Solutions Manual

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for the book

Fundamentals of Complex Analysis, 3^{rd} ed.

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CHAPTER 1: Complex Numbers

EXERCISES 1.1: The Algebra of Complex Numbers

1. $-i = a + bi \Longrightarrow a = 0$ and $b = -1 \Rightarrow$
   $(-i)^2 = (a^2 - b^2) + (2ab)i = -b^2 = -1$

2. The Commutative and Associative laws for addition follow directly from the real counterparts.
   Commutative law for multiplication:

$$\begin{aligned}(a+bi)(c+di) &= (ac-bd)+(bc+ad)i\\ &= (ca-db)+(da+cb)i\\ &= (c+di)(a+bi)\end{aligned}$$

   Associative law for multiplication:

$$\begin{aligned}&[(a+bi)(c+di)](e+fi) = [(ac-bd)+(bc+ad)i](e+fi)\\ &\quad= [(ac-bd)e-(bc+ad)f]+[(bc+ad)e+(ac-bd)f]i\\ &\quad= [a(ce-df)-b(de+cf)]+[b(ce-df)+a(de+cf)]i\\ &\quad= (a+bi)[(ce-df)+(de+cf)i]\\ &\quad= (a+bi)[(c+di)(e+fi)]\end{aligned}$$

   Distributive law:

$$\begin{aligned}&(a+bi)[(c+di)+(e+fi)] = (a+bi)[(c+e)+(d+f)i]\\ &\quad= [a(c+e)-b(d+f)]+[b(c+e)+a(d+f)]i\\ &\quad= [(ac-bd)+(bc+ad)i]+(ae-bf)+(be+af)i]\\ &\quad= (a+bi)(c+di)+(a+bi)(e+fi)\end{aligned}$$

3. a. $z_3 = z_2 - z_1 \quad \Longleftrightarrow$
   $e + fi = (c-a) + (d-b)i = (c+di) - (a+bi) \quad \Longleftrightarrow$
   $e = c - a$ and $f = d - b \quad \Longleftrightarrow$
   $e + a = c$ and $f + b = d \quad \Longleftrightarrow$
   $(e+fi) + (a+bi) = c + di \quad \Longleftrightarrow$

b. $(e+fi)(c+di) = a+bi \iff$
$ce - fd = a$ and $fc + ed = b \iff$

$$\begin{aligned}
\frac{a+bi}{c+di} &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i, \quad c+id \neq 0 \\
&= \frac{(ec-fd)c+(fc+ed)d}{c^2+d^2} \\
&\quad + \frac{(fc+ed)c-(ec-fd)d}{c^2+d^2}i \\
&= e+fi
\end{aligned}$$

4. Suppose $z_1 \neq 0$. Then $z_2 = \dfrac{z_1 z_2}{z_1} = \dfrac{0}{z_1} = 0$.

5. a. $0 + \left(-\frac{3}{2}\right)i = \frac{-3}{2}i$
   b. $3 + 0i = 3$
   c. $0 + (-2)i = -2i$

6. a. $0 + (-2)i = -2i$
   b. $6 + (-3)i = 6 - 3i$
   c. $4 + \pi i$

7. a. $8 + 1i = 8 + i$
   b. $1 + 1i = 1 + i$
   c. $0 + \left(\frac{-8}{3}\right)i = -\frac{8i}{3}$

8. $\frac{33}{25} - \frac{19}{25}i$

9. $\frac{61}{185} - \frac{107}{185}i$

10. $-\frac{253}{4225} - \frac{204}{4225}i$

11. $2 + 0i = 2$

12. $-9 + (-7)i$

13. $6+5i$

14. $z=a+bi$. $\operatorname{Re}(iz)=\operatorname{Re}(ai-b)=-b=-\operatorname{Im} z$

15. $i^{4k}=(i^4)^k=1^k=1$
$i^{4k+1}=i^{4k}\cdot i=1\cdot i=i$
$i^{4k+2}=i^{4k}\cdot i^2=1\cdot(-1)=-1$
$i^{4k+3}=i^{4k}\cdot i^3=1\cdot(-i)=-i$

16. a. $-i$
 b. -1
 c. -1
 d. $-i$

17. $3i^{2(4)+3}+6i^3+8i^{-5(4)}+i^{-1(4)+3}$
$=3(-i)+6(-i)+8(1)+(-i)=8-10i$

18. $(-1+i)^2+2(-1+i)+2=-2i+(-2+2i)+2=0$

19. The real equations are

$$\begin{aligned}\operatorname{Re}(z^3+5z^2) &= \operatorname{Re}(z+3i)\\ \operatorname{Im}(z^3+5z^2) &= \operatorname{Im}(z+3i).\end{aligned}$$

If $z=a+bi$ these can be rewritten as

$$\begin{aligned}&a^3-3ab^2+5a^2-5b^2-a=0\\ &3a^2b-b^3+10ab-b-3=0.\end{aligned}$$

20. a. $z=\dfrac{4}{2i}=-2i$
 b. $z=\dfrac{1-5i}{2-5i}=\dfrac{27}{29}-\dfrac{5i}{29}$
 c. $z=0,\quad -\dfrac{1}{4}+\dfrac{i}{8}$
 d. $z=\pm 4i$

21. $(-i)[(1-i)z_1 + 3z_2] + (1-i)[iz_1 + (1+2i)z_2]$
$= -i(2-3i) + (1-i)(1)$
$\Longrightarrow z_2 = \dfrac{-2-3i}{3-2i} = -i \Longrightarrow z_1 = 1+i$

22. $0 = z^4 - 16 = (z-2)(z+2)(z-2i)(z+2i) \Longrightarrow z = 2, -2, 2i, -2i$

23. Suppose $z = a + bi$.

$$\mathrm{Re}\left(\frac{1}{z}\right) = \mathrm{Re}\left(\frac{a-ib}{a^2+b^2}\right) = \frac{a}{a^2+b^2} > 0$$

whenever $a > 0$.

24. Suppose $z = a + bi$.

$$\begin{aligned} \mathrm{Im}\left(\frac{1}{z}\right) &= \mathrm{Im}\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i\right) \\ &= -\frac{b}{a^2+b^2} < 0 \text{ whenever } b > 0. \end{aligned}$$

25. Let $z_1 = a + bi$ and $z_2 = c + di$. The hypotheses specify that $a + c < 0$, $b + d = 0$, $ac - bd < 0$, and $ad + bc = 0$.

$b = 0 \Longrightarrow d = 0 \Longrightarrow z_1$ and z_2 are real.

$b \neq 0 \Longrightarrow d = -b$ and $ad + bc = a(-b) + bd = -b(a-c) = 0$

$\Longrightarrow a = c$, a contradiction of the fact that $z_1 z_2 < 0$.

26. By induction: The case when $n = 1$ is obvious. Assume $\mathrm{Re}\left(\sum_{j=1}^{m} z_j\right) = \sum_{j=1}^{m} \mathrm{Re}(z_j)$ for all positive integers $m < n$

$$\begin{aligned} \mathrm{Re}\left(\sum_{j=1}^{n} z_j\right) &= \mathrm{Re}\left(\sum_{j=1}^{n-1} z_j + z_n\right) \\ &= \sum_{j=1}^{n-1} \mathrm{Re}(z_j) + \mathrm{Re}(z_n) \\ &= \sum_{j=1}^{n} \mathrm{Re}(z_j) \end{aligned}$$

The corresponding result for the imaginary parts follows by replacing "Re" by "Im" in the above proof.

Disprove: $\operatorname{Re}\left(\prod_{j=1}^{n} z_j\right) = \prod_{j=1}^{n} \operatorname{Re}(z_j)$ and

$$\operatorname{Im}\left(\prod_{j=1}^{n} z_j\right) = \prod_{j=1}^{n} \operatorname{Im}(z_j).$$

$\operatorname{Re}[(a+bi)(c+di)] = ac - bd$
$\operatorname{Re}(a+bi)\operatorname{Re}(c+di) = ac$
These are not equal whenever $bd \neq 0$.

$\operatorname{Im}[(a+bi)(c+di)] = ad + bc$
$\operatorname{Im}(a+bi)\operatorname{Im}(c+di) = bd$
These are not equal whenever $ad + bc \neq bd$.
(For example, consider the pair 2 and i.)

27. By induction: The case when $n = 1$ is obvious. Assume

$$\begin{aligned}(z_1+z_2)^m &= z_1^m + \binom{m}{1} z_1^{m-1} z_2 + \cdots \\ &\quad + \binom{m}{k} z_1^{m-k} z_2^k + \cdots + z_2^m\end{aligned}$$

for all positive integers $m < n$. Recall that, for positive integers r and s with $r > s$,

$$\binom{r}{s} + \binom{r}{s+1} = \binom{r+1}{s+1} \text{ and } \binom{r}{0} = \binom{r}{r} = 1\cdot$$

$$\begin{aligned}(z_1+z_2)^n &= (z_1+z_2)^{n-1}(z_1+z_2) \\ &= z_1^{n-1}(z_1+z_2) + \binom{n-1}{1} z_1^{n-2} z_2 (z_1+z_2) \\ &\quad + \cdots + \binom{n-1}{k} z_1^{n-1-k} z_2^k (z_1+z_2) \\ &\quad + \cdots + z_2^{n-1}(z_1+z_2) \\ &= z_1^n + z_1^{n-1} z_2 + \binom{n-1}{1}(z_1^{n-1} z_2 + z_1^{n-2} z_2^2) \\ &\quad + \cdots + \binom{n-1}{k}(z_1^{n-k} z_2^k + z_1^{n-(k+1)} z_2^{k+1}) + \cdots + z_2^{n-1} z_1 + z_2^n\end{aligned}$$

$$
\begin{aligned}
&= z_1^n + \left[\binom{n-1}{0} + \binom{n-1}{1}\right] z_1^{n-1} z_2 \\
&\quad + \left[\binom{n-1}{1} + \binom{n-1}{2}\right] z_1^{n-2} z_2^2 \\
&\quad + \cdots \left[\binom{n-1}{k-1} + \binom{n-1}{k}\right] z_1^{n-k} z_2^k \\
&\quad + \cdots + z_2^n \\
&= z_1^n + \binom{n}{1} z_1^{n-1} z_2 + \binom{n}{2} z_1^{n-2} z_2^2 \\
&\quad + \cdots + \binom{n}{k} z_1^{n-k} z_2^k + \cdots + z_2^n.
\end{aligned}
$$

28. $2^5 + \binom{5}{1} 2^4(-i) + \binom{5}{2} 2^3(-i)^2 + \binom{5}{3} 2^2(-i)^3 + \binom{5}{4} 2(-i)^4 + (-i)^5$
$= 32 - 80i - 80 + 40i + 10 - i = -38 - 41i$

29. Suppose $x = \dfrac{p}{q}$, where p and q are relatively prime integers, and that $x^2 = 2$.
$\left(\dfrac{p}{q}\right)^2 = 2 \Longrightarrow p^2 = 2q^2 \Longrightarrow p^2 = 4k$ for some integer k and $q^2 = 2k$,
a contradiction (If p^2 is an even integer so is p.).

30. By contradiction. Suppose there is a nonempty subset P of the complex numbers satisfying (i), (ii), and (iii) and suppose i is in P.
Then, by (iii), $i^2 = -1$ and $(-1)i = -i$. This violates (i).
Similarly (i) is violated by assuming $-i$ belongs to P.

31. Purpose: to add, subtract, multiply and divide $z_1 = a + bi$ and $z_2 = c + di$.
Input a, b, c, d
Set sum= $(a + c, b + d)$
Print "$z1 + z2 =$ "; sum
Set diff=$(a - c, b - d)$
Print "$z1 - z2 =$ "; diff
Set prod=$(a * c - b * d, b * c + a * d)$

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Print "z1 * z2 = "; prod
Set denom = c^2 + d^2
If denom = 0, print "there is no quotient"
Else
    Set quot=((a * c + b * d)/(denom), (b * c - a * d)/(denom))
    Print "z1/z2 = "; quot
Endif
Stop
```

32. prod=$(a * c - b * d,\ (a + b) * (c + d) - a * c - b * d)$

EXERCISES 1.2: Point Representation of Complex Numbers; Absolute Value and Complex Conjugates

1. The real and imaginary parts of

$$\frac{z_1 + z_2}{2} = \frac{x_1 + x_2}{2} + i\frac{y_1 + y_2}{2}$$

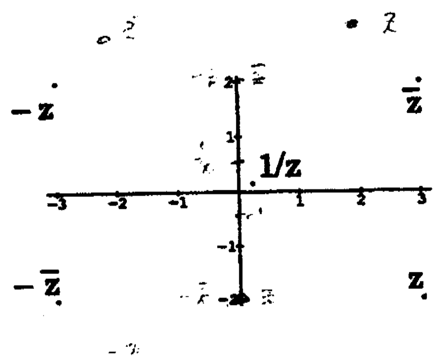
give the familiar algebra formula for the midpoint of the line segment joining two points in $\mathbf{R}^2$.

Alternatively, one could establish that $(z_1 + z_2)/2$ is a point on the line through z_1 and z_2 and that $|z_1 - (z_1 + z_2)/2| = |z_2 - (z_1 + z_2)/2|$.

2. $\hat{z} = \dfrac{2(1 + i) + (-3i) + 3(1 - 2i) + 5(-6)}{2 + 1 + 3 + 5} = -\dfrac{25}{11} - \dfrac{7}{11}i$

3. -3

4. $\left(\dfrac{1}{z} = \dfrac{3}{13} + \dfrac{2}{13}i\right)$



5. The three side lengths are equal:

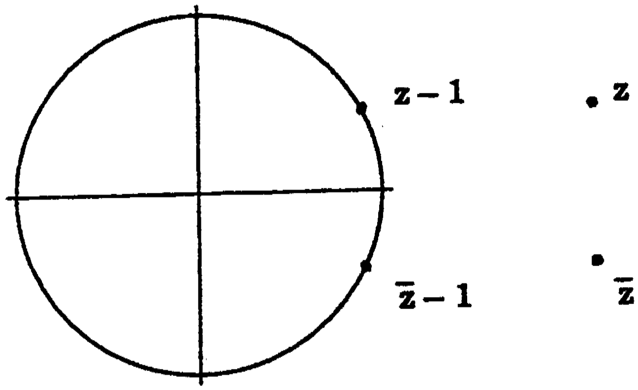
$$\left|1-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right| = \left|1-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right|$$
$$= \left|\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right| = \sqrt{3}$$

6. The Pythagorean theorem is satisfied:

$$10+10 = |(3+i)-6|^2 + |(3+i)-(4+4i)|^2 = |6-(4+4i)|^2 = 20$$

7. a. All points on the horizontal line through $z=-2i$
   b. All points on the circle of radius 3 with center at $1-i$
   c. All points on the circle of radius 2 with center at $\frac{1}{2}i$
   d. The points must be equidistant from 1 and $-i$, thus lie on the perpendicular bisector of the line through 1 and $-i$.
   e. The equation can be written as $x = \frac{1}{4}y^2 - 1$. The points lie on this parabola.
   f. The points z have the property that their distance from 1 added to their distance from -1 is always 7, so the points lie on an ellipse with foci ± 1, with x intercepts $\pm\frac{7}{2}$ and y intercepts $\pm\frac{3}{2}\sqrt{5}$.
   g. All points on the circle of radius $\frac{3}{8}$ with center at $\frac{9}{8}$
   h. All points in the half plane $x \geq 4$
   i. All points inside the circle of radius 2 centered at i
   j. All points outside the circle of radius 6 centered at the origin

8. $|(a+bi)-1| = \sqrt{(a-1)^2+b^2}$
$= \sqrt{(a-1)^2+(-b)^2}$
$= |\overline{a+bi}-1|$



9. $|rz| = |r(a+bi)| = |ra+rbi| = \sqrt{(ra)^2+(rb)^2}$
$= \sqrt{r^2(a^2+b^2)} = r\sqrt{a^2+b^2} = r|z|$

10. $|\text{Re}\, z| = |a| = \sqrt{a^2} \le \sqrt{a^2+b^2} = |z|$
$|\text{Im}\, z| = |b| = \sqrt{b^2} \le \sqrt{a^2+b^2} = |z|$

11. $a = |a+bi| = \sqrt{a^2+b^2} \Longrightarrow a \ge 0$ and $b = 0$

12. a. $\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{a_1+b_1 i}{a_2+b_2 i}\right)} = \overline{\left(\frac{(a_1a_2+b_1b_2)+(a_2b_1-a_1b_2)i}{a_2^2+b_2^2}\right)}$

$$= \frac{(a_1a_2+b_1b_2)+(-a_2b_1+a_1b_2)i}{a_2^2+b_2^2}$$

$$= \frac{a_1-b_1 i}{a_2-b_2 i} = \frac{\overline{z_1}}{\overline{z_2}}\,.$$

b. $\frac{z+\overline{z}}{2} = \frac{(a+bi)+(a-bi)}{2} = a = \text{Re}\, z$

c. $\frac{z-\overline{z}}{2i} = \frac{(a+bi)-(a-bi)}{2i} = b = \text{Im}\, z$

13. $(\overline{z})^2 - z^2 = 0 \Longrightarrow (\overline{z} - z)(\overline{z} + z) = 0 \Longrightarrow$

either: $\overline{z} - z = 0 \Longrightarrow 2i\mathrm{Im}\, z = 0 \Longrightarrow z$ is real, or
$\overline{z} + z = 0 \Longrightarrow 2\mathrm{Re}\, z = 0 \Longrightarrow z$ is pure imaginary.

14. $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 \overline{z_1})(z_2 \overline{z_2}) = |z_1|^2 |z_2|^2$

15. By induction: The case when $k = 0$ is obvious. Assume $(\overline{z})^m = \overline{(z^m)}$ for all positive integers $m < k$.

$$(\overline{z})^k = (\overline{z})^{k-1}(\overline{z}) = (\overline{z^{k-1}})\overline{z} = \overline{z^{k-1} z} = \overline{z^k}$$

Also,

$$(\overline{z})^{-k} = \frac{1}{(\overline{z})^k} = \frac{1}{\overline{z^k}} = \overline{\left(\frac{1}{z^k}\right)} = \overline{z^{-k}}$$

16. Let $z = a + bi$. Since $|z|^2 = a^2 + b^2 = 1$,

$$\mathrm{Re}\left(\frac{1}{1-z}\right) = \mathrm{Re}\left(\frac{1}{(1-a) - bi}\right) = \mathrm{Re}\left(\frac{(1-a) + bi}{2 - 2a}\right) = \frac{1}{2}.$$

17. $\overline{z_0}^{\,n} + a_1 \overline{z_0}^{\,n-1} + \cdots + a_{n-1}\overline{z_0} + a_n$
$= \overline{z_0^n + a_1 z_0^{n-1} + \cdots + a_{n-1} z_0 + a_n} = \overline{0} = 0$

18. The roots of $z^2 + a_1 z + a_2 = 0$ are $z = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$.

$a_1^2 - 4a_2 \geq 0 \Longrightarrow$ Both roots are real
$\Longrightarrow$ Each root is its own conjugate
$a_1^2 - 4a_2 < 0 \Longrightarrow \pm\sqrt{a_1^2 - 4a_2} = \pm i\sqrt{4a_2 - a_1^2}$
$\Longrightarrow$ The roots are complex conjugates.

19. The line ax+by=c can be represented in the complex plane as $z = r\cos\theta + ir\sin\theta + c/a$ where $\theta = \tan^{-1}(-a/b)$ and $-\infty < r < \infty$. By working with triangles you can obtain
$\cos\theta = -b/\sqrt{(a^2 + b^2)}$ and $\sin\theta = a/\sqrt{(a^2 + b^2)}$. To get to point z write the equation from point c/a down the line and make a turn on the perpendicular as
$z = x+iy = r\cos\theta + ir\sin\theta + c/a - s\sin\theta + is\cos\theta$ with $-\infty < s < \infty$. Equating the real and imaginary parts
$x - c/a = r\cos\theta - s\sin\theta$; $y = r\sin\theta + s\cos\theta$
Solve for s as $s = (-\sin\theta(x-c/a) + y\cos\theta) = (-ax + c - by)/\sqrt{(a^2 + b^2)}$. The distance from the point z to the line ax + by = c is s. Denote the reflected point by z_r. The reflected point lies s units on the other side of the line.
$z_r = z - 2s(-a - ib)/)/\sqrt{(a^2 + b^2)} = x + iy - 2\{(-ax + c - by)/\sqrt{(a^2 + b^2)}\}(-a - ib)/\sqrt{(a^2 + b^2)}$
$= \{[(b^2 - a^2)x - 2aby + 2ac] + i[(a^2 - b^2)y - 2abx + 2bc]\}/\sqrt{(a^2 + b^2)}$
$= [2ic + (b-ai)(x-iy)]/(b+ai)$

20. (a) Suppose $u^\dagger Au = 0$ for all n by 1 column vectors with complex entries. Let
$u = [0\ 0 \ldots 1 \,..0]^T$ with the i^{th} entry being the only nonzero entry. Then $u^\dagger Au = (a_{ii}) = 0$ for i=1 to n.

Let u be all zeros except for $½ + i\sqrt{3}/2$ 0n the ith row and $½ - i\sqrt{3}/2$ on the jth row. Now

$u^\dagger Au = (a_{ij})(½ - i\sqrt{3}/2)^2 + (a_{ji})(½ + i\sqrt{3}/2)^2 = -(1/2 - i\sqrt{3}/2)(a_{ij}) - (½ + i\sqrt{3}/2)(a_{ji}) = 0$.

Setting the real and imaginary parts equal to zero yields $a_{ij} = 0$ and $a_{ji} = 0$ for all i,j= 1 to n. Consequently A = 0.

(b) Let A = [0 1; -1 0]. Now $u^\dagger Au = 0$ for all 2 by 1 real column vectors .

21. The matrix A is *Hermitian* $A^\dagger = A$. Observe $(Au)^\dagger = u^\dagger A^\dagger = u^\dagger A$.

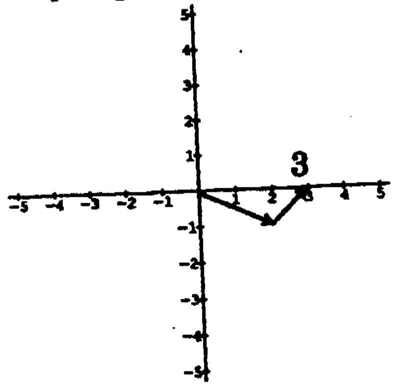
(a) $(u^\dagger Au)^\dagger$ is the conjugate transpose of the matrix $u^\dagger Au$ which is a one by one matrix, so
$(u^\dagger Au)^\dagger = u^\dagger A^\dagger u = u^\dagger Au$ because A is *Hermitian*. The conjugate is equal to the number only when the number is real.

(b) $(B^\dagger B)^\dagger = B^\dagger B$ and therefore is *Hermitian.*

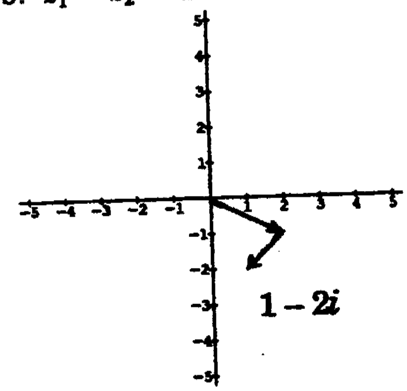
(c) $(u^\dagger B^\dagger Bu)^\dagger = (Bu)^\dagger(u^\dagger B^\dagger)^\dagger = u^\dagger B^\dagger Bu$ a real number.

EXERCISES 1.3: Vectors and Polar Forms

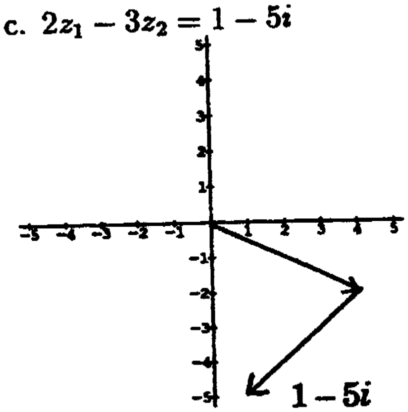
1. a. $z_1 + z_2 = 3$



b. $z_1 - z_2 = 1 - 2i$



c. $2z_1 - 3z_2 = 1 - 5i$



2. $|z_1 z_2 z_3| = |(z_1 z_2) z_3| = |z_1 z_2||z_3| = |z_1||z_2||z_3|$

3. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

4. By induction: The case when $k = 0$ is obvious. Assume $|z^m| = |z|^m$ for all positive integers $m < k$.

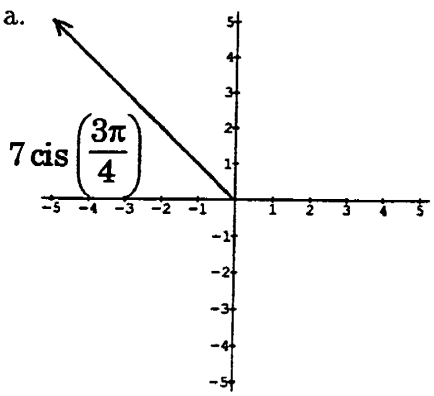
$$|z^k| = |z^{k-1}z| = |z^{k-1}||z| = |z|^{k-1}|z| = |z|^k$$

Also,

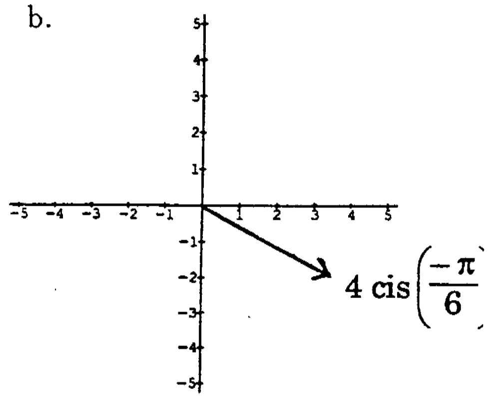
$$|z^{-k}| = \left|\frac{1}{z^k}\right| = \frac{1}{|z^k|} = \frac{1}{|z|^k} = |z|^{-k}$$

5. a. 1
   b. $5\sqrt{26}$
   c. $\dfrac{5\sqrt{5}}{2}$
   d. 1

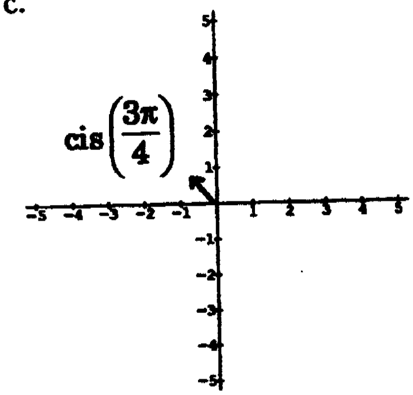
6. a.



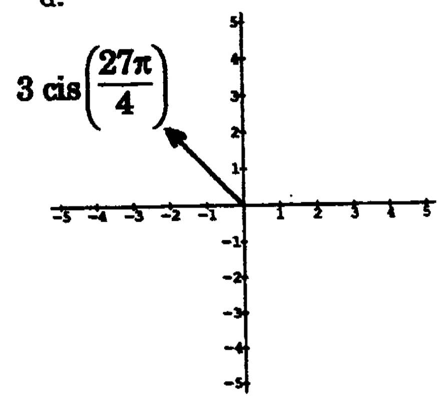
b.



c.



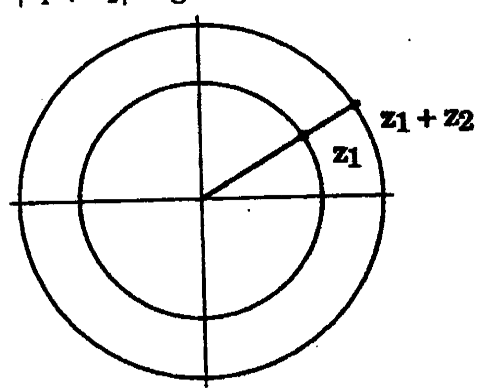
d.



7. (Only the value of $\operatorname{Arg} z$ is given for each of the following.)

a. $\frac{1}{2}\text{cis}\,\pi$ b. $3\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$
c. $\pi\text{cis}\left(-\frac{\pi}{2}\right)$ d. $4\text{cis}\left(-\frac{5\pi}{6}\right)$
e. $2\sqrt{2}\text{cis}\left(\frac{7\pi}{12}\right)$ f. $4\text{cis}\left(-\frac{\pi}{3}\right)$
g. $\frac{1}{\sqrt{2}}\text{cis}\left(\frac{5\pi}{12}\right)$ h. $\frac{\sqrt{14}}{2}\text{cis}\left(\frac{-11\pi}{12}\right)$

8. Suppose $|z_2| = r$. Then $z_1 + z_2$ lies on the circle in the figure and $|z_1 + z_2|$ is greatest when $\arg z_1 = \arg z_2$



9. It is a vector of length $|z|$ and angle of inclination $\arg z + \phi$; it is obtained by rotating z by angle ϕ in the counterclockwise direction.

10a. $\arg(z_1z_2z_3) = \arg((z_1z_2)z_3) = \arg(z_1z_2) + \arg z_3 = \arg z_1 + \arg z_2 + \arg z_3$

10b. $\arg z_1\overline{z_2} = \arg z_1 + \arg \overline{z_2} = \arg z_1 - \arg z_2$

11. $(1 + i)(5 - i)^4 = \sqrt{2}\,\text{cis}(\pi/4)\sqrt{(26)}\text{cis}(-4\tan^{-1}(1/5)) = (1 + i)(24 - i10)^2$
$= (1 + i)(24^2 - 100 - i480) = 976 - i4$
$\arg(1 + i)(5 - i)^4) = \pi/4 - 4\tan^{-1}(1/5) = -\tan^{-1}(1/239)$
$\pi/4 = 4\tan^{-1}(1/5) - \tan^{-1}(1/239)$.

12. a. $-\dfrac{3\pi}{4}$
b. π
c. $\dfrac{\pi}{2}$
d. $-\dfrac{\pi}{6}$

13. b and d always true

Counterexample for part a:

$$z_1 = z_2 = \text{cis}\,\frac{5\pi}{6} \Longrightarrow \text{Arg}\, z_1z_2 = -\frac{\pi}{3}, \quad \text{Arg}\, z_1 + \text{Arg}\, z_2 = \frac{5\pi}{3}$$

Counterexample for part c:

$$z_1 = -i, \quad z_2 = i \Longrightarrow \text{Arg}\left(\frac{z_1}{z_2}\right) = \pi, \quad \text{Arg}\, z_1 - \text{Arg}\, z_2 = -\pi$$

14. If $x > 0$ then $\tan^{-1}\left(\frac{y}{x}\right) + \frac{\pi}{2}(1 - 1) = \tan^{-1}\left(\frac{y}{x}\right)$, which corresponds to $\frac{-\pi}{2} < \arg z < \frac{\pi}{2}$.

If $x < 0$ then $\tan^{-1}\left(\frac{y}{x}\right) + \frac{\pi}{2}(1 + 1) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$, which corresponds to $\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$.

If $x = 0$ and $y > 0$, then $\frac{\pi}{2}(1) = \arg z$.

If $x = 0$ and $y < 0$, then $\frac{\pi}{2}(-1) = \arg z$.

If $x = y = 0$ then $\arg z$ is undefined.

If $y > 0$ then $1 \cdot \cos^{-1}\left(x/\sqrt{x^2 + y^2}\right)$ corresponds to $0 < \text{Arg}\, z < \pi$.

If $y < 0$ then $-\cos^{-1}\left(x/\sqrt{x^2 + y^2}\right)$ corresponds to $-\pi < \text{Arg}\, z < 0$.

If $y = 0$ and $x > 0$ then $0 = \text{Arg}\, z$.

15. $|z_1 - z_2| = |z_1 + (-z_2)| \le |z_1| + |-z_2| = |z_1| + |z_2|$

16. Apply Exercise 15 twice:

$|z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2| \Longrightarrow$

$|z_1| - |z_2| \le |z_1 - z_2|$

Similarly (beginning with $|z_2|$),

$|z_2| - |z_1| \le |z_2 - z_1| = |z_1 - z_2|$

Thus,

$-|z_1 - z_2| \le |z_1| - |z_2| \le |z_1 - z_2|$, or

$||z_1| - |z_2|| \le |z_1 - z_2|$

17. If vector z_1 is parallel to vector z_2, then $z_2 = cz_1$ for some real number $c \ne 0$, and $z_1\overline{z_2}$ is real valued since $z_1\overline{z_2} = c|z_1|^2$.

Conversely if $z_1\overline{z_2}$ is real valued,

$\arg z_1 - \arg z_2 = \arg(z_1\overline{z_2}) = k\pi, \quad k = 0, \pm 1, \pm 2, \ldots \Longrightarrow$

$\arg z_2 = \arg z_1 + k\pi \quad \Longrightarrow$ Vector z_2 is parallel to vector z_1.

18. By Example 1, the points z_2, z_1 and z lie on the same line if and only if $z - z_1 = c'(z_1 - z_2)$, which is true if and only if $z = z_1 + c(z_2 - z_1)$, where $c = -c'$. It follows that z lies strictly between z_1 and z_2 if and only if $0 < c < 1$.

19. $z_1 = cz_2$ with c real and $c > 0 \Longleftrightarrow$

$\arg z_1 = \arg c + \arg z_2 = 0 + \arg z_2 = \arg z_2$

20. The triangle with vertices z_1, z_2, and z_3 has sides represented by the vectors $z_2 - z_1$, $z_3 - z_1$, and $z_3 - z_2$. Let ϕ be the angle between $z_3 - z_1$ and $z_2 - z_1$. Then

$$\begin{aligned}\phi &= \arg(z_3 - z_1) - \arg(z_2 - z_1)\\ &= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)\end{aligned}$$

The result can now be recognized as the Law of Cosines.

21. $r_1 \text{cis}\, \theta_1 + r_2 \text{cis}\, \theta_2 = [r_1 \cos\theta_1 + r_2 \cos\theta_2] + i[r_1 \sin\theta_1 + r_2 \sin\theta_2]$

$$\begin{aligned}
\Longrightarrow\ r^2 &= [r_1 \cos\theta_1 + r_2\cos\theta_2]^2 + [r_1 \sin\theta_1 + r_2 \sin\theta_2]^2 \\
&= r_1^2 + 2r_1 r_2(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) + r_2^2 \\
&= r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) \\
\Longrightarrow\ r &= \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)} \\
\cos\theta &= \text{Re}\left(\frac{r_1 \text{cis}\,\theta_1 + r_2 \text{cis}\,\theta_2}{r}\right) \\
&= \frac{r_1 \cos\theta_1 + r_2 \cos\theta_2}{\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)}} \\
\sin\theta &= \text{Im}\left(\frac{r_1 \text{cis}\,\theta_1 + r_2 \text{cis}\,\theta_2}{r}\right) \\
&= \frac{r_1 \sin\theta_1 + r_2 \sin\theta_2}{\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)}} \\
\theta &= \tan^{-1}\left(\frac{r_1 \sin\theta_1 + r_2 \sin\theta_2}{r_1 \cos\theta_1 + r_2\cos\theta_2}\right)
\end{aligned}$$

when $r_1 \cos\theta_1 + r_2 \cos\theta_2 > 0$.

See Exercise 14 to adjust θ for the other cases.

22. By induction. The case when $n = 2$ is the standard triangle inequality. Assume

$$\left|\sum_{k=1}^{m} z_k\right| \le \sum_{k=1}^{m} |z_k|$$

for all positive integers $m < n$. Then

$$\begin{aligned}
\left|\sum_{k=1}^{n} z_k\right| &= \left|\sum_{k=1}^{n-1} z_k + z_n\right| \\
&\le \left|\sum_{k=1}^{n-1} z_k\right| + |z_n| \\
&\le \sum_{k=1}^{n-1} |z_k| + |z_n| = \sum_{k=1}^{n} |z_k|.
\end{aligned}$$

23. $\left|\dfrac{m_1z_1+m_2z_2+m_3z_3}{m_1+m_2+m_3}\right|$

$$\leq \left|\frac{m_1z_1}{m_1+m_2+m_3}\right|+\left|\frac{m_2z_2}{m_1+m_2+m_3}\right|+\left|\frac{m_3z_3}{m_1+m_2+m_3}\right|$$

$$\leq \frac{m_1}{m_1+m_2+m_3}+\frac{m_2}{m_1+m_2+m_3}+\frac{m_3}{m_1+m_2+m_3}=1$$

Physical interpretation: If three particles z_1, z_2, and z_3 lie inside or on the unit circle, then their center of mass also must be inside or on the unit circle.

24. (See Exercise 14)

```
        Input x,y
Step1 Set r = sqrt (x^2 + y^2)
Step2 If x ≤ 0 and y = 0, Set t = pi
Step3 Else Set t = sgn(y) * arccos(x/r)
Step4 Print "Polar coordinates are (r,t) = "; (r,t)
Step5 Stop
```

```
        Input r,t
Step1 Set x = r * cos(t), y = r * sin(t)
Step2 Print "Rectangular coordinates are (x,y) = "; (x,y)
Step3 Stop
```

25.

$\bar{z}_1 z_2 = (x_1 - iy_1)(x_2 + iy_2) = x_1x_2 + y_1y_2 + i(x_1y_2 - y_1x_2)$

Re $(\bar{z}_1 z_2) = x_1x_2 + y_1y_2$

26. $z_1 \bullet z_2 = x_1x_2 + y_1y_2 = 0 \Rightarrow y_2/x_2 = 1/(-y_1/x_1)$ and the vector z_1 is orthogonal to z_2. In other words z_1 leads z_2 $\pi/2$ radians so $z_1 = icz_2$.
If $z_1 = icz_2$ for some real c,
$z_1 \bullet z_2 = \mathrm{Re}(\bar{z}_1 z_2) = \mathrm{Re}(-ic(x_2 - iy_2)(x_2 + iy_2)) = -cx_2y_2 + cx_2y_2 = 0$
and z_1 is orthogonal to z_2.

27. (a) $\mathrm{Im}(\bar{z}_1 z_2) = \mathrm{Im}((x_1 - iy_1)(x_2 + iy_2)) = x_1y_2 - x_2y_2$

(b) If z_1 and z_2 are parallel $z_1 = cz_2 \Rightarrow \mathrm{Im}(z_1z_2) = cx_2y_2 - x_2cy_2 = 0$
If $\mathrm{Im}(z_1z_2) = 0$, $x_1y_2 - x_2y_1 = 0 \Rightarrow x_1/y_1 = x_2/y_2 \Rightarrow z_1 = cz_2$ for some real c.

EXERCISES 1.4: The Complex Exponential

1. a. $\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

   b. $e^2 i$

   c. $e^{\cos 1}\cos(\sin 1)+ie^{\cos 1}\sin(\sin 1)$

2. a. $\sin 3$

   b. $e^3\sqrt{3}+e^3 i$

   c. $e^2\cos 2\sqrt{3}+ie^2\sin 2\sqrt{3}$

3. a. $\frac{\sqrt{2}}{3}e^{-i\pi/4}$

   b. $16\pi e^{-i2\pi/3}$

   c. $8e^{i3\pi/2}$

4. a. $e^{i2\pi/3}$

   b. $\frac{2\sqrt{2}e^{i\pi/4}}{2e^{i5\pi/6}}=\sqrt{2}e^{-i7\pi/12}$

   c. $\frac{2e^{i\pi/2}}{3e^4e^i}=\frac{2}{3e^4}e^{i(\pi/2-1)}$

5. $|e^{x+iy}|=|e^xe^{iy}|=|e^x||e^{iy}|=e^x$

   $\arg(e^{x+iy})=\arg e^xe^{iy}=\arg e^x+\arg e^{iy}=0+y+2k\pi,\ k=0,\pm1,\dots$

6. a. $\frac{\sin\theta}{\cos\theta}=\frac{(e^{i\theta}-e^{-i\theta})/2i}{(e^{i\theta}+e^{-i\theta})/2}=\frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$

   b. $\frac{1}{\sin\theta}=\frac{2i}{e^{i\theta}-e^{-i\theta}}=\frac{2e^{i\pi/2}}{e^{i\theta}-e^{-i\theta}}=\frac{2}{e^{i(\theta-\pi/2)}-e^{-i(\theta+\pi/2)}}$

7. $$\begin{aligned} e^{z+2\pi i} &= e^{x+i(y+2\pi)}=e^x[\cos(y+2\pi)+i\sin(y+2\pi)]\\ &= e^x(\cos y+i\sin y)=e^{x+iy}=e^z\end{aligned}$$

8. a. $e^{z+\pi i}=e^x[\cos(y+\pi)+i\sin(y+\pi)]=-e^x[\cos y+i\sin y]=-e^z$

   b. $$\begin{aligned}\overline{e^z}=\overline{e^x\operatorname{cis} y}&=e^x(\cos y-i\sin y)\\ &=e^x(\cos(-y)+i\sin(-y))\\ &=e^{\bar z}\end{aligned}$$

9. $$\begin{aligned}(e^z)^n=(e^x\operatorname{cis} y)^n&=e^{nx}(\operatorname{cis} y)^n\\ &=e^{nx}\operatorname{cis} ny\\ &=e^{n(x+iy)}=e^{nz}\end{aligned}$$

   $$(e^z)^{-n}=\frac{1}{(e^z)^n}=\frac{1}{e^{nz}}=e^{-nz}$$

10. $z=x+iy$ with $x<0$. $|e^z|=e^x\le e^0=1$

11. a,c, and d are true. b is false because $e^{z+2\pi i} = e^z$.

12. a. $$\begin{aligned} \sin 3\theta &= \operatorname{Im}(\cos 3\theta + i\sin 3\theta) = \operatorname{Im}(\cos\theta + i\sin\theta)^3 \\ &= \operatorname{Im}[\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(-\sin\theta) - i\sin^3\theta] \\ &= 3\cos^2\theta\sin\theta - \sin^3\theta \end{aligned}$$

b. $$\begin{aligned} \sin 4\theta &= \operatorname{Im}(\cos 4\theta + i\sin 4\theta) = \operatorname{Im}(\cos\theta + i\sin\theta)^4 \\ &= \operatorname{Im}[\cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(-\sin^2\theta) \\ &\quad + 4\cos\theta(-i\sin^3\theta) + \sin^4\theta] \\ &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \end{aligned}$$

13. a. $$\begin{aligned} \sin^2\theta + \cos^2\theta &= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 + \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 \\ &= -\frac{1}{4}(e^{i2\theta} - 2 + e^{-i2\theta}) + \frac{1}{4}(e^{i2\theta} + 2 + e^{-i2\theta}) = 1 \end{aligned}$$

b. $$\begin{aligned} \cos(\theta_1 + \theta_2) &= \frac{e^{i(\theta_1+\theta_2)} + e^{-i(\theta_1+\theta_2)}}{2} \\ &= \frac{e^{i\theta_1}e^{i\theta_2} + e^{-i\theta_1}e^{-i\theta_2}}{2} \\ &= \frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)}{2} \\ &\quad + \frac{[\cos(-\theta_1) + i\sin(-\theta_1)][\cos(-\theta_2) + i\sin(-\theta_2)]}{2} \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \end{aligned}$$

since $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$.

14. Yes, because if $n > 0$ then

$$\begin{aligned} (\cos\theta + i\sin\theta)^{-n} &= \frac{1}{(\cos\theta + i\sin\theta)^n} \\ &= \frac{1}{\cos n\theta + i\sin n\theta} \end{aligned}$$

$$\begin{aligned} &= \cos n\theta - i \sin n\theta \\ &= \cos(-n\theta) + i \sin(-n\theta) \end{aligned}$$

15. a. $$\begin{aligned} e^{z_1} e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1) e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{x_1} e^{x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i \cos y_1 \sin y_2 \\ &\quad + i \sin y_1 \cos y_2) \\ &= e^{x_1+x_2}(\cos(y_1 + y_2) + i \sin(y_1 + y_2)) \\ &= e^{z_1+z_2} \end{aligned}$$

b. $$\begin{aligned} \frac{e^{z_1}}{e^{z_2}} &= \frac{e^{x_1}(\cos y_1 + i \sin y_1)}{e^{x_2}(\cos y_2 + i \sin y_2)} \cdot \frac{\cos y_2 - i \sin y_2}{\cos y_2 - i siny_2} \\ &= e^{x_1} e^{-x_2}(\cos y_1 \cos y_2 + \sin y_1 \sin y_2 + i \sin y_1 \cos y_2 \\ &\quad - i \cos y_1 \sin y_2) \\ &= e^{x_1-x_2}[\cos(y_1 - y_2) + i \sin(y_1 - y_2)] \\ &= e^{z_1-z_2} \end{aligned}$$

16. $\exp(\ln r + i\theta) = e^{\ln r} e^{i\theta} = re^{i\theta} = z$

17. The standard parametrization of the unit circle traversed in the counterclockwise direction is $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$, which gives $z = \cos t + i \sin t = e^{it}$.

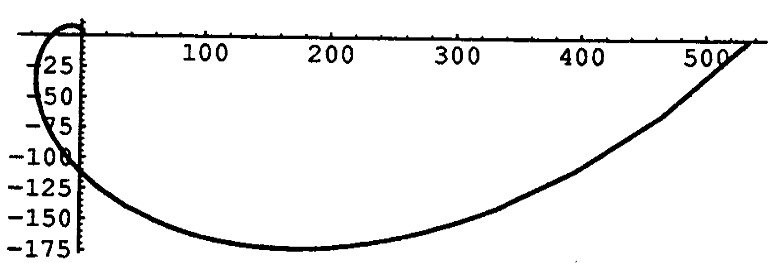
   a. The circle $|z| = 3$ traversed counterclockwise.

   b. The circle $|z - i| = 2$ traversed counterclockwise.

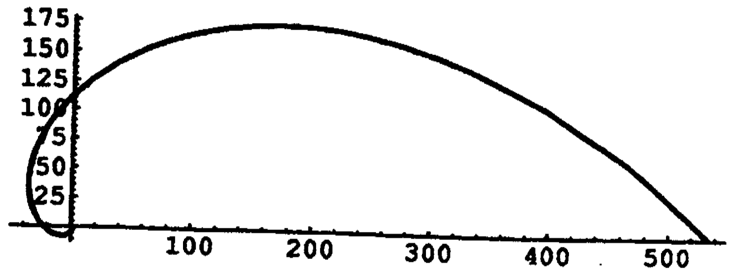
   c. The upper half of the circle $|z| = 2$ traversed counterclockwise.

   d. The circle $|z - (2 - i)| = 3$ traversed clockwise.

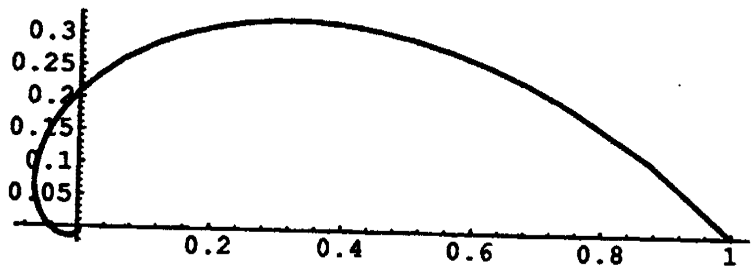
18. a.



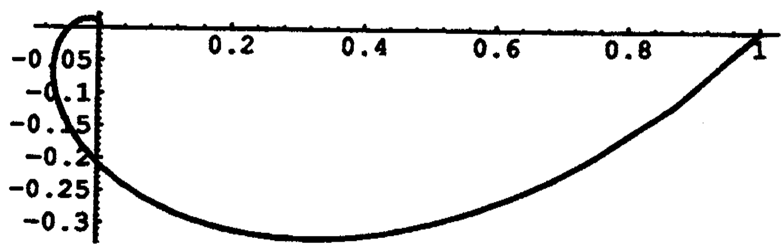
b.



c.



d.



19. $|e^{2k\pi i/n}| = 1$, $k = 0, 1, \ldots, n-1 \Longrightarrow$ The vertices lie on the unit circle.

$|e^{2k\pi i/n} - e^{2(k+1)\pi i/n}| = |1 - e^{2\pi i/n}| \Longrightarrow$ The n side lengths are equal.

20. $(z-1)(1 + z + z^2 + \cdots + z^n) = z^{n+1} - 1 \quad \Longrightarrow$

$$1 + z + z^2 + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1} \text{ when } z \neq 1$$

Suppose $z = e^{i\theta}$, $\theta \neq 0$. Then

$$\begin{aligned}1 + z + z^2 + \cdots + z^n &= 1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} \\ &= (1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta) \\ &\quad + i(\sin\theta + \sin 2\theta + \cdots + \sin n\theta)\end{aligned}$$

and

$$\begin{aligned}\frac{z^{n+1} - 1}{z - 1} &= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} \\ &= \frac{\cos(n+1)\theta - 1 + i\sin(n+1)\theta}{(\cos\theta - 1)^2 + \sin^2\theta}(\cos\theta - 1 - i\sin\theta) \\ &= \frac{\cos n\theta - \cos(n+1)\theta - \cos\theta + 1}{2 - 2\cos\theta} \\ &\quad - i\frac{\sin n\theta - \sin(n+1)\theta + \sin\theta}{2 - 2\cos\theta} \\ &= \frac{\sin(n+1/2)\theta + \sin\theta/2}{2\sin\theta/2} + i\frac{\sin(n+1)\theta/2\,\sin(n\theta/2)}{\sin\theta/2}\end{aligned}$$

a) follows by equating the real parts of both equations and b) follows by equating the imaginary parts.

21. $\left|\frac{1 - z^n}{1 - z}\right| = \left|\frac{1 - (\cos\theta + i\sin\theta)^n}{1 - (\cos\theta + i\sin\theta)}\right| = \left|\frac{(1 - \cos n\theta) + i(\sin n\theta)}{(1 - \cos\theta) + i(\sin\theta)}\right|$

$$\begin{aligned}&= \sqrt{\frac{(1 - \cos n\theta)^2 + \sin^2 n\theta}{(1 - \cos\theta)^2 + \sin^2\theta}} = \sqrt{\frac{2 - 2\cos n\theta}{2 - 2\cos\theta}} \\ &= \sqrt{\frac{(1 - \cos n\theta)/2}{(1 - \cos\theta)/2}} = \sqrt{\frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)}} = \left|\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right|\end{aligned}$$

On the other hand,

$$\left|\frac{1 - z^n}{1 - z}\right| = |1 + z + z^2 + \cdots + z^{n-1}| \leq 1 + 1 + 1 + \cdots + 1 = n.$$

22. $$\int_0^{2\pi} e^{in\theta}\, d\theta = \int_0^{2\pi} e^{0}\, d\theta = 2\pi, \; for \cdot n = 0$$

$$\int_0^{2\pi} e^{in\theta}\, d\theta = e^{i2\pi n} - 1 = 0, \; for \cdot n \neq 0$$

23 (a) $$\int_0^{2\pi} \cos^8(\theta) d\theta = \int_0^{2\pi} ((e^{i\theta} + e^{-i\theta})/2)^8 d\theta = \left(\frac{1}{256}\right) \int_0^{2\pi} \sum\nolimits_{m=0}^{8} \left(C_m^8\right) e^{i(8-2m)\theta} d\theta$$

$= 35\pi/64$

(b) $$\int_0^{2\pi} \sin^6(2\theta) d\theta = \int_0^{2\pi} \left(\frac{e^{i2\theta} - e^{-i2\theta}}{2i}\right)^6 d\theta = -20(2\pi)/(i2)^6 = 5\pi/8$$

EXERCISES 1.5: Powers and Roots

1. By induction: The case when $n = 1$ is obvious. Assume $z^m = r^m(\cos m\theta + i \sin m\theta)$ for all positive integers $m < n$.

$$\begin{aligned} z^n = z^{n-1}z &= r^{n-1}[(\cos(n-1)\theta + i\sin(n-1)\theta)][r(\cos\theta + i\sin\theta)] \\ &= r^n[\cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta \\ &\quad + i\sin(n-1)\theta\cos\theta + i\sin\theta\cos(n-1)\theta] \\ &= r^n(\cos n\theta + i\sin n\theta) \end{aligned}$$

2. Let m be a positive integer. Then

$$\begin{aligned} z^{-m} &= \frac{1}{z^m} \\ &= \frac{1}{r^m(\cos m\theta + i\sin m\theta)} \\ &= \frac{1}{r^m}(\cos m\theta - i\sin m\theta) \\ &= r^{-m}(\cos(-m\theta) + i\sin(-m\theta)) \end{aligned}$$

3. By induction: The case when $n = 1$ is obvious. Assume $\arg(z^m) = m\text{Arg}\, z + 2k\pi$, $k = 0, \pm 1, \ldots$ for all positive integers $m < n$.

$$\begin{aligned} \arg(z^n) &= \arg(z^{n-1}z) = \arg(z^{n-1}) + \arg z \\ &= (n-1)\text{Arg}\, z + \arg z + 2k\pi \\ &= n\text{Arg}\, z + 2k\pi \end{aligned}$$

4. a. $(\sqrt{3} - i)^7 = 2^7\left(\cos -\frac{7\pi}{6} + i\sin -\frac{7\pi}{6}\right) = 2^7\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -64\sqrt{3} + 64i$

b. $(1+i)^{95} = (\sqrt{2})^{95}\left(\cos\frac{95\pi}{4} + i\sin\frac{95\pi}{4}\right)$

$= (\sqrt{2})^{95}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 2^{47}(1-i)$

5. a. $(-16)^{1/4} = 2\exp\left(i\dfrac{\pi + 2k\pi}{4}\right)$, $k = 0, 1, 2, 3$

b. $1^{1/5} = \exp\left(i\dfrac{2k\pi}{5}\right)$, $k = 0, 1, 2, 3, 4$

c. $i^{1/4} = \exp\left(i\dfrac{\pi/2 + 2k\pi}{4}\right)$, $k = 0, 1, 2, 3$

d. $(1 - \sqrt{3}i)^{1/3} = \sqrt[3]{2}\exp\left(i\dfrac{-\pi/3 + 2k\pi}{3}\right)$, $k = 0, 1, 2$

e. $(i - 1)^{1/2} = \sqrt[4]{2}\exp\left(i\dfrac{3\pi/4 + 2k\pi}{2}\right)$, $k = 0, 1$

f. $\left(\dfrac{2i}{1+i}\right)^{1/6} = (1+i)^{1/6} = \sqrt[12]{2}\exp\left(i\dfrac{\pi/4 + 2k\pi}{6}\right)$, $k = 0, 1, 2, 3, 4, 5$

6. In each case one can find a root w, then construct the others as vertices of a regular pentagon inscribed in the circle $|z| = |w|$ by marking off arcs of length $|w|\dfrac{2\pi}{5}$.

a. One root is -1.

b. One root is $e^{i\pi/10}$.

c. One root is $2^{1/10}e^{i\pi/20}$.

7. a. $z = -\dfrac{1}{4} \pm \dfrac{\sqrt{23}}{4}i$

b. $z = 2 - i,\ 1 - i$

c. $z = 1 \pm \sqrt{1 - i} = 1 \pm 2^{1/4}\left(\cos\dfrac{\pi}{8} - i\sin\dfrac{\pi}{8}\right)$

8. From the quadratic formula the two solutions

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are distinct and real when $b^2 - 4ac > 0$. When $b^2 - 4ac < 0$, $\sqrt{b^2 - 4ac} = i\sqrt{-(b^2 - 4ac)}$ so the solutions are non-real complex conjugates.

9. Note that $z^3 - 3z^2 + 6z - 4 = (z-1)(z^2 - 2z + 4)$, $z = 1, 1 \pm i\sqrt{3}$

10. $z = (-1)^{1/4} = \exp\left(i\dfrac{\pi + 2k\pi}{4}\right)$, $k = 0, 1, 2, 3$
$(z - e^{(\pi/4)i})(z - e^{(7\pi/4)i}) = z^2 - \sqrt{2}z + 1 \quad (k = 0, 3)$
$(z - e^{(3\pi/4)i})(z - e^{(5\pi/4)i}) = z^2 + \sqrt{2}z + 1 \quad (k = 1, 2)$

11. $\dfrac{(z+1)^5}{z^5} = \left(1 + \dfrac{1}{z}\right)^5 = 1 \Longrightarrow 1 + \dfrac{1}{z} = q^{1/5} = w$, where $w = e^{(2k\pi/5)i}$, $k = 0, 1, 2, 3, 4$. Therefore $z = \dfrac{1}{w-1}$, $k = 1, 2, 3, 4$.

12. $z_0^{1/n} = |z_0|^{1/n} \exp\left(i\dfrac{\theta_0 + 2k\pi}{n}\right)$, $k = 0, 1, \ldots, n-1$, where $\theta_0 = \operatorname{Arg} z_0$. For each k, $z_0^{1/n}$ is the constant distance $|z_0|^{1/n}$ from the origin, and the difference in the arguments of $z_0^{1/n}$ for consecutive k is the constant $\dfrac{2\pi}{n}$. Hence the n points $z_0^{1/n}$ are equally spaced on the circle $|z| = |z_0|^{1/n}$.

13. $\omega_3 = e^{(2\pi/3)i} = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$

$1 + \omega_3 + \omega_3^2 = 1 + \left(-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right) + \left(-\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right) = 0$

$\omega_4 = e^{(\pi/2)i} = i$

$1 + \omega_4 + \omega_4^2 + \omega_4^3 = 1 + i + (-1) + (-i) = 0.$

14. $(z^m)^{1/n} = (|z|^m e^{im\theta})^{1/n}, \quad \theta = \arg z$

$$= |z|^{m/n} \exp\, i\left(\frac{m\theta + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1$$

$$= |z|^{m/n} \exp i\left(\frac{m\theta + 2km\pi}{n}\right) \quad \text{since } m \text{ and } n \text{ are relatively prime}$$

$$= |z|^{m/n} \exp\, im\left(\frac{\theta + 2k\pi}{n}\right) \quad (*)$$

$$= \left(|z|^{1/n} \exp\ i \left(\frac{\theta + 2k\pi}{n} \right) \right)^m = \left(z^{1/n} \right)^m$$

Expanding (*) gives

$$z^{m/n} = |z|^{m/n} \left(\cos \frac{m}{n}(\theta + 2k\pi) + i \sin \frac{m}{n}(\theta + 2k\pi) \right), \quad k = 0, 1, \ldots, n -$$

15. $(1 - i)^{3/2} = (\sqrt{2})^{3/2} e^{(3i/2)(-\pi/4 + 2k\pi)}, \quad k = 0, 1$

$$= 2^{3/4} e^{i(-3\pi/8 + 3k\pi)}, \quad k = 0, 1$$

16. $(z+1)^{100} = (z-1)^{100} \Rightarrow (z+1) = (z-1)e^{2\pi ki/100} \Rightarrow z(1-e^{2\pi ki/100}) = -(1+e^{2\pi ki/100})$
$z = (e^{2\pi ki/100} + 1)/(e^{2\pi ki/100} - 1) = (e^{\pi ki/100} + e^{-\pi ki/100})/(e^{\pi ki/100} - e^{\pi ki/100})$
$z = -i\cos(\pi k/100)/\sin(\pi k/100)$ for $k = 0, 1, \ldots, 99$. Because the cos and sin functions of a real variable are real z will have zero real part.

17. (Use Exercise 20 from Section 1.4)

$$1 + \omega_m^{\ell} + \omega_m^{2\ell} + \cdots + \omega_m^{(m-1)\ell} = \frac{\omega_m^{m\ell} - 1}{\omega_m - 1} = 0$$

18. Let $k = mn$. Then
$(\alpha\beta)^k = (\alpha\beta)^{mn} = \alpha^{mn}\beta^{mn} = (\alpha^n)^m (b^m)^n = 1^m 1^n = 1.$

19. (a) $F(z) = \left(1/|z - z_0|\right) e^{i\arg(z - z_0)} = \left(1/|z - z_0| e^{-i\arg(z - z_0)}\right) = 1/(\bar{z} - \bar{z}_0)$
(b) Solve $z_{o1} = 1+i$, $z_{o2} = -1+i$, $z_{o3} = 0$
$1/(\underline{z} - \underline{z}_{o1}) + 1/(\underline{z} - \underline{z}_{o2}) + 1/\underline{z} = 0 \Rightarrow z = (\pm\sqrt{2} + 2i)/3$

20. . Define subroutines called sum, diff, prod, and quot based on exercise 31, section 1.1. Also define subroutines called polar and rectangular based on exercise 24, section 1.3. Define compsqrt(x, y) as follows:

Input x, y
Set (r, t)=polar(x, y)
Set new$r$=sqrt(r), new$t$=$t/2$
Set (new$x$, new$y$)=rectangular(new$r$,new$t$)
Output (new$x$, new$y$)
Stop

Now the quadratic formula program can be written.

Input ar, ai, br, bi, cr, ci
Set (discrim r, discrim i) = prod(br, bi, br, bi)$-4 *$ prod (ar, ai, cr, ci)
Set (toproot$r$, toproot$i$)=compsqrt(discrim$r$, discrim$i$)
Set ($z1r$, $z1i$)=quot($-br +$ toproot$r$, $-bi +$ toproot$i$, $2 * ar$, $2 * ai$)
Set ($z2r$, $z2i$)=quot($-br -$ toproot$r$, $-bi -$ toproot$i$, $2 * ar$, $2 * ai$)
Print "One solution is $(x, y) =$"; ($z1r$, $z1i$); "which is $(r, t) =$";
 polar ($z1r$, $z1i$)
Print "The other solution is $(x, y) =$"; ($z2r$, $z2i$); "which is $(r, t) =$";
 polar($z2r$, $z2i$)
Stop

21. (a) $\pm(3+i)$ (b) $\pm(3+2i)$ (c) $\pm(5+i)$
(d) $\pm(2-i)$ (e) $\pm(1+3i)$ (f) $\pm(3-i)$

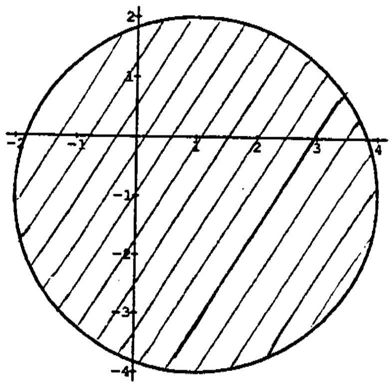
EXERCISES 1.6: Planar Sets

1. Let z_1 be in the neighborhood $|z - z_0| < \rho$ and let $R = \rho - |z_1 - z_0|$. Choose a point ω in $|z - z_1| < R$. Then

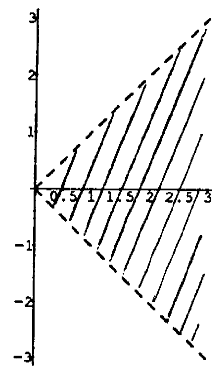
$$\begin{aligned} |z_0 - \omega| &= |z_0 - z_1 + z_1 - \omega| \\ &\leq |z_0 - z_1| + |z_1 - \omega| \\ &< |z_0 - z_1| + R = \rho \end{aligned}$$

so z_1 is an interior pont of $|z - z_0| < \rho$ and the neighborhood is an open set.

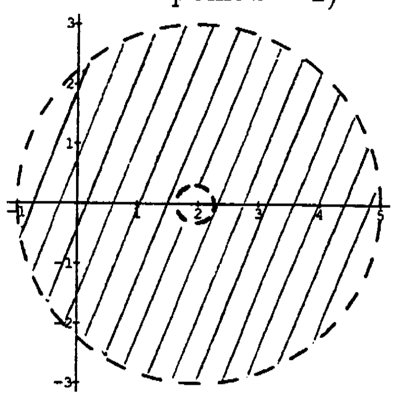
2. a. $|z-(1-i)| \le 3$



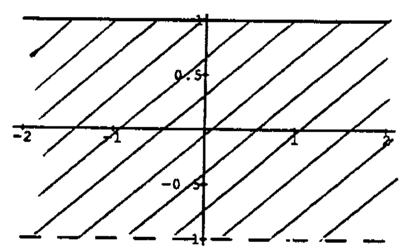
b. $|\mathrm{Arg}\, z| < \pi/4$



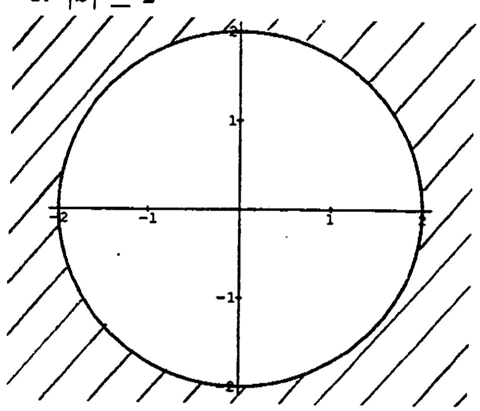
c. $0 < |z-2| < 3$ (excludes the point $z=2$)



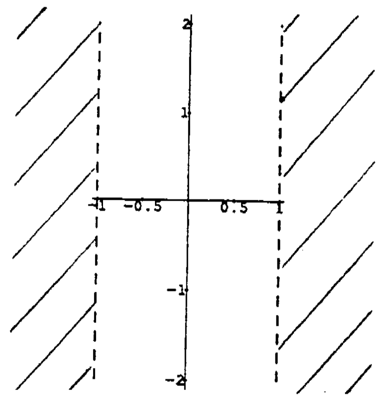
d. $-1 < \mathrm{Im}\, z \le 1$



e. $|z| \ge 2$



f. $(\mathrm{Re}\, z)^2 > 1$



3. b, c, f

4. b, c

5. a, c

6. a. $|z-(1-i)|=3$
   b. $z=re^{i\pi/4}$ and $z=re^{-i\pi/4}$
   c. $z=2$ and $|z-2|=3$
   d. $z=x+i$ and $z=x-i$ for all real x
   e. $|z|=2$
   f. $z=1+iy$ and $z=-1+iy$ for all real y

7. a, b, c, d, e

8. a, e

9. The set $S=\{z_1,z_2,\ldots,z_n\}$ is bounded by the neighborhood $|z|<\rho$, where $\rho>\max|z_j|$, $j=1,2,\ldots,n$.

10. Let $\rho_0=|z_0|$ and choose $R>\rho+\rho_0$. Then for z in $|z-z_0|\le\rho$

$$\begin{aligned}|z|=|z-z_0+z_0| &\le |z-z_0|+|z_0|\\ &\le \rho+\rho_0<R.\end{aligned}$$

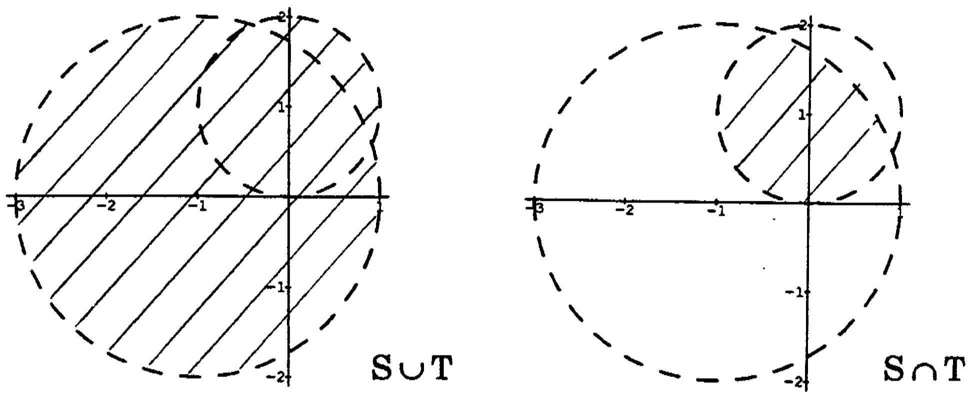
11. $S\cup\{0\}$

12. Since z_0 is not an interior point, every nighborhood of z_0 contains at least one point not in S. At the same time, every neighborhood of z_0 contains z_0, which is in S. Thus z_0 is a boundary point of S.

13. S is closed $\Longleftrightarrow$
S contains all of its boundary points. $\Longleftrightarrow$
No point of $\mathbf{C}\setminus S$ is a boundary point. $\Longleftrightarrow$
z_0 in $\mathbf{C}\setminus S$ implies that there exists a disk $|z-z_0|<\epsilon\subseteq\mathbf{C}\setminus S$. $\Longleftrightarrow$
$\mathbf{C}\setminus S$ is open.

14. By contradiction: Suppose z_0 is an accumulation point of S but that z_0 belongs to $\mathbf{C} \setminus S$. Then z_0 is a boundary point of S since each of its neighborhoods contains points in S. Because S is closed, z_0 is in $S \cap \mathbf{C} \setminus S \neq \emptyset$.

15.



16. Suppose z_0 is in $S \cup T$. If z_0 is in S, then there is a neighborhood $|z - z_0| < \rho$ that is contained in S, thus it is contained in $S \cup T$. Likewise if z_0 is in T there is a neighborhood $|z - z_0| < \rho$, (in T) that is contained in $S \cup T$. Hence z_0 is an interior point of $S \cup T$.

17. No. Counterexample:

$$S: \ 1 < |z| < 3$$
$$T: \ -1 < \operatorname{Im} z < 1$$

$S \cap T$ is not connected.

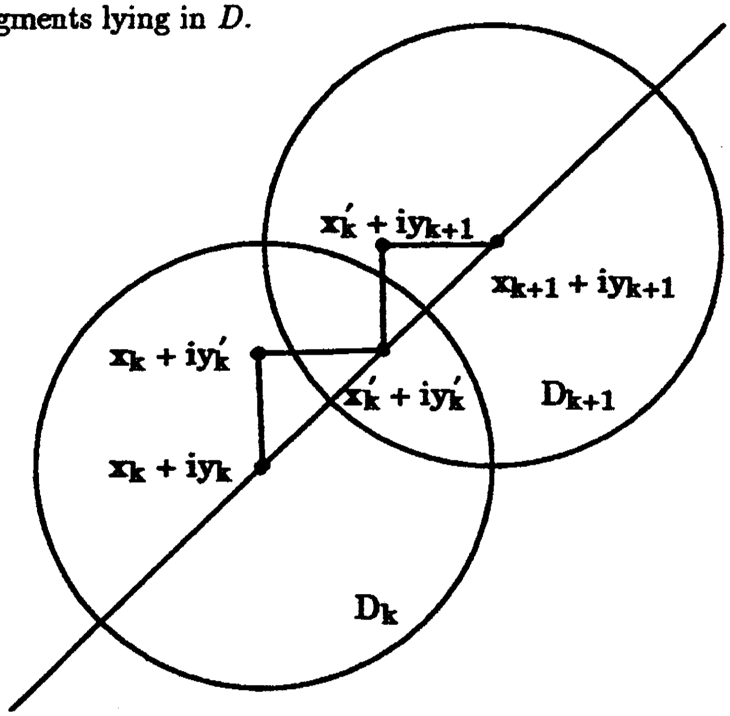
18. $S \cup T$ is open (by Exercise 16). To show that $S \cup T$ is connected, let z_0, z_1, and z be points in S, T, and $S \cap T$, respectively. Then z_0 and z can be joined by a polygonal path in S. Likewise z and z_1 can be joined by a polygonal path in T. Therefore z_0 and z_1 can be joined by a polygonal path in $S \cup T$.

19. $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$ in $\{z : |z| < 1\}$ because u is constant there and in $\{z : |z| > 2\}$ because u is constant there. Thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$ in D. Theorem 1 is not contradicted because D is not connected.

20. Let $v(x, y) = u(x, y) - xy$ at all points of D. Then $\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - y = 0$ and $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial y} - x = 0$. By Theorem 1, $v(x, y) = c$, a constant. Thus, $u(x, y) = xy + c$.

21. $\mu(x,y) = \log(x^2 + y^2) + C$ where C is a constant.

22. Let ℓ be a line segment belonging to a polygonal path connecting two points in D. Let $z_k = x_k + iy_k$ for $k = 1, 2, 3, \ldots, K$ be the centers of open disks D_k in D that cover ℓ. Let $z'_k = x'_k + iy'_k$ be any point in $D_k \cap D_{k+1}$. Then the vertical segment from $x_k + iy_k$ to $x_k + iy'_k$ is in D_k, the horizontal segment from $x_k + iy'_k$ to $x'_k + iy'_k$ is in D_k, the vertical line segment from $x'_k + iy'_k$ to $x'_k + iy_{k+1}$ is in D_{k+1}, and the horizontal line segment from $x'_k + iy_{k+1}$ to $x_{k+1} + iy_{k+1}$ is in D_{k+1}. Thus, the line segment from $x_k + iy_k$ to $x_{k+1} + iy_{k+1}$ can be replaced by these horizontal and vertical segments without leaving $D_k \cup D_{k+1}$ (and without leaving D). In this manner one can replace ℓ by horizontal and vertical line segments lying in D, and one can replace the entire polygonal path connecting the pair of points by horizontal and vertical line segments lying in D.



23. (a) The set is a continuum.
(b) The set is not a continuum.
(c) The set is not a continuum.
(d) The set is a continuum.

24. a. If $x_0 + iy_0$ and $x_1 + iy_1$ are the endpoints of the line segment then $x = (x_1 - x_0)t + x_0$, $y = (y_1 - y_0)t + y_0$ is such a parametrization.

b. $\dfrac{dU}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} = 0 \cdot (x_1 - x_0) + 0 \cdot (y_1 - y_0) = 0.$

c. Any two points z_1, z_2 in D are connected by a polygonal path lying in D. u is constant on each line segment in this path, so u is constant on the path, and $u(x_1, y_1) = u(x_2, y_2)$.

Exercises 1.7

1. (a) $i \Rightarrow (x_1, x_2, x_3) = (0, 1, 0)$
   (b) $6 - 8i \Rightarrow (x_1, x_2, x_3) = (12/101, -16/101, 99/101)$
   (b) $-3/10 + 2i/5 \Rightarrow (x_1, x_2, x_3) = (-12/25, 16/25, -3/5)$
2. (a) $z = x+iy \Rightarrow (x_1, x_2, x_3) =$
   $[2x/(x^2 + y^2 +1), 2y/(x^2 + y^2 + 1), (x^2 + y^2 - 1)/(x^2 + y^2 + 1)]$
   $1/\underline{z}^* = x/(x^2 + y^2) + iy//(x^2 + y^2) \Rightarrow (\underline{x}_1, \underline{x}_2, \underline{x}_3) =$
   $[2x/(x^2 + y^2 +1), 2y/(x^2 + y^2 + 1), (1 - x^2 - y^2)/(x^2 + y^2 + 1)]$
   $(x_{n1}, xn_2, xn_3) = (x_1, x_2, -x_3)$
   (b) $-1/\underline{z} \Rightarrow (xx_1, xx_2, xx_3) = (-x_1, -x_2, -x_3)$
   $\text{dist}(Z,W) = 2|z + 1/\underline{z}|/\surd(1+|z|^2)\surd(1 + |1/\underline{z}|^2) = 2$
3. $Z = (x_1, x_2, x_3)$, $W = (w_1, w_2, w_3)$ and $(0, 0, 0)$ define a great circle because the distance from the point Z and 0 is unity. The great circle through Z and 0 must pass through $-1/\underline{z}$ as shown in Problem 2. Example 2 showed that all lines and circles in the z-plane correspond to circles on the Riemann sphere. In Problem 10 below it will be shown that circles on the Riemann sphere correspond to lines or circles in the z-plane. Therefore, the great circle corresponds to a line or circle in the z-plane that goes through points z, $-1/\underline{z}$, w, $-1/\underline{w}$.
4. The points w and $-1/\underline{w}$ correspond to many great circles that goes through W and the center of the Riemann sphere. One of these great circles also passes through the points z and $-1/\underline{z}$.
5. (a) The hemisphere $x_1 > 0$.
   (b) The bowl $x_3 < -3/5$
   (c) The slice $0 < x_3 < 3/5$
   (d) The dome $0.8 < x_3$
   (e) The great circle $x_1 = x_2$, $1 \geq x_3 \geq -1$ or longitude 45° and longitude 225°.
6. The point Z is away from the x_3 axis a distance
   $\{[2x/(1+|z|^2)]^2 + [2y/(1+|z|^2]^2\}^{.5} = 2|z|/(1+|z|^2)$.
   The right triangle formed by $x_3 = 1$ (the point ∞) and Z and back to the x_3 axis is similar to the right triangle formed by $x_3 = 1$ and the points z and 0 in the z-plane. This gives the ratio of sides: $\chi[z,\infty]/\surd(1+|z|^2) = \{2|z|/(1+|z|^2)\}/|z|$.
   Solving yields $\chi[z,\infty] = 2/\surd(1+|z|^2)$.
7. See Figure 1.21. $|z-w|$ is related to the triangle $x_3=1$,z,w by
   $|z-w|^2 = 1+|z|^2 + 1 + |w|^2 - 2\surd(1+|z|^2)\surd(1+|w|^2)\cos\alpha$.
   $\cos\alpha = [2+|z|^2+|w|^2 - |z-w|^2]/[2\surd(1+|z|^2)\surd(1+|w|^2)]$

* ***In these solutions the complex conjugate of z is indicated by*** $\underline{z}$.

   Applying the law of cosines again yields
   $|Z-W|^2 = (2/\surd(1+|z|^2))^2 + (2/\surd(1+|w|^2))^2 - 2\{(4)/[2\surd(1+|z|^2)\surd(1+|w|^2)]\}\cos\alpha$

   Using the solution for $\cos\alpha$ in this equation gives
   $|Z-W| = 2|z-w|/\surd(1+|z|^2)\surd(1+|w|^2)$.

8. $\chi[z,w] = 2|z-w|/[\sqrt{(1+|z|^2)}\sqrt{(1+|w|^2)}]$.
   $\chi[1/z,1/w] = 2|1/z-1/w|/[\sqrt{(1+1/|z|^2)}\sqrt{(1+1/|w|^2)}]$
   $= 2(|w-z|/|z||w|)/[\sqrt{(|z|^2+1)}\sqrt{(|w|^2+1)}/|z||w|]$
   $= 2(|w-z|)/[\sqrt{(|z|^2+1)}\sqrt{(|w|^2+1)}] = \chi[z,w]$
   $\chi[-z,-w] = \chi[z,w]$ Because the projection of $-1/\underline{z}$ is on the diameter starting at Z and the projection of $-1/\underline{w}$ is on the diameter starting at W, $\chi[-1/\underline{z},-1/\underline{w}] = \chi[z,w] = \chi[1/\underline{z},1/\underline{w}]$.
9. The chords $\chi[z_1,w]$, $\chi[z_2,w]$ and $\chi[z_1,z_2]$ form a triangle. The triangle inequality (11) holds.
10. A circle on the Riemann sphere satisfies the equations
    $x_1^2 + x_2^2 + x_3^2 = 1$ and $Ax_1 + Bx_2 + Cx_3 + D = 0$.

$2xA/(1+|z|^2) + 2yB/(1+|z|^2)\ (|z|^2-1)C/(1+|z|^2) + D = 0$
$2Ax + 2By + (x^2 + y^2 - 1)C + (1+x^2 + y^2)D = 0$
$(C+D)(x^2 + y^2) + 2Ax + 2By + D-C = 0$
Let $a=C+D$, $c = 2A$, $d = 2B$ and $e = D-C$ lets you write
$a(x^2 + y^2) + cx + dy + e = 0$, an equation for a line or circle in the xy plane.

CHAPTER 2: Analytic Functions

EXERCISES 2.1: Functions of a Complex Variable

1. a. $w = (3x^2 - 3y^2 + 5x + 1) + i(6xy + 5y + 1)$

   b. $w = \dfrac{x}{x^2+y^2} + i\left(-\dfrac{y}{x^2+y^2}\right)$

   c. $w = \dfrac{1}{z-i} = \dfrac{x}{x^2+(y-1)^2} + i\dfrac{-y+1}{x^2+(y-1)^2}$

   d. $w = \dfrac{2x^2-2y^2+3}{\sqrt{(x-1)^2+y^2}} + i\dfrac{4xy}{\sqrt{(x-1)^2+y^2}}$

   e. $w = e^{3x}\cos 3y + ie^{3x}\sin 3y$

   f. $w = (e^x + e^{-x})\cos y + i(e^x - e^{-x})\sin y$
   $= 2\cosh x \cos y + i2\sinh x \sin y$

2. a. $\mathbf{C}$

   b. $\mathbf{C} \setminus \{0\}$

   c. $\mathbf{C} \setminus \{i, -i\}$

   d. $\mathbf{C} \setminus \{1\}$

   e. $\mathbf{C}$

   f. $\mathbf{C}$

3. a. $\operatorname{Re} w > 5$

   b. $\operatorname{Im} w \geq 0$

   c. $|w| \geq 1$

   d. The intersection of $|w| < 2$ and $-\pi < \operatorname{Arg} w < \pi/2$

4. a. Taking θ from 0 to 2π, the points $z = re^{i\theta}$ traverse the circle $|z| = r$ exactly once in the counterclockwise direction. For the same values of θ the points $w = \dfrac{1}{re^{i\theta}} = \dfrac{1}{r}e^{-i\theta}$ traverse the circle $|w| = \dfrac{1}{r}$ exactly once in the clockwise direction, hence the mapping is onto.

   b. For $z = re^{i\theta_0}$ on the ray $\operatorname{Arg} z = \theta_0$, $w = \dfrac{1}{re^{i\theta_0}} = \dfrac{1}{r}e^{-i\theta_0}$ is on the ray $\operatorname{Arg} w = -\theta_0$. Taking values $0 < r < \infty$ shows that this mapping goes onto the ray $\operatorname{Arg} w = -\theta_0$.

4 (c) $|z-1| = 1\ 2\pi > \theta \geq 0 \Rightarrow z = 1 + e^{i\theta}$. $F(z) = 1/z. = 1/(1+e^{i\theta})$
$= (1+e^{-i\theta})/\{2(1+\cos\theta)\} = ½\ -i(½)\sin\theta/(1+\cos\theta)$
which is a vertical line at $x = ½$.

5. a. domain: $\mathbf{C}$
   range: $\mathbf{C} \setminus \{0\}$

   b. $f(-z) = e^{-z} = \dfrac{1}{e^z} = \dfrac{1}{f(z)}$

   c. circle $|w| = e$

   d. ray $\operatorname{Arg} w = \pi/4$

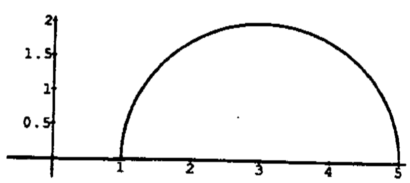
   e. infinite sector $0 \leq \operatorname{Arg} w \leq \pi/4$

6. a. $J\left(\dfrac{1}{z}\right) = \dfrac{1}{2}\left(\dfrac{1}{z} + \dfrac{1}{1/z}\right) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) = J(z)$

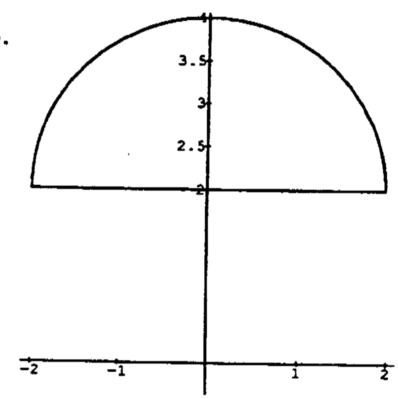
   b. For $z = e^{i\theta}$ on the unit circle $|z| = 1$, $J(z) = \dfrac{1}{2}\left(e^{i\theta} + \dfrac{1}{e^{i\theta}}\right) = \cos\theta$. For all values of θ, this ranges over the real interval $[-1, 1]$.

   c. For $z = re^{i\theta}$ on the circle $|z| = r$, $J(z) = \dfrac{1}{2}\left(re^{i\theta} + \dfrac{1}{re^{i\theta}}\right) = \dfrac{1}{2}\left(r + \dfrac{1}{r}\right)\cos\theta + i\dfrac{1}{2}\left(r - \dfrac{1}{r}\right)\sin\theta$. Setting u and v equal to the real and imaginary parts of this expression, respectively, one gets a pair of parametric equations that are equivalent to the ellipse $\dfrac{u^2}{[\frac{1}{2}(r+\frac{1}{r})]^2} + \dfrac{v^2}{[\frac{1}{2}(r-\frac{1}{r})]^2} = 1$, which has foci at ± 1.

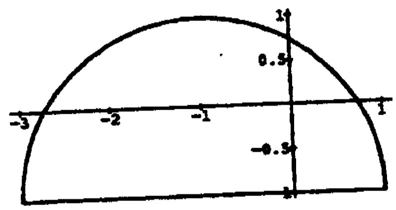
7. a.



b.

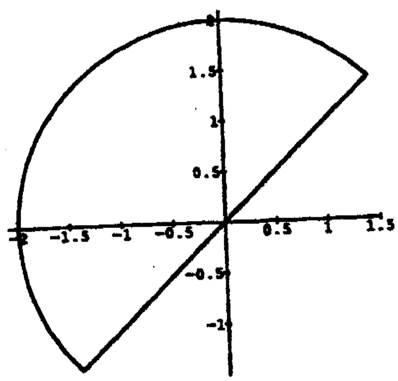


c.



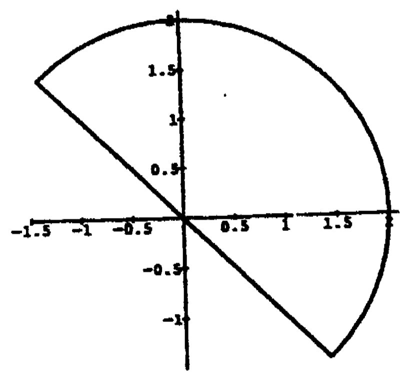
1
0.5
-3
-2
-1
1
-0.5
-1

8. a.



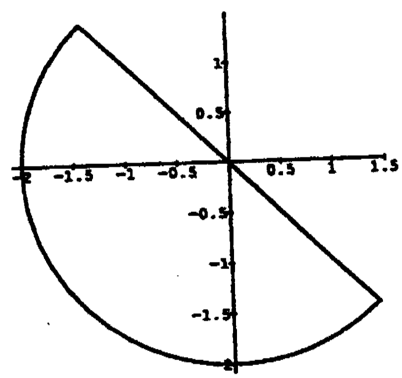
2
1.5
1
0.5
-2
-1.5
-1
-0.5
0.5
1
1.5
-0.5
-1

b.



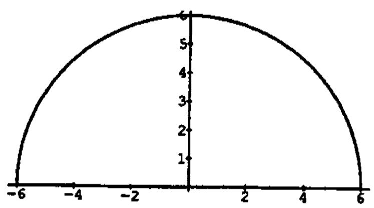
2
1.5
1
0.5
-1.5
-1
-0.5
0.5
1
1.5
2
-0.5
-1

c.

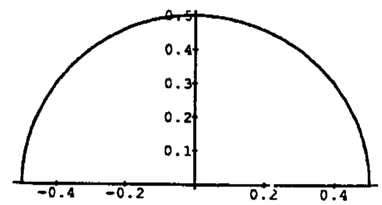


1
0.5
-2
-1.5
-1
-0.5
0.5
1
1.5
-0.5
-1
-1.5
-2

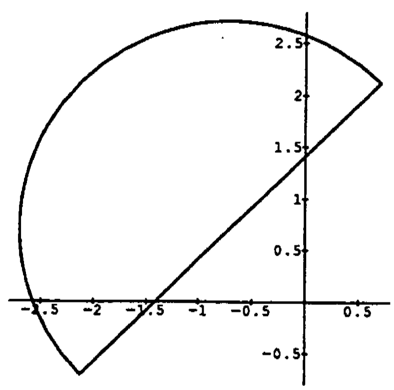
9. a.



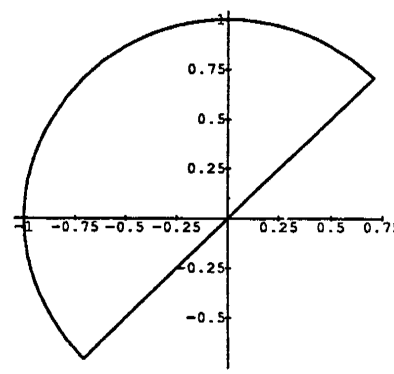
b.



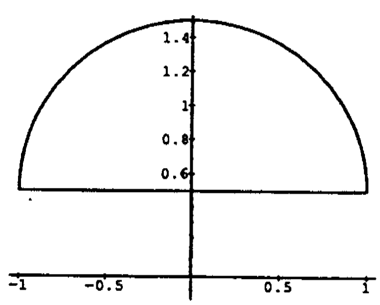
10. a. translate by i, rotate $\pi/4$



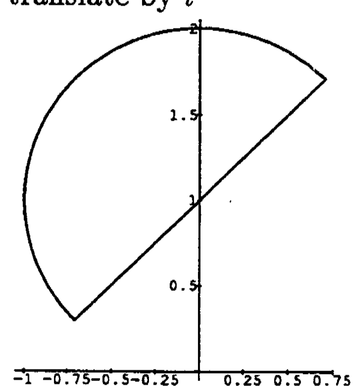
b. reduce by $1/2$, rotate $\pi/4$



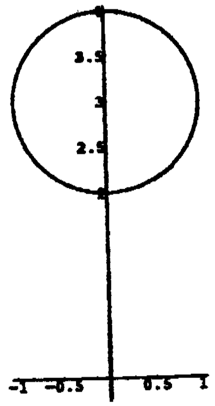
c. translate by i, reduce by $1/2$



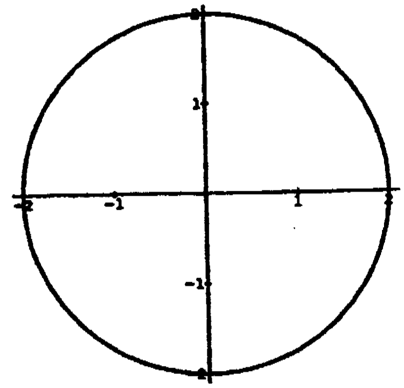
d. reduce by $1/2$, rotate $\pi/4$, translate by i



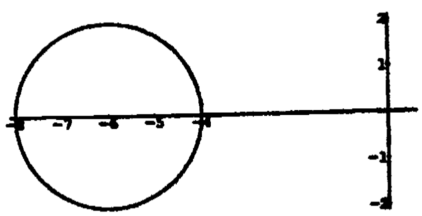
11. a. **translate by -3, rotate $-\pi/2$**



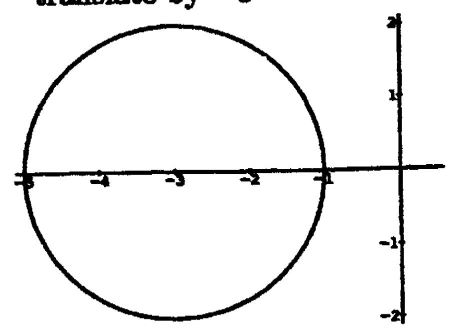
b. magnify by 2, rotate $-\pi/2$



c. **translate by -3, magnify by 2**



d. **magnify by 2, rotate $-\pi/2$, translate by -3**



12. Let $a = \rho e^{i\phi}$, $F(z) = \rho z$, $G(z) = e^{i\phi}z$, and $H(z) = z + b$. Then $H(G(F(z))) = az + b$.

13. (a) $w = u + iv = z^2 = (1 + iy)^2 = 1 - y^2 + i2y$

$u = 1 - y^2$, $v = 2y \Rightarrow y = v/2 \Rightarrow u = 1 - v^2/4$ a parabola in the w-plane.

(b) $w = u + iv = z^2 = (x + iy)^2 = (x + i/x)^2 = x^2 - 1/x^2 + 2i$

$u = x^2 - 1/x^2$, $v = 2$ a straight line in the w-plane.

(c) $w = u + iv = z^2 = (1 + e^{i\theta})^2 = (1 + 2e^{i\theta} + e^{i2\theta}) = (e^{-i\theta} + 2 + e^{i\theta})e^{i\theta}$

$= (2 + 2\cos\theta)e^{i\theta} = 2(1 + \cos\theta)e^{i\theta}$ a cardioid in the w-plane.

14. (a) $x_1 = 2x/(|z|^2 + 1)$, $x_2 = 2y/(|z|^2 + 1)$, $x_3 = (|z|^2 - 1)/(|z|^2 + 1)$

$w = e^{i\varphi}z = x\cos\varphi - y\sin\varphi + i(x\sin\varphi + y\cos\varphi)$, $|w| = |z|$

$\underline{x}_1 = (x\cos\varphi - y\sin\varphi)/(|z|^2 + 1)$, $\underline{x}_2 = (x\sin\varphi + y\cos\varphi)/(|z|^2 + 1)$, $\underline{x}_3 = x_3$

$\underline{x}_1 = (x_1\cos\varphi - x_2\sin\varphi)$, $\underline{x}_2 = (x_1\sin\varphi + x_2\cos\varphi)$, $\underline{x}_3 = x_3$ which corresponds to a rotation of an angle φ about the x_3 axis.

(b) $w = -1/z$. $|w| = 1/|z|$. $w = -1/(x+iy) = -x/|z| + iy/|z|$

$\underline{x}_1 = -x_1$, $\underline{x}_2 = x_2$, $\underline{x}_3 = -x_3$ so that $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ is obtained from (x_1, x_2, x_3) by a 180° rotation about the x_2 axis.

15. $w = (1+z)/(1 - z) = (1+x + iy)/(1 - x - iy) = (1-|z|^2 + i2y)/(1- 2x + |z|^2)$

$|w|^2 = (1 + 2x + |z|^2)/(1- 2x + |z|^2)$.

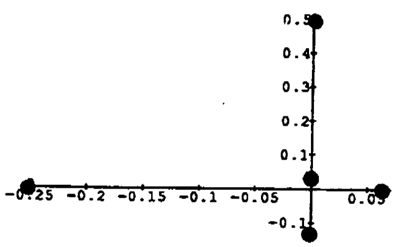
$(\underline{x}_1, \underline{x}_2, \underline{x}_3) = (-x_3, x_2, x_1)$ so that $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ is obtained by a 90° counterclockwise rotation about the x_2 axis.

16. $w = (1 - iz)/(1 + iz) = (1 - ix + y)/(1 + ix - y) = (1 - |z|^2 + i2x)/(1 - 2y + |z|^2)$
$|w|^2 = (1 + 2y + |z|^2)/(1 - 2y + |z|^2)$.
$(\underline{x}_1, \underline{x}_2, \underline{x}_3) = (-x_3, -x_1, x_2)$ so that $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ is obtained as a 90° counterclockwise rotation about the x_2 axis followed by a 90° counterclockwise rotation about the $x_3$axis.

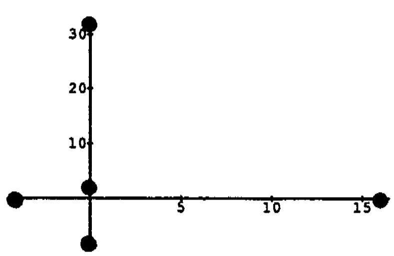
17. Any circle or line in the z-plane corresponds to a line or circle on the stenographic projection onto he Riemann sphere. The function $w=1/z$ rotates the Riemann sphere 180° about the x_1 axis. Lines and circles on the rotated sphere project to lines and circles in the w-plane. As a result lines and circles in the z-plane map to lines and circles in the w-plane.

EXERCISES 2.2: Limits and Continuity

1. The first five terms are, respectively, $\frac{i}{2}$, $-\frac{1}{4}$, $-\frac{i}{8}$, $\frac{1}{16}$, and $\frac{i}{32}$. The sequence converges to 0 in a spiral-like fashion.



2. $2i$, -4, $-8i$, 16, $32i$; divergent because terms grow in modulus without bound.



3. If $\lim_{n\to\infty} z_n = z_0$, then for any $\varepsilon>0$, there is an integer N such that $|z_n - z_0|<\varepsilon$ for all $n>N$. For the same integer N we have $|x_n - x_0|<=|z_n - z_0|<\varepsilon$ and $|y_n - y_0|<=|z_n - z_0|<\varepsilon$ for all $n>N$. Therefore, $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} y_n = y_0$.

If $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} y_n = y_0$, then for any $\varepsilon_1 >0$ and $\varepsilon_2 >0$ there are integers N_1 and N_2 such
$|x_n - x_0|<\varepsilon_1$ for all $n > N_1$ and $|y_n - y_0|<\varepsilon_2$ for all $n > N_2$. Given any $\varepsilon > 0$; let $\varepsilon_1 = \varepsilon/2$ and $\varepsilon_2 = \varepsilon/2$. Then
$|z_n - z_0|<=\varepsilon\ |x_n - x_0| + |y_n - y_0| < \varepsilon_1 + \varepsilon_2 = \varepsilon$ for all $n >$ maximum(N_1, N_2).
Thus $\lim_{n\to\infty} z_n = z_0$.

4. If $z_n = x_n + iy_n \to z_0 = x_0 + iy_0$, then $x_n \to x_0$ and $y_n \to y_0$ (see Problem 3). $\underline{z_n} = x_n - iy_n \to x_0 - iy_0 = \underline{z_0}$.

If $\underline{z_n} = x_n - iy_n \to \underline{z_0} = x_0 - iy_0$, then $x_n \to x_0$ and $y_n \to y_0$ (see Problem 3).. $z_n = x_n + iy_n \to x_0 + iy_0 = z_0$. Thus $z_n \to z_0$ if and only if $\underline{z_n} \to \underline{z_0}$.

5. $\lim_{n\to\infty} |z_n| = 0 \Longrightarrow$ **There exists an integer** N **such that** $||z_n| - 0| = |z_n| < \varepsilon$ **whenever** $n > N$. $\Longrightarrow |z_n - 0| < \varepsilon$ **whenever** $n > N$. $\Longrightarrow \lim_{n\to\infty} z_n = 0$, **and conversely.**

6. $z_0^n \to 0$ **as** $n \to \infty$ **by problem 3, since the real-valued sequence** $|z_0^n| \to 0$ **as** $n \to \infty$. **On the other hand, if** $|z_0| > 1$, **then** $|z_0^n| \to \infty$ **as** $n \to \infty$ **so** z_0^n **diverges.**

7. **a. converges to 0**
 b. does not converge
 c. converges to π
 d. converges to $2+i$
 e. converges to 0
 f. does not converge

8. **Given** $\varepsilon > 0$, **choose** $\delta = \varepsilon/6$. **Then whenever** $0 < |z - (1+i)| < \delta$,

$$|6z - 4 - (2 + 6i)| = 6|z - (1 + i)| < 6(\varepsilon/6) = \varepsilon$$

9. **Given** $\varepsilon > 0$, **choose** $\delta = \dfrac{\varepsilon}{1+\varepsilon}$. **Whenever** $0 < |z - (-i)| < \delta$ **notice that** $|z| > 1 - \delta$ **and**

$$\left|\frac{1}{z} - i\right| = \left|\left(-\frac{i}{z}\right)(i + z)\right| = \frac{1}{|z|}|z - (-i)| < \left(\frac{1}{1-\delta}\right)\delta = \varepsilon$$

10. **Given that f and g are continuous at z_0,**

$$\lim_{z\to z_0} f(z) \pm g(z) = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z) = f(z_0) \pm g(z_0)$$

$\Longrightarrow f(z) \pm g(z)$ **is continuous at z_0.**

$$\lim_{z\to z_0} f(z)g(z) = \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z) = f(z_0)g(z_0)$$

$\Longrightarrow f(z)g(z)$ **is continuous at z_0.**

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)} = \frac{f(z_0)}{g(z_0)}, \text{ provided } g(z_0) \neq 0$$

$\Longrightarrow \dfrac{f(z)}{g(z)}$ is continuous at z_0.

11. a. $-8i$
    b. $-\dfrac{7}{2}i$
    c. $6i$
    d. $-1/2$
    e. $2z_0$
    f. $4\sqrt{2}$

12. Clearly $\operatorname{Arg} z$ is discontinuous at $z = 0$. Let $a > 0$ be any real number and consider the sequence

$$z_n = -a - i/n \quad n = 1, 2, \ldots, \text{ which converges to } -a.$$

For each n, $-\pi < \operatorname{Arg} z_n < -\pi/2$, but $\operatorname{Arg}(-a) = \pi$.

13. $\lim_{z\to z_0} f(z)$ exists for all $z \neq -1$; f is continuous for all $z \neq 0, -1$; f has a removable discontinuity at $z = 0$.

14. Let z_0 be any complex number. Given $\varepsilon > 0$ choose $\delta = \varepsilon$. Then whenever $|z - z_0| < \delta$,

$$|g(z) - g(z_0)| = |\overline{z} - \overline{z}_0| = |\overline{z - z_0}| = |z - z_0| < \varepsilon.$$

15. Given $\varepsilon > 0$ choose δ so that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Then, whenever $|z - z_0| < \delta$:

    a. $|\overline{f(z)} - \overline{f(z_0)}| = |\overline{f(z) - f(z_0)}| = |f(z) - f(z_0)| < \varepsilon$
    b. $|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| = |\operatorname{Re}(f(z) - f(z_0))| \leq |f(z) - f(z_0)| < \varepsilon$
    c. $|\operatorname{Im} f(z) - \operatorname{Im} f(z_0)| = |\operatorname{Im}(f(z) - f(z_0))| \leq |f(z) - f(z_0)| < \varepsilon$
    d. $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)| < \varepsilon$

16. Given $\varepsilon > 0$, choose $\delta_0 > 0$ such that $|f(g(z)) - f(g(z_0))| < \varepsilon$ whenever $|g(z) - g(z_0)| < \delta_0$. Now choose $\delta > 0$ such that $|g(z) - g(z_0)| < \delta_0$ whenever $|z - z_0| < \delta$. Then $|f(g(z)) - f(g(z_0))| < \varepsilon$ whenever $|z - z_0| < \delta$; hence $f(g(z))$ is continuous at z_0.

17. No: Observe that although $\frac{1}{n} \to 0$ and $\frac{i}{n} \to 0$ as $n \to \infty$, $f\left(\frac{1}{n}\right) \to 1 + 2i$ and $f\left(\frac{i}{n}\right) \to 2i$; thus $\lim_{z\to 0} f(z)$ does not exist.

18. If $\lim_{z\to z_0} f(z) = w_0$, then given $\varepsilon>0$ there exists $\delta>0$ such that $|f(z)-w_0|<\varepsilon$ for all $|z-z_0|<\delta$. Notice that $|\underline{f(z)}-\underline{w_0}| = |\underline{f(z)-w_0}| = |f(z)-w_0|<\varepsilon$ for all $|z-z_0|<\delta$. So that $\lim_{z\to z_0}\underline{f(z)} = \underline{w_0}$.
$\lim_{x\to x_0, y\to y_0} \mu(x,y) = \lim_{z\to z_0} ((f(z)+\underline{f(z)})/2) = (w_0+\underline{w_0})/2 = \mu_0$.
$\lim_{x\to x_0, y\to y_0} \upsilon(x,y) = \lim_{z\to z_0} ((f(z)-\underline{f(z)})/2i) = (w_0-\underline{w_0})/2i = \upsilon_0$.
Thus, $\lim_{x\to x_0, y\to y_0} \mu(x,y) = \mu_0$ and $\lim_{x\to x_0, y\to y_0} \upsilon(x,y) = \upsilon_0$.

Conversely, if $\lim_{x\to x_0, y\to y_0} \mu(x,y) = \mu_0$ and $\lim_{x\to x_0, y\to y_0} \upsilon(x,y) = \upsilon_0$, then (by Theorem 1.) $\mu_0 + i\upsilon_0 = \lim_{x\to x_0, y\to y_0} \mu(x,y) + i\lim_{x\to x_0, y\to y_0} \upsilon(x,y) = \lim_{z\to z_0} ((f(z)+\underline{f(z)})/2) + \lim_{z\to z_0} ((f(z)-\underline{f(z)})/2) = \lim_{z\to z_0} f(z) = w_0$.
Also $\mu_0 - i\upsilon_0 = \lim_{x\to x_0, y\to y_0} \mu(x,y) - i\lim_{x\to x_0, y\to y_0} \upsilon(x,y) = \lim_{z\to z_0} ((f(z)+\underline{f(z)})/2) - \lim_{z\to z_0} ((f(z)-\underline{f(z)})/2) = \lim_{z\to z_0} f(z) = \underline{w_0}$.

Thus, $\lim_{z\to z_0} f(z) = w_0$.

19. $-\frac{1}{2} - i$, since $\lim_{\substack{x\to 1\\ y\to -1}} \frac{x}{x^2+3y} = -\frac{1}{2}$ and $\lim_{\substack{x\to 1\\ y\to -1}} xy = -1$.

20. For any z_0 in the complex plane,

$$\lim_{z\to z_0} e^z = \lim_{\substack{x\to x_0\\ y\to y_0}} e^x \cos y + i \lim_{\substack{x\to x_0\\ y\to y_0}} e^x \sin y = e^{x_0}\cos y_0 + ie^{x_0}\sin y_0 = e^{z_0}.$$

21. a. 1
    b. 0
    c. $-\pi/2 + i$
    d. 1

21

22. By contradiction: Suppose $\lim_{z\to z_0} f(z) \neq w_0$. Then there is an $\varepsilon > 0$ for which there exists a sequence $\{z_n\}$ such that $|z_n - z_0| < \frac{1}{n}$ but $|f(z_n) - w_0| > \varepsilon$. For this sequence, $\lim_{n\to\infty} z_n = z_0$ but $\lim_{n\to\infty} f(z_n) \neq w_0$, contrary to hypothesis.

23. If $z_n \to \infty$, then for any M>0 there exist an integer N such $|z_n| > M$ for al $n > N$. Consider the chordal distance $\chi(z_n,\infty) = 2/\sqrt{(|z_n|^2 + 1)} < 2/\sqrt{(|z_n|^2)} = 2/|z_n| < 2/M < \varepsilon$ for all $n > N$. Thus $z_n \to \infty$ as $n \to \infty$ is equivalent to $\chi(z_n,\infty) \to 0$ as $n \to \infty$.

24. If $\lim_{z\to z_0} f(z) = \infty$, then for any M>0 there exis $\delta > 0$ such that $|f(z)| > M$ for all $|z-z_0| < \delta$. Consider $\chi(f(z),\infty) = 2/\sqrt{(|f(z)|^2 + 1)} < 2/\sqrt{(|f(z)|^2)} = 2/|f(z)| < 2/M < \varepsilon$ for all $|z-z_0| < \delta$. Thus $\lim_{z\to z_0} f(z) = \infty$, is equivalent to $\lim_{z\to\infty} \chi(f(z),\infty) = 0$.

25. (a) ∞ (b) 3 (c) ∞ (d) ∞ (f) the limit does not exist.

EXERCISES 2.3: Analyticity

1. Let $\Delta z = z - z_0$ so that $\Delta z \to 0 \iff z \to z_0$. Then

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = L \iff$$

given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - L \right| < \varepsilon \text{ whenever} |\Delta z - 0| < \delta \iff$$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - L \right| < \varepsilon \text{whenever} |z - z_0| < \delta \iff$$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L.$$

2. If $\lambda(z) = \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$, then $\lambda(z) \to 0$ as $z \to z_0$ and $f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0) = f(z)$.

3. $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} [f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0)]$
$= f(z_0) + 0 + 0 = f(z_0)$.

4. a. $\displaystyle\lim_{\Delta z\to 0}\frac{\mathrm{Re}\,(z+\Delta z)-\mathrm{Re}\,(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{\mathrm{Re}\,(\Delta z)}{\Delta z}=\begin{cases}1, & \text{if } \Delta z=\Delta x\\ 0, & \text{if } \Delta z=i\Delta y\end{cases}$

b. $\displaystyle\lim_{\Delta z\to 0}\frac{\mathrm{Im}\,(z+\Delta z)-\mathrm{Im}(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{\mathrm{Im}\,(\Delta z)}{\Delta z}=\begin{cases}0, & \text{if } \Delta z=\Delta x\\ -i, & \text{if } \Delta z=i\Delta y\end{cases}$

c. Case 1, $z=0$.

$$\lim_{\Delta z\to 0}\frac{|0+\Delta z|-|0|}{\Delta z}=\lim_{\Delta z\to 0}\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta x+i\Delta y}=\begin{cases}\pm 1, & \text{if } \Delta z=\Delta x\\ -i, & \text{if } \Delta z=\pm i\Delta y\end{cases}$$

Case 2, $z\neq 0$.

$$\begin{aligned}
&\lim_{\Delta z\to 0}\frac{|z+\Delta z|-|z|}{\Delta z}\\
&=\lim_{\Delta z\to 0}\frac{\sqrt{(x+\Delta x)^2+(y+\Delta y)^2}-\sqrt{x^2+y^2}}{\Delta x+i\Delta y}\\
&=\lim_{\Delta z\to 0}\frac{(x+\Delta x)^2+(y+\Delta y)^2-(x^2+y^2)}{(\Delta x+i\Delta y)(\sqrt{(x+\Delta x)^2+(y+\Delta y)^2}+\sqrt{x^2+y^2})}\\
&=\lim_{\Delta z\to 0}\frac{2x\Delta x+(\Delta x)^2+2y\Delta y+(\Delta y)^2}{(\Delta x+i\Delta y)(\sqrt{(x+\Delta x)^2+(y+\Delta y)^2}+\sqrt{x^2+y^2})}\\
&=\begin{cases}\dfrac{x}{\sqrt{x^2+y^2}}, & \text{if } \Delta z=\Delta x,\ z\neq 0\\[2ex] \dfrac{y}{i\sqrt{x^2+y^2}}, & \text{if } \Delta z=i\Delta y,\ z\neq 0\end{cases}
\end{aligned}$$

5. Rule 5: $\displaystyle (f\pm g)'(z_0)=\lim_{\Delta z\to 0}\frac{(f\pm g)(z_0+\Delta z)-(f\pm g)(z_0)}{\Delta z}$

$$\begin{aligned}
&= \lim_{\Delta z\to 0}\left[\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\pm\frac{g(z_0+\Delta z)-g(z_0)}{\Delta z}\right]\\
&= f'(z_0)\pm g'(z_0)
\end{aligned}$$

Rule 7: $\displaystyle (fg)'(z_0)=\lim_{\Delta z\to 0}\frac{fg(z_0+\Delta z)-fg(z_0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \left\{ \frac{f(z_0 + \Delta z)g(z_0 + \Delta z) - f(z_0 + \Delta z)g(z_0)}{\Delta z} \right.$$

$$\left. + \frac{f(z_0 + \Delta z)g(z_0) - f(z_0)g(z_0)}{\Delta z} \right\}$$

$$= \lim_{\Delta z \to 0} \left\{ f(z_0 + \Delta z) \frac{[g(z_0 + \Delta z) - g(z_0)]}{\Delta z} \right.$$

$$\left. + g(z_0) \frac{[f(z_0 + \Delta z) - f(z_0)]}{\Delta z} \right\}$$

$$= f(z_0)g'(z_0) + g(z_0)f'(z_0)$$

6. Let $n > 0$ be an integer.
   Then $\dfrac{d}{dz}z^{-n} = \dfrac{d}{dz}\left(\dfrac{1}{z^n}\right) = \dfrac{-nz^{n-1}}{z^{2n}}$ (using Rule 8) $= -nz^{-n-1}$.

7. a. $18z^2 + 16z + i$
   b. $-12z(z^2 - 3i)^{-7}$
   c. $\dfrac{-iz^4 + (2 + 27i)z^2 + 2\pi z + 18}{(iz^3 + 2z + \pi)^2}$
   d. $\dfrac{-(z+2)^2(5z^2 + (16 + i)z - 3 + 8i)}{(z^2 + iz + 1)^5}$
   e. $24i(z^3 - 1)^3(z^2 + iz)^{99}(53z^4 + 28iz^3 - 50z - 25i)$

8. Let $z = z_0 + \Delta z$. Then

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = \left| \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right| = |f'(z_0)|.$$

$$\begin{aligned} \lim_{z \to z_0} \arg[f(z) - f(z_0)] - \arg(z - z_0) &= \lim_{z \to z_0} \arg \left[\frac{f(z) - f(z_0)}{z - z_0} \right] \\ \arg \left[\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right] &= \arg[f'(z_0)] \end{aligned}$$

9. a. $2-3i$

   b. $\pm i$

   c. $\dfrac{-1\pm i\sqrt{15}}{2}$

   d. $\dfrac{1}{2}, 1$

10. $\lim_{\Delta z\to 0}\dfrac{|z_0+\Delta z|^2-|z_0|^2}{\Delta z}$

$$= \lim_{\Delta z\to 0}\frac{(z_0+\Delta z)(\overline{z}_0+\overline{\Delta z})-z_0\overline{z}_0}{\Delta z}$$
$$= \lim_{\Delta z\to 0}\left(\overline{z}_0+\frac{\overline{\Delta z}}{\Delta z}z_0+\overline{\Delta z}\right)=\begin{cases}\overline{z_0}+z_0 & \text{if } \Delta z=\Delta x\\ \overline{z_0}-z_0 & \text{if } \Delta z=i\Delta y\end{cases}$$

If $z_0=0$, then the difference quotient is

$$\lim_{\Delta z\to 0}(0+0+\overline{\Delta z})=0.$$

11. a. nowhere analytic

   b. nowhere analytic

   c. analytic except at $z=5$

   d. everywhere analytic

   e. nowhere analytic

   f. analytic except at $z=0$

   g. nowhere analytic

   h. nowhere analytic

12. The case when $n=1$ is trivial. Assume that the result holds for all positive integers less than or equal to n and define $Q(z)=P(z)(z-z_{n+1})$. Since $Q'(z)=P'(z)(z-z_{n+1})+P(z)$, it follows that

$$\frac{Q'(z)}{Q(z)}=\frac{P'(z)}{P(z)}+\frac{1}{z-z_{n+1}}=\frac{1}{z-z_1}+\frac{1}{z-z_2}+\cdots+\frac{1}{z-z_{n+1}}$$

13. a, b, d, f, and g are always true

14. $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{[f(z) - f(z_0)]/(z - z_0)}{[g(z) - g(z_0)]/(z - z_0)} = \frac{f'(z_0)}{g'(z_0)}$

15. $\frac{3}{5}$

16. Any point on the line through z_1 and z_2 has the form $z = -\frac{1}{2} + i\sqrt{3}\left(\frac{1}{2} - t\right)$, t real (see Section 1.3, Exercise 18). However, $f(z_2) - f(z_1) = 0$ but $f'(w) = 3w^2 \neq 0$ on the line in question.

17. $F'(z_0) = f(z_0)(gh)'(z_0) + f'(z_0)gh(z_0)$
$= f(z_0)[g(z_0)h'(z_0) + g'(z_0)h(z_0)] + f'(z_0)g(z_0)h(z_0)$
$= f'(z_0)g(z_0)h(z_0) + f(z_0)g'(z_0)h(z_0) + f(z_0)g(z_0)h'(z_0)$

EXERCISES 2.4: The Cauchy-Riemann Equations

1. a. $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$

   b. $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0$

   c. $\frac{\partial u}{\partial y} = 2 \neq -\frac{\partial v}{\partial x} = 1$

2. $\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 3 = \frac{\partial v}{\partial y}$, but $\frac{\partial u}{\partial y} = 6xy = \frac{\partial v}{\partial x}$. Therefore $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ only when $x = 0$ or $y = 0$. This means h is differentiable on the axes but h is nowhere analytic since lines are not open sets in the complex plane.

3. $\frac{\partial u}{\partial x} = 6x + 2 = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -6y = -\frac{\partial v}{\partial x}$. Since these partial derivatives exist and are continuous for all x and y, g is analytic. g can be written as $g(z) = 3z^2 + 2z - 1$.

4. $\dfrac{\partial u}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{u(\Delta x, 0) - u(0,0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0}{\Delta x} = 0$

$\dfrac{\partial u}{\partial y} = \lim\limits_{\Delta y \to 0} \dfrac{u(0, \Delta y) - u(0,0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{0}{\Delta y} = 0.$

Similarly $\dfrac{\partial v}{\partial x} = 0$ and $\dfrac{\partial v}{\partial y} = 0$.

However, when $\Delta z \to 0$ through real values ($\Delta z = \Delta x$)

$$\lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = 0,$$

while along the real line $y = x$ ($\Delta z = \Delta x + i\Delta x$)

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} &= \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^{4/3}(\Delta x)^{5/3} + i(\Delta x)^{5/3}(\Delta x)^{4/3}}{2(\Delta x)^2}}{\Delta x(1+i)} \\
&= \frac{1}{2}.
\end{aligned}$$

Therefore f is not differentiable at $z = 0$.

5. $\dfrac{\partial u}{\partial x} = 2e^{x^2-y^2}[x\cos(2xy) - y\sin(2xy)] = \dfrac{\partial v}{\partial y}$

$\dfrac{\partial u}{\partial y} = -2e^{x^2-y^2}[y\cos(2xy) + x\sin(2xy)] = -\dfrac{\partial v}{\partial x}$

f is entire because these first partials exist and are continuous for all x and y.

$$\begin{aligned}
f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= 2e^{x^2-y^2}(x+iy)[\cos(2xy) + i\sin(2xy)] \\
&= 2e^{(x^2-y^2)}e^{i2xy}(x+iy) \\
&= 2ze^{z^2}
\end{aligned}$$

(This derivative could have been obtained directly, since $f(z) = e^{z^2}$.)

6. $z = re^{i\theta} \Longrightarrow x = r\cos\theta$ and $y = r\sin\theta$ and

$$f(z) = u(x(r,\theta), y(r,\theta)) + iv(x(r,\theta), y(r,\theta))$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

Similar applications of the chain rule yield

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta$$
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y} r\cos\theta$$

Replace the partial derivatives on the right sides of the equations for $\frac{\partial u}{\partial r}$ and $\frac{\partial v}{\partial r}$ by their Cauchy-Riemann counterparts to obtain:

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

7. Let $h(z) = f(z) - g(z)$. Then h is analytic in D and $h'(z) = 0$ so h is a constant function.

$$h(z) = c = f(z) - g(z) \Longrightarrow f(z) = g(z) + c$$

8. $u(x,y) = c$ in $D \Longrightarrow \frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$. Hence $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$ so f is constant in D.

9. By contradiction. If f is analytic in a domain D then $v(x,y) = 0$ (a constant) $\Rightarrow f$ is constant (by condition 8) $\Rightarrow u$ is constant. (However, there is no open set in which $u(x,y) = |z^2 - z|$ is constant).

10. $\operatorname{Im} f(z) = 0$ in $D \Longrightarrow \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \Longrightarrow \frac{\partial u}{\partial x} = 0$ $\Longrightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 \Longrightarrow f$ is constant in D.

11. $\operatorname{Re} f(z) = \frac{1}{2}[f(z) + \overline{f(z)}]$ is real valued and analytic if both f and $\overline{f}$ are analytic. Hence $\operatorname{Re} f(z)$ is constant by Exercise 10. It follows that $f(z)$ is constant by Exercise 8.

12. $|f(z)|$ constant in $D \Longrightarrow |f(z)|^2 = u^2 + v^2$ is constant in D. If $u = 0$ or $v = 0$ in D, then f is constant by Exercises 8 and 10. Otherwise,

$$\begin{aligned} \frac{\partial |f|^2}{\partial x} &= 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \\ \frac{\partial |f|^2}{\partial y} &= 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = -2u\frac{\partial v}{\partial x} + 2v\frac{\partial u}{\partial x} = 0 \end{aligned}$$

$$\begin{aligned} &\Longrightarrow \frac{1}{2}v\frac{\partial |f|^2}{\partial x} - \frac{1}{2}u\frac{\partial |f|^2}{\partial y} = 0 = (u^2 + v^2)\frac{\partial v}{\partial x} \\ &\Longrightarrow \frac{\partial v}{\partial x} = 0 \Longrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0 \\ &\Longrightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 \\ &\Longrightarrow f \text{ is constant in } D. \end{aligned}$$

13. $|f(z)|$ is analytic and real-valued, so the result follows from Exercises 10 and 12.

14. If the line is vertical then $\operatorname{Re} f(z)$ is constant and this reduces to Problem 8. If the line is not vertical, then $v(x, y) = mu(x, y) + b$, and

$$\begin{aligned} \frac{\partial v}{\partial x} &= m\frac{\partial u}{\partial x} = m\frac{\partial v}{\partial y}, \\ \frac{\partial v}{\partial y} &= m\frac{\partial u}{\partial y} = -m\frac{\partial v}{\partial x} = -m^2\frac{\partial v}{\partial y}. \end{aligned}$$

It follows that

$$\frac{\partial v}{\partial y} = 0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \text{ and } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0.$$

Hence $f(z)$ is constant.

15. $$\begin{aligned} J(x_0, y_0) &= \left.\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} \\ &= \left[\frac{\partial u}{\partial x}(x_0, y_0)\right]^2 + \left[\frac{\partial v}{\partial x}(x_0, y_0)\right]^2 \\ &= |f'(z_0)|^2 \quad \text{(using Equation (1))} \end{aligned}$$

16. a. $$\frac{\partial \tilde{f}}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{1}{2i}$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
$$\frac{\partial \tilde{f}}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(\frac{-1}{2i}\right)$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

b. $$\frac{\partial \tilde{f}}{\partial \eta} = 0 \Leftrightarrow 0 = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \text{ and } 0 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\Leftrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

EXERCISES 2.5: Harmonic Functions

1. a. $u(x,y) = x^2 - y^2 + 2x + 1,\ \dfrac{\partial^2 u}{\partial x^2} = 2 = -\dfrac{\partial^2 u}{\partial y^2} \Longrightarrow \Delta u = 0$

$v(x,y) = 2xy + 2y,\ \dfrac{\partial^2 v}{\partial x^2} = 0 = -\dfrac{\partial^2 v}{\partial y^2} \Longrightarrow \Delta v = 0$

b. $u(x,y) = \dfrac{x}{x^2+y^2},\ \dfrac{\partial^2 u}{\partial x^2} = \dfrac{2x(x^2-3y^2)}{(x^2+y^2)^3} = -\dfrac{\partial^2 u}{\partial y^2} \Longrightarrow \Delta u = 0$

$v(x,y) = -\dfrac{y}{x^2+y^2},\ \dfrac{\partial^2 v}{\partial x^2} = \dfrac{-2y(3x^2-y^2)}{(x^2+y^2)^3} = -\dfrac{\partial^2 v}{\partial y^2} \Longrightarrow \Delta v = 0$

c. $u(x,y) = e^x \cos y,\ \dfrac{\partial^2 u}{\partial x^2} = e^x \cos y = -\dfrac{\partial^2 u}{\partial y^2} \Longrightarrow \Delta u = 0$

$v(x,y) = e^x \sin y,\ \dfrac{\partial^2 v}{\partial x^2} = e^x \sin y = -\dfrac{\partial^2 v}{\partial y^2} \Longrightarrow \Delta v = 0$

2. $h(x,y) = ax^2 + bxy - ay^2$

3. a. $u = \mathrm{Re}(-iz)$, $v = -x + a$, where a is a constant

b. $u = \mathrm{Re}(-ie^z)$, $v = -e^x \cos y + a$

c. $u = \mathrm{Re}\left(\frac{-i}{2}z^2 - iz - z\right)$, $v = -\frac{1}{2}(x^2 - y^2) - (x + y) + a$

d. It is straightforward to verify that $\Delta u = 0$.

$\frac{\partial u}{\partial x} = \cos x \cosh y = \frac{\partial v}{\partial y}$

$\Rightarrow v(x,y) = \int \cos x \cosh y dy = \cos x \sinh y + \psi(x)$

$\frac{\partial u}{\partial y} = \sin x \sinh y = -\frac{\partial v}{\partial x} = \sin x \sinh y + \psi'(x) \Rightarrow \psi(x) = a$

Thus, $v(x,y) = \cos x \sinh y + a$.

e. It is straightforward to verify that $\Delta u = 0$.

$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y} \Rightarrow$

$v(x,y) = \int \frac{x}{x^2 + y^2} dy = \tan^{-1}\left(\frac{y}{x}\right) + \psi(x)$

$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2} - \psi'(x) \Rightarrow \psi(x) = a$

Thus, $v(x,y) = \tan^{-1}\left(\frac{y}{x}\right) + a$.

f. $u = \mathrm{Re}\left(-ie^{z^2}\right)$, $v = -e^{x^2 - y^2} \cos(2xy) + a$.

4. Suppose v and w are both harmonic conjugates of u, and consider $\phi(x,y) = w(x,y) - v(x,y)$. Then (using the Cauchy-Riemann equations for v and w),

$$\frac{\partial \phi}{\partial x} = \frac{\partial w}{\partial x} - \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} - \left(-\frac{\partial u}{\partial y}\right) = 0$$

and similarly $\frac{\partial \phi}{\partial y} = 0$. Hence $\phi(x,y) = a$, from which it follows that

$$w(x,y) = v(x,y) + a.$$

5. If $f(z) = u(x,y) + iv(x,y)$ is analytic then $-if(z) = v(x,y) - iu(x,y)$ is analytic. Thus $-u$ is a harmonic conjugate of v.

6. Since $f(z) = u + iv$ is analytic, $\frac{1}{2}[f(z)]^2 = \frac{1}{2}(u^2 - v^2) + iuv$ is analytic. Thus $uv = \mathrm{Im}\frac{1}{2}[f(z)]^2$is harmonic.

7. $\phi(x, y) = x + 1$

8.
   a. Yes, because $\Delta(u + v) = \Delta u + \Delta v = 0$.
   b. No. Take $u = x, v = x^2 - y^2$ as an example.
   c. Yes, because $\Delta(u_x) = u_{xxx} + u_{xyy} = u_{xxx} + u_{yyx}$

$$= \frac{\partial}{\partial x}(\Delta u) = \frac{\partial}{\partial x}(0) = 0.$$

9. $\phi(x, y) = xy - 1$ $\left(\text{this is } \mathrm{Im}\left(\frac{1}{2}z^2 - i\right)\right)$

10. Let $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}
\frac{\partial\phi}{\partial r} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial\phi}{\partial x}\cos\theta + \frac{\partial\phi}{\partial y}\sin\theta \\
\frac{\partial^2\phi}{\partial r^2} &= \frac{\partial^2\phi}{\partial x^2}\frac{\partial x}{\partial r}\cos\theta + \frac{\partial^2\phi}{\partial y\partial x}\frac{\partial y}{\partial r}\cos\theta \\
&\quad + \frac{\partial^2\phi}{\partial x\partial y}\frac{\partial x}{\partial r}\sin\theta + \frac{\partial^2\phi}{\partial y^2}\frac{\partial y}{\partial r}\sin\theta \\
&= \frac{\partial^2\phi}{\partial x^2}\cos^2\theta + \frac{\partial^2\phi}{\partial y\partial x}2\sin\theta\cos\theta + \frac{\partial^2\phi}{\partial y^2}\sin^2\theta \\
\frac{\partial\phi}{\partial\theta} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\theta} = \frac{\partial\phi}{\partial x}(-r\sin\theta) + \frac{\partial\phi}{\partial y}r\cos\theta \\
\frac{\partial^2\phi}{\partial\theta^2} &= \frac{\partial^2\phi}{\partial x^2}\frac{\partial x}{\partial\theta}(-r\sin\theta) + \frac{\partial^2\phi}{\partial y\partial x}\frac{\partial y}{\partial\theta}(-r\sin\theta) + \frac{\partial\phi}{\partial x}(-r\cos\theta) \\
&\quad + \frac{\partial^2\phi}{\partial x\partial y}\frac{\partial x}{\partial\theta}(r\cos\theta) + \frac{\partial^2\phi}{\partial y^2}\frac{\partial y}{\partial\theta}(r\cos\theta) + \frac{\partial\phi}{\partial y}(-r\sin\theta) \\
&= \frac{\partial^2\phi}{\partial x^2}r^2\sin^2\theta + \frac{\partial^2\phi}{\partial y\partial x}(-2r^2\sin\theta\cos\theta) + \frac{\partial^2\phi}{\partial y^2}r^2\cos^2\theta \\
&\quad + \frac{\partial\phi}{\partial x}(-r\cos\theta) + \frac{\partial\phi}{\partial y}(-r\sin\theta).
\end{aligned}$$

Combining these partial derivatives, one gets

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

11. $\operatorname{Im} f(z) = y - \dfrac{y}{x^2+y^2} = 0 \Longrightarrow yx^2 + y^3 - y = y(x^2+y^2-1) = 0.$
The points satisfying $x^2+y^2-1=0$ lie on the circle $|z|=1$. The points (other than $z=0$) satisfying $y=0$ lie on the real axis.

12. $f(z) = z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta) \Longrightarrow$
$\operatorname{Re} f(z) = r^n \cos n\theta$ and $\operatorname{Im} f(z) = r^n \sin n\theta$ are harmonic since f is analytic.

13. $\phi(x,y) = \operatorname{Im} z^4 = r^4 \sin 4\theta = -4xy^3 + 4x^3y$

14. Let $\phi(x,y) = \ln|f(z)| = \dfrac{1}{2}\ln(u^2+v^2)$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = \frac{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}}{u^2+v^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{(v^2-u^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2\right] - 4uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}{(u^2+v^2)^2} + \frac{u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 v}{\partial x^2}}{u^2+v^2}$$

A similar calculation yields $\dfrac{\partial^2 \phi}{\partial y^2}$. By applying Laplace's equation and the Cauchy-Riemann equations of u and v to $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}$, the sum simplifies to reveal that $\Delta\phi = 0$.

15. Consider $\varphi(z) = \operatorname{Re}(Az^n + Bz^{-n}) + C$ which is harmonic for $1 \le |z| \le 2$. Consider the polar form for z. $z = re^{i\theta}$ and select n=3 to agree with the cosine argument. $\varphi(re^{i\theta}) = Ar^3\operatorname{Re}(e^{i3\theta}) + Br^{-3}\operatorname{Re}(e^{-i3\theta}) + C$.
$\varphi(re^{i\theta}) = Ar^3\cos 3\theta + Br^{-3}\cos 3\theta + C = (Ar^3 + Br^{-3})\cos 3\theta + C$.

$r=1 \Rightarrow (A+B)\cos 3\theta + C = 0 \Rightarrow A + B = 0,\ C = 0.$
$r=2 \Rightarrow (A*8 + B/8)\cos 3\theta = 5\cos 3\theta.\ A = 40/63,\ B = -40/63$
$\varphi(re^{i\theta}) = (40/63)(r^3 - r^{-3})\cos 3\theta = (40/63)\operatorname{Re}(z^3 - z^{-3}).$

16. $\phi(x,y) = \dfrac{1}{\ln 3}\ln|z| - 1$ or $\phi(x,y) = \ln\left|\dfrac{z}{3}\right|$ are two possibilities.

17. a. $\phi(x,y) = \text{Re}(z^2+5z+1) = x^2 - y^2 + 5x + 1$

b. $\phi(x,y) = 2\text{Re}\left(\dfrac{z^2}{z+2i}\right) = \dfrac{2x(x^2+4y+y^2)}{x^2+y^2+4y+4}$

18. Let $u = \phi_x$, $v = -\phi_y$. Then

$$\frac{\partial u}{\partial x} = \phi_{xx} = -\phi_{yy} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = \phi_{xy} = -\frac{\partial v}{\partial x}$$

19. $\cos^2\theta = (½)\cos 2\theta + ½ = \varphi(z) = A\text{Re}(r^{-2}e^{-i2\theta})n + B = Ar^{-2}\cos 2\theta + B$. In the limit as $r \to \infty$ $\varphi(z) = ½ \Rightarrow B = ½$. On the circle $|z|=1$, $r = 1. \Rightarrow A = ½$. $\varphi(z) = (½)r^{-2}\cos 2\theta + ½ = \text{Re}\,[1/(2z^2)] + ½$.

20. In order that $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$, let $v(x,y) = \displaystyle\int_0^y \frac{\partial u}{\partial x}(x,\eta)d\eta + \psi(x)$. Then

$$\begin{aligned}\frac{\partial v}{\partial x} &= \int_0^y \frac{\partial^2 u}{\partial x^2}(x,\eta)d\eta + \psi'(x)\\ &= -\int_0^y \frac{\partial^2 u}{\partial y^2}(x,\eta)d\eta + \psi'(x) \quad \text{(because } u \text{ is harmonic)}\\ &= -\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x,0) + \psi'(x).\end{aligned}$$

In order that $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$, it must be true that $\psi'(x) = -\dfrac{\partial u}{\partial y}(x,0)$. Thus,

$$\psi(x) = -\int_0^x \frac{\partial u}{\partial y}(\zeta,0)d\zeta + a$$

and

$$v(x,y) = \int_0^y \frac{\partial u}{\partial x}(x,\eta)d\eta - \int_0^x \frac{\partial u}{\partial y}(\zeta,0)d\zeta + a.$$

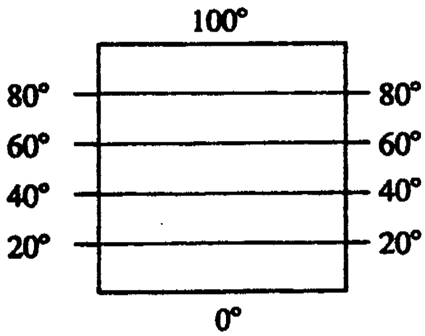
21. It is easily verified that $u = \ln|z|$ satisfies Laplace's equation on $\mathbf{C}\setminus\{0\}$ and that $u+iv = \ln|z| + i\text{Arg}(z)$ satisfies the Cauchy-Riemann equations on the domain $D = \mathbf{C}\setminus$ {nonpositive real axis}, so that

Arg (z) is a harmonic conjugate of u on D. By Problem 4, any harmonic conjugate of u has to be of the form Arg$(z)+a$ in D. It is impossible to have a harmonic conjugate of this form that is continuous on $\mathbf{C}\setminus\{0\}$.

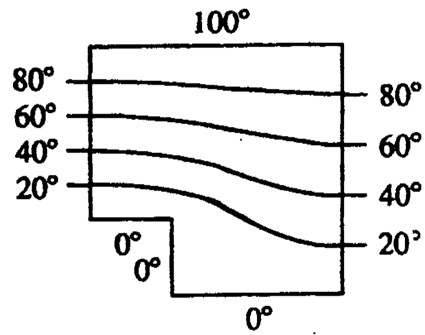
22. $$\begin{aligned}\frac{\partial u}{\partial x} &= \phi_{xx}\phi_y + \phi_x\phi_{yx} + \psi_{xx}\psi_y + \psi_x\psi_{yx}\\ &= -\phi_{yy}\phi_y + \phi_x\phi_{yx} - \psi_{yy}\psi_y + \psi_x\psi_{yx} = \frac{\partial v}{\partial y}\end{aligned}$$

EXERCISES 2.6: Steady-State Temperature as a Harmonic Function.

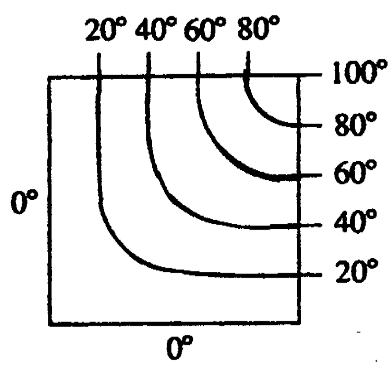
1. a.



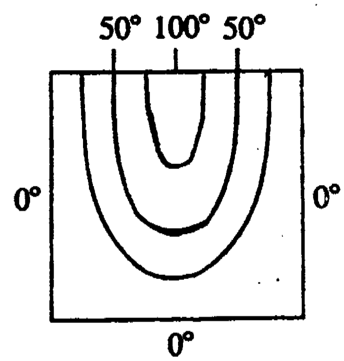
b.



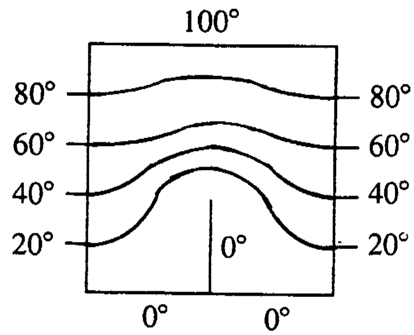
c.



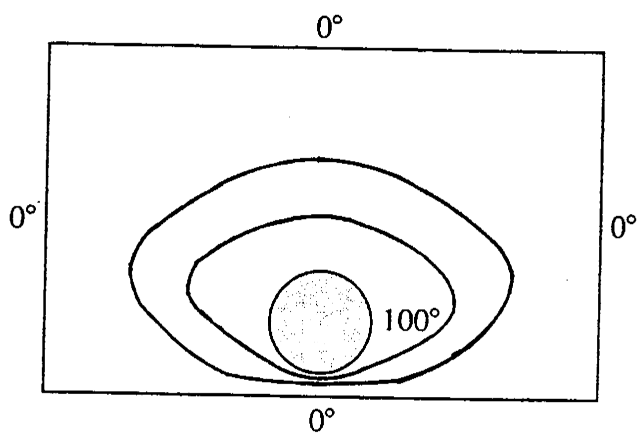
d.



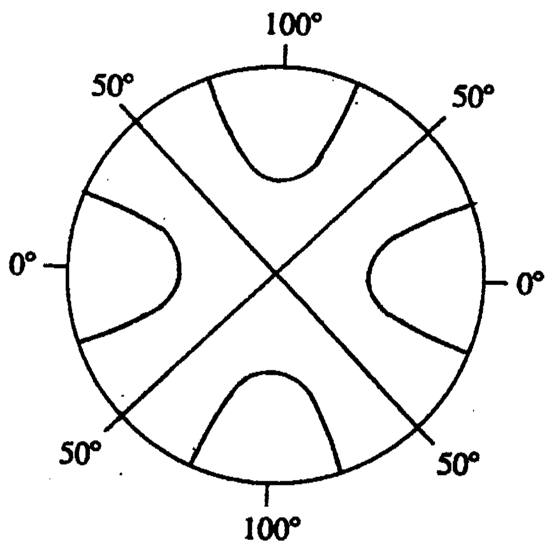
e.



2. This does not violate the maximum principle.



3. This does not violate the maximum principle.



Exercises 2.7

1. $f(z) = z^2 + c$ where c is a real constant.
$\zeta_1 = (1 + \sqrt{(1-4c)})/2$, $\zeta_2 = (1 - \sqrt{(1-4c)})/2$
Only ζ_2 is an attractor for $-3/4 < c < 1/4$.

2. $f(\varsigma) = \varsigma$ and $f'(\varsigma) > 1$ Therefore we can pick a real number ρ between 1 and $|f'(\varsigma)|$ such that $|f(z) - \varsigma| = \rho|z-\varsigma|$ for all z in a sufficiently small disk around ς. If any point z_0 in this disk is the seed for an orbit $z_1 = f(z_0)$, $z_2 = f(z_1)$, ... $z_n = f(z_{n-1})$, then we have $|z_n-\varsigma| \geq \rho|z_{n-1}-\varsigma| \geq ... \geq \rho^n|z_{n-1}-\varsigma|$. Because $\rho>1$, the point z_n moves away from ς until the magnitude of the derivative becomes 1 or less. The orbit is out of the disk.

3. (a) Fixed points are $\zeta_1 = i$, $\zeta_2 = -i$. Both are repellors.
(b) Fixed points are $\zeta_1 = 1/2$, $\zeta_2 = -1/2$, $\zeta_3 = -1$. Fixed points ζ_1 and ζ_3 are repellors, but fixed point ζ_2 is an attractor.

4. $z_0 = e^{i2\pi\alpha}$ with α an irrational real number. $z_n = e^{i2\pi\alpha 2\wedge n}$. Because $|z_n| = 1$, the trajectory will follow the unit circle. If iterations p and q coincide, $2\pi\alpha 2^p - 2\pi\alpha 2^q = 2\pi\alpha(2^p - 2^q) = 2\pi k$ for some integer k. But because $(2^p - 2^q)$ is an integer that can be represented by m, the equation $2\pi\alpha m=2\pi k$ is satisfied only if $k=\alpha m$ or $\alpha = k/m$. Because α is irrational it cannot be represented by a rational number and no iterations repeat.

5. Fixed points are $\zeta_1 = -1/2 + i\sqrt{5}/2$ (an attractor) and $\zeta_2 = -1/2 - i\sqrt{5}/2$ (a repellor).

6. $f(z) = z^2$. The seed is z_0. $z_1 = z_0^2$, $z_2 = z_0^4$, ... $z_n = z_0\wedge(2^n)$. To have an n cycle $z_n = z_0 = z_0\wedge(2^n)$. Or $z_n/z_0 = z_0\wedge(2^n-1) = 1 = e^{i2\pi}$. Solving gives
$z_0 = e\wedge(i2\pi/(2^n-1))$.

7. The cycle is 4. $2^4(2\pi/p) = 2\pi \bmod p \Rightarrow 2^4 = 1 \bmod p$. p=3,5,15. 3 will give repeated cycles of length 2. 5 and 15 will give the desired cycles of length 4.

8. Student Matlab: n=100;c=.253; zo=0;y(1)=zo;
for k=1:n-1,y(k+1)=y(k)^2+c;end
plot(y)

9. If $|\alpha| \leq 1$ the whole complex plane is the filled Julia set. If $|\alpha| \geq 1$ the origin is the filled Julia set.

10. $f(z) = z - F(z)/F'(z)$. $f(\zeta) = \zeta - F(\zeta)/F'(\zeta) = \zeta \Rightarrow F(\zeta)/F'(\zeta) = 0 \Rightarrow F(\zeta)=0$ with the possible exception of the points where $F'(\zeta) = 0$.
$f'(z) = 1 - F'(z)/F'(z) + F(z)F''(z)/(F'(z))^2 = F(z)F''(z)/(F'(z))^2$
$f'(\zeta) = F(\zeta)F''(\zeta)/(F'(\zeta))^2 = 0$ where $F'(\zeta) \neq 0$ and every zero of F(z) is an attractor as long as $F'(\zeta) \neq 0$.

Chapter 3

Exercises 3.1

1. $p(z) = 2(z+1)^2(z^2 + 9)$

2. Multiply to get $a_nz^n + a_{n-1}z^{n-1} + \ldots = a_nz^{d1+d2+\ldots+dr} - a_n(d_1z_1 + d_2z_2 + \ldots + d_rz_r) + \ldots\ a_n(-1)^nz_1^{d1}z_2^{d2}\ldots z_r^{dr}$ to get

   (a) $n = d_1+d_2+\ldots+d_r$

   (b) $a_{n-1} = -a_n\,(d_1z_1 + d_2z_2 + \ldots + d_rz_r)$

   (c) $a_0 = a_n(-1)^nz_1^{d1}z_2^{d2}\ \ldots\ z_r^{dr})$

3. (a) $z^3(z + 1+i)^2$

   (b) $(z-2)(z+2)(z+2i)(z-2i)$

   (c) $(z-\omega_7)\,(z-\omega_7^2)\ldots\,(z-\omega_7^6)$, where $\omega_7 = e^{i2\pi/7}$.

4. $a_0 = a_n(-1)^nz_1z_2\ldots z_r,\ a_n = 1 \Rightarrow |a_0| = |z_1||z_2|\ldots|z_r|>1 \Rightarrow |z_j| > 1$
   for some j, $1<j<r$ or at least one zero is outside the unit circle.

5. (a) $p(z) = 42 + 83(z-2) + 80(z-2)^2 + 40(z-2)^3 + 10(z-2)^4 + (z-2)^5$

   (b) $p(z) = \sum_{k=0}^{10} \binom{10}{k} 2^{10-k}(z-2)^k$ (binomial expansion of $[(z-2)+2]^{10}$

   (c) $p(z) = (z-2)^3 + (z-2)^4$

6. $p(z) = a_nz^n + a_{n-1}z^{n-1} + \ldots + a_0$ and $p^*(z) = \underline{a_n}z^n + \underline{a_{n-1}}z + \ldots + \underline{a_0}\,z^n$

   (a) $z^n\underline{p(1/z')} = z^n(\underline{a_n}/z^n + \underline{a_{n-1}}/z^{n-1}\ldots\underline{a_0}) = p^*(z)$

   (b) $p(z)$ has a zero at $z_0 \neq 0 \Rightarrow p^*(z_1) = z_1^n\underline{p(1/z_1')} = 0$ when $1/\underline{z_1} = z_0$ or when $z_1 = 1/\underline{z_0}$

   (c) $|z| = 1 \Rightarrow z = e^{i\theta}$ for $0<\theta<2\pi$

   $p^*(z) = \underline{a_n} + \underline{a_{n-1}}e^{i(n-1)\theta} + \ldots + \underline{a_0}e^{in\theta} = (\underline{a_n}e^{-in\theta} + \underline{a_{n-1}}e^{-i(n-1)\theta} + \ldots + \underline{a_0})e^{in\theta}$

   $|p^*(z)| = |(\underline{a_n}e^{-in\theta} + \underline{a_{n-1}}e^{-i(n-1)\theta} + \ldots + \underline{a_0})||e^{in\theta}|$

   $= |(\underline{a_n}e^{-in\theta} + \underline{a_{n-1}}e^{-i(n-1)\theta} + \ldots + \underline{a_0})| = |\underline{p(z)}| = |p(z)|$

7. If $p(z)$ has a zero of order m at z_0, then $p(z) = (z-z_0)^mf(z)$ and $f(z_0) \neq 0$
   $p'(z) = m(z-z_0)^{m-1}f(z) + (z-z_0)^mf'(z) = (z-z_0)^{m-1}(m + (z-z_0)f'(z))$ has a zero of order m-1 at z_0.

8. $p(z) = (z-z_0)^mf(z),\ q(z) = (z-z_0)^kg(z)$ and $f(z_0) \neq 0,\ g(z_0) \neq 0$.
   $p(z)q(z) = (z-z_0)^{m+k}f(z)g(z)$ and $f(z_0)g(z_0) \neq 0$
   $p(z)q(z)$ has a zero of order m+k at z_0.

9. $p(z) = (z-z_0)^dq(z)$ and $q(z)$ is a polynomial with no zeros for all z sufficiently near z_0. The function $q(z)$ is continuous in the z-plane and will have a maximum and minimum in a region near z_0. Let $c_1 = \min |q(z)|$ and $c_2 = \max |q(z)|$ in the region sufficiently near z_0. In this way we have $c_1|(z-z_0)|^d \leq |p(z)| = |(z-z_0)|^d|q(z)| \leq c_2|(z-z_0)|^d$.

10. $|p(z)| = |a_nz^n + a_{n-1}z^{n-1} + \ldots + a_0|$ ~le $|a_nz^n| + |a_{n-1}z^{n-1} + \ldots + a_0|$ and
    $|a_nz^n| - |a_{n-1}z^{n-1} + \ldots + a_0| \leq |p(z)|$ If we take $|z|$ large enough
    $|a_nz^n| - |a_{n-1}z^{n-1} + \ldots + a_0| > 0$ for $|z| > N$. Let
    $c_1 = |a_n| - |a_{n-1}N^{n-1} + \ldots + a_0|/N^n$ and
    $c_2 = |a_n| + |a_{n-1}N^{n-1} + \ldots + a_0|/N^n$ with the result
    $c_1|z|^n \leq |p(z)| \leq c_2|z|^n$ for all z such that $|z| > N$.

11. (a) Poles at 0 of order 3 and at $-(1+\sqrt{2})i$ and $-(1-\sqrt{2})i$ of order 1.

    (b) Poles at 2 of order 1 and at 3 of order 2.

(c) Pole at −2 of order 6
(d) Pole at −2 of order 1.

12. $R_{m,n}(z) = (b_m z^m + b_{m-1}z^{m-1} + \dots + b_0)/(a_n z^n + a_{n-1}z^{n-1} + \dots a_0)$
$= ((b_m/a_n)z_m + (b_{m-1}/a_n)zm^{-1} + \dots + b_0/a_n)/(z^n + (a_{n-1}/a_n)\, z_{n-1} + \dots a_0/a_n)$ is determined if the m+n+1 constants b_m/a_n, b_{m-1}/a_n, …, b_0/a_n, a_{n-1}/a_n, …, a_0/a_n are found. We would need n+m+1 separate points to determine the n+m+1 constants. Thus if $R_{m,n} = r_{m,n}$ at n + m + 1 discrete points in the z-plane, they would have the same coefficients and $R_{m,n} = r_{m,n}$ for all z.

13. (a) $((3+i)/2)/z - (3+i)/(z+1) + ((3+i)/2)/(z+2)$
(b) $i/z + (i/2)/(z+i) - (3i/2)/(z-i)$
(c) $(1/6 + i\sqrt{3}/6)/(z+1/2+i\sqrt{3}/2)^2 - i\sqrt{3}/9/(z+1/2+i\sqrt{3}/2) +$
$(1/6 - i\sqrt{3}/6)/(z+1/2-i\sqrt{3}/2)^2 + i\sqrt{3}/9)/(z+1/2-i\sqrt{3}/2)$
(d) $(5/2)z^2 - (15/4)z + 47/8 + (33/16)/(z+1/2) - 9/(z+1)$

14. $R(z) = r(z)/(z-z_0)^m \Rightarrow R'(z) = r'(z)/(z-z_0)^m - mr(z)/(z-z_0)^{m+1}$
$= (r'(z)(z-z_0) - mr(z))/(z-z_0)^{m+1}$ and $r(z_0)$ is finite.
Thus R'(z) has a pole at z_0 of order m+1.

15. (a) Res(i) = (3 - 2i)/4
(b) Res(-1) = -1/512
(c) Res(0) = 6
(d) Res(3i) = 0
(e) Res(0) = 3

16. $R_{m,n} = (b_m z^m + b_{m-1}z^{m-1} + \dots b_0)/(a_n z^n + a_{n-1}z^{n-1} + \dots a_0)$
$|R_{m,n}| \le (|b_m z^m| + |b_{m-1}z^{m-1} + \dots b_0|)/(|a_n z^n| - |a_{n-1}z^{n-1} + \dots a_0|)$
$|R_{m,n}| \le (|b_m z^m|/|a_n z^n| + |b_{m-1}z^{m-1} + \dots b_0|/|a_n z^n|)/(1 - |a_{n-1}z^{n-1} + \dots a_0|/|a_n z^n|)$
Let M be chosen so that $1 - |a_{n-1}z^{n-1} + \dots a_0|/|a_n z^n| > 0$ for $|z| > M$. Select c_2
$c_2 > (|b_m/a_n|M^{m-n} + |b_{m-1}M^{m-1} + \dots b_0|/|a_n M^n|)/(1 - |a_{n-1}M^{n-1} + \dots a_0|/|a_n M^n|)$
$(|b_m z^m|/|a_n z^n| - |b_{m-1}z^{m-1} + \dots b_0|/|a_n z^n|)/(1 + |a_{n-1}z^{n-1} + \dots a_0|/|a_n z^n|) \ge |R_{m,n}|$
Let N be chosen so that $(|b_m z^m|/|a_n z^n| - |b_{m-1}z^{m-1} + \dots b_0|/|a_n z^n|) > 0$ for $|z| >$ N. Select c_1
$c_1 < (|b_m/a_n|N^{m-n} - |b_{m-1}N^{m-1} + \dots b_0|/|a_n N^n|)/(1 + |a_{n-1}N^{n-1} + \dots a_0|/|a_n N^n|)$
$c_1|z|^{m-n} < |R_{m,n}| < c_2|z|^{m-n}$

17. $p(z) = a_n(z-z_1)^{d1}(z-z_2)^{d2} \dots (z-z_r)^{dr}$
$p'(z) = a_n d_1(z-z_1)^{d1-1}(z-z_2)^{d2} \dots (z-z_r)^{dr} + a_n d_2 z-z_1)^{d1}(z-z_2)^{d2-1} \dots (z-z_r)^{dr} +$
$\dots + a_n d_r(z-z_1)^{d1}(z-z_2)^{d2} \dots (z-z_r)^{dr-1}$
$p'(z)/p(z) = d_1/(z-z_1) + d_2/(z-z_2) + \dots + d_r/(z-z_r)$

18. $\overline{R(z)} = d_1(z-z_1)/|z-z_1|^2 + d_2(z-z_2)/|z-z_2|^2 + \dots + d_r(z-z_r)/|z-z_r|^2$ Because each d_k is real and positive, each component of a term $d_k(z-z_k)/|z-z_k|^2$ is directed down and so will have a negative imaginary part and cannot add to zero. R(z) has no zeros with negative imaginary parts.
Alternatively, $R(z) = \sum_{k=1}^{r} d_r/(z-z_k)$ take $(z-z_k) = r_k e^{-i\theta k}$, $0 \le \theta_k < \pi$. Then $R(z) = \sum_{k=1}^{r} d_r e^{i\theta k}/r_k$ and R(z) will have only positive imaginary parts when Im z < 0.

19. If all the zeros of p(z) lie in the upper half plane, then R(z) = p'(z)/p(z) has no zeros in the lower half plane (see Problems 17 and 18). Consequently, p'(z) = p(z)R(z) has zeros only in the upper half plane.

20. If all the zeros of a polynomial lie on one side of any line, the zeros of $p'(z)/p(z) = R(z)$ would have no zeros on or across the line. The same is true for $p'(z) = R(z)p(z)$.

21. The zeros of a polynomial $p(z)$ can be sheltered by a convex hull that is obtained as the common "territory" that is sheltered by every straight line that shelters every zero of $p(z)$. This convex hull also shelters every zero of $p'(z)$ by Problem 20.

22. If $p(z)$ is of degree n and $q(z)$ is of degree m, then the degree of $p(z)q(z)$ is of degree m+n. This works when the degree is zero or the polynomial is a constant. To include the zero polynomial the degree of the product of the zero polynomial and any other polynomial must add to the degree of the zero polynomial. Choosing the degree of the zero polynomial as $-\infty$ finesses this.

EXERCISES 3.2: The Exponential, Trigonometric, and Hyperbolic Functions

1. $e^z = (1+i)/\sqrt{2}$
$\iff e^x \cos y + ie^x \sin y = (1+i)/\sqrt{2}$
$\iff x = 0 \text{ and } y = \frac{\pi}{4} + 2k\pi,\ k = 0, \pm1, \pm2, \ldots$
$\iff z = \left(\frac{\pi}{4} + 2k\pi\right) i,\ k = 0, \pm1, \pm2, \ldots$

2. $f^{(3)}(0) = e^z - 1\big|_{z=0} = 0.$

3. $\dfrac{1-e^{101z}}{1-e^z}$ for $z \neq 2k\pi i$, 101 for $z = 2k\pi i$

4. $e^{\ln r + i\theta} = e^{\ln r} e^{i\theta} = re^{i\theta} = \omega$

5. a. $\dfrac{e^2}{\sqrt{2}} + i\dfrac{e^2}{\sqrt{2}}$

   b. ie^2

   c. $i \sinh 2$

   d. $\cos 1 \cosh 1 + i \sin 1 \sinh 1$

   e. $-\sinh 1$

   f. 0

6. Identity 8: $\sin^2 z + \cos^2 z = \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2$

$$= \frac{e^{2iz} - 2 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} = 1$$

Identity 9: $\sin z_1 \cos z_2 + \sin z_2 \cos z_1$

$$= \left(\frac{e^{iz_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right) + \left(\frac{e^{iz_2} - e^{-iz_2}}{2i}\right)\left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)$$
$$= \frac{1}{4i}\left[2e^{i(z_1+z_2)} - 2e^{-i(z_1+z_2)}\right] = \sin(z_1 + z_2)$$

$$\begin{aligned} \sin(z_1 - z_2) &= \sin[z_1 + (-z_2)] \\ &= \sin z_1 \cos(-z_2) + \sin(-z_2)\cos z_1 \\ &= \sin z_1 \cos z_2 - \sin z_2 \cos z_1 \end{aligned}$$

7. $\cos z + i \sin z = \dfrac{e^{iz} + e^{-iz}}{2} + i\left(\dfrac{e^{iz} - e^{-iz}}{2i}\right) = \dfrac{1}{2}(2e^{iz}) = e^{iz}$

8. $\dfrac{d}{dz}(\sinh z) = \dfrac{d}{dz}\left(\dfrac{e^z - e^{-z}}{2}\right) = \dfrac{e^z + e^{-z}}{2} = \cosh z$

$\dfrac{d}{dz}(\cosh z) = \dfrac{d}{dz}\left(\dfrac{e^z + e^{-z}}{2}\right) = \dfrac{e^z - e^{-z}}{2} = \sinh z$

9. a. $2\pi z e^{\pi z^2}$

b. $-2\sin(2z) - i\dfrac{\cos(1/z)}{z^2}$

c. $2e^{\sin 2z}\cos 2z$

d. $3\tan^2 z \sec^2 z$

e. $2\cosh z(\sinh z + 1)$

f. $\operatorname{sech}^2 z$

10. Sums and composites of entire functions are entire.

11. The function $\dfrac{\cos z}{e^z}$ is entire (e^z is never zero) so its real and imaginary parts are harmonic everywhere.

12. a. $\cosh^2 z - \sinh^2 z = \cos^2 iz - (-i \sin iz)^2$
$= \cos^2 iz + \sin^2 iz = 1$

b. $\sinh(z_1+z_2) = -i\sin(iz_1+iz_2)$
$= -i(\sin iz_1 \cos iz_2 + \sin iz_2 \cos iz_1)$
$= \sinh z_1 \cosh z_2 + \sinh z_2 \cosh z_1$

c. $\cosh(z_1+z_2) = \cos(iz_1+iz_2)$
$= \cos iz_1 \cos iz_2 + (-i)^2 \sin iz_1 \sin iz_2$
$= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

13. a. $\sin(x+iy) = \sin x \cos iy + \sin iy \cos x$
$= \sin x \cosh y + i \sinh y \cos x$

b. $\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$
$= \cos x \cosh y - i \sin x \sinh y$

14. a. $e^{i(z+2\pi)} = e^{iz}e^{2\pi i} = e^{iz}$

b. $\tan(z+\pi) = \dfrac{\sin(z+\pi)}{\cos(z+\pi)} = \dfrac{-\sin z}{-\cos z} = \tan z$

c. $\sinh(z+2\pi i) = -i\sin(iz-2\pi) = -i\sin(iz) = \sinh z$
$\cosh(z+2\pi i) = \cos(iz-2\pi) = \cos iz = \cosh z$

d. $\tanh(z+\pi i) = \dfrac{\sinh(z+\pi i)}{\cosh(z+\pi i)}$
$= \dfrac{-i\sin(iz-\pi)}{\cos(iz-\pi)} = \dfrac{i\sin iz}{-\cos iz}$
$= \dfrac{\sinh z}{\cosh z} = \tanh z$

15. $\cos z = \dfrac{e^{iz}+e^{-iz}}{2} = 0 \Leftrightarrow e^{iz}+e^{-iz} = 0$
$\Leftrightarrow e^{2iz} = -1 \Leftrightarrow 2iz = \pi i + 2k\pi i$
$\Leftrightarrow z = \pi/2 + k\pi, \quad k = 0, \pm1, \pm2, \ldots$

16. $\sin z_2 - \sin z_1 = (e^{iz_2} - e^{-iz_2} - e^{iz_1} + e^{-iz_1})/2i$

$= 2[e^{i(z_2+z_1)/2} + e^{-i(z_2+z_1)/2})/2][e^{i(z_2-z_1)/2} - e^{-i(z_2-z_1)/2})/2i]$

$= 2\cos((z_2+z_1)/2)\sin((z_2-z_1)/2)$

$= 0$ if $(z_2+z_1)/2 = \pi/2 + k\pi$ or $(z_2-z_1)/2 = k\pi$,

or if $z_2 = -z_1 + (2k+1)\pi$ or $z_2 = z_1 + 2k\pi$ where k is an integer.

17. a. $e^{4z} = 1 \Longrightarrow 4z = 2k\pi i \Longrightarrow z = \dfrac{k\pi i}{2}$, where k is an integer

b. $e^{iz} = e^{i(x+iy)} = e^{-y}e^{ix} = 3 \Longrightarrow$
$e^{-y} = 3$ and $e^{ix} = 1 \Longrightarrow$
$y = -\ln 3$ and $x = 2k\pi$, where k is an integer.
Therefore, $z = 2k\pi - i\ln 3$.

c. No solution.

18. a. $\displaystyle\lim_{z\to 0} \frac{\sin z}{z} = \lim_{z\to 0} \frac{\sin z - \sin 0}{z} = \cos z\big|_{z=0} = 1$

b. $\displaystyle\lim_{z\to 0} \frac{\cos z - 1}{z} = -\sin z\big|_{z=0} = 0$

Alternatively, use L'Hospital's rule.

19. Let z_1 and z_2 lie in an open disk of radius π. Then,

$$e^{z_1} = e^{z_2} \Longrightarrow e^{z_1 - z_2} = 1 \Longrightarrow z_1 - z_2 = 2k\pi i$$

for some integer k. But $|z_1 - z_2| < 2\pi \Longrightarrow k = 0$. Hence $z_1 = z_2$.

20. Points inside or on the rectangle are characterized by $z = x + iy$, where

$$-1 \le x \le 1 \quad \text{and} \quad 0 \le y \le \pi.$$

Hence,

$$|w| = |e^z| = e^x \Longrightarrow e^{-1} \le |w| \le e \text{ and}$$
$$\operatorname{Arg} w = \operatorname{Arg} e^{iy} \Longrightarrow 0 \le \operatorname{Arg} w \le \pi.$$

Now suppose w_0 is in the semiannulus. Choose $x_0 = \operatorname{Log}|w_0|$ and $y_0 = \operatorname{Arg} w_0$. Then, $e^{x_0+iy_0} = w_0$, $0 \le y_0 \le \pi$, and $e^{-1} \le e^{x_0} \le e \Longrightarrow -1 \le x_0 \le 1$; thus the mapping is onto. It is one-to-one as a consequence of Theorem 1(as in Prob.19).

21. (a) If $\sin z_2 = \sin z_1$, $z_2 = -z_1 + (2k+1)\pi$ or $z_2 = z_1 + 2k\pi$ where k is an integer.
If $z_2 = -z_1 + (2k+1)\pi$, $y_2 = -y_1$. Re(z) = 0 is not in the semi-infinite strip.
If $z_2 = z_1 + 2k\pi$ is in the semi-infinite strip for k=0. $z_2 = z_1$ and the mapping is one-to-one. Answer: the plane cut along the negative imaginary axis and along the interval [-1,1].
(b) $w = \sin z = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$ and the strip $-\pi/2 < x < \pi/2$ and $y > 0$ is mapped into the upper half of the w-plane.

22. (By induction) Note that $c_1 e^{\lambda_1 z} = 0 \Leftrightarrow c_1 = 0$ because $e^{\lambda_1 z} \neq 0$. Now as the induction hypothesis, assume that for

any m distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ ($\lambda_i \neq \lambda_j$ for $i \neq j$), the statement

$c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \cdots + c_m e^{\lambda_m z} = 0$ for all z
implies that $c_1, c_2, \ldots, c_m$ all are zero. Consider
$c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \cdots + c_m e^{\lambda_m z} + c_{m+1} e^{\lambda_{m+1} z} = 0$. Dividing by

$e^{\lambda_1 z}$ gives $c_1 + c_2 e^{(\lambda_2 - \lambda_1) z} + \cdots + c_m e^{(\lambda_m - \lambda_1) z} + c_{m+1} e^{(\lambda_{m+1} - \lambda_1) z} = 0$. Taking the derivative with respect to z produces

$$(\lambda_2 - \lambda_1) c_2 e^{(\lambda_2 - \lambda_1) z} + \cdots + (\lambda_m - \lambda_1) c_m e^{(\lambda_m - \lambda_1) z} + (\lambda_{m+1} - \lambda_1) c_{m+1} e^{(\lambda_{m+1} - \lambda_1) z} = 0$$

for all z. By induction, $c_k(\lambda_k - \lambda_1) = 0$ for $k = 2, \ldots, m+1$. Because $(\lambda_k - \lambda_1) \neq 0$, $c_k = 0$ for $k=2, \ldots m+1$. With these constants equal to zero in the m+1 term equation, we conclude $c_1 = 0$. Thus by induction the functions

$e^{\lambda 1 z}, e^{\lambda 2 z}, \ldots, e^{\lambda m z}$ are linearly independent on C.

23. **a. Sums and products of entire functions are entire.**

b. $f'(z) = 2 \sin z \cos z - 2 \sin z \cos z = 0$

c. Theorem 6, Section 2.4

d. $f(0) = \sin^2(0) + \cos^2(0) = 1$

e. (c) and (d)

24. a. expo $= (\exp(x) * \cos(y), \exp(x) * \sin(y))$

b. sine $= (\sin(x) * \cosh(y), \cos(x) * \sinh(y))$

c. hypcosine $= (\cosh(x) * \cos(y), \sinh(x) * \sin(y))$

25. (a) $e^{1/z} = i \Rightarrow z = 1/i(\pi/2 + 2\pi k)$.

(b) $e^{1/z} = -1 \Rightarrow z = 1/i(2k+1)\pi$.

(c) $e^{1/z} = 6.02 \times 10^{23} \Rightarrow z = 1/[\ln(6.02 \times 10^{23}) + i2k\pi]$.

(d) $e^{1/z} = 1.6 \times 10^{-19} \Rightarrow z = 1/[\ln(1.6 \times 10^{-19}) + i2k\pi]$

In all cases the choice $k=1000/2\pi$ will suffice.

EXERCISES 3.3: The Logarithmic Function

1. a. $i\left(\frac{\pi}{2} + 2k\pi\right),\ k = 0, \pm 1, \ldots$

   b. $\frac{1}{2}\mathrm{Log}\, 2 + i\left(-\frac{\pi}{4} + 2k\pi\right),\ k = 0, \pm 1, \ldots$

   c. $-\frac{\pi}{2}i$

   d. $\mathrm{Log}\, 2 + i\frac{\pi}{6}$

2. Formula 5:

$$\begin{aligned}
\log z_1 z_2 &= \mathrm{Log}\,|z_1 z_2| + i\arg(z_1 z_2) \\
&= \mathrm{Log}\,|z_1| + \mathrm{Log}\,|z_2| + i(\arg z_1 + \arg z_2) \\
&= \mathrm{Log}\,|z_1| + i\arg z_1 + \mathrm{Log}\,|z_2| + i\arg z_2 \\
&= \log z_1 + \log z_2
\end{aligned}$$

Formula 6:

$$\begin{aligned}
\log\left(\frac{z_1}{z_2}\right) &= \log(z_1 z_2^{-1}) \\
&= \log z_1 + \log z_2^{-1} \\
&= \log z_1 + \mathrm{Log}\,|z_2^{-1}| + i\arg z_2^{-1} \\
&= \log z_1 - \mathrm{Log}\,|z_2| - i\arg z_2 \\
&= \log z_1 - \log z_2
\end{aligned}$$

3. $\operatorname{Log}[i(i-1)] = \operatorname{Log}(-1-i) = \operatorname{Log}\sqrt{2} - i\frac{3\pi}{4}$

$$\begin{aligned}\operatorname{Log} i + \operatorname{Log}(i-1) &= i\frac{\pi}{2} + \operatorname{Log}\sqrt{2} + i\frac{3\pi}{4} \\ &= \operatorname{Log}\sqrt{2} + i\frac{5\pi}{4}\end{aligned}$$

4. $$\begin{aligned}\operatorname{Log} e^z &= \operatorname{Log}|e^z| + i\operatorname{Arg} e^z \\ &= \operatorname{Log} e^x + i\operatorname{Arg} e^{iy} \\ &= x + iy \iff -\pi < y \le \pi \\ &= z\end{aligned}$$

5. a. $z = \log 2i = \operatorname{Log} 2 + i\left(\frac{\pi}{2} + 2k\pi\right), \quad k = 0, \pm 1, \pm 2, \ldots$

b. $z^2 - 1 = e^{\operatorname{Log}(z^2-1)} = e^{i\pi/2} = i$

$\iff z^2 = 1 + i$

$\iff z = (1+i)^{1/2} = \sqrt[4]{2}e^{i\pi/8}, \sqrt[4]{2}e^{i9\pi/8}$

c. (First use the quadratic formula)

$$e^z = \frac{-1+\sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\begin{aligned} z &= \log\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) \\ &= \operatorname{Log}\left|-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right| + i\arg\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) \\ &= i\left(\pm\frac{2\pi}{3} + 2k\pi\right), \quad k = 0, \pm 1, \ldots\end{aligned}$$

6. $\operatorname{Log}(z^2)$ is not $2\operatorname{Log}(z)$

7. In polar form $\operatorname{Log} z = \operatorname{Log} re^{i\theta} = \operatorname{Log} r + i\theta$. If $u(r,\theta) = \operatorname{Log} r$ and $v(r,\theta) = \theta$, u and v have continuous partial derivatives in D^* and

$$\frac{\partial u}{\partial r} = \frac{1}{r} \qquad \frac{\partial v}{\partial r} = 0$$

$$\frac{\partial u}{\partial \theta} = 0 \qquad \frac{\partial v}{\partial \theta} = 1.$$

Hence the Cauchy-Riemann equations (in polar form) are satisfied and $\operatorname{Log} z$ is analytic in D^*. To determine $\frac{d}{dz}(\operatorname{Log} z)$ choose to evaluate the limit along any ray ($\theta =$ constant); thus $\Delta z = \Delta r e^{i\theta}$ in this case.

$$\begin{aligned}
&\lim_{\Delta z \to 0} \frac{\operatorname{Log}(z+\Delta z) - \operatorname{Log} z}{\Delta z} \\
&= \lim_{\Delta r \to 0} \frac{\operatorname{Log}\left(re^{i\theta} + \Delta r e^{i\theta}\right) - \operatorname{Log}\left(re^{i\theta}\right)}{\Delta r e^{i\theta}} \\
&= \frac{1}{e^{i\theta}} \lim_{\Delta r \to 0} \frac{\operatorname{Log}(r+\Delta r) + i\theta - \operatorname{Log} r - i\theta}{\Delta r} \\
&= \frac{1}{re^{i\theta}}, \text{ since the last limit is } \frac{d}{dr}\operatorname{Log} r \\
&= \frac{1}{z}
\end{aligned}$$

8. $\operatorname{Log}|z|$ is harmonic on $D^* = \mathbf{C}\backslash\{x \leq 0,\ y = 0\}$ since it is the real part of $\operatorname{Log} z$, which is analytic there. Similarly, $\operatorname{Log}|z|$ is harmonic on $\mathbf{C}\backslash\{x \geq 0,\ y = 0\}$ because it is the real part of $\mathcal{L}_0(z)$. Putting these together, $\operatorname{Log}|z|$ is harmonic on $\mathbf{C}\backslash\{0\}$.

9. Domain of analyticity $\mathbf{C} \setminus \{x \geq 4,\ y = 1\}$

   $f'(z) = \dfrac{-1}{4+i-z}$

10. $\operatorname{Log}\ (-z) + i\pi$ is $\mathcal{L}_0(z)$

11. Choose the principal branch

$$\frac{d}{dz}\operatorname{Log}(z^2+2z+3)\Big|_{z=-1} = \frac{2z+2}{z^2+2z+3}\Big|_{z=-1} = 0.$$

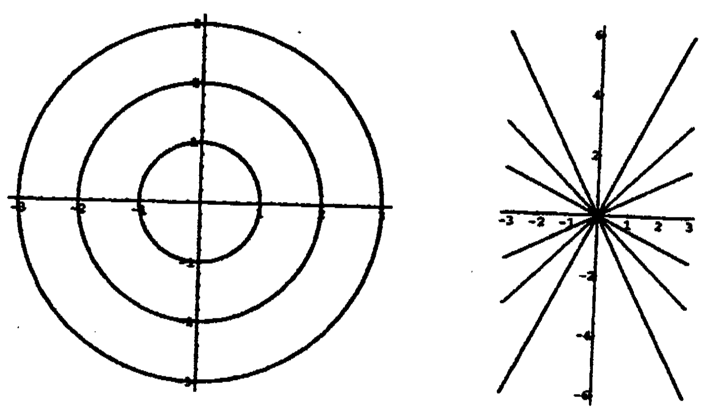
12. $\mathcal{L}_\pi(z^2+1)$

13. a. $\operatorname{Log}(2z-1)$
    b. $\mathcal{L}_0(2z-1)$
    c. $\mathcal{L}_{\pi/2}(2z-1)$

14. (By contradiction). If such a function exists, define $g(z) = F(z) - \operatorname{Log} z$. Then $g'(z) = F'(z) - \dfrac{1}{z} = 0$ except on the real segment $[-2,-1]$ so that $g(z) = c$ for some constant c. This means that $F(z) = \operatorname{Log} z + c$ throughout the annulus except along $[-2,-1]$, which contradicts the analyticity assumption.

15. $w = \dfrac{1}{\pi}\operatorname{Log} z$

16.



The level curves $x^2 + y^2 = c$ and $y = mx$ intersect at the points $(x,y) = \pm\left(\sqrt{\dfrac{c}{1+m^2}}, m\sqrt{\dfrac{c}{1+m^2}}\right)$, where the slope of the line is m and the slope of a tangent to the circle is $-\dfrac{1}{m}$.
Alternatively, show that $[u_x, u_y] \cdot [v_x, v_y] = 0$, i.e. the gradients are orthogonal.

17. If $w = f(z)$ is any branch of $\log z$ analytic on a domain D, then $e^w = z$. For $z_0 \in D$,

$$\lim_{z\to z_0} \frac{w - w_0}{z - z_0} = \frac{1}{\lim\limits_{z\to z_0} \dfrac{z - z_0}{w - w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0}.$$

18. Let $G(z)$ be another branch of $\log z$ analytic on D. Then $G'(z) - F'(z) = 0$, so $G(z) = F(z) + c$. Since the imaginary part of each branch has to be a value of $\arg z$, the constant c must be a multiple of $2\pi i$. Thus $G(z) = F(z) + 2k\pi i$ for some value of $k = 0, \pm 1, \ldots$.

19. Define $\log z = \operatorname{Log}|z| + i\theta$ with

$$\theta = \begin{cases} \text{the value of } \arg z \text{ between } \pi/2 \text{ and } 2\pi \\ \qquad \text{for } z \text{ in quad. II, III, or IV} \\ \text{the value of } \arg z \text{ between } 0 \text{ and } \pi/2 \\ \qquad \text{for } z \text{ in quad. I above the half parabola} \\ \text{the value of } \arg z \text{ between } 2\pi \text{ and } 5\pi/2 \\ \qquad \text{for } z \text{ in quad. I below the half parabola} \end{cases}$$

To make this explicit, one could find θ as a function of r on the half parabola $y = \sqrt{x}$.

$$\theta = \operatorname{Tan}^{-1}\left(\frac{y}{x}\right) = \operatorname{Tan}^{-1}\left(\frac{1}{y}\right)$$

$$= \operatorname{Tan}^{-1}\left(\frac{1}{r\sin\theta}\right) = \operatorname{Tan}^{-1}\sqrt{\frac{2}{\sqrt{1+4r^2}-1}}$$

20. Define subroutines called radius and argument based on Exercise 24, Section1.3.

a. INPUT x, y

Step1 If $x = 0$ and $y = 0$, go to step 6
Step2 If $x < 0$ and $y = 0$, set logarithm= (log($-x$), pi)
Step3 Else set logarithm = (log(radius(x, y)), argument ($x, y,$))
Step4 Print "logarithm is "; logarithm
Step5 Go to step7
Step6 Print "undefined"
Step7 Stop

b. Same as part a. but replace steps 2 and 3 as follows (and renumber steps).
Step2 If $x > 0$ and $y = -x$, set logarithm
$= (\log(\text{radius}(x, y)), 7 * \text{pi} /4)$
Step3 If $y \geq 0$ or $y > -x$, set logarithm
$= (\log(\text{radius}(x, y)), \text{argument}(x, y))$
Step4 Else set logarithm
$= (\log(\text{radius}(x, y)), \text{argument}(x, y) + 2 * \text{pi})$

c. Same as part a. but replace steps 2 and 3 as follows (and renumber steps).
Step2 If $x > 0$ and $y = 0$, set logarithm $= (\log(x), 2 * \text{pi})$
Step3 If $y > 0$ or ($x < 0$ and $y = 0$), set logarithm
$= (\log(\text{radius}(x, y)), \text{argument}(x, y))$
Step4 Else set logarithm
$= (\log(\text{radius}(x, y)), \text{argument}(x, y) + 2 * \text{pi})$

d. Same as part a. but replace steps 2 and 3 as follows (and renumber steps).
Step2 If $x > 0$ and $y = 0$, set logarithm $= (\log(x), 6 * \text{pi})$
Step3 If $y > 0$ or ($x < 0$ and $y = 0$), set logarithm
$= (\log(\text{radius}(x, y)), \text{argument}(x, y) + 4 * \text{pi})$
Step4 Else set logarithm
$= (\log(\text{radius}(x, y)), \text{argument}(x, y) + 6 * \text{pi})$

21. For a computer that uses the principal logarithm, take $z_1 = z_2 = e^{-i3\pi/4}$

Exercises 3.4

1. $\phi(0,0) = 0.5$
2. $\phi(0,0) = 5/3$
3. $\phi(1,1) = 0$
4. $\phi(2,3) = \pi/4$
5. $\phi(x,y) = [(B-A)/(\text{Log } r_1 - \text{Log } r_2)] \text{ Log } |z| + (A \text{ Log } r_1 - B \text{ Log } r_2)/(\text{Log } r_1 - \text{Log } r_2)$
6. $\phi(x,y) = 30$
7. $f(x,y) = (½)(\log z)^2 = (½)(\log |z|)^2 - (½)(\arg z)^2 + i(\arg z)(\log |z|)$ is analytic with a proper branch cut. Therefore its imaginary part is harmonic.

EXERCISES 3.5

1. a. $i^i = e^{i\log i} = e^{-\arg i} = e^{-\pi/2+2k\pi} \quad k = 0, \pm 1, \ldots$

b. $(-1)^{2/3} = e^{2/3\log(-1)} = e^{\frac{2}{3}i(2k+1)\pi} \quad k = 0, 1, 2$

$= 1, \quad -1/2 \pm i\sqrt{3}/2$

c. $2^{\pi i} = e^{\pi i \log 2} = e^{-2k\pi^2} e^{i\pi \mathrm{Log}\, 2} \quad k = 0, \pm 1, \ldots$

d.
$$\begin{aligned}(1+i)^{1-i} &= (1+i)(1+i)^{-i} \\ &= (1+i)e^{-i\log(1+i)} \\ &= (1+i)\exp\left[\frac{\pi}{4} + 2k\pi - \frac{i}{2}\mathrm{Log}\, 2\right], \quad k = 0, \pm 1, \ldots\end{aligned}$$

e.
$$\begin{aligned}(1+i)^3 = e^{3\log(1+i)} &= \exp\left[\frac{3}{2}\mathrm{Log}\, 2 + i\frac{3\pi}{4}\right] \\ &= 2^{3/2}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \\ &= -2 + 2i\end{aligned}$$

2. $z^0 = e^{0\log z} = e^0 = 1$

3. a. $4^{\frac{1}{2}} = e^{\frac{1}{2}\mathrm{Log}\, 4} = e^{\mathrm{Log}\, 2} = 2$

b. $i^{2i} = e^{2i\mathrm{Log} i} = e^{-\pi}$

c. $(1+i)^{(1+i)} = (1+i)e^{i\mathrm{Log}\,(1+i)} = (1+i)\exp\left[-\frac{\pi}{4} + \frac{i}{2}\mathrm{Log}\, 2\right]$

4. No: $1^\alpha = e^{\alpha \log 1} = e^{2\alpha k\pi i} \quad k = 0, \pm 1, \ldots$
For example, take $k = 1, \alpha = \frac{1}{2}$. $\quad 1^\alpha = e^{\pi i} = -1$

5. $[i(-1+i)]^{\frac{1}{2}} = (-1-i)^{\frac{1}{2}} = \exp\left(\frac{1}{4}\mathrm{Log}\, 2 - i\frac{3\pi}{8}\right)$
$i^{\frac{1}{2}}(-1+i)^{\frac{1}{2}} = \exp\left(i\frac{\pi}{4}\right)\exp\left(\frac{1}{4}\mathrm{Log}\, 2 + i\frac{3\pi}{8}\right) = \exp\left(\frac{1}{4}\mathrm{Log}\, 2 + i\frac{5\pi}{8}\right)$

6. a. $z^{-\alpha} = e^{-\alpha\,\mathrm{Log}\, z} = \dfrac{1}{e^{\alpha \mathrm{Log}\, z}} = \dfrac{1}{z^\alpha}$

b. $z^\alpha z^\beta = e^{\alpha \mathrm{Log}\, z} e^{\beta \mathrm{Log}\, z} = e^{(\alpha+\beta)\mathrm{Log}\, z} = z^{\alpha+\beta}$

c. $z^\alpha / z^\beta = \dfrac{e^{\alpha \mathrm{Log}\, z}}{e^{\beta \mathrm{Log}\, z}} = e^{(\alpha-\beta)\mathrm{Log}\, z} = z^{\alpha-\beta}$

7. $(1+i)i^i = (1+i)e^{-\frac{\pi}{2}}$

8.
$$\begin{aligned}z = \sin^{-1} 2 &= -i\log\left[2i + (-3)^{\frac{1}{2}}\right] \\ &= -i\mathrm{Log}\,(2 \pm \sqrt{3}) + \arg[(2 \pm \sqrt{3})i] \\ &= -i\mathrm{Log}\,(2 \pm \sqrt{3}) + \frac{\pi}{2} + 2k\pi, \quad k = 0, \pm 1, \ldots\end{aligned}$$

Now observe that since $(2+\sqrt{3})(2-\sqrt{3})=1$,
$0=\text{Log}\,[(2+\sqrt{3})(2-\sqrt{3})]=\text{Log}\,(2+\sqrt{3})+\text{Log}\,(2-\sqrt{3})$ so that

$$\text{Log}(2-\sqrt{3})=-\text{Log}(2+\sqrt{3}).$$

Therefore,

$$z=\pm i\text{Log}(2+\sqrt{3})+\frac{\pi}{2}+2k\pi \quad k=0,\pm1,\ldots$$

9. Formula 9: $z=\cos w=\dfrac{e^{iw}+e^{-iw}}{2}\Rightarrow e^{2iw}-2ze^{iw}+1=0$

$$\begin{aligned}\Rightarrow e^{iw} &= \frac{2z+(4z^2-4)^{1/2}}{2}\\ &= z+(z^2-1)^{1/2}\\ \Rightarrow w &= -i\log[z+(z^2-1)^{1/2}]\end{aligned}$$

Formula 11: choose a branch of $\cos^{-1} z$ and the same branch of the square root.

$$\begin{aligned}\frac{d}{dz}(\cos^{-1} z) &= \frac{d}{dz}\left\{-i\log\left[z+(z^2-1)^{1/2}\right]\right\}\\ &= -i\frac{1+z(z^2-1)^{-1/2}}{z+(z^2-1)^{1/2}}\\ &= \frac{-i}{(z^2-1)^{1/2}}\cdot\frac{(z^2-1)^{1/2}+z}{z+(z^2-1)^{1/2}}\\ &= \frac{-1}{(1-z^2)^{1/2}},\quad z\neq\pm1\end{aligned}$$

10. $z=\cos^{-1}(2i)=-i\log\left[2i+(-5)^{1/2}\right]=-i\log[(2\pm\sqrt{5})i]$

$$=-i\text{Log}(2+\sqrt{5})+\frac{\pi}{2}+2k\pi \text{ and } i\text{Log}(2+\sqrt{5})-\frac{\pi}{2}+2k\pi.$$

See Exercise 8 for a method to show that $\text{Log}(2-\sqrt{5})=-\text{Log}(2+\sqrt{5})$, a fact used in writing the latter value of z.

11. $\tan z = 1 \Rightarrow z = \tan^{-1}(1)$

$$= \frac{i}{2}\log\left(\frac{i+1}{i-1}\right) = \frac{i}{2}\log(-i)$$
$$= \frac{\pi}{4} + k\pi, \quad k = 0, \pm 1, \ldots$$

12. Formula 10:

$$\begin{aligned} z = \tan w &= -i\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \Rightarrow \\ e^{2iw} &= \frac{1+iz}{1-iz} \Rightarrow \\ w &= \frac{-i}{2}\log\left(\frac{1+iz}{1-iz}\right) \\ &= \frac{i}{2}\log\left(\frac{1-iz}{1+iz}\right) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right) \end{aligned}$$

Formula 12: choose a branch of the logarithm.

$$\begin{aligned} \frac{d}{dz}\left(\tan^{-1} z\right) &= \frac{d}{dz}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] \\ &= \frac{i}{2}\frac{i-z}{i+z}\frac{2i}{(i-z)^2} \\ &= \frac{1}{1+z^2}, \quad z \neq \pm i \end{aligned}$$

13. Formula 13:

$$z = \sinh w = \frac{e^w - e^{-w}}{2} \Rightarrow$$
$$e^{2w} - 2ze^w - 1 = 0 \Rightarrow$$
$$e^w = \frac{2z + (4z^2+4)^{1/2}}{2} = z + (z^2+1)^{1/2} \Rightarrow$$
$$w = \log[z + (z^2+1)^{1/2}]$$

Formula 14:

$$z = \cosh w = \frac{e^w + e^{-w}}{2} \Rightarrow$$
$$e^{2w} - 2ze^w + 1 = 0 \Rightarrow$$

$$e^w = \frac{2z + (4z^2 - 4)^{1/2}}{2} = z + (z^2 - 1)^{1/2} \Rightarrow$$
$$w = \log[z + (z^2 - 1)^{1/2}]$$

14. Choose a branch of the square root and a branch of the logarithm.

$$\begin{aligned} \frac{d}{dz}\left(\sinh^{-1} z\right) &= \frac{d}{dz}\left\{\log[z + (z^2+1)^{1/2}]\right\} \\ &= \frac{1 + z(z^2+1)^{-1/2}}{z + (z^2+1)^{1/2}} \\ &= \frac{1}{(z^2+1)^{1/2}} \cdot \frac{(z^2+1)^{1/2} + z}{z + (z^2+1)^{1/2}} \\ &= \frac{1}{(z^2+1)^{1/2}} \quad z \neq \pm i \end{aligned}$$

15. a. $i \exp\left[\frac{1}{2}\mathrm{Log}(1 - z^2)\right]$

b. $z \exp\left[\frac{1}{2}\mathrm{Log}(1 + 4/z^2)\right]$

c. $z^2 \exp\left[\frac{1}{2}\mathrm{Log}(1 - 1/z^4)\right]$

d. $z \exp\left[\frac{1}{3}\mathrm{Log}(1 - 1/z^3)\right]$

16. Choose a branch of $\log z$ that is analytic at c.

$$\begin{aligned} \frac{d}{dz}c^z &= \frac{d}{dz}e^{z \log c} \\ &= \log c\, e^{z \log c} \\ &= (\log c)\, c^z \text{ for all } z \end{aligned}$$

17. Set $w = \sec^{-1} z$. Then

$$z = \sec w = \frac{2}{e^{iw} + e^{-iw}}$$

$$ze^{iw} + ze^{-iw} = 2$$

$$ze^{2iw} - 2e^{iw} + z = 0$$

$$e^{iw} = \frac{2 + (4 - 4z^2)^{1/2}}{2z} \quad \text{(quadratic formula)}$$

$$e^{iw} = \frac{1}{z} + \left(\frac{1}{z^2} - 1\right)^{1/2}$$

$$iw = \log\left[\frac{1}{z} + \left(\frac{1}{z^2} - 1\right)^{1/2}\right]$$

$$\sec^{-1} z = w = -i\log\left[\frac{1}{z} + \left(\frac{1}{z^2} - 1\right)^{1/2}\right].$$

For the range of $\mathrm{Sec}^{-1}x$, when $x > 1$ or $x < -1$, consider

$$\begin{aligned}
\mathrm{Sec}^{-1}x &= -i\mathrm{Log}\left[\frac{1}{x} + i\sqrt{\left|\frac{1}{x^2} - 1\right|}\right] \\
&\qquad \text{(using the principal square root).} \\
&= -i\left[\mathrm{Log}\,|1| + i\mathrm{Arg}\left(\frac{1}{x} + i\sqrt{1 - \frac{1}{x^2}}\right)\right] \\
&= \mathrm{Arg}\left(\frac{1}{x} + i\sqrt{1 - \frac{1}{x^2}}\right)
\end{aligned}$$

For $x > 1$, this ranges $0 < \mathrm{Sec}^{-1}x < \pi/2$ and for $x < -1$ this ranges $\pi/2 < \mathrm{Sec}^{-1}x < \pi$. This agrees with the ranges given for $\mathrm{Sec}^{-1}x$ in a standard trigonometry text.

19. a. **Define subroutines called compprod(from Section 1.1, Problem 31), expo(from Section 3.1, Problem 22) and princlog(from Section 3.2, Problem 20a).**
Input alphar, alphai, x, y
Output expo(compprod(alphar, alphai, princlog(x,y)))

b. **Define a subroutine called compsqrt as in the body of Problem 19, Section 1.5.**
Output compprod(0, −1, princlog((−y,x)+compsqrt((1,0) −compprod(x,y,x,y))))

c. **Output compprod(0,−1,princlog(compquot(1,0,x,y) +compsqrt(compquot(1,0,compprod(x,y,x,y))−(1,0))))**

d. **Output compprod(x,y, expo(0.5 * princlog((1,0) −compquot(1,0,compprod(x,y,x,y)))))**

19. $w = 2e^z + e^{2z}$ if and only if $e^{2z} + 2e^z - w = 0$

$e^z = -1 \pm \sqrt{(1+w)}$

$z = \log(-1 \pm \sqrt{(1+w)})$;

$z = i2\pi k$ or $z = \mathrm{Log}(3) + i\pi(2k+1)$

Exercises 3.6

1. $I_s = R^{-1}\sin(\omega t - \varphi_0)$
2. I_s = Real part of

$\{a_1e^{i(\omega t+\gamma_1)} + a_2e^{i(\omega t+\gamma_2)} + \cdots + a_me^{i(\omega t+\gamma_m)}$

$- i[b_1e^{i(\omega t+\delta_1)} + b_2e^{i(\omega t+\delta_2)} + \cdots + b_ne^{i(\omega t+\delta_n)}]\}$

$I_s = \mathrm{Re}\{a_1e^{i\gamma_1} + a_2e^{i\gamma_2} + \cdots + a_me^{i\gamma_m} - i[b_1e^{i\delta_1} + b_2e^{i\delta_2} + \cdots + b_ne^{i\delta_n}]\}e^{i\omega t}$

$I_s = \mathrm{Re}(\alpha e^{i(\omega t)})$ where

3. $I_s = \mathrm{Re}\{[(1+iR\omega C)/(R(1-\omega^2LC)+i\omega L)e^{i\omega t}\}$

$I_s = \mathrm{Re}\{[(R(1-\omega^2LC)-\omega^2LCR+i(R^2\omega C(1-\omega^2LC)-\omega L)/(R^2(1-\omega^2LC)^2+\omega^2L^2)e^{i\omega t}\}$

$= \mathrm{Re}\{[(R+i(R^2\omega C(1-\omega^2LC)-\omega L)/(R^2(1-\omega^2LC)^2+\omega^2L^2)]e^{i\omega t}\}$

$I_s = \{R\cos\omega t-(R^2\omega C(1-\omega^2LC)-\omega L)\sin\omega t\}/(R^2(1-\omega^2LC)^2+\omega^2L^2)$

4. (a) $I_s = \mathrm{Re}\{[i\omega C/(1+i\omega CR)]Ae^{i\omega t)}\}$

$= \mathrm{Re}\{[(-\omega^2RC+i\omega C)/(1+\omega^2C^2R^2)]Ae^{i\omega t)}\}$

$= A(-\omega^2RC\cos\omega t - \omega C\sin\omega t)/(1+\omega^2C^2R^2)$

(b) $I_s = \mathrm{Re}\{[i\omega C + 1/R + 1/i\omega L]Ae^{i(\omega t+\beta)}\}$

$= \mathrm{Re}\{[1/R + i(\omega C - 1/\omega L)]Ae^{i(\omega t+\beta)}\}$

$I_s = A\cos(\omega t+\beta)/R + (1/\omega L-\omega C)A\sin(\omega t+\beta)$

(c) $I_s = \mathrm{Re}\{[1/(1/i\omega C + 1/(1/R_2 + 1/(R_1 + i\omega L)]Ae^{i\omega t}\}$

$I_s = \mathrm{Re}\{[(-\omega^2LC/R_2+i(1+R_1\omega C/R_2)/(1+R_1/R_2 - \omega^2LC + i(\omega L/R_2 + R_1\omega C)]Ae^{i\omega t}\}$

$I_s = \mathrm{Re}\{[(\omega^4L^2C^2R_2 + (R_2+R_1)R_2R_1C^2\omega^2+i\omega[(R_2+R_1)^2C^2 - R_2^2LC^2\omega^2 + L^2C\omega^2)/$

$((R_2+R_1-\omega^2LC)^2 + \omega^2(L + R_1R_2C)^2)]Ae^{i\omega t}\}$

$I_s = A\{[(\omega_1{}^4L^2C^2R_2 + (R_2+R_1)R_2R_1C^2\omega_1{}^2]\cos\omega_1t$

$-\omega_1[(R_2+R_1)^2C^2 - R_2^2LC^2\omega_1{}^2 + L^2C\omega_1{}^2]\sin\omega_1t\}/$

$((R_2+R_1-\omega_1{}^2LC)^2 + \omega_1{}^2(L + R_1R_2C)^2)$ $+ B\{[(\omega_2{}^4L^2C^2R_2 + (R_2+R_1)R_2R_1C^2\omega_2{}^2]\sin\omega_2t$

$-\omega_2[(R_2+R_1)^2C^2 - R_2^2LC^2\omega_2{}^2 + L^2C\omega_2{}^2]\cos\omega_2t\}/$

$((R_2+R_1-\omega_2{}^2LC)^2 + \omega_2{}^2(L + R_1R_2C)^2)$

5. $I_s \to 1/R$.
6. $I_s \to 0$.
7. (a) In the z-plane $\phi = \cos\omega t + e^{i2\pi/3}\cos(\omega t-2\pi/3) + e^{i4\pi/3}\cos(\omega t-4\pi/3)$

$\phi = (e^{i\omega t}+e^{-i\omega t})/2 + e^{i2\pi/3}(e^{i(\omega t-2\pi/3)} + e^{i(\omega t-2\pi/3)})/2 + e^{i4\pi/3}(e^{i(\omega t-4\pi/3)} + e^{i(\omega t-4\pi/3)})/2$

$\phi = (3/2)e^{i\omega t}$

(b) 2 inductors: $\phi = \cos\omega t + e^{i\pi}\cos(\omega t-\pi) = 2\cos\omega t$ (*fails*).

4 inductors: $\phi = \cos \omega t + e^{i2\pi/4}\cos(\omega t\text{-}\pi/2) + e^{i\pi}\cos(\omega t\text{-}\pi)$
$+ e^{i6\pi/4}\cos(\omega t\text{-}6\pi/4) = (4/2)e^{i\omega t}$
n(>2) inductors: $\phi = \Sigma_{k=0}{}^{n-1}e^{i2\pi k/n}\cos(\omega t\text{-}2\pi k/n) = (n/2)e^{i\omega t}$
2 inductors at 90 deg with 90 deg phase shift:
$\phi = \cos \omega t + e^{i\pi/2}\cos(\omega t\text{-}\pi/2) = e^{i\omega t}$.

8. For convenience we write Ω instead of ω_m :
(a) $s_{FM}(t) = \mathrm{Re}(Ae^{i\int(\omega+\beta\cos(\Omega t)dt}) = \mathrm{Re}(Ae^{i(\omega t+(\beta/\Omega)\sin(\Omega t)})$
$= \mathrm{Re}(Ae^{i(\omega t}e^{i(\beta/\Omega)\sin(\Omega t)}) = \mathrm{Re}[Ae^{i(\omega t}(1+(\beta/\Omega)\sin(\Omega t))]$
$s_{FM}(t) = \mathrm{Re}[e^{i(\omega t}(A+(A\beta/\Omega)\sin(\Omega t))]$
$s_{AM}(t) = \mathrm{Re}[e^{i(\omega t}(A+\beta\cos(\Omega t))] = (A+\beta\cos(\Omega t))\cos(\omega t)$
(b) Am: bottom figure
FM: top figure
(c) (You plot.)

CHAPTER 4: Complex Integration

EXERCISES 4.1: Contours

1. a. $z(t) = (1+i) + t[(-2-3i) - (1+i)]$
$= (1-3t) + i(1-4t), \qquad 0 \le t \le 1$

   b. $z(t) = 4e^{-it} + 2i, \qquad 0 \le t \le 2\pi$

   c. $z(t) = Re^{it}, \qquad \pi/2 \le t \le \pi$

   d. $z(t) = t + it^2, \qquad 1 \le t \le 3$

2. $z(t) = t^2 + it^3$ is a parametrization for $x = y^{2/3}$, which has a cusp at the origin, where $t = 0$ and $z'(t)$ vanishes.

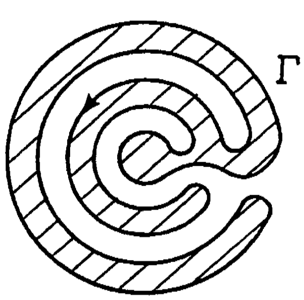
3. $z(t) = a\cos t + ib\sin t, \quad 0 \le t \le 2\pi$ is an admissible parametrization of the ellipse.

4. $z(t) = t^3 + it^6, \quad -1 \le t \le 1$, is a parametrization of the real function $y = x^2$. The alternate parametrization

$$\hat{z}(t) = t + it^2, \qquad -1 \le t \le 1$$

is admissible, hence $z(t)$ is smooth.

5. Yes.



4 inductors: $\phi = \cos \omega t + e^{i2\pi/4}\cos(\omega t-\pi/2) + e^{i\pi}\cos(\omega t-\pi)$
$+ e^{i6\pi/4}\cos(\omega t-6\pi/4) = (4/2)e^{i\omega t}$

n(>2) inductors: $\phi = \Sigma_{k=0}{}^{n-1} e^{i2\pi k/n}\cos(\omega t-2\pi k/n) = (n/2)e^{i\omega t}$

2 inductors at 90 deg with 90 deg phase shift:
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(a) $s_{FM}(t) = \mathrm{Re}(Ae^{i\int(\omega+\beta\cos(\Omega t)dt}) = \mathrm{Re}(Ae^{i(\omega t+(\beta/\Omega)\sin(\Omega t)})$
$= \mathrm{Re}(Ae^{i(\omega t}e^{i(\beta/\Omega)\sin(\Omega t)}) = \mathrm{Re}[Ae^{i(\omega t}(1+(\beta/\Omega)\sin(\Omega t))]$
$s_{FM}(t) = \mathrm{Re}[e^{i(\omega t}(A+(A\beta/\Omega)\sin(\Omega t))]$
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(c) (You plot.)

CHAPTER 4: Complex Integration

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   d. $z(t) = t + it^2, \qquad 1 \le t \le 3$

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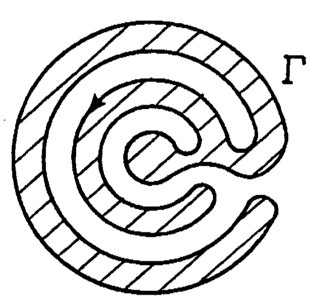
is admissible, hence $z(t)$ is smooth.

5. Yes.



6. Define $f(t) = \frac{b-a}{d-c}t + \frac{ad-bc}{d-c}$, $c \neq d$. f is linear, thus is 1-1, and has a derivative for $c \leq t \leq d$. Also $f(c) = a$, $f(d) = b$ and $f'(t) \neq 0$ for $c \leq t \leq d$. Since $z_1(t) = z(f(t))$ it follows that $z_1(c) = z(a)$, $z_1(d) = z(b)$ and $z_1(t)$ satisfies the three defining properties of Definition 1 for an admissible parametrization.

7.
$$z(t) = \begin{cases} -1-i+8t & 0 \leq t \leq \frac{1}{4} \\ 1-i+8i\left(t-\frac{1}{4}\right) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ 1+i-8\left(t-\frac{1}{2}\right) & \frac{1}{2} \leq t \leq \frac{3}{4} \\ -1+i-8i\left(t-\frac{3}{4}\right) & \frac{3}{4} \leq t \leq 1 \end{cases}$$

$l(\Gamma) = 8$

8. Γ is parametrized by

$$z(t) = \begin{cases} -2+2i+t(1-2i) & 0 \leq t \leq 1 \\ e^{-\pi i t} & 1 \leq t \leq 2 \end{cases}$$

$-\Gamma$ is parametrized by $z(-t)$ $\quad -2 \leq t \leq 0$.

9.
$$z(t) = \begin{cases} -2+e^{-\pi i t} & -2 \leq t \leq 0 \\ -1+2t & 0 \leq t \leq 1 \\ 2+e^{\pi i t} & 1 \leq t \leq 3 \end{cases}$$

10. a. $\gamma : z(t) = z_1 + t(z_2 - z_1) \quad 0 \leq t \leq 1$

$l(\gamma) = \int_0^1 |z_2 - z_1|\, dt = |z_2 - z_1|$

b. $\gamma : z(t) = z_0 + re^{it} \quad 0 \leq t \leq 2\pi$

$l(\gamma) = \int_0^{2\pi} \left|rie^{it}\right| dt = \int_0^{2\pi} r dt = 2\pi r$

11. $l(\Gamma) = \int_0^{\pi} \left|15ie^{3it}\right| dt = \int_0^{\pi} 15 dt = 15\pi$

12. Yes

13. a. $z'(t)$ is the velocity of the moving particle at time t.

b. $|z'(t)|$ is the instantaneous speed of the particle at time t.

c. $|z'(t)dt|$ is an increment of distance traveled by the particle during the time interval dt.

d. $\int_a^b |z'(t)|\,dt$ is the distance traveled by the particle as it moved along the contour from $z(a)$ to $z(b)$.

14. Let $t = \phi(s)$. Then $dt = \phi'(s)ds$. Since $z_2(s) = z_1(\phi(s))$, one gets $\dfrac{dz_2}{ds} = \dfrac{dz_1}{dt}\phi'(s)$ and $|z_2'(s)| = |z_1'(t)|\,\phi'(s)$.

Thus

$$\int_a^b |z_1'(t)|\,dt = \int_c^d |z_2'(s)|\,\frac{1}{\phi'(s)}\phi'(s)ds = \int_c^d |z_2'(s)|\,ds$$

EXERCISES 4.2: Contour Integrals

1. Let $\mathcal{P}_n$ be a partition of γ with $\alpha = z_0$ and $\beta = z_n$. Let $c_1, c_2, \ldots, c_n$ be points on γ such that c_i lies on γ between z_{i-1} and z_i.
Then

$$S(\mathcal{P}_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} c(z_k - z_{k-1})$$

$$= -cz_0 + cz_n \;=\; c(\beta - \alpha).$$

Since this telescoping property holds for every partition,

$$\lim_{n\to\infty} S(\mathcal{P}_n) = c(\beta - \alpha) \text{ whenever } \lim_{n\to\infty} \mu(\mathcal{P}_n) = 0.$$

Yes. The telescoping property applies to every curve in an arbitrary contour.

2. Consider any sequence $\mathcal{P}_n$ of partitions of γ with $\lim_{n\to\infty} \mu(\mathcal{P}_n) = 0$.

Property 1 : $\int_\gamma [f(z) \pm g(z)]dz$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [f(c_i) \pm g(c_i)]\,(z_i - z_{i-1})$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,(z_i - z_{i-1}) \pm \lim_{n\to\infty} \sum_{i=1}^{n} g(c_i)\,(z_i - z_{i-1})$$
$$= \int_\gamma f(z)dz \pm \int_\gamma g(z)dz$$

Property 2 : $\int_\gamma cf(z)dz = \lim_{n\to\infty} \sum_{i=1}^{n} cf(c_i)\,(z_i - z_{i-1})$

$$= c \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)(z_i - z_{i-1})$$
$$= c\int_\gamma f(z)dz$$

Property 3 : $-\int_\gamma f(z)dz = -\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)(z_i - z_{i-1})$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)(z_{i-1} - z_i)$$
$$= \int_{-\gamma} f(z)dz$$

(Since a partition of γ can be reversed to give a partition of $-\gamma$)

3. a. $1 + \frac{1}{3}i$

b. $(1+i)(-i\sin it)|_{-2}^{0} = (1+i)\sinh 2$

c. $\left.\frac{(1+2it)^6}{12i}\right|_0^1 = \frac{11-29i}{3}$

d. $\frac{-1}{2}\left.(t^2+i)^{-1}\right|_0^2 = \frac{-2-8i}{17}$

4. Let $F(t) = U(t) + iV(t)$ so that $F'(t) = U'(t) + iV'(t)$.
If $F'(t) = f(t) = u(t) + iv(t)$
then

$$\int_a^b f(t)dt = \int_a^b [u(t) + iv(t)]dt$$

$$= \int_a^b u(t)dt + i\int_a^b v(t)dt$$

$$= U(t)|_a^b + i\,V(t)|_a^b$$

$$= [U(b) + iV(b)] - [U(a) + iV(a)]$$

$$= F(b) - F(a)$$

5. $6(0) + 2(2\pi i) + 1(0) - 3(0) = 4\pi i$

6. a. $\displaystyle\int_0^{2\pi} 2e^{-i\theta} i2e^{i\theta}d\theta = 8\pi i$

   b. $-8\pi i$

   c. $-24\pi i$

7. Using $z(t) = t(1+2i)$ for $0 \le t \le 1$,
   $\displaystyle\int_\gamma \mathrm{Re}(z)dz = \int_0^1 t(1+2i)dt = \frac{1}{2} + i$

8. Let $C = C_1 + C_2 + C_3 + C_4$, where

$$\begin{array}{ll} C_1 : z(t) = t & 0 \le t \le 1 \\ C_2 : z(t) = 1 + it & 0 \le t \le 1 \\ C_3 : z(t) = 1 - t + i & 0 \le t \le 1 \\ C_4 : z(t) = i(1-t) & 0 \le t \le 1 \end{array}$$

$$\int_C e^z dz = \int_0^1 e^t dt + i\int_0^1 e^{(1+it)}dt - \int_0^1 e^{1-t+i}dt - i\int_0^1 e^{i(1-t)}dt = 0$$

9. $\displaystyle\int_\Gamma (x - 2xyi)dz = \int_0^1 (t - 2t^3 i)(1+2it)dt = \frac{13}{10} + \frac{1}{6}i$

10. $\displaystyle\int_C \overline{z}^2 dz = \int_0^1 t^2 dt + \int_0^1 (1-it)^2(i)dt$
$\displaystyle\qquad + \int_0^1 (1-t-i)^2(-1)dt + \int_0^1 (-i)^2(1-t)^2(-i)dt$

$= 2(1+i)$

11. a. $3+i$, using the parametrization $z(t) = t + i(-1+t), \quad 0 \le t \le 1$
    b. $3+i$, using the parametrizations $z_1(t) = i(-1+t), \quad 0 \le t \le 1$ and $z_2(t) = t, \quad 0 \le t \le 1.$
    c. $3+i$

12. True: both are $2\pi i$

13. $\int_0^{\pi} (i - e^{it}) ie^{it} dt = -2i$

14. a. $\left| \int_C \frac{dz}{z^2 - i} \right| \le \max_{z \text{ on } C} \left| \frac{1}{z^2 - i} \right| l(C) = \frac{1}{8} \cdot 6\pi = \frac{3\pi}{4}$
    b. $\left| \int_\gamma \frac{e^{3z}}{1+e^z} dz \right| \le \max_{z \text{ on } \gamma} \left| \frac{e^{3z}}{1+e^z} \right| l(\gamma) \le \frac{e^{3R}}{e^R - 1} \cdot 2\pi$
    c. $\left| \int_\Gamma \text{Log} z dz \right| \le \max_{z \text{ on } \Gamma} \left| \text{Log}|z| + i\text{Arg } z \right| l(\Gamma) \le \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$
    d. $\left| \int_\gamma e^{\sin z} dz \right| \le \max_{z \text{ on } \gamma} \left| e^{\sin z} \right| l(\gamma) = 1 \cdot 1 = 1.$

15. Let $\mathcal{P}_n$ be a partition of $[a, b]$.
Then

$$\left| \int_a^b f(t)dt \right| = \left| \lim_{\substack{n\to\infty \\ \mu(\mathcal{P}_n)\to 0}} S(\mathcal{P}_n) \right|$$

$$= \left| \lim_{\substack{n\to\infty \\ \mu(\mathcal{P}_n)\to 0}} \sum_{i=1}^{n} f(c_i)(t_i - t_{i-1}) \right| \text{ where } c_i \in (t_{i-1}, t_i)$$

$$\le \lim_{\substack{n\to\infty \\ \mu(\mathcal{P}_n)\to 0}} \sum_{i=1}^{n} |f(c_i)(t_i - t_{i-1})|$$

$$= \lim_{\substack{n\to\infty \\ \mu(\mathcal{P}_n)\to 0}} \sum_{i=1}^{n} |f(c_i)|\,(t_i - t_{i-1})$$

$$= \int_a^b |f(t)|\,dt.$$

16. Let $z(t)$ be an admissible parametrization of γ with $z(a) = \alpha$ and $z(b) = \beta$.

$$\int_\gamma z\,dz = \int_a^b z(t)z'(t)dt = \left.\frac{[z(t)]^2}{2}\right|_{t=a}^{b} = \frac{\beta^2 - \alpha^2}{2}$$

17. The result follows immediately since $\alpha = \beta$.

18. $$\int_{\Gamma_1} f(z)dz = \int_a^b f(z_1(t))z_1'(t)dt$$

$$= \int_a^c f(z_1(t))z_1'(t)dt + \int_c^b f(z_1(t))z_1'(t)dt$$

$$= \int_b^{b-a+c} f(z_1(s+a-b))z_1'(s+a-b)ds + \int_c^b f(z_1(t))z_1'(t)dt$$

(change of variables $s = b - a + t$)

$$= \int_b^{b-a+c} f(z_2(s))z_2'(s)ds + \int_c^b f(z_2(t))z_2'(t)dt$$

$$= \int_{\Gamma_2} f(z)dz.$$

EXERCISES 4.3: Independence of Path

1. Theorem 7 allows us to antidifferentiate.

   a. $-3 + 2i$

   b. $e^{-1} - e = -2\sinh 1$

   c. πi

   d. 0

   e. $\dfrac{\sin^3 i}{3} = -i\dfrac{\sinh^3 1}{3}$

   f. $\dfrac{1}{2}e^i(\sin i + \cos i) + \dfrac{1}{2}e^\pi = \dfrac{1}{2}(\cos 1\cosh 1 - \sin 1\sinh 1 + e^\pi)$
   $+\dfrac{1}{2}i(\sin 1\cosh 1 + \cos 1\sinh 1)$

g. $\frac{1}{3}(-\sqrt{2}-2\pi^{3/2})+i\frac{\sqrt{2}}{3}$

h. $\pi-2+i(2-\frac{\pi^2}{4})$

i. $\frac{\pi}{4}-\frac{1}{2}\text{Arctan}\,2+\frac{i}{4}\text{Log}\,5$

2. $P(z)$ is continuous and has an antiderivative everywhere in the complex plane.

3. a. $\left|\int_\Gamma f(z)dz\right| = \left|\int_\Gamma [f(z)-P(z)]dz\right|$

$$\leq \epsilon l(\Gamma) \quad \text{(by Theorem 5)}$$

Taking $\epsilon \to 0$ it is clear that $\int_\Gamma f(z)dz = 0$.

b. By Theorem 7, f has an antiderivative F in **C**. $F'(z) = f(z)$ in **C**, so F is entire.

4. False: $\int_{|z|=1} \frac{1}{z}dz = 2\pi i$.

5. If the contour in Figure 4.21 is named Γ_1 and the contour in 4.22 is Γ_2, then $\Gamma_1 - \Gamma_2$ forms a closed loop. According to Example 2

$$\int_{\Gamma_1-\Gamma_2} \frac{1}{z}dz = \pi i - (-\pi i) = 2\pi i$$

If $f(z) = \frac{1}{z}$ had an antiderivative throughout $\mathbf{C}\backslash\{0\}$ this integral would be zero, by Corollary 2 to Theorem 6.

6. Let C_0 be the portion of C extending from α to β. Then,

$$\begin{aligned}\int_C \frac{1}{z-z_0}dz &= \lim_{\alpha,\beta\to\tau}\int_{C_0}\frac{1}{z-z_0}dz \\ &= \lim_{\alpha,\beta\to\tau}[\text{Log}(\beta-z_0)-\text{Log}(\alpha-z_0)] \\ &= \text{Log}\,|\tau-z_0| + i\pi - (\text{Log}\,|\tau-z_0| - i\pi) \\ &= 2\pi i\end{aligned}$$

7. Choose a branch cut (for the log) that does not intersect C. Then $\log(z - z_0)$ is an antiderivative of $\dfrac{1}{z - z_0}$ in the slit domain, and $\int_C \dfrac{1}{z - z_0} dz = 0$ by Theorem 7.

8. Let Γ be any loop in D. Write Γ as the sum of two contours $\Gamma_1 + \Gamma_2$. Then Γ_1 and $-\Gamma_2$ both begin and end at the same points z_I and z_T. Assuming Theorem 7(iii),

$$\int_{\Gamma_1} f(z)dz = \int_{-\Gamma_2} f(z)dz, \quad \text{so}$$

$$0 = \int_{\Gamma_1} f(z)dz - \int_{-\Gamma_2} f(z)dz = \int_{\Gamma} f(z)dz = 0$$

9. Different choices of z_0 will produce antiderivatives $F(z)$ that differ by a constant.

10. Choose $\epsilon > 0$. By continuity there is a $\delta > 0$ such that $|f(z + t\Delta z) - f(z)| < \epsilon$ when $|\Delta z| < \delta$. Thus,

$$\left|\int_0^1 f(z + t\Delta z)dt - f(z)\right| = \left|\int_0^1 f(z + t\Delta z)dt - \int_0^1 f(z)dt\right|$$

$$\leq \int_0^1 |f(z + t\Delta z) - f(z)|\, dt < \epsilon$$

whenever $|\Delta z| < \delta$.

11. $$\int_\Gamma [f'(z)g(z) + f(z)g'(z)]\, dz = \int_\Gamma \frac{d}{dz}[f(z)g(z)]\, dz$$

$$= f(z)g(z)\Big|_{z_I}^{z_T} \text{ by Theorem 6.}$$

In other words, $\int_\Gamma f'(z)g(z)dz = f(z)g(z)\Big|_{z_I}^{z_T} - \int_\Gamma f(z)g'(z)dz$

12. Let γ be the line segment joining z_1 and z_2. Applying Theorem 5, Section 4.2 we have

$$|f(z_2) - f(z_1)| = \left|\int_\gamma f'(z)dz\right| \leq Ml(\gamma) = M|z_2 - z_1|$$

EXERCISES 4.4: Cauchy's Integral Theorem

1. a and c

2. Let $z_0(s,t)$ and $z_1(s,t)$, $\quad 0 \le s \le 1, \quad 0 \le t \le 1$ be continuous deformations of Γ_0 and Γ_1 to respective points z_0 and z_1. Let $z_2(s,t)$ be a continuous deformation taking z_0 to z_1. Then $z_1(1-s,t)$ is a continuous deformation from z_1 to Γ_1 and

$$z(t) = \begin{cases} z_0(3s,t) & 0 \le s \le \frac{1}{3} \\ z_2(3s-1,t) & \frac{1}{3} \le s \le \frac{2}{3} \\ z_1(3-3s,t) & \frac{2}{3} \le s \le 1 \end{cases}$$

is a continuous deformation from Γ_0 to Γ_1.

3. a, b, d and e

4. $z(s,t) = \begin{cases} e^{4\pi i(1-s)t} & 0 \le t \le \frac{1}{2} \\ e^{4\pi i(1-s)(1-t)} & \frac{1}{2} \le t \le 1. \end{cases}$

$\displaystyle\int_{\Gamma_0} f(z)dz = \oint_{|z|=1} f(z)dz - \oint_{|z|=1} f(z)dz = 0.$

$\displaystyle\int_{z=1} f(z)dz = 0$ (integral over a single point)

5. $z(s,t) = (2-s)\cos 2\pi t + i(3-2s)\sin 2\pi t, \quad 0 \le s \le 1, \quad 0 \le t \le 1.$

6. Assuming path independence (condition (iii)), the vector field $\overline{f(z)} = u - iv$ has a potential $\phi(x,y)$ with $\dfrac{\partial \phi}{\partial x} = u$ and $\dfrac{\partial \phi}{\partial y} = -v$ and the vector field $i\overline{f(z)} = v + iu$ has a potential $\psi(x,y)$ with $\dfrac{\partial \psi}{\partial x} = v$ and $\dfrac{\partial \psi}{\partial y} = u$. Define $F(z) = \phi(x,y) + i\psi(x,y)$.

Then $F'(z) = \dfrac{\partial \phi}{\partial x} + i\dfrac{\partial \psi}{\partial x} = u + iv = f(z)$, so $F(z)$ is an antiderivative of $f(z)$.

7. a. For $f(z) = u + iv$ the vector field for $\overline{f(z)}$ is $(V_1, V_2) = (u, -v)$. Then, using the Cauchy-Riemann equations,

$$\frac{\partial V_1}{\partial y} = u_y = -v_x = \frac{\partial V_2}{\partial x}$$

$$\text{and } \frac{\partial V_1}{\partial x} = u_x = v_y = -\frac{\partial V_2}{\partial y}$$

b. This is true by Theorem 5, Section 2.4, since

$$u_x = \frac{\partial V_1}{\partial x} = -\frac{\partial V_2}{\partial y} = v_y$$

$$\text{and } u_y = \frac{\partial V_1}{\partial y} = \frac{\partial V_2}{\partial x} = -v_x.$$

8. Using the result of Problem 7, for $f(z)$ analytic both $\overline{f(z)}$ and $\overline{-if(z)} = i\overline{f(z)}$ are irrotational vector fields, so they have potentials $\phi(x,y)$ and $\psi(x,y)$ respectively. Then $F(z) = \phi(x,y) + i\psi(x,y)$ is an antiderivative of $f(z)$ as in Problem 6.

9. a, c, d, f

10. a. $\mathbf{C}\backslash\{-5i, 5i\}$

 b. $\mathbf{C}$

 c. $\mathbf{C}\backslash\{3-i, 3+i\}$

 d. $\mathbf{C}\backslash\{x+iy : x \leq -3, y = 0\}$

 e. $\mathbf{C}\backslash\{(1+2k)\pi, \quad k = 0, \pm 1, \pm 2, \ldots\}$

 In each case the integral is 0 by Cauchy's Integral Theorem (Theorem 9,12) because the functions are analytic in a simply connected domain containing $|z| = 2$.

11. e^{z^2} is entire so it has an antiderivative in $\mathbf{C}$ (which is simply connected) by Theorem 10,13.

12. If D *were* simply connected the result would violate the contrapositive statement of Cauchy's Integral Theorem (Theorem 9,12).

13. $I = \int \frac{1}{z^2+1} dz = \frac{i}{2} \int \frac{1}{z+i} dz - \frac{i}{2} \int \frac{1}{z-i} dz$

Along Γ_1: $\quad I = \frac{i}{2}(0) - \frac{i}{2}(2\pi i) = \pi$

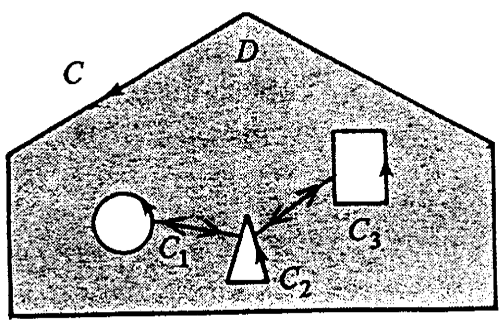
Along Γ_2: $\quad I = \frac{i}{2}(2\pi i) - \frac{i}{2}(2\pi i) = 0$

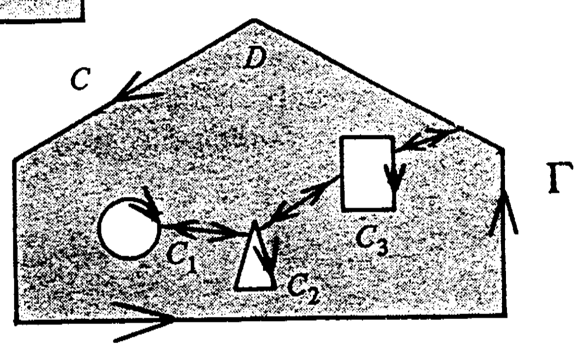
Along Γ_3: $\quad I = \frac{i}{2}(2\pi i) - \frac{i}{2}(0) = -\pi$

14. For readers of 4.4a: C can be continuously deformed to the contour of the figure on the left below. In evaluating the integral over the contour shown, the portions along the straight line segments cancel, leaving the stated result.

For readers of 4.4b: Refer to the figure on the right. Γ is a positively oriented contour inside which $f(z)$ is analytic, so

$0 = \int_\Gamma f(z)dz = \int_C f(z)dz - \int_{C_1} f(z)dz - \int_{C_2} f(z)dz - \int_{C_3} f(z)dz$





15. $\displaystyle\int_\Gamma \frac{z}{(z+2)(z-1)}dz = \frac{2}{3}\int_\Gamma \frac{1}{z+2}dz + \frac{1}{3}\int_\Gamma \frac{1}{z-1}dz$

$$= \frac{2}{3}(-4\pi i) + \frac{1}{3}(-4\pi i) = -4\pi i$$

16. $\displaystyle\oint_{|z|=1} f(z)dz = \oint_{|z|=1} \frac{A_k}{z^k}dz + \oint_{|z|=1} \frac{A_{k-1}}{z^{k-1}}dz + \cdots + \oint_{|z|=1} \frac{A_1}{z}dz + \oint_{|z|=1} g(z)dz$

$$= 0 + 0 + \cdots + 2\pi i A_1 + 0$$

The last integral is zero by Cauchy's integral theorem. The other integrals follow from Example 2, Section 4.2.

17. $\displaystyle\int_\Gamma \frac{2z^2-z+1}{(z-1)^2(z+1)}dz = \int_\Gamma \frac{1}{(z-1)^2}dz + \int_\Gamma \frac{1}{z-1}dz + \int_\Gamma \frac{1}{z+1}dz$

$$= 0 - 2\pi i + 2\pi i = 0$$

18. a. The singularities of the integrand all lie inside $|z| = 2$.

b. $\displaystyle\left|\oint_{|z|=R} \frac{dz}{z^2(z-1)^3}\right| \le \frac{1}{R^2(R-1)^3}2\pi R = \frac{2\pi}{R(R-1)^3}$

c. $\displaystyle\lim_{R\to\infty} \frac{2\pi}{R(R-1)^3} = 0$

d. $\displaystyle I = \lim_{R\to\infty} I(R) = 0$

19. Let $\displaystyle I := \oint_{|z|=r} \frac{1}{P(z)}dz$ where $P(z) := a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$.

Then $\displaystyle I = I(R) := \oint_{|z|=R} \frac{1}{P(z)}dz$ for $R > r$ (since all zeros of $P(z)$ are within $|z| = r$).

and

$$|I(R)| \le \frac{1}{\min\limits_{|z|=R} |P(z)|}2\pi R \quad \text{(by Theorem 5)}$$

$$\le \frac{2\pi R}{|a_n|R^n - |a_{n-1}|R^{n-1} - \cdots - |a_0|}$$

Since $n \ge 2$, $\displaystyle\lim_{R\to\infty} I(R) = 0 = I$.

20. (i) Partial Fractions: $\int_\Gamma dz/(z^4-1)$
$= \int_\Gamma\{(1/4)/(z-1) - 1/4)/(z+1) + i/4)/(z-i) - i/4)/(z+i)\}dz$
$= 2\pi i(1/4 - 1/4 + i/4 - i/4) = 0$
(ii) The contour Γ n Figure 4.51 can be continuously deformed into any circle of radius r>0. If r>1, $\int_\Gamma dz/(z^4-1) = 0$ as shown in Prob. 19.

EXERCISES 4.5: Cauchy's Integral Formula and Its Consequences

1. 0, since $f(z)/(z-z_0)$ is analytic inside and on Γ.

2. For z_0 inside Γ,
$$f(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\oint_\Gamma \frac{g(z)}{z-z_0}dz = g(z_0)$$

3. a. $f(z) = \sin 3z$ $\qquad 2\pi i f(\pi/2) = -2\pi i$

   b. $f(z) = \frac{1}{2}ze^z$ $\qquad 2\pi i f(3/2) = \frac{3\pi i}{2}e^{3/2}$

   c. $f(z) = \frac{\cos z}{z^2+9}$ $\qquad 2\pi i f(0) = \frac{2\pi i}{9}$

   d. $f(z) = 5z^2+2z+1$ $\qquad \frac{2\pi i}{2!}f''(i) = 10\pi i$

   e. $f(z) = e^{-z}$ $\qquad 2\pi i f'(-1) = -2\pi e i$

   f. $f(z) = \frac{\sin z}{z-4}$ $\qquad 2\pi i f'(0) = -\frac{\pi i}{2}$

4. a. $f(z) = \frac{z+i}{z+2}$ $\qquad 2\pi i f'(0) = \frac{\pi}{2} + \pi i$

   b. $f(z) = \frac{z+i}{z^2}$ $\qquad 2\pi i f(-2) = -\frac{\pi}{2} - \pi i$

   c. 0

5. Let $g(\zeta) = \zeta^2 - \zeta + 2$

Then $G(1) = 2\pi i g(1) = 4\pi i$
$G'(i) = 2\pi i g'(i) = -4\pi - 2\pi i$
$G''(-i) = 2\pi i g''(-i) = 4\pi i$

6. $$\oint_{|z-i|=r} \frac{e^{iz}/(z+i)^2}{(z-i)^2}dz + \oint_{|z+i|=r} \frac{e^{iz}/(z-i)^2}{(z+i)^2}dz \text{ where } r < 2$$
$$= 2\pi i\left(-\frac{e^{-1}i}{2}\right) + 2\pi i(0) = \frac{\pi}{e}$$

7. $f(z) = \dfrac{\cos z}{z-3} \quad 2\pi i f'(0) = -\dfrac{2\pi i}{9}$

8. If Γ is the positively oriented circle $|z - z_0| = r$ then Γ is parametrized by $z(\theta) = z_0 + re^{i\theta}, 0 \leq \theta \leq 2\pi$ and

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

More generally,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} ire^{i\theta}d\theta$$
$$= \frac{n!}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta}d\theta$$

9. $$|f^{(n)}(0)| = \left|\frac{n!}{2\pi i}\oint_{|z|=1} \frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right| \qquad \text{(by Cauchy's integral formula.)}$$
$$\leq \frac{n!}{2\pi}M2\pi = n!M \qquad \text{(by Theorem 5)}$$
It follows that $|f(0)| \leq M$ and $|f'(0)| \leq M$.

10. For z_0 inside Γ,
$2\pi i f'(z_0) = \oint_\Gamma \dfrac{f'(z)}{z-z_0}dz$ by Theorem 14.

$2\pi i f'(z_0) = \oint_\Gamma \frac{f(z)}{(z-z_0)^2}dz$ by Theorem 19.
For z_0 outside Γ both integrals are zero by Cauchy's integral theorem (Theorem 9, 12 of Section 4.4)

11. $\frac{\partial^2 u}{\partial x^2}$ is the real part of the analytic function $f''(z)$ (Theorem 16), so it is harmonic (Theorem 7, Section 2.5)

12. Let z be inside Γ. As in the proof of Theorem 15, it suffices to show that $J(z)$ approaches zero as $\Delta z \to 0$, where

$$J(z) := \frac{H(z+\Delta z) - H(z)}{\Delta z} - 2\int_\Gamma \frac{g(\zeta)}{(\zeta - z)^3}d\zeta$$

$$J(z) = \int_\Gamma \left[\frac{1}{\Delta z(\zeta - z - \Delta z)^2} - \frac{1}{\Delta z(\zeta - z)^2} - \frac{2}{(\zeta - z)^3}\right] g(\zeta)d\zeta$$

$$= \int_\Gamma \Delta z \left[\frac{3\zeta - 3z - 2\Delta z}{(\zeta - z - \Delta z)^2(\zeta - z)^3}\right] g(\zeta)d\zeta$$

Let d =the shortest distance $|\zeta - z|$ from z to Γ and let r be the largest. Let Δz be smaller than $d/2$. Let $M = \max_{\zeta \text{ on } \Gamma} |g(\zeta)|$.

Then $|J(z)| \le \int_\Gamma |\Delta z| \left[\frac{3|\zeta - z| + 2|\Delta z|}{(|\zeta - z| - |\Delta z|)^2 |\zeta - z|^3}\right] |g(\zeta)|d\zeta$

$$\le |\Delta z| \frac{3r + d}{(d/2)^2 d^3} M l(\Gamma) \to 0 \text{ as } \Delta z \to 0$$

13. For $|z| < 1$,

$$G(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{1}{\zeta(\zeta - z)}d\zeta = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{1}{z}\left[\frac{1}{\zeta - z} - \frac{1}{\zeta}\right] d\zeta$$

$$= \frac{1}{2\pi i z}(2\pi i - 2\pi i) = 0$$

Here $G(z)$ is zero inside Γ, while on the boundary $|g(\zeta)| = 1$, so that $\lim_{z\to\zeta} G(z) \neq g(\zeta)$. This does not violate Cauchy's formula because $g(\zeta) = \frac{1}{\zeta}$ is not analytic on any simply connected domain containing $\Gamma : |\zeta| = 1$.

14. a. $\cos(2+3i)$

 b. 0

15. By Theorem 15, $G(z) = \dfrac{1}{2\pi i}\displaystyle\oint_{|\zeta|=1} \dfrac{f(\zeta)}{\zeta(\zeta - z)} d\zeta$ is analytic for all z not on $|z| = 1$.

$G(0) = \dfrac{1}{2\pi i}\displaystyle\oint_{|\zeta|=1} \dfrac{f(\zeta)}{\zeta^2} d\zeta = f'(0) = F(0)$ (Theorem 19)

$$\text{For } z \neq 0, G(z) = \frac{1}{z}\left[\frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta} d\zeta\right]$$

$$= \frac{1}{z}(f(z) - f(0)) = F(z)$$

Therefore $G(z) = F(z)$ for any z in $|z| < 1$, and $F(z)$ is analytic on $|z| \leq 1$.

16. a. By Theorem 16, $f'(z)$ is analytic in D. By Theorem 3, Section 2.3 the quotient of analytic functions is analytic when the denominator is not zero.

 b. Theorem 10(13), Section 4.4

 c. $H'(z) = \dfrac{f'(z)}{f(z)}$, so

$$\frac{d}{dz}\left[f(z)e^{-H(z)}\right] = f'(z)e^{-H(z)} + f(z)e^{-H(z)}\left[\frac{-f'(z)}{f(z)}\right] = 0.$$

 Thus $f(z)e^{-H(z)} \equiv c$, since its derivative is zero. (Theorem 6, Section 2.4). Thus $f(z) = ce^{H(z)}$.

 d. $H(z) + \alpha$ is an analytic function in D and $f(z) = e^{H(z)+\alpha}$, so $H(z) + \alpha$ is a branch of $\log f(z)$.

17. z^3-1 is never zero in $|z|<1$, so there is a single-valued branch of $\log(z^3-1)$ analytic in $|z|<1$ by Problem 16. Using this branch define $(z^3-1)^{1/2} = \exp\left[\frac{1}{2}\log(z^3-1)\right]$.

EXERCISES 4.6: Bounds for Analytic Functions

1. On the circle $|z|=R$,
$\left|\frac{1}{(1-z)^2}\right| \le \frac{1}{(1-|z|)^2} \le \frac{1}{(1-R)^2}$, and this maximum is attained when $z=R$.

   $f^{(n)}(z) = (n+1)!(1-z)^{-n-2}$ gives $f^{(n)}(0) = (n+1)!$ so by the Cauchy estimate (Theorem 20), $(n+1)! = \left|f^{(n)}(0)\right| \le \frac{n!}{R^n(1-R)^2}$

2. For $|z| \le R$, $|f(z)| \le 1/(1-R)$. By the Cauchy estimate (Theorem 20), $|f^{(n)}(0)| \le \frac{n!}{R^n(1-R)}$.
To find the minimum value of this estimate use elementary calculus.

$$0 = \frac{d}{dR}\left[\frac{n!}{R^n(1-R)}\right] = \frac{n!R^n - n!nR^{n-1}(1-R)}{R^{2n}(1-R)^2}$$
$$\Longrightarrow 0 = n!R^{n-1}[(n+1)R-n] \Longrightarrow R = \frac{n}{n+1}.$$

   This is the minimum because the upper bound approaches ∞ as R approaches 0 or 1.

3. Fix z. The circle $|\zeta - z| = r - |z|$ lies inside the disk $|\zeta| \le r$, so $\max\limits_{|\zeta-z|=r-|z|} |f(\zeta)| \le M$ (Theorem 24). Applying Theorem 20 to the circle of radius $r-|z|$ centered at z yields
$\left|f^{(n)}(z)\right| \le \frac{n!M}{(r-|z|)^n}$

4. Applying Theorem 20,
$\left|p^{(k)}(0)\right| = |k!a_k| \le \frac{k!M}{1}$
Hence $|a_k| \le M$.

5. $|e^{f(z)}| = e^{Re(f)} \le e^M$
e^f is a bounded entire function, so it is constant (and f is constant) by Liouville's Theorem.

6. By Liouville's theorem, $f^{(4)}$ is a constant function. One can antidifferentiate four times to obtain f equal to a polynomial of degree at most five. (Theorem 6, Sec. 2.4 guarantees that each antiderivative is unique up to a constant of integration.)

7. Fix z_0 and choose $R > |z_0| + r_0$. Then the circle $|z - z_0| = R$ lies outside the circle $|z| = r_0$, and one can apply Cauchy's estimate on the circle $|z - z_0| = R$ to get

$$|f^{(n)}(z_0)| \le \frac{n! \max\limits_{|z-z_0|=R} |z|^2}{R^n} = \frac{n!(|z_0| + R)^2}{R^n}$$

For $n \ge 3$, letting R approach ∞ yields $f^{(n)}(z_0) = 0$, so that $f^{(n)} \equiv 0$. Hence f is a polynomial of degree at most 2.

In general, if f is entire and $|f(z)| \le M|z|^n$ for $|z| > r_0$, where n is a nonnegative integer, then f must be a polynomial of degree at most n.

8. $f(z)/3z^2$ is analytic in $1 \le |z| \le 2$ with $|f(z)/3z^2| \le 1$ on both $|z| = 1$ and $|z| = 2$. By the maximum modulus principle (Theorem 24), $|f(z)/3z^2| \le 1$ for $1 \le |z| \le 2$, which yields $|f(z)| \le 3|z|^2$ for $1 \le |z| \le 2$.

9. Suppose $|f(z_0)| > |f(z_1)|$ for some z_1 on C_R. By the continuity of f there is an $\epsilon > 0$ such that $|f(z_0 + Re^{it})| \le |f(z_0)| - \epsilon$ over some interval $t_0 \le t \le t_1$. Then,

$$\begin{aligned}|f(z_0)| &\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + Re^{it})|dt \\ &= \frac{1}{2\pi}\int_0^{t_0} |f(z_0 + Re^{it})|dt + \frac{1}{2\pi}\int_{t_0}^{t_1} |f(z_0 + Re^{it})|dt \\ &\quad + \frac{1}{2\pi}\int_{t_1}^{2\pi} |f(z_0 + Re^{it})|dt\end{aligned}$$

$$\leq \frac{1}{2\pi}\left[|f(z_0)|t_0 + (|f(z_0)| - \epsilon)(t_1 - t_0) + |f(z_0)|(2\pi - t_1)\right]$$

$$< |f(z_0)|.$$

This contradiction $|f(z_0)| < |f(z_0)|$ shows that no such z_1 can exist.

10. $f(z) \equiv i$ by Theorem 23.

11. Suppose $f(z_0) = 0$ for some z_0 inside C. Let $g(z) = f(z) - 1$. Then $|g(z_0)| = 1$. By Theorem 23, $g(z) \equiv -1$. This contradicts $|g(z)| < 1$ for all z on C. Thus f can have no zeros inside C.

12. Suppose $|f(z)|$ achieves its maximum value at a point z_0 in the domain D where f is analytic. Then $f(z_0)$ is not an interior point of $f(D)$, so $f(D)$ is not open and f must be constant.

13. $f(z)/g(z)$ is analytic in D, so it attains its maximum value of 1 on the boundary B. Thus $|f(z)| \leq |g(z)|$ for all z in D.

14. If f is nonzero then $1/f(z)$ is analytic in D and $|1/f(z)|$ attains its maximum value on the boundary of D, so that $|f(z)|$ attains its minimum value on the boundary of D.

    To see why f must be nonzero, consider $f(z) = z$ on the unit disk $|z| < 1$. The modulus $|f(z)|$ attains its minimum at $z = 0$, not on the boundary $|z| = 1$.

15. If f does not have a zero in D then $|f(z)|$ attains both its maximum and minimum on B (Theorem 24, Problem 14). Thus $|f(z)|$ is constant in D, and $f(z)$ is constant (Problem 12, Section 2.4). This is a contradiction, so f must have a zero in D.

16. Certainly $\max\limits_{|z|\leq 1} |az^n + b| \leq |a|1^n + |b|$.

    To see that this maximum is attained, let $a = |a|e^{i\theta_1}$ and $b = |b|e^{i\theta_2}$. Substituting $z_0 = e^{i(\theta_2 - \theta_1)/n}$ gives

$$|az_0^n + b| = \left||a|e^{i\theta_1}e^{i(\theta_2-\theta_1)} + |b|e^{i\theta_2}\right| = |a| + |b|.$$

17. The maximum M is on the boundary, so one can substitute $z = e^{i\theta}$ to get

$$M^2 = \max_{0\leq\theta\leq 2\pi}\left|(e^{i\theta} - 1)(e^{i\theta} + \frac{1}{2})\right|^2$$

$$= \max_{0\leq\theta\leq 2\pi}\left(-2\cos^2\theta - \frac{1}{2}\cos\theta + \frac{5}{2}\right).$$

Using calculus, we find the critical points from
$0 = 4\cos\theta\sin\theta + \frac{1}{2}\sin\theta = \sin\theta\left(4\cos\theta + \frac{1}{2}\right).$
Thus, $\theta = 0, \pi, 2\pi, \arccos\left(-\frac{1}{8}\right)$ and $2\pi - \arccos\left(-\frac{1}{8}\right)$. The maximum $M = \frac{9\sqrt{2}}{8}$ is attained at $\theta = \arccos\left(-\frac{1}{8}\right)$ and $\theta = 2\pi - \arccos\left(-\frac{1}{8}\right)$.

18. With $P(z) = c(z - z_1)(z - z_2)\cdots(z - z_n)$, apply the product rule to get

$$\frac{P'(z)}{P(z)} = \frac{\sum_{k=1}^{n} c(z - z_1)\cdots(z - z_{k-1})(z - z_{k+1})\cdots(z - z_n)}{c(z - z_1)(z - z_2)\cdots(z - z_n)}$$

$$= \sum_{k=1}^{n} \frac{1}{z - z_k}$$

Now $\frac{1}{2\pi i}\int_\Gamma \frac{dz}{z - z_k} = \begin{cases} 1 & \text{if } z_k \text{ is inside } \Gamma \\ 0 & \text{otherwise} \end{cases}$

Thus

$$\frac{1}{2\pi i}\int_\Gamma \frac{P'(z)}{P(z)}dz = \frac{1}{2\pi i}\sum_{k=1}^{n}\int_\Gamma \frac{dz}{z - z_k}$$

$= \text{the number of zeros } z_k \text{ (counting multiplicity) that lie inside } \Gamma.$

19.
$$
\begin{aligned}
Q(z) &= z^n P(1/z) \\
&= z^n\left(\frac{1}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \cdots + \frac{a_1}{z} + a_0\right) \\
&= 1 + a_{n-1}z + \cdots + a_0 z^n
\end{aligned}
$$

Note that $Q(0) = 1$ and $\max\limits_{|z|=1}|Q(z)| = \max\limits_{|z|=1}|P(1/z)| = \max\limits_{|w|=1}|P(w)|$.
By Theorem 24, $\max\limits_{|z|=1}|Q(z)| \geq |Q(0)| = 1$, so $\max\limits_{|z|=1}|P(z)| \geq 1$.

EXERCISES 4.7: Applications to Harmonic Functions

1. $\phi(z) \equiv -5$

2. Consider $\mathrm{Log}\,|z|$. By Problem 20, Section 2.5, this is harmonic on $\mathbf{C}\backslash\{0\}$ but it is not the real part of a function analytic in $\mathbf{C}\backslash\{0\}$.

3. Theorem: *If ϕ is harmonic in a domain D and it achieves its maximum (or minimum) value at a point z_0 in D, then ϕ is constant in D.*

   Proof of maximum principle (minimum is similar):

   Suppose ϕ is not constant. Then there must be a point z_1 in D such that $\phi(z_1) < \phi(z_0)$. Let γ be a path in D from z_0 to z_1.

   There is a point w on γ where ϕ begins to decrease. That is,
   (i) $\phi(z) = \phi(z_0)$ for all z preceding w on γ.
   and (ii) There are points z on γ, arbitrarily close to w, where $\phi(z) < \phi(z_0)$.

   By the continuity of ϕ, $\phi(w) = \phi(z_0)$. There is a disk centered at w which lies entirely within D. By Theorem 26, ϕ is constant in this disk. However, this contradicts part (ii) of the definition of w. Hence ϕ must be constant on all of D.

4. The highest and lowest temperatures of a solid in a steady-state occur on its surface (boundary).

5. $\phi_1(x, y) = y$ and $\phi_2(x, y) \equiv 0$ are two functions that are harmonic in the upper half-plane; each vanishes on the real axis.

6. Using Poisson's integral formula (Theorem 29),

$$\phi(0) = \frac{\rho^2}{2\pi}\int_0^{2\pi}\frac{\phi(\rho e^{it})}{\rho^2}dt = \frac{1}{2\pi}\int_0^{2\pi}\phi(\rho e^{it})dt$$

7. $$\int_0^R \phi(0)\rho d\rho \;= \frac{1}{2\pi}\int_0^R\int_0^{2\pi}\phi(\rho e^{it})\rho dt d\rho$$

$$\implies \phi(0)\frac{R^2}{2} = \frac{1}{2\pi}\iint_D \phi(x,y)dxdy$$

$$\implies \phi(0) \;= \frac{1}{\pi R^2}\iint_D \phi(x,y)dxdy$$

8. This is Poisson's integral formula (Theorem 29) with $\phi \equiv 1$, which is harmonic in the domain $|z| < R$.

9. Since $|\cos x| \le 1$,

$$R^2 - 2rR + r^2 \le R^2 + r^2 - 2rR\cos(t-\theta) \le R^2 + 2rR + r^2$$

and $\dfrac{1}{(R+r)^2} \le \dfrac{1}{R^2+r^2-2rR\cos(t-\theta)} \le \dfrac{1}{(R-r)^2}$.

Since ϕ is nonnegative,

$$\frac{\phi(Re^{it})}{(R+r)^2} \le \frac{\phi(Re^{it})}{R^2+r^2-2rR\cos(t-\theta)} \le \frac{\phi(Re^{it})}{(R-r)^2}$$

Multiplying by $(R^2-r^2)/2\pi$ and integrating with respect to t yields

$$\frac{R-r}{R+r}\frac{1}{2\pi}\int_0^{2\pi}\phi(Re^{it})dt \le \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{\phi(Re^{it})}{R^2+r^2-2rR\cos(t-\theta)}dt$$

$$\le \frac{R+r}{R-r}\frac{1}{2\pi}\int_0^{2\pi}\phi(Re^{it})dt.$$

Now apply Problem 6 to the ends and Poisson's integral formula (Theorem 29) to the middle to obtain

$$\phi(0)\frac{R-r}{R+r} \le \phi(re^{i\theta}) \le \phi(0)\frac{R+r}{R-r}.$$

10. Suppose ϕ is bounded below by M. Then $\phi + M$ is a nonnegative harmonic function, and Harnack's inequality can be applied to get

$$(\phi(0) + M)\frac{R-r}{R+r} \le \phi(re^{i\theta}) + M \le (\phi(0) + M)\frac{R+r}{R-r} \text{ when } 0 \le r < R$$

Letting $R \to \infty$ gives
$\phi(0) + M \le \phi(re^{i\theta}) + M \le \phi(0) + M$ for any r.
That is, $\phi(z) \equiv \phi(0)$.
The case when ϕ is bounded above can be handled by replacing ϕ with $-\phi$.

11. Using Problem 6,
$$\phi(0) = \frac{1}{2\pi}\left[\frac{\pi}{2}(0) + \frac{\pi}{2}(2) + \frac{\pi}{2}(4) + \frac{\pi}{2}(6)\right] = 3.$$

12 a. Substituting $\zeta = Re^{it}$ and $z = re^{i\theta}$ yields

$$P(\zeta, z) = \frac{R^2 - r^2}{|Re^{it} - re^{i\theta}|^2}$$

$$= \frac{R^2 - r^2}{(R\cos t - r\cos\theta)^2 + (R\sin t - r\sin\theta)^2}$$

$$= \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(t-\theta)}$$

The result follows.

b. $$\frac{\zeta + z}{\zeta - z} = \frac{(\zeta + z)(\overline{\zeta} - \overline{z})}{|\zeta - z|^2}$$

$$= \frac{|\zeta|^2 - |z|^2 + z\overline{\zeta} - \overline{z}\zeta}{|\zeta - z|^2}$$

$$= P(\zeta, z) + 2i\frac{\mathrm{Im} z\overline{\zeta}}{|\zeta - z|^2}$$

c. $H(z) = \dfrac{1}{2\pi i}\oint_{|\zeta|=R}\dfrac{u(\zeta)}{\zeta - z}d\zeta + \dfrac{z}{2\pi i}\oint_{|\zeta|=R}\dfrac{u(\zeta)/\zeta}{\zeta - z}d\zeta$

Since $u(\zeta)$ and $u(\zeta)/\zeta$ are continuous on $|\zeta| = R$, Theorem 15 can be applied to show that $H(z)$ is analytic.

d. $$\text{Re}[H(z)] = \text{Re}\left[\frac{1}{2\pi i}\int_0^{2\pi}\frac{Re^{it}+z}{Re^{it}-z}\frac{u(Re^{it})}{Re^{it}}iRe^{it}dt\right]$$

$$= \text{Re}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{Re^{it}+z}{Re^{it}-z}u(Re^{it})dt\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi}P(Re^{it},z)u(Re^{it})dt$$

e. Since the real parts of $f(z)$ and $H(z)$ coincide, $f(z) - H(z)$ is a purely imaginary analytic function. By Problem 8 in Sec. 2.4, $f(z) - H(z)$ is a constant iC.

Now $$H(0) = \frac{1}{2\pi i}\oint_{|\zeta|=R}\frac{u(\zeta)}{\zeta}d\zeta$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\frac{u(Re^{it})}{Re^{it}}Rie^{it}dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}u(Re^{it})dt$$

$= u(0)$ by the circumferential mean-value theorem.

Plug $z = 0$ into $f(z) = H(z) + iC$ to get $u(0) + iv(0) = u(0) + iC$. Hence $C = v(0)$.
Then

$$f(z) = H(z) + iv(0)$$
$$= \frac{1}{2\pi i}\oint_{|\zeta|=R}\frac{\zeta+z}{\zeta-z}\frac{u(\zeta)}{\zeta}d\zeta + iv(0) \quad \text{for } |z| < R$$

f. $v(z) = \operatorname{Im} f(z)$

$$= \operatorname{Im}\left[\frac{1}{2\pi i}\oint_{|\zeta|=R}\frac{\zeta+z}{\zeta-z}\frac{u(\zeta)}{\zeta}d\zeta + iv(0)\right]$$

$$= \operatorname{Im}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{Re^{it}+z}{Re^{it}-z}u(Re^{it})dt + iv(0)\right] \text{ (as in part d.)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{2\operatorname{Im} z\,Re^{-it}}{|Re^{it}-z|^2}u(Re^{it})dt + v(0) \text{ (using b.)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} Q(Re^{it}, z)u(Re^{it})dt + v(0)$$

13. a. $|f(z)| = \left|\frac{1}{2\pi i}\oint_{|\zeta|=2|z|}\frac{\zeta+z}{\zeta-z}\frac{u(\zeta)}{\zeta}d\zeta\right|$

$$\leq \frac{1}{2\pi}\frac{2|z|+|z|}{2|z|-|z|}\frac{M|z|^2}{2|z|}4\pi|z| \qquad \text{(by Theorem 5, Sec. 4.2)}$$

$$= 3M|z|^2$$

b. By the proof of Prob. 7, Sec. 4.6 (which works for $|f(z)| \leq C|z|^2$), f is a polynomial of degree at most 2.

14. a. Cauchy's formula (Theorem 14, Section 4.5)

b. $\overline{z}$ lies outside Γ_R, so Cauchy's theorem (Theorem 9,12 of Section 4.4) applies.

c. $f(z) = \frac{1}{2\pi i}\int_{\Gamma_R} f(\zeta)\left[\frac{1}{\zeta-z} - \frac{1}{\zeta-\overline{z}}\right]d\zeta$

$$= \frac{1}{2\pi i}\int_{\Gamma_R} f(\zeta)\frac{2i\operatorname{Im} z}{(\zeta-z)(\zeta-\overline{z})}d\zeta$$

$$= \frac{1}{2\pi i}\int_{-R}^{R} f(\xi)\frac{2i\operatorname{Im} z}{|\xi-z|^2}d\xi + \frac{1}{2\pi i}\int_{C_R^+} f(\zeta)\frac{2i\operatorname{Im} z}{(\zeta-z)(\zeta-\overline{z})}d\zeta$$

(Note that $\xi = \overline{\xi}$ on the real axis).

d. $$\left|\frac{1}{2\pi i}\int_{C_R^+} f(\zeta)\ \frac{2i\,\mathrm{Im}\, z}{(\zeta - z)(\zeta - \overline{z})}d\zeta\right|$$

$$\leq \frac{1}{\pi}\max_{\zeta \text{ on } C_R^+}\left|\frac{f(\zeta)\ \mathrm{Im}\ z}{(\zeta - z)(\zeta - \overline{z})}\right|\pi R$$

$$\leq \frac{1}{\pi}\ \mathrm{Im}\ z\frac{\max_{\zeta \text{ on } C_R^+}|f(\zeta)|}{(R - |z|)^2}\pi R$$

$$\leq \frac{K}{\pi}\frac{\mathrm{Im}\ z}{(R - |z|)^2}\pi R$$

e. Since the bound in part d approaches zero as R approaches ∞,

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)y}{|\xi - (x + iy)|^2}d\xi.$$

The real part of this is

$$\phi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\phi(\xi,0)}{(\xi - x)^2 + y^2}d\xi \quad (y > 0).$$

15. $$\phi(x,y) = \frac{y}{\pi}\int_{-1}^{1}\frac{1}{(\xi - x)^2 + y^2}d\xi$$

$$= \frac{1}{\pi}\arctan\left(\frac{\xi - x}{y}\right)\Bigg|_{\xi=-1}^{1}$$

$$= \frac{1}{\pi}\arctan\left(\frac{1 - x}{y}\right) - \frac{1}{\pi}\arctan\left(\frac{-1 - x}{y}\right)$$

16. a. $$f(z) = \frac{1}{2\pi i}\int_{\Gamma_R} f(\zeta)\left[\frac{1}{\zeta - z} + \frac{1}{\zeta - \overline{z}}\right]d\zeta$$

$$= \frac{1}{2\pi i}\int_{\Gamma_R} f(\zeta)\frac{(2\zeta - 2\mathrm{Re}\ z)}{(\zeta - z)(\zeta - \overline{z})}d\zeta$$

$$= \frac{1}{2\pi i}\int_{-R}^{R} f(\xi)\frac{2(\xi - x)}{|\xi - z|^2}d\xi + \frac{1}{2\pi i}\int_{C_R^+} f(\zeta)\frac{2(\zeta - x)}{(\zeta - z)(\zeta - \overline{z})}d\zeta$$

b. $$\left|\frac{1}{2\pi i}\int_{C_R^+} f(\zeta)\ \frac{2(\zeta - x)}{(\zeta - z)(\zeta - \overline{z})}d\zeta\right|$$

$$\leq \frac{1}{\pi}\max_{\zeta \text{ on } C_R^+}\left|\frac{f(\zeta)(\zeta - x)}{(\zeta - z)(\zeta - \overline{z})}\right|\pi R$$

$$\leq \frac{2R}{(R - |z|)^2}\max_{\zeta \text{ on } C_R^+}|f(\zeta)|\, R$$

$$\leq \frac{2R^2}{(R - |z|)^2}\frac{K}{R^\alpha}$$

As R approaches ∞, this bound approaches zero.

Thus $f(z) = \dfrac{1}{\pi i}\displaystyle\int_{-\infty}^{\infty} f(\xi)\frac{\xi - x}{|\xi - z|^2}d\xi$ for $\operatorname{Im} z > 0$.

c. Taking the imaginary part of the last equation in part b. gives
$v(x,y) = \operatorname{Im}\left[\dfrac{1}{\pi i}\displaystyle\int_{-\infty}^{\infty}(u + iv)\frac{\xi - x}{|\xi - z|^2}d\xi\right] = -\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} u(\xi,0)\frac{\xi - x}{|\xi - z|^2}d\xi$
for $\operatorname{Im} z > 0$.

d. Combining the results of Problem 14 and part c of this problem,

$f(z) = u(x,y) + iv(x,y)$

$$= \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{u(\xi,0)}{|\xi - z|^2}d\xi - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{u(\xi,0)(\xi - x)}{|\xi - z|^2}d\xi$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(\xi,0)(\xi - x + iy)(-i)}{|\xi - z|^2}d\xi$$

$$= \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{u(\xi,0)}{\xi - z}d\xi \quad \text{for } \operatorname{Im} z > 0$$

17. a. $-\overline{f(\overline{z})} = \overline{\dfrac{-1}{\pi i}\displaystyle\int_{-\infty}^{\infty}\frac{u(\xi,0)}{\xi - \overline{z}}\,d\xi}$

$$= \frac{-1}{-\pi i}\int_{-\infty}^{\infty} \frac{u(\xi,0)}{\xi - \overline{\overline{z}}}\, d\xi$$

$$= \tilde{f}(z)$$

To show analyticity, let $f(z) = u(x,y) + iv(x,y)$ and $\tilde{f}(z) = \tilde{u}(x,y) + i\tilde{v}(x,y)$. Then $\tilde{f}(z) = -\overline{f(\overline{z})}$ implies $\tilde{u}(x,y) = -u(x,-y)$ and $\tilde{v}(x,y) = v(x,-y)$.

It follows that

$$\left.\frac{\partial \tilde{u}}{\partial x}\right|_{(x,y)} = -\left.\frac{\partial u}{\partial x}\right|_{(x,-y)} = -\left.\frac{\partial v}{\partial y}\right|_{(x,-y)} = \left.\frac{\partial \tilde{v}}{\partial y}\right|_{(x,y)}$$

and

$$\left.\frac{\partial \tilde{u}}{\partial y}\right|_{(x,y)} = \left.\frac{\partial u}{\partial y}\right|_{(x,-y)} = -\left.\frac{\partial v}{\partial x}\right|_{(x,-y)} = -\left.\frac{\partial \tilde{v}}{\partial x}\right|_{(x,y)}$$

Since $\tilde{f}(z)$ satisfies the Cauchy-Riemann equations when $\operatorname{Im} z < 0$, it is analytic there.

b. The change is

$$\lim_{y\to 0^+}[\tilde{f}(x-iy) - f(x+iy)] = \lim_{y\to 0^+}\left[-\overline{f(x+iy)} - f(x+iy)\right]$$

$$= -\overline{f(x+i0)} - f(x+i0)$$

$$= -2u(x,0)$$

18. Taking $f(z) = \dfrac{1}{z+i} = \dfrac{x - i(y+1)}{x^2+(y+1)^2}$, the formula in Problem 16, part d. gives

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\xi/(\xi^2+(0+1)^2)}{\xi - z}\, d\xi$$

$$= \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\xi}{(\xi^2+1)(\xi - z)}\, d\xi \text{ for } \operatorname{Im} z > 0$$

From Problem 17, taking $\tilde{f}(z) = -\overline{f(\overline{z})} = \dfrac{-1}{z-i}$ yields

$$\frac{-1}{z-i} = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\xi}{(\xi^2+1)(\xi - z)}\, d\xi \text{ for } \operatorname{Im} z < 0$$

By Prob. 17, part b, the jump in the value of the integral is $-2x/(x^2+1)$.

CHAPTER 5: Series Representations for Analytic Functions

EXERCISES 5.1: Sequences and Series

1. a. $\dfrac{1}{1-i/3}=\dfrac{9+3i}{10}$

   b. $\dfrac{3}{1-\left(\frac{1}{1+i}\right)}=3-3i$

   c. $\dfrac{1}{1+2/3}=\dfrac{3}{5}$

   d. $\left(\dfrac{1}{2i}\right)^{14}\left(\dfrac{1}{1-\frac{1}{2i}}\right)=\dfrac{-2+i}{2^{13}\cdot 5}$

   e. $\dfrac{1}{1-\frac{1}{9}}=\dfrac{9}{8}$

   f. -1 (telescoping series)

2. a. $\displaystyle\lim_{j\to\infty}\left|\frac{1}{(j+1)!}\frac{j!}{1}\right|=\lim_{j\to\infty}\frac{1}{j+1}=0$

   b. $\displaystyle\lim_{k\to\infty}\left|\frac{(3+i)^{k+1}}{(k+1)!}\frac{k!}{(3+i)^k}\right|=\lim_{k\to\infty}\left|\frac{3+i}{k+1}\right|=0$

   c. $\displaystyle\lim_{j\to\infty}\left|\frac{(j+1)^2}{4^{j+1}}\frac{4^j}{j^2}\right|=\lim_{j\to\infty}\left|\frac{(j+1)^2}{4j^2}\right|=\frac{1}{4}$

   d. $\displaystyle\lim_{k\to\infty}\left|\frac{(k+1)!}{(k+1)^{k+1}}\frac{k^k}{k!}\right|=\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k=\lim_{k\to\infty}\frac{1}{\left(1+\frac{1}{k}\right)^k}=\frac{1}{e}$

3. Suppose $\lim\limits_{n\to\infty}z_n=L$. Furthermore, suppose that given $\epsilon>0$, $n>N$ implies $|z_n-L|<\epsilon/2$.

   Then whenever $n>N+1$,

   $$|z_n-z_{n-1}|=|(z_n-L)-(z_{n-1}-L)|\le|z_n-L|+|z_{n-1}-L|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

   Therefore $\lim\limits_{n\to\infty}(z_n-z_{n-1})=0$

4. For $|z| = |z_0|, \quad z \neq z_0$,

$$\lim_{n\to\infty} \left| \left(\frac{z}{z_0}\right)^n - \left(\frac{z}{z_0}\right)^{n-1} \right| = \left| \frac{z}{z_0} - 1 \right| > 0.$$

The sequence diverges as a result of Problem 3.

For $|z| > |z_0|, \quad \lim_{n\to\infty} \left| \left(\frac{z}{z_0}\right)^n \right| = \infty$ so the sequence diverges.

5. If $S_n = \sum_{j=0}^{n} c_j$ converges, then $\lim_{n\to\infty} c_n = \lim_{n\to\infty} (S_n - S_{n-1}) = 0$ by Problem 3.

6. $\lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} c^n \neq 0.$

7. a. diverges.
   b. converges.
   c. diverges.
   d. diverges.
   e. converges.
   f. diverges.

8. a. Given $\epsilon > 0$ there is an N so that $n > N$ implies $\left| \sum_{j=0}^{n} c_j - S \right| < \epsilon$.

   Then $\left| \sum_{j=0}^{n} \overline{c_j} - \overline{S} \right| = \left| \sum_{j=0}^{n} c_j - S \right| < \epsilon$ when $n > N$.

   Thus $\sum_{j=0}^{\infty} \overline{c_j} = \overline{S}$.

   b. Given $\epsilon > 0$ there is an N so that $n > N$ implies $\left| \sum_{j=0}^{n} c_j - S \right| < \frac{\epsilon}{|\lambda|}$.

   Then $\left| \sum_{j=0}^{n} \lambda c_j - \lambda S \right| = |\lambda| \left| \sum_{j=0}^{n} c_j - S \right| < \epsilon$ when $n > N$.

   Thus $\sum_{j=0}^{\infty} \lambda c_j = \lambda S$.

c. Given $\epsilon > 0$ there is an N so that $n > N$ implies
$\left|\sum_{j=0}^{n} c_j - S\right| < \frac{\epsilon}{2}$ and $\left|\sum_{j=0}^{n} d_j - T\right| < \frac{\epsilon}{2}$.
Then
$$\left|\sum_{j=0}^{n}(c_j + d_j) - (S+T)\right| \le \left|\sum_{j=0}^{n} c_j - S\right| + \left|\sum_{j=0}^{n} d_j - T\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$
Thus $\sum_{j=0}^{\infty}(c_j + d_j) = S + T$.

9. Suppose $\sum_{j=0}^{\infty} z_j = S$.

Then $\sum_{j=0}^{\infty} \operatorname{Re}(z_j) = \sum_{j=0}^{\infty} \frac{1}{2}(z_j + \overline{z_j}) = \operatorname{Re}(S)$ by Problem 8.

Similarly, $\sum_{j=0}^{\infty} \operatorname{Im}(z_j) = \sum_{j=0}^{\infty} \frac{1}{2i}(z_j - \overline{z_j}) = \operatorname{Im}(S)$.

Conversely, if $\sum_{j=0}^{\infty} \operatorname{Re}(z_j) = P$ and $\sum_{j=0}^{\infty} \operatorname{Im}(z_j) = Q$ then

$\sum_{j=0}^{\infty} z_j = \sum_{j=0}^{\infty}[\operatorname{Re}(z_j) + i\operatorname{Im}(z_j)] = P + iQ$ by Problem 8.

10. For $|z| < 3$, $\lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} \frac{z^n/3^n}{z^n/3^n - 1} = 0$.

For $|z| > 3$, $\lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} \frac{1}{1 - 3^n/z^n} = 1$.

11. a. $\lim_{j\to\infty} \left|\frac{(j+1)z^{j+1}}{jz^j}\right| = |z|$. The series converges when $|z| < 1$.

b. $\lim_{k\to\infty} \left|\frac{(z-i)^{k+1}}{2^{k+1}} \frac{2^k}{(z-i)^k}\right| = \left|\frac{z-i}{2}\right|$. The series converges when $|z - i| < 2$.

c. $\lim_{j\to\infty} \left|\frac{z^{j+1}}{(j+1)!} \frac{j!}{z^j}\right| = \lim_{j\to\infty} \left|\frac{z}{j+1}\right| = 0$. The series converges for all z.

d. $\lim_{k\to\infty}\left|\dfrac{(z+5i)^{2k+2}}{(z+5i)^{2k}}\dfrac{(k+2)^2}{(k+1)^2}\right| = \left|(z+5i)^2\right|$. The series converges when $|z+5i|<1$.

12. Choose $\epsilon > 0$ and suppose $|z| \le R$. For $n > \dfrac{R+3}{\epsilon}$,

$$|F_n(z) - z| = \left|\frac{nz}{n+1} + \frac{3}{n} - z\right| = \left|\frac{-z}{n+1} + \frac{3}{n}\right| \le \frac{|z|}{n} + \frac{3}{n} \le \frac{R+3}{n} < \epsilon.$$

Thus $F_n(z)$ converges uniformly to $F(z) = z$ on $|z| \le R$.

13. For $j \ge 2$,
$\dfrac{1}{j^p} \le \displaystyle\int_{j-1}^{j} \frac{dx}{x^p}.$
Moreover, $\displaystyle\sum_{j=2}^{n}\int_{j-1}^{j}\frac{dx}{x^p} = \int_1^n \frac{dx}{x^p} = \left.\frac{x^{-p+1}}{-p+1}\right]_1^n = \frac{n^{-p+1}-1}{-p+1}.$
These partial sums converge to $\dfrac{1}{p-1}$ as $n \to \infty$. By the comparison test, $\displaystyle\sum_{j=1}^{\infty}\frac{1}{j^p}$ also converges.

14. a. $\left|\dfrac{1}{j(j+i)}\right| \le \dfrac{1}{j^2}$

b. $\left|\dfrac{\sin(k^2)}{k^{3/2}}\right| \le \dfrac{1}{k^{3/2}}$

c. $\left|\dfrac{k^2 i^k}{k^4+1}\right| \le \dfrac{1}{k^2}$

d. $\left|(-1)^k\dfrac{5k+8}{k^3-1}\right| \le \dfrac{6k}{\frac{1}{2}k^3} = \dfrac{12}{k^2}$ for $k \ge 3$

15. If $L > 1$ then $\lim_{j\to\infty} c_j \ne 0$ and the series diverges by Problem 5. If $L < 1$ then take ϵ so that $0 < \epsilon < 1 - L$ and choose J such that $j > J$ implies
$\left|\dfrac{c_{j+1}}{c_j}\right| < L + \epsilon < 1.$
Then $|c_{J+1}| < |c_J|\,(L+\epsilon)$.
$|c_{J+2}| < |c_{J+1}|\,(L+\epsilon) < |c_J|\,(L+\epsilon)^2$ and, proceeding inductively,

$|c_k| < |c_J|\,(L+\epsilon)^{k-J}$ for $k > J$.

Now $\sum_{k=J+1}^{\infty} c_J(L+\epsilon)^{k-J}$ is a convergent geometric series (with ratio $L+\epsilon < 1$)

Thus $\sum_{k=0}^{\infty} c_k$ converges by the comparison test.

16. $\sum_{k=1}^{n-1}(z_{k+1}-z_k) = z_n - z_1$ (telescoping sum). Thus, these partial sums converge if and only if z_n converges.

17. $|F_n(x) - F(x)| < \frac{1}{2} \iff |x^n| < \frac{1}{2} \iff 2 < x^{-n} \iff \frac{\text{Log } 2}{\text{Log } x^{-1}} < n$

18. Let $\epsilon = \frac{\rho}{2}$.
Then there exists an integer N such that $n > N$ implies

$$|F_n(z) - F(z)| < \epsilon = \frac{\rho}{2}.$$

It follows that

$$|F_n(z)| = |F(z) + F_n(z) - F(z)| \geq |F(z)| - |F_n(z) - F(z)| > \rho - \frac{\rho}{2} = \frac{\rho}{2}.$$

19. $|F(z)| = |e^z| = e^x \geq e^{-5}$ for z in the disk $|z| \leq 5$. By Problem 18, there is an integer N such that for $n > N$,

$$|F_n(z)| = \left|\sum_{k=0}^{n} \frac{z^k}{k!}\right| > \frac{e^{-5}}{2}$$

for all z in the disk $|z| \leq 5$. Thus $F_n(z)$ has no zeros on $|z| \leq 5$.

20. $\left|\sum_{j=0}^{n} z^j - \frac{1}{1-z}\right| = \left|\frac{z^{n+1}}{1-z}\right|$

Given any $\epsilon > 0$, there is no N so that $n > N$ implies $\left|\frac{z^{n+1}}{1-z}\right| < \epsilon$ for all $z : |z| < 1$. Hence the convergence is not uniform on $|z| < 1$.

21. The final point is $z_f = \Sigma_{k=0}^{\infty} e^{i\alpha k}/2^k = 1/(1-e^{i\alpha}/2)$.
$|z_f - 4/3|^2 = \{1/(1-e^{i\alpha}/2)-4/3\}\{1/(1-e^{-i\alpha}/2)-4/3\} = 4/9$

EXERCISES 5.2: Taylor Series

1. a. $f^{(j)}(z) = (-1)^j e^{-z} \Longrightarrow \dfrac{f^{(j)}(0)}{j!} = \dfrac{(-1)^j}{j!}$

   b. $f^{(2j)}(z) = \cosh z \Longrightarrow \dfrac{f^{(2j)}(0)}{(2j)!} = \dfrac{1}{(2j)!}$

   $f^{(2j+1)}(z) = \sinh z \Longrightarrow \dfrac{f^{(2j+1)}(0)}{(2j+1)!} = 0$

   c. $f^{(2j)}(z) = \sinh z \Longrightarrow \dfrac{f^{(2j)}(0)}{(2j)!} = 0$

   $f^{(2j+1)}(z) = \cosh z \Longrightarrow \dfrac{f^{(2j+1)}(0)}{(2j+1)!} = \dfrac{1}{(2j+1)!}$

   d. $f^{(j)}(z) = \dfrac{j!}{(1-z)^{j+1}} \Longrightarrow \dfrac{f^{(j)}(i)}{j!} = \dfrac{1}{(1-i)^{j+1}}$

   e. $f^{(j)}(z) = \dfrac{-(j-1)!}{(1-z)^j}$ for $j \geq 1 \Longrightarrow \dfrac{f^{(j)}(0)}{j!} = \dfrac{-1}{j}$ for $j \geq 1$

   $f(0) = 0$

   f. $f(z) = z^3$ $\qquad f(1) = 1$

   $f'(z) = 3z^2$ $\qquad f'(1) = 3$

   $f''(z) = 6z$ $\qquad \dfrac{f''(1)}{2} = 3$

   $f'''(z) = 6$ $\qquad \dfrac{f'''(1)}{3!} = 1$

   $f^{(j)}(z) = 0$ for $j \geq 4$ $\qquad \dfrac{f^{(j)}(1)}{j!} = 0$ for $j \geq 4$

2. a. C

   b. C

   c. C

d. $|z-i| < |1-i| = \sqrt{2}$

e. $|z| < 1$

f. $\mathbf{C}$

3. Note that the uniqueness of Taylor series (Theorem 11 of Section 5.3) will allow us to make these substitutions directly without resorting to the derivative calculations below.

a. $\sum_{j=0}^{\infty} a_j z^{2j}$ is the Maclaurin expansion of $g(z) := f(z^2)$.

It can be shown that $g^{(n)}(z) = \sum_{m=0}^{n} f^{(m)}(z^2) \sum_{\substack{a+2b=n \\ a+b=m}} \frac{n!}{a!b!}(2z)^a$. (This is a special case of Faá di Bruno's formula for derivatives of a composition of functions.)

If n is odd then $a \geq 1$ in this expansion and $g^{(n)}(0) = 0$.

If $n = 2j$ is even then

$$\begin{aligned} g^{(2j)}(0) &= f^{(j)}(0)\frac{(2j)!}{(j)!} \qquad \text{(with } a=0,\ b=j,\ m=j\text{)} \\ &= a_j(2j)! \end{aligned}$$

Thus $g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n = \sum_{j=0}^{\infty} \frac{g^{(2j)}(0)}{(2j)!} z^{2j} = \sum_{j=0}^{\infty} a_j z^{2j}$

b. $\sum_{j=0}^{\infty} a_j c^j z^j$ is the Maclaurin expansion of $h(z) := f(cz)$.

$$h^{(j)}(z) = c^j f^{(j)}(cz)$$

$$h(z) = \sum_{j=0}^{\infty} \frac{c^j f^{(j)}(0)}{j!} z^j = \sum_{j=0}^{\infty} a_j c^j z^j$$

c. $\sum_{j=0}^{\infty} a_j z^{m+j}$ is the Maclaurin expansion of $H(z) := z^m f(z)$.

$$H^{(j)}(z) = \sum_{k=0}^{j} \frac{j!}{(j-k)!k!} \frac{d^k}{dz^k}(z^m) f^{(j-k)}(z)$$

$$H^{(j)}(0) = 0 \qquad \text{if} \quad j < m$$

$$H^{(j)}(0) = a_{j-m}j! \quad \text{if} \quad j \geq m.$$

Hence $\displaystyle H(z) = \sum_{j=0}^{\infty} \frac{H^{(j)}(0)}{j!} z^j = \sum_{j=m}^{\infty} a_{j-m} z^j = \sum_{p=0}^{\infty} a_p z^{m+p}.$

d. $\displaystyle\sum_{j=0}^{\infty} a_j (z-z_0)^j$ is the Taylor expansion of $G(z) := f(z-z_0)$ around z_0.
$G^{(j)}(z) = f^{(j)}(z - z_0)$
$G^{(j)}(z_0) = f^{(j)}(0) = a_j j!$
$\displaystyle G(z) = \sum_{j=0}^{\infty} \frac{G^{(j)}(z_0)}{j!}(z - z_0)^j = \sum_{j=0}^{\infty} a_j (z - z_0)^j$

4. $f(z) = (1 + z)^\alpha$ is analytic on $|z| < 1$, with
$f(0) = 1$ and, for $j \geq 1$,
$f^{(j)}(z) = \alpha(\alpha - 1) \cdots (\alpha - j + 1)(1 + z)^{\alpha - j}$
$f^{(j)}(0) = \alpha(\alpha - 1) \cdots (\alpha - j + 1)$
The Maclaurin series for $f(z)$ on $|z| < 1$ is

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j = 1 + \sum_{j=1}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - j + 1)}{j!} z^j$$

5. a. $\displaystyle\sum_{j=0}^{\infty} (-z)^j$ valid for $|z| < 1$

b. $\displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{j!}$ valid for all z

c. $\displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j 3^{2j+1} z^{2j+4}}{(2j+1)!}$ valid for all z

d. $\displaystyle\sum_{k=0}^{\infty} \frac{2(-1)^k z^{2k}}{(2k)!} - \sum_{j=0}^{\infty} \frac{iz^j}{j!} = \sum_{j=0}^{\infty} \frac{i^j + (-i)^j - i}{j!} z^j$ valid for all z

e. $\displaystyle\frac{1+z}{1-z} = \left(\frac{1+i}{1-i} + \frac{z-i}{1-i}\right)\left(1 - \frac{z-i}{1-i}\right)^{-1}$

$$= \frac{1+i}{1-i} \sum_{j=0}^{\infty} \left(\frac{z-i}{1-i}\right)^j + \sum_{j=0}^{\infty} \left(\frac{z-i}{1-i}\right)^{j+1}$$

$$= i + \frac{2}{1-i}\sum_{j=1}^{\infty}\left(\frac{z-i}{1-i}\right)^j$$

(reindex second series and extract $j = 0$ term from first series)
The series is valid for $|z-i| < \sqrt{2}$

f. $\cos z = \frac{\sqrt{2}}{2}\left[1 - \left(z-\frac{\pi}{4}\right) - \frac{\left(z-\frac{\pi}{4}\right)^2}{2!} + \frac{\left(z-\frac{\pi}{4}\right)^3}{3!} + \frac{\left(z-\frac{\pi}{4}\right)^4}{4!} - \cdots\right]$

$$= \sum_{j=0}^{\infty}\frac{\cos\left(\frac{\pi}{4}(1+2j)\right)}{j!}\left(z-\frac{\pi}{4}\right)^j \text{ valid for all } z.$$

g. $\frac{z}{(1-z)^2} = z\frac{d}{dz}\left(\frac{1}{1-z}\right) = z\sum_{j=0}^{\infty}\frac{d}{dz}\left(z^j\right) = \sum_{j=1}^{\infty} jz^j$ valid for $|z| < 1$.

6. $f^{(j)}(z) = \frac{j!}{(\zeta - z)^{j+1}}$,
so $\frac{1}{\zeta - z} = \sum_{j=0}^{\infty}\frac{f^{(j)}(z_0)}{j!}(z-z_0)^j = \sum_{j=0}^{\infty}\frac{(z-z_0)^j}{(\zeta - z_0)^{j+1}}$
This is a geometric series convergent for
$\left|\frac{z-z_0}{\zeta - z_0}\right| < 1$, or $|z-z_0| < |\zeta - z_0|$.

7. $\operatorname{Log}\left(\frac{1+z}{1-z}\right) = \operatorname{Log}\left|\frac{1+z}{1-z}\right| + i\operatorname{Arg}\left(\frac{1+z}{1-z}\right)$

$$= \operatorname{Log}|1+z| - i\operatorname{Log}|1-z| + i\operatorname{Arg}\left(\frac{1+z}{1-z}\right)$$

Now $|z| < 1$ insures that $1+z$ and $1-z$ lie in the right half plane so this equals
$\operatorname{Log}|1+z| - \operatorname{Log}|1-z| + i\operatorname{Arg}(1+z) - i\operatorname{Arg}(1-z)$

$$= \operatorname{Log}(1+z) - \operatorname{Log}(1-z)$$

It follows that $\operatorname{Log}\left(\frac{1+z}{1-z}\right) = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}z^j}{j} - \sum_{j=1}^{\infty}\frac{-z^j}{j}$

$$= 2\sum_{k=0}^{\infty}\frac{z^{2k+1}}{2k+1}$$

8. a. $\sin(-z) = \sum_{j=0}^{\infty}\frac{(-1)^j(-z)^{2j+1}}{(2j+1)!} = -\sum_{j=0}^{\infty}\frac{(-1)^j z^{2j+1}}{(2j+1)!} = -\sin z$

b. $\dfrac{de^z}{dz} = \displaystyle\sum_{j=0}^{\infty} \frac{d}{dz}\left(\frac{z^j}{j!}\right) = \sum_{j=0}^{\infty} \frac{jz^{j-1}}{j!} = \sum_{j=1}^{\infty} \frac{z^{j-1}}{(j-1)!} = \sum_{j=0}^{\infty} \frac{z^j}{j!} = e^z$

c. $e^{-iz} = \displaystyle\sum_{j=0}^{\infty} \frac{(-iz)^j}{j!} = \sum_{k=0}^{\infty} \frac{(-i)^{2k} z^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-i)^{2k+1} z^{2k+1}}{(2k+1)!}$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} - i \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$= \cos z - i \sin z$$

d. $e^z e^z = \left(1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots\right)\left(1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots\right)$

$$= 1 + 2z + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \cdots$$

(Cauchy product)

$$= e^{2z}$$

9. a. Let $h(z) = cf(z)$. Then $h^{(j)}(z) = cf^{(j)}(z)$ so the Taylor series for $cf(z)$ is

$$\sum_{j=0}^{\infty} \frac{h^{(j)}(z_0)}{j!}(z - z_0)^j = \sum_{j=0}^{\infty} \frac{cf^{(j)}(z_0)}{j!}(z - z_0)^j = \sum_{j=0}^{\infty} ca_j (z - z_0)^j$$

b. Let $h(z) = f(z) \pm g(z)$. Then $h^{(j)}(z) = f^{(j)}(z) \pm g^{(j)}(z)$ and the result follows as in part a.

10. The case $j = 1$ is obvious. Assume that the formula is true up to the jth derivative of fg. Recall the binomial coefficient notation
$\dbinom{j}{l} = \dfrac{j!}{(j-l)!l!}$.
Then

$$\begin{aligned}(fg)^{(j+1)} &= \frac{d}{dz}(fg)^{(j)} \\ &= \sum_{l=0}^{j} \frac{j! \frac{d}{dz}\left[f^{(j-l)} g^{(l)}\right]}{(j-l)!l!}\end{aligned}$$

$$= \sum_{l=0}^{j} \binom{j}{l} \left[f^{(j+1-l)} g^{(l)} + f^{(j-l)} g^{(l+1)} \right]$$

$$= f^{(j+1)} g + f g^{(j+1)} + \sum_{l=1}^{j} \binom{j}{l} f^{(j+1-l)} g^{(l)} + \sum_{l=0}^{j-1} \binom{j}{l} f^{(j-l)} g^{(l+1)}$$

$$= f^{(j+1)} g + f g^{(j+1)} + \sum_{l=1}^{j} \binom{j}{l} f^{(j+1-l)} g^{(l)} + \sum_{l=1}^{j} \binom{j}{l-1} f^{(j+1-l)} g^{(l)}$$

$$= f^{(j+1)} g + f g^{(j+1)} + \sum_{l=1}^{j} \binom{j+1}{l} f^{(j+1-l)} g^{(l)}$$

$$\left(\text{because } \binom{j}{l} + \binom{j}{l-1} = \binom{j+1}{l} \right)$$

$$= \sum_{l=0}^{j+1} \binom{j+1}{l} f^{(j+1-l)} g^{(l)}$$

11. a. $e^z \cos z = \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right)\left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots\right)$

$$= 1 + z - \frac{1}{3}z^3 + \cdots$$

b. $e^z \left(\frac{-1}{1-z}\right) = \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right)\left(-1 - z - z^2 - z^3 - \cdots\right)$

$$= -1 - 2z - \frac{5}{2}z^2 + \cdots$$

c. $(\cos z)(\sec z) = 1$

$$\left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots\right)\left(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots\right) = 1$$

$b_0 = 1$

$b_1 z = 0\, z \Longrightarrow b_1 = 0$

$\left(-\frac{1}{2}b_0 + b_2\right) z^2 = 0\, z^2 \Longrightarrow b_2 = \frac{1}{2}$

$\left(-\frac{1}{2}b_1 + b_3\right) z^3 = 0\, z^3 \Longrightarrow b_3 = 0$

$\left(\frac{1}{24}b_0 - \frac{1}{2}b_2 + b_4\right) z^4 = 0\, z^4 \Longrightarrow b_4 = \frac{5}{24}$

$\sec z = 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \cdots$

d. $\cosh z \tanh z = \sinh z$

$$\left(1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots\right)\left(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots\right)$$

$$= z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots$$

$b_0 = 0$

$b_1 z = z \Longrightarrow b_1 = 1$

$$\left(\frac{1}{2}b_0 + b_2\right) z^2 = 0\ z^2 \Longrightarrow b_2 = 0$$

$$\left(\frac{1}{2}b_1 + b_3\right) z^3 = \frac{1}{6}z^3 \Longrightarrow b_3 = -\frac{1}{3}$$

$$\left(\frac{1}{24}b_0 + \frac{1}{2}b_2 + b_4\right) z^4 = 0\ z^4 \Longrightarrow b_4 = 0$$

$$\left(\frac{1}{24}b_1 + \frac{1}{2}b_3 + b_5\right) z^5 = \frac{1}{120}\ z^5 \Longrightarrow b_5 = \frac{16}{120} = \frac{2}{15}$$

$$\tanh z = z - \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots$$

12. Suppose there is another such polynomial q_n. Then $F = p_n - q_n$ is a polynomial of degree at most n for which $F^{(j)}(z_0) = p_n^{(j)}(z_0) - q_n^{(j)}(z_0) = 0$ as long as $0 \le j \le n$. Examining the Taylor series for F reveals that $F \equiv 0$ and $p_n \equiv q_n$.

13. $(1-z)^{-1} = \sum_{k=0}^{\infty} z^k$.

    Differentiate to get $(1-z)^{-2} = \sum_{k=0}^{\infty} kz^{k-1}$

    Multiply by z to get $z(1-z)^{-2} = \sum_{k=0}^{\infty} kz^k$

    Differentiate to get $(1+z)(1-z)^{-3} = \sum_{k=0}^{\infty} k^2 z^{k-1}$

    Multiply by z to get $\dfrac{z(1+z)}{(1-z)^3} = \sum_{k=0}^{\infty} k^2 z^k$

14. The Taylor series for $f(z)$ converges to 0 for all z in D. By Theorem 3, $f(z) \equiv 0$ in D.

15. $f(-z) = \sum_{j=0}^{\infty} a_{2j}(-z)^{2j} = \sum_{j=0}^{\infty} a_{2j}z^{2j} = f(z)$

16. $p(1) = \sum_{j=0}^{n} a_j$

$$p'(1) = \sum_{j=1}^{n} ja_j$$

$$p''(1) = \sum_{j=2}^{n} j(j-1)a_j$$

$$p^{(k)}(1) = \sum_{j=k}^{n} \frac{j!}{(j-k)!}a_j$$

$$p(z) = \sum_{k=0}^{n}\left[\sum_{j=k}^{n} \frac{j!}{(j-k)!k!}a_j\right](z-1)^k$$

$$= \sum_{j=0}^{n} a_j + (z-1)\sum_{j=1}^{n} ja_j + \frac{(z-1)^2}{2}\sum_{j=2}^{n} j(j-1)a_j + \cdots + (z-1)^n a_n$$

17. $|z| < 1$ is not a closed disk, and attempting to close it includes $z = 1$, where $f(z) = (1-z)^{-1}$ is not analytic.

18. a. $\left|e^z - \sum_{k=0}^{n} \frac{z^k}{k!}\right| = \left|\sum_{k=n+1}^{\infty} \frac{z^k}{k!}\right| \le \sum_{k=n+1}^{\infty} \left|\frac{z^k}{k!}\right| \le \sum_{k=n+1}^{\infty} \frac{1}{k!}$

$$= \sum_{k=0}^{\infty} \frac{1}{(n+k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(n+k+1)(n+k)(n+k-1)\cdots(n+2)(n+1)!}$$

$$\le \sum_{k=0}^{\infty} \frac{1}{(n+1)!}\left(\frac{1}{n+2}\right)^k$$

$$= \frac{1}{(n+1)!}\left(\frac{1}{1-\frac{1}{n+2}}\right) = \frac{1}{(n+1)!}\frac{n+2}{n+1}$$

$$= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1}\right)$$

b. Note that

$$\frac{4n^2+18n+20}{4n^2+18n+19} = \frac{(2n+5)(2n+4)}{(2n+5)(2n+4)-1}$$
$$= \frac{1}{1-\frac{1}{(2n+5)(2n+4)}} = \sum_{k=0}^{\infty}\left[\frac{1}{(2n+5)(2n+4)}\right]^k.$$

Then

$$\left|\sin z - \sum_{k=0}^{n}\frac{(-1)^k z^{2k+1}}{(2k+1)!}\right|$$
$$= \left|\sum_{k=n+1}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!}\right| \le \sum_{k=n+1}^{\infty}\frac{1}{(2k+1)!}$$
$$= \sum_{k=0}^{\infty}\frac{1}{(2k+2n+3)!}$$
$$= \sum_{k=0}^{\infty}\frac{1}{(2k+2n+3)(2k+2n+2)\cdots(2n+5)(2n+4)(2n+3)!}$$
$$\le \frac{1}{(2n+3)!}\sum_{k=0}^{\infty}\left[\frac{1}{(2n+5)(2n+4)}\right]^k = \frac{1}{(2n+3)!}\left(\frac{4n^2+18n+20}{4n^2+18n+19}\right)$$

19. To get $\frac{1}{(n+1)!}\left(1+\frac{1}{n+1}\right) \le 10^{-5}$ requires $n = 8$, yielding nine terms of the series.

20. $f(z) - \sum_{j=0}^{n}\frac{f^{(j)}(0)}{j!}z^j$

$$= \frac{1}{2\pi i}\oint_{|\zeta|=R}\frac{f(\zeta)}{\zeta - z}d\zeta - \sum_{j=0}^{n}\frac{j!}{j!2\pi i}\oint_{|\zeta|=R}\frac{f(\zeta)}{\zeta^{j+1}}d\zeta\, z^j$$

(Cauchy integral formula)

$$= \frac{1}{2\pi i}\oint_{|\zeta|=R} f(\zeta)\left[\sum_{j=0}^{\infty}\frac{z^j}{\zeta^{j+1}} - \sum_{j=0}^{n}\frac{z^j}{\zeta^{j+1}}\right]d\zeta$$

$$= \frac{1}{2\pi i}\oint_{|\zeta|=R}\frac{z^{n+1}}{\zeta^{n+1}}\frac{f(\zeta)}{\zeta - z}d\zeta$$

EXERCISES 5.3: Power Series

1. a. $\lim_{j\to\infty} z^j \neq 0$ when $|z| = 1$, so the series does not converge by Problem 5, Section 5.1.

   b. $\left|\frac{z^j}{j^2}\right| = \frac{1}{j^2}$, so the series converges by the comparison test.

2. $\lim_{j\to\infty}\left|\frac{a_{j+1}(z-z_0)^{j+1}}{a_j(z-z_0)^j}\right| = \lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right||z-z_0| = L|z-z_0|$.

   The ratio test implies that the series is convergent for $L|z-z_0| < 1$, yielding a radius of convergence of $1/L$.

3. a. $|z| = 1$

   b. $|z-1| = 1/2$

   c. $z = 0$

   d. $|z-i| = 3$

   e. $|z+2| = 1/\sqrt{10}$

   f. $|z| = 2$

4. No, since $|2+3i-0| = \sqrt{13}$ is greater than $|3-i-0| = \sqrt{10}$.

5. a. $f^{(6)}(0) = 6!\,a_6 = \frac{6!6^3}{3^6}$

   b. $\oint_{|z|=1}\frac{f(z)}{z^4}dz = \frac{2\pi i}{3!}f^{(3)}(0) = \frac{2\pi i(3!a_3)}{3!} = 2\pi i$

   c. $e^z f(z)$ is analytic inside and on $|z| = 1$, so the integral is 0.

   d. $$\oint_{|z|=1}\frac{f(z)\sin z}{z^2}dz = 2\pi i\,\frac{d}{dz}[f(z)\sin z]\Big|_{z=0}$$

   $$= 2\pi i\,[f'(z)\sin z + f(z)\cos z]\big|_{z=0} = 0$$

6. a. $f(0) = 1$ agrees with the series at $z = 0$. At any other value of z the result follows by dividing each term of the Maclaurin series for $\sin z$ by z.

b. $f(z) = \dfrac{\sin z}{z}$ clearly has a derivative for any $z \neq 0$. At $z = 0$,
$f'(0) = \lim_{z\to 0}\left(\dfrac{\sin z}{z} - 1\right) = \lim_{z\to 0}\left(\dfrac{\sin z - z}{z}\right) = \lim_{z\to 0}(\cos z - 1) = 0.$
Thus $f(z)$ has a derivative in a neighborhood of the origin, and is analytic there.

c. $f^{(3)}(0) = 3!\, a_3 = 0$
$f^{(4)}(0) = 4!\, a_4 = 4!\,(1/5!) = 1/5$

7. $\displaystyle\int_0^z e^{\zeta^2} d\zeta = \sum_{k=0}^{\infty} \int_0^z \frac{\zeta^{2k}}{k!} d\zeta = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{k!(2k+1)}$

The first three nonzero terms are $z + \dfrac{z^3}{3} + \dfrac{z^5}{10}$

8. $\displaystyle f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j = f(0) + f'(0)z + \sum_{j=2}^{\infty} \frac{f^{(j)}(0)}{j!} z^j$
$\displaystyle = z^2 \sum_{j=0}^{\infty} \frac{f^{(j+2)}(0)}{(j+2)!} z^j = z^2 g(z).$
$g(z)$ is analytic at $z = 0$ since $f(z)$ is.

9. Let $\{p_n(z)\}$ be such a sequence of functions. Since each p_n is entire, $\int_C p_n(z)dz = 0$ for all n. By Theorem 8, $\int_C g(z)dz = \lim_{n\to\infty} \int_C p_n(z)dz = 0.$

10. Applying Problem 2, $R = \lim_{k\to\infty} \left|\dfrac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} \left|\dfrac{k a_k}{(k+1)a_{k+1}}\right|$

11. a. $f(z) = \sum_{k=0}^{\infty} (a_k - b_k) z^k \equiv 0$ in some neighborhood of the origin, so $f^{(k)}(0) = 0$ for $k = 0, 1, 2, \ldots$ and $a_k - b_k = 0$.

b. The complex series $\sum_{k=0}^{\infty} a_k z^k$ must be convergent on a disk that includes the interval on which $\sum_{k=0}^{\infty} a_k x^k$ converges (similarly for b_k). Thus this reduces to part a.

12. $\displaystyle f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = f(0) + \frac{f'(0)}{1} z + \frac{f''(0)}{2} z^2 + \cdots$

$$= f(0) + f(0)z + \frac{f(0)}{2}z^2 + \cdots$$
$$= f(0)\left(1 + z + \frac{z^2}{2} + \cdots\right)$$
$$= 1 + z + \frac{z^2}{2} + \cdots = e^z$$

13. a. $f'' = zf' + f \Longrightarrow f''(0) = 0 \cdot f'(0) + f(0) = 1$
$f^{(3)} = zf'' + 2f' \Longrightarrow f^{(3)}(0) = 0$
In general,
$f^{(j)} = zf^{(j-1)} + (j-1)f^{(j-2)}$
and

$$f^{(2j+1)}(0) = 0$$
$$f^{(2j)}(0) \quad = (2j-1)(2j-3)\cdots 3 \cdot 1$$

$$f(z) = \sum_{j=0}^{\infty} \frac{(2j-1)(2j-3)\cdots 3\cdot 1}{(2j)!} z^{2j}$$
$$= \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{z^2}{2}\right)^j = e^{z^2/2}$$

b. $f'' + 4f = 0 \Longrightarrow f^{(j)} = -4f^{(j-2)}$
$f^{(2k+1)}(0) = (-4)^k f'(0) = (-4)^k$
and $f^{(2k)}(0) = (-4)^k f(0) = (-4)^k$

$$f(z) = \sum_{k=0}^{\infty} \frac{(-4)^k z^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{(-4)^k}{(2k)!} z^{2k}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k (2z)^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k (2z)^{2k}}{(2k)!}$$

$$= \frac{1}{2}\sin 2z + \cos 2z$$

c. Assuming $f(z) = \sum_{j=0}^{\infty} a_j z^j$, obtain the recurrence relation

$$a_{j+2} = \frac{j+4}{j+2} a_j$$

by substituting the series into the differential equation.

$$a_0 = f(0) = 1 \implies a_2 = 2a_0 = 2, a_4 = \frac{6}{4} \cdot 2 = 3, \text{ etc.}$$
$$\implies a_{2k} = k+1$$
$$a_1 = f'(0) = 0 \implies a_{2k+1} = 0$$

$$f(z) = \sum_{k=0}^{\infty}(k+1)z^{2k}, \text{ which is the series for } \frac{1}{(1-z^2)^2}.$$

14. Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$.

$f'' + f = 0 \implies f^{(j+2)}(0) = -f^{(j)}(0)$.

$$a_{2k+1} = \frac{f^{(2k+1)}(0)}{(2k+1)!} = \frac{(-1)^k f'(0)}{(2k+1)!} = \frac{(-1)^k}{(2k+1)!}$$

and $\quad a_{2k} = \dfrac{f^{(2k)}(0)}{(2k)!} = \dfrac{(-1)^k f(0)}{(2k)!} = 0.$

Thus $\quad f(z) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sin z.$

15. a. $F(z) = \displaystyle\int_{-1}^{2} g(t) \left[\sum_{k=0}^{\infty} \frac{(-1)^k (zt)^{2k+1}}{(2k+1)!}\right] dt$

$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{(2k+1)!} \int_{-1}^{2} t^{2k+1} g(t)dt\right] z^{2k+1}$$

This series expansion is valid because the series for $\sin(zt)$ converges uniformly on any disk in $\mathbf{C}$. It follows that the series expansion for $F(z)$ converges uniformly to $F(z)$ on any disk in $\mathbf{C}$, and $F(z)$ is entire by Theorem 9.

b. $F'(z) = \dfrac{d}{dz} \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left[\int_{-1}^{2} t^{2k+1} g(t)dt\right] z^{2k+1}$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left[\int_{-1}^{2} t^{2k+1} g(t)dt\right] (2k+1)z^{2k}$$

$$= \int_{-1}^{2} tg(t)\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k}z^{2k}\right]dt$$

$$= \int_{-1}^{2} tg(t)\cos(zt)dt$$

16. $H(z) = \int_0^1 g(t)\left[\sum_{k=0}^{\infty}(zt^2)^k\right]dt$ convergent for $|zt^2| < 1$

$$= \sum_{k=0}^{\infty}\left[\int_0^1 g(t)t^{2k}dt\right]z^k$$

Since $0 \le t \le 1$, this series is convergent on $|z| < 1$, and uniformly convergent on any disk in $|z| < 1$. It follows from Theorem 9 that $H(z)$ is analytic on $|z| < 1$.

17. a. ${}_2F_1(1,1;2;z) = \sum_{j=0}^{\infty}\frac{(1)_j(1)_j}{(2)_j}\frac{z^j}{j!} = \sum_{j=0}^{\infty}\frac{j!j!z^j}{(j+1)!j!} = \sum_{j=0}^{\infty}\frac{z^j}{j+1}$

$\frac{z^n}{n!}\ {}_1F_1(1;n+1;z)$

$$= \frac{z^n}{n!}\sum_{j=0}^{\infty}\frac{(1)_j}{(n+1)_j}\frac{z^j}{j!} = \frac{z^n}{n!}\sum_{j=0}^{\infty}\frac{j!n!}{(n+j)!}\frac{z^j}{j!}$$

$$= \sum_{j=0}^{\infty}\frac{z^{n+j}}{(n+j)!} = \sum_{j=0}^{\infty}\frac{z^j}{j!} - \sum_{j=0}^{n-1}\frac{z^j}{j!} = e^z - \sum_{j=0}^{n-1}\frac{z^j}{j!}$$

b. Applying Problem 2, the radius of convergence is

$$R = \lim_{j\to\infty}\left|\frac{a_j}{a_{j+1}}\right| = \lim_{j\to\infty}\left|\frac{(b)_j(c)_j}{(d)_j j!}\frac{(d)_{j+1}(j+1)!}{(b)_{j+1}(c)_{j+1}}\right|$$

$$= \lim_{j\to\infty}\left|\frac{(d+j)(j+1)}{(b+j)(c+j)}\right| = 1.$$

Thus the series converges for $|z| < 1$.

$$z(1-z)\,{}_2F_1'' + [d-(b+c+1)z]{}_2F_1' - bc\,{}_2F_1$$

$$= z(1-z)\sum_{j=2}^{\infty}\frac{(b)_j(c)_j}{(d)_j}\frac{z^{j-2}}{(j-2)!}$$

$$+[d-(b+c+1)z]\sum_{j=1}^{\infty}\frac{(b)_j(c)_j}{(d)_j}\frac{z^{j-1}}{(j-1)!} - bc\sum_{j=0}^{\infty}\frac{(b)_j(c)_j}{(d)_j}\frac{z^j}{j!}$$

$$= z^0\left[d\frac{(b)_1(c)_1}{(d)_1} - bc\frac{(b)_0(c)_0}{(d)_0}\right]$$

$$+z\left[\frac{(b)_2(c)_2}{(d)_2} + d\frac{(b)_2(c)_2}{(d)_2} - (b+c+1)\frac{(b)_1(c)_1}{(d)_1} - bc\frac{(b)_1(c)_1}{(d)_1}\right]$$

$$+\sum_{j=2}^{\infty} z^j\left[\frac{(b)_{j+1}(c)_{j+1}}{(d)_{j+1}(j-1)!} - \frac{(b)_j(c)_j}{(d)_j(j-2)!} + d\frac{(b)_{j+1}(c)_{j+1}}{(d)_{j+1}j!} - (b+c+1)\frac{(b)_j(c)_j}{(d)_j(j-1)!} - bc\frac{(b)_j(c)_j}{(d)_j j!}\right]$$

$$= \left[\frac{dbc}{d} - bc\right]$$

$$+z\left[(1+d)\left(\frac{b(b+1)c(c+1)}{d(d+1)}\right) - (b+c+1+bc)\left(\frac{bc}{d}\right)\right]$$

$$+\sum_{j=2}^{\infty}\frac{z^j(b)_j(c)_j}{(d)_j j!}\left[\frac{(b+j)(c+j)j}{d+j} - j(j-1) + \frac{d(b+j)(c+j)}{d+j} - (b+c+1)j - bc\right]$$

$$= 0$$

c. Applying Problem 2, the radius of convergence is

$$R = \lim_{j\to\infty}\left|\frac{a_j}{a_{j+1}}\right| = \lim_{j\to\infty}\left|\frac{(c)_j}{(d)_j j!}\frac{(d)_{j+1}(j+1)!}{(c)_{j+1}}\right|$$

$$= \lim_{j\to\infty}\left|\frac{(d+j)(j+1)}{(c+j)}\right| = \infty.$$

Thus the series converges for all z.

$$z\,{}_1F_1'' + (d-z)\,{}_1F_1' - c\,{}_1F_1$$

$$= z\sum_{j=2}^{\infty}\frac{(c)_j}{(d)_j}\frac{z^{j-2}}{(j-2)!} + (d-z)\sum_{j=1}^{\infty}\frac{(c)_j}{(d)_j}\frac{z^{j-1}}{(j-1)!} - c\sum_{j=0}^{\infty}\frac{(c)_j}{(d)_j}\frac{z^j}{j!}$$

$$= z^0\left[d\frac{(c)_1}{(d)_1} - c\frac{(c)_0}{(d)_0}\right]$$

$$+\sum_{j=1}^{\infty} z^j\left[\frac{(c)_{j+1}}{(d)_{j+1}(j-1)!} + d\frac{(c)_{j+1}}{(d)_{j+1}j!} - \frac{(c)_j}{(d)_j(j-1)!} - c\frac{(c)_j}{(d)_j j!}\right]$$

$$= \left[\frac{dc}{d} - c\right] + \sum_{j=1}^{\infty}\frac{z^j(c)_j}{(d)_j j!}\left[\frac{(c+j)j}{(d+j)} + \frac{d(c+j)}{(d+j)} - j - c\right]$$

$$= 0$$

18. $$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{\displaystyle\sum_{j=0}^{\infty}\frac{f^{(j)}(z_0)}{j!}(z-z_0)^j}{\displaystyle\sum_{j=0}^{\infty}\frac{g^{(j)}(z_0)}{j!}(z-z_0)^j}$$

$$= \lim_{z\to z_0}\frac{\displaystyle\sum_{j=m}^{\infty}\frac{f^{(j)}(z_0)}{j!}(z-z_0)^j}{\displaystyle\sum_{j=m}^{\infty}\frac{g^{(j)}(z_0)}{j!}(z-z_0)^j}$$

$$= \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}$$

19 (a) $\iint_{\underline{D}} z^j \underline{z}^k dxdy = \iint_{\underline{D}} r^j r^k e^{ij\theta} e^{-ik\theta} r dr d\theta$

$= \int_0^1 \int_0^{2\pi} r^{j+k+1} e^{i(j-k)\theta} dr d\theta = \pi/(k+1)$, if $k = j$

$= 0$, if $k \neq j$

(b) $\iint_{\underline{D}} f(z)\underline{z}^k dxdy = \Sigma_{j=0}^{\infty} \iint_{\underline{D}} a_k z^j \underline{z}^k dxdy = a_k \pi/(k+1)$

$a_k = [(k+1)/\pi] \iint_{\underline{D}} f(z)\underline{z}^k dxdy$, $k = 0, 1, 2, \ldots$

(c) $f(\zeta) = \Sigma_{k=0}^{\infty} a_k \zeta^k = \Sigma_{k=0}^{\infty} [(k+1)/\pi] \iint_{\underline{D}} f(z)\underline{z}^k dxdy\ \zeta^k$

$= (1/\pi) \iint_{\underline{D}} \Sigma_{k=0}^{\infty} (k+1) f(z)\underline{z}^k \zeta^k\ dxdy$

$= (1/\pi) \iint_{\underline{D}} f(z) 1/(1-\underline{z}\zeta)^2\ dxdy$

EXERCISES 5.4

1. a. 2 b. ∞

c. 0

d. ∞

2. Let $A = \limsup \dfrac{x_{n+1}}{x_n}$. If $A = \infty$ then there is nothing to prove, so suppose $A < \infty$. Choose $\epsilon > 0$. Then there is an integer N so that $n \geq N$ implies $\dfrac{x_{n+1}}{x_n} \leq A + \epsilon$. It follows that $x_{N+k} \leq x_N(A+\epsilon)^k$ and (replacing $N + k$ by m)

$$\limsup_{m\to\infty} \sqrt[m]{x_m} \leq \limsup_{m\to\infty} \sqrt[m]{\frac{x_N(A+\epsilon)^m}{(A+\epsilon)^N}} = A + \epsilon.$$

Therefore, $\limsup\limits_{m\to\infty} \sqrt[m]{x_m} \leq A = \limsup\limits_{m\to\infty} \dfrac{x_{m+1}}{x_m}$

3. a. $\limsup \sqrt[j]{|a_j|} = \limsup \left[\dfrac{2^j}{3^j + 4^j}\right]^{1/j} = \dfrac{1}{2}$, so $R = 2$

b. $a_n = \begin{cases} 2^{\sqrt{n}} \text{ if } n \text{ is a perfect square} \\ 0 \text{ otherwise} \end{cases}$

$\limsup \sqrt[n]{|a_n|} = \limsup \left(2^{\sqrt{n}}\right)^{1/n} = 1$, so $R = 1$

c. $\limsup \sqrt[j]{|a_j|} = \limsup \left[2 + (-1)^j\right] = 3$, so $R = \dfrac{1}{3}$

d. $R = 1/\lim \left|\dfrac{a_{j+1}}{a_j}\right| = \lim \left|\dfrac{j!}{j^j}\dfrac{(j+1)^{j+1}}{(j+1)!}\right| = \lim \left(\dfrac{j+1}{j}\right)^j = e$

e. $a_n = \begin{cases} \dfrac{4}{3n} \text{ if } n \text{ is even} \\ 0 \text{ otherwise} \end{cases}$

$\limsup \sqrt[n]{a_n} = \limsup \left[\dfrac{4}{3n}\right]^{1/n} = 1$, so $R = 1$

f. $a_n = \begin{cases} 1 \text{ if } n = j! \\ 0 \text{ otherwise} \end{cases}$

$\limsup \sqrt[n]{a_n} = \lim \sqrt[n]{1} = 1$, so $R = 1$

4. Each of the three series converges for $|z| < 1$. For $|z| = 1$,
$\left|\dfrac{z^j}{j^2}\right| = \dfrac{1}{j^2}$, so $\displaystyle\sum_{j=1}^{\infty} \frac{z^j}{j^2}$ converges by the comparison test.

$\sum_{j=1}^{\infty} \frac{z^j}{j}$ converges for some values of z on $|z| = 1$ and diverges for others, since $\sum_{j=1}^{\infty} \frac{1^j}{j} = \sum_{j=1}^{\infty} \frac{1}{j}$ diverges, while $\sum_{j=1}^{\infty} \frac{(-1)^j}{j}$ converges.

$\sum_{j=0}^{\infty} z^j$ diverges for all $|z| = 1$ since the terms of this series do not approach zero (Problem 5, section 5.1)

5. a. $\limsup \left[j^3|a_j|\right]^{1/j} = \limsup j^{3/j}|a_j|^{1/j} = \frac{1}{R}$. The radius of convergence is R.

   b. $\limsup \sqrt[j]{|a_j|^4} = 1/R^4$. The radius of convergence is R^4.

   c. $b_n = \begin{cases} a_{n/2} \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$

   $\limsup \sqrt[n]{|b_n|} = \limsup \sqrt[2n]{|a_n|} = \frac{1}{\sqrt{R}}$. The radius of convergence is $\sqrt{R}$.

   d. $z^7 \sum_{j=0}^{\infty} a_j z^j$. The radius of convergence is R.

   e. $\limsup \sqrt[j]{j^{-j}|a_j|} = \limsup \frac{\sqrt[j]{|a_j|}}{j} = \begin{cases} 0 \text{ if } R > 0 \\ \text{indeterminate if } R = 0 \end{cases}$
   The radius of convergence is ∞ if $R > 0$, uncertain if $R = 0$.

6. $|\operatorname{Re}(a_j)| \leq |a_j|$, so $1/\limsup \sqrt[j]{|\operatorname{Re}(a_j)|} \geq 1/\limsup \sqrt[j]{|a_j|} = R$

7. $J_0''(z) + \frac{1}{z}J_0'(z) + J_0(z)$

$$= \sum_{j=1}^{\infty} \frac{(-1)^j(2j)(2j-1)z^{2j-2}}{2^{2j}(j!)^2} + \sum_{j=1}^{\infty} \frac{(-1)^j 2j z^{2j-2}}{2^{2j}(j!)^2} + J_0(z)$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^j}{2^{2(j-1)}(j-1)!} z^{2(j-1)} + J_0(z)$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{2^{2j}(j!)^2} z^{2j} + J_0(z) = 0$$

8. $R = \lim_{j\to\infty} \left|\frac{a_j}{a_{j+1}}\right| = \lim_{j\to\infty} \left|\frac{(-1)^j}{j!(n+j)!2^{2j+n}} \frac{(j+1)!(n+j+1)!2^{2j+2+n}}{(-1)^{j+1}}\right|$

$$= \lim_{j\to\infty} |4(j+1)(n+j+1)| = \infty$$

The series for $J_n(z)$ converges for all z, so $J_n(z)$ is entire.

$$J_n''(z) + \frac{1}{z}J_n'(z) + \left(1 - \frac{n^2}{z^2}\right) J_n(z)$$
$$= \sum_{j=0}^{\infty} \frac{(-1)^j(2j+n)(2j+n-1)}{j!(n+j)!2^{2j+n}} z^{2j+n-2}$$
$$+ \sum_{j=0}^{\infty} \frac{(-1)^j(2j+n)}{j!(n+j)!2^{2j+n}} z^{2j+n-2}$$
$$+ \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!2^{2j+n}} z^{2j+n}$$
$$- \sum_{j=0}^{\infty} \frac{(-1)^j n^2}{j!(n+j)!2^{2j+n}} z^{2j+n-2}$$
$$= \sum_{j=0}^{\infty} \frac{(-1)^j\left[(2j+n)(2j+n-1) + (2j+n) - n^2\right]}{j!(n+j)!2^{2j+n}} z^{2j+n-2}$$
$$+ \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!2^{2j+n}} z^{2j+n}$$
$$= \sum_{j=1}^{\infty} \frac{(-1)^j 4j(n+j)}{j!(n+j)!2^{2j+n}} z^{2j+n-2} + \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!2^{2j+n}} z^{2j+n}$$
$$= -\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!2^{2j+n}} z^{2j+n} + \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!2^{2j+n}} z^{2j+n}$$
$$= 0$$

9. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

$$\sum_{n=0}^{\infty} a_n z^n - z - \sum_{n=0}^{\infty} a_n z^{2n} = 0$$

$(a_0 - a_0) + (a_1 - 1)z + \sum_{n=2}^{\infty} b_n z^n = 0$ where $0 = b_n = \begin{cases} a_n \text{ if } n \text{ is odd} \\ a_n - a_{n/2} \text{ if } n \text{ is even} \end{cases}$

Thus a_0 is arbitrary, $a_1 = 1$, and

$$a_n = \begin{cases} 0 \text{ if the prime factorization of } n \text{ contains any odd number} \\ 1 \text{ if } n = 2^j \end{cases}$$

Thus $f(z) = a_0 + \sum_{j=0}^{\infty} z^{2^j}$. This series converges for $|z| < 1$.

10. $$\begin{aligned} f(z) = \sum_{n=0}^{\infty} a_n z^n &= 1 + z + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) z^n \\ &= 1 + z + \sum_{n=1}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} a_n z^{n+2} \\ &= 1 + z f(z) + z^2 f(z) \end{aligned}$$

It follows that $f(z) = \dfrac{1}{1 - z - z^2}$ (an analytic function).

Taking $c_1 = \dfrac{-1 - \sqrt{5}}{2}$ and $c_2 = \dfrac{-1 + \sqrt{5}}{2}$ yields.

$$\begin{aligned} f(z) &= \frac{-1}{(c_1 - z)(c_2 - z)} \\ &= \frac{-1}{\sqrt{5}} \left(\frac{1}{c_1 - z} \right) + \frac{1}{\sqrt{5}} \left(\frac{1}{c_2 - z} \right) \text{ (partial fractions)} \\ &= \frac{-1}{\sqrt{5} c_1} \sum_{j=0}^{\infty} \left(\frac{z}{c_1} \right)^j + \frac{1}{\sqrt{5} c_2} \sum_{j=0}^{\infty} \left(\frac{z}{c_2} \right)^j \\ &= \sum_{j=0}^{\infty} \frac{1}{\sqrt{5}} \left[\frac{c_1^{j+1} - c_2^{j+1}}{(c_1 c_2)^{j+1}} \right] z^j \\ &= \sum_{j=0}^{\infty} \frac{1}{\sqrt{5}} \left[\frac{c_1^{j+1} - c_2^{j+1}}{(-1)^{j+1}} \right] z^j \end{aligned}$$

$$= \sum_{j=0}^{\infty} \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{j+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{j+1} \right] z^j$$

Thus $a_j = \dfrac{1}{\sqrt{5}} \left[\left(\dfrac{1+\sqrt{5}}{2} \right)^{j+1} - \left(\dfrac{1-\sqrt{5}}{2} \right)^{j+1} \right]$

11. Let
$$\begin{aligned} f(z) &= (1-2\zeta z + z^2)^{-1/2} \\ f'(z) &= (1-2\zeta z + z^2)^{-3/2}(\zeta - z) \\ f''(z) &= (1-2\zeta z + z^2)^{-5/2} 3(\zeta - z)^2 - (1-2\zeta z + z^2)^{-3/2} \\ f'''(z) &= (1-2\zeta z + z^2)^{-7/2} 15(\zeta - z)^3 + (1-2\zeta z + z^2)^{-5/2}(-9\zeta + 9z) \end{aligned}$$

$$\begin{aligned} P_0(\zeta) &= f(0) = 1 \\ P_1(\zeta) &= f'(0) = \zeta \\ P_2(\zeta) &= \frac{f''(0)}{2} = \frac{3\zeta^2 - 1}{2} \\ P_3(\zeta) &= \frac{f'''(0)}{3!} = \frac{15\zeta^3 - 9\zeta}{6} = \frac{5\zeta^3 - 3\zeta}{2} \end{aligned}$$

To show that $P_j(\zeta)$ is a polynomial in ζ of degree j, we can show by induction that $f^{(j)}(z) = (1-2\zeta z + z^2)^{-\frac{1}{2}-j} Q_j(\zeta, z)$ where $Q_j(\zeta, z)$ is a polynomial of degree less than or equal to j in ζ. When $j = 0$ this is obvious. Assume this is true for j.

Then
$$f^{(j+1)}(z) = (1-2\zeta z + z^2)^{-\frac{1}{2}-j-1} \left[\left(-\frac{1}{2} - j \right) (-2\zeta + 2z)\, Q_j(\zeta, z) + \left(1 - 2\zeta z + z^2 \right) \frac{\partial Q_j}{\partial z}(\zeta, z) \right],$$
which is $(1-2\zeta z + z^2)^{-\frac{1}{2}-(j+1)}$ times a polynomial of degree less than or equal to $j+1$ in ζ. It follows that $P_j(\zeta) = \dfrac{f^{(j)}(0)}{j!} = Q_j(\zeta, 0)$, which is a polynomial in ζ of degree j.

12. $\left| \dfrac{1}{j^z} \right| = |\exp[-z \operatorname{Log} j]| = \exp[-\operatorname{Re}(z) \operatorname{Log} j] \leq \exp[-\lambda \operatorname{Log} j] = \dfrac{1}{j^\lambda}$

By the Weierstrass M-test (Theorem 13), $\zeta(z)$ converges uniformly to an analytic function on $\operatorname{Re}(z) \geq \lambda > 1$. Since this is true for any $\lambda > 1$, $\zeta(z)$ is analytic on $\operatorname{Re}(z) > 1$.

13. On one hand,

$$\lim_{n\to\infty}\lim_{r\to 1^-}\left|f(r)-\sum_{j=0}^{n}a_jr^j\right| = \lim_{n\to\infty}\left|A-\sum_{j=0}^{n}a_j\right| = \left|A-\sum_{j=0}^{\infty}a_j\right|.$$

On the other hand,

$$\lim_{n\to\infty}\lim_{r\to 1^-}\left|f(r)-\sum_{j=0}^{n}a_jr^j\right| \le \lim_{n\to\infty}(M_{n+1}+M_{n+2}) = 0.$$

Thus $A=\sum_{j=0}^{\infty}a_j$.

14. $\operatorname{Log}(1+z)=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}z^j}{j}$ is analytic on $|z|<1$. Problem 13 yields

$$\operatorname{Log} 2 = \lim_{r\to 1^-}\operatorname{Log}(1+r) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

EXERCISES 5.5: Laurent Series

1. a. $\frac{1}{z}\frac{1}{1-(-z)} = \sum_{j=-1}^{\infty}(-1)^{j+1}z^j$

   b. $\frac{1}{z^2}\left[\frac{1}{1-\left(-\frac{1}{z}\right)}\right] = \sum_{j=2}^{\infty}(-1)^j z^{-j}$

   c. $\frac{-1}{z+1}\left[\frac{1}{1-(z+1)}\right] = -\sum_{j=-1}^{\infty}(z+1)^j$

   d. $\frac{1}{(z+1)^2}\left[\frac{1}{1-\frac{1}{z+1}}\right] = \sum_{j=2}^{\infty}(z+1)^{-j}$

2. No. Theorem 14 does not apply because $\sqrt{z}$ has a branch cut.

3. Let $f(z)=\frac{z}{(z+1)(z-2)}$.

   a. $f(z)=\frac{1}{3}\left[\frac{1}{z+1}\right]+\frac{2}{3}\left[\frac{1}{z-2}\right]=\frac{1}{3}\left[\frac{1}{1-(-z)}\right]-\frac{1}{3}\left[\frac{1}{1-\frac{z}{2}}\right]$

$$= \frac{1}{3}\sum_{j=0}^{\infty}(-1)^j z^j - \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{z}{2}\right)^j = \frac{1}{3}\sum_{j=0}^{\infty}\left[(-1)^j - \left(\frac{1}{2}\right)^j\right] z^j$$

b. $$f(z) = \frac{1}{3z}\left[\frac{1}{1-\left(-\frac{1}{z}\right)}\right] - \frac{1}{3}\left[\frac{1}{1-\frac{z}{2}}\right] = \frac{1}{3}\sum_{j=1}^{\infty}(-1)^{j-1}z^{-j} - \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j z^j$$

c. $$f(z) = \frac{1}{3z}\left[\frac{1}{1-\left(-\frac{1}{z}\right)}\right] + \frac{2}{3z}\left[\frac{1}{1-\frac{2}{z}}\right]$$

$$= \frac{1}{3}\sum_{j=1}^{\infty}(-1)^{j-1}z^{-j} + \frac{2}{3}\sum_{j=1}^{\infty}2^{j-1}z^{-j}$$

$$= \frac{1}{3}\sum_{j=1}^{\infty}\left[(-1)^{j-1} + 2^j\right] z^{-j}$$

4. $$\sum_{j=0}^{\infty}\frac{(-1)^j 2^{2j+1}}{(2j+1)!} z^{2j-2}$$

5. $$\frac{1}{(z-4)^3}\left[1 + \frac{1}{4}\left[\frac{1}{1+\left(\frac{z-4}{4}\right)}\right]\right] = \frac{1}{(z-4)^3} + \frac{1}{4(z-4)^3}\sum_{j=0}^{\infty}(-1)^j\left(\frac{z-4}{4}\right)^j$$

$$= \frac{5}{4(z-4)^3} + \sum_{j=-2}^{\infty}\frac{(-1)^{j+1}}{4^{j+4}}(z-4)^j$$

6. $$\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!3^{2j}} z^{-2j+2}$$

7. a. $$e^{1/z}\frac{1}{z^2}\left[\frac{1}{1-\left(\frac{1}{z^2}\right)}\right] = \left[\sum_{j=0}^{\infty}\frac{1}{j!z^j}\right]\left[\sum_{j=0}^{\infty}\left(\frac{1}{z^2}\right)^{j+1}\right]$$

$$= \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots\right)\left(\frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{3}{2z^4} + \cdots$$

b. $\dfrac{1}{e^z-1} = \dfrac{1}{z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots} = \dfrac{1}{z}\dfrac{1}{1+\frac{1}{2}z+\frac{1}{6}z^2+\cdots} = \dfrac{1}{z}\sum_{j=0}^{\infty} a_j z^j$
where
$(a_0 + a_1 z + a_2 z^2 + \cdots)(1 + \dfrac{1}{2}z + \dfrac{1}{6}z^2 + \cdots) = 1.$
Coefficient of $1 : a_0 = 1$
Coefficient of $z : a_1 + \frac{1}{2}a_0 = 0, \quad a_1 = -\frac{1}{2}$
Coefficient of $z^2 : a_2 + \frac{1}{2}a_1 + \frac{1}{6}a_0 = 0, \quad a_2 = \frac{1}{12}$
$\dfrac{1}{e^z-1} = \dfrac{1}{z} - \dfrac{1}{2} + \dfrac{z}{12} + \cdots$

c. $\csc z = \dfrac{1}{\sin z} = \dfrac{1}{z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots}$

$= \dfrac{1}{z}\dfrac{1}{1-\frac{1}{6}z^2+\frac{1}{120}z^4-\cdots} = \dfrac{1}{z}\sum_{j=0}^{\infty} a_j z^j$ where
$\left(a_0 + a_1 z + a_2 z^2 + \cdots\right)\left(1 - \dfrac{1}{6}z^2 + \dfrac{1}{120}z^4 - \cdots\right) = 1$
Coefficient of $1 : a_0 = 1$
Coefficient of $z : a_1 = 0$
Coefficient of $z^2 : a_2 - \frac{1}{6}a_0 = 0, \quad a_2 = \frac{1}{6}$
Coefficient of $z^3 : a_3 - \frac{1}{6}a_1 = 0, \quad a_3 = 0$
Coefficient of $z^4 : a_4 - \frac{1}{6}a_2 + \frac{1}{120}a_0 = 0, \quad a_4 = \frac{7}{360}$
$\csc z = \dfrac{1}{z} + \dfrac{z}{6} + \dfrac{7z^3}{360} + \cdots$

d. $\dfrac{1}{e^{1-z}} = \dfrac{1}{e}e^z = \dfrac{1}{e}\left[1 + z + \dfrac{z^2}{2!} + \cdots\right]$
(This series is valid for all z.)

8. For $g_1(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j$ and $g_2(z) = \sum_{j=1}^{\infty} c_{-j}(z-z_0)^{-j}$ the function $f(z) = g_1(z) + g_2(z)$ is analytic in $r < |z - z_0| < R$.
Let C be $|z - z_0| = \rho$ for some $\rho : r < \rho < R$. Then, by Theorem 14, the Laurent coefficients of $f(z)$ are
$\dfrac{1}{2\pi i}\oint_C \dfrac{f(\zeta)}{(\zeta - z_0)^{j+1}}d\zeta = \dfrac{1}{2\pi i}(c_j 2\pi i) = c_j$, so the Laurent series for $f(z)$

is $\sum_{j=-\infty}^{\infty} c_j(z-z_0)^j$.

9. $\sum_{j=-\infty}^{\infty} \frac{z^j}{2^{|j|}} = \sum_{j=1}^{\infty} \frac{1}{(2z)^j} + \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j$

These geometric series converge when $\left|\frac{1}{2z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$, or $\frac{1}{2} < |z| < 2$.

10. $a_k = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta^{k+1}} d\zeta = \frac{1}{2\pi i} \oint_C \frac{\exp\left[\frac{\lambda}{2}\left(\zeta - \frac{1}{\zeta}\right)\right]}{\zeta^{k+1}} d\zeta$

$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{\exp\left[\frac{\lambda}{2}\left(e^{i\theta} - e^{-i\theta}\right)\right]}{e^{i(k+1)\theta}} ie^{i\theta} d\theta$ (using $\zeta = e^{i\theta}$)

$= \frac{1}{2\pi} \int_0^{2\pi} \exp\left[i\left(\lambda \sin\theta - k\theta\right)\right] d\theta$

$= \frac{1}{2\pi} \int_0^{2\pi} \cos\left(\lambda \sin\theta - k\theta\right) d\theta + \frac{i}{2\pi} \int_0^{2\pi} \sin\left(\lambda \sin\theta - k\theta\right) d\theta$

$= \frac{1}{2\pi} \int_0^{2\pi} \cos(k\theta - \lambda \sin\theta) d\theta + 0 = J_k(\lambda)$

11. $f_1(z) = \frac{1}{z}\left(\frac{1}{1-\alpha/z}\right) = \sum_{k=0}^{\infty} \alpha^k z^{-k-1}$

$f_n(z) = \frac{(-1)^{n-1} f_1^{(n-1)}(z)}{(n-1)!} = \sum_{j=n}^{\infty} \alpha^{j-n} \frac{(j-1)!}{(j-n)!(n-1)!} z^{-j}$

12. By Theorem 7, Section 5.3, this series converges for $|z| < R$ where R is a radius of convergence at least as big as ρ. Thus $\lim_{z\to 0} f(z)$ exists and equals a_0.

13. Let C be the positively oriented circle $|z - z_0| = \rho$ with $r < \rho < R$. Then for $j = 0, 1, 2 \ldots$

$$|a_j| = \left|\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}d\zeta\right|$$

$$\leq \frac{1}{2\pi}\int_C \left|\frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\right| d\zeta$$

$$\leq \frac{1}{2\pi}\frac{M}{\rho^{j+1}}2\pi\rho = \frac{M}{\rho^j} \text{ (Theorem 5, Section 4.2)}$$

This inequality is true for any ρ with $r < \rho < R$, so $|a_j| \leq \dfrac{M}{R^j}$
Similarly

$$|a_{-j}| = \left|\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{-j+1}}d\zeta\right|$$

$$\leq \frac{1}{2\pi}\int_C \left|f(\zeta)(\zeta - z_0)^{j-1}\right| d\zeta$$

$$\leq \frac{1}{2\pi}M\rho^{j-1}\ 2\pi\rho = M\rho^j$$

This inequality is true for any ρ with $r < \rho < R$, so $|a_{-j}| \leq Mr^j$

EXERCISES 5.6: Zeros and Singularities

1. a. pole of order 2 at 0, removable singularity at -1
   b. essential singularity at 0
   c. simple poles at $\pm i$
   d. simple poles at $2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$
   e. simple poles at $\dfrac{(2n+1)\pi}{2} \quad (n = 0, \pm 1, \pm 2, \ldots)$
   f. essential singularity at 0
   g. removable singularity at 0
   h. essential singularity at 0, simple poles at $\dfrac{1}{n\pi} \quad (n = \pm 1, \pm 2, \ldots)$

2. 8 (since $1/f(z)$ has a zero of order 8 at $z = 0$)

3. a. $\dfrac{(z-i)^2}{(z-2+3i)^5}$

   b. $z\exp[1/(z-1)]$

   c. $\dfrac{(\sin z)\exp[1/(z-i)]}{z(z-1)^6}$

   d. $\dfrac{\exp[1/(z^2-z)]}{(z-1-i)^2}$

4. Suppose that f has a zero of order m at z_0. Then by Theorem 16 there is a function $g(z)$ analytic at z_0 with $g(z_0) \neq 0$ and $f(z) = (z-z_0)^m g(z)$. Consequently $\dfrac{1}{f(z)} = \dfrac{1/g(z)}{(z-z_0)^m}$ and $1/f$ has a pole of order m at z_0 by Lemma 7.

   Conversely, if f has a pole of order m at z_0 it follows from the properties of g given in Lemma 7 that $\dfrac{1}{f(z)} = \dfrac{(z-z_0)^m}{g(z)}$ has a removable singularity at z_0. Defining $1/f(z_0) = 0$ results in $1/f$ having a zero of order m at z_0.

5. a. false b. true c. true d. false e. true

6. By Lemma 7 there is a function $g(z)$ analytic at z_0 with $g(z_0) \neq 0$ and $f(z) = (z-z_0)^{-m}g(z)$. Then

$$f'(z) = (z-z_0)^{-(m+1)}[-mg(z) + (z-z_0)g'(z)]$$

   and Lemma 7 applies. Thus, $f'(z)$ has a pole of order $m+1$.

7. an essential singularity

8. Let c be any complex number.

   $\cos(1/z) = \dfrac{e^{i/z} + e^{-i/z}}{2} = c \Longrightarrow$

   $e^{2i/z} - 2ce^{i/z} + 1 = 0 \Longrightarrow z = i/\log[c \pm \sqrt{c^2-1}]$.

   Values of the logarithm can be chosen to make $|z| < \epsilon$ for any positive ϵ, so that $\cos(1/z)$ achieves the value c in any neighborhood of $z_0 = 0$.

9. Yes; $e^{1/z}$ is bounded by one on the negative real axis.

10. By repeated applications of Theorem 16,

$$\begin{aligned} f(z) &= (z-z_1)^{m_1} g_1(z) = (z-z_1)^{m_1}(z-z_2)^{m_2} g_2(z) = \dots \\ &= (z-z_1)^{m_1}(z-z_2)^{m_2} \dots (z-z_n)^{m_n} g_n(z). \end{aligned}$$

11. (By contradiction.) Suppose Re $f(z) \le M$ in some neighborhood of z_0. Then $|e^{f(z)}| = e^{\mathrm{Re}\, f(z)} \le e^M$ in this neighborhood so that $e^{f(z)}$ has a removable singularity at z_0 by Theorem 18, and $g(z) = e^{f(z)}, z \ne z_0$ can be suitably defined at z_0 so that $g(z)$ is analytic at z_0. This means, using Problem 16 of Section 4.5, that there is a branch of $\log g(z)$ that is analytic at z_0. But then $f(z) = \log g(z) + c$ is bounded at z_0.

    A similar argument shows that Re $f(z)$ cannot be bounded from below. Use $if(z)$ to establish the result for Im $f(z)$.

12. Let $f(z) = (z-z_0)^{-m} h(z)$ as in Lemma 7.
    Then

$$\begin{aligned} g(z) = \frac{f'(z)}{f(z)} &= \frac{(z-z_0)^{-m-1}[-mh(z) + (z-z_0)h'(z)]}{(z-z_0)^{-m} h(z)} \\ &= \frac{h'(z)}{h(z)} - \frac{m}{(z-z_0)} \end{aligned}$$

    It follows that $g(z)$ has a simple pole at $z = z_0$ and that the coefficient of $(z-z_0)^{-1}$ in the Laurent series for $g(z)$ is $-m$.

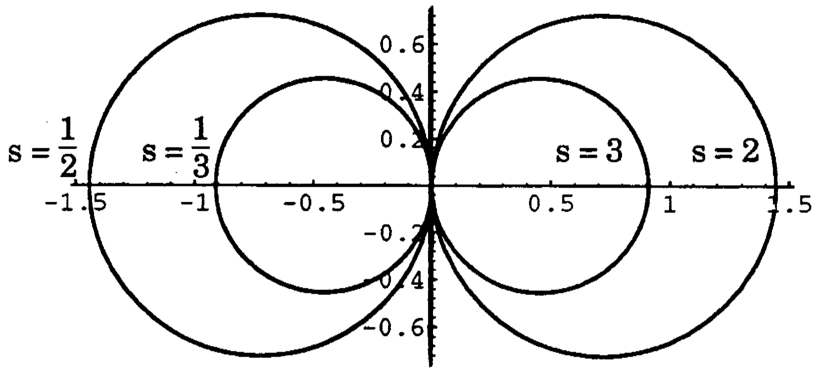
13. Suppose $|f(z)| < M$ for $0 < |z-z_0| < \rho < R$ and define C to be the positively oriented circle $|z-z_0| = \rho$.
    Then $|a_{-j}| = \left|\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{-j+1}} d\zeta\right| \le \frac{1}{2\pi}\frac{M}{\rho^{-j+1}} 2\pi\rho = \rho^j M$
    Letting $\rho \to 0$ shows that $a_{-j} = 0$.

14. (By contradiction) Assume that $|f(z) - c| \ge \delta > 0$ in $0 < |z-z_0| < \rho$ and define $g(z) = \dfrac{1}{f(z)-c}$. Then $|g(z)| \le \dfrac{1}{\delta}$ in the punctured disk so that $g(z)$ has a removable singularity at z_0 by Problem 13. If $g(z_0)$ could

be defined to be 0 then $f(z)$ would have a pole at z_0. If $g(z_0)$ could be defined to be nonzero then $f(z)$ would have a removable singularity at z_0. Either case contradicts the fact that $f(z)$ has an essential singularity at z_0.

15. Since $f(z)$ has an essential singularity at z_0, there are sequences $z_n \to z_0$ and $z'_n \to z_0$ with $f(z_n) \to \infty$ (real) and $f(z'_n) \to w_0$ (a constant). Then $e^{f(z_n)} \to \infty$ and $e^{f(z'_n)} \to e^{w_0}$. Thus $e^{f(z)}$ is neither bounded nor does it tend to infinity as $z \to z_0$. It follows that $e^{f(z)}$ has an essential singularity at z_0.

16. $\left|e^{1/z}\right| = s \Longrightarrow (x^2+y^2)\text{Log } s - x = 0$



s = 1 (y axis)

17. a. $F(z)$ has the Maclaurin series $\sum_{j=0}^{\infty} \frac{f^{(j+1)}(0)}{(j+1)!} z^j$, which converges whenever the series for $f(z)$ converges. Thus F is analytic in U.

b. $|F(\zeta)| \le \max_{z \text{ on } C_r} \left|\frac{f(z)}{z}\right| = \max_{z \text{ on } C_r} \frac{|f(z)|}{r} \le \frac{1}{r}$

c. For $\zeta \ne 0, \left|\frac{f(\zeta)}{\zeta}\right| = |F(\zeta)| \le \lim_{r \to 1^-} \frac{1}{r} = 1$. Thus $|f(\zeta)| \le |\zeta|$.

18. $|F(z_0)| = \left|\frac{f(z_0)}{z_0}\right| = 1$ implies that $F(z)$ is a constant $e^{i\theta}$ in the disk, by Theorem 23 of Section 4.6 (the maximum modulus principle). Then

for $z \neq 0$, $\frac{f(z)}{z} = e^{i\theta}$ so that $f(z) = e^{i\theta}z$.
(For $z = 0, f(0) = 0 = e^{i\theta}0$.)

Now if $|f'(0)| = 1$, then $|F(0)| = |f'(0)| = 1$ and the same proof shows that $f(z) = e^{i\theta}z$.

19. (a) $h(z) = (1/\sin z) - 1/z + 2z/(z^2-\pi^2)$

$\lim_{z\to 0}(h(z)) = \lim_{z\to 0}[(1/\sin z) - 1/z] = 0$

$\lim_{z\to \pi}(h(z)) = \lim_{z\to \pi}[(1/\sin z) + 1/(z-\pi)] = 0$

$\lim_{z\to -\pi}(h(z)) = \lim_{z\to -\pi}[(1/\sin z) + 1/(z+\pi)] = 0$

(b) $h(z) = (1/\sin z) - 1/z + 2z/(z^2-\pi^2)$

$= 0 + (1/6 - 2/\pi^2)z + (0)z^2 + (7/360 - 2/\pi^4)z^3 + (0)z^4 + \ldots,$

(radius $= 2\pi$)

(c) $\csc(z) = 1/\sin z = 1/z - 2z/(z^2-\pi^2) + h(z) = 1/z - (2/z)(1/(1-\pi^2/z^2))\} + h(z)$

$= \ldots - 2\pi^2/z^3 - 1/z + (1/6 - 2/\pi^2)z + (7/360 - 2/\pi^4)z^3 + \ldots$

EXERCISES 5.7: The Point at Infinity

1. a. essential singularity
   b. essential singularity
   c. analytic
   d. zero of order 2
   e. pole of order 2
   f. essential singularity
   g. essential singularity
   h. essential singularity
   i. analytic

2. $f\left(\frac{1}{w}\right)$ is analytic at $w = 0$.

   By Theorem 3 of Section 5.2, $f\left(\frac{1}{w}\right) = \sum_{n=0}^{\infty} a_n w^n$ with radius of convergence R and this series converges uniformly in any closed subdisk $|w| \le R' < R$. It follows by the substitution $z = \frac{1}{w}$ that $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ converges uniformly for $|z| \ge 1/R'$.

3. Let $z = \frac{1}{w}$

   a. $$\frac{z-1}{z+1} = \frac{1-w}{1+w} = (1-w)\sum_{j=0}^{\infty}(-w)^j = 1 + 2\sum_{j=1}^{\infty}(-1)^j w^j$$

   $$= 1 + 2\sum_{j=1}^{\infty}\frac{(-1)^j}{z^j},$$ convergent for $|w| < 1$, or $|z| > 1$.

   b. $$\frac{z^2}{z^2+1} = \frac{1}{1+w^2} = \sum_{j=0}^{\infty}(-1)^j w^{2j} = \sum_{j=0}^{\infty}\frac{(-1)^j}{z^{2j}},$$ convergent for $|w| < 1$, or $|z| > 1$.

c. $\dfrac{1}{z^3 - i} = \dfrac{w^3}{1 - iw^3} = w^3 \sum_{j=0}^{\infty} \left(iw^3\right)^j = \sum_{j=0}^{\infty} i^j w^{3j+3}$

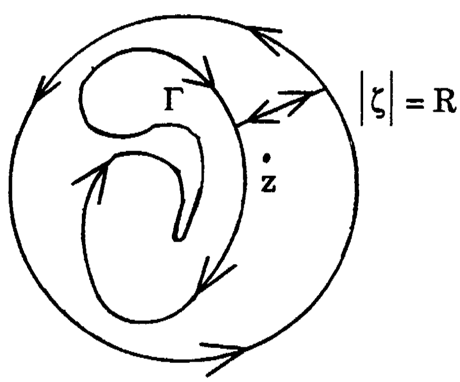
$$= \sum_{j=0}^{\infty} \frac{i^j}{z^{3j+3}}, \text{ convergent for } |w| < 1, \text{ or } |z| > 1.$$

4. A function with an essential singularity at ∞ assumes every complex number, with possibly one exception, as a value in any neighborhood $|z| > R$.

   To verify this for e^z, notice that Example 3 in Section 5.6 shows that $e^{1/w}$ attains all values but 0 in any neighborhood $|w| < \epsilon$ of 0. Thus, by substituting $z = 1/w$, it follows that e^z attains all values but 0 on the set $|z| > 1/\epsilon$.

5. $\deg Q - \deg P$

6. Let C be the simple closed contour given by Γ together with $|\zeta| = R$ and two line segments as shown. Note that Γ is negatively oriented but C is positively oriented.



By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \oint_{|\zeta| = R} \frac{f(\zeta)}{\zeta - z} d\zeta$$

The last integral goes to zero as $R \to \infty$, so $f(z) = \dfrac{1}{2\pi i} \oint_\Gamma \dfrac{f(\zeta)}{\zeta - z} d\zeta$

7. Applying Cauchy's integral theorem (Theorems 9 and 12, Section 4.4) on the contour C shown in Problem 6 yields

$$0 = \oint_C f(z)dz = \oint_\Gamma f(z)dz + \oint_{|z|=R} f(z)dz$$

Now $$\left|\oint_{|z|=R} f(z)dz\right| = \left|\oint_{|z|=R} \frac{zf(z)}{z} dz\right|$$
$$\leq 2\pi \max_{|z|=R} |zf(z)|,$$

which tends to 0 as $R \to \infty$, since $f(z)$ has a zero of order 2 or more at ∞. This leaves
$$0 = \oint_\Gamma f(z)dz.$$

If f has only a simple zero at ∞ then this integral may not vanish. As an example, take $f(z) = \dfrac{1}{z}$ and $\Gamma : |z| = 1$. Then $\oint_\Gamma \dfrac{1}{z} dz = -2\pi i$ by the Cauchy integral formula.

8. a. By Theorem 16, Section 5.6, one can express $f(z)$ as $(z-z_0)^m g(z)$ where g is analytic in a neighborhood of z_0 and $g(z_0) \neq 0$. Let $c = g(z_0)$ and $\epsilon(z) = g(z) - g(z_0)$. Then $f(z) = (z-z_0)^m[c+\epsilon(z)]$ and one can use the continuity of g at z_0 to find a neighborhood where $|\epsilon(z)| < |c|/100$.

b. By Lemma 7, Section 5.6, $f(z) = (z-z_0)^{-m} g(z)$. Let $c = g(z_0)$ and $\epsilon(z) = g(z) - g(z_0)$, and the result follows as in part a.

9. a. Let $z = x + iy$ with $x > 0$. Then

$$\mathrm{Re}\left[\frac{R/zC}{R+1/zC} + zL\right] = \mathrm{Re}\left[\frac{R(xRC+1-iyRC)}{(xRC+1)^2+(yRC)^2} + (x+iy)L\right]$$

$$= \frac{xR^2C+R}{(xRC+1)^2+(yRC)^2} + xL > 0.$$

b. $$\mathrm{Re}\left[\frac{1}{R_{eff}}\right] = \mathrm{Re}\left[\frac{zRC+1}{R+z^2LRC+zL}\right]$$

$$= \frac{(xRC+1)\left[R+x^2LRC+xL\right]+xy^2LR^2C^2}{\left[R+(x^2-y^2)LRC+xL\right]^2+\left[2xyLRC+yL\right]^2} > 0.$$

c. Let $f(z) = u(z) + iv(z)$ with $u(z) > 0$ when $\mathrm{Re}\, z > 0$. Then $\mathrm{Re}\left[\frac{1}{f(z)}\right] = \frac{u}{u^2+v^2} > 0$ when $\mathrm{Re}\, z > 0$.

10. a. f cannot have a zero at z_0 in the right half-plane because in such a case $f(z_0) = 0$ so $\mathrm{Re}\, f(z_0)$ is not positive. f cannot have a pole in the right half-plane because in such a case $f(z) \to \infty$ with all arguments as $z \to z_0$ (yielding negative values of Re $[f(z)]$ for z near z_0).

b. Let z_0 be a pure imaginary zero or pole of f and write $z - z_0 = re^{i\theta}$. Then by Problem 11,

$$f(z) = (z-z_0)^{\pm m}(c+\epsilon(z)) = r^{\pm m}e^{\pm i\theta m}(c+\epsilon(z)).$$

Since $|\epsilon(z)| < \dfrac{|c|}{100}$ for z near z_0, $f(z) \approx r^{\pm m} e^{\pm im\theta} c$. Because f is a positive function, $-\dfrac{\pi}{2} < \pm m\theta + \arg c < \dfrac{\pi}{2}$ (that is, $\operatorname{Re} f(z) > 0$) whenever $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ (that is, $\operatorname{Re} z > 0$). This inequality holds only when $m = 1$ (simple pole or zero) and c is a positive real constant.

11 Note that $\operatorname{Re} z > 0 \iff \operatorname{Re}[1/z] > 0$, so that $f(z)$ is a positive function $\iff f\left(\dfrac{1}{z}\right)$ is a positive function. By Problem 13b, the function $f\left(\dfrac{1}{z}\right)$ is analytic and nonzero at $z = 0$ or has a simple pole or zero. It follows that $f(z)$ exhibits the same behavior at ∞.

If $f(z)$ is expressed as a polynomial quotient $\dfrac{P(z)}{Q(z)}$ then

$$\deg P = \begin{cases} \deg Q + 1 & \text{for a simple pole at } \infty \\ \deg Q & \text{for analyticity at } \infty \\ \deg Q - 1 & \text{for a simple zero at } \infty \end{cases}$$

12. a. By Theorem 27, Section 4.7, the minimal value of $\operatorname{Re}[f(z)]$ over $\{z : \operatorname{Re} z \geq 0, |z| \leq R\}$ is attained on the boundary. Letting $R \to \infty$ one sees that the minimal value of $\operatorname{Re}[f(z)]$ over the closed right half-plane occurs on the imaginary axis or at ∞ (which can be considered as being on the imaginary axis).

b. From the proof of Problem 13b, near an imaginary pole $\operatorname{Re} f(z) \approx r^{-1} \cos\theta(c + \operatorname{Re}\epsilon(z))$. As $r \to 0$ these values go to $+\infty$. Thus the minimum of $\operatorname{Re} f$ must be away from a pole, and the contour argument of part a shows that the minimum is attained on the imaginary axis.

c. For z in the right half plane near any pole, $\operatorname{Re} f$ is large and positive. Reasoning as in parts a and b, the minimum of $\operatorname{Re} f(z)$ for $\operatorname{Re} z > 0$ is attained on the imaginary axis and is greater than or equal to 0. Thus $\operatorname{Re} f(z) \geq 0$ when $\operatorname{Re} z \geq 0$.

13. a. Suppose $f(z) = (z^2 - 1)^{1/3}$ is such a branch. Then $f(z)$ cannct be finite at ∞ or have an essential singularity there, since then $[f(z)]^3 = z^2 - 1$ would show the same behavior. If $f(z)$ has a

pole of order m at ∞ then it follows that $g(w) = w^m f\left(\frac{1}{w}\right)$ with $g(0) \neq 0$. However, $|g(0)| = \left|\lim_{z\to\infty} \frac{(z^2-1)^{1/3}}{z^m}\right| = 0$. Hence no such $f(z)$ can exist.

b. $f(z) = z\exp\left[\frac{1}{3}\text{Log}\left[\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right)\right]\right]$ is such a branch, since $\text{Arg}\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right) \neq \pi$ for $|z| > 3$.

EXERCISES 5.8: Analytic Continuation

1. $f(z) = z^2$, by Corollary 5.
2. Apply Corollary 5 of Theorem 22.
3. Yes; for example $\sin\left(\frac{1}{1-z}\right)$ vanishes at $z = 1 - \frac{1}{n\pi}$, $n = 1, 2, 3, \ldots$
4. f has an isolated singularity at $z = 0$. If this singularity is removable then $f(z) \equiv 0$ by Theorem 22. Otherwise f must have an essential singularity at $z = 0$, since Theorem 18, Section 5.6 shows that f cannot have a pole at $z = 0$ (since $\left|f\left(\frac{1}{n}\right)\right| \not\to \infty$ as $n \to \infty$).
5. All values except real numbers $0 \leq \alpha < 1$, since the disk of convergence is $|z - \alpha| < |1 - \alpha|$.
6. Taking $f(z) = \text{Log}\, z$ with D and z_1 as depicted in Figure 5.18, let $f_1(z)$ be the Taylor series of f centered at z_1. This series is convergent on the disk $|z - z_1| < |z_1|$. Hence $f_1(z)$ is clearly continuous across the negative real axis, so it cannot agree with $f(z) = \text{Log}\, z$.
7. Both series sum to $f(z) = \frac{1}{1-z}$, so $f(z)$ is a direct analytic continuation of both. $\Bigg($The analytic continuation of $\sum_{j=0}^{\infty} z^j$ along the top half

of the circle $|z-1|=1$ yields $\left. -\sum_{j=0}^{\infty}(2-z)^j \right)$.

8. a. $\Gamma(z+1) = \int_0^\infty e^{-t}t^z dt$

$$= -e^{-t}t^z\Big|_{t=0}^{\infty} + \int_0^\infty e^{-t}zt^{z-1}dt$$

$$= 0 + z\Gamma(z)$$

b. Repeated use of part a. yields

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n)} \text{ for } \operatorname{Re} z > 0.$$

The right side is defined and analytic whenever $\operatorname{Re}(z+n+1) > 0$ and $z \neq 0, -1, -2, \ldots, -n$.
Since this equality holds for any positive integer n, $\Gamma(z)$ can be analytically continued for any z except $z = 0, -1, -2, \ldots$.

c. $\Gamma(1) = \int_0^\infty e^{-t}dt = 1 = 0!$

For $n > 1, \Gamma(n) = (n-1)\Gamma(n-1) = \ldots = (n-1)!\Gamma(1) = (n-1)!$

9. a. No. $3 \cdot 2^{2/3} \neq 3 \cdot 2^{2/3}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

b. Yes.

c. Yes.

d. No. $\sin\sqrt{2} \neq \sin(-\sqrt{2})$

e. Yes.

f. Yes.

10. Let $\frac{p}{q}$ be any rational number with p and q relatively prime integers.

Then $\exp\left(i\frac{p}{q}\pi j!\right) = 1$ for $j \geq q$, and

$$\lim_{r\to 1^-}\left|f\left(r\exp\left(i\frac{p}{q}\pi\right)\right)\right| = \left|\lim_{r\to 1^-}\sum_{j=1}^{\infty} r^{j!}\exp\left(i\frac{p}{q}\pi j!\right)\right|$$

$$\geq \lim_{r\to 1^-}\left|\sum_{j=q}^{\infty} r^{j!}\right| - \lim_{r\to 1^-}\left|\sum_{j=1}^{q-1} r^{j!}\exp\left(i\frac{p}{q}\pi j!\right)\right|$$

$$= \infty$$

Thus $f(z)$ cannot be analytically continued to the point $z = \exp\left(i\frac{p}{q}\pi\right)$. These points are dense in the unit circle (that is, in every neighborhood of a point on the unit circle there is a point $\exp\left(i\frac{p}{q}\pi\right)$ of this form), so $f(z)$ cannot be analytically continued to any point on the unit circle or any point outside the unit circle.

11. $f(z) = zg'(z)$, so if $g(z)$ could be analytically continued beyond the unit circle, so could $f(z)$.

12. $f(z) = z + f(z^2) = z + z^2 + \cdots + z^{2^{n-1}} + f(z^{2^n})$ for any $n = 1, 2, 3, \ldots$. Since $\lim_{z\to 1}|f(z)| = \infty$ it follows that $\lim_{z\to\omega}|f(z)| = \infty$ when $\omega^{2^n} = 1$. These points ω are dense in the unit circle (that is, in every neighborhood of a point on the unit circle there is a point ω of this form), so $f(z)$ cannot be analytically continued to any point on the unit circle or any point outside the unit circle.

13. a. Let $U(x,y) = u(x,-y)$ and $V(x,y) = -v(x,-y)$.
Then $U_x(x,y) = u_x(x,-y) = v_y(x,-y) = V_y(x,y)$ and $U_y(x,y) = -u_y(x,-y) = v_x(x,-y) = -V_x(x,y)$. Thus $F(z) = U(x,y) + iV(x,y)$ satisfies the Cauchy-Riemann equations in D'.

b. From the definition of $U(x)$, $F(z)$ is continuous on $D\cup\gamma$. Now, $v(x,y)\to 0$ as (x,y) approaches γ so that $u(x,y)\to U(x)$ and $u(x,-y)\to U(x)$ as (x,y) approaches γ. Thus $F(z)$ is continuous on $D\cup\gamma\cup D'$.

c. $\int_\Gamma F(z)dz = \int_{\Gamma_1} F(z)dz + \int_{\Gamma_2} F(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\overline{\Gamma_2}} f(z)dz = 0$ since f is analytic in D.

By Morera's Theorem (Theorem 18, Section 4.5), parts b and c imply that $F(z)$ is analytic on $D \cup \gamma \cup D'$. (Alternatively, one can apply Theorem 5, Section 2.4, using parts a and b. The calculation in part a shows that the first partial derivatives are continuous in $D \cup \gamma \cup D'$.)

14 Suppose that $f(z) = u(x,y) + iv(x,y)$ is analytic in a simply connected domain D which lies on one side of a line $z = z_0 + re^{i\theta_0}$ and which has a segment γ of that line as part of its boundary. Suppose furthermore that $f(z)$ approaches $w_0 + \rho e^{i\phi_0}$ as z approaches $z_0 + re^{i\theta_0}$ within D. Then the function f can be analytically continued across γ into the domain D', which is the reflection of D across the line $z = z_0 + re^{i\theta_0}$.

Proof: Apply Problem 13 to $g(z) = \dfrac{f(z_0 + e^{i\theta_0} z) - w_0}{e^{i\phi_0}}$ to analytically continue $g(z)$ to a function $G(z)$ analytic on a domain symmetric with respect to the real axis. Then $F(z) = e^{i\phi_0} G\left(\dfrac{z - z_0}{e^{i\theta_0}}\right) + w_0$ is the desired analytic continuation of $f(z)$.

15. As in Problem 13 for the case where $\phi(x,y) \to 0$ on the real axis, a harmonic continuation on $D \cup \gamma \cup D'$ can be defined by

$$\Phi(x,y) = \begin{cases} \phi(x,y) & \text{for } z = x + iy \text{ in } D \\ 0 & \text{for } z = x \text{ on } \gamma \\ -\phi(x,-y) & \text{for } z = x + iy \text{ in } D' \end{cases}$$

Proof: By Problem 19, Section 2.5, $\phi(x,y)$ is the imaginary part of a function $f(z)$ analytic in D. Applying Problem 13 of this section yields an analytic continuation $F(z)$ whose imaginary part is the harmonic continuation $\Phi(x,y)$ that we seek.

In the case where $\dfrac{\partial \phi}{\partial y} \to 0$ on the real axis, a harmonic continuation on $D \cup \gamma \cup D'$ can be defined by

$$\Phi(x,y) = \begin{cases} \phi(x,y) & \text{for } z = x + iy \text{ in } D \\ \lim\limits_{y \to 0} \phi(x,y) & \text{for } z = x \text{ on } \gamma \\ \phi(x,-y) & \text{for } z = x + iy \text{ in } D' \end{cases}$$

Proof: Again, $\phi(x,y)$ is the imaginary part of a function $f(z)$ analytic in D. Applying Problem 13 of this section to $g(z) = if'(z)$ yields an analytic continuation $G(z)$ on $D \cup \gamma \cup D'$. By Theorem 10, Section 4.4a, $G(z)$ has an antiderivative $iF(z)$ from which $\Phi(x,y)$ can be recovered.

16. The *local* maximum of $|f(z)|$ at z_0 is a true maximum in some neighborhood of z_0, so $f(z)$ is constant in this neighborhood by Theorem 23 in Sec. 4.6. Use Corollary 5 to extend this to all of D.

CHAPTER 6: Residue Theory

EXERCISES 6.1: The Residue Theorem

1. a. $\text{Res}(2) = e^6$

   b. $\text{Res}(1) = -2$
   $\text{Res}(2) = 3$

   c. $\text{Res}(0) = 0$

   d. $\text{Res}(-1) = -6$

   e. $\text{Res}(0) = 1$
   $\text{Res}(-1) = -\dfrac{5}{2e}$

   f. $\sin\left(\dfrac{1}{3z}\right) = \dfrac{1}{3z} - \dfrac{1}{3!}\left(\dfrac{1}{3z}\right)^3 + \cdots$
   $\text{Res}(0) = \dfrac{1}{3}$

   g. $\text{Res}\left(\dfrac{\pi}{2} + n\pi\right) = \left.\dfrac{\sin z}{\frac{d}{dz}(\cos z)}\right|_{\pi/2+n\pi} = -1$ (using Example 2)

   h. $\text{Res}(n\pi) = \left.\dfrac{z-1}{\frac{d}{dz}(\sin z)}\right|_{n\pi} = (-1)^n(n\pi - 1)$ (using Example 2)

   i. $\text{Res}\ (1) = \lim\limits_{z\to 1} \dfrac{(z-1)z^2}{1-\sqrt{z}} = \lim\limits_{z\to 1} -\left(1+\sqrt{z}\right) z^2 = -2$

2. When $f(z)$ is analytic inside a simple closed positively oriented contour $\Gamma, g(z) = f(z)/(z - z_0)$ has a simple pole at z_0 (inside Γ) so
   $f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{f(z)}{z - z_0}dz = \text{Res}(g; z_0)$

3. a. $2\pi i\ \text{Res}(2) + 2\pi i\ \text{Res}(-2) = 2\pi i \dfrac{\sin 2}{4} + 2\pi i \dfrac{\sin(-2)}{-4} = \pi i \sin 2$

   b. $2\pi i\ \text{Res}(0) + 2\pi i\ \text{Res}(2) = 2\pi i\left(-\dfrac{1}{8}\right) + 2\pi i\left(\dfrac{1}{8}e^2\right) = \dfrac{\pi i}{4}\left(e^2 - 1\right)$

   c. See Problem 1, part g.
   $2\pi i\ \text{Res}(\pi/2) + 2\pi i\ \text{Res}(3\pi/2) + 2\pi i\ \text{Res}(-\pi/2) + 2\pi i\ \text{Res}(-3\pi/2)$
   $= 2\pi i(-1) + 2\pi i(-1) + 2\pi i(-1) + 2\pi i(-1) = -8\pi i$

d. $2\pi i \operatorname{Res}(0) + 2\pi i \operatorname{Res}(2) = 2\pi i\left(\dfrac{-12+5i}{100}\right) + 2\pi i \dfrac{e^{2i}}{4(2+5i)}$

$$= \pi i\left[\frac{-12+5i}{50} + \frac{(2-5i)e^{2i}}{58}\right]$$

e. $2\pi i \operatorname{Res}(0) = 2\pi i\left(\dfrac{1}{6}\right) = \dfrac{\pi i}{3}$

f. $2\pi i \operatorname{Res}\left(e^{i\pi/4}\right) + 2\pi i \operatorname{Res}\left(e^{i3\pi/4}\right) + 2\pi i \operatorname{Res}\left(e^{i5\pi/4}\right) + 2\pi i \operatorname{Res}\left(e^{i7\pi/4}\right)$

$$= 2\pi i\left[\frac{3e^{i\pi/4}+2}{4e^{i3\pi/4}} + \frac{3e^{i3\pi/4}+2}{4e^{i9\pi/4}} + \frac{3e^{i5\pi/4}+2}{4e^{i15\pi/4}} + \frac{3e^{i7\pi/4}+2}{4e^{i21\pi/4}}\right] = 0$$

g. $2\pi i \operatorname{Res}\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right) + 2\pi i \operatorname{Res}\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)$

$$= 2\pi i\left(\frac{1}{i\sqrt{3}}\right) + 2\pi i\left(\frac{1}{-i\sqrt{3}}\right) = 0$$

4. $f(z) = \cdots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$
   $f'(z) = \cdots - 2a_{-2}(z-z_0)^{-3} - a_{-1}(z-z_0)^{-2} + a_1 + 2a_2(z-z_0) + \cdots$
   $f'(z)$ has no $(z-z_0)^{-1}$ term so $\operatorname{Res}(f'; z_0) = 0$.

5. No. A simple pole requires a nonzero coefficient of $(z-z_0)^{-1}$.
   Yes. For example $\operatorname{Res}\left(\dfrac{1}{z^2}; 0\right) = 0$.

6. Write $f(z) = (z-z_0)^m h(z)$ as in Theorem 16, Section 5.6. Then

$$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1}h(z) + (z-z_0)^m h'(z)}{(z-z_0)^m h(z)}$$

$$= \frac{m}{z-z_0} + \frac{h'(z)}{h(z)}, \text{ with } \frac{h'(z)}{h(z)} \text{ analytic at } z_0.$$

   Thus $\dfrac{f'(z)}{f(z)}$ has a simple pole at z_0 with residue m.

7. $e^{1/z}\sin(1/z) = \left(1 + \dfrac{1}{z} + \dfrac{1}{2z^2} + \cdots\right)\left(\dfrac{1}{z} - \dfrac{1}{6z^3} + \cdots\right) = \left(\dfrac{1}{z} + \cdots\right)$

$$\oint_{|z|=1} e^{1/z}\sin(1/z)dz = 2\pi i \text{ Res}(0) = 2\pi i(1) = 2\pi i$$

EXERCISES 6.2: Trigonometric Integrals Over $[0, 2\pi]$

In this and the following exercises, let C denote the positively oriented unit circle $|z| = 1$ and let I denote the integral in question.

1. $I = \int_C \frac{2dz}{z^2 + 4iz - 1}$

$$= 2\pi i \text{ Res}\left(\frac{2}{z^2 + 4iz - 1}; i(-2 + \sqrt{3})\right)$$

$$= 2\pi i\left(\frac{1}{i\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$

2. $I = \frac{1}{2}\int_0^{2\pi} \frac{8}{5 + 2\cos\theta} = -4i\int_C \frac{dz}{z^2 + 5z + 1}$

$$= -4i(2\pi i) \text{ Res}\left(-\frac{5}{2} + \frac{\sqrt{21}}{2}\right) = \frac{8\pi}{\sqrt{21}}$$

3. $I = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(3 + 2\cos\theta)^2} = \frac{1}{2i}\int_C \frac{zdz}{(z^2 + 3z + 1)^2}$

$$= \frac{2\pi i}{2i} \text{ Res}\left(-\frac{3}{2} + \frac{\sqrt{5}}{2}\right) = \pi\left(\frac{3}{5\sqrt{5}}\right) = \frac{3\pi\sqrt{5}}{25}$$

4. $I = 4i\int_C \frac{zdz}{(z^2 + 2z - 1)(z^2 - 2z - 1)}$

$$= 4i(2\pi i)\left[\text{ Res}(-1 + \sqrt{2}) + \text{ Res}(1 - \sqrt{2})\right]$$

$$= -8\pi\left(-\frac{1}{8\sqrt{2}} - \frac{1}{8\sqrt{2}}\right) = \pi\sqrt{2}$$

(Use Example 2, Section 6.1 to find the residues).

5. $I = \frac{2}{ai}\int_C \frac{dz}{z^2 + \frac{2}{a}z + 1} = \frac{2}{ai}(2\pi i)\ \text{Res}\left(\frac{-1+\sqrt{1-a^2}}{a}\right)$

$$= \frac{4\pi}{a}\left(\frac{a}{2\sqrt{1-a^2}}\right) = \frac{2\pi}{\sqrt{1-a^2}}$$

6. $I = \frac{-1}{2bi}\int_C \frac{(z^2-1)^2 dz}{z^2\left(z^2 + \frac{2a}{b}z + 1\right)}$

$$= \frac{-2\pi i}{2bi}\left[\text{Res}(0) + \text{Res}\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)\right]$$

$$= -\frac{\pi}{b}\left[\frac{-2a}{b} + \frac{2\sqrt{a^2-b^2}}{b}\right]$$

$$= \frac{2\pi}{b^2}\left(a - \sqrt{a^2-b^2}\right)$$

7. $I = \frac{8}{i}\int_C \left[\frac{z^3}{\left(z-\sqrt{a}-\sqrt{a+1}\right)^2\left(z-\sqrt{a}+\sqrt{a+1}\right)^2} \cdot \frac{1}{\left(z+\sqrt{a}-\sqrt{a+1}\right)^2\left(z+\sqrt{a}+\sqrt{a+1}\right)^2}\right] dz$

$$= \frac{8}{i}(2\pi i)\left[\text{Res}\left(\sqrt{a}-\sqrt{a+1}\right) + \text{Res}\left(-\sqrt{a}+\sqrt{a+1}\right)\right]$$

$$= 16\pi\left[\frac{2a+1}{64\sqrt{(a^2+a)^3}} + \frac{2a+1}{64\sqrt{(a^2+a)^3}}\right] \text{ (tedious calculation)}$$

$$= \frac{\pi(2a+1)}{2\sqrt{(a^2+a)^3}}$$

8. $I = \frac{4}{i(b^2-a^2)}\int_C \frac{z\,dz}{\left(z^2 - \frac{a+b}{a-b}\right)\left(z^2 - \frac{a-b}{a+b}\right)}$

$$= \frac{4(2\pi i)}{i(b^2-a^2)}\left[\text{Res}\left(\sqrt{\left|\frac{a-b}{a+b}\right|}\right) + \text{Res}\left(-\sqrt{\left|\frac{a-b}{a+b}\right|}\right)\right]$$

$$= \frac{8\pi}{b^2-a^2}\left[\frac{b^2-a^2}{8ab}+\frac{b^2-a^2}{8ab}\right] = \frac{2\pi}{ab}$$

9. $I = \dfrac{1}{2^{2n}i}\displaystyle\int_C \frac{1}{z}\left(z+\frac{1}{z}\right)^{2n} dz$

   By the binomial theorem (see Problem 27 in Section 1.1) the coefficient of the z^{-1} term of the integrand is $\dfrac{(2n)!}{n!n!}$.

   $I = \dfrac{1}{2^{2n}i}2\pi i\dfrac{(2n)!}{(n!)^2} = \dfrac{\pi(2n)!}{2^{2n-1}(n!)^2}$

10. $I = \dfrac{1}{2}\displaystyle\int_0^{2\pi} e^{\cos\theta}\left(e^{i(n\theta-\sin\theta)}+e^{-i(n\theta-\sin\theta)}\right) d\theta$

$$= \frac{1}{2}\int_0^{2\pi}\left(e^{in\theta}e^{\cos\theta-i\sin\theta}+e^{-in\theta}e^{\cos\theta+i\sin\theta}\right) d\theta$$

$$= \frac{1}{2i}\int_C \frac{z^n e^{1/z}+z^{-n}e^z}{z} dz$$

$$= \frac{1}{2i}\int_C z^{n-1}\left(1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3!z^3}+\cdots\right) dz + \frac{1}{2i}\int_C z^{-n-1}\left(1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\cdots\right) dz$$

$$= \frac{2\pi i}{2i}\left(\frac{1}{n!}\right)+\frac{2\pi i}{2i}\left(\frac{1}{n!}\right) = \frac{2\pi}{n!}$$

11. $I = \dfrac{1}{2i}\displaystyle\int_0^{2\pi}\frac{e^{i(\theta+ia)}-e^{-i(\theta+ia)}}{e^{i(\theta+ia)}+e^{-i(\theta+ia)}} d\theta$

$$= \frac{-1}{2}\int_C \frac{ze^{-a}-z^{-1}e^a}{ze^{-a}+z^{-1}e^a}\frac{dz}{z}$$

$$= \frac{-1}{2}\int_C \frac{z^2 - e^{2a}}{z(z^2 + e^{2a})}dz.$$

If $a > 0$ then $I = -\frac{1}{2}(2\pi i)\ \mathrm{Res}(0) = -\pi i(-1) = \pi i$

If $a < 0$ then $I = -\frac{1}{2}(2\pi i)\left[\ \mathrm{Res}(0) + \ \mathrm{Res}(ie^a) + \ \mathrm{Res}(-ie^a)\right]$

$$= -\pi i\left[-1 + 1 + 1\right] = -\pi i$$

$I = (\text{sign } a)\pi i$

EXERCISES 6.3: Improper Integrals of Certain Functions Over $(-\infty, \infty)$

1. $I = 2\pi i\ \mathrm{Res}(-1 + i) = \pi$

2. $I = 2\pi i\ \mathrm{Res}(3i) = 2\pi i\left(\frac{1}{12i}\right) = \frac{\pi}{6}$

3. $I = \frac{1}{2}(2\pi i)\left[\ \mathrm{Res}\left(e^{\pi i/4}\right) + \ \mathrm{Res}\left(e^{3\pi i/4}\right)\right]$

$$= \pi i\left[\frac{1+i}{4\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)} + \frac{1-i}{4\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)}\right] = \frac{\pi}{\sqrt{2}}$$

(Use Example 2, Section 6.1 to evaluate the residues)

4. $I = 2\pi i\left[\ \mathrm{Res}(i) + \ \mathrm{Res}(2i)\right] = 2\pi i\left[\frac{1}{6i} - \frac{1}{12i}\right] = \frac{\pi}{6}$

5. $I = 2\pi i\ \mathrm{Res}(-2 + 3i) = 2\pi i\left(\frac{-1}{54i}\right) = -\frac{\pi}{27}$

6. $I = \frac{1}{2}(2\pi i)\left[\ \mathrm{Res}(i) + \ \mathrm{Res}(2i)\right] = \pi i\left[\frac{-1}{6i} + \frac{1}{3i}\right] = \frac{\pi}{6}$

7. $I = \frac{1}{2}(2\pi i)\left[\text{ Res }\left(e^{i\pi/4}\right) + \text{ Res }\left(e^{i3\pi/4}\right)\right]$

$$= \pi i\left[\frac{d}{dz}\left(\frac{z^6}{\left(z - e^{i3\pi/4}\right)^2\left(z - e^{i5\pi/4}\right)^2\left(z - e^{i7\pi/4}\right)^2}\right)\Bigg|_{z=e^{i\pi/4}}\right.$$

$$\left. + \frac{d}{dz}\left(\frac{z^6}{\left(z - e^{i\pi/4}\right)^2\left(z - e^{i5\pi/4}\right)^2\left(z - e^{i7\pi/4}\right)^2}\right)\Bigg|_{z=e^{i3\pi/4}}\right]$$

$$= \pi i\left[\frac{3\sqrt{2} - 3\sqrt{2}i}{32} + \frac{-3\sqrt{2} - 3\sqrt{2}i}{32}\right] = \frac{3\pi\sqrt{2}}{16}$$

8. Let C_ρ^- be the half circle $z = \rho e^{it}$, $\pi \le t \le 2\pi$, and let γ_ρ be the real segment $z = -t$, $-\rho \le t \le \rho$. Let $\Gamma_\rho = \gamma_\rho + C_\rho^-$ (which is positively oriented). $\int_{C_\rho^-} f(z)dz$ approaches zero as ρ approaches ∞, as in the proof of Lemma 1.

Thus p.v. $\int_{-\infty}^{\infty} f(x)dx = -\lim_{\rho\to\infty}\int_{\gamma_\rho} f(z)dz = -\lim_{\rho\to\infty}\int_{\Gamma_\rho} f(z)dz$

$= -2\pi i \sum$ [residues of $f(z)$ at the

poles in the lower half-plane]

9. Let $\Gamma_\rho = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where

$$\begin{array}{ll} \gamma_1 : z = t, & -\rho \le t \le \rho \\ \gamma_2 : z = \rho + it, & 0 \le t \le 1 \\ \gamma_3 : z = -t + i, & -\rho \le t \le \rho \\ \gamma_4 : z = -\rho + i(1-t), & 0 \le t \le 1 \end{array}$$

$\frac{e^{2z}}{\cosh(\pi z)} = \frac{2e^{(2+\pi)z}}{e^{2\pi z}+1}$ has simple poles at $(1+2k)\frac{i}{2}$, $k = 0, \pm 1, \pm 2 \ldots.$

Thus $\int_{\Gamma_\rho} \frac{2e^{(2+\pi)z}}{e^{2\pi z}+1}dz = 2\pi i \text{ Res}\left(\frac{i}{2}\right) = 2e^i$

Looking at the component integrals,

$$\lim_{\rho\to\infty}\left|\int_{\gamma_2}\frac{2e^{(2+\pi)z}}{e^{2\pi z}+1}dz\right| \le \lim_{\rho\to\infty}\frac{2e^{(2+\pi)\rho}}{e^{2\pi\rho}-1} = 0$$

$$\lim_{\rho\to\infty}\left|\int_{\gamma_4} f(z)dz\right| \le \lim_{\rho\to\infty}\frac{2e^{-(2+\pi)\rho}}{1-e^{-2\pi\rho}} = 0$$

$$\lim_{\rho\to\infty}\int_{\gamma_1} f(z)dz = \text{ p.v. } \int_{-\infty}^{\infty} f(x)dx$$

$$\lim_{\rho\to\infty}\int_{\gamma_3} f(z)dz = \text{ p.v. } \int_{-\infty}^{\infty}\frac{2e^{(2+\pi)(-t+i)}}{e^{2\pi(-t+i)}+1}(-1)dt$$

$$= e^{2i}\text{ p.v. }\int_{-\infty}^{\infty}\frac{2e^{-(2+\pi)t}}{e^{-2\pi t}+1}dt = e^{2i}\text{ p.v. }\int_{-\infty}^{\infty} f(x)dx$$

Summarizing,

$$2e^{i} = \lim_{\rho\to\infty}\int_{\Gamma_\rho} f(z)dz = \left(1+e^{2i}\right)\text{ p.v. }\int_{-\infty}^{\infty} f(x)dx.$$

Hence $\text{p.v. }\displaystyle\int_{-\infty}^{\infty} f(x)dx = \frac{2e^{i}}{1+e^{2i}} = \frac{2}{e^{-i}+e^{i}} = \sec 1$

10. Let $\Gamma_\rho = \gamma_1+\gamma_2+\gamma_3+\gamma_4$ where

$\gamma_1 : z = t, \quad 0 \le t \le \rho$

$\gamma_2 : z = \rho + it, \quad 0 \le t \le \lambda$

$\gamma_3 : z = -t + i\lambda, \quad -\rho \le t \le 0$

$\gamma_4 : z = i(\lambda - t), \quad 0 \le t \le \lambda$

e^{-z^2} is entire, so $\displaystyle\int_{\Gamma_\rho} e^{-z^2}dz = 0$

$$\lim_{\rho\to\infty}\int_{\gamma_1} e^{-z^2}dz = \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

$$\lim_{\rho\to\infty}\left|\int_{\gamma_2} e^{-z^2}dz\right| = \lim_{\rho\to\infty}\left|\int_0^{\lambda} e^{-(\rho+it)^2}idt\right| \le \lim_{\rho\to\infty} e^{-\rho^2+\lambda^2}\lambda = 0$$

$$\lim_{\rho\to\infty}\int_{\gamma_3} e^{-z^2}dz = \int_{-\infty}^{0} e^{-(-t+i\lambda)^2}(-1)dt = -e^{\lambda^2}\int_0^{\infty} e^{-t^2-2\lambda t i}dt$$

$$= -e^{\lambda^2}\int_0^{\infty}\left[e^{-x^2}\cos(2\lambda x) - ie^{-x^2}\sin(2\lambda x)\right]dx$$

$$\lim_{\rho\to\infty}\int_{\gamma_4} e^{-z^2}dz = \int_0^{\lambda} e^{(\lambda-t)^2}(-i)dt = -i\int_0^{\lambda} e^{y^2}dy$$

Summarizing,

$$0 = \int_{\Gamma_\rho} e^{-z^2} dz$$

$$= \frac{\sqrt{\pi}}{2} - e^{\lambda^2} \int_0^\infty \left[e^{-x^2} \cos(2\lambda x) - ie^{-x^2} \sin(2\lambda x)\right] dx - i \int_0^\lambda e^{y^2} dy$$

Take the real and imaginary parts to obtain

$\int_0^\infty e^{-x^2} \cos(2\lambda x) dx = \frac{\sqrt{\pi}}{2} e^{-\lambda^2}$

and $\int_0^\infty e^{-x^2} \sin(2\lambda x) dx = e^{-\lambda^2} \int_0^\lambda e^{y^2} dy$

11. $S_\rho = \gamma_1 + \gamma_2 + \gamma_3$ where

$\gamma_1 : z = t, \quad 0 \le t \le \rho$

$\gamma_2 : z = \rho e^{it}, \quad 0 \le t \le 2\pi/3$

$\gamma_3 : z = -te^{2\pi i/3}, \quad -\rho \le t \le 0$

$\int_{S_\rho} \frac{dz}{z^3+1} = 2\pi i \text{ Res } \left(e^{i\pi/3}\right) = \frac{2\pi i}{3} e^{-2\pi i/3}$

$\lim_{\rho\to\infty} \int_{\gamma_1} \frac{dz}{z^3+1} = \int_0^\infty \frac{dx}{x^3+1}$

$\lim_{\rho\to\infty} \left|\int_{\gamma_2} \frac{dz}{z^3+1}\right| = \lim_{\rho\to\infty} \left|\int_0^{2\pi/3} \frac{i\rho e^{it}}{\rho^3 e^{i3t}+1} dt\right| \le \lim_{\rho\to\infty} \frac{\rho}{\rho^3-1} \frac{2\pi}{3} = 0$

$\lim_{\rho\to\infty} \int_{\gamma_3} \frac{dz}{z^3+1} = \int_{-\infty}^0 \frac{-e^{2\pi i/3}}{-t^3+1} dt = -e^{2\pi i/3} \int_0^\infty \frac{dx}{x^3+1}$

Summarizing,

$\frac{2\pi i}{3} e^{-2\pi i/3} = \lim_{\rho\to\infty} \int_{S_\rho} \frac{dz}{z^3+1} = \left(1 - e^{2\pi i/3}\right) \int_0^\infty \frac{dx}{x^3+1}$

Hence, $\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi i}{3e^{2\pi i/3}\left(1-e^{2\pi i/3}\right)} = \frac{2\pi\sqrt{3}}{9}$

12. Let $f(\xi) = \frac{\xi}{(\xi^2+1)(\xi-z)}$. Integrate over a semicircular arc in the upper half-plane.

For $\operatorname{Im} z > 0$,

$$\frac{1}{\pi i} \int_{-\infty}^\infty f(\xi) d\xi = \frac{2\pi i}{\pi i} \left[\text{ Res}(i) + \text{ Res}(z)\right]$$

$$= 2\left[\frac{i}{2i(i-z)} + \frac{z}{z^2+1}\right] = \frac{1}{z+i}$$

For $\operatorname{Im} z < 0, \frac{1}{\pi i}\int_{-\infty}^{\infty} f(\xi)d\xi = \frac{2\pi i}{\pi i}\operatorname{Res}(i) = \frac{-1}{z-i}$

13. $\int_{-\infty}^{\infty}(1+x^2)^{-n-1}dx = 2\pi i\operatorname{Res}(i)$

$$= 2\pi i \lim_{z\to i}\frac{1}{n!}\frac{d^n}{dz^n}\left[(z+i)^{-n-1}\right]$$

$$= \frac{2\pi i}{n!}(-1)^n(n+1)(n+2)\cdots(n+n)(2i)^{-2n-1}$$

$$= \frac{\pi(2n)!}{2^{2n}(n!)^2}$$

14. a. $g(z)$ has simple poles at $z = k,\ \ k = 0, \pm 1, \pm 2, \ldots$

$$\operatorname{Res}(g;k) = \lim_{z\to k}\frac{\pi f(z)\cos(\pi z)}{\frac{d}{dz}(\sin(\pi z))} = f(k)$$

b. On the parts of Γ_N where $|y| \geq \frac{1}{2}$,

$$|\cot(\pi z)| = \left|\frac{e^{\pi iz}+e^{-\pi iz}}{e^{\pi iz}-e^{-\pi iz}}\right| \leq \left|\frac{e^{-\pi y}+e^{\pi y}}{e^{-\pi y}-e^{\pi y}}\right| \leq \frac{1+e^{-\pi}}{1-e^{-\pi}}$$

On the parts of Γ_N where $|y| \leq \frac{1}{2}$,

$$|\cot(\pi z)| = \left|\cot\pi\left(\pm N \pm \frac{1}{2} + iy\right)\right| = |\tan(\pi y i)|$$

$$= |\tanh \pi y| \leq \tanh\left(\frac{\pi}{2}\right).$$

Thus $|\cot(\pi z)|$ is bounded by some M on Γ_N, independent of N.

c. Since $f(z)$ is a rational function $P(z)/Q(z)$ with $\deg Q \geq 2 + \deg P$

one can form a bound $|f(z)| \le \dfrac{m}{\left(N+\frac{1}{2}\right)^2}$ for z on Γ_N, N large.

Then

$$\lim_{N\to\infty}\left|\int_{\Gamma_N}\pi f(z)\cot(\pi z)dz\right| \le \lim_{N\to\infty}\max_{z \text{ on } \Gamma_N}|f(z)\pi\cot(\pi z)|\, l(\Gamma_N)$$

$$= \lim_{N\to\infty}\frac{m}{\left(N+\frac{1}{2}\right)^2}M(8N+4) = 0$$

d. $0 = \lim_{N\to\infty}\int_{\Gamma_N}\pi f(z)\cot(\pi z)dz$

$= 2\pi i \sum$(residues of $\pi f(z)\cot(\pi z)$ at poles of $\cot(\pi z)$ and poles of $f(z)$)

$$= 2\pi i\left[\sum_{k=-\infty}^{\infty} f(k) + \sum(\text{residues at poles of } f(z))\right]$$

Thus $\displaystyle\sum_{k=-\infty}^{\infty} f(k) = -\sum$(residues of $\pi f(z)\cot(\pi z)$ at poles of $f(z)$)

15. a. $$\sum_{k=-\infty}^{\infty}\frac{1}{k^2+1} = -\text{ Res}\left(\frac{\pi\cot(\pi z)}{z^2+1}; i\right) - \text{ Res}\left(\frac{\pi\cot(\pi z)}{z^2+1}; -i\right)$$

$$= \frac{-\pi\cot(\pi i)}{2i} - \frac{\pi\cot(-\pi i)}{-2i} = \pi\coth(\pi)$$

b. $$\sum_{k=-\infty}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2} = -\text{ Res}\left(\frac{\pi\cot(\pi z)}{\left(z-\frac{1}{2}\right)^2}; \frac{1}{2}\right) = -\frac{d}{dz}(\pi\cot(\pi z))\Big|_{z=\frac{1}{2}} = \pi^2$$

c. $$0 = \sum_{k=-\infty}^{\infty}\text{ Res}\left(\frac{\pi\cot(\pi z)}{z^2}; k\right)$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{k^2} + \text{ Res}\left(\frac{\pi\cot(\pi z)}{z^2}; 0\right).$$

Since

$$\frac{\pi\cot\pi z}{z^2} = \frac{\pi}{z^2}\left(\frac{1}{\pi z} - \frac{\pi z}{3} - \frac{(\pi z)^3}{45} + \cdots\right) = \frac{1}{z^3} - \frac{\pi^2}{3z} - \cdots,$$

$$\text{Res}\left(\frac{\pi\cot\pi z}{z^2}; 0\right) = -\frac{\pi^2}{3}.$$

Thus , $0 = 2\sum_{k=1}^{\infty}\frac{1}{k^2} - \frac{\pi^2}{3}$ and $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$

16. $\pi z\cot(\pi z) = \pi z i\dfrac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = \pi z i\dfrac{e^{i2\pi z} + 1}{e^{i2\pi z} - 1}$

$$= \frac{1}{2}\left[\frac{-2\pi z i}{e^{-2\pi z i} - 1} + \frac{2\pi z i}{e^{2\pi z i} - 1}\right]$$

$$= \frac{1}{2}\sum_{j=0}^{\infty}\frac{B_j}{j!}(-2\pi z i)^j + \frac{1}{2}\sum_{j=0}^{\infty}\frac{B_j}{j!}(2\pi z i)^j$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{B_{2k}}{(2k)!}(2\pi z)^{2k} \text{ (odd powers cancel)}$$

$$0 = \sum_{k=-\infty}^{\infty} \text{Res}\left(\frac{\pi\cot(\pi z)}{z^{2n}}; k\right)$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{k^{2n}} + \text{Res}\left(\frac{\pi z\cot(\pi z)}{z^{2n+1}}; 0\right)$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{k^{2n}} + (-1)^n\frac{B_{2n}}{(2n)!}(2\pi)^{2n}$$

Thus $\sum_{k=1}^{\infty}\frac{1}{k^{2n}} = (-1)^{n-1}\pi^{2n}\frac{2^{2n-1}}{(2n)!}B_{2n}$

17. a. $\sum_{k=-\infty}^{\infty}\dfrac{1}{(k+a)^2} = -\text{Res}\left(\dfrac{\pi\cot(\pi z)}{(z+a)^2}; -a\right)$(Problem 14)

$$= \pi^2\csc^2(\pi a)$$

b. $$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+a^2} = -\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2+a^2}; ai\right) - \operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2+a^2}; -ai\right)$$

$$= \frac{-\pi\cot(\pi ai)}{2ai} - \frac{\pi\cot(-\pi ai)}{-2ai} = \frac{\pi}{a}\coth(\pi a)$$

c. $$\sum_{k=-\infty}^{\infty} \frac{k^2-a^2}{(k^2+a^2)^2} = -\operatorname{Res}\left(\frac{(z^2-a^2)\pi\cot(\pi z)}{(z^2+a^2)^2}; ai\right)$$

$$-\operatorname{Res}\left(\frac{(z^2-a^2)\pi\cot(\pi z)}{(z^2+a^2)^2}; -ai\right)$$

$$= \frac{-\pi^2}{2}\operatorname{csch}^2(\pi a) - \frac{\pi^2}{2}\operatorname{csch}^2(\pi a) = -\pi^2\operatorname{csch}^2(\pi a)$$

d. $$\sum_{k=-\infty}^{\infty} \frac{1}{(k-r)^2+a^2}$$

$$= -\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{(z-r)^2+a^2}; r+ai\right) - \operatorname{Res}\left(\frac{\pi\cot(\pi z)}{(z-r)^2+a^2}; r-ai\right)$$

$$= \frac{-\pi\cot(\pi r+\pi ai)}{2ai} + \frac{\pi\cot(\pi r-\pi ai)}{2ai}$$

$$= \frac{\pi i}{2a}\left[\frac{\cos(\pi r+\pi ai)\sin(\pi r-\pi ai)}{\sin(\pi r+\pi ai)\sin(\pi r-\pi ai)}\right.$$

$$\left.-\frac{\sin(\pi r+\pi ai)\cos(\pi r-\pi ai)}{\sin(\pi r+\pi ai)\sin(\pi r-\pi ai)}\right]$$

$$= \frac{\pi i}{2a}\left[\frac{\sin(-2\pi ai)}{\sin^2(\pi r)\cos^2(\pi ai)-\cos^2(\pi r)\sin^2(\pi ai)}\right]$$

$$= \frac{\pi}{2a}\left[\frac{\sinh(2\pi a)}{\sin^2(\pi r)+\sinh^2(\pi a)}\right]$$

e. $$\sum_{k=-\infty}^{\infty} \frac{(k-r)^2-a^2}{[(k-r)^2+a^2]^2}$$

$$= -\operatorname{Res}\left(\frac{\pi\cot(\pi z)[(z-r)^2-a^2]}{[(z-r)^2+a^2]^2}; r+ai\right)$$

$$-\operatorname{Res}\left(\frac{\pi\cot(\pi z)[(z-r)^2-a^2]}{[(z-r)^2+a^2]^2}; r-ai\right)$$

$$= \frac{\pi^2}{2}\csc^2(\pi r+\pi ai) + \frac{\pi^2}{2}\csc^2(\pi r-\pi ai)$$

$$= \frac{\pi^2}{2}\left[\frac{\sin^2(\pi r-\pi ai)+\sin^2(\pi r+\pi ai)}{\sin^2(\pi r+\pi ai)\sin^2(\pi r-\pi ai)}\right]$$

$$= \frac{\pi^2}{2}\left[\frac{1-\frac{1}{2}\cos(2\pi r-2\pi ai)-\frac{1}{2}\cos(2\pi r+2\pi ai)}{\left[\sin^2(\pi r)+\sinh^2(\pi a)\right]^2}\right]$$

$$= \frac{\pi^2}{2}\left[\frac{1-\cos(2\pi r)\cosh(2\pi a)}{[\sin^2(\pi r)+\sinh^2(\pi a)]^2}\right]$$

f. a noninteger in part a.
ai noninteger in parts b,c.
$r \pm ai$ noninteger in parts d, e.

18. Follow the structure of Problem 14.

a. $\pi f(z)\csc(\pi z)$ has simple poles at $z = k, k = 0, \pm 1, \pm 2 \ldots$
$\operatorname{Res}(\pi f(z)\csc(\pi z); k) = (-1)^k f(k)$

b. On the parts of Γ_N where $|y| \geq \frac{1}{2}$,
$$|\csc(\pi z)| = \left|\frac{2}{e^{\pi iz}-e^{-\pi iz}}\right| \leq \left|\frac{2}{e^{\pi y}-e^{-\pi y}}\right| \leq \frac{2}{e^{\pi/2}-e^{-\pi/2}}$$
On the parts of Γ_N where $|y| \leq \dfrac{1}{2}$,
$$|\csc(\pi z)| = \left|\csc\pi(\pm N \pm \frac{1}{2} + iy)\right| = |\operatorname{sech}(\pi y)| \leq 1.$$
Thus $|\csc(\pi z)|$ is bounded on Γ_N, independent of N.

c. $\displaystyle\lim_{N\to\infty}\int_{\Gamma_N}\pi f(z)\csc(\pi z)dz = 0$ as in Problem 14.

d. $0 = \lim_{N\to\infty} \int_{\Gamma_N} \pi f(z)\csc(\pi z)dz$

$= 2\pi i \sum$(residues of $\pi f(z)\csc(\pi z)$ at poles of $\csc(\pi z)$ and poles of $f(z)$)

$$= 2\pi i \left[\sum_{k=-\infty}^{\infty} (-1)^k f(k) + \sum \text{ (residues at poles of } f(z))\right]$$

Thus $\sum_{k=-\infty}^{\infty} (-1)^k f(k)$

$= -\sum$(residues of $\pi f(z)\csc(\pi z)$ at poles of $f(z)$)

19. $0 = \sum_{k=-\infty}^{\infty} \text{ Res}\left(\frac{\pi\csc(\pi z)}{z^2}; k\right)$

$$= 2\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} + \text{ Res}\left(\frac{\pi\csc(\pi z)}{z^2}; 0\right)$$

Since $\frac{\pi}{z^2}\csc(\pi z) = \frac{\pi}{z^2}\left(\frac{1}{\pi z} + \frac{\pi z}{6} + \frac{7(\pi z)^3}{360} + \cdots\right) = \frac{1}{z^3} + \frac{\pi^2}{6z} + \cdots,$

$\text{Res}\left(\frac{\pi\csc(\pi z)}{z^2}; 0\right) = \frac{\pi^2}{6}.$

Thus, $0 = 2\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} + \frac{\pi^2}{6}$ and $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = \frac{-\pi^2}{12}$

EXERCISES 6.4: Improper Integrals Involving Trigonometric Functions; Jordan's Lemma

1. $I = \text{ Re }\left[2\pi i \text{ Res}\left(\frac{e^{2iz}}{z^2+1}; i\right)\right] = \frac{\pi}{e^2}$

2. $I = \text{ Im }\left[2\pi i \text{ Res}\left(\frac{ze^{iz}}{z^2-2z+10}; 1+3i\right)\right] = \text{ Im }\left[2\pi i\frac{(1+3i)e^{i(1+3i)}}{6i}\right]$

$$= \frac{\pi}{3e^3}(3\cos 1 + \sin 1)$$

3. $I = \dfrac{1}{2}\,\text{Re}\left[2\pi i\,\text{Res}\left(\dfrac{e^{iz}}{(z^2+1)^2}; i\right)\right] = \dfrac{\pi}{2e}$

4. $I = 2\pi i\,\text{Res}\left(\dfrac{e^{3iz}}{z-2i}; 2i\right) = \dfrac{2\pi i}{e^6}$

5. $$\begin{aligned} I &= \text{Im}\left[2\pi i\,\text{Res}\left(\frac{ze^{3iz}}{z^4+4}; 1+i\right) + 2\pi i\,\text{Res}\left(\frac{ze^{3iz}}{z^4+4}; -1+i\right)\right] \\ &= \text{Im}\left[2\pi i\left(\frac{e^{-3+3i}}{8i} - \frac{e^{-3-3i}}{8i}\right)\right] \\ &= \frac{\pi \sin 3}{2e^3} \end{aligned}$$

6. $I = -2\pi i\,\text{Res}\left(\dfrac{e^{-2iz}}{z^2+4}; -2i\right) = \dfrac{\pi}{2e^4}$ (use contour in lower half plane)

7. $$\begin{aligned} I &= \text{Re}\left[2\pi i\,\text{Res}\left(\frac{e^{iz}}{(z^2+1)(z^2+4)}; i\right) + 2\pi i\,\text{Res}\left(\frac{e^{iz}}{(z^2+1)(z^2+4)}; 2i\right)\right] \\ &= \text{Re}\left[2\pi i\left(\frac{e^{-1}}{6i} + \frac{e^{-2}}{-12i}\right)\right] = \frac{\pi}{3e}\left(1 - \frac{1}{2e}\right) \end{aligned}$$

8. $I = \dfrac{1}{2}\,\text{Im}\left[2\pi i\,\text{Res}\left(\dfrac{z^3e^{2iz}}{(z^2+1)^2}; i\right)\right] = \dfrac{1}{2}\,\text{Im}\,(0) = 0$

9. $$\begin{aligned} I &= \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{2ix} + e^{-2ix}}{2(x-3i)}\,dx \\ &= \lim_{\rho\to\infty} \int_{C_\rho^+} \frac{e^{2iz}}{2(z-3i)}\,dz - \lim_{\rho\to\infty} \int_{C_\rho^-} \frac{e^{-2iz}}{2(z-3i)}\,dz \\ &= 2\pi i\,\text{Res}\left(\frac{e^{2iz}}{2(z-3i)}; 3i\right) = \frac{\pi i}{e^6} \end{aligned}$$

10. $I = \text{p.v.} \displaystyle\int_{-\infty}^{\infty} \frac{e^{ix} + e^{-ix}}{2(x-w)}\,dx$

$$= \lim_{\rho\to\infty} \int_{C_\rho^+} \frac{e^{iz}}{2(z-w)} dz - \lim_{\rho\to\infty} \int_{C_\rho^-} \frac{e^{-iz}}{2(z-w)} dz$$

$$= \begin{cases} 2\pi i \operatorname{Res}\left(\dfrac{e^{iz}}{2(z-w)}; w\right) = \pi i e^{iw} & \text{if } \operatorname{Im} w > 0 \\ -2\pi i \operatorname{Res}\left(\dfrac{e^{-iz}}{2(z-w)}; w\right) = -\pi i e^{-iw} & \text{if } \operatorname{Im} w < 0 \end{cases}$$

11. $m > 0, \deg P < \deg Q$ (Jordan's Lemma)

12. Let $\Gamma_\rho = \gamma_1 + \gamma_2 + \gamma_3$ where

$\gamma_1 : z(t) = t, \qquad 0 \le t \le \rho$
$\gamma_2 : z(t) = \rho e^{it} \qquad 0 \le t \le \pi/4$
$\gamma_3 : z(t) = -te^{\pi i/4} \qquad -\rho \le t \le 0$

e^{iz^2} is entire, so $\int_{\Gamma_\rho} e^{iz^2} dz = 0$.

$$\lim_{\rho\to\infty} \int_{\gamma_1} e^{iz^2} dz = \int_0^\infty e^{ix^2} dx.$$

$$\lim_{\rho\to\infty} \left| \int_{\gamma_2} e^{iz^2} dz \right| = \lim_{\rho\to\infty} \left| \int_0^{\pi/4} \exp\left(i\rho^2 e^{2it}\right) i\rho e^{it} dt \right|$$

$$\le \lim_{\rho\to\infty} \rho \int_0^{\pi/4} e^{-\rho^2 \sin 2t} dt$$

$$\le \lim_{\rho\to\infty} \rho \int_0^{\pi/4} e^{-\rho^2 4t/\pi} dt \quad \left(\text{since } \sin 2t \ge \frac{4t}{\pi}\right)$$

$$= \lim_{\rho\to\infty} \frac{\rho\pi}{-\rho^2 4} \left(e^{-\rho^2} - 1\right) = 0.$$

$$\lim_{\rho\to\infty} \int_{\gamma_3} e^{iz^2} dz = \int_{-\infty}^0 e^{-t^2} \left(-e^{\pi i/4}\right) dt = -\frac{\sqrt{2\pi}}{4}(1+i)$$

Summarizing,

$$0 = \lim_{\rho\to\infty} \int_{\Gamma_\rho} e^{iz^2}\,dz = \int_0^\infty e^{ix^2}\,dx + 0 - \frac{\sqrt{2\pi}}{4}(1+i).$$

Thus $\displaystyle\int_0^\infty e^{ix^2}\,dx = \frac{\sqrt{2\pi}}{4}(1+i)$

EXERCISES 6.5: Indented Contours

1. a. $i\left(\frac{\pi}{2} - 0\right)\ \text{Res}(0) = \frac{\pi i}{2}$

   b. $i\left(\pi - \frac{\pi}{4}\right)\ \text{Res}(1) = \frac{3\pi i e^{3i}}{8}$

   c. $-i\,(2\pi - \pi)\ \text{Res}\,(1) = 0$

   d. $-i(2\pi - \pi)\ \text{Res}(0) = -\pi i$

2. $\pi i\ \text{Res}(-1) = \pi i e^{-2i}$

3. $\pi i\ \text{Res}(1) + \pi i\ \text{Res}(2) = \pi i\left(e^{2i} - e^{i}\right)$

4. Let $f(z) = \dfrac{e^{2iz}}{z(z^2+1)^2}$

$$I = \frac{1}{2}\ \text{Im}\ [2\pi i\ \text{Res}\,(i) + \pi i\ \text{Res}(0)]$$

$$= \frac{1}{2}\ \text{Im}\left[\frac{-2\pi i}{e^2} + \pi i\right] = \pi\left(\frac{1}{2} - \frac{1}{e^2}\right)$$

5. Let $f(z) = \dfrac{(e^{iz} - 1)}{z^2}$

$$I = \frac{1}{2}\ \text{Re}\ [\pi i\ \text{Res}(0)] = \frac{-\pi}{2}$$

6. Let $f(z) = \dfrac{e^{iz}}{(z^2+4)(z-1)}$

$$I = \ \text{Im}\ [2\pi i\ \text{Res}(2i) + \pi i\ \text{Res}(1)]$$

$$= \ \text{Im}\left[\frac{2\pi i e^{-2}}{4i(2i-1)} + \frac{\pi i e^{i}}{5}\right] = \frac{\pi}{5}\left(\cos(1) - \frac{1}{e^2}\right)$$

7. Let $f(z) = \dfrac{ze^{iz}}{(z-1)(z-2)}$

$$I = \text{Re}\left[\pi i \text{ Res}(1) + \pi i \text{ Res}(2)\right]$$

$$= \text{Re}\left(-\pi i e^{i} + 2\pi i e^{2i}\right) = \pi\left(\sin(1) - 2\sin(2)\right)$$

8. Let $f(z) = \dfrac{e^{2iz}}{z^3+1}$

$$I = \text{Re}\left[2\pi i \text{ Res}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \pi i \text{ Res}(-1)\right]$$

$$= \text{Re}\left[2\pi i \frac{e^{-\sqrt{3}+i}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)}{3} + \pi i \frac{e^{-2i}}{3}\right]$$

$$= \frac{\pi}{3}e^{-\sqrt{3}}\left[\sin(1) + \sqrt{3}\cos(1)\right] + \frac{\pi \sin(2)}{3}$$

9. Let $f(z) = \dfrac{3e^{iz} - e^{3iz} - 2}{4z^3}$

$$= \frac{3\left(1 + iz - \frac{z^2}{2} + \cdots\right) - \left(1 + 3iz - \frac{9z^2}{2} + \cdots\right) - 2}{4z^3} = \frac{3}{4z} + \cdots$$

$$I = \text{Im}\left[\pi i \text{ Res}(0)\right] = \frac{3\pi}{4}$$

10. Let $f(z) = \dfrac{1 - e^{2iz}}{2z^2} = \dfrac{1 - (1 + 2iz + \cdots)}{2z^2} = \dfrac{-i}{z} + \cdots$

$$I = \frac{1}{2}\text{Re}\left[\pi i \text{ Res}(0)\right] = \frac{\pi}{2}$$

11. As in Example 3 on page 261, taking limits of the integrals around the indented contour gives

$$0 = \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x - 1}dx - e^{2\pi a i}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x - 1}dx - \pi i \text{ Res}(0) - \pi i \text{ Res}(2\pi i)$$

so $I = \dfrac{\pi i + \pi i e^{2\pi a i}}{1 - e^{2\pi a i}} = -\pi\cot(a\pi)$

12. Let $f(z) = \dfrac{e^{aiz}}{z(z^2+b^2)}$

$$I = \frac{1}{2}\,\mathrm{Im}\,[2\pi i\,\mathrm{Res}(bi) + \pi i\,\mathrm{Res}(0)]$$

$$= \frac{1}{2}\,\mathrm{Im}\left[\frac{2\pi i e^{-ab}}{-2b^2} + \frac{\pi i}{b^2}\right] = \frac{\pi}{2b^2}\left(1 - e^{-ab}\right)$$

EXERCISES 6.6: Integrals Involving Multiple-Valued Functions

1. $\left(1 - e^{\pi i}\right) I = 2\pi i\left[\mathrm{Res}\left(\dfrac{\sqrt{z}}{z^2+1}; i\right) + \mathrm{Res}\left(\dfrac{\sqrt{z}}{z^2+1}; -i\right)\right]$

$$I = \pi i\left[\frac{1}{2i}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) + \frac{1}{-2i}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\right] = \frac{\pi}{\sqrt{2}}$$

2. $\left(1 - e^{(\alpha-1)2\pi i}\right) I = 2\pi i\,\mathrm{Res}\left(\dfrac{z^{\alpha-1}}{z+1}; -1\right)$

$$I = \frac{-2\pi i\,(e^{\alpha\pi i})}{1 - e^{\alpha 2\pi i}} = \frac{\pi}{\sin(\pi\alpha)}$$

3. $\left(1 - e^{\alpha 2\pi i}\right) I = 2\pi i\,\mathrm{Res}\left(\dfrac{z^{\alpha}}{(z+9)^2}; -9\right)$

$$I = \frac{2\pi i\alpha(-9)^{\alpha-1}}{1 - e^{\alpha 2\pi i}} = \frac{9^{\alpha-1}\pi\alpha}{\sin(\pi\alpha)}$$

4. $\left(1 - e^{\alpha 2\pi i}\right) I = 2\pi i\left[\mathrm{Res}\left(\dfrac{z^{\alpha}}{(z^2+1)^2}; i\right) + \mathrm{Res}\left(\dfrac{z^{\alpha}}{(z^2+1)^2}; -i\right)\right]$

$$I = \frac{2\pi i}{1 - e^{\alpha 2\pi i}}\left[\frac{(1-\alpha)e^{\alpha\pi i/2}}{4i} - \frac{(1-\alpha)e^{3\alpha\pi i/2}}{4i}\right] = \frac{\pi(1-\alpha)e^{\alpha\pi i/2}\left(1 - e^{\alpha\pi i}\right)}{2\left(1 + e^{\alpha\pi i}\right)\left(1 - e^{\alpha\pi i}\right)}$$

$$= \frac{\pi(1-\alpha)}{2\left(e^{\alpha\pi i/2} + e^{-\alpha\pi i/2}\right)} = \frac{\pi(1-\alpha)}{4\cos(\alpha\pi/2)}$$

5. $$\left(1-e^{(\alpha-1)2\pi i}\right) I = 2\pi i\left[\operatorname{Res}\left(\frac{z^{\alpha-1}}{z^2+z+1}; e^{2\pi i/3}\right)\right.$$
$$\left. + \operatorname{Res}\left(\frac{z^{\alpha-1}}{z^2+z+1}; e^{4\pi i/3}\right)\right]$$

$$I = \frac{2\pi i}{1-e^{\alpha 2\pi i}}\left[\frac{e^{(\alpha-1)2\pi i/3}}{i\sqrt{3}} - \frac{e^{(\alpha-1)4\pi i/3}}{i\sqrt{3}}\right]$$

$$= \frac{2\pi}{\sqrt{3}}\left[\frac{e^{-\alpha\pi i/3}\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) - e^{\alpha\pi i/3}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}{-\left(e^{\alpha\pi i}-e^{-\alpha\pi i}\right)}\right]$$

$$= \frac{2\pi}{\sqrt{3}}\csc(\alpha\pi)\left[-\frac{1}{2}\sin\left(\frac{\alpha\pi}{3}\right) + \frac{\sqrt{3}}{2}\cos\left(\frac{\alpha\pi}{3}\right)\right]$$

$$= \frac{2\pi}{\sqrt{3}}\csc(\alpha\pi)\left[-\sin\frac{\pi}{6}\sin\frac{\alpha\pi}{3} + \cos\frac{\pi}{6}\cos\frac{\alpha\pi}{3}\right]$$

$$= \frac{2\pi}{\sqrt{3}}\csc(\alpha\pi)\cos\left(\frac{2\alpha\pi+\pi}{6}\right)$$

6. $$\left(1-e^{\alpha 2\pi i}\right) I = 2\pi i \operatorname{Res}\left(\frac{z^\alpha}{z^2-1}; -1\right) + \pi i\left(1+e^{\alpha 2\pi i}\right) \operatorname{Res}\left(\frac{z^\alpha}{z^2-1}; 1\right)$$

$$I = \frac{2\pi i}{1-e^{\alpha 2\pi i}}\left(\frac{e^{\alpha\pi i}}{-2}\right) + \frac{\pi i\left(1+e^{\alpha 2\pi i}\right)}{\left(1-e^{\alpha 2\pi i}\right)}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{2\sin(\pi\alpha)}\left[1-\cos(\pi\alpha)\right]$$

7. $$\left(1-e^{\alpha 2\pi i}\right) I = 2\pi i\left[\operatorname{Res}\left(\frac{z^\alpha}{(z+e^{\phi i})(z+e^{-\phi i})}; -e^{\phi i}\right)\right.$$
$$\left. + \operatorname{Res}\left(\frac{z^\alpha}{(z+e^{\phi i})(z+e^{-\phi i})}; -e^{-\phi i}\right)\right]$$

$$I = \frac{2\pi i}{1-e^{\alpha 2\pi i}}\left[\frac{e^{\alpha\pi i}e^{\phi\alpha i}}{-e^{\phi i}+e^{-\phi i}} + \frac{e^{\alpha\pi i}e^{-\phi\alpha i}}{-e^{-\phi i}+e^{\phi i}}\right] = \frac{\pi}{\sin(\pi\alpha)}\frac{\sin(\phi\alpha)}{\sin\phi}$$

8. Following the hint, the integrals along the arcs C_ρ^+ and S_r go to zero as $\rho \to \infty$ and $r \to 0$, so

$$I = \text{Re}\left[2\pi i \,\text{Res}\left(\frac{\text{Log}\, z}{z^2+4}; 2i\right)\right]$$

$$= \text{Re}\left[2\pi i\left(\frac{1}{4i}\text{Log}\, 2 + \frac{\pi}{8}\right)\right] = \frac{\pi}{2}\text{Log}\, 2$$

9. (a) As shown in Example 4 $\int_0^\infty x\,dx/[(x+1)(x^2+2x+2)]$
$= -\Sigma\text{Res}(zL_0(z)/[(z+1)(z^2+2z+2)])$ if $\lim_{\varepsilon\to 0}\int_{\Gamma\varepsilon} zL_0(z)dz/[(z+1)(z^2+2z+2)]$ $=0$ and $\lim_{\rho\to 0}\int_{C\rho} zL_0(z)dz/[(z+1)(z^2+2z+2)] = 0$ as can be shown using arguments similar to those in Example 4. Then,
$I = -\text{Log}(\sqrt{2}) + \pi/4$
(b) $\int_0^\infty dx/(x^3+1) = -\Sigma\text{Res}(L_0(z)/(z^3+1)) = 2\pi/(3\sqrt{3})$

10. $I = \frac{1}{2}\text{Re}\left[2\pi i\,\text{Res}\left(\frac{\text{Log}\, z}{z^2+1}; i\right)\right] = \frac{1}{2}\text{Re}\left[2\pi i\left(\frac{\pi}{4}\right)\right] = 0$

11. $I = \frac{1}{2}\text{Re}\left[2\pi i\,\text{Res}\left(\frac{\text{Log}\, z}{(z^2+1)^2}; i\right)\right] = \text{Re}\left[\pi i\left(\frac{i}{4} + \frac{\pi}{8}\right)\right] = \frac{-\pi}{4}$

12. Let $\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where

$\gamma_1 : z(t) = t, \quad \epsilon \le t \le \rho$

$\gamma_2 : z(t) = \rho e^{it}, \quad 0 \le t \le \frac{\pi}{2}$

$\gamma_3 : z(t) = -ti, \quad -\rho \le t \le -\epsilon$

$\gamma_4 : z(t) = \epsilon e^{-it}, \quad \frac{-\pi}{2} \le t \le 0$

$\int_\Gamma e^{-z}z^{\alpha-1}dz = 0$. Along the contours we have:

$$\lim_{\substack{\rho\to\infty\\ \epsilon\to 0}} \int_{\gamma_1} e^{-z}z^{\alpha-1}dz = \int_0^\infty e^{-x}x^{\alpha-1}dx = \Gamma(\alpha)$$

$$\lim_{\rho\to\infty}\left|\int_{\gamma_2} e^{-z}z^{\alpha-1}dz\right| \le \lim_{\rho\to\infty}\frac{\rho^{\alpha-1}}{e^\rho}\frac{\pi}{2}\rho = 0$$

$$\lim_{\substack{\rho\to\infty\\ \epsilon\to 0}} \int_{\gamma_3} e^{-z}z^{\alpha-1}dz = \int_{-\infty}^0 e^{ti}(-ti)^{\alpha-1}(-i)dt = -e^{\alpha\pi i/2}\int_0^\infty e^{-xi}x^{\alpha-1}dx$$

$$\lim_{\epsilon\to 0}\left|\int_{\gamma_4} e^{-z}z^{\alpha-1}dz\right| \le \lim_{\epsilon\to 0}\frac{\epsilon^{\alpha-1}\pi}{e^\epsilon 2}\epsilon = 0$$

Summarizing,

$$0 = \lim_{\substack{\rho\to\infty\\ \epsilon\to 0}} \int_\Gamma e^{-z}z^{\alpha-1}dz = \Gamma(\alpha) - e^{\alpha\pi i/2}\int_0^\infty e^{-xi}x^{\alpha-1}dx.$$

Hence $\int_0^\infty e^{-xi}x^{\alpha-1}dx = e^{-\alpha\pi i/2}\Gamma(\alpha)$

Now take the imaginary part to obtain

$$\int_0^\infty x^{\alpha-1}\sin x\,dx = \sin\left(\frac{\pi\alpha}{2}\right)\cdot\Gamma(\alpha)$$

13. The function $f(z) = (z^2 - z^3)^{(-1/3)}$ is single valued for $|z| > 1$. The branch cut of [0, 1] in the z-plane permits calculating the integral $\int_0^1 f(x)dx$ by first calculating the integral $\int f(z)dz$ on an circle of radius R and taking the limit as $R \to \infty$. First note that $\int_{CR} f(z)dz$ approaches $\int_{CR}(e^{-i\pi/3}/z)dz = 2\pi i e^{-i\pi/}3$ as $|z| = R \to \infty$ because
$|\int_{CR}[f(z) - e^{-i\pi/3}/z]dz| = |\int_{CR}[1/(1/z - 1)^{1/3} - e^{-i\pi/3}]/z]dz|$
$\leq 2\pi R\ (1/R)\max_{|z|=R}|1/(|1/z - 1|e^{i\pi/3}) - e^{-i\pi/3}| = 0$
Deforming the circle of radius R down to the barbell composed of small circles about the points z = 0 and 1 permits writing
$\int_{below(0,1)}\int_{|z-1|=\varepsilon}\int_{above(0,1)}\int_{|z|=\delta}[1/(z^2 - z^3)^{(1/3)}]dz$
On the circle $|z-1| = \varepsilon$ we have the estimate for the integral $2\pi\varepsilon/(1^{2/3}\varepsilon^{1/3}) \to 0$. On the circle $|z| = \delta$ we have the estimate for the integral $2\pi\delta/(\delta^{2/3}1^{1/3}) \to 0$. On the line below (0, 1) $\arg(1/(z^2 - z^3)^{(1/3)})$ is zero, while wrapping around the $|z-1| = \varepsilon$ circle $\arg((z^2 - z^3)^{(1/3)})$ increases by $2\pi/3$ with the result the integral above the line (1, 0) is multiplied by $-e^{-i2\pi/3}$. Taking the total gives

$$[1-e^{-i2\pi/3}]\int_0^1(z^2 - z^3)^{(-1/3)}dz = i2\pi e^{-i\pi/3}$$
$$\int_0^1(z^2 - z^3)^{(-1/3)}dz = i2\pi e^{-i\pi/3}/[1-e^{-i2\pi/3}] = i2\pi/[e^{i\pi/3}-e^{-i\pi/3}] = \pi/\sin(\pi/3)$$
$$\int_0^1(z^2 - z^3)^{(-1/3)}dz = 2\pi/\sqrt{3}$$

EXERCISES 6.7: The Argument Principle and Rouché's Theorem

1. a, c, e, f

2. For sufficiently large R all n of the zeros of $P(z)$ will be within $|z| = R$, and the result follows from Corollary 1 of the Argument Principle.

3. $N_0(f) - N_p(f) = 5 - 4 = 1$

4. Let $g(z) = f(z) - w_0$.

$$I = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{g'(z)}{g(z)} dz = N_0(g) - 0$$

$$= (\text{number of solutions of } f(z) = w_0 \text{ inside } |z| = \rho)$$

5. Since f is one-to-one on C, $f(C)$ is a simple closed curve, and for w_0 inside $f(C)$,

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z) - w_0} dz = \frac{1}{2\pi} \Delta_C \arg(f(z) - w_0) = 1.$$

By the Argument Principle, this implies that $f(z) - w_0$ has one zero inside C. That is, $f(z)$ is one-to-one inside C.

6. Let $f(z) = 4z^2$ and $h(z) = z^6 - 1$. On $|z| = 1$, $|h(z)| < 2 < 4 = |f(z)|$. Since $f(z) = 4z^2$ has two zeros inside $|z| = 1$, it follows that $f(z) + h(z) = z^6 + 4z^2 - 1$ also has two zeros inside $|z| = 1$ by Rouché's theorem.

7. Let $f(z) = 27$ and $h(z) = z^3 + 9z$. On $|z| = 2$, $|h(z)| < |2|^3 + 9|2| = 26 < 27 = |f(z)|$. Since $f(z) = 27$ has no zeros inside $|z| = 2$, it follows that $f(z) + h(z) = z^3 + 9z + 27$ has no zeros inside $|z| = 2$ by Rouché's theorem.

8. Let $f_1(z) = 10$ and $h_1(z) = z^6 - 5z^2$. On $|z| = 1$, $|h_1(z)| \leq 1 + 5 = 6 < 10 = |f_1(z)|$. Since $f_1(z) = 10$ has no zeros inside $|z| = 1$, it follows from Rouché's theorem that $f_1(z) + h_1(z) = z^6 - 5z^2 + 10$ has no zeros inside $|z| = 1$.

Let $f_2(z) = z^6$ and $h_2(z) = -5z^2+10$. On $|z| = 2$, $|h_2(z)| \le 20+10 = 30 < 64 = |f_2(z)|$. Since $f_2(z) = z^6$ has all six zeros inside $|z| = 2$, it follows from Rouché's theorem that $f_2(z) + h_2(z) = z^6 - 5z^2 + 10$ has all six zeros inside $|z| = 2$ (but outside $|z| = 1$).

9. 4. Take $f(z) = 6z^4$ and $h(z) = z^3 - 2z^2 + z - 1$. On $|z| = 1$, $|h(z)| \le 1+2+1+1 = 5 < 6 = |f(z)|$. By Rouché's theorem, $f(z) = 6z^4$ and $f(z)+h(z) = 6z^4+z^3-2z^2+z-1$ both have four zeros inside $|z| = 1$.

10. Let $f(z) = z - 2$ and $h(z) = e^{-z}$.
For $\rho > 4$ consider $\Gamma = \gamma_1 + \gamma_2$ where
$\gamma_1 : z = \rho e^{it}, \qquad -\pi/2 \le t \le \pi/2,$
$\gamma_2 : z = -ti, \qquad -\rho \le t \le \rho.$
On $\Gamma, |h(z)| = e^{-x} \le 1 < 2 \le |z-2| = |f(z)|$. By Rouché's theorem, both $f(z)$ and $f(z)+h(z) = z-2+e^{-z}$ have one zero inside Γ. Letting $\rho \to \infty$ yields one zero in the right half-plane.
The root is real because $\operatorname{Im}(z - 2 + e^{-z}) = 0 \Longrightarrow y - \sin y = 0 \Longrightarrow y = 0$.

11. It is easy to see that $P(iy) = (y^2-2)(y^2-1) + iy(1-2y^2)$. It follows

that $\lim\limits_{R\to\infty} \arg P(iy)\Big|_{-R}^{R} = \lim\limits_{R\to\infty} 2 \operatorname{Arctan}\left(\dfrac{R(1-2R^2)}{(R^2-2)(R^2-1)}\right) = 0$

Consider $P(z)$ on the contour $\Gamma = \gamma_1 + \gamma_2$ where
$\gamma_1 : z = Re^{it}, \qquad -\pi/2 \le t \le \pi/2,$
$\gamma_2 : z = -it, \qquad -R \le t \le R.$
Then the number of zeros of $P(z)$ in the right half-plane is

$$\lim_{R\to\infty} \frac{1}{2\pi}\Delta_\Gamma \arg P(z) = \lim_{R\to\infty}\left(\frac{1}{2\pi}\Delta_{\gamma_1} \arg P(z) + \frac{1}{2\pi}\Delta_{\gamma_2} \arg P(z)\right)$$

$$= \lim_{R\to\infty} \frac{1}{2\pi} \arg R^4 e^{4it}\Big|_{t=-\frac{\pi}{2}}^{\frac{\pi}{2}} - \lim_{R\to\infty} \frac{1}{2\pi} \arg P(iy)\Big|_{y=-R}^{R}$$

$$= 2 - 0$$

12. a. Let $f_1(z) = -z$ and $h_1(z) = f(z)$. On $|z| = 1$, $|h_1(z)| = |f(z)| < 1 = |f_1(z)|$. By Rouché's theorem, $-z$ and $f(z) - z$ both have one zero in $|z| < 1$.

b. Assume $|f(z)| \le M < 1$ for $|z| = 1$. Define the iterates of f by $f_1(z) = f(z),\ \ f_n(z) = f(f_{n-1}(z))$.

Case 1: The fixed point is zero. The proof of Schwarz's lemma (Problem 17, Section 5.6) can be modified to show that $|f(z)| \le M|z|$ for $|z| \le 1$. Then
$|z_n| = |f_n(z_0)| \le M|f_{n-1}(z_0)| \le \ldots \le M^n|z_0|$.
It follows that $\lim\limits_{n\to\infty} z_n = 0$.

Case 2: The fixed point is $a \ne 0$. It will be shown in Section 7.4 that $\phi(z) = \dfrac{z-a}{1-\overline{a}z}$ is a one-to-one mapping of $|z| \le 1$ onto itself, with inverse $\phi^{-1}(z) = \dfrac{z+a}{1+\overline{a}z}$. Apply the generalized Schwarz lemma to the composite function
$g_n(z) = \left(\phi \circ f \circ \phi^{-1}\right)_n(z) = \phi\left(f_n\left(\phi^{-1}(z)\right)\right)$ to get
$\left|\dfrac{f_n(z)-a}{1-\overline{a}f_n(z)}\right| \le M^n \left|\dfrac{z-a}{1-\overline{a}z}\right|$.
It follows from this inequality that $\lim\limits_{n\to\infty} f_n(z_0) = a$.

13. Let $h(z) = -f(z)$. Then $|h(z)| \le |f(z)|$ for z on C, but $f+h$ is identically zero no matter how many zeros f has.

14. Theorem: *If f and h are each functions that are meromorphic inside and on a simple closed contour C and if the strict inequality $|h(z)| < |f(z)|$ holds at each point on C, then $N_0(f) - N_p(f) = N_0(f+h) - N_p(f+h)$.*

Proof: As in the leash discussion preceding Rouché's theorem and applying the Argument Principle,

$$\begin{aligned} N_0(f) - N_p(f) &= \Delta_C \arg f(z) = \Delta_C \arg\left[f(z) + h(z)\right] \\ &= N_0(f+h) - N_p(f+h). \end{aligned}$$

15. Poles of $f+h$ occur exactly at the poles of f, so $N_p(f) = N_p(f+h)$. Now apply Problem 14 to get $N_0(f) = N_0(f+h)$.

16. a. $\tan(x+iy) = \dfrac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}$ (Problem 13, Section 3.1)

$$= \frac{\sin x \cos x(\cosh^2 y - \sinh^2 y)}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2}$$

$$+ \frac{i \sinh y \cosh y(\cos^2 x + \sin^2 x)}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

$$= \frac{\frac{1}{2}\sin 2x + i\frac{1}{2}\sinh 2y}{\cos^2 x \cosh^2 y + \sinh^2 y - \cos^2 x \sinh^2 y}$$

$$= \frac{\sin 2x + i \sinh 2y}{2\sinh^2 y + 1 + 2\cos^2 x - 1}$$

$$= \frac{\sin 2x + i \sinh 2y}{\cosh 2y + \cos 2x}$$

b. $\Gamma_n = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where
$\gamma_1 : z(t) = n\pi(1 + it), \quad -1 \le t \le 1$
$\gamma_2 : z(t) = n\pi(-t + i), \quad -1 \le t \le 1$
$\gamma_3 : z(t) = n\pi(-1 - it), \quad -1 \le t \le 1$
$\gamma_4 : z(t) = n\pi(t - i), \quad -1 \le t \le 1$
On the vertical segments

$|\tan z| = |\tan(\pm n\pi \pm in\pi t)| = |\tan(in\pi t)|$ (periodicity)

$$= |\tanh(n\pi t)| = \left|\frac{e^{n\pi t} - e^{-n\pi t}}{e^{n\pi t} + e^{-n\pi t}}\right| \le 1$$

On the horizontal segments

$$|\tan z| = |\tan(\pm n\pi t \mp in\pi)| = \left|\frac{\sin(\pm 2n\pi t) + i\sinh(\mp 2n\pi)}{\cosh(\mp 2n\pi) + \cos(\pm 2n\pi t)}\right|$$

$$\le \frac{2 + |e^{2n\pi} - e^{-2n\pi}|}{|e^{2n\pi} + e^{-2n\pi}| - 2}$$

For n large this quantity is ≤ 2.

c. Choose n large enough so that $2 \le \lambda|z|$ for z on Γ_n and so that part a can be applied. Then on Γ_n

$$|g(z) + \lambda z| = |\tan z| \le 2 \le \lambda|z|$$

d. There are $2n$ poles inside Γ_n located at $z = \pm\left(\frac{\pi}{2} + k\pi\right)$, $k = 0, 1, 2, \ldots, n-1$ (the poles of $\tan z$).

e. Inside $\Gamma_n, N_0(\lambda z) - N_p(\lambda z) = 1 - 0 = 1$
By the general form of Rouché's theorem, $1 = N_0(g(z)+\lambda z-\lambda z) - N_p(g(z) + \lambda z - \lambda z) = N_0(g(z)) - 2n$. Thus $N_0(g(z)) = 2n + 1$.

17. Choose ϵ small enough so that z_0 is the only zero of $f(z) - f(z_0)$ in $D_\epsilon : |z - z_0| < \epsilon$. By the open mapping theorem, f maps D_ϵ to an open set containing $f(z_0)$. Choose δ so that the disk $|w - f(z_0)| < \delta$ is contained in $f(D_\epsilon)$.

Let $F(z) = f(z) - f(z_0)$ and $H(z) = f(z_0) - w$. On $|z - z_0| = \epsilon$ one gets $|H(z)| = |f(z_0) - w| < \delta \leq |f(z) - f(z_0)| = |F(z)|$

It follows from Rouché's theorem that $F(z)$ and $F(z) + H(z)$ have the same number of zeros, so $f(z) - f(z_0)$ and $f(z) - w$ both have n zeros in $|z - z_0| < \epsilon$.

18. The conditions given in parts a, b, and c amount to $f(D)$ being a subset of a vertical line, a horizontal line, and a circle, respectively. In each case $f(D)$ cannot be open. By (the contrapositive of) the open mapping property, f is constant.

19. Let $\min\limits_{|z|=\delta} |f(z)| = \epsilon > 0$. For n sufficiently large, $|f_n(z) - f(z)| < \epsilon$ for all z in $|z| \leq \delta$. Let $h(z) = f_n(z) - f(z)$. On $|z| = \delta$, $|h(z)| < \epsilon \leq |f(z)|$. It follows from Rouché's theorem that $f_n(z)$ and $f(z)$ have the same number of zeros inside $|z| = \delta$.

20. Notice that for $z = e^{i\theta}, |P^*(z)| = \left|e^{in\theta}\overline{P(e^{i\theta})}\right| = |P(z)|$. Now let $h(z) = -a_n P^*(z)$ and $f(z) = \overline{a_0} P(z)$. On $|z| = 1$,

$$|h(z)| = |a_n|\,|P^*(z)| < |a_0||P(z)| = |f(z)|.$$

It follows from Rouché's theorem that $\overline{a_0}P(z) - a_n P^*(z)$ has the same number of zeros in $|z| < 1$ as $\overline{a_0}P(z)$ (and hence $P(z)$).

21. Applying the Argument Principle on Γ_r for r large,
$N_0(F) - n = -\Delta_{\Gamma_r} \arg(1 + P(z)) = -m$.

When $m = n$, $N_0(F) = 0$ in the right half-plane so all zeros of F would lie in the left half-plane.

22. a. If $|h(z)| < |f(z)| + |f(z) + h(z)|$ then the dog never crosses the line τ and $N_0(f) = \Delta_C \arg(f(z)) = \Delta_C \arg(f(z) + h(z)) = N_0(f + h)$.

b. $f/(f + h)$ is never negative or zero because that would cause $f(z)$ and $f(z) + h(z)$ to point in opposite directions, yielding the contradiction $|h(z)| = |f(z)| + |f(z) + h(z)|$.

Thus $0 = \displaystyle\int_C \frac{d}{dz}\left[\operatorname{Log}\frac{f}{f+h}\right] dz = \int_C \left[\frac{f'}{f} - \frac{(f+h)'}{(f+h)}\right] dz$

$= N_0(f) - N_0(f + h)$

CHAPTER 7: Conformal Mapping

EXERCISES 7.1: Invariance of Laplace's Equation

1. From the notation $e^z = e^x e^{iy}$ it can be seen that $|e^z| = e^x > 1$ when $x > 0$. Also, since $y = \arg e^z$, $\frac{-\pi}{2} <$ Arg $e^z < \frac{\pi}{2}$ when $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
   Let $w = f(z) = e^z$. Then
   $z = \operatorname{Log} w = \operatorname{Log}|w| + i \operatorname{Arg} w = x + iy$
   $\phi(z) = x + y$ corresponds to

   $\psi(w) = x(w) + y(w) = \operatorname{Log}|w| + \operatorname{Arg}(w)$

   For the next part write $w = e^z = e^x \cos y + ie^x \sin y = u + iv$
   $\psi(w) = u + v$ corresponds to
   $\phi(z) = u(z) + v(z) = e^x \cos y + e^x \sin y$

2. a. $\frac{\partial \psi}{\partial u} + i\frac{\partial \psi}{\partial v}$

$$= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + i\left(\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial \phi}{\partial x}\left(\frac{\partial x}{\partial u} + i\frac{\partial x}{\partial v}\right) + i\frac{\partial \phi}{\partial y}\left(-i\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial \phi}{\partial x}\left(\frac{\partial x}{\partial u} - i\frac{\partial y}{\partial u}\right) + i\frac{\partial \phi}{\partial y}\left(\frac{\partial x}{\partial u} - i\frac{\partial y}{\partial u}\right)$$

$$= \left(\frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}\right)\overline{\frac{dz}{dw}}$$

   b. $$\frac{\partial^2 \psi}{\partial u^2} = \frac{\partial \phi}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial x}{\partial u}\frac{\partial}{\partial u}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial \phi}{\partial y}\frac{\partial^2 y}{\partial u^2} + \frac{\partial y}{\partial u}\frac{\partial}{\partial u}\left(\frac{\partial \phi}{\partial y}\right)$$

$$= \frac{\partial \phi}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial x}{\partial u}\left[\frac{\partial^2 \phi}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 \phi}{\partial y \partial x}\frac{\partial y}{\partial u}\right]$$

$$+\frac{\partial \phi}{\partial y}\frac{\partial^2 y}{\partial u^2}+\frac{\partial y}{\partial u}\left[\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial x}{\partial u}+\frac{\partial^2 \phi}{\partial y^2}\frac{\partial y}{\partial u}\right]$$

Similarly

$$\frac{\partial^2 \psi}{\partial v^2}=\frac{\partial \phi}{\partial x}\frac{\partial^2 x}{\partial v^2}+\frac{\partial x}{\partial v}\left[\frac{\partial^2 \phi}{\partial x^2}\frac{\partial x}{\partial v}+\frac{\partial^2 \phi}{\partial y \partial x}\frac{\partial y}{\partial v}\right]$$

$$+\frac{\partial \phi}{\partial y}\frac{\partial^2 y}{\partial v^2}+\frac{\partial y}{\partial v}\left[\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial x}{\partial v}+\frac{\partial^2 \phi}{\partial y^2}\frac{\partial y}{\partial v}\right]$$

Adding,

$$\frac{\partial^2 \psi}{\partial u^2}+\frac{\partial^2 \psi}{\partial v^2}$$

$$=\frac{\partial \phi}{\partial x}\left(\frac{\partial^2 x}{\partial u^2}+\frac{\partial^2 x}{\partial v^2}\right)+\frac{\partial \phi}{\partial y}\left(\frac{\partial^2 y}{\partial u^2}+\frac{\partial^2 y}{\partial v^2}\right)$$

$$+\frac{\partial^2 \phi}{\partial x^2}\left[\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial x}{\partial v}\right)^2\right]+\frac{\partial^2 \phi}{\partial y^2}\left[\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2\right]$$

$$+\frac{\partial^2 \phi}{\partial x \partial y}\left(2\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}+2\frac{\partial x}{\partial v}\frac{\partial y}{\partial v}\right)$$

$$=0+0+\frac{\partial^2 \phi}{\partial x^2}\left[\left(\frac{\partial x}{\partial u}\right)^2+\left(-\frac{\partial y}{\partial u}\right)^2\right]$$

$$+\frac{\partial^2 \phi}{\partial y^2}\left[\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial x}{\partial u}\right)^2\right]+0$$

$$=\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)\left|\frac{dz}{dw}\right|^2$$

c. Using part b it is clear that if $\phi(x, y)$ satisfies Laplace's equation so does $\psi(u, v)$.

d. The result follows directly from part b.

3. $\phi(z) = \text{Arg}(z-x_1) + B\text{Arg}(z-x_2) + C\text{Arg}(z-x_3) + D.$

$a_1 = A\pi + B\pi + C\pi + D$

$a_2 = 0 + B\pi + C\pi + D$

$a_3 = 0 + 0 + C\pi + D$

$a_4 = 0 + 0 + 0 + D$

$\phi(z) = (1/\pi)[(a_1-a_2)\text{Arg}(z-x_1) + (a_2-a_3)\text{Arg}(z-x_2) + (a_3-a_4)\text{Arg}(z-x_3)] + a_4$

4. a. $\psi\left(e^{i\theta}\right) = \phi\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right) = \phi\left(\frac{i\sin\theta}{1-\cos\theta}\right) = \phi\left(0, \frac{\sin\theta}{1-\cos\theta}\right)$

$$= \frac{\frac{\sin\theta}{1-\cos\theta}}{1+\frac{\sin^2\theta}{(1-\cos\theta)^2}} = \frac{\sin\theta}{2}$$

b. $w = e^{i\theta} = \cos\theta + i\sin\theta = u + iv$ so $\psi(e^{i\theta}) = \frac{v}{2} = \frac{1}{2}\,\text{Im}\, w$ on the circle and throughout the disk.

c. $\phi(z) = \frac{1}{2}\,\text{Im}\left(\frac{z-1}{z+1}\right) = \frac{1}{2}\,\text{Im}\,\frac{z-1}{z+1}\cdot\frac{\overline{z}+1}{\overline{z}+1}$

$$= \frac{1}{2}\,\text{Im}\,\frac{x^2+y^2+2iy-1}{y^2+(x+1)^2}$$

$$= \frac{y}{y^2+(x+1)^2}$$

5. $\psi\left(e^{i\theta}\right) = \phi\left(\frac{i\sin\theta}{1-\cos\theta}\right) = \frac{1}{\frac{\sin^2\theta}{(1-\cos\theta)^2}+1}$

$$= \frac{1}{2} - \frac{\cos\theta}{2}$$

$$= \frac{1}{2} - \frac{1}{2}\,\mathrm{Re}\,w \text{ throughout the disk and}$$

$$\phi(z) = \frac{1}{2} - \frac{1}{2}\,\mathrm{Re}\left(\frac{z-1}{z+1}\right)$$

$$= \frac{1}{2} - \frac{1}{2}\left(\frac{x^2+y^2-1}{(1+x)^2+y^2}\right)$$

6. Suppose $\frac{\partial\phi}{\partial n} = 0$ on Γ. Since this is the length of the projection of the gradient $(\partial\phi/\partial x) + i\,(\partial\phi/\partial y)$ onto the normal, it follows that the gradient is orthogonal to the normal. That is, the gradient is tangent to Γ. From calculus, the gradient is orthogonal to the level curves $\phi(x, y) = c$, so Γ is orthogonal to these level curves.

   $w = f(z)$ takes the level curves of ϕ to level curves of ψ. By the preservation of angles in a conformal mapping, the image of Γ is orthogonal to the level curves of ψ. Therefore $\partial\psi/\partial n = 0$ on Γ'.

7. They are the Cauchy-Riemann equations for the inverse mapping.

EXERCISES 7.2: Geometric Considerations

1. a. $m = 1 \quad f(-1-a) = f(-1+a) = a^2$

   b. $m = 1 \quad f(n\pi - a) = f(n\pi + a)$

   c. $m = 2 \quad f(a) = f(ae^{2\pi i/3})$

2. $f(z)$ has an inverse $f^{-1}(w)$ near w_0 since there is a neighborhood η of z_0 in which f is one-to-one by Theorem 1. Since $f'(z_0) \neq 0$, f is non-constant showing that $f(\eta)$ is an open set by Theorem 3. Thus for w in $f(\eta), f^{-1}(w)$ is in η so that $f^{-1}(w)$ is continuous in this neighborhood. Also, for all such w

$$\lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{w \to w_0} \frac{1}{\frac{w - w_0}{f^{-1}(w) - f^{-1}(w_0)}}$$

$$= \lim_{z \to z_0} \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}} = \frac{1}{\frac{df}{dz}}$$

3. For $\alpha > 1$ the angles increase.
   For $0 < \alpha < 1$ the angles decrease.

4. (see Theorem 23 in Section 4.6).
   Since $|f(z)| \leq |f(z_0)|$ for all z in D, $f(z_0)$ is a boundary point of $f(D)$. Consequently $f(D)$ is not an open set so f is constant by Theorem 3.

5. $f(z) = ia$, where a is a real number. (since $f(D)$ is not an open set)

6. As in the proof of Theorem 2, except that $w'(t_0) = \overline{f'(z_0)z'(t_0)}$. The angles which the tangent vectors $z'(t_0)$ and $w'(t_0)$ make with the horizontal are related by

   $\arg w'(t_0) = -\arg f'(z_0) - \arg z'(t_0)$.

   Hence the angle between two curves through z_0 is reversed in orientation and is rotated by $-\arg f'(z_0)$ (preserving the magnitude of the angle).

7. Parametrize the circle by $z(t) = \rho e^{it}$, $\quad 0 \leq t \leq 2\pi$.
   Then $w(t) = \rho e^{it} + \dfrac{1}{\rho} e^{-it} = \left(\rho + \dfrac{1}{\rho}\right) \cos t + i \left(\rho - \dfrac{1}{\rho}\right) \sin t$.
   $u(t) =$ Re(w) and $v(t) =$ Im(w) are the parametric equations for the ellipse.

8. For points very close to z_0

   $$\frac{w - w_0}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0} \approx f'(z_0).$$

   Therefore, for such points

   $|w - w_0| \approx |z - z_0| \, |f'(z_0)|$

which shows that the segment length $|z - z_0|$ is scaled by $|f'(z_0)|$.

9. The area of D' is measured using infinitesimal rectangles with dimensions $du \approx |f'(z)|dx$ and $dv \approx |f'(z)|dy$. Therefore

$$A' = \iint\limits_{D'} du dv = \iint\limits_{D} |f'(z)|^2 dx dy$$

10. If D were the entire plane, its image under some analytic mapping f would be bounded by the unit circle as a result of the Riemann mapping theorem. Liouville's theorem implies that f must be constant in such a case.

11. a. $\operatorname{Im} w > 0$

    b. $\mathbf{C} \setminus \left\{ w = e^{a(1+i)}, \; -\infty \le a < \infty \right\}$

    c. $|w| < 1$ and $\operatorname{Im} w > 0$

    d. $|w| > 1$ and $\operatorname{Im} w > 0$

    e. $e < |w| < e^2$ and $\operatorname{Im} w > 0$

    f. $\mathbf{C} \setminus \{|w| = 1\}$

12. $P'(z) = 2z - (\alpha + \beta)$ so $P'(z) = 0$ only when $z = \dfrac{\alpha+\beta}{2}$. By Theorem 1, P is one-to-one in open neighborhoods that exclude $\dfrac{\alpha+\beta}{2}$, hence in the open half planes bounded by L.

13. a. $\operatorname{Im} w > 0$

    b. $\operatorname{Im} w < 0$ and $\operatorname{Re} w > 0$

    c. $\mathbf{C} \setminus \{w : w \text{ is real and } \; w \le -1 \text{ or } w \ge 1\}$

    d. The interior of the ellipse parametrized by
    $u = \cosh 1 \cos\theta$
    $v = \sinh 1 \sin\theta, \quad 0 \le \theta \le 2\pi$
    excluding the real segments
    $-\cosh 1 < w < -1$ and $1 < w < \cosh 1$

14. If f has a simple pole at z_0 then $\dfrac{1}{f(z)} = (z - z_0)g(z)$, where g is analytic and $g(z_0) \neq 0$ by Theorem 18, Section 5.6. In this case $\displaystyle\lim_{z\to z_0} \frac{d}{dz}\left(\frac{1}{f(z)}\right) = g(z_0) \neq 0$ so there exists a punctured neighborhood of z_0 in which $\dfrac{1}{f}$, and consequently f, is one-to-one.

15. For any distinct points z_1 and z_2 in D, parametrize the line segment between z_1 and z_2 by $\gamma : z(t) = z_1 + t(z_2 - z_1), \quad 0 \leq t \leq 1$. Then $(z_2 - z_1)\displaystyle\int_0^1 f'(z(t))dt = f(z_2) - f(z_1) \neq 0$ since $\operatorname{Re} f'(z) > 0$. Therefore f is one-to-one in D.

16. Let f be a one-to-one analytic function on the simply connected domain D and let Γ be any loop in D. There is a continuous deformation $z(s,t), \quad 0 \leq s \leq 1, 0 \leq t \leq 1$ that shrinks Γ to a point z_0 in D. Then $f(z(s,t))$ is a continuous deformation from $f(\Gamma)$ to $f(z_0)$ (a single point). Any loop in $f(D)$ can be expressed as $f(\Gamma)$ for some loop Γ in D, so any loop in $f(D)$ can be shrunk to a point. Thus $f(D)$ is simply connected.

EXERCISES 7.3: Möbius Transformations

1. $w = 3\left(e^{i\pi/2}z\right) + 5 = 3iz + 5$

2. $\{w : \operatorname{Re} w < 1\} \setminus \left\{w : \left|w - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$

3. a. $\{w : |w - 2 + 2i| \leq 1\}$
   b. $\{w : |w - 6i| \leq 3\}$
   c. $\left\{w : \operatorname{Re} w \leq \frac{1}{2}\right\}$

d. $\left\{w : \operatorname{Re} w \geq \frac{3}{2}\right\}$

e. $\left\{w : \left|w - \frac{2}{3}\right| \leq \frac{1}{3}\right\}$

4. $w_1 = iz;\ \ w_2 = \dfrac{w_1 - 1}{w_1 + 1};\ \ w = w_2 - 1 = \dfrac{-2}{iz+1}$

5. $w = e^{i3\pi/4}\left(\dfrac{z+i}{z-1}\right)$

6. $z = \dfrac{az+b}{cz+d} \Longrightarrow cz^2 + (d-a)z - b = 0$
This quadratic equation can have at most two distinct solutions except when $b = c = 0$ and $a = d$ in which case $f(z) \equiv z$.

7. a. $w = iz$

b. $w = \dfrac{2z}{z+1}$

c. $w = \dfrac{z+i}{z-1}$

d. $w = \dfrac{z+1}{z-1}$

8. $\{w : |w| < 1 \text{ and } \operatorname{Im} w < 0\}$

9. The line $y = x$ maps to $\left|w - \dfrac{1-i}{2}\right| = \dfrac{1}{\sqrt{2}}$ and the line $y = -x$ maps to $\left|w - \dfrac{1+i}{2}\right| = \dfrac{1}{\sqrt{2}}$. The image of the sector is the exterior of the union of these circles.

10. $w = \left(\dfrac{z+1}{z-1}\right)^2$

11. $w_1 = z - 2;\ \ w_2 = \dfrac{1}{w_1};\ \ w_3 = w_2 + \dfrac{1}{4};\ \ w_4 = e^{-i\pi/2}w_3;\ \ w_5 = 4\pi w_4$
w_5 maps the shaded region in Figure 7.25 to the strip $0 < \operatorname{Im} w_5 < \pi$. Finally,

$$w = e^{w_5} = \exp\left[-\pi i\left(\frac{z+2}{z-2}\right)\right]$$

maps the shaded region onto the upper half plane.

12. $w_1 = z - 1; \quad w_2 = e^{-3\pi i/4} w_1; \quad w_3 = \dfrac{w_2 - 1}{w_2 + 1} = \dfrac{e^{-3\pi i/4}(z-1) - 1}{e^{-3\pi i/4}(z-1) + 1}$

13. $W = \dfrac{Z-1}{Z+1} \Longrightarrow Z = R + iB = \dfrac{-W-1}{W-1}.$

When $W = u + iv$ this simplifies to

i) $R = \dfrac{1 - u^2 - v^2}{(u-1)^2 + v^2}$ and

ii) $B = \dfrac{2v}{(u-1)^2 + v^2}$

a. i) can be rewritten as

$$\left(u - \frac{R}{1+R}\right)^2 + v^2 = \frac{1}{(1+R)^2}$$

b. ii) can be rewritten as

$$(u-1)^2 + \left(v - \frac{1}{B}\right)^2 = \frac{1}{B^2}$$

14. $W' = \dfrac{Z'-1}{Z'+1} = \dfrac{Z\cos\beta l + i\sin\beta l - \cos\beta l - iZ\sin\beta l}{Z\cos\beta l + i\sin\beta l + \cos\beta l + iZ\sin\beta l}$

$$= \frac{Ze^{-i\beta l} - e^{-i\beta l}}{Ze^{i\beta l} + e^{i\beta l}} = e^{-i2\beta l}\frac{Z-1}{Z+1} = e^{-i2\beta l}W$$

15. $z = x + iy = 1/\omega = 1/(u+iv) = u/(u^2 + v^2) - iv/(u^2 + v^2)$
$Ax + By = [Au - Bv]/(u^2 + v^2) = C$ and $C \neq 0$
$Au - Bv = C(u^2 + v^2) \Leftrightarrow (u^2 + v^2) - (A/C)u + (B/C)v = 0$
$(u^2 + v^2) - (A/C)u + (B/C)v = 0$ is the equation of a circle through the origin of the ω-plane.

16. $(u^2 + v^2) + Au + Bv = 0$ is a circle passing through the origin in the ω-plane. Dividing by $(u^2 + v^2)$ gives
$1 + Au/(u^2 + v^2) + Bv/(u^2 + v^2) = 1 + Ax + By = 0$ which is an equation of a line not passing through zero in the z-plane.

17. In the z-plane circles not passing through the origin are represented as $x^2 + y^2 + Ax + By = C$, with $C \neq 0$.
Substituting $x = u/(u^2 + v^2)$ and $y = v/(u^2 + v^2)$ gives
$[u/(u^2 + v^2)]^2 + [v/(u^2 + v^2)]^2 + Au/(u^2 + v^2) + Bv/(u^2 + v^2) = C \Leftrightarrow$
$1 + Au + Bv = C(u^2 + v^2) \Leftrightarrow u^2 + v^2 - (A/C)u + (B/C)v = 1/C$
which are circles in the ω-plane not passing through zero.

EXERCISES 7.4: Möbius Transformations, Continued

1. $f_1^{-1}(f_2(z)) = \dfrac{3\left(\frac{z}{z+1}\right)-2}{-\left(\frac{z}{z+1}\right)+1} = z-2$

2. The left region of $(w,-i,1,i)$ is $|w|<1$ and the left region of $(z,-i,i,1)$ is $|z|>1$. Therefore $|z|<1$ maps to $|w|>1$.

3. a. $\alpha^* = \dfrac{4-3i}{25}$

   b. $\alpha^* = \dfrac{7-i}{6}$

   c. $\alpha^* = \dfrac{5-2i}{3}$

4. Let $f(z) = z+c,\;\; g(z) = az,\;\; h(z) = \dfrac{1}{z}$

   Then for z_1, z_2, z_3, z_4 finite,

$$(f(z_1), f(z_2), f(z_3), f(z_4)) = \frac{(z_1+c-z_2-c)(z_3+c-z_4-c)}{(z_1+c-z_4-c)(z_3+c-z_2-c)}$$

$$= (z_1, z_2, z_3, z_4)$$

$$(g(z_1), g(z_2), g(z_3), g(z_4)) = \frac{(az_1-az_2)(az_3-az_4)}{(az_1-az_4)(az_3-az_2)}$$

$$= (z_1, z_2, z_3, z_4)$$

$$(h(z_1), h(z_2), h(z_3), h(z_4)) = \frac{(1/z_1-1/z_2)(1/z_3-1/z_4)}{(1/z_1-1/z_4)(1/z_3-1/z_2)}$$

$$= \frac{(z_2-z_1)(z_4-z_3)}{(z_4-z_1)(z_2-z_3)}$$

$$= (z_1, z_2, z_3, z_4)$$

   Similarly, one can use Formula 5 to verify these equations in case one of the z_j is ∞. Since any Möbius transformation T can be represented as the composition of translations f, magnifications and rotations g, and inversions h, it follows that

$$(T(z_1), T(z_2), T(z_3), T(z_4)) = (z_1, z_2, z_3, z_4).$$

5. $\lambda > 0$

6. $(z, -3i, 3/2, \infty) = (w, 2, 4+2i, 6)$, or

$$\frac{2(z+3i)}{3+6i} = \frac{(w-2)(-2+2i)}{(w-6)(2+2i)}$$

maps the half plane below the line to the interior of the circle. For the map to the exterior use
$(z, -3i, 3/2, \infty) = (w, 6, 4+2i, 2)$

7. No; i and $-i$ are symmetric with respect to the real line but their images 2 and $\frac{-1}{2}$ are not symmetric with respect to the unit circle.

8. If $S(z_1) = w_1, S(z_2) = w_2$ and $S(z_3) = w_3$ then $(T^{-1} \circ S)(z)$ has the three fixed points z_1, z_2 and z_3. Therefore, according to Problem 6 in Section 7.3, $(T^{-1} \circ S)(z) \equiv z$, or $T = S$.

9. 1 and -1 are symmetric with respect to the imaginary axis, so $f(1)$ and $f(-1)$ are symmetric with respect to $|w| = 1$. Hence $f(-1) = 0$.

10. (a) A line L and a circle C have no points in common. A lateral shift and a rotation places the line on the real axis ($f_1(z) = e^{i\theta}(z-\alpha)$ where $\alpha$is real and $-\pi < \theta < \pi$). the mapping $f_1(z)$ keeps the circle C entirely in the upper or lower half-plane. If it is in the lower half-plane, use $f_2(z) = -z$ to put it in the upper half-plane. Next shift the circle by a real number to put the center on the imaginary axis with a center at $i\beta$. Because the circle does not have points in common with the real axis, the radius $r < \beta$. Scale the center to $i\lambda$ with the mapping λ/β. The real axis is remains the real axis under these transformations. The radius of the circle centered at $i\lambda$ is $R=(\lambda/\beta)r$ and $R < \lambda$.
(b) The points ω_1 and ω_2 are symmetric with respect to the real axis when $\omega_2 = \underline{\omega}_1$. The points ω_1 and ω_2 are symmetric with respect to the circle with radius R centered at $i\lambda$ when

$$\omega_2 = R^2/(\underline{\omega}_1 + i\lambda) + i\lambda \text{ (Formula 11)}$$

Substituting $\omega_2 = R^2/(\underline{\omega}_2 + i\lambda) + i\lambda$ can be solved as $\omega_2{}^2 = R^2 - \lambda^2$.
Choose $\omega_1 = -i\sqrt{(\lambda^2 - R^2)}$ and $\omega_2 = i\sqrt{(\lambda^2 - R^2)}$
(c) Let T(z) be the Möbius transformation obtained as the composite of the transformations in Part (a). By the symmetry $z_1 = T^{-1}(\omega_1)$ and $z_2 = T^{-1}(\omega_2)$ are symmetric to the line L and the circle C.

11. The circles C_1 and C_2 have no points in common. The radius of C_1 is R_1 and the center is a_1. The mapping $f_1(z) = (z-a_1)/R_1$ maps C_1 to the unit circle and $f_2 = i\,(1 + z)/(1 - z)$ maps the unit circle to the real axis. Because the mapping $f_2 o f_1$ is one-to-one, the image of circle C_2 has no points in common with the real line. From Problem 10 the line and the circle have points ω_1 and ω_2 that are symmetric simultaneously. By the symmetry principle C_1 and C_2 have points z_1 and z_1 that are symmetric to both.

12. Let $S(z) = \dfrac{z - z_1}{z - z_2}$. Then $S(z_1) = 0$ and $S(z_2) = \infty$ are two points symmetric with respect to $S(C_1)$ and $S(C_2)$ simultaneously. It follows that $S(C_1)$ and $S(C_2)$ both have center zero, so that they are concentric.

13. Let T be a Möbius transformation that maps C to the real line. Then $T(z_1), T(z_2)$, and $T(z_3)$ are real and z and z^* are symmetric if and only if $T(z^*) = \overline{T(z)}$ (symmetry with respect to the real line). By Problem 4, this occurs if and only if

$$(z^*, z_1, z_2, z_3) = (T(z^*), T(z_1), T(z_2), T(z_3)) = \left(\overline{T(z)}, T(z_1), T(z_2), T(z_3)\right)$$

$$= \overline{(T(z), T(z_1), T(z_2), T(z_3))} = \overline{(z, z_1, z_2, z_3)}$$

14. Since $(w_1, w_2, w_3, w_4) = (z, 0, 1, \infty) \equiv z$ the left side is real if and only if z is real. This is true if and only if w_1, w_2, w_3 and w_4 lie on the image of the real line, which must be a circle or line in the w-plane.

15. $|f(-1)| = \left|e^{i\theta}\right| \left|\dfrac{-1-\alpha}{-\overline{\alpha}-1}\right| = 1$

$|f(1)| = \left|e^{i\theta}\right| \left|\dfrac{1-\alpha}{\overline{\alpha}-1}\right| = 1$

$|f(i)| = \left|e^{i\theta}\right| \left|\dfrac{i-\alpha}{i\overline{\alpha}-1}\right| = 1$

Consequently, the images of three points on $|z| = 1$ lie on $|w| = 1$. Since $|f(0)| = \left|e^{i\theta}\alpha\right| < 1$, f maps the interior to the interior.

16. The proof is similar to Example 3. Let f(z) be a Möbius transformation that maps $|z| < 1$ onto $|w| > 1$. Then f(z) maps the circle $|z| = 1$ onto $|w| = 1$. Furthermore, there must be some point β, $|\beta| < 1$, that is mapped to ∞, i.e., $f(\beta) = \infty$. According to formula (11) (with $a = 0$ and $R = 1$) the point

$\beta^* = 1^2/(\infty - a) + 0 = 0 = 1/\underline{\beta}$

is symmetric to β with respect to the circle $|z| = 1$. But the point symmetric to ∞ is the center of the circle $a = 0$.

$f(1/\underline{\beta}) = 0$

Consequently, f has a zero at $1/\underline{\beta}$ and a pole at β, so f is of the form

$f = k(z-1/\underline{\beta})/(z-\beta)$ with $|\beta| < 1$.

for some constant k. Let $\alpha = 1/\underline{\beta}$. Then,

$f = k(z-\alpha)/(z-1/\underline{\alpha}) = k\underline{\alpha}(z-\alpha)/(\underline{\alpha}z-1)$

Moreover, since f(1) lies on the circle $|w| = 1$, we have

$1 = |f(1)| = |k\underline{\alpha}|\,|(1-\alpha)/(\underline{\alpha}-1)| = |k\underline{\alpha}|$

Thus $k\underline{\alpha} = e^{i\theta}$ for some real θ and

$f = e^{i\theta}(z-\alpha)/(\underline{\alpha}z-1)$ with $|\alpha| > 1$ is the desired transformation.

17. $w = e^{i\theta}\dfrac{2z-i}{-iz-2}$ for any real θ.

18. Expand the determinant by cofactors across the first row to get the equation

$$\begin{vmatrix} z_1 & w_1 & z_1w_1 \\ z_2 & w_2 & z_2w_2 \\ z_3 & w_3 & z_3w_3 \end{vmatrix} - z\begin{vmatrix} 1 & w_1 & z_1w_1 \\ 1 & w_2 & z_2w_2 \\ 1 & w_3 & z_3w_3 \end{vmatrix} + w\begin{vmatrix} 1 & z_1 & z_1w_1 \\ 1 & z_2 & z_2w_2 \\ 1 & z_3 & z_3w_3 \end{vmatrix} - zw\begin{vmatrix} 1 & z_1 & w_1 \\ 1 & z_2 & w_2 \\ 1 & z_3 & w_3 \end{vmatrix} = 0$$

Since all of the 3×3 determinants contain only constants this equation can be written as

$A - zB + wC - zwD = 0$

Solving for w yields $w = \dfrac{Bz-A}{-Dz+C}$, which is a Möbius transformation whenever $AD \neq BC$. To see that this Möbius transformation takes z_i to w_i, substitute $z = z_i$ and $w = w_i$ in the determinant. The determinant is zero because it has two identical rows.

19. Let x_1, x_2, x_3, X_1, X_2, and X_3 be real numbers such that $x_1 < x_2 < x_3$ and $X_1 < X_2 < X_3$. Then the cross ratio

$(z, x_1, x_2, x_3) = (w, X_1, X_2, X_3)$

characterizes all Möbius transformations of the upper half-plane onto itself. This can be rearranged to have the form $w = \dfrac{az+b}{cz+d}$, where a, b, c, d are real and $ad - bc > 0$ (so that $f(i)$ is in the upper half-plane).

20. Let $w = f(z) = \dfrac{az+b}{cz+d} = \dfrac{a}{c}\dfrac{z+b/a}{z+d/c}$ $\quad a, c \neq 0$ be a transformation of the upper half-plane onto $|w| < 1$.

$$|f(\infty)| = \left|\frac{a}{c}\right| = 1 \Longrightarrow \frac{a}{c} = e^{i\theta}$$

$$|f(0)| = 1 \Longrightarrow |b| = |d| \text{ and } \left|\frac{b}{a}\right| = \left|\frac{d}{c}\right|$$

$$|f(1)| = 1 \Longrightarrow \left|\frac{a+b}{c+d}\right| = 1 \Longrightarrow \text{Re}\left(\frac{b}{a}\right) = \text{Re}\left(\frac{d}{c}\right)$$

Since $\left|\dfrac{b}{a}\right| = \left|\dfrac{d}{c}\right|$ as well, it follows that $\dfrac{b}{a} = \dfrac{\overline{d}}{\overline{c}}$

$\left(\dfrac{b}{a}\text{ could equal } \dfrac{d}{c}, \text{ but then } f(z) \text{ degenerates}\right)$

Letting $z_0 = \dfrac{-b}{a}$, it follows that $f(z) = e^{i\theta}\dfrac{z-z_0}{z-\overline{z_0}}$.

$\text{Im}\,(z_0) > 0$ since f maps z_0 to zero inside the disk.

21. Let $f_1(z) = \dfrac{az+b}{cz+d}$, $f_2(z) = \dfrac{ez+f}{gz+h}$ and $f_3(z) = \dfrac{kz+l}{mz+n}$ be Möbius transformations.

a. We verify the group properties:

(i) $(f_1 \circ f_2)(z) = \dfrac{a\left(\frac{ez+f}{gz+h}\right)+b}{c\left(\frac{ez+f}{gz+h}\right)+d}$

$$= \frac{(ae+bg)z+(af+bh)}{(ce+dg)z+(cf+dh)}$$

(ii) $[(f_1 \circ f_2) \circ f_3](z) = \dfrac{(ae+bg)\left(\frac{kz+l}{mz+n}\right)+(af+bh)}{(ce+dg)\left(\frac{kz+l}{mz+n}\right)+(cf+dh)}$

$$[f_1 \circ (f_2 \circ f_3)](z) = \frac{a\left[\frac{(ek+fm)z+(el+fn)}{(gk+hm)z+(gl+hn)}\right]+b}{c\left[\frac{(ek+fm)z+(el+fn)}{(gk+hm)z+(gl+hn)}\right]+d}$$

Both $(f_1 \circ f_2) \circ f_3$ and $f_1 \circ (f_2 \circ f_3)$ simplify to

$$\frac{(aek+afm+bgk+bhm)z+(ael+afn+bgl+bhn)}{(cek+cfm+dgk+dhm)z+(cel+cfn+dgl+dhn)}$$

(iii) $I(z) = z$ is the identity because

$$(I \circ f_1)(z) = I\left(\frac{az+b}{cz+d}\right) = f_1(z) = (f_1 \circ I)(z)$$

(iv) $f_1^{-1}(z) = \dfrac{dz-b}{-cz+a}$ because

$$(f_1 \circ f_1^{-1})(z) = \frac{a\left(\frac{dz-b}{-cz+a}\right)+b}{c\left(\frac{dz-b}{-cz+a}\right)+d} = z = I(z) \text{ and}$$

$$(f_1^{-1} \circ f_1)(z) = I(z)$$

b. No; for example if $f_1(z) = z+1$ and $f_2(z) = \dfrac{1}{2}z$ then

$(f_1 \circ f_2)(z) = \dfrac{1}{2}z + 1$ but $(f_2 \circ f_1)(z) = \dfrac{z+1}{2}$

22. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, C = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$ with determinants $|A| = |B| = |C| = 1$.

a. (i) $AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$ is a two-by-two matrix with $|AB| = |A||B| = 1$.

(ii) $A(BC) = (AB)C$

$$= \begin{pmatrix} aek+afm+bgk+bhm & ael+afn+bgl+bhn \\ cek+cfm+dgk+dhm & cel+cfn+dgl+dhn \end{pmatrix}$$

(iii) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity.

(iv) $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

b. $\dfrac{az+b}{cz+d} \cdot \dfrac{1/\sqrt{ad-bc}}{1/\sqrt{ad-bc}} = \dfrac{\alpha z+\beta}{\gamma z+\delta} \Longrightarrow$

$\alpha\delta - \beta\gamma = \dfrac{ad}{ad-bc} - \dfrac{bc}{ad-bc} = 1$

c. $(T_1 \circ T_2)(z) = \dfrac{(\alpha_1\alpha_2+\beta_1\gamma_2)z + (\alpha_1\beta_2+\beta_1\delta_2)}{(\gamma_1\alpha_2+\delta_1\gamma_2)z + (\gamma_1\beta_2+\delta_1\delta_2)}$

$S_1S_2 = \begin{pmatrix} \alpha_1\alpha_2+\beta_1\gamma_2 & \alpha_1\beta_2+\beta_1\delta_2 \\ \gamma_1\alpha_2+\delta_1\gamma_2 & \gamma_1\beta_2+\delta_1\delta_2 \end{pmatrix}$

The association is clear; see the statement of part b.

23. Let $z = x+iy$ and $w = u+iv$ with $x > 0, u > 0$. Then

$$T_0(w) = \frac{a_0}{(x+a_0+u)+(iy+b_1+iv)}$$

$$= \frac{a_0(x+a_0+u) - a_0(iy+b_1+iv)}{(x+a_0+u)^2 + |iy+b_1+iv|^2}$$

For w in the right half-plane, $T_0(w)$ has a positive real part. Similarly, $T_k(w)$ has a positive real part for $k = 1, 2, 3, \ldots$ Thus $T_k(RHP) \subseteq RHP$.

To see more precisely how T_0 maps the right half-plane, calculate $T_0(0), T_0(i)$, and $T_0(\infty)$. It is straightforward to show that each of these points lies in the right half-plane within $\left|\zeta - \frac{1}{2}\right| \le \frac{1}{2}$ (and $T_0(\infty) = 0$). It follows that T_0 maps the right half-plane to an open disk inside $\left|\zeta - \frac{1}{2}\right| = \frac{1}{2}$. Then any number of applications of T_k maps the RHP inside the RHP, and $T_0 \circ T_1 \circ T_2 \circ \cdots \circ T_{n-2} \circ T_{n-1}(RHP) \subseteq T_0(RHP) \subseteq \left|\zeta - \frac{1}{2}\right| < \frac{1}{2}$

24. $\dfrac{Q(z)}{P(z)} = T_0 \circ T_1 \circ \cdots \circ T_{n-1}(0)$ as defined in Problem 21, so $Q(z)/P(z)$ maps the closed right half-plane into $\left|\zeta - \frac{1}{2}\right| \le \frac{1}{2}$. Thus all the poles of $Q(z)/P(z)$ (corresponding to zeros of $P(z)$) are in the left half-plane.

25. $\dfrac{Q(z)}{P(z)} = \dfrac{3z^2+6}{z^3+3z^2+6z+6} = \dfrac{3}{z+3+\frac{4z}{z^2+2}} = \dfrac{3}{z+3+\frac{4}{z+2/z}}$

By Problem 22, $P(z)$ has all its zeros in the left half-plane.

EXERCISES 7.5: The Schwarz-Christoffel Transformation

1. At the corner $w_1 = -1$ the polygon takes a right turn of θ_1 with $\theta_1 \to \pi$. For any x_1 chosen as the preimage of w_1,

$$f(z) = \lim_{\theta \to \pi} A \int_0^z (\zeta - x_1)^{\theta_1/\pi} d\zeta + B$$

$$= A(z - x_1)^2 + B$$

(These are not the same A and B, but they are still constants that we have yet to determine, so we will not create new notation like A' and B' in this and the following problems.)
$f(x_1) = -1 \Longrightarrow B = -1$
$f(\pm\infty) = -\infty \Longrightarrow A < 0$
$f(z) = A(z - x_1)^2 - 1$ with $A < 0$

2. At $w_1 = 0$ we have a turn of $\theta_1 = -\pi/2$. Map $x_1 = 0$ to $w_1 = 0$. Then
$f(z) = A\int_0^z \zeta^{-1/2} d\zeta + B = Az^{1/2} + B$.
$f(0) = 0 \Longrightarrow B = 0$
A can be any positive value. $A = 1$ yields $f(z) = \sqrt{z}$ as one mapping of the upper half-plane onto the first quadrant.

3. At $w_1 = i$ and $w_2 = 0$ turn $\theta_1 = -\pi/2$, $\theta_2 = -\pi/2$. Choose $x_1 = -$ and $x_2 = 1$.

$$f(z) = A\int_0^z (\zeta+1)^{-1/2}(\zeta-1)^{-1/2} d\zeta + B$$

$$= A\sin^{-1} z + B$$

$$f(-1) = i \Longrightarrow \quad A\left(\frac{-\pi}{2}\right) + B = i$$

$$f(1) = 0 \Longrightarrow \quad A\left(\frac{\pi}{2}\right) + B = 0$$

$$\Longrightarrow B = i/2, \quad A = -i/\pi$$

$$f(z) = \frac{-i}{\pi}\sin^{-1} z + \frac{i}{2}$$

4. This is a Schwarz-Christoffel transformation taking $x_1 = -1$ w_1, taking $x_2 = 1$ to some w_2, and taking $x_3 = \infty$ to some u the integrand is $(\zeta-1)^{-2/3}(\zeta+1)^{-2/3}$ there are turns of -2τ and w_2. It follows immediately from a sketch that w_3 is w_1, w_2, w_3 are the corners of an equilateral triangle. E

5. At $w_1 = 1$ and $w_2 = -1$ turn $\theta_1 = \pi/2, \theta_2 = \pi/2$.
Choose $x_1 = -1$ and $x_2 = 1$.

$$f(z) = A\int_0^z (\zeta+1)^{1/2}(\zeta-1)^{1/2} d\zeta + B$$

$$= A\left(\sin^{-1} z + z\sqrt{1-z^2}\right) + B$$

$$f(-1) = 1 \Longrightarrow A(-\pi/2) + B = 1$$
$$f(1) = -1 \Longrightarrow A(\pi/2) + B = -1$$

$\Longrightarrow B = 0, A = -2/\pi$

$f(z) = \dfrac{-2}{\pi}\left(\sin^{-1} z + z\sqrt{1-z^2}\right)$

6. Let $w_1 = 1 - z, w_2 = \sqrt{w_1}, w_3 = \dfrac{w_2 - \sqrt{2}}{w_2 + \sqrt{2}}$.
   Then Upper half-plane $\xrightarrow{w_1}$ Lower half-plane $\xrightarrow{w_2}$ Fourth quadrant $\xrightarrow{w_3} \{|w_3| < 1 \text{ and } \operatorname{Im} w_3 < 0\}$.
   As $x \to -\infty, w_2 \to +\infty$ and $w_3 \to 1$,
   so $f(x) = \dfrac{\sqrt{2}}{\pi} w_2 + \dfrac{1}{\pi}\log w_3 + i \to +\infty$
   As $x \to \infty, w_2 \to -i\infty$ and $w_3 \to 1$, so $f(x) \to -i\infty$
   As $x \to -1^-, w_2 \to \sqrt{2}^+$ and $w_3 \to 0^+$, so $f(x) \to -\infty$
   As $x \to -1^+, w_2 \to \sqrt{2}^-$ and $w_3 \to 0^-$, so $f(x) \to -\infty$
   $f(1) = \dfrac{\sqrt{2}}{\pi}(0) + \dfrac{1}{\pi}\log(-1) + i = 0$

7. At $w_1 = 0$ and $w_2 = 1$ turn $\theta_1 = \pi/2, \theta_2 = -\pi/2$.
   Choose $x_1 = -1$ and $x_2 = 1$.

$$f(z) = A\int_0^z \frac{(\zeta+1)^{1/2}}{(\zeta-1)^{1/2}}d\zeta + B = A\int_0^z \frac{\zeta+1}{(\zeta^2-1)^{1/2}}d\zeta + B$$

$$= A\left(\sqrt{z^2-1} - i\sin^{-1} z\right) + B$$

$f(-1) = 0 \Longrightarrow A(\pi i/2) + B = 0$
$f(1) = 1 \Longrightarrow A(-\pi i/2) + B = 1$
$\Longrightarrow B = 1/2, A = i/\pi$
$f(z) = \dfrac{i}{\pi}\sqrt{z^2-1} + \dfrac{1}{\pi}\sin^{-1} z + \dfrac{1}{2}$

8. From Formula 8, it is clear that $f(z)$ maps $x_1 = -k, x_2 = -1$, $x_3 = 1, x_4 = k$ to the four corners w_i of a polygon with angles $\theta_i = -\pi/2$. The result is a rectangle. (The corner corresponding to $x_5 = \infty$ has a turn of $\theta_5 = 0$, according to Formula 6; thus it is not a corner at all.) The upper half-plane maps to the interior of this rectangle since it is positively oriented.

   To locate the corners of the rectangle,

$$f(k) = \int_0^k \frac{dx}{\sqrt{(1-x^2)(k^2-x^2)}}$$

$$= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(k^2-x^2)}} + \int_1^k \frac{dx}{\sqrt{(1-x^2)(k^2-x^2)}}$$

$$= \frac{1}{k}\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-x^2/k^2)}} + \frac{i}{k}\int_1^k \frac{dx}{\sqrt{(x^2-1)(1-x^2/k^2)}}$$

The real and imaginary parts of $f(k)$ are both positive. Denote $f(k) = b + ic$.

To see how the other corners relate to this, compute

$$f(-1) = \int_0^{-1} \frac{dx}{\sqrt{(1-x^2)(k^2-x^2)}} = \frac{-1}{k}\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-x^2/k^2)}} = -b$$

$$f(1) = \frac{1}{k}\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-x^2/k^2)}} = b$$

$$f(-k) = \int_0^{-k} \frac{dx}{\sqrt{(1-x^2)(k^2-x^2)}}$$

$$= \frac{-1}{k}\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-x^2/k^2)}} + \frac{i}{k}\int_1^k \frac{dx}{\sqrt{(x^2-1)(1-x^2/k^2)}}$$

$$= -b + ic$$

9. At the corners $w_1 = i, w_2 \to -\infty, w_3 \to \infty$ the turns are $\theta_1 \to 0^-, \theta_2 \to -\pi^+, \theta_3 \to -\pi^+$.
Choose preimages $x_1 < x_2$ and $x_3 = \infty$.

$$f(z) = A\int_0^z (\zeta - x_1)^0(\zeta - x_2)^{-1} d\zeta + B$$

$$= A\,\mathrm{Log}(z - x_2) + B$$

For $x_2 < x < x_3 = +\infty, f(x)$ is real, so A and B are real.

$$f(x_1) = i \implies A\,\mathrm{Log}(x_1 - x_2) + B = i$$
$$\implies A\,\mathrm{Log}(x_2 - x_1) + Ai\pi + B = i$$
$$\implies A = \frac{1}{\pi}, B = \frac{-1}{\pi}\,\mathrm{Log}(x_2 - x_1)$$

$$f(z) = \frac{1}{\pi}\,\mathrm{Log}(z - x_2) - \frac{1}{\pi}\,\mathrm{Log}(x_2 - x_1) = \frac{1}{\pi}\,\mathrm{Log}\left(\frac{z - x_2}{x_2 - x_1}\right) \text{ for } x_1 < x_2$$

10. The Mōbius transformation $t = i\left(\frac{1-z}{1+z}\right)$ maps $|z| < 1$ to the upper half-plane and maps the points z_j to points x_j on the real line. Then

$$f(t) = A\int_0^t (\tau - x_1)^{\theta_1/\pi}(\tau - x_2)^{\theta_2/\pi}\cdots(\tau - x_n)^{\theta_n/\pi}d\tau$$

is the Schwarz-Christoffel transformation from the upper half-plane to the polygon. Composing with $\tau = i\left(\frac{1-\zeta}{1+\zeta}\right)$ yields

$$\tau - x_j = i\left(\frac{1-\zeta}{1+\zeta} - \frac{1-z_j}{1+z_j}\right) = -2i\left(\frac{\zeta - z_j}{(1+\zeta)(1+z_j)}\right).$$

$$d\tau = \frac{-2i}{(1+\zeta)^2}d\zeta$$

Thus

$$f(z) = A\int_0^z \left(\frac{-2i}{1+\zeta}\right)^{\sum\limits_{j=1}^{n}\theta_j/\pi}\left(\frac{\zeta - z_1}{1+z_1}\right)^{\theta_1/\pi}\cdots$$
$$\cdot\left(\frac{\zeta - z_n}{1+z_n}\right)^{\theta_n/\pi}\left(\frac{-2i}{(1+\zeta)^2}\right)d\zeta + B$$

Observe that $\sum\limits_{j=1}^{n}\theta_j/\pi = -2$ and many constant factors can be absorbed into A, so that

$$f(z) = A\int_0^z (\zeta - z_1)^{\theta_1/\pi}\cdots(\zeta - z_n)^{\theta_n/\pi}d\zeta + B$$

11. By the generalization of the Schwarz reflection principle given in Problem 14, Section 5.8, the analytic continuation of the Schwarz-Christoffel transformation across the interval (x_{j-1}, x_j) maps the lower half-plane

onto the figure obtained by reflecting the polygon across the line segment from $f(x_{j-1})$ to $f(x_j)$. Note that different line segments (x_{j-1}, x_j) produce different analytic continuations, dependent on the location of branch cuts.

Applying this to Example 1, the analytic continuation of the Schwarz-Christoffel transformation across the interval $(-1, 1)$ maps the lower half-plane to the interior of the triangle with corners $1, 1+i$, and i, while the analytic continuation across the interval $(1, \infty)$ maps the lower half-plane to the interior of the triangle with corners $0, i$, and -1.

EXERCISES 7.6: Applications in Electrostatics, Heat Flow, and Fluid Mechanics

1. $w = \dfrac{1+z}{1-z}$ maps the semidisk conformally to the first quadrant with the line segment $-1 \le z \le 1$ mapping to the positive u axis and the semicircular arc mapping to the positive v axis.

   $\psi(w) = \dfrac{2}{\pi} \operatorname{Arg} w$ is a harmonic function that is zero on the positive u axis and one on the positive v axis.
   $\phi(z) = \dfrac{2}{\pi} \operatorname{Arg}\left(\dfrac{1+z}{1-z}\right)$

2. $w = -i\dfrac{1+z}{1-z}$ maps the region conformally to the first quadrant with the line segments $z \le -1$ and $z \ge 1$ mapping to the positive v axis and the semicircular arc mapping to the positive u axis. $\psi(w) = 1 - \dfrac{2}{\pi} \operatorname{Arg} w$ is a harmonic function that is zero on the positive v axis and one on the positive u axis.

   $$\phi(z) = 1 - \frac{2}{\pi} \operatorname{Arg}\left(-i\frac{1+z}{1-z}\right)$$

3. $w_1 = z^2$ maps the region to the semidisk $\{|w_1| < 1, \operatorname{Im} w_1 > 0\}$ and $w = \dfrac{1+w_1}{1-w_1} = \dfrac{1+z^2}{1-z^2}$ maps the region to the first quadrant with the line segments mapping to the positive u axis and the circular arc mapping to the positive v axis.

$\psi(w) = \frac{2}{\pi} \operatorname{Arg} w - 1$ is a harmonic function that is -1 on the positive u axis and 0 on the positive v axis.

$$T(z) = \frac{2}{\pi} \operatorname{Arg}\left(\frac{1+z^2}{1-z^2}\right) - 1$$

4. $w = i\dfrac{1+z}{1-z}$ maps the disk to the upper half-plane with the arcs in the first, second, third, and fourth quadrants mapping to the line segments $-\infty \le w \le -1$, $-1 \le w \le 0$, $0 \le w \le 1$, and $1 \le w \le \infty$ respectively.

   Applying Problem 3, Section 7.1,

$$\psi(w) = 3 + \frac{1}{\pi}(0-1) \operatorname{Arg}(w+1) + \frac{1}{\pi}(1-2) \operatorname{Arg}(w) + \frac{1}{\pi}(2-3) \operatorname{Arg}(w-1)$$

$$= 3 - \frac{1}{\pi}[\operatorname{Arg}(w+1) + \operatorname{Arg}(w) + \operatorname{Arg}(w-1)]$$

   is a harmonic function with the right boundary values.

$$T(z) = 3 - \frac{1}{\pi}\left[\operatorname{Arg}\left(i\frac{1+z}{1-z}+1\right) + \operatorname{Arg}\left(i\frac{1+z}{1-z}\right) + \operatorname{Arg}\left(i\frac{1+z}{1-z}-1\right)\right]$$

5. By Example 6, $w = (z^2+1)^{1/2}$ takes the slit z-plane to the upper half of the w plane with the slit from zero to i mapping to the line segment $-1 \le w \le 1$ and with the real axis mapping to the rays $w \le -1, w \ge 1$. Applying Problem 3, Section 7.1,

$$\psi(w) = \frac{1}{\pi}[\operatorname{Arg}(w-1) - \operatorname{Arg}(w+1)]$$

$$\phi(z) = \frac{1}{\pi}\left[\operatorname{Arg}(\sqrt{z^2+1}-1) - \operatorname{Arg}(\sqrt{z^2+1}+1)\right]$$

6. $w = \dfrac{z}{z-2}$ maps the region to the strip $0 < \operatorname{Re} w < 1/2$ with the inner circle mapping to $\operatorname{Re} w = 0$ and the outer circle mapping to $\operatorname{Re} w = \dfrac{1}{2}$.

$\psi(w) = 2 \text{ Re}\, w$ is a harmonic function that is zero on $\text{Re}\, w = 0$ and one on $\text{Re}\, w = 1/2$.

$$\phi(z) = 2 \text{ Re} \left(\frac{z}{z-2}\right)$$

7. a. By Problem 3, Section 7.5 take the mapping $z = \frac{i}{2} - \frac{i}{\pi} \sin^{-1} w$, or $w = \sin(\pi i z + \pi/2) = \cos(\pi i z)$

   This maps the region shown to the upper half-plane, with $\{x \geq 0, y = i\}$ mapping to the line segment $-\infty < w \leq -1$, with $\{x = 0, 0 \leq y \leq 1\}$ mapping to $-1 \leq w \leq 1$, and with $\{x \geq 0, y = 0\}$ mapping to $1 \leq w < \infty$.

   Applying Problem 3, Section 7.1,

   $$\psi(w) = 1 + \frac{1}{\pi}[- \text{Arg}(w+1) - \text{Arg}(w-1)]$$

   $$T(z) = 1 - \frac{1}{\pi}[\text{Arg}(\cos(\pi i z) + 1) + \text{Arg}(\cos(\pi i z) - 1)]$$

   b. $w = e^{\pi z}$ maps the region to the upper half-plane, with $\{x \geq 0, y = 1\}$ mapping to the real ray $-\infty < w \leq -1$, with $\{x \leq 0, y = 1\}$ mapping to the line segment $-1 \leq w \leq 0$, and with $\{y = 0\}$ mapping to the real ray $0 \leq w < \infty$.

   Applying Problem 3, Section 7.1,

   $$\psi(w) = \frac{1}{\pi}[2 \text{ Arg}(w+1) - \text{Arg}(w)]$$

   $$T(z) = \frac{2}{\pi} \text{Arg}(e^{\pi z} + 1) - \frac{1}{\pi} \text{Arg}(e^{\pi z})$$

8. $w_1 = z + 2$ shifts the region two units to the right. From the solution of Problem 11, Section 7.4 it would help to find two points u_1 and u_2 (necessarily real) simultaneously symmetric with respect to $|w_1| = 1$ and $|w_1 - 5| = 2$. The symmetry equations (Formula 11, Section 7.4) are

   $$u_2 = \frac{1}{u_1},\ u_2 = \frac{4}{u_1 - 5} + 5$$

   Solving, $u_1 = \dfrac{11 - 4\sqrt{6}}{5}$ and $u_2 = \dfrac{11 + 4\sqrt{6}}{5}$ are simultaneously symmetric.

Now $w = \dfrac{w_1 - u_2}{w_1 - u_1} = \dfrac{5w_1 - 11 - 4\sqrt{6}}{5w_1 - 11 + 4\sqrt{6}}$ maps

$|w_1| = 1$ to $|w| = \dfrac{11 + 4\sqrt{6}}{5} = 4.15959$ and maps $|w_1 - 5| = 2$ to $|w| = \dfrac{7 - 2\sqrt{6}}{5} = 0.4202$. The original region maps to the annulus between these concentric circles.

$\psi(w) = A \operatorname{Log}|w| + B$ is a harmonic function that will equal zero on $|w| = \dfrac{7-2\sqrt{6}}{5}$ and one on $|w| = \dfrac{11+4\sqrt{6}}{5}$ with the proper choice of A and B.

$$0 = A \operatorname{Log}\left|\frac{7-2\sqrt{6}}{5}\right| + B$$

$$1 = A \operatorname{Log}\left|\frac{11+4\sqrt{6}}{5}\right| + B$$

$$\Longrightarrow A = 1/\operatorname{Log}\left|\frac{11+4\sqrt{6}}{7-2\sqrt{6}}\right| = 1/\operatorname{Log}(5+2\sqrt{6})$$

$$B = \frac{-\operatorname{Log}\left|\frac{7-2\sqrt{6}}{5}\right|}{\operatorname{Log}\left|5+2\sqrt{6}\right|}$$

$$\psi(w) = \frac{\operatorname{Log}|w| - \operatorname{Log}\left|\frac{7-2\sqrt{6}}{5}\right|}{\operatorname{Log}(5+2\sqrt{6})} = \frac{\operatorname{Log}\left|\frac{7+2\sqrt{6}}{5}w\right|}{\operatorname{Log}(5+2\sqrt{6})}$$

$$\phi(z) = \frac{1}{\operatorname{Log}(5+2\sqrt{6})} \operatorname{Log}\left|\frac{(5z-1-4\sqrt{6})\left(\frac{7+2\sqrt{6}}{5}\right)}{5z-1+4\sqrt{6}}\right|$$

$$= \frac{1}{\operatorname{Log}(5+2\sqrt{6})} \operatorname{Log}\left|\frac{(7+2\sqrt{6})z - 11 - 6\sqrt{6}}{5z-1+4\sqrt{6}}\right|$$

$$= 0.43622 \operatorname{Log}\left|\frac{11.89898z - 25.69694}{5z + 8.79796}\right|$$

9. Find two points x_1 and x_2 (both real) simultaneously symmetric with respect to $|z| = 4$ and $|z-2| = 1$.

The symmetry equations are

$$x_2 = \frac{16}{x_1},\ x_2 = \frac{1}{x_1 - 2} + 2$$

Solving, $x_1 = \dfrac{19 - \sqrt{105}}{4}$ and $x_2 = \dfrac{19 + \sqrt{105}}{4}$ are simultaneously symmetric.

Now $w = \dfrac{z - x_2}{z - x_1} = \dfrac{4z - 19 - \sqrt{105}}{4z - 19 + \sqrt{105}}$ maps $|z| = 4$ to $|w| = \dfrac{19 + \sqrt{105}}{16} = 1.82793$ and maps $|z-2| = 1$ to $|w| = \dfrac{11 + \sqrt{105}}{4} = 5.31174$. The original region maps to the annulus between these concentric circles.

$\psi(w) = A \operatorname{Log}|w| + B$ is a harmonic function that will equal zero o $|w| = \dfrac{19 + \sqrt{105}}{16}$ and one on $|w| = \dfrac{11 + \sqrt{105}}{4}$ with the proper cho of A and B.

$$0 = A \operatorname{Log}\left|\frac{19 + \sqrt{105}}{16}\right| + B$$

$$1 = A \operatorname{Log}\left|\frac{11 + \sqrt{105}}{4}\right| + B$$

$$\Longrightarrow A = 1/\operatorname{Log}\left|\frac{44 + 4\sqrt{105}}{19 + \sqrt{105}}\right| = 1/\operatorname{Log}\left|\frac{13 + \sqrt{105}}{8}\right|$$

$$B = -\operatorname{Log}\left|\frac{19 + \sqrt{105}}{16}\right| / \operatorname{Log}\left|\frac{13 + \sqrt{105}}{8}\right|$$

$$\psi(w) = \left[\operatorname{Log}\left|\frac{13 + \sqrt{105}}{8}\right|\right]^{-1} \operatorname{Log}\left|\frac{16w}{19 + \sqrt{105}}\right|$$

Let $s = \dfrac{13 - \sqrt{105}}{8}$ and $\lambda = \dfrac{19 + \sqrt{105}}{16}$

$$\text{Then } \phi(z) = [\operatorname{Log}(1/s)]^{-1} \operatorname{Log}\left|\frac{z - 4\lambda}{\lambda(z - 4/\lambda)}\right|$$

$$= [\operatorname{Log} s]^{-1} \operatorname{Log}\left|\frac{4 - \lambda z}{4\lambda - z}\right| = -0$$

9. Find respec

10. The circles intersect at $e^{\pi i/6}$ and $e^{5\pi i/6}$. $w = \dfrac{z - e^{5\pi i/6}}{z - e^{\pi i/6}}$ maps the region to the unbounded sector $-2\pi/3 < \phi < -\pi/3$ in the w-plane with the arc of $|z| = 1$ mapping to the ray $\phi = -2\pi/3$ and the arc of $|z - i| = 1$ mapping to the ray $\phi = -\pi/3$. $\psi(w) = \dfrac{6}{\pi}\,\mathrm{Arg}\,w + 3$ is a harmonic function that equals -1 on $\phi = -2\pi/3$ and equals 1 on $\phi = -\pi/3$.

$$T(z) = \frac{6}{\pi}\,\mathrm{Arg}\left(\frac{z - e^{5\pi i/6}}{z - e^{\pi i/6}}\right) + 3$$

11. By Problem 7, Section 7.5, $w = f(z) = \dfrac{i}{\pi}\sqrt{z^2 - 1} + \dfrac{1}{\pi}\sin^{-1} z + \dfrac{1}{2}$ relates the region to the upper half of the z-plane, where the streamlines are $\mathrm{Im}\, z = c$. Thus the streamlines in the crooked region are $\mathrm{Im}\left[f^{-1}(w)\right] = c$.

EXERCISES 7.7: Further Physical Applications of Conformal Mapping

1. Consider the half-strip $-\pi/2 < u < \pi/2, v > 0$ in the w-plane. The mapping $w_1 = \sin w$ transforms this to the upper half of the w_1-plane, and the Schwarz-Christoffel transformation from Problem 7, Section 7.5,

$$z = \frac{i}{\pi}\sqrt{w_1^2 - 1} + \frac{1}{\pi}\sin^{-1} w_1 + \frac{1}{2} = \frac{-1}{\pi}\cos w + \frac{w}{\pi} + \frac{1}{2} = g(w)$$

maps to the region in Figure 7.81.
$\{u = -\pi/2, v \geq 0\}$ maps to $\{x = 0, y \leq 0\}$, where $T = -10$
$\{-\pi/2 \leq u \leq \pi/2, v = 0\}$ maps to $\{0 \leq x \leq 1, y = 0\}$, where the boundary is insulated.
$\{u = \pi/2, v \geq 0\}$ maps to $\{x = 1, y \geq 0\}$, where $T = 10$
$\psi(w) = \dfrac{20}{\pi}\,\mathrm{Re}\,(w)$ satisfies these boundary conditions. Isotherms in the z plane can be parametrized by $z(v) = g(u_0 + iv)$ where u_0 is constant, $-\pi/2 < u_0 < \pi/2$ and $v > 0$.

2. On the obstacle, $z = re^{i\theta}$ with $r \leq 1$ and $\theta = \alpha$ or $\alpha + \pi$. Substituting into ψ.

$$\psi\left(re^{i\alpha}\right) = \operatorname{Im}\left[r\left(\cos\alpha + i\sin\alpha\sqrt{1-\frac{1}{r^2}}\right)\right]$$

$$= \operatorname{Im}\left[r\cos\alpha - r\sin\alpha\sqrt{\frac{1}{r^2}-1}\right] = 0$$

$$\psi\left(re^{i(\alpha+\pi)}\right) = \operatorname{Im}\left[re^{i\pi}\left(\cos\alpha + i\sin\alpha\sqrt{1-\frac{1}{r^2}}\right)\right]$$

$$= \operatorname{Im}\left[-r\cos\alpha + r\sin\alpha\sqrt{\frac{1}{r^2}-1}\right] = 0$$

$\psi \equiv 0$ on the obstacle. For z large, the streamlines $\psi(x,y) = c$ approach $\operatorname{Im}\left[e^{-i\alpha}z\left(\cos\alpha + i\sin\alpha\right)\right] = c \implies \operatorname{Im} z = c$, a horizontal line.

3. $w = \sin^{-1}(z^2)$ maps this region to the half-strip
$-\pi/2 \le u \le \pi/2,\ v \ge 0$ and
$\{x = 0, y \ge 1\}$ maps to $\{u = -\pi/2,\ v \ge 0\}$, where $T = -1$
$\{x = 0,\ 0 \le y \le 1\}$ maps to $\{-1 \le u \le 0,\ v = 0\}$, where the boundary is insulated.
$\{0 \le x \le 1,\ y = 0\}$ maps to $\{0 \le u \le 1,\ v = 0\}$, where the boundary is insulated.
$\{x \ge 1,\ y = 0\}$ maps to $\{u = \pi/2,\ v \ge 0\}$, where $T = 1$.

$\psi(w) = \dfrac{2}{\pi}\operatorname{Re} w$ satisfies these boundary conditions.

$$T(z) = \frac{2}{\pi}\operatorname{Re}\left[\sin^{-1}(z^2)\right]$$

4. $w_1 = \dfrac{z+R}{z-R}$ maps the region to the fourth quadrant.
$w = (iw_1)^2 = -\left(\dfrac{z+R}{z-R}\right)^2$ maps to the upper half of the w-plane. The boundary in Figure 7.84 maps to the real line, where the stream function is zero.

$$\psi(z) = \operatorname{Im} w = \operatorname{Im}\left[-\left(\frac{z+R}{z-R}\right)^2\right]$$

$$= \operatorname{Im}\left[-\left(\frac{x+R+iy}{x-R+iy}\right)^2\right] = \operatorname{Im}\left[-\frac{[x^2-R^2+y^2-2yRi]^2}{[(x-R)^2+y^2]^2}\right]$$

$$= \frac{4yR(x^2+y^2-R^2)}{[(x-R)^2+y^2]^2}$$

5. $w_1 = e^{\pi z}$ maps this strip to the upper half-plane.
   $w_2 = 1/w_1$ preserves the upper half-plane, but rearranges the boundary.
   $w = \sin^{-1} w_2$ maps to the half-strip $-\pi/2 \le u \le \pi/2$, $v \ge 0$.
   $\{x \le 0,\ y = 1\}$ maps to $\{u = -\pi/2,\ v \ge 0\}$, where $T = -10$
   $\{x \ge 0,\ y = 1\}$ maps to $\{-\pi/2 \le u \le 0,\ v = 0\}$, where the boundary is insulated.
   $\{x \le 0,\ y = 0\}$ maps to $\{u = \pi/2,\ v \ge 0\}$, where $T = 10$
   $\{x \ge 0,\ y = 0\}$ maps to $\{0 \le u \le \pi/2,\ v = 0\}$, where the boundary is insulated.

$$T(z) = \frac{20}{\pi}\operatorname{Re} w = \frac{20}{\pi}\operatorname{Re}\left[\sin^{-1}(e^{-\pi z})\right]$$

6. $w = f(z) = z + R^2/z$
   $f(Re^{i\theta}) = R(e^{i\theta} + e^{-i\theta}) = 2R\cos\theta$,
   which traces the line segment from $w = -2R$ to $w = 2R$.
   $f(R'e^{i\theta}) = R'e^{i\theta} + \frac{R^2}{R'}e^{-i\theta} = 2\left(R' + \frac{R^2}{R'}\right)\cos\theta + 2i\left(R' - \frac{R^2}{R'}\right)\sin\theta$,
   which is an ellipse surrounding the line segment. The equipotentials are ellipses $u = 2(r_0 + R^2/r_0)\cos\theta$, $v = 2(r_0 - R^2/r_0)\sin\theta$ with $R < r_0 < R'$.

7. $w_1 = z^{\pi/\alpha}$ maps the region in Figure 7.87 to that of Figure 7.84, with $R = 1$. $w = w_1 + 1/w_1$ maps to the upper half-plane.
   The streamlines are
   $\operatorname{Im}(w) = c$ constant
   $\operatorname{Im}\left(z^{\pi/\alpha} + z^{-\pi/\alpha}\right) = $ constant

8. a. If $z_1 = w_a + \frac{1}{w_a} = w_b + \frac{1}{w_b}$ with $w_a \ne w_b$ then $w_a - w_b = \frac{1}{w_a} - \frac{1}{w_b} = \frac{w_a - w_b}{w_a w_b}$, so $w_a w_b = 1$. Furthermore if $|w_a| = 1$ then $w_a = e^{i\theta}, w_b = e^{-i\theta}$, and $z = 2\cos\theta$. Thus every z not in [-2,2] corresponds to exactly one w with $|w|>1$.
   b. $z_1^2 - 2 = (w_1^2 + \frac{1}{w_1^2}) - 2 = w_1^2 + \frac{1}{w_1^2} = w_2 + \frac{1}{w_2} = z_2$.
   c. z_1 ($\notin$[-2,2]), $z_2, z_3, \ldots$ corresponds to w_1 (>1), $w_1^2, w_1^4, \ldots$; $w_1^n \to \infty$ so $z_n \to \infty$.

CHAPTER 8: The Transforms of Applied Mathematics

EXERCISES 8.1: Fourier Series (The Finite Fourier Transform)

1. a. $\left(\dfrac{e^{it}-e^{-it}}{2i}\right)^3 = \dfrac{i}{8}\left(-e^{-3it}+3e^{-it}-3e^{it}+e^{3it}\right)$

   b. $F(t)$ has a period of 3π

$$c_n = \frac{1}{3\pi}\int_{-3\pi/2}^{3\pi/2}\left|\cos^3\left(\frac{t}{3}\right)\right| e^{-i2nt/3}dt$$

$$= \frac{1}{3\pi}\int_{-3\pi/2}^{3\pi/2}\left(\frac{3}{4}\cos\left(\frac{t}{3}\right)+\frac{1}{4}\cos t\right) e^{-i2nt/3}dt$$

$$= \frac{1}{3\pi}\left[\frac{3e^{-i2nt/3}}{4\left(-4n^2/9+1/9\right)}\left(\frac{-i2n}{3}\cos\left(\frac{t}{3}\right)+\frac{1}{3}\sin\left(\frac{t}{3}\right)\right)\right.$$

$$\left.+\frac{e^{-i2nt/3}}{4\left(-4n^2/9+1\right)}\left(\frac{-i2n}{3}\cos t+\sin t\right)\right]\Bigg|_{\frac{-3\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \frac{1}{3\pi}\left[\frac{9(-1)^n}{2(-4n^2+1)}-\frac{9(-1)^n}{2(-4n^2+9)}\right]$$

$$= \frac{12(-1)^n}{\pi(1-4n^2)(9-4n^2)}$$

The Fourier series is

$$\sum_{n=-\infty}^{\infty}\frac{12(-1)^n}{\pi(1-4n^2)(9-4n^2)}e^{i2nt/3}$$

   c. $c_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} t^2 dt = \dfrac{\pi^2}{3}$

For $n \neq 0$,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-int}\,dt$$

$$= \frac{1}{2\pi}\left[\frac{e^{-int}}{in^3}\left(-n^2t^2 + 2int + 2\right)\right]\Bigg|_{-\pi}^{\pi} = \frac{2(-1)^n}{n^2}$$

d. $c_0 = 0$

For $n \neq 0$,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} t|t|e^{-int}dt = -\frac{1}{2\pi}\int_{-\pi}^{0} t^2e^{-int}dt + \frac{1}{2\pi}\int_{0}^{\pi} t^2e^{-int}dt$$

$$= -\frac{1}{2\pi}\left[\frac{e^{-int}}{in^3}\left(-n^2t^2 + 2int + 2\right)\right]\Bigg|_{-\pi}^{0}$$

$$+\frac{1}{2\pi}\left[\frac{e^{-int}}{in^3}\left(-n^2t^2 + 2int + 2\right)\right]\Bigg|_{0}^{\pi}$$

$$= \frac{-1}{\pi in^3} + \frac{(-1)^n}{2\pi in^3}\left(-n^2\pi^2 - 2in\pi + 2\right)$$

$$+\frac{(-1)^n}{2\pi in^3}(-n^2\pi^2 + 2in\pi + 2) - \frac{1}{\pi in^3}$$

$$= (-1)^n\left[\frac{i\pi}{n} - \frac{2i}{\pi n^3}\right] + \frac{2i}{\pi n^3}$$

2. a. $c_{\pm 1} = -\frac{1}{2\pi}\int_{-\pi}^{0}\left(\sin t\cos t \mp i\sin^2 t\right)dt$

$$+\frac{1}{2\pi}\int_{0}^{\pi}\left(\sin t\cos t \mp i\sin^2 t\right)dt$$

$$= -\frac{1}{2\pi}\left[\frac{\sin^2 t}{2} \mp i\left(\frac{t}{2} - \frac{1}{4}\sin 2t\right)\right]\Bigg|_{-\pi}^{0}$$

$$+\frac{1}{2\pi}\left[\frac{\sin^2 t}{2} \mp i\left(\frac{t}{2} - \frac{1}{4}\sin 2t\right)\right]\Bigg|_{0}^{\pi} = 0$$

For $n \neq \pm 1$,

$$c_n = -\frac{1}{2\pi}\int_{-\pi}^{0} \sin t e^{-int} dt + \frac{1}{2\pi}\int_{0}^{\pi} \sin t e^{-int} dt$$

$$= -\frac{1}{2\pi}\left[\frac{e^{-int}}{-n^2+1}(-in\sin t - \cos t)\right]\Bigg|_{-\pi}^{0}$$

$$+\frac{1}{2\pi}\left[\frac{e^{-int}}{-n^2+1}(-in\sin t - \cos t)\right]\Bigg|_{0}^{\pi}$$

$$= \frac{1}{2\pi(1-n^2)} + \frac{(-1)^n}{2\pi(1-n^2)} + \frac{(-1)^n}{2\pi(1-n^2)} + \frac{1}{2\pi(1-n^2)}$$

$$= \frac{1+(-1)^n}{\pi(1-n^2)}$$

$$= \begin{cases} \dfrac{-2}{\pi(n^2-1)} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

The series converges uniformly to $|\sin t|$ on $[-\pi, \pi]$.

b. $c_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{t-int} dt = \dfrac{1}{2\pi}\dfrac{e^{t-int}}{1-in}\Bigg|_{-\pi}^{\pi}$

$$= \frac{(-1)^n e^{\pi} - (-1)^n e^{-\pi}}{2\pi(1-in)} = \frac{(-1)^n \sinh\pi}{\pi(1-in)}$$

This series converges pointwise to e^t on $(-\pi, \pi)$ and converges to $\frac{1}{2}\left(e^{\pi} + e^{-\pi}\right)$ at $t = -\pi$ or π.

c. $c_{\pm 1} = \dfrac{1}{2\pi}\displaystyle\int_{0}^{\pi}\left(\sin t\cos t \mp i\sin^2 t\right) dt$

$$= \frac{1}{2\pi}\left[\frac{\sin^2 t}{2} \mp i\left(\frac{t}{2} - \frac{1}{4}\sin 2t\right)\right]\Bigg|_{0}^{\pi} = \mp\frac{i}{4}$$

For $n \neq \pm 1$,

$$c_n = \frac{1}{2\pi}\int_0^\pi \sin t e^{-int}dt$$

$$= \frac{1}{2\pi}\left[\frac{e^{-int}}{-n^2+1}(-in\sin t - \cos t)\right]\Big|_0^\pi$$

$$= \frac{(-1)^n + 1}{2\pi(1-n^2)}$$

The Fourier series is

$$\left[\sum_{n=-\infty}^{-2} \frac{(-1)^n+1}{2\pi(1-n^2)}e^{int}\right] + \frac{i}{4}e^{-it} + \frac{1}{\pi} - \frac{i}{4}e^{it} + \left[\sum_{n=2}^{\infty}\frac{(-1)^n+1}{2\pi(1-n^2)}e^{int}\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin t + \sum_{n=2}^{\infty}\frac{(-1)^n+1}{2\pi(1-n^2)}\left(e^{int}+e^{-int}\right)$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin t - \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{2\cos nt}{\pi(n^2-1)}$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kt}{4k^2-1}$$

This series converges uniformly on $[-\pi, \pi]$.

d. $c_0 = \dfrac{1}{2\pi}\displaystyle\int_0^\pi t\,dt = \frac{\pi}{4}$

For $n \neq 0$,

$$c_n = \frac{1}{2\pi}\int_0^\pi te^{-int}dt$$

$$= \frac{1}{2\pi}\frac{e^{-int}}{-n^2}(-int-1)\Big|_0^\pi = \frac{(-1)^n}{2\pi n^2}(in\pi+1) - \frac{1}{2\pi n^2}$$

The Fourier series is

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n-1}{2\pi n^2}\left(e^{int}+e^{-int}\right) + i\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\left(e^{int}-e^{-int}\right)$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} \cos nt - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nt$$

This series converges pointwise on $(-\pi, \pi)$ and converges to $\pi/2$ at $t = -\pi$ or π.

3. Theorem 3 applies to none of the series in Problem 2. It *does* apply to (a) and (b) in Problem 1, however.

4. $1 + 2\sum_{n=1}^{\infty} r^n \cos n(\theta - \phi)$

$$= \operatorname{Re}\left[1 + 2\sum_{n=1}^{\infty} r^n e^{in(\theta-\phi)}\right]$$

$$= \operatorname{Re}\left[1 + \frac{2re^{i(\theta-\phi)}}{1 - re^{i(\theta-\phi)}}\right] \text{ (geometric series)}$$

$$= \operatorname{Re}\left[1 + \frac{2re^{i(\theta-\phi)} - 2r^2}{[1 - re^{i(\theta-\phi)}]\,[1 - re^{-i(\theta-\phi)}]}\right]$$

$$= \operatorname{Re}\left[\frac{1 - 2r\cos(\theta-\phi) + r^2 + 2r\cos(\theta-\phi) + i2r\sin(\theta-\phi) - 2r^2}{1 - 2r\cos(\theta-\phi) + r^2}\right]$$

$$= \frac{1 - r^2}{1 - 2r\cos(\theta-\phi) + r^2}$$

5. $c_0 + \sum_{n=1}^{\infty} c_n e^{int} + \sum_{n=1}^{\infty} c_{-n} e^{-int}$

$$= c_0 + \sum_{n=1}^{\infty} c_n(\cos nt + i\sin nt) + \sum_{n=1}^{\infty} c_{-n}(\cos nt - i\sin nt)$$

$$= c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n})\cos nt + \sum_{n=1}^{\infty} i(c_n - c_{-n})\sin nt$$

The series is a real function if c_0 is real and $c_n = \overline{c_{-n}}$ for $n \geq 1$.

6. a. $c_n = \frac{1}{2\pi}\int_{-\pi}^{0} -F(-t)e^{-int}dt + \frac{1}{2\pi}\int_0^{\pi} F(t)e^{-int}dt$

$$= \frac{1}{2\pi}\int_0^{\pi} F(t)\left(-e^{int} + e^{-int}\right) dt$$

$$= \frac{-i}{\pi}\int_0^{\pi} F(t)\sin nt dt$$

The Fourier series is

$$\sum_{n=-\infty}^{\infty}\left[\frac{-i}{\pi}\int_0^{\pi} F(t)\sin nt dt\right] e^{int}$$

$$= \sum_{n=1}^{\infty}\left[\frac{-i}{\pi}\int_0^{\pi} F(t)\sin nt dt\right]\left(e^{int} - e^{-int}\right)$$

$$= \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} F(t)\sin nt dt\right]\sin(nt)$$

This series converges for $0 \le t \le \pi$ if $F(t)$ is piecewise continuous, and it converges to $F(t)$ if $F(\tau_j) = \frac{1}{2}[F(\tau_j+) + F(\tau_j-)]$ at the subdivision points.

b. $c_n = \frac{1}{2\pi}\int_{-\pi}^{0} F(-t)e^{-int}dt + \frac{1}{2\pi}\int_0^{\pi} F(t)e^{-int}dt$

$$= \frac{1}{2\pi}\int_0^{\pi} F(t)\left(e^{int} + e^{-int}\right) dt$$

$$= \frac{1}{\pi}\int_0^{\pi} F(t)\cos nt dt$$

The Fourier series is

$$\sum_{n=-\infty}^{\infty}\left[\frac{1}{\pi}\int_0^{\pi} F(t)\cos nt dt\right] e^{int}$$

$$= \frac{1}{\pi}\int_0^{\pi} F(t)dt + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\int_0^{\pi} F(t)\cos nt dt\right]\left(e^{int} + e^{-int}\right)$$

$$= \frac{1}{\pi}\int_0^{\pi} F(t)dt + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} F(t)\cos ntdt\right]\cos nt$$

The convergence properties are as in part a.

7. a. $$\sum_{n=-\infty}^{\infty}\left(-n^2c_n + 3c_n\right)e^{int} = \frac{1}{16}\left(e^{-i4t} - 4e^{-i2t} + 6 - 4e^{i2t} + e^{i4t}\right)$$

$$c_{\pm 4} = \frac{-1}{208},\quad c_{\pm 2} = \frac{1}{4},\quad c_0 = \frac{1}{8}$$

$$f(t) = \frac{-1}{208}e^{-i4t} + \frac{1}{4}e^{-i2t} + \frac{1}{8} + \frac{1}{4}e^{i2t} - \frac{1}{208}e^{i4t}$$

$$= \frac{1}{8} + \frac{1}{2}\cos 2t - \frac{1}{104}\cos 4t$$

b. $$\sum_{n=-\infty}^{\infty}(-n^2c_n + inc_n + c_n)e^{int} = \frac{\pi^2}{3} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{2(-1)^n}{n^2}e^{int}$$

(from Problem 1c)

$$c_0 = \frac{\pi^2}{3}$$

For $n \neq 0$, $c_n = \dfrac{2(-1)^n}{(-n^2 + in + 1)n^2}$

c. $$\sum_{n=-\infty}^{\infty}(-n^2c_n + 4inc_n + 2c_n)e^{int} = \frac{i}{\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n - 1}{n}e^{int}$$

(from Example 4)

$c_0 = 0$

For $n \neq 0$, $c_n = \dfrac{i\left[(-1)^n - 1\right]}{(-n^2 + 4in + 2)\pi n}$

8. Fourier sine series

$$\beta_n = \frac{2}{\pi}\int_0^{\pi} t\sin ntdt = \frac{2}{\pi}\left(\frac{-t}{n}\cos nt + \frac{1}{n^2}\sin nt\right)\Bigg|_0^{\pi} = \frac{2(-1)^{n+1}}{n}$$

Fourier cosine series

$$\alpha_0 = \frac{1}{\pi}\int_0^{\pi} tdt = \frac{\pi}{2}$$

$$\alpha_n = \frac{2}{\pi}\int_0^\pi t\cos nt\,dt = \frac{2}{\pi}\left(\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right)\Big|_0^\pi$$

$$= \frac{2\left[(-1)^n - 1\right]}{\pi n^2} = \begin{cases} \dfrac{-4}{\pi n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

9. $u(z) = \mathrm{Re}\left(\sum_{n=0}^{\infty} \alpha_n z^n\right) + \mathrm{Re}\left(-i\sum_{n=1}^{\infty} \beta_n z^n\right)$
$u(z)$ is harmonic because it is the real part of an analytic function. Since the boundary values of u match $U(\theta)$, it follows that u is the unique solution to this Dirichlet problem.

10. $\dfrac{\partial T(x,t)}{\partial t} = \sum_{n=1}^{\infty} a_n \sin nx e^{-n^2 t}(-n^2) = \dfrac{\partial^2 T(x,t)}{\partial x^2}$
$T(0,t) = T(\pi,t) = \sum_{n=1}^{\infty} a_n 0 e^{-n^2 t} = 0$
$T(x,0) = \sum_{n=1}^{\infty} a_n \sin nx e^0 = f(x)$ (Fourier sine series)
The limiting value of $T(x,t)$ as $t \to \infty$ can be interpreted as the steady-state temperature distribution along this rod.
$\lim_{t\to\infty} T(x,t) = 0$

11. $\dfrac{\partial^2 u(x,t)}{\partial x^2} = \sum_{n=1}^{\infty} b_n(-n^2)\sin nx \cos nt = \dfrac{\partial^2 u(x,t)}{\partial t^2}$
$u(0,t) = u(\pi,t) = \sum_{n=1}^{\infty} b_n 0 \cos nt = 0$
$u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx \cos 0 = f(x)$ (Fourier sine series)
$\dfrac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} -nb_n \sin nx \sin 0 = 0$
With the modified initial conditions use
$u(x,t) = \sum_{n=1}^{\infty} \dfrac{b_n}{n} \sin nx \sin nt$
For the more general set of initial conditions construct the solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \cos nt + \sum_{n=1}^{\infty} c_n \sin nx \sin nt$$

where $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f_1(\xi) \sin n\xi \, d\xi$

and $c_n = \dfrac{2}{n\pi}\displaystyle\int_0^{\pi} f_2(\xi) \sin n\xi \, d\xi$

12. a. $e^{-i2\pi n(j+N_1)/N} = e^{-i2\pi nj/N - i2\pi n/2} = (-1)^n e^{-i2\pi nj/N}$

c. Case 1: n even.
Since $n = 2n_1$, and $N = 2N_1$, $n/N = n_1/N_1$
and the two summations are equal.

Case 2: n odd.
Since $n = 2n_1 + 1$ and $N = 2N_1$,

$$\frac{n}{N} = \frac{2n_1}{2N_1} + \frac{1}{N}.$$

The result follows by using
$C_j = B_j e^{-i2\pi j/N}$

EXERCISES 8.2: The Fourier Transform

1. a. $G(\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{0} e^{t-i\omega t} dt + \dfrac{1}{2\pi}\int_0^{\infty} e^{-t-i\omega t} dt$

$$= \frac{1}{2\pi}\left(\frac{e^{t-i\omega t}}{1-i\omega}\right)\Bigg|_{-\infty}^{0} + \frac{1}{2\pi}\left(\frac{e^{-t-i\omega t}}{-1-i\omega}\right)\Bigg|_{0}^{\infty}$$

$$= \frac{1}{2\pi}\left(\frac{1}{1-i\omega} + \frac{1}{1+i\omega}\right) = \frac{1}{\pi}\left(\frac{1}{1+\omega^2}\right)$$

The inversion formula is

$$F(t) = e^{-|t|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{1+\omega^2} d\omega$$

b. $G(\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-t^2-i\omega t} dt$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(t+i\omega/2)^2} e^{-\omega^2/4} dt = \frac{1}{2\sqrt{\pi}} e^{-\omega^2/4}$$

The inversion formula is

$$F(t) = e^{-t^2} = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\omega^2/4+i\omega t} d\omega$$

c. $G(\omega)= \frac{1}{2\pi}\int_{-\infty}^{\infty} te^{-t^2-i\omega t} dt$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (t + i\omega/2 - i\omega/2) e^{-(t+i\omega/2)^2} e^{-\omega^2/4} dt$$

$$= \frac{-e^{-\omega^2/4}}{4\pi} e^{-(t+i\omega/2)^2}\Big|_{t=-\infty}^{\infty} - \frac{i\omega e^{-\omega^2/4}}{4\pi}\int_{-\infty}^{\infty} e^{-(t+i\omega/2)^2} dt$$

$$= \frac{-i\omega e^{-\omega^2/4}}{4\sqrt{\pi}}$$

The inversion formula is

$$F(t) = te^{-t^2} = \frac{-i}{4\sqrt{\pi}}\int_{-\infty}^{\infty} \omega e^{-\omega^2/4+i\omega t} d\omega$$

d. $G(\omega)= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin t}{t} e^{-i\omega t} dt$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin W\pi}{W\pi} e^{-i\omega W\pi}(\pi) dW$$

(change of variables $t = W\pi$)

$$= \begin{cases} \frac{1}{2} & \text{for } |-\omega\pi| < \pi, \quad \text{or } |\omega| < 1 \\ 0 & \text{for } |-\omega\pi| > \pi, \quad \text{or } |\omega| > 1 \\ \frac{1}{4} & \text{for } -\omega\pi = \pm\pi, \quad \text{or } \omega = \pm 1 \end{cases}$$

The inversion formula is

$$F(t) = \frac{\sin t}{t} = \frac{1}{2}\int_{-1}^{1} e^{i\omega t} d\omega$$

e. $$G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin \pi t}{1-t^2} e^{-i\omega t} dt = I_1 + I_2$$

where $I_1 = \text{ p.v. } \dfrac{-1}{4\pi i}\displaystyle\int_{-\infty}^{\infty} \frac{e^{it(\pi-\omega)}}{(t-1)(t+1)} dt$

and $I_2 = \text{ p.v. } \dfrac{1}{4\pi i}\displaystyle\int_{-\infty}^{\infty} \frac{e^{-it(\pi+\omega)}}{(t-1)(t+1)} dt$

For $\omega \leq \pi$, use a contour in the upper half-plane to get

$$I_1 = \frac{-1}{4\pi i}(\pi i)\left[\text{Res }(t=-1) + \text{Res}(t=1)\right]$$

$$= \frac{-1}{4}\left[\frac{-e^{i(\omega-\pi)}}{2} + \frac{e^{i(\pi-\omega)}}{2}\right] = \frac{-i}{4}\sin(\pi-\omega) = \frac{-i}{4}\sin\omega$$

For $\omega \geq \pi$, use a contour in the lower half-plane to get

$$I_1 = \frac{-1}{4\pi i}(-\pi i)\left[\text{Res }(t=-1) + \text{Res }(t=1)\right] = \frac{i}{4}\sin\omega$$

For $\omega \leq -\pi$, use a contour in the upper half-plane to get

$$I_2 = \frac{1}{4\pi i}(\pi i)\left[\text{Res }(t=-1) + \text{Res }(t=1)\right]$$

$$= \frac{1}{4}\left[\frac{-e^{i(\pi+\omega)}}{2} + \frac{e^{-i(\pi+\omega)}}{2}\right] = \frac{-i}{4}\sin(\pi+\omega) = \frac{i}{4}\sin\omega$$

For $\omega \geq -\pi$, use a contour in the lower half-plane to get

$$I_2 = \frac{1}{4\pi i}(-\pi i)\left[\text{Res }(t=-1) + \text{Res }(t=1)\right] = \frac{-i}{4}\sin\omega$$

Summarizing,

$$G(\omega) = I_1 + I_2 = \begin{cases} \dfrac{-i}{2}\sin\omega & \text{for } |\omega| \leq \pi \\ 0 & \text{for } |\omega| \geq \pi \end{cases}$$

The inversion formula is

$$F(t) = \frac{\sin \pi t}{1 - t^2} = \frac{-i}{2} \int_{-\pi}^{\pi} \sin \omega e^{i\omega t} d\omega$$

2. $$f(t) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{e^{i6\pi\omega} - e^{-i6\pi\omega}}{2i(1-\omega^2)} \left(\frac{e^{i\omega t}}{-\omega^2 + 2i\omega + 2} \right) d\omega = I_1 + I_2$$

where $I_1 = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \frac{e^{i(6\pi\omega + \omega t)}}{(\omega^2 - 1)(\omega^2 - 2i\omega - 2)} d\omega$

and $I_2 = \dfrac{-1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \frac{e^{i(-6\pi\omega + \omega t)}}{(\omega^2 - 1)(\omega^2 - 2i\omega - 2)} d\omega$

For $t \leq -6\pi$, use a contour in the lower half-plane to get

$$I_1 = \frac{1}{2\pi} [-\pi i \text{Res}(\omega = -1) - \pi i \text{Res}(\omega = 1)]$$

$$= \frac{-i}{2} \left[\frac{e^{-i(6\pi + t)}}{(-2)(-1 + 2i)} + \frac{e^{i(6\pi + t)}}{2(-1 - 2i)} \right]$$

$$= \frac{-1}{10} \sin t + \frac{1}{5} \cos t$$

For $t \geq -6\pi$, use a contour in the upper half-plane to get

$$I_1 = \frac{1}{2\pi} [\pi i \text{Res}(\omega = -1) + \pi i \text{Res}(\omega = 1)$$

$$+ 2\pi i \text{Res}(\omega = 1 + i) + 2\pi i \text{Res}(\omega = -1 + i)]$$

$$= \frac{1}{10} \sin t - \frac{1}{5} \cos t + i \left(\frac{(2-i)e^{it} e^{-6\pi - t}}{10i} \right) + i \left(\frac{(2+i)e^{-it} e^{-6\pi - t}}{10i} \right)$$

$$= \frac{1}{10} \sin t - \frac{1}{5} \cos t + e^{-6\pi - t} \left(\frac{2}{5} \cos t + \frac{1}{5} \sin t \right)$$

For $t \leq 6\pi$, use a contour in the lower half-plane to get

$$I_2 = \frac{-1}{2\pi} [-\pi i \text{Res}(\omega = -1) - \pi i \text{Res}(\omega = 1)]$$

$$= \frac{i}{2}\left[\frac{e^{i(6\pi - t)}}{(-2)(-1+2i)} + \frac{e^{i(-6\pi+t)}}{2(-1-2i)}\right]$$

$$= \frac{-1}{5}\cos t + \frac{1}{10}\sin t$$

For $t \geq 6\pi$, use a contour in the upper half-plane to get

$$I_2 = \frac{-1}{2\pi}\left[\pi i \operatorname{Res}(\omega = -1) + \pi i \operatorname{Res}(\omega = 1)\right.$$

$$\left. + 2\pi i \operatorname{Res}(\omega = 1+i) + 2\pi i \operatorname{Res}(\omega = -1+i)\right]$$

$$= \frac{1}{5}\cos t - \frac{1}{10}\sin t - i\left(\frac{(2-i)e^{it}e^{6\pi - t}}{10i}\right) - i\left(\frac{(2+i)e^{-it}e^{6\pi - t}}{10i}\right)$$

$$= \frac{1}{5}\cos t - \frac{1}{10}\sin t + e^{6\pi - t}\left(\frac{-2}{5}\cos t - \frac{1}{5}\sin t\right)$$

Summarizing, $G(\omega) = I_1 + I_2$

$$= \begin{cases} 0 & \text{for } t \leq -6\pi \\ \frac{2}{5}\left(e^{-6\pi - t} - 1\right)\cos t + \frac{1}{5}\left(e^{-6\pi - t} + 1\right)\sin t & \text{for } -6\pi \leq t \leq 6\pi \\ -\left(e^{6\pi} - e^{-6\pi}\right)e^{-t}\left(\frac{2}{5}\cos t + \frac{1}{5}\sin t\right) & \text{for } 6\pi \leq t \end{cases}$$

3. a. $G(\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{t^4 + 1} dt$

For $\omega \geq 0$,

$$G(\omega) = \frac{1}{2\pi}\left[-2\pi i \operatorname{Res}\left(t = e^{5\pi i/4}\right) - 2\pi i \operatorname{Res}\left(t = e^{7\pi i/4}\right)\right]$$

$$= -i\left[\frac{e^{i\omega\left(1/\sqrt{2} + i/\sqrt{2}\right)}}{2\sqrt{2} - 2\sqrt{2}i} + \frac{e^{i\omega\left(-1/\sqrt{2} + i/\sqrt{2}\right)}}{-2\sqrt{2} - 2\sqrt{2}i}\right]$$

$$= \frac{e^{-\omega/\sqrt{2}}}{2\sqrt{2}} \left(\cos\left(\frac{\omega}{\sqrt{2}}\right) + \sin\left(\frac{\omega}{\sqrt{2}}\right) \right)$$

For $\omega \leq 0$,

$$G(\omega) = \frac{1}{2\pi} \left[2\pi i \operatorname{Res}\left(t = e^{\pi i/4}\right) + 2\pi i \operatorname{Res}\left(t = e^{3\pi i/4}\right) \right]$$

$$= i \left[\frac{e^{i\omega(-1/\sqrt{2}-i/\sqrt{2})}}{-2\sqrt{2}+2\sqrt{2}i} + \frac{e^{i\omega(1/\sqrt{2}-i/\sqrt{2})}}{2\sqrt{2}+2\sqrt{2}i} \right]$$

$$= \frac{e^{\omega/\sqrt{2}}}{2\sqrt{2}} \left(\cos\left(\frac{\omega}{\sqrt{2}}\right) - \sin\left(\frac{\omega}{\sqrt{2}}\right) \right)$$

Combining these cases,

$$G(\omega) = \frac{e^{-|\omega|/\sqrt{2}}}{2\sqrt{2}} \left(\cos\left(\frac{|\omega|}{\sqrt{2}}\right) + \sin\left(\frac{|\omega|}{\sqrt{2}}\right) \right)$$

The inversion formula is

$$F(t) = \frac{1}{t^4+1} = \frac{1}{2\sqrt{2}} \int_{-\infty}^{\infty} e^{-|\omega|/\sqrt{2}+i\omega t} \left(\cos\left(\frac{|\omega|}{\sqrt{2}}\right) + \sin\left(\frac{|\omega|}{\sqrt{2}}\right) \right) d\omega$$

To confirm this, the integral is

$$\frac{1}{2\sqrt{2}} \int_0^{\infty} e^{-\omega/\sqrt{2}} \left(e^{i\omega t} + e^{-i\omega t} \right) \left(\cos\left(\frac{\omega}{\sqrt{2}}\right) + \sin\left(\frac{\omega}{\sqrt{2}}\right) \right) d\omega$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\infty} \left(\frac{1-i}{2}\right) \left(e^{-\omega/\sqrt{2}+i\omega t+i\omega/\sqrt{2}} + e^{-\omega/\sqrt{2}-i\omega t+i\omega/\sqrt{2}} \right)$$

$$+ \left(\frac{1+i}{2}\right) \left(e^{-\omega/\sqrt{2}+i\omega t-i\omega/\sqrt{2}} + e^{-\omega/\sqrt{2}-i\omega t-i\omega/\sqrt{2}} \right) d\omega$$

$$= \frac{1-i}{4\sqrt{2}} \left(\frac{-1}{-1/\sqrt{2}+it+i/\sqrt{2}} + \frac{-1}{-1/\sqrt{2}-it+i/\sqrt{2}} \right)$$

$$+ \frac{1+i}{4\sqrt{2}} \left(\frac{-1}{-1/\sqrt{2}+it-i/\sqrt{2}} + \frac{-1}{-1/\sqrt{2}-it-i/\sqrt{2}} \right)$$

$$= \frac{1}{t^4+1}$$

b. $G(\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{te^{-i\omega t}}{t^4+1}dt$

For $\omega \geq 0$,

$$G(\omega) = \frac{1}{2\pi}\left[-2\pi i \operatorname{Res}\left(t = e^{5\pi i/4}\right) - 2\pi i \operatorname{Res}\left(t = e^{7\pi i/4}\right)\right]$$

$$= -i\left[\frac{e^{i\omega\left(1/\sqrt{2}+i/\sqrt{2}\right)}}{4i} + \frac{e^{i\omega\left(-1/\sqrt{2}+i/\sqrt{2}\right)}}{-4i}\right]$$

$$= \frac{-i}{2}e^{-\omega/\sqrt{2}}\sin\left(\frac{\omega}{\sqrt{2}}\right)$$

For $\omega \leq 0$,

$$G(\omega)= \frac{1}{2\pi}\left[2\pi i \operatorname{Res}\left(t = e^{\pi i/4}\right) + 2\pi i \operatorname{Res}\left(t = e^{3\pi i/4}\right)\right]$$

$$= i\left[\frac{e^{i\omega\left(-1/\sqrt{2}-i/\sqrt{2}\right)}}{4i} + \frac{e^{i\omega\left(1/\sqrt{2}-i/\sqrt{2}\right)}}{-4i}\right]$$

$$= \frac{-i}{2}e^{\omega/\sqrt{2}}\sin\left(\frac{\omega}{\sqrt{2}}\right)$$

Combining these cases,

$$G(\omega) = \frac{-i}{2}e^{-|\omega|/\sqrt{2}}\sin\left(\frac{\omega}{\sqrt{2}}\right)$$

The inversion formula is

$$F(t) = \frac{t}{t^4+1} = \frac{-i}{2}\int_{-\infty}^{\infty} e^{-|\omega|/\sqrt{2}+i\omega t}\sin\left(\frac{\omega}{\sqrt{2}}\right)d\omega$$

To confirm this, the integral is

$$\frac{-i}{2}\int_0^{\infty} e^{-\omega/\sqrt{2}}\left(e^{i\omega t} - e^{-i\omega t}\right)\left(\frac{e^{i\omega/\sqrt{2}} - e^{-i\omega/\sqrt{2}}}{2i}\right)d\omega$$

$$= \frac{-1}{4}\int_0^{\infty}\left(e^{-\omega/\sqrt{2}+i\omega t+i\omega/\sqrt{2}} - e^{-\omega/\sqrt{2}-i\omega t+i\omega/\sqrt{2}}\right.$$

$$\left. -e^{-\omega/\sqrt{2}+i\omega t-i\omega/\sqrt{2}} + e^{-\omega/\sqrt{2}-i\omega t-i\omega/\sqrt{2}}\right) d\omega$$

$$= \frac{-1}{4}\left(\frac{-1}{-1/\sqrt{2}+it+i/\sqrt{2}} + \frac{1}{-1/\sqrt{2}-it+i/\sqrt{2}}\right.$$

$$\left. +\frac{1}{-1/\sqrt{2}+it-i/\sqrt{2}} + \frac{-1}{-1/\sqrt{2}-it-i/\sqrt{2}}\right)$$

$$= \frac{t}{t^4+1}$$

c. **From Problem 1b, $G(\omega) = \frac{1}{2\sqrt{\pi}}e^{-\omega^2/4}$ and the inversion formula is**

$$F(t) = e^{-t^2} = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\omega^2/4+i\omega t}d\omega$$

To confirm this, the integral is

$$\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(\omega/2-it)^2}e^{-t^2}d\omega = e^{-t^2}$$

4. If $G(\omega)$ is an even function, $G(-\omega) = G(\omega)$. $F(t)$ is real. The real part of the Fourier transform is
$\mathrm{Re}[G(\omega)] = \mathrm{Re}[(1/2\pi)\int_{-\infty}^{\infty}F(t)e^{-i\omega t}d\omega]$
$= \mathrm{Re}[(1/2\pi)\int_{-\infty}^{\infty}F(t)(\cos\omega t - i\sin\omega t)d\omega] = (1/2\pi)\int_{-\infty}^{\infty}F(t)(\cos\omega t)d\omega$.
$\mathrm{Re}[G(\omega)]$ is an even function.

If $G(\omega)$ is an odd function, $G(-\omega)= -G(\omega)$. The imaginary part of the Fourier transform is
$\mathrm{Im}[G(\omega)] = \mathrm{Im}[(1/2\pi)\int_{-\infty}^{\infty}F(t)e^{-i\omega t}d\omega] = -i(1/2\pi)\int_{-\infty}^{\infty}F(t)(\sin\omega t)d\omega$.
$G(\omega)$ is an odd function.

5. Given $G(\omega)] = (1/2\pi)\int_{-\infty}^{\infty}F(t)e^{-i\omega t}d\omega$. The Fourier transform of $F(t)e^{i\Omega t}$ is
$(1/2\pi)\int_{-\infty}^{\infty}F(t)e^{i\Omega t}e^{-i\omega t}d\omega = (1/2\pi)\int_{-\infty}^{\infty}F(t)e^{-i(\omega-\Omega)t}d\omega = G(\omega-\Omega)$.

6. a. **A solution to $f'' + f' + f = e^{i\omega t}$ is given by $\frac{e^{i\omega t}}{-\omega^2+i\omega+1}$.**
Thus $f(t) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-\omega^2/4+i\omega t}}{-\omega^2+i\omega+1}d\omega$
(using the inversion formula from Problem 1b)

b. $G(\omega) = \dfrac{1}{2\pi}\int_0^\infty e^{-t-i\omega t}dt = \dfrac{1}{2\pi(1+i\omega)}$

The inversion formula is $F(t) = \dfrac{1}{2\pi}\int_{-\infty}^\infty \dfrac{e^{i\omega t}}{1+i\omega}d\omega$

A solution to the differential equation is given by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{i\omega t}}{(1+i\omega)(-\omega^2+4i\omega+1)}d\omega$$

c. $G(\omega) = \dfrac{1}{2\pi}\int_{-1}^1 e^{-i\omega t}dt = \dfrac{1}{\omega\pi}\sin\omega t$

The inversion formula is $F(t) = \dfrac{1}{\pi}\int_{-\infty}^\infty \dfrac{e^{i\omega t}\sin\omega t}{\omega}d\omega$

A solution to the differential equation is given by

$$f(t) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^{i\omega t}\sin\omega t}{\omega(-\omega^2+2i\omega+3)}d\omega$$

7. $\dfrac{\partial T(x,t)}{\partial t} = \int_{-\infty}^\infty G(\omega)e^{i\omega x}(-\omega^2)e^{-\omega^2 t}d\omega = \dfrac{\partial^2 T(x,t)}{\partial x^2}$

$T(x,0) = \int_{-\infty}^\infty G(\omega)e^{i\omega x}d\omega = f(x)$ (Fourier inversion formula)

$$T(x,t) = \int_{-\infty}^\infty \left[\frac{1}{2\pi}\int_{-\infty}^\infty f(\xi)e^{-i\omega\xi}d\xi\right]e^{i\omega x}e^{-\omega^2 t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(\xi)\left[\int_{-\infty}^\infty e^{-i\omega\xi+i\omega x-\omega^2 t}d\omega\right]d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(\xi)\left\{\int_{-\infty}^\infty \exp\left[-\left(\omega\sqrt{t}+\frac{i(\xi-x)}{2\sqrt{t}}\right)^2\right]\right.$$

$$\left.\cdot\exp\left[\frac{-(\xi-x)^2}{4t}\right]d\omega\right\}d\xi$$

$$= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^\infty f(\xi)\exp\left[\frac{-(\xi-x)^2}{4t}\right]d\xi$$

(using the hint from Problem 1b)

8. $\dfrac{\partial^2 u(x,t)}{\partial x^2} = \int_{-\infty}^\infty G(\omega)(-\omega^2)e^{i\omega x}\cos\omega t\,d\omega = \dfrac{\partial^2 u(x,t)}{\partial t^2}$

$u(x,0) = \int_{-\infty}^\infty G(\omega)e^{i\omega x}d\omega = f(x)$ (Fourier inversion formula)

$\dfrac{\partial u}{\partial t}(x,0) = \int_{-\infty}^\infty G(\omega)e^{i\omega x}(-\omega)\sin\omega t\,d\omega\Big|_{t=0} = 0$

With the modified initial conditions the solution becomes

$$u(x,t) = \int_{-\infty}^{\infty} G(\omega) e^{i\omega x} \frac{\sin \omega t}{\omega} d\omega$$

For the more general set of initial conditions construct the solution

$$u(x,t) = \int_{-\infty}^{\infty} e^{i\omega x} \left(G_1(\omega) \cos \omega t + G_2(\omega) \frac{\sin \omega t}{\omega} \right) d\omega$$

9. a. $f(r) = F(x) = \int_{-\infty}^{\infty} G(\omega) e^{i\omega x} d\omega = \int_{-\infty}^{\infty} g(\omega) e^{-i\omega \operatorname{Log} r} d\omega$

$$g(\omega) = G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{-i\omega x} dx = \frac{1}{2\pi} \int_0^{\infty} f(r) e^{i\omega \operatorname{Log} r} r^{-1} dr$$

(Change of variables $x = -\operatorname{Log} r, r = e^{-x}$)

b. $e^{-i\omega \operatorname{Log} r} = r^{-i\omega}$ and $g(\omega) = \frac{1}{2\pi} M[i\omega; f]$

The result follows from the first formula in part a.

10. a. $\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M[i\omega; f]}{\sinh \omega\theta_0} \left(-\omega^2\right) r^{-i\omega-2} \sinh \omega\theta d\omega$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M[i\omega; f]}{\sinh \omega\theta_0} r^{-i\omega-2} (\omega^2) \sinh \omega\theta d\omega$$

$$= 0$$

$$\phi(r,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M[i\omega; f]}{\sinh \omega\theta_0} r^{-i\omega} \sinh 0 d\omega = 0$$

$$\phi(r,\theta_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M[i\omega; f]}{\sinh \omega\theta_0} r^{-i\omega} \sinh \omega\theta_0 d\omega = f(r)$$

(using Problem 7b)

b. $\phi(r,\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M[i\omega; f]}{\sinh \omega\theta_0} r^{-i\omega} \sinh(\omega\theta_0 - \omega\theta) d\omega$

c. $\phi(r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{r^{-i\omega}}{\sinh \omega\theta_0} \{M[i\omega; f_1] \sinh(\omega\theta_0 - \omega\theta)$

$+M[i\omega; f_2] \sinh \omega\theta\} \, d\omega$

d. $\phi(r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{M[i\omega; f]}{\cosh \omega\theta_0} r^{-i\omega} \cosh \omega\theta d\omega$

EXERCISES 8.3: The Laplace Transform

1. a. $g(s) = \int_0^{\infty} 3\cos 2te^{-st} - 8e^{-2t-st}dt$

$$= \frac{3e^{-st}}{s^2+4}(-s\cos 2t + 2\sin 2t) - \frac{8e^{-2t-st}}{-2-s}\Bigg|_{t=0}^{\infty}$$

$$= \frac{3s}{s^2+4} - \frac{8}{s+2}$$

b. $g(s) = \int_0^{\infty} 2e^{-st} - e^{4t-st}\sin \pi t dt$

$$= \frac{2e^{-st}}{-s} - \frac{e^{4t-st}}{(4-s)^2+\pi^2}((4-s)\sin \pi t - \pi\cos \pi t)\Bigg|_{t=0}^{\infty}$$

$$= \frac{2}{s} - \frac{\pi}{(4-s)^2+\pi^2}$$

c. $g(s) = \int_0^1 e^{-st}dt = \frac{e^{-st}}{-s}\Bigg|_{t=0}^{1} = \frac{-e^{-s}}{s} + \frac{1}{s}$

d. $g(s) = \int_1^2 e^{-st}dt = \frac{e^{-st}}{-s}\Bigg|_{t=1}^{2} = \frac{-e^{-2s}}{s} + \frac{e^{-s}}{s}$

e. $g(s) = \int_0^{\infty} e^{-st}\sin^2 tdt = \int_0^{\infty} \frac{1}{2}e^{-st} - \frac{1}{2}e^{-st}\cos 2tdt$

$$= \frac{e^{-st}}{-2s} - \frac{e^{-st}}{2(s^2+4)}(-s\cos 2t + 2\sin 2t)\Bigg|_{t=0}^{\infty}$$

$$= \frac{1}{2s} - \frac{s}{2(s^2+4)} = \frac{2}{s(s^2+4)}$$

f. $g(s) = \int_0^\infty \frac{e^{-st}}{\sqrt{t}} dt = \frac{2}{\sqrt{s}} \int_0^\infty e^{-\chi^2} d\chi = \sqrt{\frac{\pi}{s}}$

2. The case $n = 1$ is done as Formula 6 in the text. Assume that the result holds for all positive integers less than n.

Then $\mathcal{L}\left\{F^{(n)}\right\}(s) = \int_0^\infty F^{(n)}(t)e^{-st}dt$

$= F^{(n-1)}(t)e^{-st}\Big|_0^\infty + \int_0^\infty F^{(n-1)}(t)se^{-st}dt$ (integration by parts)

$= -F^{(n-1)}(0) + s\mathcal{L}\left\{F^{(n-1)}\right\}(s)$

(using the hypothesized inequality)

$= -F^{(n-1)}(0) - sF^{(n-2)}(0) - \cdots - s^{n-1}F(0) + s^n\mathcal{L}\{F\}(s)$

By induction on n, the theorem is proved.

3. Use the table of Laplace transforms in this section.

a. $F(t) = \frac{1}{2}\sin 2t$ (entry iv)

b. $F(t) = 4te^t$ (entry ix)

c. $\frac{s+1}{s^2+4s+4} = \frac{1}{s+2} - \frac{1}{(s+2)^2}$

$F(t) = e^{-2t} - te^{-2t}$ (entries i, ix)

d. $\frac{1}{s^3+3s^2+2s} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$

$F(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$ (entries i, ii)

e. $\frac{s+3}{s^2+4s+7} = \frac{s+2}{(s+2)^2+3} + \frac{1}{(s+2)^2+3}$

$F(t) = e^{-2t}\cos\sqrt{3}t + \frac{1}{\sqrt{3}}e^{-2t}\sin\sqrt{3}t$ (entries vii, viii)

4. $\mathcal{L}\{f_\tau(t)\} = \int_\tau^\infty f(t-\tau)e^{-st}dt$

$$= e^{-s\tau}\int_0^\infty f(x)e^{-sx}dx$$

(change of variables $x = t - \tau$)

$$= e^{-s\tau}\mathcal{L}\{f(t)\}$$

5. a. $s\mathcal{L}\{f\} - f(0) - \mathcal{L}\{f\} = \dfrac{1}{s-3}$

$$\mathcal{L}\{f\} = \frac{3}{s-1} + \frac{1}{(s-1)(s-3)} = \frac{5}{2(s-1)} + \frac{1}{2(s-3)}$$

$$f(t) = \frac{5}{2}e^t + \frac{1}{2}e^{3t}$$

b. $s^2\mathcal{L}\{f\} - sf(0) - f'(0) - 5s\mathcal{L}\{f\} + 5f(0) + 6\mathcal{L}\{f\} = 0$

$$\mathcal{L}\{f\} = \frac{s-6}{s^2-5s+6} = \frac{4}{s-2} - \frac{3}{s-3}$$

$$f(t) = 4e^{2t} - 3e^{3t}$$

c. $s^2\mathcal{L}\{f\} - sf(0) - f'(0) - s\mathcal{L}\{f\} + f(0) - 2\mathcal{L}\{f\} = \dfrac{2}{(s+1)^2+4}$

$$\mathcal{L}\{f\} = \frac{2}{s^2-s-2} + \frac{2}{(s^2-s-2)[(s+1)^2+4]}$$

$$= \frac{28}{39(s-2)} - \frac{5}{6(s+1)} + \frac{3s-1}{26[(s+1)^2+4]}$$

$$f(t) = \frac{28}{39}e^{2t} - \frac{5}{6}e^{-t} + \frac{3}{26}e^{-t}\cos 2t - \frac{1}{13}e^{-t}\sin 2t$$

d. The Laplace transform of the right side is

$$\int_3^6 e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_{t=3}^6 = \frac{-e^{-6s}}{s} + \frac{e^{-3s}}{s}$$

Thus $s^2\mathcal{L}\{f\} - sf(0) - f'(0) - 3s\mathcal{L}\{f\} + 3f(0) + 2\mathcal{L}\{f\} = \dfrac{e^{-3s} - e^{-6s}}{s}$

$$\mathcal{L}\{f\} = \frac{e^{-3s} - e^{-6s}}{s(s^2 - 3s + 2)} = \left(e^{-3s} - e^{-6s}\right)\left(\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right)$$

$$f(t) = \begin{cases} \frac{1}{2}(0-0) - (0-0) + \frac{1}{2}(0-0) & \text{if } 0 \le t < 3 \\ \frac{1}{2}(1-0) - \left(e^{t-3} - 0\right) + \frac{1}{2}\left(e^{2t-6} - 0\right) & \text{if } 3 \le t < 6 \\ \frac{1}{2}(1-1) - \left(e^{t-3} - e^{t-6}\right) + \frac{1}{2}\left(e^{2t-6} - e^{2t-12}\right) & \text{if } 6 \le t < \infty \end{cases}$$

$$= \begin{cases} 0 & \text{if } 0 \le t < 3 \\ \frac{1}{2} - e^{t-3} + \frac{1}{2}e^{2t-6} & \text{if } 3 \le t < 6 \\ -e^{t-3} + e^{t-6} + \frac{1}{2}e^{2t-6} - \frac{1}{2}e^{2t-12} & \text{if } 6 \le t < \infty \end{cases}$$

6. a. e^{-t} is continuously differentiable and integrable, so the inversion formula is

$$F(t) = e^{-t} = \frac{1}{2\pi}\int_{-i\infty}^{i\infty} \frac{e^{st}}{s+1} ds$$

In order to keep the integrand bounded, consider the contour $\Gamma = \gamma_1 + \gamma_2$ in the left half-plane, where

$$\gamma_1 : z(\tau) = \tau i \qquad -\rho \le \tau \le \rho$$
$$\gamma_2 : z(\tau) = \rho e^{i\tau} \qquad \pi/2 \le \tau \le 3\pi/2$$

Then $\frac{1}{2\pi}\int_{-i\infty}^{i\infty} \frac{e^{st}}{s+1} ds = \frac{2\pi i}{2\pi i} \operatorname{Res}\left(\frac{e^{zt}}{z+1}; z = -1\right) = e^{-t}$

b. By Theorem 6, the inversion formula is

$$F(t) = 1 = \frac{1}{2\pi}\int_{1-i\infty}^{1+i\infty} \frac{e^{st}}{s} ds$$

In order to keep the integrand bounded, consider the contour $\Gamma = \gamma_1 + \gamma_2$ in the left half-plane $x \le 1$, where

$\gamma_1 : z(\tau) = 1 + \tau i \qquad -\rho \le \tau \le \rho$
$\gamma_2 : z(\tau) = 1 + \rho e^{i\tau} \quad \pi/2 \le \tau \le 3\pi/2$

Then $\dfrac{1}{2\pi}\displaystyle\int_{1-i\infty}^{1+i\infty} \frac{e^{st}}{s} ds = \frac{2\pi i}{2\pi i} \operatorname{Res}\left(\frac{e^{zt}}{z}; z = 0\right) = 1$

7. a. $a_n\mathcal{L}\left\{f^{(n)}\right\} + a_{n-1}\mathcal{L}\left\{f^{(n-1)}\right\} + \cdots + a_1\mathcal{L}\{f'\} + a_0\mathcal{L}\{f\} = \mathcal{L}\{u\}$

Using Equation 8, this reduces to

$\left(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)\mathcal{L}\{f\} = \mathcal{L}\{u\} + P(s)$

That is, $F(s) = \dfrac{U(s) + P(s)}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$

b. The poles of the transfer function either are real or come in conjugate pairs z and $\overline{z}$, so that $F(s) = \dfrac{P(s)}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$ can be decomposed into partial fractions with denominators of the form $s - x_i$ and $(s - z_j)(s - \overline{z}_j) = (s - \operatorname{Re} z_j)^2 + (\operatorname{Im} z_j)^2$.
It follows that $F(t)$ is the sum of terms $b_i e^{x_i t}$ and $c_j e^{\operatorname{Re}(z_j)t} \cos \operatorname{Im}(z_j)t$ and $d_j e^{\operatorname{Re}(z_j)t} \sin \operatorname{Im}(z_j)t$ (from entries i, vii, and viii in the Laplace transform table).
The poles lie in the left half-plane, so $x_i < 0$ and $\operatorname{Re} z_j < 0$, and the terms of $F(t)$ go to zero as $t \to \infty$.

8.

a.	$1/(s-1)$	pole $s = 1$	not stable
b.	$1/(s^2 - 5s + 6)$	poles $s = 2, 3$	not stable
c.	$1/(s^2 - s - 2)$	poles $s = -1, 2$	not stable
d.	$1/(s^2 - 3s + 2)$	poles $s = 1, 2$	not stable

9. a. Spring 1 is stretched by x, spring 2 is stretched by $y - x$, and spring 3 is stretched by $-y$ (consider compression to be negative stretching). By considering each mass separately,

$m_1\dfrac{d^2x}{dt^2} = -Kx - K(x - y)$ (springs 1 and 2)

$m_2\dfrac{d^2y}{dt^2} = -Ky + K(x - y)$ (springs 3 and 2)

b. $\dfrac{d^2x}{dt^2} + 2x - y = 0$

$\dfrac{d^2y}{dt^2} - x + 2y = 0$

In each case $x'(0) = 0,\ \ y'(0) = 0.$
Taking Laplace transforms,

$$\begin{cases} s^2\mathcal{L}\{x\} - sx(0) - x'(0) + 2\mathcal{L}\{x\} - \mathcal{L}\{y\} = 0 \\ s^2\mathcal{L}\{y\} - sy(0) - y'(0) - \mathcal{L}\{x\} + 2\mathcal{L}\{y\} = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} (s^2+2)\mathcal{L}\{x\} - \mathcal{L}\{y\} = sx(0) \\ -\mathcal{L}\{x\} + (s^2+2)\mathcal{L}\{y\} = sy(0) \end{cases}$$

$$\Longrightarrow [(s^2+2)^2 - 1]\mathcal{L}\{x\} = (s^2+2)sx(0) + sy(0)$$

(Gaussian elimination)

$$\Longrightarrow \begin{cases} \mathcal{L}\{x\} = \dfrac{[x(0)+y(0)]s}{2(s^2+1)} + \dfrac{[x(0)-y(0)]s}{2(s^2+3)} \text{(partial fractions)} \\ \mathcal{L}\{y\} = \dfrac{[x(0)+y(0)]s}{2(s^2+1)} + \dfrac{[y(0)-x(0)]s}{2(s^2+3)} \text{(symmetry)} \end{cases}$$

$$\Longrightarrow \begin{cases} x(t) = \dfrac{1}{2}[x(0)+y(0)]\cos t + \dfrac{1}{2}[x(0)-y(0)]\cos\sqrt{3}t \\ y(t) = \dfrac{1}{2}[x(0)+y(0)]\cos t + \dfrac{1}{2}[y(0)-x(0)]\cos\sqrt{3}t \end{cases}$$

(Laplace transform table, entry iii)

Now substitute the given initial conditions.

(i) $x(t) = \cos\sqrt{3}t, \quad y(t) = -\cos\sqrt{3}t$

(ii) $x(t) = \cos t, \quad y(t) = \cos t$

(iii) $x(t) = \dfrac{1}{2}\cos t + \dfrac{1}{2}\cos\sqrt{3}t, \quad y(t) = \dfrac{1}{2}\cos t - \dfrac{1}{2}\cos\sqrt{3}t$

c. (i) and (ii)

In (i) the motion is mirror-symmetric with respect to the midpoint.

In (ii) the spring connecting m_1 and m_2 remains rigid.

EXERCISES 8.4: The z-Transform

1. The z-transform of $\{\alpha a(n)+\beta b(n)\}$ is

$$\sum_{n=-\infty}^{\infty}\left[\alpha a(n)+\beta b(n)\right]z^{-n}=\alpha\sum_{n=-\infty}^{\infty}a(n)z^{-n}+\beta\sum_{n=-\infty}^{\infty}b(n)z^{-n}$$

$$=\alpha A(z)+\beta B(z)$$

in the common region of convergence.

2. $-zA'(z)=-z\dfrac{d}{dz}\displaystyle\sum_{n=-\infty}^{\infty}a(n)z^{-n}=z\sum_{n=-\infty}^{\infty}na(n)z^{-n-1}=\sum_{n=-\infty}^{\infty}na(n)z^{-n}$,
which is the z-transform of $na(n)$.
By Formula 8, this converges in the annulus

$$\limsup_{n>0}\sqrt[n]{n|a(n)|}<|z|<\frac{1}{\limsup_{n>0}\sqrt[n]{n|a(-n)|}}$$

or $\displaystyle\limsup_{n>0}\sqrt[n]{|a(n)|}<|z|<\frac{1}{\limsup_{n>0}\sqrt[n]{|a(-n)|}}$

3. $A(z/\alpha)=\displaystyle\sum_{n=-\infty}^{\infty}a(n)(z/\alpha)^{-n}=\sum_{n=-\infty}^{\infty}\alpha^{n}a(n)z^{-n}$,
which is the z-transform of $\alpha^{n}a(n)$, with annulus of convergence
$\displaystyle|\alpha|\limsup_{n>0}\sqrt[n]{|a(n)|}<|z|<\frac{1}{|\alpha|\limsup_{n>0}\sqrt[n]{|a(-n)|}}$
(multiply the radii by $|\alpha|$)

4. a. $A(z)=z^{0}+\displaystyle\sum_{n=1}^{\infty}0z^{-n}=1$

   b. $A(z)=\displaystyle\sum_{n=0}^{\infty}z^{-n}=\frac{1}{1-\frac{1}{z}}=\frac{z}{z-1}$

   c. $A(z)=\displaystyle\sum_{n=0}^{\infty}nz^{-n}=-z\frac{d}{dz}\sum_{n=0}^{\infty}z^{-n}=-z\frac{d}{dz}\left(\frac{z}{z-1}\right)=\frac{z}{(z-1)^{2}}$

d. $A(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \dfrac{1}{1-\alpha/z} = \dfrac{z}{z-\alpha}$

e. $A(z) = \sum_{n=0}^{\infty} \sin(n\omega) z^{-n} = \dfrac{1}{2i} \sum_{n=0}^{\infty} e^{in\omega} z^{-n} - e^{-in\omega} z^{-n}$

$$= \frac{1}{2i}\left[\frac{z}{z-e^{i\omega}} - \frac{z}{z-e^{-i\omega}}\right] = \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}$$

f. $A(z) = \sum_{n=0}^{\infty} \cos(n\omega) z^{-n} = \dfrac{1}{2} \sum_{n=0}^{\infty} e^{in\omega} z^{-n} + e^{-in\omega} z^{-n}$

$$= \frac{1}{2}\left[\frac{z}{z-e^{i\omega}} + \frac{z}{z-e^{-i\omega}}\right] = \frac{z(z-\cos\omega)}{z^2 - 2z\cos\omega + 1}$$

5. a. $\dfrac{1}{1+1/(3z)} = \dfrac{3z}{1+3z} = 3z\sum_{n=0}^{\infty}(-3z)^n = \sum_{n=-\infty}^{-1} -(-3)^{-n} z^{-n}$

$a(n) = -(-3)^{-n}$ for $n \leq -1$, zero otherwise

b. $\dfrac{1}{1+1/(3z)} = \sum_{n=0}^{\infty}\left(\dfrac{-1}{3z}\right)^n = \sum_{n=0}^{\infty}\left(\dfrac{-1}{3}\right)^n z^{-n}$

$a(n) = \left(\dfrac{-1}{3}\right)^n$ for $n \geq 0$, zero otherwise

c. $\dfrac{z^4}{z+2} = \dfrac{z^4}{2}\sum_{n=0}^{\infty}\left(\dfrac{-z}{2}\right)^n = \sum_{n=-\infty}^{-4}(-1)^n 2^{n+3} z^{-n}$

$a(n) = (-1)^n 2^{n+3}$ for $n \leq -4$, zero otherwise

d. $\dfrac{z^4}{z+2} = z^3\sum_{n=0}^{\infty}\left(\dfrac{-2}{z}\right)^n = \sum_{n=-3}^{\infty}(-2)^{n+3} z^{-n}$

$a(n) = (-2)^{n+3}$ for $n \geq -3$, zero otherwise

e. $\dfrac{z+2}{2z^2-7z+3} = \dfrac{1}{z-3} - \dfrac{1}{2z-1} = \dfrac{-1}{3}\sum_{n=0}^{\infty}\left(\dfrac{z}{3}\right)^n - \dfrac{1}{2z}\sum_{n=0}^{\infty}\left(\dfrac{1}{2z}\right)^n$

$$= \sum_{n=-\infty}^{0} -3^{n-1} z^{-n} + \sum_{n=1}^{\infty} -2^{-n} z^{-n}$$

$a(n) = -3^{n-1}$ for $n \leq 0$, -2^{-n} for $n \geq 1$

f. $\dfrac{1-1/(2z)}{1+3/(4z)+1/(8z^2)} = \dfrac{4}{1+1/(2z)} - \dfrac{3}{1+1/(4z)}$

$$= 4\sum_{n=0}^{\infty}(-2)^{-n}z^{-n} - 3\sum_{n=0}^{\infty}(-4)^{-n}z^{-n}$$

$a(n) = 4(-2)^{-n} - 3(-4)^{-n}$ for $n \geq 0$, zero otherwise

g. $$\frac{z}{\left(z-\frac{1}{2}\right)(z-1)^2} = \frac{2}{z-\frac{1}{2}} - \frac{2}{z-1} + \frac{2}{(z-1)^2}$$

$$= \frac{\frac{2}{z}}{1-\frac{1}{2z}} - \frac{\frac{2}{z}}{1-\frac{1}{z}} - 2\frac{d}{dz}\left(\frac{1}{1-\frac{1}{z}}\right)$$

$$= \frac{2}{z}\sum_{n=0}^{\infty}2^{-n}z^{-n} - \frac{2}{z}\sum_{n=0}^{\infty}z^{-n} - 2\frac{d}{dz}\sum_{n=0}^{\infty}z^{-n}$$

$$= \sum_{n=1}^{\infty}2^{-n+2}z^{-n} + \sum_{n=1}^{\infty}-2z^{-n} + \sum_{n=1}^{\infty}2(n-1)z^{-n}$$

$a(n) = 2^{-n+2} + 2n - 4$ for $n \geq 1$, zero otherwise

h. $$\frac{1-\alpha/z}{\alpha-1/z} = \frac{1/\alpha - 1/z}{1-1/(\alpha z)} = \left(\frac{1}{\alpha} - \frac{1}{z}\right)\sum_{n=0}^{\infty}\alpha^{-n}z^{-n}$$

$$= \sum_{n=0}^{\infty}\alpha^{-n-1}z^{-n} - \sum_{n=1}^{\infty}\alpha^{-n+1}z^{-n}$$

$a(n) = \alpha^{-n-1} - \alpha^{-n+1}$ for $n \geq 1$, $1/\alpha$ for $n = 0$, zero otherwise

6. The series converges for $\limsup_{n>0}\sqrt[n]{|a(n)|} < |z| < \infty$ and

$$\lim_{z\to\infty}A(z) = \lim_{z\to\infty}\left(a(0) + \frac{a(1)}{z} + \frac{a(2)}{z^2} + \cdots\right) = a(0)$$

7. a. $zA^+(z) - za(0) = 0.5A^+(z)$

$$A^+(z) = \frac{2z}{z-\frac{1}{2}} = \frac{2}{1-\frac{1}{2z}} = 2\sum_{n=0}^{\infty}2^{-n}z^{-n}$$

$a(n) = 2^{-n+1}$

b. $zA^+(z) - za(0) + 2A^+(z) = \frac{z}{z-1}$ (Problem 4b)

$$A^+(z) = \frac{z^2}{(z+2)(z-1)} = \frac{z}{3}\left[\frac{1}{1-\frac{1}{z}} - \frac{1}{1+\frac{2}{z}}\right]$$

$$= \frac{z}{3}\sum_{n=0}^{\infty} z^{-n} - \frac{z}{3}\sum_{n=0}^{\infty}(-2)^n z^{-n} = \sum_{n=-1}^{\infty}\left[\frac{1}{3} - \frac{1}{3}(-2)^{n+1}\right] z^{-n}$$

$$a(n) = \frac{1}{3} - \frac{1}{3}(-2)^{n+1}$$

c. $z^2A^+(z) - z^2a(0) - za(1) - 5zA^+(z) + 5za(0) + 6A^+(z) = \dfrac{z}{z-1}$

$$A^+(z) = \frac{2z^2 - 7z + \frac{z}{z-1}}{z^2 - 5z + 6} = \frac{2z^3 - 9z^2 + 8z}{(z-1)(z-2)(z-3)}$$

$$= z\left(\frac{1}{2(z-1)} + \frac{2}{z-2} - \frac{1}{2(z-3)}\right) = \frac{\frac{1}{2}}{1-\frac{1}{z}} + \frac{2}{1-\frac{2}{z}} - \frac{\frac{1}{2}}{1-\frac{3}{z}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} z^{-n} + 2\sum_{n=0}^{\infty} 2^n z^{-n} - \frac{1}{2}\sum_{n=0}^{\infty} 3^n z^{-n}$$

$$a(n) = \frac{1}{2} + 2^{n+1} - \frac{3^n}{2}$$

8. $\displaystyle\sum_{n=0}^{\infty} a(n-N)z^{-n} = \sum_{n=N}^{\infty} a(n-N)z^{-n} = \sum_{n=0}^{\infty} a(n)z^{-n-N} = z^{-N}A^+(z)$

EXERCISES 8.5: Cauchy Integrals and the Hilbert Transform

1. It suffices to show that $\phi(x)$ and $\psi(x)$ are the real and imaginary parts of a function $f(z)$ analytic in the upper half-plane with $z = x$ substituted and with rapid convergence to 0 as $z \to \infty$ (as in the discussion at the top of Page 419).

   a. For $\omega > 0$, $e^{i\omega z}\Big|_{z=x} = \cos\omega x + i\sin\omega x$

   $\left|e^{i\omega(x+iy)}\right| = e^{-\omega y} \to 0$ as $y \to \infty$

   b. For $\omega < 0$, $e^{-i\omega z}\Big|_{z=x} = \cos\omega x - i\sin\omega x$ (converges as in part a)

   c. For $\omega > 0$, $-ie^{i\omega z}\Big|_{z=x} = \sin\omega x - i\cos\omega x$ (converges as in part a)

   d. For $\omega < 0$, $ie^{-i\omega z}\Big|_{z=x} = \sin\omega x + i\cos\omega x$ (converges as in part a)

e. For $a > 0$, let $f(z) = \dfrac{i}{z + ai}$. Then $f(x) = \dfrac{a}{a^2 + x^2} + i\dfrac{x}{a^2 + x^2}$

$|f(z)| \leq K/|z|$

f. For $a > 0$, let $f(z) = \dfrac{e^{iaz} - 1}{iz}$. Then $f(x) = \dfrac{\sin ax}{x} + i\dfrac{1 - \cos ax}{x}$

$|f(z)| \leq K\left|e^{iaz}\right|$

2. a. $$\frac{-1}{\pi} \text{ p.v. } \int_{-\infty}^{\infty} \frac{\cos \omega\xi}{\xi - x} d\xi = \frac{-1}{\pi} \text{Re} \left[\text{p.v. } \int_{-\infty}^{\infty} \frac{e^{i\omega\xi}}{\xi - x} d\xi \right]$$

$$= \frac{-1}{\pi} \text{Re} \left[\pi i \text{ Res} \left(\frac{e^{i\omega z}}{z - x}; z = x \right) \right] = \sin \omega x$$

b,c,d. Similar to a.

e. $$\frac{-1}{\pi} \text{ p.v. } \int_{-\infty}^{\infty} \frac{a}{(a^2 + \xi^2)(\xi - x)} d\xi$$

$$= \frac{-1}{\pi} \text{Re} \left[\text{p.v. } \int_{-\infty}^{\infty} \frac{i}{(\xi + ai)(\xi - x)} d\xi \right]$$

$$= \frac{-1}{\pi} \text{Re} \left[\pi i \text{ Res} \left(\frac{i}{(z + ai)(z - x)}; z = x \right) \right]$$

$$= \frac{-1}{\pi} \text{Re} \left[\frac{-\pi}{x + ai} \right] = \text{Re} \left[\frac{x - ai}{a^2 + x^2} \right] = \frac{x}{a^2 + x^2}$$

f. $$\frac{-1}{\pi} \text{ p.v. } \int_{-\infty}^{\infty} \frac{\sin a\xi}{\xi(\xi - x)} d\xi = \frac{-1}{\pi} \text{ Im} \left[\text{p.v. } \int_{-\infty}^{\infty} \frac{e^{ia\xi}}{\xi(\xi - x)} d\xi \right]$$

$$= \frac{-1}{\pi} \text{ Im} \left[\pi i \text{ Res} \left(\frac{e^{iaz}}{z(z - x)}; z = 0 \right) + \pi i \text{ Res} \left(\frac{e^{iaz}}{z(z - x)}; z = x \right) \right]$$

$$= \frac{-1}{\pi} \text{ Im} \left[\frac{-\pi i}{x} + \frac{\pi i e^{iax}}{x} \right] = \frac{1 - \cos ax}{x}$$

3. S'_ϵ and S''_ϵ cover arc lengths of $3\pi i/2$ and $-\pi i/2$ respectively. Lemma 4, Section 6.5 shows that

$$\int_{S'_\epsilon} \frac{f(\zeta)}{\zeta - z_0} d\zeta = \frac{3\pi i}{2} \text{ Res} \left(\frac{f(\zeta)}{\zeta - z_0}; z_0 \right) = \frac{3\pi i}{2} f(z_0)$$

(similar for S''_ϵ)

4. Let $f(z) = \frac{1}{2}\left(e^{i\omega z} + e^{-i\omega z}\right) e^{i\Omega_1 z}$
$f(x) = \cos\omega x \cos\Omega_1 x + i\cos\omega x \sin\Omega_1 x$
The condition $|\omega| < \Omega_1$ implies that $|f(z)| \le \left|e^{i(\Omega_1 - |\omega|)z}\right| = \left|e^{-(\Omega_1-|\omega|)y}\right|$
in the upper half-plane.

5. $\psi(x) = -(1/\pi)\text{p.v.}\int_{-\infty}^{\infty}\{g(x)/(t-x)\}dt$
$\psi(x) = -(1/\pi)\text{p.v.}\int_{-\infty}^{\infty}\{[f(e^{i\omega t}) + f(e^{-i\omega t})]/(t-x)\}dt$
$= -(1/\pi)\text{p.v.}\int_{-\infty}^{\infty}\{\Sigma_{k=0}^{\infty}(1/k!)f^{(k)}(1)[(e^{i\omega t} - 1)^k + (e^{-i\omega t} - 1)^k\ ]/(t-x)\}dt$
which can be integrated term by term using Jordan's lemma and Cauchy's theorem to give
$= -(1/\pi)\{\Sigma_{k=0}^{\infty}(1/k!)f^{(k)}(1)i\pi[(e^{i\omega x} - 1)^k - (e^{-i\omega x} - 1)^k\ ]\}dt$
$= -i\ [\ f(e^{i\omega x}) - f(e^{-i\omega x})]$

6, $$\frac{-1}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty}\frac{\operatorname{Im}k(\eta)}{\eta-\omega}d\eta = \frac{-1}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty}\frac{-\eta L}{(R^2+\eta^2L^2)(\eta-\omega)}d\eta$$

$$= \frac{L}{\pi}\left[-2\pi i\ \text{Res}\left(\frac{\eta}{(R^2+\eta^2L^2)(\eta-\omega)}; \eta = \frac{-iR}{L}\right)\right.$$

$$\left. -\pi i\ \text{Res}\left(\frac{\eta}{(R^2+\eta^2 L)(\eta-\omega)}; \eta = \omega\right)\right]$$

$$= \frac{L}{\pi}\left[-2\pi i\left(\frac{-1}{2L(iR+\omega L)}\right) - \pi i\left(\frac{\omega}{R^2+\omega^2L^2}\right)\right]$$

$$= \frac{R}{R^2+\omega^2L^2} = \text{Re}\ k(\omega)$$

7, a. $$\lim_{\substack{z\to z_0\\ |z|>1}}\oint_{|\zeta|=1}\frac{1}{\zeta(\zeta-z)}d\zeta$$

$$= 2\pi i\ \text{Res}\left(\frac{1}{\zeta(\zeta-z_0)}; \zeta = 0\right) - \pi i\ \text{Res}\left(\frac{1}{\zeta(\zeta-z_0)}; \zeta = z_0\right)$$

$$= \frac{-2\pi i}{z_0} - \frac{\pi i}{z_0} = \frac{-3\pi i}{z_0}$$

$$\lim_{\substack{z\to z_0\\ |z|<1}}\oint_{|\zeta|=1}\frac{1}{\zeta(\zeta-z)}d\zeta$$

$$= 2\pi i \text{ Res}\left(\frac{1}{\zeta(\zeta - z_0)}; \zeta = 0\right) + \pi i \text{ Res}\left(\frac{1}{\zeta(\zeta - z_0)}; \zeta = z_0\right)$$

$$= -\frac{2\pi i}{z_0} + \frac{\pi i}{z_0} = \frac{-\pi i}{z_0}$$

The difference of the limits is $\dfrac{2\pi i}{z_0}$

b. For $\operatorname{Im} z > 0$,

$$\lim_{\substack{z \to x_0 \\ \operatorname{Im} z > 0}} \int_{-\infty}^{\infty} \frac{1}{\xi - z} d\xi = \pi i \text{ Res}\left(\frac{1}{\xi - x_0}; x_0\right) = \pi i$$

For $\operatorname{Im} z < 0$,

$$\lim_{\substack{z \to x_0 \\ \operatorname{Im} z < 0}} \int_{-\infty}^{\infty} \frac{1}{\xi - z} d\xi = -\pi i \text{ Res}\left(\frac{1}{\xi - x_0}; x_0\right) = -\pi i$$

The difference of the limits is $2\pi i$

c. Using Problem 10, Section 6.4,

$$\lim_{\substack{z \to x_0 \\ \operatorname{Im} z > 0}} \int_{-\infty}^{\infty} \frac{\cos \xi}{\xi - z} d\xi - \lim_{\substack{z \to x_0 \\ \operatorname{Im} z < 0}} \int_{-\infty}^{\infty} \frac{\cos \xi}{\xi - z} d\xi = \lim_{\substack{z \to x_0 \\ \operatorname{Im} z > 0}} \pi i e^{iz} - \lim_{\substack{z \to x_0 \\ \operatorname{Im} z < 0}} -\pi i e^{-iz}$$

$$= \pi i \left(e^{ix_0} + e^{-ix_0}\right) = 2\pi i \cos x_0$$

d. Using Problem 18, Section 4.7,

$$\lim_{\substack{z \to x_0 \\ \operatorname{Im} z > 0}} \int_{-\infty}^{\infty} \frac{\xi}{(\xi^2 + 1)(\xi - z)} d\xi - \lim_{\substack{z \to x_0 \\ \operatorname{Im} z < 0}} \int_{-\infty}^{\infty} \frac{\xi}{(\xi^2 + 1)(\xi - z)} d\xi$$

$$= \frac{\pi i}{x_0 + i} - \frac{-\pi i}{x_0 - i} = 2\pi i \frac{x_0}{x_0^2 + 1}$$

8. a. Let $\Gamma = \Gamma' + C_\rho^+$.

$$\lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0 - i\epsilon} dx$$

$$= \lim_{\epsilon \downarrow 0} \int_{\Gamma} \frac{f(z)}{z - x_0 - i\epsilon} dz = \text{ p.v. } \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx + \pi i \text{ Res}\left(\frac{f(z)}{z - x_0}; x_0\right)$$

$$= \text{ p.v. } \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx + \pi i f(x_0)$$

b. p.v. $\dfrac{1}{x - x_0} - i\pi\delta(x - x_0)$