Solutions to homework 9:

1. Suppose $f : A \to A$ such that $f \circ f$ is bijective. Is f necessarily bijective?

   - If $f \circ f$ is injective, for $a_1, a_2 \in A$ where $a_1 \neq a_2$, $f(f(a_1)) \neq f(f(a_2))$.
   - This implies that $f(a_1) \neq f(a_2)$ for $a_1 \neq a_2$. Therefore, f is injective.
   - When $f \circ f$ is surjective. $\forall a \in A, \exists b \in A$ such that $f(f(b)) = a$.
   - This also means that $\forall a \in A$, there exists $f(b) \in A$ where $f(f(b)) = a$.
   - Thus, f is surjective and hence f is bijective.

2. Suppose that $f : A \to B$ is a surjection and $Y \subseteq B$. Show that

$$f(f^{-1}(Y)) = Y.$$

   - Let $y \in f(f^{-1}(Y))$, because f is surjective $\exists x \in f^{-1}(Y)$ where $f(x) = y$.
   - When $x \in f^{-1}(Y)$ then $f(x) \in Y$ so $y \in f(Y)$, therefore $f(f^{-1}(Y)) \subseteq Y$.
   - Now let $y \in Y$, because f is surjective $\exists x \in f^{-1}(Y)$ such that $f(x) = y$ is satisfied.
   - When $x \in f^{-1}(Y), f(x) \in f(f^{-1}(Y))$ hence $Y \subseteq f(f^{-1}(Y))$.
   - Therefore, $f(f^{-1}(Y)) = Y$.

3. Let $f : E \to F$ be a function. We recall that for any $A \subseteq E$, the image $f(A)$ of A by f is defined as

$$f(A) = \{f(x) : x \in A\}.$$

   Show that f is surjective if and only if

$$\forall A \subseteq E, F - f(A) \subseteq f(E - A).$$

   - Let f be surjective and let $y \in F - f(A)$, then $\exists x \in E$ such that $f(x) = y$.
   - As we know that $y \in F - f(A)$ we can see that $x \notin A$ making $x \in E - A$.
   - Therefore, $f(x) = y \in f(E - A)$. Hence $F - f(A) \subseteq f(E - A)$.
   - Now let $F - f(A) \subseteq f(E - A)$. And let's choose $y \in F - f(A)$.
   - Thus y also satisfies $y \in f(E - A)$,$\forall A \subseteq E$.
   - This means that $\exists x \in E - A$ such that $f(x) = y$.
   - As we know that $\forall A \subseteq E$, f becomes surjective.

4. (a) Prove that the function $g : \mathbb{R}^2 \to \mathbb{R}, g(x, y) = x^2 - y^2$, is surjective.
   (b) Find $g^{-1}(\{0\})$.
   (c) Let $A := \{a \in \mathbb{R}, a \geq 0$ and consider the function $h : A \to A, h(x) = x^4 + 3$. Find $h^{-1}(\{c\})$ for each c in the codomain.

- (a)
  - For $\forall z \in \mathbb{R}$, choose $x, y \in \mathbb{R}$ such that $x^2 - z \geq 0$ and $y = \sqrt{x^2 - z}$.
  - Then $g(x, y) = g(x, \sqrt{x^2 - z}) = x^2 - (\sqrt{x^2 - z})^2 = x^2 - (x^2 - z) = z$.
  - Hence, g is surjective.
- (b)
  - $g^{-1}(\{0\})$ is the set that satisifies $g(x, y) = 0$.
  - Hence we need to find $(x, y) \in \mathbb{R}^2$ where $x^2 - y^2 = 0$.
  - As $x, y \in \mathbb{R}$,

$$\begin{aligned} x^2 - y^2 &= 0 && (1) \\ x^2 &= y^2 && (2) \\ x &= \pm y && (3) \end{aligned}$$

  - Thus $g^{-1}(\{0\}) = \{(y, y), (-y, y)\} = \{(x, y) \in \mathbb{R}^2 : x = y \text{ or } x = -y\}$.
- (c)
  - We need to find a set X in A that contains x where $h(x) = x^4 + 3 = c$.

$$\begin{aligned} x^4 + 3 &= c && (4) \\ x^4 &= c - 3 && (5) \\ x^2 &= \pm\sqrt{c - 3} && (6) \\ &= \sqrt{c - 3} \qquad (x^2 \geq 0) && (7) \end{aligned}$$

  - Case 1: $c < 3$, We know that $x \in \mathbb{R}$ hence $c - 3 > 0$ needs to be satisfied.
  - Therefore, when $c < 3$ then $h^{-1}(\{c\}) = \emptyset$.
  - Case 2: $c = 3$, then $x^2 = 0$.
  - Therefore $x = 0$ making $h^{-1}(\{c\}) = \{0\}$.
  - Case 3: $c > 3$, then $x = \pm\sqrt[4]{c - 3}$
  - As we know that $x \in A, x > 0$ hence $x = \sqrt[4]{c - 3}$.
  - Hence $h^{-1}(\{c\}) = \{\sqrt[4]{c - 3}\}$.
  - Thus when $c < 3$, $h^{-1}(\{c\}) = \emptyset$
  - And when $c \geq 3$, $h^{-1}(\{c\}) = \{x \in \mathbb{R} : x = \sqrt[4]{c - 3}\}$

5. For a function $f : A \to B$ and subsets E and F of B, prove

$$f^{-1}(E - F) = f^{-1}(E) - f^{-1}(F).$$

- Let $x \in f^{-1}(E - F)$, then $f(x) \in E - F$. Meaning that $f(x) \in E$ and $f(x) \notin F$.
- Hence, $x \in f^{-1}(E)$ and $x \notin f^{-1}(F)$. Which means that $x \in f^{-1}(E) - f^{-1}(F)$.
- Therefore, $f^{-1}(E - F) \subseteq f^{-1}(E) - f^{-1}(F)$.

- Now let $x \in f^{-1}(E) - f^{-1}(F)$, Then we know that $x \in f^{-1}(E)$ and $x \notin f^{-1}(F)$.
- Thus, $f(x) \in E$ and $f(x) \notin F$ therefore $f(x) \in E - F$.
- Showing that $x \in f^{-1}(E - F)$, therefore $f^{-1}(E) - f^{-1}(F) \subseteq f^{-1}(E - F)$.
- To conclude, $f^{-1}(E - F) = f^{-1}(E) - f^{-1}(F)$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + ax + b$, where $a, b \in \mathbb{R}$. Determine whether f is injective and/or surjective.
   - First, we can see that $f(x) = x^2 + ax + b = (x + \frac{a}{2})^2 + b - \frac{a^2}{4}$.
   - Let's choose $y \in \mathbb{R}$ such that $y = b - \frac{a^2}{4} + 1$ and $f(x) = y$.
   - Then

$$f(x) = y = b - \frac{a^2}{4} + 1 \tag{8}$$

$$(x + \frac{a}{2})^2 + b - \frac{a^2}{4} = b - \frac{a^2}{4} + 1 \tag{9}$$

$$(x + \frac{a}{2})^2 = 1 \tag{10}$$

$$x = \pm 1 - \frac{a}{2} \tag{11}$$

   - Hence as we know that $1 - \frac{a}{2} \neq -1 - \frac{a}{2}$ and $f(1 - \frac{a}{2}) = f(-1 - \frac{a}{2})$ is proves that f is not injective.
   - Not let's choose $y = b - \frac{a^2}{4} - 1 \in \mathbb{R}$.
   - Then

$$f(x) = y = b - \frac{a^2}{4} - 1 \tag{12}$$

$$(x + \frac{a}{2})^2 + b - \frac{a^2}{4} = b - \frac{a^2}{4} - 1 \tag{13}$$

$$(x + \frac{a}{2})^2 = -1 \tag{14}$$

   - We know that $x \in \mathbb{R}$, thus no x exists in $\mathbb{R}$ that satisfies this relation.
   - Hence for the chosen y no x exists such that $f(x) = y$.
   - Therefore, f is not surjective.

7. For $n \in \mathbb{N}$, let $A = \{a_1, a_2, a_3, \ldots, a_n\}$ be a fixed set and let F be the set of all functions $f : A \to \{0, 1\}$.
   - (a) What is $|F|$, the cardinality of F?
     - For each element of A we can choose between $0, 1$, hence the cardinality of F is 2^n.

- (b) Let $g : F \to \mathcal{P}(A)$ be defined as $g(f) = \{a \in A : f(a) = 1\}$. We prove that the function is both surjective and injective.
  - Let $B \subseteq \mathcal{P}(A)$, then we know that $B \subseteq A$.
  - Thus, $f_B : A \to \{0, 1\}$, defined such that if $x \in B$, $f(x) = 1$ and if $x \notin B, f(x) = 0$.
  - Therefore, f is well-defined and also we have $g(f_B) = B$, hence g is surjective.
  - Now let $f_1, f_2 \in F$ and assume that $g(f_1) = g(f_2)$, then $\{a \in A : f_1(a) = 1\} = \{a \in A : f_2(a) = 1\}$.
  - Let's call this set X, let $x \in A$ then either $x \in X$ or $x \notin X$.
  - When $x \in X$, we have by definition $f_1(x) = 1 = f_2(x)$. Then when $x \notin X$, $f_1(x) = 0 = f_2(x)$.
  - Thus $\forall x \in A$, we have $f_1(x) = f_2(x)$, hence $f_1 = f_2$.
  - Which implies that g is injective.

8. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ that are injective and such that for all $n \in \mathbb{N}$ we have $f(n) \leq n$.

   - Let's start from $n = 1$, as $f(n) \leq n$ thus $f(1) \leq 1$ and $f : \mathbb{N} \to \mathbb{N}$ therefore $f(1) = 1$.
   - Then for $f(2)$, $f(2) \leq 2$ and we know that f is injective and $f(1) = 1$.
   - Thus $f(2) = 2$.
   - Therefore we can conclude that $f(n) = n$ for all $n \in \mathbb{N}$ when $f(n) \leq n$ and f is injective.