MATH 340, 2024/25, Term 2, Assignment 2

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January 26, 2025

3. For a nonempty $S \subset \mathbf{R}^n$ and a positive real number $r \in \mathbb{R}$ (and $r > 0$), define the set rS as follows:

$$rS := \{z \in \mathbf{R}^n \mid z = rx, x \in S\}$$

Here rx is the multiplication of the vector $x \in \mathbf{R}^n$ by the scalar $r \in \mathbf{R}$; in your more familiar notation, $r\vec{x}$. The set rS is the set of all points that are obtained by multiplying r with the vectors $x \in S$.
For a given nonempty $S \subset \mathbf{R}^n$ and a given positive number $r > 0$, prove that if S is a convex set then rS is a convex set as well.

Solution:

Since S is convex, for all $x, y \in S$ and $t \in [0, 1]$, we have

$$(1-t)x + ty \in S.$$

Let $w = (1-t)x + ty \in S$. For some $r \in \mathbf{R}^+$, we consider:

$$rw = r(1-t)x + rty = (1-t)(rx) + t(ry).$$

By the definition of rS, if $x, y, w \in S$, then $rx, ry, rw \in rS$.

Therefore, rS is also a convex set.

4. For given two nonempty sets $S_1, S_2 \subset \mathbf{R}^n$, define the operation $S_1 + S_2$ as follows:

$$S_1 + S_2 := \{z \in \mathbf{R}^n \mid \text{ there exist some } x \in S_1 \text{ and some } y \in S_2 \text{ such that } z = x + y\}$$

that is, each point $z \in S_1 + S_2$ is the one that can be expressed as the sum $x + y$ for some $x \in S_1$, and $y \in S_2$; here the sum $x + y$ is the vector sum between the two vectors. One does this for all $x \in S_1$ and $y \in S_2$ and get the set $S_1 + S_2$.

(a) Consider $S_1 = \left\{(x_1, x_2) \in \mathbf{R}^2 \mid |x_1 - 1| \leq 1 \& |x_2 - 2| \leq 1\right\}$
and $S_2 = \left\{(x_1, x_2) \in \mathbf{R}^2 \mid |x_1| \leq 2 \& |x_2| \leq 1\right\}$. Sketch the set $S_1 + S_2$. You do not need to explain your solution for this question. But, your sketch should be neat and very clear, indicating all the relevant coordinate values.
Solution:

(b) Is it true that $S_1 + S_2$ must be convex for **any** nonempty convex sets S_1 and S_2 in $\mathbf{R}^n$? Justify your answer carefully. **[This problem is independent of part (a). The sets S_1, S_2 are arbitrary convex sets in this question, not the particular example**

given in part (a). If you do this problem only for the sets of part (a) or a particular example, you will get zero mark.]

Solution:

Let $z_1, z_2 \in S_1 + S_2$ such that $z_1 = x_1 + y_1, z_2 = x_2 + y_2$ and $x_1, x_2 \in S_1,\ y_1, y_2 \in S_2$. First, let

$$\begin{aligned} w &= w_1 + w_2 \\ &= (1-t)z_1 + tz_2 \\ &= (1-t)(x_1 + y_1) + t(x_2 + y_2) \\ &= \{(1-t)x_1 + tx_2\} + \{(1-t)y_1 + ty_2\}. \end{aligned}$$

for some $t \in [0, 1]$.
Since S_1 and S_2 are non-empty convex sets, we have:

$$w_1 = (1-t)x_1 + tx_2 \in S_1 \quad \text{and} \quad w_2 = (1-t)y_1 + ty_2 \in S_2.$$

By the definition of $S_1 + S_2$, we can write:

$$w = w_1 + w_2 = (1-t)z_1 + tz_2 \in S_1 + S_2, \quad t \in [0, 1],$$

proving that $S_1 + S_2$ is also a convex set.