1. $(1+x^3)y'' - 6xy = 0$

(a)

Guess $y=\sum_{n=0}^{\infty} a_n \cdot x^n$, $(1+x^3)\cdot\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1)\cdot x^{n-2} - 6x \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$

$$= \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n+1} + \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2} - 6 \sum_{n=0}^{\infty} a_n \cdot x^{n+1}$$

$$= \sum_{n=3}^{\infty} a_{n-1} \cdot (n-1)(n-2) \cdot x^n + \sum_{n=0}^{\infty} a_{n+2} \cdot (n+2)(n+1) \cdot x^n - 6 \cdot \sum_{n=1}^{\infty} a_{n-1} \cdot x^n$$

$$= \sum_{n=3}^{\infty} \left[a_{n-1} \cdot \{(n-1)(n-2) - 6\} + a_{n+2} \cdot (n+2)(n+1) \right] x^n$$

$$+ 2a_2 + 6a_3 x + 12a_4 \cdot x^2 - 6a_0 x - 6a_1 x^2$$

x^0 : $2a_2 = 0 \rightarrow a_2 = 0$

x^1 : $(6a_3 - 6a_0)x = 0 \rightarrow a_0 = a_3$

x^2 : $(12a_4 - 6a_1)x^2 = 0 \rightarrow a_1 = 2a_4$

x^n, $n \geq 3$: $a_{n-1} \cdot \{(n-1)(n-2) - 6\} + a_{n+2} \cdot (n+2) \cdot (n+1) = 0$

$$a_{n+2} = \frac{6-(n-1)(n-2)}{(n+1)(n+2)} \cdot a_{n-1}$$

Using the recurrence relation.

$$y(x) = \sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 x + a_0 x^3 + \frac{1}{2} a_1 x^4 - \frac{1}{14} a_1 x^7 + \cdots$$

$$= a_0 (1 + x^3 + \cdots) + a_1 (x + \frac{1}{2} x^4 - \frac{1}{14} x^7 + \cdots)$$

(b)

$$\lim_{n\to\infty} \left| \frac{a_{n+2} \cdot x^{n+2}}{a_{n-1} \cdot x^{n-1}} \right| = \lim_{n\to\infty} \left| \frac{6-(n-1)(n-2)}{(n+1)(n+2)} \right| \cdot |x^3|, \quad |x^3| < 1 \longrightarrow \rho = 1.$$

When expanded by x_0, the radius of convergence, ρ, is at least the distance from x_0 to the closest singular point.

The singular points for this ODE is when $x^3+1=0$, hence $(e^{r\theta})^3 = 1 \cdot e^{i\cdot(\pi+2n\pi)}$ thus $r=1$, $\theta = \frac{\pi+2n\pi}{3}$, $n \in \mathbb{Z}$. Since $r=1$ all singular points are at distance 1 away from $x_0=0$.

Thus, it shows that $\rho \geq 1$.

2. $(1+x^2)y'' + xy' - y = 0$

(a) Guess $y(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$, then $(1+x^2)y'' + xy' - y = (1+x^2) \cdot \sum_{n=2}^{\infty} a_n \cdot n(n-1) x^{n-2} + x \cdot \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} - \sum_{n=0}^{\infty} a_n \cdot x^n$

$$= \sum_{n=2}^{\infty} a_n \cdot n(n-1) \cdot x^{n-2} + \sum_{n=2}^{\infty} a_n \cdot n(n-1) \cdot x^n + \sum_{n=1}^{\infty} a_n \cdot n \cdot x^n - \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$= \sum_{n=0}^{\infty} a_{n+2} \cdot (n+2)(n+1) \cdot x^n + \sum_{n=2}^{\infty} a_n \cdot n(n-1) \cdot x^n + \sum_{n=1}^{\infty} a_n \cdot n \cdot x^n - \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$= 2a_2 + 6a_3 x + a_1 x - a_0 - a_1 x + \sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1) \cdot x^n + (n+1)(n-1) a_n x^n$$

$$= -a_0 + 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} x^n \cdot (n+1) \cdot \{(n+2) a_{n+2} + (n-1) a_n\}$$

$x^0: -a_0 + 2a_2 = 0$, $x^1: 6a_3x = 0 \rightarrow a_3 = 0$. a_1 is arbitrary, $\underset{n\geq 2}{x^n}: a_{n+2} = -\frac{a_n(n-1)}{(n+2)}$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \frac{1}{2}a_0 x^2 + 0\cdot x^3 + \left(-\frac{1}{4}\right)\cdot\left(\frac{1}{2}a_0\right)x^4 + \cdots = a_1 x + a_0\left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots\right)$$

(b) $\lim_{n\to\infty}\left|\frac{a_{n+2}\cdot x^{n+2}}{a_n \cdot x^n}\right| = \lim_{n\to\infty}\left|\frac{-(n-1)}{n+2}\cdot x^2\right| = |x^2| < 1 \Rightarrow -1 < x < 1$, $\therefore \rho = 1$

Similar to Q(1)-b, the singular points are when $x^2+1=0$, thus $x = \pm i$. $i, -i$ are both distance 1 from $x_0 = 0$, hence $\rho \geq 1$. which also agrees with our calculations above.

(c) $y(0) = a_0 = 1$, $y'(0) = a_1 = 0$

$\therefore y(x) = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots$