概率论与数理统计第六次作业

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4.1

Solution: Because of $P(\mu-\sigma<X\leq\mu-\sigma)=0.68$

So for X N(0,1),$P(0<X\leq 1)=0.34$

$$P(X\geq\varepsilon)=\int_{\varepsilon}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$$

Let $u=t-\varepsilon$

$$\begin{aligned}
P(X\geq\varepsilon)=&\int_0^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(u+\varepsilon)^2}{2}}du\\
=&\int_0^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2+2u\varepsilon+(\varepsilon+1)^2-2\varepsilon-1}{2}}du\\
=&e^{-\frac{(\varepsilon+1)^2}{2}}\int_0^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2+2u(\varepsilon-1)-1}{2}}du\\
\geq&e^{-\frac{(\varepsilon+1)^2}{2}}\int_0^{1}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2+2\varepsilon(u-1)-1}{2}}du\\
&While\ u\in[0,1],u-1\leq 0\\
P(X\geq\varepsilon)\geq&e^{-\frac{(\varepsilon+1)^2}{2}}\int_0^{1}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du\\
\geq&\frac{1}{3}e^{-\frac{(\varepsilon+1)^2}{3}}
\end{aligned}$$

4.2

Solution: From the definition,

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_{0}^{\infty} \frac{x}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} dx
\end{aligned}
$$

let $u = \frac{x}{\beta}$,So

$$
\begin{aligned}
E(X) =& \frac{\beta}{\Gamma(\alpha)} \int_{0}^{\infty} u^{\alpha} e^{-u} du \\
=& \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha+1) \\
=& \frac{\beta}{\Gamma(\alpha)} \alpha\Gamma(\alpha) \\
=& \alpha\beta
\end{aligned}
$$

As the same:

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
&= \int_{0}^{\infty} \frac{x^2}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} dx \\
&= \frac{\beta^2}{\Gamma(\alpha)} \int_{0}^{\infty} u^{\alpha+1} e^{-u} du \\
&= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha+2) \\
&= \frac{\beta^2}{\Gamma(\alpha)} (\alpha+1)\alpha\Gamma(\alpha) \\
&= \alpha(\alpha+1)\beta^2
\end{aligned}
$$

So

$$
D(X) = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2
$$

4.3

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解: 将 $N(\mu,\sigma)$ 转化为标准正态分布 $N(0,1)$, 有:

$$P(a \le X \le b) = P(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}) = \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})$$

1

$$\begin{aligned}
P(2 < X \le 5) =& \Phi(\frac{5-3}{2}) - \Phi(\frac{2-3}{2}) \\
=& \Phi(1) - \Phi(-0.5) \\
=& \Phi(1) + \Phi(0.5) - 1 = 0.8413 - 1 + 0.6915 \\
=& = 0.5328
\end{aligned}$$

2

$$\begin{aligned}
P(-4 < X \le 10) =& \Phi(\frac{10-3}{2}) - \Phi(\frac{-4-3}{2}) \\
=& \Phi(3.5) - \Phi(-3.5) \\
=& 2\Phi(3.5) - 1 = 2 \times 0.9998 - 1 \\
=& 0.9996
\end{aligned}$$

3

$$\begin{aligned}
P(|X| > 2) =& 1 - P(-2 \le X \le 2) = 1 - (\Phi(\frac{2-3}{2}) - \Phi(\frac{-2-3}{2})) \\
=& 1 - \Phi(-0.5) + \Phi(-2.5) \\
=& \Phi(0.5) - \Phi(2.5) + 1 \\
=& 0.6915 + 1 - 0.9938 = 0.6977
\end{aligned}$$

4

$$\begin{aligned} P(X > 3) =& 1 - \Phi(\frac{3-3}{2}) \\ =& 1 - \Phi(0) \\ =& 0.5 \end{aligned}$$

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解: 由于 $f(x), g(x)$ 都是概率密度函数, 所以有:

$$f(x) \geq 0, g(x) \geq 0, \int_{-\infty}^{\infty} f(x)dx = 1, \int_{-\infty}^{\infty} g(x)dx = 1$$

由于 $\alpha \in [0,1]$, 所以 $\alpha f(x) \geq 0, (1-\alpha)g(x) \geq 0$
故 $h(x) = \alpha f(x) + (1-\alpha)g(x) \geq 0$
且 $\int_{-\infty}^{\infty} h(x) = \alpha \int_{-\infty}^{\infty} f(x)dx + (1-\alpha)\int_{-\infty}^{\infty} g(x)dx = \alpha + (1-\alpha) = 1$
故 $h(x)$ 同样是一个概率密度函数.

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X 的概率密度函数为

$$f(x) = \begin{cases} 1, & ,x \in (0,1) \\ 0, & ,x \notin (0,1) \end{cases}$$

记 X,Y 的分布函数为 $F_X(x), F_Y(y)$

1

$Y = e^X > 0$. 故当 $y \leq 0$ 时,$F_Y(y) = 0, f(y) = 0$. 当 $y > 0$ 时,

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y) = F_X(\ln y).$$

两边同时求导, 可得

$$f(y)=f(\ln y)\cdot\frac{1}{y}=\begin{cases}\frac{1}{y} & ,1<y<e\\ 0 & ,y\in(0,1)or(e,+\infty)\end{cases}$$

2

当 $X\in(0,1)$ 时,$Y>0$, 故当 $y\leq 0$ 时,$F_Y(y)=0$, 即 $f(y)=0$. 当 $y>0$ 时:

$$\begin{aligned}F_Y(y)=&P(Y\leq y)=P(y\geq -2\ln X)=P(X\geq e^{-\frac{y}{2}})\\ =&1-F_X(e^{-\frac{y}{2}})\end{aligned}$$

$$f(y)=-f(e^{-\frac{y}{2}})(-\frac{1}{2}e^{-\frac{y}{2}})=\begin{cases}\frac{1}{2}e^{-\frac{y}{2}}, & ,y>0\\ 0, & ,y\leq 0\end{cases}$$

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1

因为 $Y=e^X$, 故 $Y\geq 0$, 当 y<0 时,f(y)=0. 当 $y>0$ 时, 有:

$$F_Y(y)=P(Y\leq y)=P(0<Y\leq y)=P(-\infty<X\leq \ln y)=\Phi(\ln y)$$

此时:

$$f(y)=\frac{d}{dx}\Phi(x)\bigg|_{x=\ln y}\cdot\frac{1}{y}=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\ln y)^2}\cdot\frac{1}{y}$$

故 $Y=e^X$ 的概率密度为

$$f(y)=\begin{cases}\frac{1}{\sqrt{2\pi}y}e^{-\frac{1}{2}(\ln y)^2}, & y>0\\ 0, & y\leq 0\end{cases}$$

2

因为 $Y=e^X$, 故 $Y\geq 1$, 当 y<1 时,f(y)=0. 当 $y>0$ 时, 有:

$$\begin{aligned}F_Y(y)=&P(Y\leq y)=P(2X^2+1\leq y)\\=&P(-\sqrt{\frac{y-1}{2}}\leq X\leq\sqrt{\frac{y-1}{2}})\\=&\Phi(\sqrt{\frac{y-1}{2}})-\Phi(-\sqrt{\frac{y-1}{2}})\\=&2\Phi(\sqrt{\frac{y-1}{2}})-1\end{aligned}$$

故 $y>1$ 时:

$$\begin{aligned}f(y)=&\frac{d}{dy}(2\Phi(\sqrt{\frac{y-1}{2}})-1)\\=&\frac{1}{2\sqrt{\pi(y-1)}}e^{-\frac{y-1}{4}}\end{aligned}$$

所以 $Y=2X^2+1$ 的概率密度为

$$f(y)=\begin{cases}\frac{1}{2\sqrt{\pi(y-1)}}e^{-\frac{y-1}{4}}, & y>1\\0, & y\leq 1\end{cases}$$

3

因为 $Y=|X|$, 故 $Y\geq 0$, 当 y<0 时,f(y)=0. 当 $y\geq 0$ 时, 有:

$$\begin{aligned}F_Y(y)=&P(0\leq Y\leq y)=P(-y\leq X\leq y)\\=&\Phi(y)-\Phi(-y)\\=&2\Phi(y)-1\end{aligned}$$

故 $y>0$ 时:

$$\begin{aligned}f(y)=&\frac{d}{dy}(2\Phi(y)-1)\\=&\frac{2}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\end{aligned}$$

所以 $Y=|X|$ 的概率密度为

$$f(y)=\begin{cases}\frac{2}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}, & y>0\\ 0, & y\leq 0\end{cases}$$

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1

$Y=X^3$, 故 $y=g(x)=x^3$, 严格单调递增, 解得 $x=h(y)=y^{1/3}, h'(y)=\frac{1}{3}y^{-2/3}$.
所以 $Y=X^3$ 的概率密度为

$$f_Y(y)=\frac{1}{3}y^{-2/3}f(y^{1/3}), y\neq 0$$

2

$Y=X^2$, 故 $y=g(x)=x^2$, 严格单调递增, 解得 $x=h(y)=y^{1/2}, h'(y)=\frac{1}{2}y^{-1/2}$.
所以 $Y=X^2$ 的概率密度为

$$f_Y(y)=\begin{cases}\frac{1}{2}y^{-1/2}e^{-\sqrt{y}}, & y>0\\ 0, & y\leq 0\end{cases}$$

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当 $X\in(0,\pi), Y=\sin X\in(0,1)$. 所以当 y<0 或 y>1 时 $f_Y(y)=0$. 当 $0\leq y\leq 1$ 时:

$$\begin{aligned}F_Y(y)=&P(0\leq Y\leq y)=P(0\leq\sin X\leq y)\\ =&P((0\leq X\leq\arcsin y)\cup(\pi-\arcsin y\leq X\leq\pi))\\ =&P(0\leq X\leq\arcsin y)+P(\pi-\arcsin y\leq X\leq\pi)\\ =&\int_0^{\arcsin y}\frac{2x}{\pi^2}dx+\int_{\pi-\arcsin y}^{\pi}\frac{2x}{\pi^2}dx\\ =&\frac{1}{\pi^2}(\arcsin y)^2+1-\frac{1}{\pi^2}(\pi-\arcsin y)^2\\ =&\frac{2}{\pi}\arcsin y\end{aligned}$$

所以当 $0 < y < 1$ 时,

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{2}{\pi\sqrt{1-y^2}}$$

所以所求的概率密度为

$$f_Y(y) = \begin{cases} \frac{2}{\pi\sqrt{1-y^2}}, & 0 < y < 1 \\ 0, & y \leq 0 \ or \ y \geq 1 \end{cases}$$

4.4

Solution:$F(x,y) = P(X \leq x, Y \leq y)$.

Let $A = \{X \leq x\}, B = \{Y \leq y\}$

And let $F_X(x) = P(X \leq x) = \lim_{y\to\infty} F(x,y), F_Y(y) = P(Y \leq y) = \lim_{x\to\infty} F(x,y)$

So

$$\begin{aligned} P(X > x, Y > y) =& P(\overline{AB}) = 1 - P(A) - P(B) + P(AB) \\ =& 1 - P(X > x) - P(Y > y) + P(X \leq x, Y \leq y) \\ =& 1 - F_X(x) - F_Y(y) + F(x,y) \end{aligned}$$