Lecture 5

The arc lenght parameter "s":

let a curve C be parametrization of any curve as $\vec{r}(t)$, $t \in [a,b]$

Def: The arc length function associated to the parametrization $\vec{r}(t)$ is the function $S(t)$ Satisfying:

$$\begin{cases} \frac{ds}{dt} = \| r'(t) \| \longrightarrow S(t) = \int \| r'(t) \| \, dt \\ S(a) = 0 \end{cases}$$

If $\| \vec{r}_t(t) \| \neq 0$ for all t, then $S(t)$ has an inverse $t(S)$, and

$\rightarrow S(t) \rightarrow$ is an increasing function and it has an inverse

Then $r(t(S))$, $S \in [0, L]$ $\longrightarrow$ ($[0, L]$ is the range of $S(t)$)

is the reparametrization of C Satisfying: $\| r_s \| = 1$ for all S.

$$\left(\| r_s \| = \| r_t \frac{dt}{ds} \| = \| r_t \| \left| \frac{dt}{ds} \right| = \| r_t \| \left(\frac{1}{\frac{ds}{dt}} \right) = 1 \right)$$

$\uparrow$ change rule

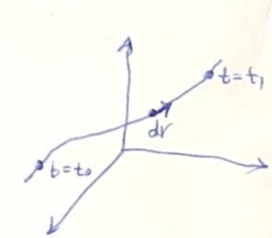
Def: In general, a parametrization $\vec{r}(s)$ of a curve C is called parametrization by arc length if "$\|r_s\| = 1$ for all s" is satisfied.

- $\vec{r}(s)$ can always be obtained from any of a $\vec{r}(t)$ in above way.
- s is called the arc. length parameter of curve C.

* Length along curves:

The length of a curve segment from $t = t_0$ to $t = t_1$ for a given parametrization $\vec{r}(t)$ is defined as:

$$\text{length} = \int_{t_0}^{t_1} \|dr\| = \underbrace{\int_{t_0}^{t_1} \left\| \frac{dr}{dt} \right\| dt}_{} = \int_{t_0}^{t_1} s'(t)\, dt = s(t_1) - s(t_0)$$



hence the term " arc length function $s(t)$" gives the length of curve.

Note: when calculated relative to any other parametrization, $\vec{r}(u)$, will get same thing
i.e. length is paramertrization invariant.

since $\longrightarrow$ $\int \|r_t\|\, dt = \int \|r_u \cancel{u_t}\| \frac{\cancel{dt}}{du}\, du = \int \|r_u\|\, du$

Note: when calculated relative to arc length param. $\vec{r}(s) \rightarrow$ get $\rightarrow$ $\int_{s_0}^{s_1} \|r_s\|\, ds = \int_{s_0}^{s_1} ds = s_1 - s_0$

Exp) $\vec{r}(t) = \langle t^2, t^3 \rangle$ $t \in (0, \infty)$, find a.l.f.

$$\frac{ds}{dt} = \|r_t\| = \sqrt{(2t)^2 + 3(t^2)^2} = \sqrt{4t^2 + 9t^4} = 2t\sqrt{1 + \frac{9}{4}t^2}$$

$$\rightarrow s(t) = \int 2t\sqrt{1 + 9/4\, t^2} = (1 + \frac{9}{4}t^2)^{3/2} \left(\frac{2}{3}\right)\left(\frac{2}{9}\right) \cdot 2 + C$$

$s(0) = 0 \longrightarrow$ so $C = -8/27$

Exp) find arc length paramation $\vec{r}(s)$ in prev. Exp:

- isolating t from $s(t)$:

$$t(s) = \left[\left[(s + \tfrac{8}{27}) \cdot \tfrac{27}{8}\right]^{2/3} - 1\right]^{1/2} \left(\tfrac{4}{9}\right)^{1/2}$$

Arc length parametrization is thus:

$$\vec{r}(s) = \langle t(s)^2, t(s)^3 \rangle, \quad s \in [0, \infty]$$

* finding $\vec{r}(s)$ from some $\vec{r}(t)$ involves:

1. Solving $s'(t) = \|r_t(t)\|$
2. Inverting $s(t)$ to get $t(s)$ $\longrightarrow$ sometimes can not do this