MATH 101 2023W2 Learning Objectives

Week 1

(a) Understand how to interpret sigma notation. (This is review from high school – see WeBWorK, not lecture).
- Express sums using sigma notation.
- Manipulate sums using arithmetic properties: constant sums, factoring and addition.

(b) Interpret the definite integral $\int_a^b f(x), \mathrm{d}x$ as signed area when $a < b$.

(c) Understand what an area function of the form $\int_a^x f(t)\,\mathrm{d}t$ is, and compute them for simple functions using geometry

(d) Understand that the areas of curved shapes can be approximated by cutting up those shapes in to many small rectangles and/or triangles.

(e) Evaluate certain definite integrals using geometry and the interpretation of definite integral as "area under the curve."

(f) Given a function, sketch the area function $A(x)$.

(g) Explain using a picture how to approximate area using left or right Riemann sums either theoretically or concretely for a small number of rectangles.

(h) Explain the Trapezoidal rule for approximating areas.

(i) Find approximations of areas using the Trapezoidal rule.

(j) Understand the definition of a definite integral as the limit of a Riemann sum.

(k) Understand why the definite integral sometimes gives negative numbers, even though areas cannot be negative.

Small class:

(l) Given a function, sketch the area function $A(x)$. (review from large class)

(m) Produce a compelling argument that $A(x) = \int_a^x f(t)\ dt$ should satisfy $A'(x) = f(x)$ if f is continuous at x; illustrate what can go wrong if f has a simple jump discontinuity at x.

(n) State the fundamental theorem of calculus part 1.

(o) Use the fundamental theorem of calculus part 1 to differentiate a function defined as a definite integral (area function).

Week 2

(a) Explain using pictures, words, equations, and inequalities, the arithmetic of integrals as well as properties involving the endpoints a and b.

(b) Define the indefinite integral and explain how it differs from the definite integral.

(c) Use FTC1 to prove FTC2

(d) Use the fundamental theorem of calculus part 2 to compute definite integrals.

(e) Explain why anti-derivatives are non unique.

(f) Find anti-derivatives of polynomials using the power rule.

(g) Find anti-derivatives of basic functions by inspection, in particular, those important integrals listed in Theorem 1.3.16.

Small class:

(h) Apply knowledge of integration (approximation via rectangles, anti-differentiation, the fundamental theorem of calculus) in context (i.e. word problems).

Week 3 (a) Explain how the chain rule for derivatives corresponds to the substitution method for antiderivatives.

(b) Use a given substitution to evaluate an indefinite integral.

(c) Show how a given substitution affects the bounds of integration when used with a definite integral.

(d) Recognize when a substitution will simplify a given integral (definite or indefinite), and determine the form of an effective substitution.

(e) Compute integrals where the integrand requires manipulation to reveal an effective substitution.

(f) Compute integrals using a sequence of substitutions.
E.g., $\int \sin^2(x^2)\cos(x^2)[2x]\,dx$.

Small class:

(g) Compute integrals involving powers of sine and cosine by utilizing an appropriate substitution.

(h) Use trigonometric identities, notably, the half-angle formulas to compute integrals involving even powers of sine and cosine. That is, you must know: $\sin^2 x + \cos^2 x = 1$, $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, $\cos^2 x = \frac{1}{2}(1+\cos(2x))$, $\sin(2x) = 2\sin x\cos x$.

(i) Use the definitions of different trigonometric functions to convert integrals into an easier form, where appropriate.

Week 4 (a) Recognize when it's appropriate to use the method of trigonometric substitution when computing an integral.

(b) Identify which substitution and which trig identity is required during trig substitution. In particular, you must know: $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, and $\sec^2 x - 1 = \tan^2 x$.

(c) Compute integrals using trig substitution.

(d) FLAVOUR

- Flavour A: Compute anti-derivatives for functions of the form $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials with the degree of p less than the degree of q and where q can be factored into distinct linear terms.

- Flavour B: TBD
- Flavour C: define surplus, understand definition

Small class: FLAVOUR

(e) Use an integral to represent the volume of a 3D object (providing it has some symmetry). Explain using a picture what each piece of the integral represents.

(f) Find the volume of surfaces of revolution using disks.

(g) Find volume by integrating over cross sectional areas.

Week 5 (a) Explain how the product rule for derivatives corresponds to integration by parts for integrals.

(b) Use integration by parts to compute definite and indefinite integrals.

(c) Identify when integration by parts is an appropriate method to use.

(d) While performing integration by parts, identify which portion of the integral should be "u" and which part should be "$\mathrm{d}v$." This includes the case where $\mathrm{d}x = \mathrm{d}v$.

Small class:

(e) TEST 1

Week 6 (a) Explain why we need numerical methods for integration citing examples of problems we cannot solve with the fundamental theorem of calculus.

(b) Use Simpson's rule to approximate integrals. You are not required to reproduce the derivation of the formula but should be able to explain why n must be an even number.

(c) Explain why we expect Simpson's method to, in general, produce a more accurate results (with the same n) than either of the previous methods.

(d) Given the true value of an integral, compute the error and relative error produced by a numerical calculation.

(e) Given an integral to compute numerically with either the trapezoidal method or Simpson's rule, compute the max error given a particular n. (recall trapezoid rule from Week 1)

(f) When computing an integral numerically with either the trapezoidal method or Simpson's rule, determine a sufficient number of intervals, n that guarantees a desired level of accuracy.

(g) Compare error estimates with true error using a spreadsheet

Small class:

(h) Use numerical integration to compute approximations to definite integrals where the function is not defined explicitly or where the indefinite integral cannot be represented using standard functions.

Week 7 (a) Check whether a given function satisfies a differential equation.

(b) Identify and solve separable differential equations. In particular, find the general solution.

(c) Given an initial condition, find a particular solution that satisfies a separable differential equation (that is, solve the initial value problem).

(d) Notice that the solutions for first-order linear DEs, which we found in Math 100, can be found using separation of variables

(e) Set up and solve a differential equation describing a physical process appropriate to your discipline, such as mixing problems, logistic growth, mortgages. Use the solution to make statements about the original application.

Small class:

(f) Interpret a differential equation in context and use the results to make inferences about your application.

(g) Identify important parameter values that change the qualitative result of your calculations.

Week 8 (a) State the different ways an integral can be improper.

(b) Define what it means to *evaluate* an improper integral. In particular, explain using a picture, what area is being computed and what limit is being taken.

(c) Define what it means for an improper integral to converge or diverge.

(d) Demonstrate the convergence/divergence of $\int \frac{1}{x^p}\,\mathrm{d}x$ for general $p > 0$, with domains $(0, 1]$ and $[1, \infty)$.

(e) Evaluate an improper integral (or prove it diverges) by explicitly writing and computing the appropriate limit.

(f) Use the comparison test to determine convergence/divergence for improper integrals without finding their antiderivatives.

(g) Use the limit comparison test to determine convergence/divergence of improper integrals without finding their antiderivatives.

Small class:

(h) Define and explain the terms: probability, event, value.

(i) (in class, not learning objective: give an example where it's not reasonable to use a discrete variable, but it is reasonable to ask whether a variable takes a value inside a particular range)

(j) Define Probability Density Function (PDF) as the function $f(t)$ such that $Pr(a \leq X \leq b) = \int_a^b f(t)\,\mathrm{d}t$.

(k) Use a PDF to compute probabilities.

(l) Use that definition to conclude properties of PDFs: $f(t) \geq 0$ and $\int_{-\infty}^{\infty} f(t)\mathrm{d}t = 1$

Week 9

(a) Use the properties of PDFs to find unknown parameters in its definition.

(b) Interpret the PDF in terms of relative likelihoods of different regions

(c) Explain what is meant by a "long-term average" and contrast this with the outcome of finitely many experiments.

(d) Define expected value for continuous systems.

(e) Compute the expected value for continuous systems.

(f) For an increasing or decreasing PDF use an intuitive argument to check whether the expected value is more or less than the halfway point of the space.

(g) Define variance and standard deviation

(h) Explain in plain(ish) language what these quantities represent, in reference to their definitions.

(i) Compute standard deviation and variance either using the conventional definition or the alternative formulation: $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$.

Small class:

(j) TEST 2

Week 10

(a) Define sequences and series and, in particular, explain the difference between the two.

(b) Find the limit of a sequence (providing the limit exists) by taking an appropriate limit.

(c) Define partial sum

(d) Explain what it means for a series to converge.

(e) Determine whether a given series is geometric.

(f) Given a geometric series, determine whether it converges or diverges.

(g) Apply the divergence test to determine the divergence of applicable series.

(h) Explain in words why the divergence test works.

(i) Explain why and how the test can be inconclusive.

Small class:

(j) State the conditions required to apply the integral test.

(k) Explain in words and with a picture why the integral test works.

(l) Use the integral test to to determine convergence or divergence of applicable series. In particular, use the integral test to derive the p-test.

(m) Use the integral test to achieve a bound on the tail of a series.

Week 11 (a) State the comparison test and explain why it works.

(b) Given a series, decide if the comparison test is appropriate. If so, determine a good series to use as a comparison.

(c) Apply the comparison test to determine the convergence or divergence of series.

(d) State the limit comparison test and explain why it follows naturally from the comparison test.

(e) Use the limit comparison test to determine whether a series converges or diverges. Supply good candidate series for comparison.

(f) State the ratio test and explain its connection with geometric series.

(g) Apply the ratio test to series when appropriate. In particular, to series involving factorials and/or exponentials.

(h) State when the ratio test is inconclusive and explain what that means.

Small class:

(i) Use the alternating series test to determine convergence of series.

(j) Give a heuristic explanation to justify the alternating series test.

(k) Define both absolute convergence and conditional convergence.

(l) Use absolute convergence to determine the convergence of some series. Explain why "absolute convergence implies conditional convergence" only works one way.

Week 12 (a) Define power series for a function centred at a point.

(b) Explain what is meant by "radius of convergence."

(c) Compute the radius of convergence given a power series.

(d) Translate the radius of convergence together with the centre of a power series to determine the interior of its interval of convergence.

(e) Perform operations on power series as per Theorem 3.5.13 keeping in mind the radius of convergence.

(f) Manipulate known series (for example, the geometric series) to derive power series for difficult-to-evaluate functions (for example, the logarithm) possibly using variable substitution.

(g) Define Taylor series and recognize Taylor series of classical functions.

(h) Explain the utility of representing complicated functions $\left(\text{eg. } \arctan x \text{ or } \int_0^x \sin t^2 \, \mathrm{d}t\right)$ as an infinite sum of polynomials.

(i) Find Taylor series of common functions via the definition.

Small class:

(j) Use Taylor series to efficiently compute limits which have an indeterminate form.

Week 13 (large only)

(a) Estimate the error in approximating a function by finitely many terms in the series.

(b) Use Taylor series to find a series representation of particular values of functions (eg. $\log(1/2)$ or $\arctan 1$).

(c) Use the error estimation formula for alternating series to establish a bound on your approximation (using finitely many terms from the series).