CPSC 303, 2024/25, Term 2, Assignment 2

Released Wednesday, January 29, 2025
Due Wednesday, February 12, 2025, 11:59pm

Note: Questions 2 through 10 are taken from the textbook, and are provided below in full, for your convenience. At the beginning of each of them we state where they can be found in the book. For example, "AG 10.8" means that we are referring to Ascher & Greif, Chapter 10, Exercise 8.

1. Consider the function

$$f(x) = \sin(\alpha x),$$

where the values of α to be considered in this question are

$$\alpha = 0.1 \ ; \ \alpha = 1 \ ; \ \alpha = 12.$$

Suppose we wish to interpolate the function (for the three different values of α) over the interval $[0, \pi]$, using the 11 data values $\{(\frac{\pi j}{10}, \sin(\frac{\alpha \pi j}{10}))\}, \ j = 0, 1, \ldots, 10$.

   (a) Write MATLAB programs to generate approximations using

      (i) a global 10th degree polynomial interpolant;
      (ii) a not-a-knot cubic spline
      (iii) a piecewise cubic Hermite polynomial
      (iv) a piecewise linear polynomial
      (v) a piecewise constant polynomial

   Either write your own code or use code provided with the book, and for (ii) and (iii) you may use the MATLAB commands `spline` and `pchip` (check out `pchip` carefully before you use it). We recommend that you write your own code – your best way to learn. Generate plots of the function and the approximating polynomials in one graph (non-logarithmic scale, using `plot`) and plots of the error (logarithmic scale, using `semilogy`), for the three values of α: a total of six graphs, with several curves on each of them. You may use the command `subplot` for a compact representation of your results.

   (b) Write down the error bounds for the five approaches. You may use information on bounds in the textbook or in other sources. (If your splint is not-a-knot, which is what the MATLAB command `spline` does, there is a constant c in the error bound which you may ignore.) Comment on the quality of the approximations and the errors, and observe how well the error bounds predict the actual errors. Specifically, make sure to explain how the value of α (which determines how oscillatory the function is) makes an effect.

2. (AG 10.8) A secret formula for eternal youth, $f(x)$, was discovered by Dr. Quick, who has been working in our biotech company. However, Dr. Quick has disappeared and is rumored to be negotiating with a rival organization.

   From the notes that Dr. Quick left behind in his hasty exit it is clear that $f(0) = 0$, $f(1) = 2$, and that $f[x_0, x_1, x_2] = 1$ for *any* three points x_0, x_1, x_2. Find $f(x)$.

3. (AG 10.9) Joe had decided to buy stocks of a particularly promising Internet company. The price per share was \$100, and Joe subsequently recorded the stock price at the end of each week. With the abscissae measured in days, the following data were acquired:
$(0, 100), (7, 98), (14, 101), (21, 50), (28, 51), (35, 50)$.

   In attempting to analyze what happened, it was desired to approximately evaluate the stock price a few days before the crash.

   (a) Pass a linear interpolant through the points with abscissae 7 and 14. Then add to this data set the value at 0 and (separately) the value at 21 to obtain two quadratic interpolants. Evaluate all three interpolants at $x = 12$. Which one do you think is the most accurate? Explain.

   (b) Plot the two quadratic interpolants above, together with the data (without a broken line passing through the data) over the interval $[0, 21]$. What are your observations?

4. (AG 10.19) Interpolate the Runge function of Example 10.6 at Chebyshev points for n from 10 to 170 in increments of 10. Calculate the maximum interpolation error on the uniform evaluation mesh `x = -1:.001:1` and plot the error vs. polynomial degree as in Figure 10.8 using `semilogy`. Observe spectral accuracy.

5. (AG 10.22) For some function f, you have a table of extended divided differences of the form

i	z_i	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
0	5.0	$f[z_0]$			
1	5.0	$f[z_1]$	$f[z_0, z_1]$		
2	6.0	4.0	5.0	-3.0	
3	4.0	2.0	$f[z_2, z_3]$	$f[z_1, z_2, z_3]$	$f[z_0, z_1, z_2, z_3]$

   Fill in the unknown entries in the table.

6. (AG 10.23) For the data in Exercise 10.22 (Question 5 in this assignment) what is the osculating polynomial $p_2(x)$ of degree at most 2 that satisfies

$$p_2(5.0) = f(5.0),\ p_2'(5.0) = f'(5.0),\ p_2(6.0) = f(6.0)?$$

7. (AG 11.3) Let $f \in C^3[a, b]$ be given at equidistant points $x_i = a + ih,\ i = 0, 1, \ldots, n$, where $nh = b - a$. Assume further that $f'(a)$ is given as well.

   (a) Construct an algorithm for C^1 piecewise quadratic interpolation of the given values. Thus, the interpolating function is written as

$$v(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2, \quad x_i \le x \le x_{i+1},$$

   for $i = 0, \ldots, n - 1$, and your job is to specify an algorithm for determining the $3n$ coefficients $a_i,\ b_i$ and c_i.

   (b) How accurate do you expect this approximation to be as a function of h? Justify.

8. (AG 11.4) Verify that the Hermite cubic interpolating $f(x)$ and its derivative at the points t_i and t_{i+1} can be written explicitly as

$$\begin{aligned} s_i(x) &= f_i + \left(h_i f_i'\right)\tau + \Big(3(f_{i+1} - f_i) - h_i(f_{i+1}' + 2f_i')\Big)\tau^2 \\ &+ \Big(h_i(f_{i+1}' + f_i') - 2(f_{i+1} - f_i)\Big)\tau^3, \end{aligned}$$

where $h_i = t_{i+1} - t_i$, $f_i = f(t_i)$, $f'_i = f'(t_i)$, $f_{i+1} = f(t_{i+1})$, $f'_{i+1} = f'(t_{i+1})$, and $\tau = \frac{x-t_i}{h_i}$.

9. (AG 11.6, with a minor revision of part (d)) The *gamma function* is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0.$$

It is known that for integer numbers the function has the value

$$\Gamma(n) = (n-1)! = 1 \cdot 2 \cdot 3 \cdots (n-1).$$

(We define $0! = 1$.) Thus, for example, $(1,1), (2,1), (3,2), (4,6), (5,24)$ can be used as data points for an interpolating polynomial.

   (a) Write a MATLAB script that computes the polynomial interpolant of degree four that passes through the above five data points.

   (b) Write a program that computes a cubic spline to interpolate the same data. (You may use MATLAB's `spline`, or your own code.)

   (c) Plot the two interpolants you found on the same graph, along with a plot of the gamma function itself, which can be produced using the MATLAB command `gamma`.

   (d) Plot the errors in the two interpolants on the same graph. What are your observations? You may separate your observations into two parts: the errors for smaller values of x, e.g., on the interval $[0,3]$, and the errors for larger values, say on the interval $[3,5]$.

10. (AG 11.15) Consider interpolating the data $(x_0, y_0), \dots, (x_6, y_6)$ given by

x	0.1	0.15	0.2	0.3	0.35	0.5	0.75
y	3.0	2.0	1.2	2.1	2.0	2.5	2.5

Construct the five interpolants specified below (you may use available software for this), evaluate them at the points $0.05 : 0.01 : 0.8$, plot and comment on their respective properties.

   (a) A polynomial interpolant.

   (b) A cubic spline interpolant.

   (c) The interpolant

$$v(x) = \sum_{j=0}^{n} c_j\phi_j(x) = c_0\phi_0(x) + \cdots + c_n\phi_n(x),$$

   where $n = 7$, $\phi_0(x) \equiv 1$,

$$\phi_j(x) = \sqrt{(x - x_{j-1})^2 + \varepsilon^2} - \varepsilon, \quad j = 1, \dots, n.$$

   In addition to the n interpolation requirements, the condition

$$c_0 = -\sum_{j=1}^{n} c_j$$

   is imposed. Construct this interpolant with (i) $\varepsilon = 0.1$, (ii) $\varepsilon = 0.01$ and $\varepsilon = 0.001$. Make as many observations as you can. What will happen if we let $\varepsilon \to 0$?