No. 30

MATH 317 — HOMEWORK 1 — Due date: July 9

- Write your name, student number, section and signature below.
- WRITE YOUR "STUDENT LIST" NUMBER in the upper left corner of this cover page. This number is just your position on the student list. To find your student list number, please go to the canvas file folder and look at the file called "student list numbers".
- Print these pages (doube sided) and write your answers to each question in the spaces provided (exactly as if you were writing a test). Submit the printed pages with your answers in Canvas page or in class by the due date (do not submit any additional pages).

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Section: 951

1. $\boxed{\text{5 marks}}$ Let C be the circle of intersection of the plane $2x + y + z = 4$ and the sphere of radius 6 centered at the point $(2, 2, 2)$ in space. Find a parametrization $\mathbf{r}(t)$ (with $t \in [a, b]$) for the "upper half" of C.

Solution:

sphere: $(x-2)^2+(y-2)^2+(z-2)^2=6^2$

plane: $2x+y+z=4$

Let $x=t$, $z=4-2x-y=4-2t-y$.

$(t-2)^2+(y-2)^2+(2-2t-y)^2=6^2 \Rightarrow t^2-4t+4+y^2-4y+4+y^2+4t^2+4-8t-4y+4ty=36$

$\Rightarrow 2y^2+5t^2-12t-8y+4ty-24=0 \Rightarrow 2y^2+(4t-8)y+(5t^2-12t-24)=0$

$$\Rightarrow y=\frac{-(4t-8)\pm\sqrt{(4t-8)^2-8(5t^2-12t-24)}}{4}=\frac{-(4t-8)\pm\sqrt{16t^2-64t+64-40t^2+96t+192}}{4}=\frac{-(4t-8)\pm\sqrt{-24t^2+32t+256}}{4}$$

$$=\frac{-(4t-8)\pm2\sqrt{-6t^2+8t+64}}{4}$$

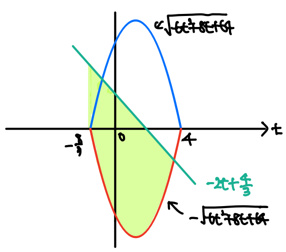
$$=\frac{-(2t-4)\pm\sqrt{-6t^2+8t+64}}{2}$$

$z=4-2x-y=4-2t-\dfrac{-(2t-4)\pm\sqrt{-6t^2+8t+64}}{2}$, we want to satisfy $z \geq$ center of intersecting circle to get the upper half of the circle with respect to the z-axis.

Let L_1 pass $(2,2,2)$ with $\vec{n}=\langle 2,1,1\rangle$, L_1: $\langle 2+2t, 2+t, 2+t\rangle \longrightarrow 2(2+2t)+(2+t)+(2+t)=4 \rightarrow t=-\frac{2}{3}$, hence the center of the circle is $(\frac{2}{3},\frac{4}{3},\frac{4}{3})$.

$z=4-2t-\dfrac{-(2t-4)\pm\sqrt{-6t^2+8t+64}}{2}\geq\frac{4}{3} \rightarrow 8-4t+(2t-4)\mp\sqrt{-6t^2+8t+64}\geq\frac{8}{3} \rightarrow -2t+\frac{4}{3}\geq\pm\sqrt{-6t^2+8t+64}$

$(-6t^2+8t+64\geq0$ so $-\frac{8}{3}\leq t\leq4)$



the coloured region satisfies where the above condition is satisfied.

when $-2t+\frac{4}{3}\geq\sqrt{-6t^2+8t+64}$, $t\in[-\frac{8}{3},\frac{2}{3}(1-\sqrt{15})]$

when $-2t+\frac{4}{3}\geq-\sqrt{-6t^2+8t+64}$, $t\in[-\frac{8}{3},\frac{2}{3}(1+\sqrt{15})]$

$$\therefore\ r(t)=\begin{cases}\left\langle t,\ \dfrac{-(2t-4)+\sqrt{-6t^2+8t+64}}{2},\ 4-2t-\dfrac{-(2t-4)+\sqrt{-6t^2+8t+64}}{2}\right\rangle;\ t\in[-\frac{8}{3},\frac{2}{3}(1-\sqrt{15})] \\ \left\langle t,\ \dfrac{-(2t-4)-\sqrt{-6t^2+8t+64}}{2},\ 4-2t-\dfrac{-(2t-4)-\sqrt{-6t^2+8t+64}}{2}\right\rangle;\ t\in[-\frac{8}{3},\frac{2}{3}(1+\sqrt{15})]\end{cases}$$

https://www.desmos.com/3d/u6xlgay4zx ⟵ plot for parameterization above.

2. Consider the curve C parametrized as $\mathbf{r}(t) = \langle \cos t, t, t^3 \rangle$. Consider the point $Q = (1, 0, 0)$ on C.

(a) $\boxed{\text{1 mark}}$ Parametrize the line L_1 containing the point Q and the vector $\mathbf{r}'(0)$.

Solution: $r'(t) = \langle -\sin t, 1, 3t^2 \rangle$

$r'(0) = \langle 0, 1, 0 \rangle$ $\quad$ $Q = (1, 0, 0)$

$L_1 = t \cdot \langle 0, 1, 0 \rangle + (1, 0, 0)$

$= \langle 1, t, 0 \rangle \; ; \; t \in \mathbb{R}.$

(b) $\boxed{\text{1 mark}}$ Parametrize the line L_2 containing the point Q and the vector $\mathbf{r}''(0)$.

Solution: $r''(t) = \langle -\cos t, 0, 6t \rangle$

$r''(0) = \langle -1, 0, 0 \rangle$ $\quad$ $Q = (1, 0, 0)$

$L_2 = t \cdot \langle -1, 0, 0 \rangle + (1, 0, 0)$

$= \langle 1-t, 0, 0 \rangle \; ; \; t \in \mathbb{R}.$

(c) $\boxed{\text{2 marks}}$ Find the equation of the plane Π containing the point Q and the lines L_1, L_2.

Solution:

$\vec{d}_1 = \langle 0, 1, 0 \rangle$ (direction vector of L_1)

$\vec{d}_2 = \langle -1, 0, 0 \rangle$ (direction vector of L_2)

$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \langle 0, 0, 1 \rangle$

$\Pi : 0 \cdot (x-1) + 0 \cdot (y-0) + 1 \cdot (z-0) = 0 \Rightarrow z = 0$

$\therefore \; \Pi : z = 0$

3. $\boxed{\text{5 marks}}$ Determine a parametrization $\mathbf{r}(u)$ for the unit circle C in the plane such that $u = 0, \frac{\pi}{4}, \frac{\pi}{2}$ correspond to the points $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), (0,1), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on C. Specify the corresponding "re-parametrization function" $u(\theta)$ with θ as the standard radians parameter for C.

Solution:

Let $r(u) = \langle \cos(u+\frac{\pi}{4}), \sin(u+\frac{\pi}{4}) \rangle ; u \in [-\frac{\pi}{4}, \frac{7\pi}{4})$ and $u(\theta) = \theta - \frac{\pi}{4}$

$u=0$, $r(0) = \langle \cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}) \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$u = \frac{\pi}{4}$, $r(\frac{\pi}{4}) = \langle \cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}) \rangle = \langle 0, 1 \rangle$

$u = \frac{\pi}{2}$, $r(\frac{\pi}{2}) = \langle \cos(\frac{3}{4}\pi), \sin(\frac{3}{4}\pi) \rangle = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$. which is a valid choice.

Given $u(\theta) = \theta - \frac{\pi}{4}$, $r(u(\frac{\pi}{4})) = \langle \cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}) \rangle$

$r(u(\frac{\pi}{2})) = \langle \cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}) \rangle$

$r(u(\frac{3}{4}\pi)) = \langle \cos(\frac{3}{4}\pi), \sin(\frac{3\pi}{4}) \rangle$, which also shows that $u(\theta) = \theta - \frac{\pi}{4}$ is the corresponding reparamization function.

4. $\boxed{\text{6 marks}}$ Determine the curvature κ and torsion τ of the curve C given by $\mathbf{r}(t) = \langle \sin t, t, t^3 \rangle$ when $t = \pi$.

Solution:

$r'(t) = \langle \cos t, 1, 3t^2 \rangle$

$r''(t) = \langle -\sin t, 0, 6t \rangle$

$r'''(t) = \langle -\cos t, 0, 6 \rangle$

$r'(\pi) = \langle -1, 1, 3\pi^2 \rangle$ $\quad r'''(\pi) = \langle 1, 0, 6 \rangle$

$r''(\pi) = \langle 0, 0, 6\pi \rangle$

$r'(\pi) \times r''(\pi) = \langle 6\pi, 6\pi, 0 \rangle$

$\|r'(\pi) \times r''(\pi)\| = 6\sqrt{2}\pi$ $\quad \therefore \kappa(\pi) = \dfrac{6\sqrt{2}\pi}{(9\pi^4+2)^{\frac{3}{2}}}$

$\|r'(\pi)\|^3 = (9\pi^4+2)^{\frac{3}{2}}$

$(r'(\pi) \times r''(\pi)) \cdot r'''(\pi) = \dfrac{\langle 6\pi, 6\pi, 0 \rangle \cdot \langle 1, 0, 6 \rangle}{\| \langle 6\pi, 6\pi, 0 \rangle \|^2} = \dfrac{6\pi}{72\pi^2} = \dfrac{1}{12\pi}$

$\therefore \tau(\pi) = \dfrac{1}{12\pi}$