No.30

MATH 317 — HOMEWORK 3 — Due date: 26th July - 15 Marks

- Write your name, student number, and signature below.
- Write your answers to each question in the spaces provided. Submit your answers in Canvas on the due date (do not submit any additional pages).

Name: Mercury Mcindoe

Student #: 85594505

Signature:

1. $\boxed{\text{5 marks}}$ Let D be a region where Green's Theorem applies. Show that for any scalar field ψ in D, the following identity holds:

$$\int_{\partial D} \psi \frac{\partial \psi}{\partial y}\, dx - \psi \frac{\partial \psi}{\partial x}\, dy = \int\int_D \left(-\psi \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) - \nabla \psi \cdot \nabla \psi \right) dA$$

Solution:

$$\int_{\partial D} \psi \cdot \frac{\partial \psi}{\partial y}\, dx - \psi \cdot \frac{\partial \psi}{\partial x}\, dy = \iint_D \left\{ \frac{\partial}{\partial x}\left(-\psi \cdot \frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\psi \cdot \frac{\partial \psi}{\partial y}\right) \right\} dA$$

$$= \iint_D \left\{ -\psi \cdot \frac{\partial^2 \psi}{\partial x^2} - \left(\frac{\partial \psi}{\partial x}\right)^2 - \psi \cdot \frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial \psi}{\partial y}\right)^2 \right\} dA$$

$$= \iint_D \left(-\psi \cdot \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) - \left\langle \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right\rangle \cdot \left\langle \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right\rangle \right) dA$$

$$= \iint_D \left(-\psi \cdot \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) - \nabla \psi \cdot \nabla \psi \right) dA \quad \square$$

as required $\square$

2. $\boxed{\text{5 marks}}$ Compute the line integral of $F = \langle y, -x \rangle$ around the curve C defined by $x^2+y^2 = 1$ oriented clockwise. Relate your answer to the area enclosed by C using Green's theorem.

Solution:

Let $r(t) = \langle \cos(2\pi - t), \sin(2\pi - t) \rangle$; $t \in [0, 2\pi]$. Then, $r'(t) = \langle \sin(2\pi - t), -\cos(2\pi - t) \rangle$

$$\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt = \int_0^{2\pi} \langle \sin(2\pi - t), -\cos(2\pi - t) \rangle \cdot \langle \sin(2\pi - t), -\cos(2\pi - t) \rangle\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

Now, let's apply Green's Theorem.

Since C is oriented clockwise, $\int_C F \cdot dr = -\iint_D \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y) \right) dA = \iint_D 2\, dA.$

And 'D' denotes the region of the circle $x^2+y^2=1$.

Hence, $\iint_D 2\, dA = 2 \cdot \pi = 2\pi$. Or equivalently, $\iint_D 2\, dA = \int_0^{2\pi}\int_0^1 2r\, dr\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$

The two methods give the same result!

3. $\boxed{\text{5 marks}}$ Show that for any constants a, b and any closed simple curve C, the line integral of $a\,dx + b\,dy$ over a closed curve C is zero.

Solution:

Since C is a simple closed curve, we can apply Green's Theorem.

Let's denote D as the region that C covers, then

$$\int_C a\,dx + b\,dy = \iint_D \left(\frac{\partial}{\partial x}(b) - \frac{\partial}{\partial y}(a) \right) dA = \iint_D 0\, dA = 0.$$

as required for any simple closed curve C.