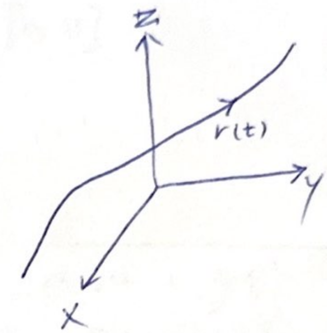
Lecture 8



Invariance of def's:

$$\int_a^b f(r(t))\,\left\|\frac{dr}{dt}(t)\right\|\,dt$$ $\longrightarrow$ integrate by substituision

$$\begin{cases} u = u(t) \\ du = \left(\frac{du}{dt}\right) dt \end{cases}$$

$$= \int_{u(a)}^{u(b)} f(r(u))\left\|\frac{dr}{du}\cancel{\frac{du}{dt}}\right\|\left(\cancel{\frac{dt}{du}}\right)du$$

$$= \int_{u(a)}^{u(b)} f(r(u))\left\|\frac{dr}{du}\right\|\,du$$ $\longrightarrow$ same expression!

$\longrightarrow$ f is Scalar.

(So $\int_C f\,ds$ is parametr. invariant formula).

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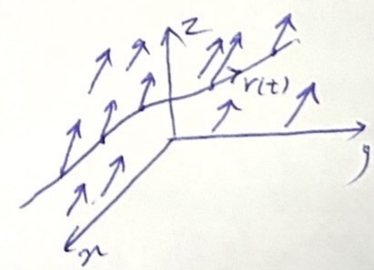
if f is vector field:

$$\int_a^b F(r(t))\cdot\frac{dr}{dt}\,dt$$ $\longrightarrow$ again use subst. above $\longrightarrow$ $$= \int_{u(a)}^{u(b)} F(r(u))\cdot\frac{dr}{du}\left(\cancel{\frac{du}{dt}}\right)\cancel{\frac{dt}{du}}\,du$$



$$= \int_{u(a)}^{u(b)} F(r(u))\,\frac{dr}{du}\,du$$ $\longrightarrow$ same expression!

Exp) $\int_C \sqrt{x^2+y^2}\,ds$, C paramet. as $r(t) = \langle t\cos t, t\sin t, t^2 \rangle$ $t \in [0, \pi]$

$$f(r(t)) = \sqrt{t^2\cos^2 t + t^2 \sin^2 t} = \sqrt{t^2} = t$$

$$\|r'(t)\| = \| \langle \cos t - t\sin t, \sin t + t\cos t, 2t \rangle \| = \sqrt{\underbrace{\cos^2 t}_{1} - \cancel{2t\sin t\cos t} + t^2\sin^2 t + 4t^2 + \sin^2 + t^2\cos^2 t + \cancel{2t\sin t\cos t}}$$

$$= \sqrt{1+5t^2}$$

$$\Longrightarrow \int_C \sqrt{x^2+y^2}\,ds = \int_0^{\pi} t\sqrt{1+5t^2}\,dt = (1+5t^2)^{3/2}\left(\frac{3}{2}\right)\left(\frac{1}{10}\right)\Big|_0^{\pi}$$

Exp) Find $\int_C F \cdot dr$, $F = \langle x-z, y-z, -x-y \rangle$

C = start line from $(0,0,0)$ to $(1,0,0)$

Then stratline from $(1,0,0)$ to $(1,1,1)$

(C is only piecewise differentiable)

integral along one piece + ing. alonge other piece:

$\begin{cases} r(t) = \langle t,0,0 \rangle; & t \in [0,1] \\ r'(t) = \langle 1,0,0 \rangle; & \end{cases}$ + $\begin{cases} r(t) = \langle 1,t,t \rangle & t \in [0,1] \\ r'(t) = \langle 0,1,1 \rangle; & \end{cases}$

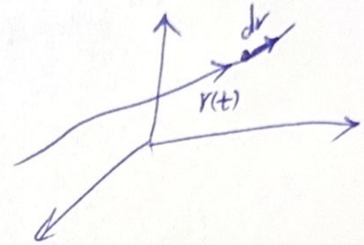
$= \int_0^1 F(r(t)) \cdot r'(t)\,dt = \int_0^1 [\langle t,0,-t \rangle \cdot \langle 1,0,0 \rangle = t]\,dt = \frac{t^2}{2}\Big|_0^1 = 1/2$ $+ \int_0^1 F(r(t)) \cdot r'(t)\,dt = \int_0^1 [\langle 1-t,0,-1-t \rangle \cdot \langle 0,1,1 \rangle]\,dt$

$-t - \frac{t^2}{2}\Big|_0^1 = -3/2$

$= -3/2 + 1/2 = -1$

* Can also be as: $\int (x-z)\,dx + (y-z)\,dy + (-x-y)\,dz$

Interpretations:



$$\int_C f\,ds = \int_a^b f(r(t))\,\|r'(t)\|\,dt$$

C = material wire

f = density of wire varies along C (Mass/Lenght)

at t:

Chang dt $\longrightarrow$ change dr

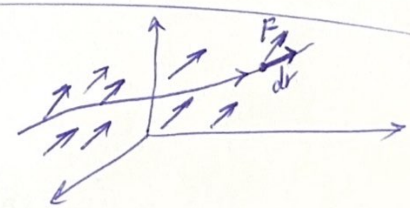
length is: $\|dr\| = \left\|\frac{dr}{dt}\right\| dt$

So mass is: $\cancel{\|dr\|}$

$$dM = f(r(t)) \cdot \left\|\frac{dr}{dt}\right\| dt$$

$\Longrightarrow \int_C f\,ds = \int_C dM$ = Total mass of curve m.

Note: if $f \equiv 1$, then, integral is infact lenght of C.



$$\int_C F.dr = \int_a^b F(r(t)).r'(t)\,dt$$

C = path of displacement (motion)

F = force field

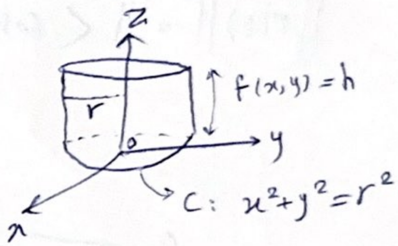
at t: Change dt $\longrightarrow$ change dr

work done by F for the change = $dW = F(r(t)).dr = F(r(t))\frac{dr}{dt}dt$

$\rightarrow = \int_C dW = W$ = Total work done by F in displacing along C.

Exp) Use a line integral to show that the lateral surface area (A) of a right circular cylinder of radius r and height h is $2\pi r h$? (5)

Sol: we use right circular cylinder with base circle C given by $x^2+y^2=r^2$ & with height h in the positive z direction:



Parametrize C: $\langle r\cos t, r\sin t\rangle$, $0 \le t \le 2\pi$

let $f(x,y)=h$ for all (x,y), then,

$$A=\int_C f(x,y)\,ds=\int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt$$

$$=\int_0^{2\pi} h\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt$$

$$=h\int_0^{2\pi} r\sqrt{\sin^2 t+\cos^2 t}\,dt$$

$$=rh\int_0^{2\pi} 1\,dt$$

$$=2\pi r h$$

Note: for line integral of real-valued functions (Scalar ~~function~~ fields):

Reversing the direction in which the integral is taken along the curve does not change the value of line integral:

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

But line integrals of vector fields changes.

let: $f(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field: P & Q are continuously differentiable

let C param. by $\langle x(t), y(t)\rangle$, $t \in [a,b]$

Then the curve $(-C)$ traversed in the opposite direction is param. by

$$\langle x(a+b-t), y(a+b-t)\rangle \; ; t \in [a,b]$$

$$\Rightarrow \int_{-C} P(x,y)\,dx = \int_a^b P(x(a+b-t)), y(a+b-t))\, \frac{d}{dt}(x(a+b-t))\,dt$$

$$= \int_a^b \quad \sim \quad \sim \quad (-x'(a+b-t))\,dt \qquad \text{(by the chain Rule)}$$

$$= \int_b^a p(x(u), y(u))\,(-x'(u))\,(-du) \qquad \text{(by letting } u = a+b-t)$$

$$= \int_b^a P(x(u), y(u))\,x'(u)\,du = -\int_a^b P(x(u), y(u))\,x'(u)\,du \qquad \left(\text{since } \int_b^a = -\int_a^b\right)$$

$\Rightarrow \int_{-C} P(x,y)\,dx = -\int_C P(x,y)\,dx$, similarly $\int_{-C} Q(x,y)\,dy = -\int_C Q(x,y)\,dy \Rightarrow \boxed{\int_{-C} f\cdot dr = -\int_C f\cdot dr}$