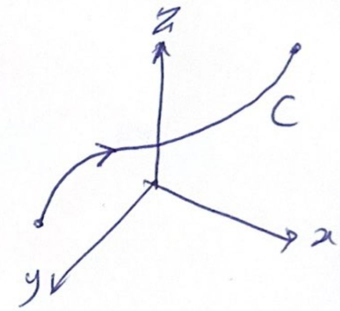
Lecture 3

Differential Geometry of curves:

Given $\vec{r}(t) = \langle X(t), Y(t), Z(t) \rangle$ $t \in [a,b]$

Parametrizing curve C.



We want an expression

$Q(X_t, Y_t, Z_t, X_{tt}, \dots)$

derivative

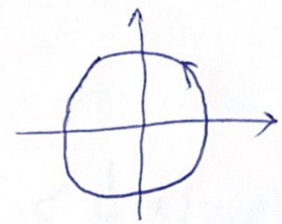
Such that when evaluated at some point at the carve C get same numbr or quntity regardless of which parametrization is used.

i.e. $Q(X_u, Y_u, \dots) = Q(X_v, Y_v, \dots)$

Same expression/function Q, but relative to 2 different parametrization $\vec{r}(u)$ & $\vec{r}(v)$ for C.

If we find such Q, it must reflect geometry of C at point!

Exp)
$$\begin{cases} \vec{r}(u) = \langle \cos u, \sin u \rangle \\ \|\vec{r}'(u)\| = 1, \quad \|\vec{r}''(u)\| = 1 \end{cases}$$



$$\begin{cases} \vec{r}(v) = \langle \cos(2v), \sin(2v) \rangle \\ \|\vec{r}'(v)\| = 2, \quad \|\vec{r}''(v)\| = 4 \end{cases}$$

Shows,

$Q = \|r_t\|$ or $Q = \|r_{tt}\|$ Doesn't work!

Note:

$Q = \dfrac{\|r_{tt}\|}{\|r_t\|^2}$ looks like it might be such a guntity (Works)!

((An check, this also fails in general))

To better understand, let $r(t)$, $r(u)$ be parametrizations of same curve C.

Then at common point p on C:

$$r_t = \left\langle \frac{d}{dt} x(t),\ \frac{d}{dt} Y(t), \dots \right\rangle = \left\langle \frac{d}{dt} \underset{\downarrow}{x}(u(t)), \dots \right\rangle$$

Component of $r(t)$ ⟵ ... component of $r(u)$

$$= \left\langle \frac{d}{du} x(u) \frac{du}{dt}, \dots \right\rangle = \left\langle \frac{d}{du} x(u), \dots \right\rangle du/dt$$

velocity vector $\rightarrow \boxed{r_t = r_u \cdot u_t}$ (1)

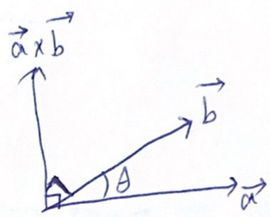
iterating this gives: $r_{tt} = (r_u u_t)_t = r_{ut} u_t + r_u u_{tt} = (r_{uu} u_t) u_t + r_u u_{tt}$

$$\boxed{r_{tt} = r_{uu} u_t^2 + r_u u_{tt}} \quad (2)$$

and $r_{ttt} = (r_{uu} u_t^2 + r_u u_{tt})_t = r_{uuu} u_t^3 + r_{uu}(2) u_t u_{tt} + r_{uu} u_t u_{tt} + r_u u_{ttt}$

$$\Longrightarrow \boxed{r_{ttt} = r_{uuu} u_t^3 + 3 r_{uu} u_t u_{tt} + r_u u_{ttt}} \quad (3)$$

Recall:



* $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

  ($\vec{a}\cdot\vec{b} = 0$ when $\theta = \pi/2$)

* $\vec{a}\times\vec{b}$ as in diagram

  ($\vec{a}\times\vec{b}$ perpendicular to $\vec{a}$ & $\vec{b}$)

* $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$

$$r_{tt}\times r_t = (r_{uu}u_t^2 + r_u u_{tt})\times r_u \cdot u_t$$

$$= r_{uu}\times r_u\,(u_t^3) + \cancelto{0}{r_u\times r_u u_{tt}u_t}$$

$$\frac{\|r_{tt}\times r_t\|}{\|r_t\|^3}$$ is a parameter invariant!

It is called Curvature "K" of C at point where being calculated.

Quick Check: $$\frac{\|r_{tt} \times r_t\|}{\|r_t\|^3} = \frac{\|r_{uu} \times r_u\| \cancel{\|u_t\|^3}}{\|r_u\|^3 \cancel{\|u_t\|^3}}$$

Note also:

$$(r_t \times r_{tt}) \cdot r_{ttt} = \left[(r_u \times r_{uu}) u_t^3\right] \cdot \left[r_{uuu} u_t^3 + \sim r_{uu} + \sim r_u\right]$$

$$= (r_u \times r_{uu}) \cdot r_{uuu} u_t^6 + \underbrace{0 + 0}_{\text{since } r_u \times r_{uu} \perp r_u \,\&\, r_{uu}}$$

$$\Rightarrow \frac{(r_t \times r_{tt}) \cdot r_{ttt}}{\|r_t \times r_{tt}\|^2}$$ → is also the same for any parameter

→ is also a parameter invariant, called torsion "τ" of C at point where calculated.

Exp) $\vec{r}(t) = \langle (\cos t + t\sin t), (\sin t - t\cos t), 0 \rangle$, $t > 0$

$\Rightarrow r_t = \langle t\cos t, t\sin t, 0 \rangle$ $\Longrightarrow \|r_{tt} \times r_t\| = \|\langle 0, 0, t^2 \rangle\| = t^2$

$\Rightarrow r_{tt} = \langle \cos t - t\sin t, \sin t + t\cos t, 0 \rangle$

$$k(t) = \frac{\|r_{tt} \times r_t\|}{\|r_t\|^3} = \frac{t^2}{t^3} = \boxed{\frac{1}{t}}$$