lecture 7

We also have: $\boxed{\dfrac{dN}{ds} = -\kappa T + \tau B}$ (*)

Proof: unit vector $N \cdot N = 1$ all ; $N \cdot B = 0$

$\rightarrow (N \cdot N)_s = 0$ $\qquad \rightarrow (N \cdot B)_s = 0$

$\rightarrow 2 N_s \cdot N = 0$ $\qquad \rightarrow N_s \cdot B + N \cdot \underbrace{B_s}_{(-\tau N)} = 0$

(N component of $N_s = 0$) 1

$\rightarrow N_s \cdot B - \tau = 0 \Rightarrow N_s \cdot B = \tau$

(B component of N_s is τ) 2

$\rightarrow$ similarly use

$N \cdot T = 0$, $T_s = \kappa N$

$\Rightarrow$ (T component of N_s is $-\kappa$) 3

$\rightarrow$ combining 1, 2, 3 gives (*)

In summary:

Frenet-~~[illegible]~~Serret Formulae:

If $r(s)$ is a.l. parametrization, $\{T(s), N(s), B(s)\}$ is frenet-serret frame:

$$\begin{cases} T_s = \kappa N \\ N_s = -\kappa T + \tau B \\ B_s = -\tau N \end{cases}$$

$\longrightarrow$ The rate of change of T, N, B can be expressed by frame itself i.e. T, N, B and κ and τ.

What is the point? If I tell you $\kappa(s)$ & $\tau(s) \neq 0$, $s \in [0, L]$, and let's say I also tell you $r(0)$, $r_s(0)$, $r_{ss}(0)$, then you can use F.S. Formula to reconstruct $r(s)$, $s \in [0, L]$!

* Means that if we know $\kappa(s)$ & $\tau(s)$ and some initial condition we can have $r(s)$. and determine the actual curve C.

Proof: This is beyond the scope of this course, which involves ODE theory.

Basically:
$$\left\{\begin{array}{l}\text{initial cond.}\\ T(0), N(0), B(0)\end{array}\right\} + \{\text{F-S formula}\}$$

$$\Downarrow$$

Can solve for $T(s)$, $N(s)$, $B(s)$ for $s \in [0, L]$, κ & τ are given functions

$$\Downarrow$$

Can then solve for $r(s)$ as antiderivative of $T(s)$

It is similar to $f' = cf$, $f(0) = 5 \longrightarrow f(x) = 5e^{cx}$ (Example)

* Given some param. $\vec{r}(t)$, do we really need to find a.l. para. $r(s)$ to determine T, N, B at given point on C? Not really, ..., but can be messy.

These give T, N, B
$$\begin{cases} r_s = r_t t_s = r_t\left(\dfrac{1}{s_t}\right) = \dfrac{r_t}{\|r_t\|} \\ r_{ss} = \left(\dfrac{r_t}{s_t}\right)_s = \left(\dfrac{r_t}{s_t}\right)_t t_s = \left(\dfrac{s_t r_{tt} - s_{tt} r_t}{s_t^2}\right)\left(\dfrac{1}{s_t}\right) \end{cases}$$

$s_{tt} = \dfrac{d}{dt}\|r_t\|$

$\Longrightarrow$ Right hand sides all are calculated relative to $r(t)$ without find a.l. function.

Line Integrals (Integral along curves)

From this point on, only need to know how to pramorize curves, and possibly re-parametriz

(will not need curvature, torsion, !)

let C be a curve parametrized as $\vec{r}(t)$, $t \in [a,b]$:

Def: Given scalar function $f(x,y,z)$, line integral of f along C is:

$$\int_C f\,ds := \int_a^b f(r(t))\,\|r_t(t)\|\,dt$$

Apears that it depends on parametrization but it gives the same result for different param. of a curve.

Def: Given vector field

$$\vec{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$$

(a vector at each point (x,y,z) in space)

Line integral of $\vec{F}$ along C is:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz := \int_a^b (\vec{F}(r(t)) \cdot r_t(t))\,dt$$

Just alterative notations for same thing: $\vec{F}\cdot d\vec{r} = \langle F_1, F_2, F_3 \rangle \cdot \langle dx, dy, dz \rangle$

Supp: work done by a force $F(r)$ moving a particle along a path $r(t)$: 4

from t to $t+dt$ $\longrightarrow$ particle moves from $r(t)$ to $r(t)+dr$ with $dr = \frac{dr}{dt}(t)\,dt$

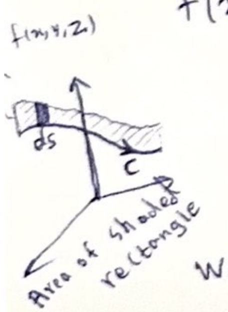
$\Rightarrow$ Work done during this time interal:

$$F(r(t)).dr = F(r(t)).\frac{dr}{dt}(t)\,dt$$

total work done during time interval from t_0 to t_1 =

$$\text{Work} = \int_{t_0}^{t_1} F(r(t)).\frac{dr}{dt}(t)\,dt$$

Def: For a real-valued function $f(x,y,z)$ and a curve C in $\mathbb{R}^3$, parametrized by $\langle x(t), y(t), z(t)\rangle$; $a \leq t \leq b$, the line integral of $f(x,y,z)$ along C with respect to arc length s is:



$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt = \int_a^b f(r(t))\,\|r_t\|\,dt$$

We know $s = s(t) = \int_a^t \sqrt{x'(u)^2+y'(u)^2+z'(u)^2}\,du$ $\qquad ds = s'(t)\,dt = \sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$