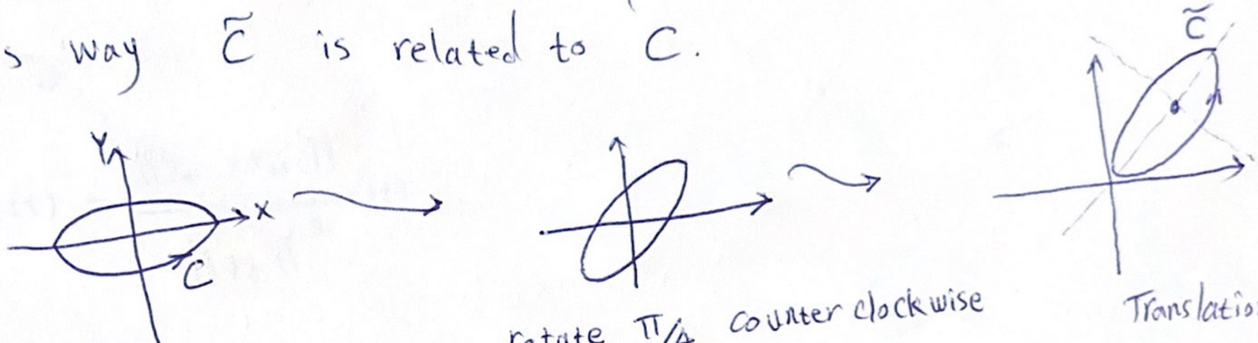
Lecture 4

* let $\tilde{C}$ be a curve obtained by $\left\{ \begin{array}{l} \text{rotation} \\ + \\ \text{translation} \end{array} \right.$ of other curve C

So in this way $\tilde{C}$ is related to C.

Exp) in $\mathbb{R}^2$



rotate $\pi/4$ counter clockwise

Translation by $\vec{V} = \langle 1, 1 \rangle$

Then parametrization $\tilde{r}$ of $\tilde{C}$ is obtained by $\left\{ \begin{array}{l} \text{rotation} \\ + \\ \text{translation} \end{array} \right.$ of Parametrization $r(t)$ of C

simply by application of matrix transformation applied to the components of $r(t)$.

And $\begin{array}{l} \tilde{r}_t(t) \\ \tilde{r}_{tt}(t) \\ \vdots \end{array}$ are obtained by rotation of $\begin{array}{l} r_t(t) \\ r_{tt}(t) \\ \vdots \end{array}$

Translation doesn't change the relation.

Follows that $\tilde{k}(t)$ & $\tilde{\tau}(t)$ are same as $k(t)$ & $\tau(t)$.

i.e. $k(t)$ & $\tau(t)$ are invariant, also, under rigid motions of C.

* rotation & translation doesn't change the geometry (rigid motion).
* To get $\tilde{k}(t)$ from $k(t)$: Just take the formula of $k(t)$, and rotate every veltor of

by the same angle.

* Remember: $k(t) = \dfrac{\|r_t \times r_{tt}\|}{\|r_t\|^3}(t)$

* Definition of $k(t)$ requires $r'(t) \neq \vec{0}$
  $\tau(t)$ " $r'(t) \times r''(t) \neq \vec{0}$

Interpretations: $k(t_0) = 0 \Rightarrow r_t \times r_{tt}(t_0) = \vec{0} \Rightarrow r_t, r_{tt}$ are parallel at t_0.

* when C is a sraight line $\longrightarrow$ r_t & r_{tt} are parallel

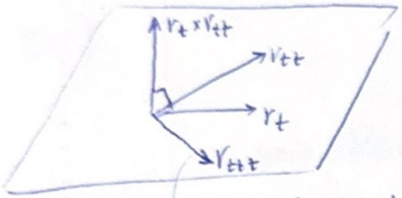
How to verify $\longrightarrow$

$$\begin{cases} r(t) = \langle a f(t), b, c f(t) + d, e f(t) + F \rangle \\ r_t = \langle a, c, e \rangle f' \\ r_{tt} = \langle a, c, e \rangle f'' \end{cases}$$

So $k(t)$ is ~~sm~~ some measure of deviation from being straight.

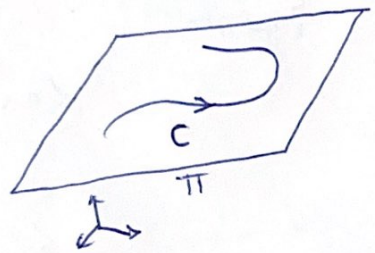
$\tau(t) = 0 \implies (r_t \times r_{tt}) \cdot r_{ttt} = 0$

$\implies r_t, r_{tt}, r_{ttt}$ are coplanar!



→ must be in the same plane.

* Model case: C lies on some plane



$r(t) = \langle F_1(t), F_2(t), F_3(t) \rangle$

$aF_1(t) + bF_2(t) + cF_3(t) = d$ → some a, b, c, d

↳ when this equation is satisfied for all t:

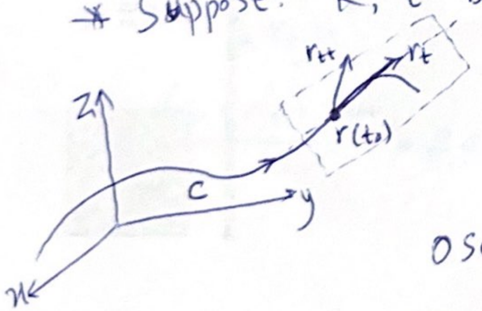
→ $\langle a, b, c \rangle \cdot r_t = 0$

→ $\langle a, b, c \rangle \cdot r_{tt} = 0$

⋮

So $\tau(t_0)$ is some measure of deviation of C from being in a plane around $r(t_0)$.

* Suppose: κ, τ both defined at $r(t_0)$ for some parametrization $r(t)$ of some C:



Then the following are parameter invariants of C at $r(t_0)$:

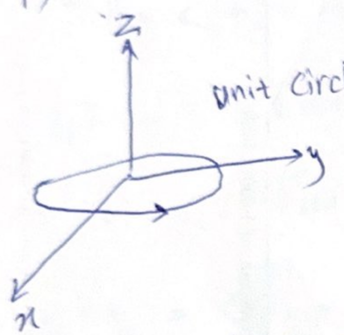
Tangent line: line containing $r(t_0)$ (point) and $\vec{r_t}(t_0)$

Oscillating plane: plane containing $r(t_0)$ (point), $\vec{r_t}(t_0)$, $\vec{r_{tt}}(t_0)$

$k(t_0) \rightarrow$ deviation from tangent line

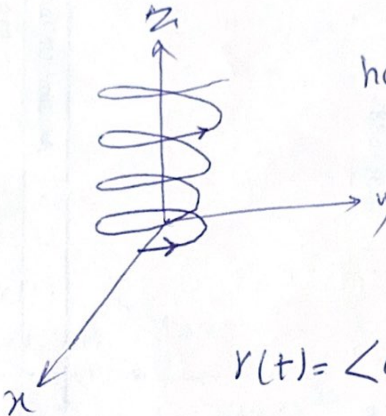
$\tau(t_0) \longrightarrow$ " " osc. plane

Exp)



unit circle

Can any segment of circle be cut out and placed on helix with overlap?



helix

$r(t) = \langle \cos t, \sin t, 0 \rangle$

$r_t(t) = \langle -\sin t, \cos t, 0 \rangle$

$r_{tt}(t) = \langle -\cos t, -\sin t, 0 \rangle$

$r_{ttt}(t) = \langle \sin t, -\cos t, 0 \rangle$

the same

$r_t \times r_{tt} = \langle 0, 0, 1 \rangle$

$$k(t) = \frac{\| \langle 0,0,1 \rangle \|}{\| r_t \|^3} = 1$$

$$\tau(t) = \frac{(r_t \times r_{tt}) \cdot r_{ttt}}{\cdots} = \frac{0}{\cdots}$$

$r(t) = \langle \cos t, \sin t, t \rangle$

$r_t(t) = \langle -\sin t, \cos t, 1 \rangle$

$r_{tt}(t) = \langle \qquad \rangle$

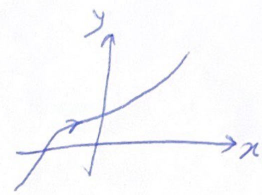
$r_{ttt}(t) = \langle \qquad \rangle$

$r_t \times r_{tt} = \langle \sin t, -\cos t, 1 \rangle$

$$k(t) = \frac{\| \langle \sin t, -\cos t, 1 \rangle \|}{\| r_t \|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{\sqrt{2}}$$

$$\tau = \frac{(r_t \times r_{tt}) \cdot r_{ttt}}{\| r_t \times r_{tt} \|^2} = \frac{1}{2}$$

Exp) Calculation of curvature of plain graph $y = f(x)$



$r(t) = \langle t, f(t), 0 \rangle$

$r_t(t) = \langle 1, f'(t), 0 \rangle$

$r_{tt}(t) = \langle 0, f''(t), 0 \rangle$

$r_t \times r_{tt} = \langle 0, 0, f'' \rangle$ $\Longrightarrow$ $k = \dfrac{\| r_t \times r_{tt} \|}{\| r_t \|^3} = \dfrac{\| f'' \|}{(1+(f')^2)^{3/2}}$

If $f'' = 0 \longrightarrow k = 0$

$\tau = 0$ for a plain curve (2D plain) since $r_{ttt} = 0$