lecture 17

Converse to theorem? $\nabla \times \vec{F} = \vec{0} \Rightarrow \vec{F}$ conservative?

Will stablish by:

$$\left\{\begin{array}{l} \nabla \times \vec{F} = \vec{0} \\ \text{at all points} \\ \text{in } U \subseteq \mathbb{R}^3 \end{array}\right\} \overset{A}{\Longleftrightarrow} \left\{\begin{array}{l} \int_C \vec{F} \cdot dr \text{ is path independent in } U: \\ \text{depends only on start point and} \\ \text{end point of } C \text{ in } U. \end{array}\right\} \overset{B}{\Longleftrightarrow} F \text{ conservative in } U$$

means that F has a potential in domain U

* A assumes that $U \subseteq \mathbb{R}^3$ is "simply connected" open set.

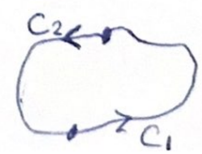
Proof of A is related to end course (Stokes theorems & Green)

* B assumes that $U \subseteq \mathbb{R}^3$ is "path connected" open set; a set in space is path connected if any two points in that set can be joined by a continuous path in that set.

* For $\vec{F} = \langle F_1, F_2 \rangle$, replace $\mathbb{R}^3$ with $\mathbb{R}^2$, replace curl $\vec{F} = \vec{0}$ with $(F_1)_y = (F_2)_x$

* $\int_C \vec{F} \cdot dr$ path indep. (all C in U) $\underset{II}{\overset{I}{\Longleftrightarrow}}$ $\int_{\bar{C}} \vec{F} \cdot dr = 0$ (all closed curves in U)

(I) $\rightarrow$ let $\bar{C}$ be closed curve (same start and end point)
Divid $\bar{C}$ into C_1 & C_2



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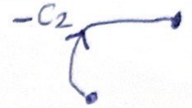


$\Rightarrow$ parametrize the same as C_2 but with opposite direction

$$\int_C \vec{F}\cdot dr = \int_{C_1} \vec{F}\cdot dr + \int_{C_2} \vec{F}\cdot dr = \int_{C_1} \vec{F}\cdot dr - \int_{-C_2} \vec{F}\cdot dr \underset{\Downarrow}{=} 0$$

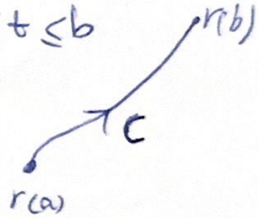
by assumption of path independence.

(II) similar ideas.

$\Leftarrow$

(Proof of B)

($\Leftarrow$) Suppose $\vec{F} = \nabla\phi$, some ϕ in U. for any C in U param as $r(t)$, $a \le t \le b$



$$\int_C \vec{F}\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt = \int_a^b \underbrace{\nabla\phi(r(t))\cdot r'(t)}_{= \frac{d}{dt}[\phi(r(t))]}\,dt = \phi(r(t))\Big|_{r(a)}^{r(b)} = \phi(r(b)) - \phi(r(a))$$

chain [redacted] rule

$$\frac{d}{dt}[\phi(x(t), y(t), \ldots)] = \phi_x \cdot x_t + \phi_y \cdot y_t + \cdots = \nabla\phi \cdot r'(t)$$

i.e. $\int_C \vec{F}\cdot dr$ is path indep!

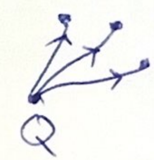
because the final value depends on start and end point only.

($\Rightarrow$) Suppose $\int_C \vec{F} \cdot dr$ is path indep. for all c in U:

define function ϕ on U as:

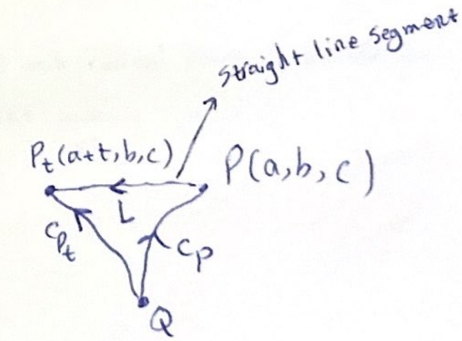
Fix some point ~~P~~ Q in U, for all P in U, let C_P be some curve from Q to P:

$$\phi(P) = \int_{C_P} \vec{F} \cdot dr \quad \text{for all } P \text{ in } U$$



* Can always join Q to any P in U by assumption of "U is path connected & open".

Now must prove $\vec{F} = \nabla\phi$! ie) $\begin{cases} \phi_x = F_1 \\ \phi_y = F_2 \\ \phi_z = F_3 \end{cases}$



fix some $P = (a,b,c)$ in U

Consider $P_t = (a+t, b, c)$ for small t

L is line segment from P to P_t

$\vec{r}(u) = \langle a+u, b, c \rangle \quad u \in [0,t]$

$r'(u) = \langle 1, 0, 0 \rangle$

$$\phi(P_t) \underset{\text{by definition}}{=} \int_{C_{P_t}} \vec{F} \cdot dr = \underbrace{\int_{C_P} \vec{F} \cdot dr + \int_L \vec{F} \cdot dr}_{\text{by path independence}} = \phi(P) + \int_0^t F_1(a+u,b,c) \cdot 1 \, du$$

$$\Rightarrow \frac{d}{dt}\phi(P_t)\Big|_{t=0} = \frac{d}{dt}\big(\phi(p)\big) + \underbrace{F_1(a,b,c)}_{\text{Fundamental Theorem Calculus}}$$

by definition

sinic doesn't dependont

0

Fundamental Theorem calculus → If $F(x) = \int_a^x f(t)\,dt \Rightarrow$ then $F'(x) = f(x)$

$$\phi_x(p) = F_1(p)$$

likwise for ϕ_y, ϕ_z ⟹ Therefore: $F = \nabla\phi$ means that F is conservative!

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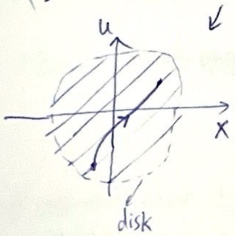
* illustrations of path connected sets in $\mathbb{R}^2$ ($\widehat{\text{sets}}$ = U)

(you can joint every two points)



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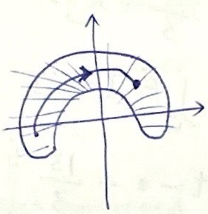

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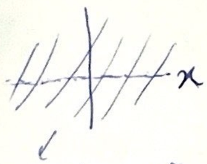
→ disk with removed point at the center like $\frac{\quad}{x^2+y^2}$



✗ is not path connected

Excluded x-axis

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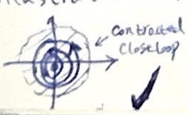
* illustration of simply connectedness of U in $\mathbb{R}^2$ → means any closed loop in U can be contracted continuosly to a point, all within U



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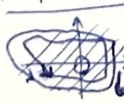

✗

$U = \mathbb{R}^2 - \{\text{origin}\}$ = excluded origin

→ Can not be contracted

Exp) Show that the line integral $\int_C (x^2+y^2)dx + 2xy\,dy$ is path independent. 5

Also, show that this line integral along any Curve C going from (0,0) to (1,2) will always be $13/3$:

Sol:

F is Path independent $\rightarrow F = \nabla\phi$

$$\begin{cases} \phi_x = x^2+y^2 \rightarrow \phi = \frac{1}{3}x^3 + xy^2 + g(y) \\ \phi_y = 2xy \Rightarrow 2xy = \left(\frac{1}{3}x^3 + xy^2 + g(y)\right)_y = 2xy + g'(y) \rightarrow g(y) = C \end{cases}$$

for some function $g(y)$

(any constant like zero)

$\Rightarrow$ potential exists for F: $\phi = \frac{x^3}{3} + xy^2$

$\Rightarrow \int_C \vec{F}.dr$ is path independent

$* \int_C F.dr = \phi(1,2) - \phi(0,0) = \frac{1}{3} + 4 - 0 = \boxed{\frac{13}{3}}$