Lecture 6

Rules of diffrentiation :

let $\vec{a}(t) = \langle F_1(t), F_2(t), F_3(t) \rangle$, $\vec{b}(t) = \langle G_1(t), \ldots \rangle$ be 2 vector value function of t

Then

$$\frac{d}{dt}\left[\vec{a}(t) \cdot \vec{b}(t)\right] = \vec{a}'(t) \cdot \vec{b}(t) + \vec{a} \cdot \vec{b}'(t)$$

$$\frac{d}{dt}\left[\vec{a}(t) \times \vec{b}(t)\right] = \vec{a}'(t) \times \vec{b}(t) + \vec{a}(t) \times \vec{b}'(t)$$

Scaling a vector →

$$\frac{d}{dt}\left[\lambda(t)\,\vec{a}(t)\right] = \lambda'(t)\,\vec{a}(t) + \lambda(t)\,\vec{a}'(t)$$

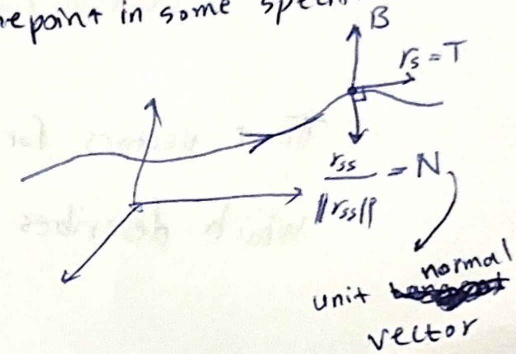
($\lambda(t)$ a scalar function)

Also $\frac{d}{dt}\|\vec{a}(t)\| = \frac{1}{\cancel{2}\|\vec{a}\|}\,\cancel{2}\,\vec{a}' \cdot \vec{a}$ Assuming $(\vec{a}(t) \neq 0)$

* Tedious, but straight foward to check!

Exp) $\frac{d}{dt}\|\vec{a}\| = \frac{d}{dt}\sqrt{F_1^2 + F_2^2 + F_3^2} = \frac{1}{2}\left(F_1^2 + F_2^2 + \cdots\right)^{-1/2}\left(2F_1F_1' + 2F_2F_2' + \cdots\right) = \frac{1}{\|\vec{a}\|}\,\vec{a} \cdot \vec{a}'$

let $\vec{r}(s)$ be arc. lenght parametrization of some C, starting from some point in some specified direction



$$\|r_s\|^2 = r_s \cdot r_s = 1$$

$$0 = (r_s \cdot r_s)_s = r_{ss} \cdot r_s + r_s \cdot r_{ss} = 2 r_{ss} \cdot r_s$$

r_s, r_{ss} are perpendicular

Def: Assume r_s, $r_s \times r_{ss}$ both nonzero for all s. let

$$T(s) = \vec{r_s}(s)$$

$$N(s) = \vec{r}_{ss}(s) \Big/ \|\vec{r}_{ss}(s)\|$$

$$B(s) = T(s) \times N(s)$$

serret formula

$\{T(s), N(s), B(s)\}$ are called Frenet (frame) of C at "point $\vec{r}(s)$".

It is an orthogonal frame at $\vec{r}(s)$, consisting of unit vectors.

Note: $B(s)$ is unit normal vector for oscillating plane! It is also called ~~[illegible]~~ Binormal vector.

It is orthogonal to both $T(s)$ & $N(s)$

(or tangent plane) plane of r_s and r_{ss}

$$* \boxed{\frac{dT}{ds}(s) = k(s)\, N(s)} \qquad T(s) = \vec{r}_s(s)$$

Curvature is rate of chang of unit tangent (N(s) is the unit vector).

Proof: $\frac{dT}{ds} = \frac{d}{ds}(r_s) = r_{ss} = \|r_{ss}\| \left(\frac{r_{ss}}{\|r_{ss}\|}\right) = \|r_{ss}\|\, N$

$$k(s) = \frac{\|r_s \times r_{ss}\|}{\underbrace{\|r_s\|^3}_{1}} = \cancel{\|r_s\|}^{1} \|r_{ss}\| \cancel{\sin \pi/2}^{1} = \|r_{ss}\|$$

* Somewhat equivalent ways of stating above:

$$K(s) = \left\| \frac{dT}{ds}(s) \right\| = \|r_{ss}\|$$

$$* \boxed{\frac{dB}{ds} = -\tau N}$$

$\tau(s)$ is rate of change of unit normal of osculating plane (how the curve is deviating from osculating plane).

Let's Proof:

B is a vector function of s

$$\left.\begin{array}{l} B \cdot B = 1 \\ 0 = (B \cdot B)_s = 2 B_s \cdot B \\ B_s \text{ is perpendicular to } B \end{array}\right\} \quad \begin{array}{l} B \cdot T = 0 \\ 0 = (B \cdot T)_s = B_s \cdot T + \underbrace{\cancel{B \cdot T_s}}_{0 \text{ since } T_s = kN} \\ B_s \text{ is perpendicular to } T \end{array}$$

$\Longrightarrow B_s$ must be parallel to $N \longrightarrow B_s = \lambda N$ for some λ

a vector function of s and it is a unit vector

Multiply both side with $\cdot N \Rightarrow \lambda = B_s \cdot N = (T \times N)_s \cdot N = (T_s \times N + T \times N_s) \cdot N = \Big(\underbrace{(kN) \times N}_{0} + \underbrace{T \times \Big(\big(\tfrac{1}{\|r_{ss}\|}\big)_s r_{ss}}_{\text{becomes 0 when} \cdot N} + \big(\tfrac{1}{\|r_{ss}\|}\big) r_{sss}\Big)\Big) \cdot N$

becomes 0 when $\cdot N$ since $T \times r_{ss}$ is perpend. r_{ss} and N parallel to r_{ss}

$$= -\frac{N \cdot (T \times r_{sss})}{\|r_{ss}\|} = -\frac{r_{ss} \cdot (r_s \times r_{sss})}{\|r_{ss}\|^2} = -\frac{r_{ss} \cdot (r_s \times r_{sss})}{\|r_s \times r_{ss}\|^2} = -\tau$$