Optimization Methods **Fall 2021**

Homework 1

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Notice

- The submission email is: **zhangzhenyao@lamda.nju.edu.cn**.
- Please use the provided LaTeX file as a template.
- If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Inequalities

Let $x \in \mathbb{R}^n, y \in \mathbb{R}^n$, where n is a positive integer. Let $\|\cdot\|$ denote the Euclidean norm.

a) Prove the triangle inequality $\|x+y\| \leq \|x\| + \|y\|$.

b) Prove $\|x+y\|^2 \leq (1+\epsilon)\|x\|^2 + (1+\frac{1}{\epsilon})\|y\|^2$ for any $\epsilon > 0$.

Hint: You may need the Young's inequality for products, i.e. if a and b are nonnegative real numbers and p and q are real numbers greater than 1 such that $1/p + 1/q = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Problem 2: Convex sets

a) Show that a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m\}$ is convex.

b) Show that if $S \subseteq \mathbb{R}^n$ is convex, and $A \in \mathbb{R}^{m\times n}$, then $A(S) = \{Ax : x \in S\}$, is convex.

c) Show that if $S \subseteq \mathbb{R}^m$ is convex, and $A \in \mathbb{R}^{m\times n}$, then $A^{-1}(S) = \{x : Ax \in S\}$, is convex.

Problem 3: Hyperplane

What is the distance between two parallel hyperplanes, i.e.,$\{x|a^\top x = b\}$ and $\{x|a^\top x = c\}$?

Problem 4: Examples

Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbb{R}^n | x^T Ax + b^T x + c \leqslant 0\}$$

with $A \in \mathbb{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

a) Show that C is convex if $A \succeq 0$.

b) Is the following statement true? The intersection of C and the hyperplane defined by $g^T x + h = 0$ is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

Problem 5: Generalized Inequalities

Let K^* be the dual cone of a convex cone K. Prove the following,

a) K^* is indeed a convex cone.

b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.