No. 30

MATH 317 — HOMEWORK 2 — Due date: 17th July - 25 Marks

- Write your name, student number, and signature below.
- Write your answers to each question in the spaces provided. Submit your answers in Canvas on the due date (do not submit any additional pages).

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1. $\boxed{\text{5 marks}}$ Consider the curve C parametrized by $\mathbf{r}(t) = \langle t, 4sin(2t), 4cos(2t) \rangle$ for $t \in [0, 2\pi]$.

(a) Determine the arc length function $s(t)$ associated with $\mathbf{r}(t)$.

(b) Find the length of C.

Solution:

$r'(t) = \langle 1, 8\cos 2t, -8\sin 2t \rangle, \ t \in [0, 2\pi]$

(a) $S(t) = \int_0^t \|r'(u)\| \, du = \int_0^t \sqrt{65} \, du = \sqrt{65}\, t$

(b) $S(2\pi) = 2\sqrt{65}\,\pi$

2. $\boxed{\text{4 marks}}$ Calculate $\int_C (x+y^2)ds$ where C is the curve parameterized as $\mathbf{r}(t) = \langle t, \sin t, \cos t \rangle$ for $t \in [0, \pi]$.

Solution:

$F(x,y) = x+y^2$, $F(x(t),y(t)) = t+\sin^2 t$.

$r'(t) = \langle 1, \cos t, -\sin t \rangle$, $\|r'(t)\| = \sqrt{2}$

$$\int_C (x+y^2)\, ds = \int_0^{\pi} \sqrt{2}\cdot(t+\sin^2 t)\, dt$$

$$= \sqrt{2}\cdot\int_0^{\pi}\left(t+\frac{1-\cos 2t}{2}\right)dt$$

$$= \sqrt{2}\cdot\left[\frac{1}{2}t^2+\frac{1}{2}t-\frac{1}{4}\sin 2t\right]_0^{\pi}$$

$$= \frac{\sqrt{2}\pi}{2}(\pi+1)_\square \qquad \therefore \frac{\sqrt{2}\pi}{2}(\pi+1)$$

3. $\boxed{\text{4 marks}}$ Calculate

$$\int_C e^y\,dx + \ln(x)\,dy$$

where C is the line segment from the point $(1,1)$ to $(4,2)$.

Solution:

$C:\ t\cdot\langle 3,1\rangle + \langle 1,1\rangle \ ;\ t\in[0,1] \longrightarrow \langle 3t+1, t+1\rangle \ ;\ t\in[0,1].$ $\quad dx = 3dt,\ dy = dt$

$\int_C e^y dx + \ln(x)\,dy = \int_C \langle e^y, \ln x\rangle \cdot \langle dx, dy\rangle = \int_C \langle e^{t+1}, \ln(3t+1)\rangle \cdot \langle 3,1\rangle\, dt$

$= \int_0^1 3e^{t+1} + \ln(3t+1)\, dt = \left[\, 3e^{t+1} + \frac{(3t+1)}{3}(\ln(3t+1) - 1)\,\right]_0^1$

$= 3(e^2 - e) + \frac{4}{3}(\ln 4 - 1) - \frac{1}{3}(-1)$

$= 3(e^2 - e) + \frac{8}{3}\ln 2 - 1.$

$\therefore\ 3(e^2 - e) + \frac{8}{3}\ln 2 - 1$.

4. $\boxed{\text{4 marks}}$.

For $\mathbf{F} = \langle 3xy, x^3 + 4y \rangle$:Determine whether each of the following vectors is conservative. If so find the potential $\phi(x, y)$

(a) $\mathbf{F} = \langle 3xy, x^3 + 4y \rangle$

(b) $\mathbf{F} = \langle ye^{xy}, e^{xy} + y^2 \rangle$

Solution:

(a) $\phi_x = 3xy$, $\phi_y = x^3 + 4y$.

$\phi_x = 3xy \longrightarrow \phi = \frac{3}{2}x^2y + f(y) \longrightarrow \phi_y = \frac{3}{2}x^2 + f'(y) = x^3 + 4y$

$f'(y) = x^3 - \frac{3}{2}x^2 + 4y$, $f(y)$ shouldn't include an x term.

$\therefore$ Not conservative.

(b) $\phi_x = ye^{xy}$, $\phi_y = e^{xy} + y^2$

$\phi_x = ye^{xy} \longrightarrow \phi = e^{xy} + f(y)$

$\longrightarrow \phi_y = x \cdot e^{xy} + f'(y)$

$f'(y) = (1 - x)e^{xy} + y^2$

$f'(y)$ should not have at x-term. $\therefore$ Not conservative.

5. $\boxed{\text{4 marks}}$ Consider the vector field $\mathbf{F}(x, y) = \langle x, y \rangle$. Let C be the curve parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for t in the interval $[0, 2\pi]$. Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ using only the definition of line integrals, without assuming any facts about conservative vector fields.

Solution:

$F(x,y) = \langle x, y \rangle \longrightarrow F(x(t), y(t)) = \langle \cos t, \sin t \rangle$

$r'(t) = \langle -\sin t, \cos t \rangle$.

$\int_C F \cdot dr = \int_0^{2\pi} \langle \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{2\pi} 0 \, dt = 0$.

$\therefore 0$

6. $\boxed{\text{4 marks}}$ Given that $\mathbf{F} = \langle x^2y^3, x^3y^2 \rangle$ and $C : \mathbf{r}(t) = \langle t^3 - 2t, t^3 + 2t \rangle$ for t in the interval $[0, 1]$.

(a) Find a function f such that $\mathbf{F} = \nabla f$.

(b) Hence, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C

Solution:

(a) $f_x = x^2y^3 \longrightarrow f = \frac{1}{3}x^3y^3 + g(y) \rightarrow f_y = x^3y^2 + g'(y) = x^3y^2$. There, $g(y) = C$, $C \in \mathbb{R}$.

$\therefore f = \frac{1}{3}x^3y^3 + C$, $C \in \mathbb{R}$.

(b) $\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(x(1), y(1)) - f(x(0), y(0))$

$= f(-1, 3) - f(0,0) = \left(\frac{1}{3}(-1)(27) + C\right) - (0 + C) = -9$

$\therefore -9$