MATH 317 — HOMEWORK 6 — Due date: 18th August - 15 Marks

- Write your name, student number, and signature below.
- Write your answers to each question in the spaces provided. Submit your answers in Canvas on the due date (do not submit any additional pages).

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1. $\boxed{\text{7 marks}}$ Verify Stokes' theorem (i.e. show that the line integral of F over curve C = Double integral of curl F.$\hat{n}$ over the surface) if $F = < y, z, x >$ and S is the portion of the plane $x + y + z = 0$ cut out by the cylinder $x^2 + y^2 = 1$, and C is its boundary (an ellipse).

Solution:

Let S denote the surface. $\iint_S \nabla \times F \cdot \hat{n}\, dS \Rightarrow \quad z = f(x,y) = -x-y \quad$ so $\quad \hat{n}\, dS = \langle 1,1,1 \rangle\, dxdy$

$\nabla \times F = \nabla \times \langle y,z,x \rangle = \langle -1,-1,-1 \rangle$. $\iint_R \langle -1,-1,-1 \rangle \cdot \langle 1,1,1 \rangle\, dxdy = -3\iint_R dxdy = -3\pi$.

$\int_C F \cdot dr \Rightarrow r(t) = \langle \cos t, \sin t, -\cos t - \sin t \rangle \quad ; \ t \in [0, 2\pi]$

$r'(t) = \langle -\sin t, \cos t, \sin t - \cos t \rangle$

$\int_0^{2\pi} \langle \sin t, -\cos t - \sin t, \cos t \rangle \cdot \langle -\sin t, \cos t, \sin t - \cos t \rangle\, dt$

$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t - \sin t\cos t + \sin t\cos t - \cos^2 t)\, dt = \int_0^{2\pi} (-1 - \cos^2 t)\, dt = -2\pi - \pi = -3\pi$.

Thus, $\int_C F \cdot dr = \iint_S \nabla \times F \cdot \hat{n}\, dS$.

$\operatorname{div} F = 0 \Leftrightarrow$ vector potential $F = \nabla \times G$

2. $\boxed{\text{8 marks}}$ Find a vector potential for

(a) $F =< -x - y, 2z, z - y >$

(b) $F =< -\sin(x), \cos(y), z\cos(x) + z\sin(y) >$

Solution:

(a) $\operatorname{div} F = \underset{x}{-1} + \underset{y}{0} + \underset{z}{1} = 0.$ has vector potential.

Let $G = \langle G_1, G_2, G_3 \rangle$ s.t. $F = \nabla \times G$.

$$\begin{cases} (G_3)_y - (G_2)_z = -x - y \\ (G_1)_z - (G_3)_x = 2z \\ (G_2)_x - (G_1)_y = z - y \end{cases}$$

Setting $G_3 = 0$, $\begin{cases} (G_2)_z = x + y \\ (G_1)_z = 2z \\ (G_2)_x - (G_1)_y = z - y \end{cases}$, Then, $G_2 = xz + yz + \phi(x,y)$ & $G_1 = z^2 + \psi(x,y)$.

$$(G_2)_x - (G_1)_y = (z + \phi_x) - (\psi_y) = z - y$$

$$\phi = -xy, \psi = 0$$

$$\Rightarrow G_1 = z^2, \ G_2 = xz + yz - xy$$

$$\therefore G = \langle z^2, xz + yz - xy, 0 \rangle$$

(b) $F = \langle -\sin x, \cos y, z \cdot \cos x + z \sin y \rangle$

$\operatorname{div}(F) = -\cos x - \sin y + \cos x + \sin y = 0$

$-\sin x = (G_3)_y - (G_2)_z$

$\cos y = (G_1)_z - (G_3)_x$

$z \cdot \cos x + z \cdot \sin y = (G_2)_x - (G_1)_y$

Letting $G_3 = 0$, $(G_2)_z = \sin x$ so $G_2 = \sin x \cdot z + \psi(x,y)$

$(G_1)_z = \cos y$ so $G_1 = \cos y \cdot z + f(x,y)$

$(G_2)_x - (G_1)_y = z \cdot \cos x + \psi_x - (-z \cdot \sin y + f_y)$

$= z \cdot \cos x + z \cdot \sin y + \psi_x - f_y$

$= z \cdot \cos x + z \cdot \sin y$ so $\psi_x = f_y$ let $\psi = 0, f = 0$

then $G = \langle z \cdot \cos y, z \cdot \sin x, 0 \rangle$

$\therefore G = \langle z \cdot \cos y, z \cdot \sin x, 0 \rangle$