CPSC 303, 2024/25, Term 2, Assignment 3

Released Wednesday, February 12, 2025
Due Wednesday, March 5, 2025, 11:59pm

Note: All questions are based on exercises from the textbook. In a few places they have been slightly edited to provide additional instructions and/or additional clarity. At the beginning of each of the questions we state where they can be found in the book. For example, "AG 12.3" means that we are referring to Ascher & Greif, Chapter 12, Exercise 3.

1. (AG 12.1, slightly edited) Using the MATLAB instruction `cond`, find the condition numbers of Hilbert matrices for $n = 4, 5, \ldots, 12$. Plot these condition numbers as a function of n using `semilogy`. What are your observations? Comment specifically on how accurate we may expect the numerical solution of a linear system with a Hilbert matrix for a given n to be in a double precision floating point system.

2. (AG 12.2, edited) Construct the second degree polynomial $q_2(t)$ that approximates $g(t) = \sin(\pi t)$ on the interval $[0, 1]$ by minimizing

$$\int_0^1 [g(t) - q_2(t)]^2 \, dt.$$

   Some useful integrals that you may use without proving their correctness:

$$\int_0^1 (6t^2 - 6t + 1)^2 \, dt = \frac{1}{5}, \quad \int_0^1 \sin(\pi t) \, dt = \frac{2}{\pi},$$
$$\int_0^1 t \sin(\pi t) \, dt = \frac{1}{\pi}, \quad \int_0^1 t^2 \sin(\pi t) \, dt = \frac{\pi^2 - 4}{\pi^3}.$$

   (a) Find, using pen and paper, the polynomial in two ways: once, using a monomial basis, and once using the Legendre polynomials. Confirm that the two polynomials you have obtained are identical.

   (b) Write a MATLAB script to plot the function $g(t)$ and the polynomial you have found, on the same graph (use the command `hold`). For this part, there is no need to write a program that computes the polynomial. You may directly plug in the polynomial you found in part (a).

3. (AG 12.3) The Legendre polynomials satisfy

$$\int_{-1}^1 \phi_j(x)\phi_k(x) dx = \begin{cases} 0 & j \neq k \\ \frac{2}{2j+1} & j = k. \end{cases}$$

   Suppose that the best fit problem is given on the interval $[a, b]$.

Show that with the transformation $t = \frac{1}{2}[(b-a)x + (a+b)]$ and a slight change of notation, we have

$$\int_a^b \phi_j(t)\phi_k(t)dt = \begin{cases} 0 & j \neq k \\ \frac{b-a}{2j+1} & j = k. \end{cases}$$

4. (AG 12.5, minimally edited)

   (a) Using an orthogonal polynomial basis, find the best least squares polynomial approximations, $q_2(t)$ of degree at most 2 and $q_3(t)$ of degree at most 3, to $f(t) = e^{-3t}$ over the interval $[0, 3]$.
   [Hint: For a polynomial $p(x)$ of degree n and a scalar $a > 0$ we have $\int e^{-ax}p(x)dx = \frac{-e^{-ax}}{a}(\sum_{j=0}^{n}(\frac{p^{(j)}(x)}{a^j}))$, where $p^{(j)}(x)$ is the j^{th} derivative of $p(x)$. Alternatively, just use numerical quadrature, e.g., the MATLAB function `integral`.]

   (b) Plot the error functions $f(t) - q_2(t)$ and $f(t) - q_3(t)$ on the same graph on the interval $[0, 3]$. Compare the errors of the two approximating polynomials. In the least squares sense, which polynomial provides the better approximation?
   [Hint: In each case you may compute the *norm* of the error,
   $\left[\int_a^b (f(t) - q_n(t))^2 dt\right]^{1/2}$, using the MATLAB function `integral`.]

   (c) Without any computation, prove that $q_3(t)$ generally provides a least squares fit, which is never worse than with $q_2(t)$.

5. (AG 12.6, harder) Let $\phi_0(x), \phi_1(x), \phi_2(x), \ldots$ be a sequence of orthogonal polynomials on an interval $[a, b]$ with respect to a positive weight function $w(x)$. Let $x_1, \ldots, x_n$ be the n zeros of $\phi_n(x)$; it is known that these roots are real and $a < x_1 < \cdots < x_n < b$.

   a) Show that the Lagrange polynomials of degree $n-1$ based on these points are orthogonal to each other, so we can write

   $$\int_a^b w(x)L_j(x)L_k(x)dx = 0, \quad j \neq k,$$

   where

   $$L_j(x) = \prod_{k=1, k\neq j}^{n} \frac{(x - x_k)}{(x_j - x_k)}, \quad 1 \leq j \leq n.$$

   [Recall Section 10.3.]

   b) For a given function $f(x)$, let $y_k = f(x_k)$, $k = 1, \ldots, n$. Show that the polynomial $p_{n-1}(x)$ of degree at most $n-1$ that interpolates the function $f(x)$ at the zeros $x_1, \ldots, x_n$ of the orthogonal polynomial $\phi_n(x)$ satisfies

   $$\|p_{n-1}\|^2 = \sum_{k=1}^{n} y_k^2 \|L_k\|^2,$$

   in the weighted least squares norm. This norm is defined by

   $$\|g\|^2 = \int_a^b w(x)[g(x)]^2 dx,$$

   for any suitably integrable function $g(x)$.

6. (AG 12.8) Using the recursion formula for Chebyshev polynomials, show that $T_n(x)$ can be written as

$$T_n(x) = 2^{n-1}(x - x_1)(x - x_2) \cdots (x - x_n),$$

where x_i are the n roots of T_n.

7. (AG 12.9) Jane works for for a famous bioinformatics company. Last year she was required by Management to approximate an important but complicated formula, $g(x)$, defined on the interval $[-1, 1]$, by a polynomial of degree $n + 1$. She did so, and called the result $f(x) = p_{n+1}(x)$.

   Last week, Management decided that they really needed a polynomial of degree n, not $n+1$, to represent g. Alas, the original g had been lost by this time and all that was left was $f(x)$. Therefore, Jane is looking for the polynomial of degree n which is closest (in the maximum norm) to f on the interval $[-1, 1]$. Please help her find it.

8. (AG 13.3, slightly edited) Find the continuous Fourier transform for $l = 50$, i.e., the coefficients of (13.1b)–(13.1c), for the function $f(x) = \cos(3x) - .5\sin(5x) + .05\cos(54x)$ on the interval $[-\pi, \pi]$. Let $v(x)$ denote the corresponding approximation. Note that most of the Fourier coefficients vanish for this example. Plot $f(x)$ and $v(x)$ one on top of each other (use `subplot`) and make observations.

9. (AG 13.7) Write a short program to interpolate the hat function of Example 13.3 by trigonometric polynomials for $l = 2, 4, 8, 16$ and 32. Measure maximum absolute error on the mesh `0:.01*pi:2*pi` and observe improvement that is only linear in l (or $n = 2l - 1$).

10. (Ag 13.9) Consider using a DFT to interpolate the function $f(x) = \log(x + 1)$ on the interval $[0, 2\pi]$ as in the examples of Section 13.2.

    (a) Construct and plot the interpolant on $[0, 2\pi]$ for $l = 16$ and $l = 32$.
    Explain why the results look unsatisfactory.

    (b) Consider an even extension of $f(x)$, defining

    $$g(t) = \begin{cases} f(t) & 0 \le t < 2\pi \\ f(4\pi - t) & 2\pi \le t < 4\pi \end{cases}.$$

    Apply DFT interpolation to $g(t)$ and plot the results on $[0, 2\pi]$. Find maximum errors for $l = 16$ and $l = 32$. Are they better than before? Why?