No. 30

MATH 317 — HOMEWORK 5 — Due date: 8th August - 25 Marks

- Write your name, student number, and signature below.
- Write your answers to each question in the spaces provided. Submit your answers in Canvas on the due date (do not submit any additional pages).

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1. $\boxed{\text{10 marks}}$ Use the Divergence Theorem to evaluate the surface integral (flux) over surface S:

$$\int\int_S F.\hat{n}\, dS$$

Where:

a) $F(x,y,z) = < x, 2y, 3z >$, and S is: $x^2 + y^2 + z^2 = 9$

b) $F(x,y,z) = < x, y, z >$, and S is boundary of the solid cube: $0 \le x, y, z \le 1$)

Solution:

a) $\iint_S F\cdot\hat{n}\,ds = \iiint_V \nabla\cdot F\,dV = \iiint_V (1+2+3)\,dV = 6\iiint_V dV = 6\cdot\frac{4}{3}\pi(3)^3 = 216\pi.$ $\therefore 216\pi$

b) $\iint_S F\cdot\hat{n}\,ds = \iiint_V \nabla\cdot F\,dV = \iiint_V (1+1+1)\,dV = 3\iiint_V dV = 3$ $\therefore 3$

2. $\boxed{\text{6 marks}}$ Let $c \in \Re$ be a constant. Let $F = (cx + cosz, y, cxz)$ be a vector field in $\Re^3$. Let S_1 be the upper hemisphere with upward normal: $x^2 + y^2 + z^2 = 1$ and $z > 0$. Let S_2 be the lower hemisphere with upward normal: $x^2 + y^2 + z^2 = 1$ and $z < 0$. Use divergence theorem to find c such that

$$\int\int_{S_1} F.\hat{n}\, dS = \int\int_{S_2} F.\hat{n}\, dS.$$

Solution:

$\nabla \cdot F = c+1+cx$

$$\iint_{S_1} F\cdot\hat{n}\, dS = \iiint_{V_1} (c+1+cx)\, dV = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 \rho^2\sin\phi\,(c+1+c\cdot\rho\sin\phi\cos\theta)\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 (c+1)\cdot\rho^2\cdot\sin\phi + c\cdot\rho^3\cdot\sin^2\phi\cos\theta\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 (c+1)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = (c+1)\cdot\frac{1}{3}\cdot 2\pi\cdot[-\cos\phi]_0^{\frac{\pi}{2}} = \frac{2\pi}{3}(c+1).$$

$$\iint_{S_2} F\cdot\hat{n}\, dS = \iiint_{V_1} (c+1+cx)\, dV = \int_{-\frac{\pi}{2}}^{0}\int_0^{2\pi}\int_0^1 (c+1)\rho^2\sin\phi + c\cdot\rho^3\cdot\sin^2\phi\cos\theta\, d\rho\, d\theta\, d\phi$$

$$= \int_{-\frac{\pi}{2}}^{0}\int_0^{2\pi}\int_0^1 (c+1)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = (c+1)\cdot\frac{1}{3}\cdot[-\cos\phi]_{-\frac{\pi}{2}}^{0} = -\frac{2\pi}{3}(c+1).$$

$$\iint_{S_1} F\cdot\hat{n}\, dS = \iint_{S_2} F\cdot\hat{n}\, dS \Longrightarrow \frac{2\pi}{3}(c+1) = -\frac{2\pi}{3}(c+1) \Rightarrow \frac{4\pi}{3}(c+1) = 0 \quad \text{thus } c=-1.$$

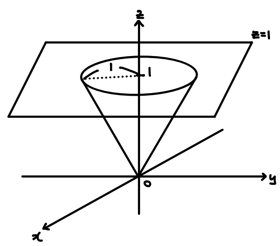
$\therefore c=-1.$

3. $\boxed{\text{5 marks}}$ Evaluate

$$\int\int_S F.\hat{n}\, dS$$

over the closed surface S formed below by a piece of the cone $z^2 = x^2 + y^2$ and above by a circular disc in the plane $z = 1$; take F to be the field of $< 0, 0, z >$; use the divergence theorem.

Solution:



$$\nabla \cdot F = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot \langle 0, 0, z \rangle = 1.$$

$$\iint_S F \cdot \hat{n}\, ds = \iiint_V 1\, dV = \text{Volume of Cone.}$$

$$\iiint_V 1\, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta = 2\pi \cdot \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} \rho^2 \sin\phi\, d\rho\, d\phi = 2\pi \int_0^{\frac{\pi}{4}} \frac{1}{3} \frac{1}{\cos^3\phi} \sin\phi\, d\phi = \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \tan\phi \cdot \sec^2\phi\, d\phi$$

$$= \frac{2\pi}{3} \cdot \left[\frac{1}{2} (\tan\theta)^2 \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{3}$$

$$\therefore \frac{\pi}{3}.$$

4. $\boxed{\text{4 marks}}$ Show that the flux of the position vector $F = xi + yj + zk$ outward through a closed surface S is three times the volume contained in that surface.

Solution:

$\iint_S F \cdot \hat{n}\, dS = \iiint_V \nabla \cdot F\, dV = \iiint_V (1+1+1)\, dV = 3 \cdot \iiint_V dV$ $\square$ as required.