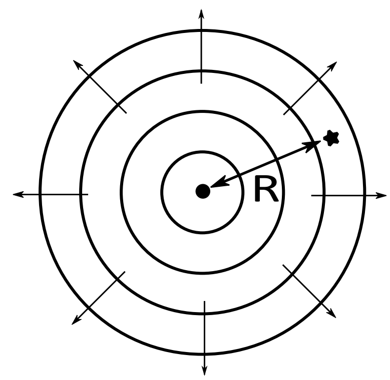
Physics 200 Homework 8, Part Two

1. In this question, we will revisit the diffraction problem from Tutorial 7. You will want to consult that Tutorial. Note the difference in notation though!

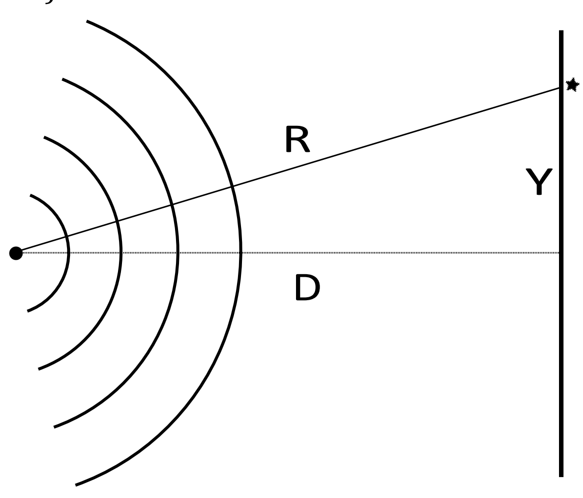
A complex wavefunction is emitted by a point source. A distance R from the source, the wavefunction is

$$\psi(R, t) = e^{2\pi i\left(\frac{R}{\lambda} - ft\right)}$$

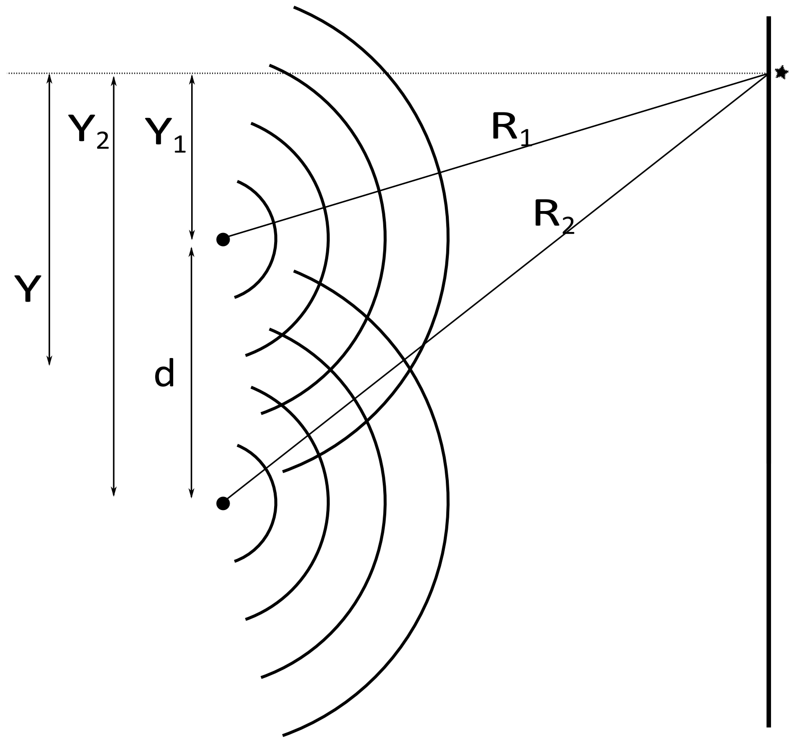


We will no longer be taking a real part of the wave—we are talking about genuinely complex wavefunction in this question. However, we are ignoring an over-all normalization factor, so none of the wavefunctions and probability densities we get will be properly normalized.

(a) Following a reasoning similar to that in the Tutorial, what is the wavefunction due to the source described above on a screen a distance D away, shown below?
(Write the wavefunction as a function of Y, D and t, and use the approximation $R \approx D + \frac{1}{2}\frac{Y^2}{D}$ given in the Tutorial.)



(b) Now consider producing a quantum superposition of particles produced in two different places, by adding the wavefunctions due to two different sources, as shown:



What is the total wavefunction at the point marked with a star, as a function of d, D, Y and t?

(c) What is the resulting probability density, as a function of Y, d, D and λ? Sketch its shape along the screen (as a function of Y).

(d) Electron wavefunctions have wavelengths similar to the size of an atom. In a single electron double-slit experiment, the electron wavefunction had wavelength = 10^{-9}m. On a screen 20m away, the bright fringes were observed to have spacing of 0.1mm. What was the distance between the slits?

2. In chemistry, resonance is a phenomenon where two or more molecular structures contribute to the actual physical form of a molecule. For example, the molecule nitrogen dioxide (one atom of nitrogen and two atoms of oxygen) can be thought of as having a double bond between the nitrogen and one of the atoms of oxygen and a single bond between the nitrogen and the other atom of oxygen, so there are two possible ways it can look:

We will call the state of the molecule with the double bond on the left $|L>$ and with the double bond on the right $|R>$. It turns out that neither of these two states has a well-defined energy! Since the actual molecule would like to exist in a state of well-defined energy (preferably the lowest energy possible), in nature the molecule exists in what the chemists call a 'resonance state', which is just a quantum superposition of the two states. It turns out that the quantum superposition proportional to $|L> +|R>$ the lowest energy E_0. The other state with well-defined energy is proportional to $|L> -|R>$ and its energy is $E_1 > E_0$.

(a) Normalize the quantum superposition $|L> +|R>$ properly to obtain a quantum state $|0>$ corresponding to energy E_0. *Normalization means simple ensure that the probability of observing it in any state is 1.* Also normalize the other quantum superposition $|L> -|R>$ to obtain a quantum state $|1>$ corresponding to energy E_1.

(b) What is $|L>$ in terms of $|0>$ and $|1>$?

(c) Through some magic (involving lasers), we prepare a molecule of nitrogen dioxide in state $|L>$. We then perform a measurement of energy of this molecule. What is the probability that we will measure energy E_0? How about energy E_1?

(d) We now prepare a molecule of nitrogen dioxide in state $0.6|L> +0.8i|R>$ and perform a measurement of energy in this molecule. What is the probability that we will measure energy E_0? How about energy E_1?

(e) Finally, we prepare a molecule of nitrogen dioxide in state $0.6|L> +0.8|R>$ and perform a measurement of energy. What is the probability that we will measure energy E_0? How about energy E_1?

To read more about resonance in chemistry, see this Wikipedia article:

http://en.wikipedia.org/wiki/Resonance (chemistry)

It explains, among other things, how the difference between energies, $E_1 - E_0$, called the resonance energy, contributes to enthalpies of chemical reactions. Many other concepts in valence bond theory are applications of quantum superpositions (for example, hybridization of orbitals).

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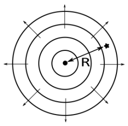

$$R^2 = D^2 + Y^2 \Rightarrow R = \sqrt{D^2+Y^2} \simeq D + \frac{Y^2}{2D}$$

$$\psi(R,t) = e^{2\pi i\cdot\left(\frac{R}{\lambda} - ft\right)} = e^{2\pi i\cdot\left(\frac{D}{\lambda} + \frac{Y^2}{2D\lambda} - ft\right)}$$

$$\therefore \psi(D,Y,t) = e^{2\pi i\cdot\left(\frac{D}{\lambda} + \frac{Y^2}{2D\lambda} - ft\right)}$$

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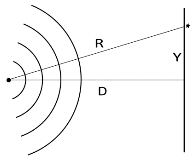

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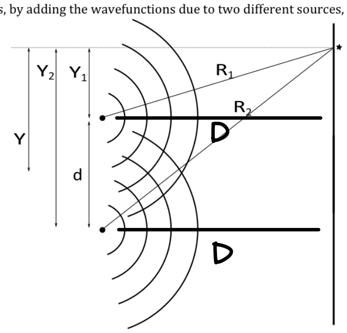


What is the total wavefunction at the point marked with a star, as a function of d, D, Y and t?

$$R_1 \simeq D + \frac{Y^2}{2D} = D + \frac{Y_1^2}{2D}$$

$$R_2 \simeq D + \frac{Y^2}{2D} = D + \frac{Y_2^2}{2D}$$

Also, $Y = \frac{Y_1+Y_2}{2} = Y_1 + \frac{d}{2}$

$\Rightarrow Y_1 = Y - \frac{d}{2}$, $Y_2 = Y + \frac{d}{2}$

$$\psi_1(d,D,Y,t) = e^{2\pi i\cdot\left(\frac{D}{\lambda} + (Y-\frac{d}{2})^2/2D\lambda - ft\right)},\quad \psi_2(d,D,Y,t) = e^{2\pi i\cdot\left(\frac{D}{\lambda} + (Y+\frac{d}{2})^2/2D\lambda - ft\right)}$$

$$\therefore \psi(d,D,Y,t) = \psi_1(d,D,Y,t) + \psi_2(d,D,Y,t) = e^{2\pi i\cdot\left(\frac{D}{\lambda} + (Y-\frac{d}{2})^2/2D\lambda - ft\right)} + e^{2\pi i\cdot\left(\frac{D}{\lambda} + (Y+\frac{d}{2})^2/2D\lambda - ft\right)}$$

(c) What is the resulting probability density, as a function of Y, d, D and λ? Sketch its shape along the screen (as a function of Y).

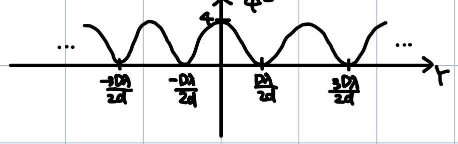
$$\psi = e^{2\pi i\cdot(\frac{R}{\lambda}-ft)}\cdot\left(e^{\pi i/D\lambda\,(Y-\frac{d}{2})^2}+e^{\pi i/D\lambda\,(Y+\frac{d}{2})^2}\right)$$

$$= e^{2\pi i\cdot(\frac{R}{\lambda}-ft)}\cdot e^{\pi i/D\lambda\cdot Y^2}\cdot e^{\pi i/D\lambda\cdot\frac{d^2}{4}}\cdot\left(e^{\pi i/D\lambda\cdot(-Yd)}+e^{\pi i/D\lambda\cdot(Yd)}\right)$$

$$= e^{2\pi i\cdot(\frac{R}{\lambda}-ft)}\cdot e^{\pi i/D\lambda\cdot Y^2}\cdot e^{\pi i/D\lambda\cdot\frac{d^2}{4}}\cdot 2\cos\left(\frac{\pi Yd}{D\lambda}\right)$$

$$\psi^2 = \left|e^{2\pi i\cdot(\frac{R}{\lambda}-ft)}\cdot e^{\pi i/D\lambda\cdot Y^2}\cdot e^{\pi i/D\lambda\cdot\frac{d^2}{4}}\cdot 2\cos\left(\frac{\pi Yd}{D\lambda}\right)\right|^2$$

$$= \left|2\cos\left(\frac{\pi Yd}{D\lambda}\right)\right|^2 = 4\cdot\cos^2\left(\frac{\pi Yd}{D\lambda}\right) = 2+2\cos\left(\frac{2\pi Yd}{D\lambda}\right)$$



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$\lambda = 10^{-9}m$, $D = 20m$, bright fringe spacing $= 0.1mm$. $\rightarrow d?$

Separation between bright fringe is π. $\frac{\pi y d}{\lambda D} = \pi \Rightarrow d = \frac{D\lambda}{y} = \frac{20m\cdot 10^{-9}m}{10^{-4}m} = 2\cdot 10^{-4}m = 0.2mm$ $\quad\therefore 0.2mm$

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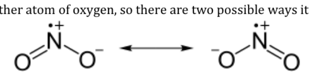

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$|0\rangle = a(|L\rangle + |R\rangle) \Rightarrow ||0\rangle|^2 = 1 = a^2(1+1) \Rightarrow a^2 = \frac{1}{2} \quad \therefore a = \frac{1}{\sqrt{2}}$

$\therefore |0\rangle = \frac{1}{\sqrt{2}}|L\rangle + \frac{1}{\sqrt{2}}|R\rangle$

$|1\rangle = b(|L\rangle - |R\rangle) \Rightarrow ||1\rangle|^2 = 1 = b^2(1+1) \Rightarrow b^2 = \frac{1}{2} \quad \therefore b = \frac{1}{\sqrt{2}}$

$\therefore |1\rangle = \frac{1}{\sqrt{2}}|L\rangle - \frac{1}{\sqrt{2}}|R\rangle$

(b) What is $|L>$ in terms of $|0>$ and $|1>$?

Since $|0\rangle = \frac{1}{\sqrt{2}}|L\rangle + \frac{1}{\sqrt{2}}|R\rangle$, $|1\rangle = \frac{1}{\sqrt{2}}|L\rangle - \frac{1}{\sqrt{2}}|R\rangle$

then $|0\rangle + |1\rangle = \sqrt{2}|L\rangle \Rightarrow |L\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ $\therefore |L\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

(c) Through some magic (involving lasers), we prepare a molecule of nitrogen dioxide in state $|L>$. We then perform a measurement of energy of this molecule. What is the probability that we will measure energy E_0? How about energy E_1?

Since $|L\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, $P(E_0) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$P(E_1) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

(d) We now prepare a molecule of nitrogen dioxide in state $0.6|L> +0.8i|R>$ and perform a measurement of energy in this molecule. What is the probability that we will measure energy E_0? How about energy E_1?

We know that $|L\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ then similarly, $|R\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$0.6|L\rangle + 0.8i|R\rangle = 0.6\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + 0.8i\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$

$= \left(\frac{0.6+0.8i}{\sqrt{2}}\right)|0\rangle + \left(\frac{0.6-0.8i}{\sqrt{2}}\right)|1\rangle$

$P(E_0) = \left|\frac{0.6+0.8i}{\sqrt{2}}\right|^2 = \frac{1}{2}(0.6^2+0.8^2) = \frac{1}{2}$

$P(E_1) = \left|\frac{0.6-0.8i}{\sqrt{2}}\right|^2 = \frac{1}{2}(0.6^2+0.8^2) = \frac{1}{2}$

(e) Finally, we prepare a molecule of nitrogen dioxide in state $0.6|L> +0.8|R>$ and perform a measurement of energy. What is the probability that we will measure energy E_0? How about energy E_1?

$0.6|L\rangle + 0.8|R\rangle$, we know that $|L\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ then similarly, $|R\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$0.6|L\rangle + 0.8|R\rangle = 0.6\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + 0.8\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$

$= \frac{1.4}{\sqrt{2}}|0\rangle - \frac{0.2}{\sqrt{2}}|1\rangle$

Then, $P(E_0) = \left(\frac{1.4}{\sqrt{2}}\right)^2 = \frac{1.96}{2} = 0.98$

$P(E_1) = \left(\frac{0.2}{\sqrt{2}}\right)^2 = \frac{0.04}{2} = 0.02$