(\mathbf{x};\phi) = -\frac{N}{2}\ln(2\pi\sigma^2) + \left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - A\cos(2\pi f_0 n + \phi))^2\right]$$

Differentiation the log-likelihood function

$$\frac{\partial \ln p(\mathbf{x};\phi)}{\partial \phi} = -\frac{A}{\sigma^2}\sum_{n=0}^{N-1}\left[x[n]\sin(2\pi f_0 n + \phi) - \frac{A}{2}\sin(4\pi f_0 n + 2\phi)\right]$$

and

$$\frac{\partial^2 \ln p(\mathbf{x};\phi)}{\partial^2 \phi} = -\frac{A}{\sigma^2}\sum_{n=0}^{N-1}[x[n]\cos(2\pi f_0 n + \phi) - A\cos(4\pi f_0 n + 2\phi)]$$

From Fisher information and CRLB, we know

$$\begin{aligned}
I(\phi) &= -\mathrm{E}\left[\frac{\partial^2 \ln p(\mathbf{x};\phi)}{\partial^2 \phi}\right] \\
&= \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1}\left[\frac{1}{2} - \frac{1}{2}\cos(4\pi f_0 n + 2\phi)\right] \\
&\approx \frac{NA^2}{2\sigma^2} \\
\mathrm{var}(\hat{\phi}) &\geq \frac{1}{I(\phi)} \\
&\geq \frac{2\sigma^2}{NA^2}
\end{aligned}$$

(b) (8 points) Now letting $\phi = 0$, assume f_0 and σ^2 are known. Find the MVU estimator of A. (Hint: find sufficient statistic and assumed it is complete)

Sol:

$$\begin{aligned}
p(\mathbf{x};\phi) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A\cos(2\pi f_0 n))^2\right] \\
&= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\Bigg[-\frac{1}{2\sigma^2}(\sum_{n=0}^{N-1}x[n]^2-2\sum_{n=0}^{N-1}x[n]A\cos(2\pi f_0 n) \\
&+ \sum_{n=0}^{N-1}A^2\cos^2(2\pi f_0 n))\Bigg]
\end{aligned}$$

Using Neyman-fisher factorization ,the sufficient statistic is

$$\mathbf{T}(\mathbf{x})=\sum_{n=0}^{N-1}x[n]\cos(2\pi f_0 n)$$

Find some function g so that $\hat{A}=g(\mathbf{T})$ is an unbiased estimator of A

$$\because \mathrm{E}\left[\mathbf{T}(\mathbf{x})\right]=\sum_{n=0}^{N-1}A\cos^2(2\pi f_0 n)$$

$$\therefore \mathit{Let}\quad g(\mathbf{T}(\mathbf{x}))=\frac{\sum_{n=0}^{N-1}x[n]\cos(2\pi f_0 n)}{\sum_{n=0}^{N-1}\cos^2(2\pi f_0 n)}\quad \mathit{such}\quad \mathit{that}\quad \mathrm{E}\{g(\mathbf{T}(\mathbf{x}))\}=A$$