# 最优化作业二

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## 1 Problem 1

### 1.1 a

**解:**

$f(x) = -\sum_{i=1}^{n} \log x_i$

$f^{'}(x) = -\sum_{i=1}^{n} \frac{1}{x_i}$

$f^{''}(x) = \sum_{i=1}^{n} \frac{1}{x_i^2}$

$f^{''}(x) > 0$

所以 $f(x)$ 为严格凸函数.

### 1.2 b

**解:**

如果 $f$ 二阶可微且为凸函数, 则:

$$f(y) \geq f(x) + \nabla f(x)^T (y-x)$$

$$f(x) \geq f(y) + \nabla f(y)^T (x-y)$$

$$\therefore (\nabla f(x) - \nabla f(y))^T (x-y) > 0$$

如果 $\nabla f$ 是单调的, 那么

令

$$
\begin{aligned}
&g(t)=f(x+t(y-x)),t\in[0,1]\\
&g^{'}(t)=\nabla f(x+t(y-x))^{T}(y-x)\\
&g^{'}(1)-g^{'}(0)>0\\
\therefore\, &g^{'}(t)-g^{'}(0)\leq 0\\
\therefore\, &f(y)=g(1)=g(0)+\int_{0}^{1}g^{'}(x)dx\geq g(0)+g^{'}(0)=f(x)+\nabla f(x)^{T}(y-x)
\end{aligned}
$$

因此 $f$ 为凸函数

## 1.3 c

**证明:** 即需要证明:$dom(g)$ 是凸集，且 $\forall(x,t),(y,s)\in dom(g),0<\theta<1$，有 $g(\theta x+(1-\theta)y,\theta t+(1-\theta)s)\leq\theta g(x,t)+(1-\theta)g(y,s)$

$$
\begin{aligned}
&g(x,t)=tf(\frac{x}{t}),f(x)\text{是凸函数}\\
&g(\theta x+(1-\theta)y,\theta t+(1-\theta)s)=(\theta t+(1-\theta)s)f(\frac{\theta x+(1-\theta)y}{\theta t+(1-\theta)s})\\
\because\, &g(x,t)=tf(\frac{x}{t}),g(y,s)=sf(\frac{y}{s})\\
\therefore\, &f(\frac{\theta x+(1-\theta)y}{\theta t+(1-\theta)s})=\frac{f(\theta t(\frac{x}{t})+(1-\theta)s(\frac{y}{s}))}{\theta t+(1-\theta)s}\\
\therefore\, &g(\theta x+(1-\theta)y,\theta t+(1-\theta)s)=f(\theta t(\frac{x}{t})+(1-\theta)s(\frac{y}{s}))\\
&\leq\theta tf(\frac{x}{t})+(1-\theta)sf(\frac{y}{s})=\theta g(x,t)+(1-\theta)g(y,s)
\end{aligned}
$$

又因定义域 $dom(g)$ 是 $dom(f)$ 在透视函数 $P:R^{n+1}\rightarrow R^{n},P(x,t)=x/t(t>0)$ 下的逆象，$dom(f)$ 为凸集，故 $dom(g)$ 是凸集， $g$ 是凸函数。

# 2 Problem 2

$$
\begin{aligned}
&\frac{\partial f(x)}{\partial x_i} = (\sum_{i=1}^{n} x_i^p)^{\frac{1-p}{p}} x_i^{p-1} = (\frac{f(x)}{x_i})^{1-p} \\
&\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{1-p}{x_i}(\frac{f(x)}{x_i})^{-p}(\frac{f(x)}{x_j})^{1-p} = \frac{1-p}{f(x)}(\frac{f^2(x)}{x_i x_j})^{1-p} \\
&while\ i \neq j, \frac{\partial^2 f(x)}{\partial x_i^2} = \frac{1-p}{f(x)}(\frac{f(x)^2}{x_i})^{1-p} - \frac{1-p}{x_i}(\frac{f(x)}{x_i})^{1-p} \\
\therefore &\nabla^2 f(x) = \frac{1-p}{f(x)}((\sum_{i=1}^{n} \frac{f(x)^{1-p}}{x_i^{1-p}})^2 - \sum_{i=1}^{n} \frac{f(x)^{2-p}}{x_i^{2-p}}) \\
&(\sum_{i=1}^{n} \frac{f(x)^{1-p}}{x_i^{1-p}})^2 = (\sum_{i=1}^{n} (\frac{f(x)}{x_i})^{-p/2} (\frac{f(x)}{x_i})^{1-p/2})^2 \\
&\leq \sum_{i=1}^{n} \frac{f(x)}{x_i}^{-p} \sum_{i=1}^{n} \frac{f(x)}{x_i}^{2-p} \\
&= \sum_{i=1}^{n} (\frac{f(x)}{x_i})^{2-p} \\
\therefore &\nabla^2 f(x) \leq 0 \\
\therefore &\ f(x) \text{ 是凹函数}
\end{aligned}
$$

# 3 Problem 3

## 3.1 a

$$\Delta\psi(x,y) = \psi(x) - \psi(y) - < \nabla\psi(y), x-y >= \psi(x) - \psi(y) - \nabla\psi(y)^T(x-y)$$

因为 $\psi$ 严格凸，连续可微所以有：

$$
\begin{aligned}
&\psi(x) \geq \psi(y) + \nabla\psi(y)^T(x-y) \\
&\Delta\psi(x,y) \geq 0
\end{aligned}
$$

当 $x = y$ 时，$\psi(x) = \psi(y) = 0, x = y$, 即 $\Delta\psi(x,y) = 0$

## 3.2 b

$$
\begin{aligned}
f(x) &= L(x) + \Delta_{\psi}(x, x_0) \\
\nabla f(x) &= \nabla L(x) + \nabla \psi(x) - \nabla \psi(x_0) \\
\therefore \nabla f(x^*) &= 0 \\
g(y) &= f(y) - f(x^*) - \Delta_{\psi}(y, x^*) \\
\nabla g(y) &= \nabla f(y) - \nabla \psi(x^*) - \nabla \psi(x^*) = \nabla L(y) + \nabla \psi(x^*) - \nabla \psi(x_0) \\
\nabla^2 g(y) &= \nabla^2 L(y) \geq 0 \\
g(x^*) &= \nabla g(x^*) = 0
\end{aligned}
$$

因此函数单调递增，原式得证。

# 4 Problem 4

## 4.1 a

证明：

$$
\begin{aligned}
&< \prod_X(x) - \prod_X(y), x-y > - \| \prod_X(x) - \prod_X(y) \| \\
&= < \prod_X(x) - \prod_X(y), (x - \prod_X(x)) + (\prod_X(y) - y) > \\
&x^* = argmin \frac{1}{2} \|x-y\|_2^2 \\
&\therefore < x - x^*, \prod_X(x) - \prod_X(y) >= 0 \\
&if x^* \in X, < \prod_X(x) - \prod_X(y), x - \prod_X(x) >= 0 \\
&else < \prod_X(x) - \prod_X(y), x - \prod_X(x) > \geq 0 \\
&\therefore < \prod_X(x) - \prod_X(y), x - \prod_X(x) > \geq 0 \\
&\therefore < \prod_X(x) - \prod_X(y), \prod_X(y) - y > \geq 0 \\
&\therefore < \prod_X(x) - \prod_X(y), (x - \prod_X(x)) + (\prod_X(y) - y > \geq 0
\end{aligned}
$$

得证

## 4.2 b

$$
\begin{aligned}
\because < \prod_X(x) - \prod_X(y), x - \prod_X(x) > \geq 0 \\
< \prod_X(x) - \prod_X(y), \prod_X(y) - y > \geq 0
\end{aligned}
$$

$x,y$ 在以 $\prod_X(x) - \prod_X(y)$ 为法向量，$\prod_X(x)$ 为交点的超平面两侧，同理 $x,y$ 在以 $\prod_X(x) - \prod_X(y)$ 为法向量，$\prod_X(y)$ 为交点的超平面两侧。

$\therefore \| \prod_X(x) - \prod_X(y) \|_2 \leq \|x-y\|_2$

# 5 Problem 5

## 5.1 a

$$
\begin{aligned}
f^*(y) &= sup(yx - max\{0, 1-x\}) \\
if\ x < 1, \frac{\partial f^*}{\partial x} &= 1 + y \\
else\ \frac{\partial f}{\partial x} &= y
\end{aligned}
$$

为了保证有上界 $y \in [-1, 0]$

$\therefore f^*(y) = y, y \in [-1, 0]$

## 5.2 b

$$
\begin{aligned}
f^*(y) &= sup(yx - \ln(1 + e^{-x})) \\
\frac{\partial f^*}{\partial x} &= y + \frac{1}{1 + e^x} \\
\frac{1}{1 + e^x} &\in (0, 1)
\end{aligned}
$$

为保证有上界, 则 $y \in (-1, 0)$

$$\therefore f^*(y) = (y+1)\ln(y+1) - y\ln(-y)$$