INSTRUCTOR NOTES

This small class sets up material that will be revisited in MATH 101 (integral calculus). Teams will learn an *ad hoc* definition of geometric series. Remember that it is okay not to provide all the details. A working definition with some big questions left to answer is a reasonable accomplishment.

Begin by getting students to sit in their teams. If possible, members should sit facing each other. Do not allow students to change teams.

If only 2 members show up for a team, they may work with another small team, but must submit their own worksheet. If only 1 member shows up for a team, they must work with another team, and may submit a blank worksheet with only their name on it for attendance-taking purposes.

Finally, pass out the handouts and announce the first question. Remember to have a routine to close questions (*e.g.* countdowns).

At the end of the class, remember to collect worksheets.

This week's tip: ***pay attention to everyone***. Some members of a team may have excellent understanding of the material while other members of the team profess complete ignorance. Pay attention to all members. It may be useful to ask some students on a team to explain an answer to other students.

NOTES ON QUESTIONS

The large lecture prior to this small class is on higher degree approximations, including Taylor polynomials. In this small class, the exploration of geometric series begins with determining the Maclaurin polynomial of $f(x) = \frac{a}{1-x}$.

1. **5 minutes**.
   To close the question, ask a team that finishes early to write their simplified answers on the board.
   $f'(x) = \dfrac{a}{(1-x)^2}$, $f^{(2)}(x) = \dfrac{2a}{(1-x)^3}$, $f^{(3)}(x) = \dfrac{6a}{(1-x)^4}$.

2. **2 minutes**.
   To close the question, ask a team to write their simplified answer on the board. If they use factorial notation, you should also write the answer without factorial notation.
   $f^{(n)}(x) = \dfrac{n!a}{(1-x)^{n+1}} = \dfrac{(n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1)\, a}{(1-x)^{n+1}}$.

3. **5 minutes**.
   Teams may need a reminder of the formula for Maclaurin polynomials. If a few teams ask, write it on the board.
   To close the question, write the answer on the board.
   $a + ax + ax^2 + ax^3 + \cdots + ax^n$.

4. **5 minutes**.
   To close the question, have a team write the answers on the board. Avoid summation notation.
   $S_n = a + ax + ax^2 + ax^3 + \cdots + ax^{n-1}$ and $xS_n = ax + ax^2 + ax^3 + \cdots + ax^{n-1} + ax^n$.

5. **5 minutes**.
   To close the question, have a team write the answer on the board.
   $S_n = \dfrac{a(1-x^n)}{1-x}$.

6. **5 minutes** of instructor-led class discussion.
   There is some sleight of hand in this question; in particular, the precise definition of what it means for a sequence of partial sums to converge is obscured. Ignore it unless students explicitly ask about it.
   To close the question, write the answer on the board.
   $\lim_{n\to\infty} S_n = \dfrac{a}{1-x} = f(x)$.

7. **Remaining time**. Teams may work at their own pace. Remember to visit often to make sure they are on the right track.

   (a) $\dfrac{1}{1-\frac{1}{2}} = 2$

   (b) $\dfrac{4}{1+\frac{1}{3}} = 3$

   (c) $\dfrac{5}{1+\frac{2}{3}} = 2$

SMALL CLASS: Geometric series

In this class, you will use knowledge of higher degree approximations to learn about geometric series, an important topic in integral calculus (MATH 101) as well as in its own right.

Contributing team members

Student number	Last name	First name

Small class questions

Let $f(x) = \dfrac{a}{1-x}$, where $a \neq 0$ is a constant and the variable x has the restriction $|x| < 1$.

1. (★☆☆☆) Calculate $f'(x)$, $f^{(2)}(x)$ and $f^{(3)}(x)$.

Answer:
Scribe:

2. (★☆☆☆) Write down a formula for the n^{th} derivative $f^{(n)}(x)$.

Answer:

Scribe:

3. (★★☆☆) What is the n^{th} degree Maclaurin polynomial of $f(x)$?

Answer:

Scribe:

4. Let S_n denote the $(n-1)^{\text{th}}$ degree Maclaurin polynomial of $f(x)$. (S_n is an n^{th} *partial sum*, a term that is used extensively in integral calculus.) Write down two equations: one for S_n, and one for xS_n.

Answer:

Scribe:

5. Take the difference of the two equations in the previous question and solve for S_n.

Answer:

Scribe:

6. (★☆☆☆) What is $\lim_{n\to\infty} S_n$?

Answer:

Scribe:

7. (★★☆☆) Use the result from question 6 to calculate the following limits.

(a) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \lim_{n\to\infty}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^{n-1}\right)$

(b) $4-\frac{4}{3}+\frac{4}{9}-\frac{4}{27}+\cdots = \lim_{n\to\infty}\left(4+4\left(-\frac{1}{3}\right)+4\left(-\frac{1}{3}\right)^2+4\left(-\frac{1}{3}\right)^3+\cdots+4\left(-\frac{1}{3}\right)^{n-1}\right)$

(c) $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\frac{80}{81}-\frac{160}{243}+\cdots$

Answer:

Scribe:

Practice questions

The questions below are for practice. They do not contribute to your grade, and it is not expected that you complete them during your small class. However, you are strongly encouraged to work through them.

8. (★★☆☆) Use the result from question 6 to calculate the limit

$$2 + 0.5 + 0.125 + 0.03125 + \cdots.$$

9. (★★★☆) Use the result from question 6 to write the number $0.878787\ldots$ as a fraction of integers.
Hint: First write the number as an infinite sum of fractions.

10. (★★★☆) Does the limit of the sum

$$\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

exist? Justify your answer carefully.
Note: For a positive integer n, $n!$ denotes the product of all integers from 1 through n; for example, $4! = 4 \times 3 \times 2 \times 1 = 24$. By convention, $0! = 1$.