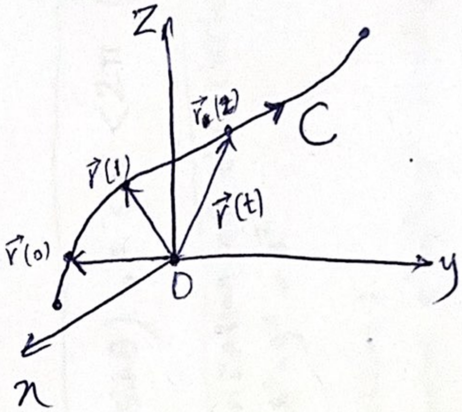
Lecture 1

Curves

We are going to study functions that assign to each real number "t" (typically in some interval) a vector $\vec{r}(t)$.

For example

3-eqns of form
$$\begin{cases} x = F_1(t) \\ y = F_2(t) \\ z = F_3(t) \end{cases}$$



might be the position of a particle at time t.

* describe a curve or path in xyz space as t increases.

* t does not have to be "time", Can be simply a parameter that is used to label different point on the curve that $\vec{r}(t)$ sweeps out.

* Call these parametrization eqns of C, i.e. $\vec{r}(t)$ provides a parametrization of the curve:

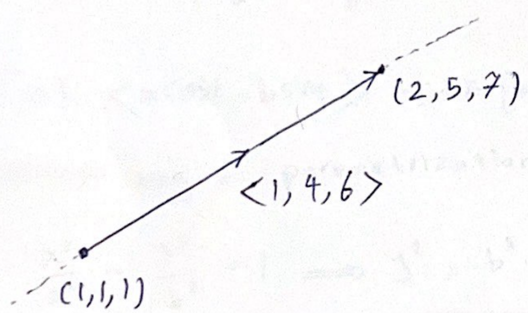
we describe a vector: $\boxed{\vec{r}(t) = \langle F_1(t), F_2(t), F_3(t) \rangle ; \ t \in [a,b]}$

(Called a parametrization of C)

* By a plane curve or curve in $\mathbb{R}^2 \longrightarrow$ Case $F_3(t) = 0$

* We may parametrize each curve in different ways!

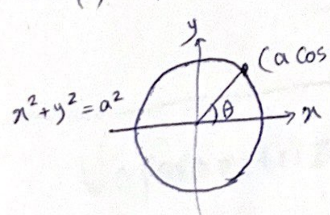
Exp) Parametrize line segment from $(1,1,1)$ to $(2,5,7)$?



$\vec{r}(t) = \langle 1+t, 1+4t, 1+6t \rangle$

$t \in [0,1]$

Exp) Parametrize Circle: $x^2 + y^2 = a^2$



parameter: $t \longrightarrow \theta$

$\vec{r}(\theta) = (a\cos\theta, a\sin\theta)$, $0 \le \theta < 2\pi$

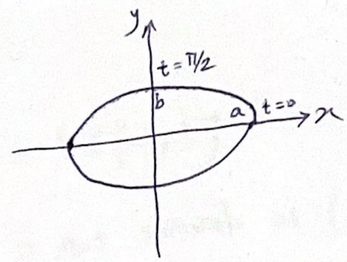
is a parametrization of the circle $x^2+y^2=a^2$

* Another way to come up with this parametrization:

we can the trig identity $\cos^2 t + \sin^2 t = 1$ into the equation $x^2+y^2=a^2$ by multiplying the trig identity by a^2 & setting $a^2\cos^2 t = x^2$ and $a^2\sin^2 t = y^2$, which turns $\rightsquigarrow$ $a\cos t = x$ & $a\sin t = y$

$\Rightarrow \vec{r}(t) = \langle a\cos t, a\sin t \rangle$

EXP) Ellipse: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$



$\vec{r}(t) = \langle a\cos t, b\sin t \rangle$, $t \in [0, 2\pi]$

Another way of parametrization:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Longrightarrow y^2 = b^2\left(1-\frac{x^2}{a^2}\right) \longrightarrow y = \pm\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$$

$x \equiv t \Longrightarrow \boxed{\vec{r}(t) = \left\langle t, +\sqrt{b^2\left(1-\frac{t^2}{a^2}\right)} \right\rangle, \; t \in [-a, a]}$ Para. of upper half of ellipse

$\boxed{\vec{r}(t) = \left\langle \sqrt{a^2\left(1-\frac{t^2}{b^2}\right)}, t \right\rangle, \; t \in [-a, a]}$ Para. of right hallf of ellipse

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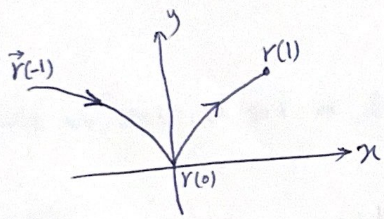
Unparametrization: Undo the parametrization $\vec{r}(t) \longrightarrow$ find the Cartesian equation of the curve

EXP) Unparametrization of $\vec{r}(t) = (\cos t, 7-t)$:

$$\left.\begin{array}{l} x = \cos t \\ y = 7-t \longrightarrow t = 7-y \end{array}\right\} \Longrightarrow \boxed{x = \cos(7-y)}$$ Cartesian Eq.

* Smoothness



$r(t) = \langle t^3, t^2 \rangle$, $t \in [-1, 1]$

$\longrightarrow$ Curve is not smooth at $(0, 0)$

* Unparametrization can help!

$$\begin{cases} x = t^3 \\ y = t^2 \end{cases} \longrightarrow y^3 = x^2 \longrightarrow y = x^{2/3} \longrightarrow \frac{dy}{dx} = \frac{2}{3} x^{-1/3}$$

$\downarrow$

Undefined $(= \infty)$ at $x = 0$!

* Therefore, Smoothess of curves is ~~not always~~ apparent from the smoothness of corresponding parametrization.

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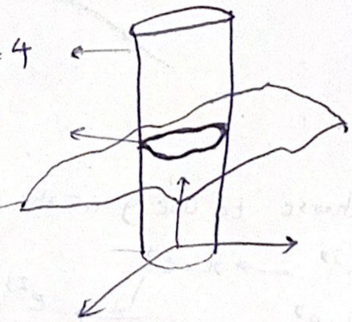
* Curves often arise as the intersection of two surfaces. One way to parametrize such curves is to choose one of the three coordinates x, y, z as the parameter, and solve the two given equations for the remaining two coordinates, as functions of the parameter.

* Parametrize curve of intersection:



✓ Could let $x=t$ (or $y=t$ or $z=t$)

✓ Then solve for the remaining variables:

to get → $x=F_1(t)$, $y=F_2(t)$, $z=F_3(t)$

OR → Use the trig identity: $\sin^2 t+\cos^2 t=1$ → $4\sin^2 t+4\cos^2 t=4$ & $x^2+y^2=4$

$\Rightarrow x=2\cos t$, $y=2\sin t$ $\Rightarrow z=-x^2+y^4=-4\cos^2 t+16\sin^2 t$

$$\vec{r}(t)=\langle 2\cos t, 2\sin t, -4\cos^2 t+16\sin^2 t\rangle, \quad t\in[0,2\pi]$$

✓ Another way: let $x=t$ → $y^2=4-t^2$ → $y=\pm\sqrt{4-t^2}$

→ $z=y^4-t^2=(4-t^2)^2-t^2=16+t^4-8t^2-t^2=t^4-9t^2+16$

$$\Rightarrow \boxed{\vec{r}(t)=\langle t, +\sqrt{4-t^2}, t^4-9t^2+16\rangle, \quad t\in[-2,2]}$$

→ This should be an actual function with single value. So we choose on sign (+ or −).

Param. of a part of C. other part → choose $-\sqrt{\cdots}$

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EXP) The set of all (x, y, z) obeying $\begin{cases} x^3 - e^{3y} = 0 \\ x^2 - e^y + z = 0 \end{cases}$ is a curve?

* We can choose to use y as the parameter:

$$\begin{cases} x^3 = e^{3y} \longrightarrow x = e^y \\ x^2 + z = e^y \end{cases} \longrightarrow e^{2y} + z = e^y \Rightarrow z = e^y - e^{2y}$$

$\Rightarrow r(y) = (e^y, y, e^y - e^{2y})$ is a parametriz. for the given curve.