Lecture 9

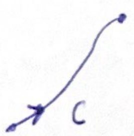
Def: A vector field $\vec{F}(x,y,z) = \langle F_1, F_2, F_3 \rangle (x,y,z)$ is called conservative if $\vec{F}(x,y,z) = \nabla \phi(x,y,z)$ for some scalar function ϕ. i.e.) $F_1 = \phi_x$, $F_2 = \phi_y$, $F_3 = \phi_z$

(ϕ is called a potential of F)

* Obvious function for 2D vector field:

$\vec{F}(x,y) \langle F_1(x,y), F_2(x,y) \rangle$ $\qquad \begin{cases} F_1 = \phi_x \\ F_2 = \phi_y \end{cases}$ $\quad \phi(x,y)$

* Very important class of vector fields:



$$\int_C \nabla\phi \cdot dr = \text{Value of } \phi \text{ (at end point)} - \text{Value of } \phi \text{ (at start point)}$$

Is there a test to determine whether a given $\vec{F}$ is conservative? Find potential?

If we find a potential, then it is conservative.

Exp) Find potential for $\vec{F} = (x+y,\ x-y)$:

Suppose: $\begin{cases} x+y = \phi_x & ① \\ x-y = \phi_y & ② \end{cases}$ Solve for ϕ

$\phi(x,y) = \int (x+y)\,dx$ (where y is treated constant)

① $\phi(x,y) = \frac{x^2}{2} + xy + f(y)$ → some $f(y)$ depends only on y

② $x - y = \phi_y = \left(x^2/2 + xy + f(y)\right)_y = x + f'(y)$.

$\Rightarrow -y = f'(y) \Rightarrow f(y) = -y^2/2 + C$

$\Longrightarrow$ $\boxed{F = \nabla\phi \text{ where } \phi = x^2/2 + xy - y^2/2 + C}$

Exp) Is $F(x,y) = \langle x+y,\ x^2-y \rangle$ Conservative? a potential?

$\begin{cases} x+y = \phi_x & ① \\ x^2-y = \phi_y & ② \end{cases}$ Solve for ϕ

Same way in previous Exp $\xrightarrow{①}$ $\phi(x,y) = \frac{x^2}{2} + xy + f(y)$

② $x^2 - y = \phi_y = x + f'(y)$

$\Rightarrow \boxed{x^2 - x = y + f'(y)}$ → No solution $f(y)$!! → RHS depends only on y, LHS does not.

$\boxed{F \text{ is not conservative!}}$