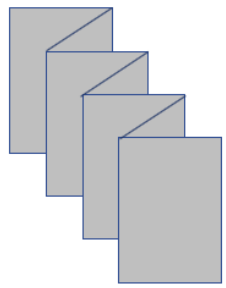
CPSC 320 2023S2: Tutorial 10 Solutions

1 Accordian Folds

Let $\mathcal{P}$ denote a thin strip of paper of length ℓ with n vertical folds, located at distances $f_1, f_2, ..., f_n$ from the left end of P. If you alternately fold $\mathcal{P}$ to the left and right, you get an *accordian fold*, as in the figure below, which has six folds.



1. Suppose $\mathcal{P}$ has length $\ell = 10$ (say in cm) and has folds at $f_1 = 2$, $f_2 = 5$, and $f_3 = 6$. What is the distance (in cm) between the left and right ends when the strip is in an accordian fold that uses all of these folds?

   SOLUTION: Once the strip is folded, if you follow it from the left end it goes right a distance of 2, then left a distance of $5 - 2 = 3$, then right a distance of $6 - 5 = 1$, and finally left a distance of $10 - 6 = 4$ to the right end. So the net total distance between the left and right ends after the strip is folded is

   $$|2 - 3 + 1 - 4| = 4.$$

2. For the same strip as in part 1, suppose that the fold f_2 is not used. Now, what is the distance between the left and right ends when the strip accordian folded along f_1 and f_3?

   SOLUTION: In this case the net total distance is

   $$|2 - 3 - 1 + 4| = 2.$$

3. Is there an accordian fold for this example, that uses a subset of the folds $\{f_1, f_2, f_3\}$, such that the two ends of the strip are aligned, i.e., the net total distance is 0?

   SOLUTION: Yes, simply fold once at $f_2 = 5$.

4. An instance of the Accordian Folds (AF) problem is the length ℓ of the paper strip, plus the positions $f_1, f_2, ..., f_n$ of the folds. The problem is to determine whether there is an accordian fold along a subset of the folds, such that the two ends are aligned. Show that AF is in NP.

   (a) First, come up with a prescise representation of a potential solution, or certificate, for an instance of AF (that can serve as input to a certification algorithm).

   SOLUTION: There are several promising options here. We could describe a certificate as a bit vector $(d_1, d_2, ..., d_n)$, where d_i is either 0, indicating that there should not be a fold at f_i, or 1, indicating that there should be a fold at f_i.

   Alternatively, the entries of certificate vector could be $+1$ or -1, with the ith entry indicating whether the ith term in the net total sum is positive or negative. That is, the net total distance is

$$\sum_{i=0}^{n} d_i(f_{i+1} - f_i), \tag{1}$$

   where we define $d_0 = 1$, $f_0 = 0$, and $f_{n+1} = \ell$. For $1 \leq i \leq n$, there is a fold at f_i if and only if $d_i \neq d_{i-1}$. We'll use this specification of a certificate in what follows; you might have found very different, but equally valid specifications.

   (b) Describe an efficient (polynomial time) certification algorithm for AF.

   SOLUTION: The algorithm takes as input an instance $I = (f_1, f_2, ..., f_n, \ell)$ of AF, and a certificate, or potential solution $(d_1, d_2, ..., d_n)$ where each $d_i \in \{+1, -1\}$. To check whether the certificate is good, the algorithm computes the net total sum using Equation **??**. The algorithm outputs "Yes" if the net total sum is 0, and "No" otherwise.

2 Reductions: Vertex Cover and Dominating Sets

Your task is to find a reduction from Vertex Cover to Dominating Sets, two famous graph problems:

- *Vertex Cover:* An instance is a graph $G = (V, E)$ and an integer K. The problem asks: Is there a vertex cover with at most K vertices in G? Here, a vertex cover is a subset W of V such that $|W| \leq K$, such that every edge in E has at least one endpoint in W.
- *Dominating Set:* An instance is a graph $G = (V, E)$ and an integer K. The problem asks: Is there a dominating set with at most K vertices in G? Here, a dominating set is a subset W of V such that $|W| \leq K$, such that every element of $V - W$ is joined by an edge to an element of W.

1. Give a reduction from Vertex Cover to Dominating Set. Explain why your reduction is correct, and runs in polynomial time.

   SOLUTION: Given an instance (G, K) of Vertex Cover, construct an instance (G', K') of Dominating Set by adding one node v_e for each edge $e \in E$, and connecting this node to the endpoints of e. That is, if $e = \{x, y\}$ then we add the edges $\{v_e, x\}$ and $\{v_e, y\}$.

   Choose K' to be K plus the number of isolated nodes of G.

   The time to generate the new nodes and edges is $O(m)$.

   Suppose that G has a vertex cover W of size at most K. Since W is a vertex cover of G, (i) all non-isolated nodes in $V - W$ are joined by an edge to the nodes of W and (ii) all nodes v_e are also joined by an edge to the nodes of W. So the set W' which contains all nodes of W, plus the isolated nodes of G, is a dominating set of G' and has size K'.

   Conversely, let W' be a dominating set of G'. Let W be obtained by W' by discarding isolated nodes, and replacing any v_e node in W' by one of e's endpoints. Then the nodes of W are all nodes of G, and have an edge to all of the v_e nodes. This means that every edge of G has at least one endpoint in W, and so W must be a vertex cover of G. [Note: Without the extra v_e nodes, this part of the reduction correctness would not hold.]

2. Suppose someone tells you that they have an $O(n^6)$ algorithm to solve the Dominating Set problem on a graph with n vertices. What can you say about the time complexity $T(n, m)$ of the Vertex Cover problem on graphs with n nodes and m edges? Choose all answers that apply.

   SOLUTION:

○ $T(n, m) \in O((n + m)^2)$	● $T(n, m) \in O((n + m)^6)$
○ $T(n, m) \in \Omega((n + m)^2)$	○ $T(n, m) \in \Omega((n + m)^6)$

   This is because the graph G' has $n + m$ nodes.