1. Let $S = \{1, \{\}, c\}$ be a sample space. List all possible events.

The list of all possible events is the list of all subsets of the sample space.

Hence, $\{\{\}, \{1\}, \{\{\}\}, \{c\}, \{1, \{\}\}, \{1, c\}, \{\{\}, c\}, \{1, \{\}, c\}\}$ would be the answer.

2. The odds that the number of tosses being odd will be $P(\text{tos, odd}) = \bigcup_{n \in \text{odd}} P(n)$ and since all are disjoint sets we get $\bigcup_{n \in \text{odd}} P(n) = \sum_{n \in \text{odd}} P(n)$

Let's consider the first few cases, ① $P(1) = \frac{1}{6}$ ② $P(3) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$ ③ $P(5) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$ we can see that for n tosses that $P(n) = \left(\frac{5}{6}\right)^{n-1} \cdot \left(\frac{1}{6}\right)$.

$$\sum_{n \in \text{odd}} P(n) = \sum_{n \in \text{odd}} \left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right) = \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{(2k-1)-1} \cdot \left(\frac{1}{6}\right) = \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{2k-2} \cdot \frac{1}{6} = \frac{1}{6} \cdot \frac{1}{1 - \left(\frac{5}{6}\right)^2} = \frac{1}{6} \cdot \frac{1}{\frac{11}{36}} = \frac{1}{6} \cdot \frac{36}{11} = \frac{6}{11} \qquad \therefore \frac{6}{11}$$

3.

(a) Royal flush

- A royal flush is when we have an ace, king, queen, jack, 10 in on hand. Since the sequence of numbers is given, all we need to chose is the symbol of each.

As there are four symbols we get $\binom{4}{1}$ possibilities, and the sample space is $\binom{52}{5}$, the probability is $\frac{\binom{4}{1}}{\binom{52}{5}}$. $P(\text{royal flush}) = \frac{\binom{4}{1}}{\binom{52}{5}} = \frac{4 \cdot 5! \cdot 47!}{52!}$ $\qquad \therefore P(\text{royal flush}) = \frac{4 \cdot 5! \cdot 47!}{52!}$

(b) Straight flush

- This is the case where we five cards of sequential rank, all of the same suit. Since we exclude a royal flush the cases are (5,4,3,2,A), (6,5,4,3,2), (7,6,5,4,3), (8,7,6,5,4), (9,8,7,6,5), (10,9,8,7,6), (J,10,9,8,7), (Q,J,10,9,8), (K,Q,J,10,9) a total of 9 cases. Similar to the above as there are four suits we get $P(\text{Straight flush}) = \frac{9 \cdot \binom{4}{1}}{\binom{52}{5}} = \frac{36 \cdot 47! \cdot 5!}{52!}$.

(c) Flush

A flush is a hand containing five cards of the same suit. Since we have four suits and each have 13 cards we get $_4C_1 \cdot {}_{13}C_5$ instances. Excluding straight · royal flushes there exists $\binom{4}{1} \cdot \binom{13}{5} - (4+36) = 4 \cdot 1287 - 40$

Hence, $P(\text{flush}) = \frac{\binom{4}{1} \cdot 1287 - 40}{\binom{52}{5}}$

(d) Straight

A straight is a hand of 5 cards of sequential rank. First, the cases of five sequential cards is (5,4,3,2,A), (6,5,4,3,2), (7,6,5,4,3), (8,7,6,5,4), (9,8,7,6,5), (10,9,8,7,6), (J,10,9,8,7), (Q,J,10,9,8), (K,Q,J,10,9), (A,K,Q,J,10). Since each card can have one of the four suits we get $10 \times \left(\binom{4}{1}\right)^5$ possibilities. Excluding straight and royal flushes $P(\text{Straight}) = \frac{10 \times \left(\binom{4}{1}\right)^5 - 40}{\binom{52}{5}} = \frac{10200 \cdot 5! \cdot 47!}{52!}$.

(e) two pair

A two pair is a hand of two pairs of a different rank and a card of a third rank. Thus, we have to chose three different ranks each being two cards, two cards and one card, also the suit doesn't matter.

First, let's chose the rank for the one card, $\binom{13}{1}$. Now out of the left over 12 ranks, all we need to choose is two and since we chose pairs of the same size, $\binom{12}{2}$. Also, since the rank can are different within a two card pair $P(\text{two pair}) = \frac{\binom{13}{1} \cdot \binom{4}{1} \cdot \binom{12}{2} \cdot \binom{4}{2} \cdot \binom{4}{2}}{\binom{52}{5}}$.

4.

(a) The ways are simply distributing 13 cards each. Hence, $|A| = \binom{52}{13} \cdot \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13}$

(b) $|A| = \binom{52}{13} \cdot \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13} = 5.36 \cdot 10^{28}$ Hence, it takes $5.36 \cdot 10^{28}$ seconds $= 8.94 \cdot 10^{26}$ minutes $= 1.49 \cdot 10^{25}$ hours $= 6.21 \cdot 10^{23}$ days $= 1.70 \cdot 10^{21}$ years. Much larger than 13.79 billion years.

5.

To get all different numbers we get the permutation of three numbers, $|A| = {}_6P_3$. And $|\Omega| = 6^3$, thus $P = \frac{{}_6P_3}{6^3} = \frac{5}{9}$.

7

We can see that P(exist a pair of same colour) = 1 − P(all different colours). Let's try to find P(all different colours).

When $n<8$, since there are more objects than colours P(all different colours) = 0. Hence, $n \geq 8$. For such n, $|A| = \binom{n}{8} 8!$ and $|\Omega| = n^8$. Therefore, P(all different colours) $= \frac{\binom{n}{8} \cdot 8!}{n^8} = \frac{{}_nP_8}{n^8}$.

Thus, P(exist a pair of same colour) $= 1 - \frac{{}_nP_8}{n^8}$. $P(8) = 1 - \frac{{}_8P_8}{8^8} = 0.998 > 0.25$, $P(16) = 1 - \frac{{}_{16}P_8}{16^8} = 0.879 > 0.25$, $P(32) = 1 - \frac{{}_{32}P_8}{32^8} = 0.614 > 0.25$, $P(64) = 1 - \frac{{}_{64}P_8}{64^8} = 0.366 > 0.25$, $P(128) = 1 - \frac{{}_{128}P_8}{128^8} = 0.2 < 0.25$, $P(96) = 1 - \frac{{}_{96}P_8}{96^8} = 0.2589 > 0.25$

$P(112) = 1 - \frac{{}_{112}P_8}{112^8} = 0.2257 < 0.25$, $P(104) = 1 - \frac{{}_{104}P_8}{104^8} = 0.241 < 0.25$, $P(100) = 1 - \frac{{}_{100}P_8}{100^8} = 0.2469 < 0.25$, $P(98) = 1 - \frac{{}_{98}P_8}{98^8} = 0.254 > 0.25$, $P(99) = 1 - \frac{{}_{99}P_8}{99^8} = 0.2519 > 0.25$. Since, $P(100) < 0.25 < P(99)$ $n \geq 100$.

$n = 100$ would be where $P(n) < 0.25$ first.