lecture 12

Def:

① A set U in $\mathbb{R}^n$ ($n = 2$ or 3) is open if for each $P \in U$, some disk/ball centered at P is also in U.

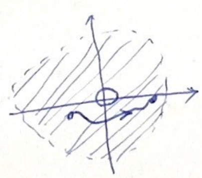
(Disk ↑ 2, ball ↑ 3)

② An open set U, is called path connected in $\mathbb{R}^n$ if any 2 points in U are joining by continuous curve also in U.

③ A path connected set U is simply connected in $\mathbb{R}^n$ if every closed curve in U can be continuously contracted/deformed to a single point in U.

$\underbrace{\text{to a single point}}_{\text{in } U}$

Exp) $\{(x,y) \mid 0 < x^2+y^2 < 1\}$



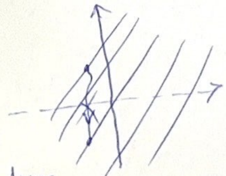
open disk in the plane, not including the boundary, open dot at origin

open ✓ $\xrightarrow{\text{based on ①}}$ if in this we choose a point the point has a distance from boundary and from the origin, and we can a disk around that point

P.C. ✓ Path. Connected → can connect ~~two~~ points

S.C. ✓ is not simply connected → can not continuously shrink it at origin

Exp) $\{(x,y) \mid y \neq 0\}$



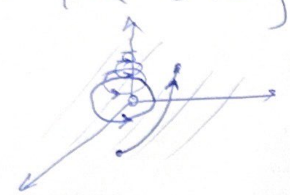
split the plane into 2 ~~blocks~~ halves

open ✓

P.C. X → we can not connect a point in top half to a point in lower half

S.C. X → we can shrink a close loop to a point but by definition, it is not P.C. so it is not S.C.

iii) $\{(x,y,z) \mid 0 < x^2+y^2+z^2\} \rightarrow \mathbb{R}^3 - \text{origin}$



open ✓

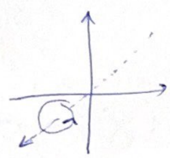
P.C. ✓

S.C. ✓ → e.g. if we have a loop around the origin in x-y plane we can contract the loop in the space to a point by lift up the loop in x-y plane and then shrink it.

(2)

iv) $\{(x,y,z) \mid y^2+z^2 \neq 0\}$

x-axis deleted

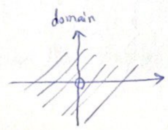


open ✓

P.C. ✓

S.C. X → if a loop be around x-axis can never contract into a point in the set without cutting through x-axis

exp1) $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$;



P.C. ✓

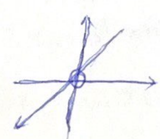
S.C. X

Exp2) $\vec{F} = \left\langle \frac{-x}{(x^2+y^2)^{3/2}}, \frac{-y}{(x^2+y^2)^{3/2}} \right\rangle$

P.C. ✓

S.C. X

Exp3) $\vec{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(\quad)}, \frac{-z}{(\quad)} \right\rangle$



P.C. ✓

S.C. ✓

in case 1 & 2 → Can easily check $(F_1)_y = (F_2)_x$

in case 3 → " " " $(\text{curl}\,\vec{F} = \vec{0})$

Then Theorem (A & B) tells us:

1) No conclusion (will see, not conservative)
2) " " (" ", is conservative)
3) F is conservative

Closer look:

for 1) $\longrightarrow$ can do $\phi_x = \frac{-y}{x^2+y^2} \Rightarrow \phi = -\arctan\left(\frac{x}{y}\right) + f(y)$

$$\Rightarrow F_2 = \frac{x}{x^2+y^2} = \phi_y = \frac{-1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right) + f'(y) = \frac{x}{x^2+y^2} + f'(y)$$

$$\Rightarrow f'(y) = 0$$

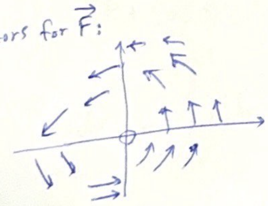
$\Longrightarrow$ $\phi(x,y) = -\arctan\left(x/y\right) + C$ is potential for $\vec{F}$, but not on domain of F!

The domain for this potential is the domain − x-axis
while the domain of F is the whole domain — origin
$\Rightarrow \vec{F}$ has no potential on all its domain

$\Longrightarrow$ F is not conservative on its domain.

Could also see by looking $\vec{F}$ as:

Few vectors for $\vec{F}$:



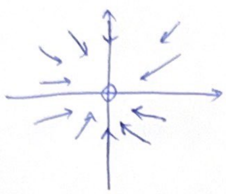
Looks like going circularly (around a circle) conter clockwis.

If C is a circle goes around origin counteeclockwise

Then, $\underbrace{\int_C \vec{F} \cdot dr}_{\neq 0} > 0$ (integrand $F(r(t)) \cdot r'(t) > 0$ as seen from the vectors)
$\cos\theta \neq 0, \theta < 90$

$\Longrightarrow$ $\boxed{\text{F cannot be conservative by theorem (B) since } U \text{ is p.c.}}$

For Exp2)



* the dirction of the field is $(-x)$ and $(-y)$ → at any point the field turns back towards the origin.
* the denomniator ($(x^2+y^2)^{3/2}$) shows the size of vectors
* Can tell you too much from pic
* ϕ : $\phi_x = \frac{-x}{(x^2+y^2)^{3/2}} \Rightarrow \phi = \frac{1}{(x^2+y^2)^{3/2}} + f(y)$

$\int_C \vec{F}\cdot dr = 0$, $\theta = 90$

$\longrightarrow \phi(x,y) = \frac{1}{(x^2+y^2)^{3/2}}$ is potential for F!

$\Longrightarrow$ F is conserative!

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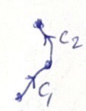
laws of line integration:

* $\int_C (F+G)\cdot dr = \int_C F\cdot dr + \int_C G\cdot dr$
* $\int_{C_1+C_2} F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr$



* $\int_C F\cdot dr = -\int_{-C} F\cdot dr$

→ All are rather obvious, but when combined with notions of "Connectivity" (Path con. or simply con.), get rather striking results!