Exp) Is $\vec{F} = \langle e^{z^2},\ 2yz^3,\ 2xze^{z^2} + 3y^2z^2 \rangle$ Conservative?

If yes, Find possible potential for F?

Solution: We need to solve for ϕ such that $F = \nabla\phi$. This implies:

$$\frac{\partial\phi}{\partial x} = e^{z^2} \Rightarrow \phi(x,y,z) = \int e^{z^2}dx = xe^{z^2} + g(y,z)$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left(xe^{z^2} + g(y,z)\right) = 2yz^3 \Rightarrow g'(y,z) = 2yz^3 \Rightarrow g(y,z) = y^2z^3 + G(z)$$

$$\Rightarrow \phi(x,y,z) = xe^{z^2} + y^2z^3 + G(z)$$

$$\Rightarrow \frac{\partial\phi}{\partial z} = 2xze^{z^2} + 3y^2z^2 + \frac{dG(z)}{dz} = F_3 = 2xze^{z^2} + 3y^2z^2 \Rightarrow \frac{\partial G(z)}{\partial z} = 0 \rightarrow G(z) = C$$

$$\Rightarrow \phi(x,y,z) = xe^{z^2} + y^2z^3 + C$$

we can ignore the constant C, because the potential function is determined up to an arbitrary constant. ~~~~

Therefore, F is conservative.

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* Would be nice to have a quicker way to determine when ~~the~~ a given field is Conservative or not, before actually solving for potential ϕ.

Theorem: (a necessary condition for $\vec{F}$ to be conservative)

If $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is conservative, then
$$\begin{cases} (F_1)_y = (F_2)_x \\ (F_2)_z = (F_3)_y \\ (F_3)_x = (F_1)_z \end{cases} \quad (1)$$

* Assume $F_1, \dots, F_3$ have continous partial derivatives

(IF $\vec{F} = \langle F_1, F_2 \rangle$ Is conservative, then $\rightarrow (F_1)_y = (F_2)_x$ $\leftarrow$ For 2D vector field)

Condition ① is neccessary, but have not yet claimed sufficient. ②

Condition ① fails $\longrightarrow \vec{F}$ is non conservative!

" ① holds $\longrightarrow \vec{F}$ may or may not be conservative!

Will show sufficiency later on.

Proof: easy $\rightarrow F = \nabla\phi = \langle \phi_x, \phi_y, \phi_z \rangle$

$\Rightarrow (F_1)_y = \phi_{xy} = \phi_{yx} = (F_2)_x$ interchangibility of partial derivative

$(F_2)_z = \phi_{yz} = \phi_{zy} = (F_3)_y$ similar for 2D Case.

$(F_3)_x = \phi_{zx} = \phi_{xz} = (F_1)_z$

(Switching 2nd order partial derivatives)

* condition (1) is curiously related to cross products!

condition (1) can be expressed as:

(2) $\langle (F_3)_y - (F_2)_z, (F_1)_z - (F_3)_x, (F_2)_x - (F_1)_y \rangle = \vec{0}$

Def: Given a field $\vec{F} = \langle F_1, F_2, F_3 \rangle$, define $\nabla \times \vec{F}$ = vector field in LHS (2)

$\nabla \times \vec{F}$ also known as $(\text{Curl } \vec{F})$.

* Actual connections to geometry of cross product?

Yes, will see later (maybe much later)!

* Notation $\nabla \times \vec{F}$ comes from fact that vector can be obtained by symbolicly doing following

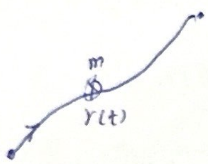
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2\right)\vec{i} + \cdots (\quad)\vec{j} + (\quad)\vec{k}$$

Exp) $\vec{F} = \left\langle x^2yz + x^z,\ x^3z/3 + y,\ x^3y/3 + x^2/2 + y \right\rangle$

$\nabla \times \vec{F} = \left\langle (x^3/3 + 1) - (x^3/3), \ldots, \ldots \right\rangle = \langle 1, \ldots, \ldots \rangle \neq \vec{0}$

$\Rightarrow \vec{F}$ is not conservative!

In general: sppse: $F = \nabla \phi$, and $\vec{F}$ displaces mass m along the path $\vec{r}(t)$ as the only force acting on m:



So that $F(r(t)) = m \underbrace{\vec{a}(t)}_{\vec{r}''(t)}$ "F=ma"

let $E(t) = \dfrac{m \|V(t)\|^2}{2} - \phi(r(t))$ ⟶ total energy at time t

$\dfrac{m\|V(t)\|^2}{2}$ ↙ kinetic energy

$\phi(r(t))$ ↓ potental energy

$$\frac{dE}{dt} = \frac{m}{2}(r' \cdot r')_t - (\phi(\vec{r}(t)))_t \quad \text{or} = \frac{d}{dt}\frac{m}{2}(V \cdot V) - \underbrace{\frac{d}{dt}\phi(r(t))}_{\text{Just by chain rule}}$$

↓ $\phi(x,y,z)$ compose with $x(t), y(t), z(t)$

$\frac{d}{dt}[\phi(x(t), y(t), \dots)]$

$\phi_x x'_t + \phi_y \cdot y'_t + \phi_z \cdot z'_t$

$$= \frac{m}{2}(2r'' \cdot r') - \nabla\phi \cdot r'(t) \qquad \boxed{r'' = \frac{F}{m}}$$

⟶ Just chain rule

$$= \cancel{m}\frac{F}{\cancel{m}} \cdot r'(t) - F \cdot r'(t)$$

$$= 0$$

Energy $E(t)$ is preseved along $r(t)$ $\equiv$ Conservation of $E(t)$, energy