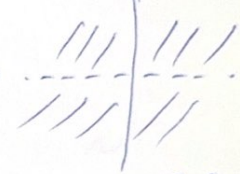
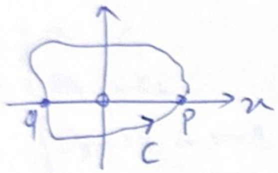
lecture 13

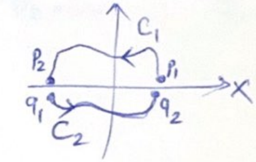
Exp) $\vec{F} = \dfrac{\langle -y, x\rangle}{(x^2+y^2)}$, show $\int_C \vec{F}\cdot d\vec{r} = 2\pi$?

Can we use potential $\phi = -\arctan\left(\frac{x}{y}\right)$ on  ? Yes!



→ This potental can be use everywhere on the curve except point p & q.

Consider we have points: p_1, p_2, q_1, q_2 (close to p & q) →



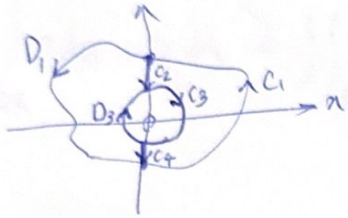
Then C_1, C_2 are both in domain of ϕ, so we can use it :

$$\int_{C_1} \vec{F}\, d\vec{r} = \phi(p_2) - \phi(p_1) \simeq \left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) = \pi$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \phi(q_2) - \phi(q_1) \simeq \left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) = \pi$$

$$\Longrightarrow \int_C F\cdot dr = \underbrace{\lim \int_{C_1} F\cdot dr}_{\text{as } p_1, q_2 \to p} + \underbrace{\lim \int_{C_2} F\cdot dr}_{\text{as } p_2, q_1 \to q} = \pi + \pi = 2\pi$$

Another way:



Then we have ① $\displaystyle\int_{C_1+C_2+C_3+C_4} \vec{F}\cdot dr = 0$ since curve lies in a simply connected domain where $(F_1)_y = (F_2)_x$!

② $\displaystyle\int_{D_1+D_2+D_3+D_4} \vec{F}\cdot dr = 0$ ~ ~

$*\begin{cases} D_2 = -C_4 \\ D_4 = -C_2 \\ D_3 + C_3 \text{ is small circle around origin} \\ C_1 + D_1 = C \text{ original loop} \end{cases}$

$$\Longrightarrow \int_{C_1+C_3} \vec{F}\cdot\vec{dr} \overset{①}{=} -\int_{C_2+C_4} \vec{F}\cdot dr \overset{*}{=} \int_{D_2+D_4} \vec{F}\cdot\vec{dr} \overset{②}{=} -\int_{D_1+D_3} \vec{F}$$

$$\Longrightarrow \int_{C_1+D_1} F\cdot dr = -\int_{C_3+D_3} F\cdot dr = 2\pi$$

by using parametrization:

$r(t) = \langle \cos t, \sin t\rangle$

$r'(t) = \langle -\sin t, \cos t\rangle$

...

We now state our most general 2D Theory:

Theorem (Greens theorem)

let $R \subset \mathbb{R}^2$ be a region (plane) with boundary ∂R consisting of finitely many simple closed curves

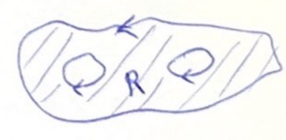
let $F = \langle F_1, F_2\rangle$ have cont. partials. Then,

$$\int_{\partial R} \vec{F}\cdot dr = \iint_R \left((F_2)_x - (F_1)_y \right) dA$$

(∂R positive oriented boundary) (a double integral)

$\begin{cases} \partial R = \text{consist of 3 simple closed curves} \\ \text{(closed + non self intersecting)} \end{cases}$

eg)



$\begin{cases} \text{Positively oriented:} \\ \text{outside loop (Counter clock)} \\ \text{inside } \text{"} \text{ (clock wise)} \end{cases}$

Alternative notation for statement of Greens theorem:

$$\int_{\partial R} P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA. \quad \text{or} \quad \int_{\partial R} \vec{F}\cdot dr = \iint_R \left(\text{"2D curl } \vec{F}\text{"}\right) dA$$

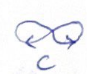
Greens theorem gives:

$$\left\{ \begin{array}{c} \text{2D curl } \vec{F} = 0 \\ \text{in } U \end{array} \right\} \overset{A}{\Longleftrightarrow} \left\{ \begin{array}{l} \int_C \vec{F}\cdot dr = 0 \\ C \text{ closed loop in } U \end{array} \right\}$$

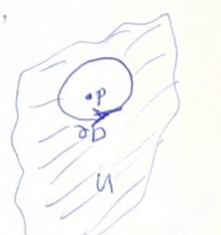
(when U is open + simply connected)

Proof: ($\Rightarrow$) • If C is simple closed loop  then $\xrightarrow[\text{theorem}]{\text{Greens}}$ $\int_C F\cdot dr = \iint_R (\text{2D curl } F)\,dA = 0$

0 since R mus lies in U by assumption U is S.C.

• If C is not simple:  Then apply above to each simple closed part of C.

Proof ($\Leftarrow$): Suppose not, so that 2D curl $\vec{F}(p) \neq 0$ for some point p in U.



Then there is a small disk D around p so that
i) D in U
ii) $|\text{2D curl } \vec{F}| > 0$ in D

if ∂D is boundary of D,

Greens theorem gives $\int_{\partial D} \vec{F}\cdot dr = \iint_D (\text{2D curl } F)\,dA \neq 0$ (contradiction assumptions)

Therefore, we could not suppose 2D curl $F(p) \neq 0 \Rightarrow$ $\boxed{\text{2D curl } \vec{F} = 0 \text{ at all points in } U}$