1 $M_X(t) = E[e^{Xt}] = \frac{1}{4}e^{-3t} + \frac{1}{2} + \frac{1}{4}e^{t}$

(a) $E[X] = M^{(1)}(0)$, $M_X'(t) = -\frac{3}{4}e^{-3t} + \frac{1}{4}e^{t} \Rightarrow M_X'(0) = E[X] = -\frac{1}{2}$ $\quad \therefore E[X] = -\frac{1}{2}$

$E[X^2] = M^{(2)}(0)$, $M_X''(t) = \frac{9}{4}e^{-3t} + \frac{1}{4}e^{t} \Rightarrow M_X''(0) = E[X^2] = \frac{5}{2}$, $Var(X) = E[X^2] - E[X]^2 = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}$ $\quad \therefore Var(X) = \frac{9}{4}$

(b) $E[e^{Xt}] = \frac{1}{4}e^{-3t} + \frac{1}{2} + \frac{1}{4}e^{t} = \frac{1}{4}e^{-3t} + \frac{1}{2}\cdot e^{0t} + \frac{1}{4}e^{t}$

p.m.f

$$P(X=k) = \begin{cases} \frac{1}{4}, & k=-3 \\ \frac{1}{2}, & k=0 \\ \frac{1}{4}, & k=1 \end{cases}$$

$E[X] = -\frac{3}{4} + 0 + \frac{1}{4} = -\frac{1}{2}$

$E[X^2] = \frac{9}{4} + 0 + \frac{1}{4} = \frac{5}{2}$

$Var(X) = E[X^2] - E[X]^2 = \frac{9}{4}$ as required.

2 Both X_1 and X_2 are independent

(a) $P_{X_1,X_2}(m,n) = P(X_1=m, X_2=n) = P(X_1=m)\cdot P(X_2=n) = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$ $(m \in \{0,1,2\}, n \in \{0,1,2,3\})$

(b) $Y_1 = X_1 \cdot X_2$

$Y_1 = 0$ when $(X_1, X_2) = (0,0), (0,1), (0,2), (0,3), (1,0), (2,0)$

$Y_1 = 1$ when $(X_1, X_2) = (1,1)$

$Y_1 = 2$ when $(X_1, X_2) = (1,2), (2,1)$

$Y_1 = 3$ when $(X_1, X_2) = (1,3)$

$Y_1 = 4$ when $(X_1, X_2) = (2,2)$

$Y_1 = 6$ when $(X_1, X_2) = (2,3)$

$Y_2 = \max\{X_1, X_2\}$

$Y_2 = 0$ when $(X_1, X_2) = (0,0)$

$Y_2 = 1$ when $(X_1, X_2) = (0,1), (1,0), (1,1)$

$Y_2 = 2$ when $(X_1, X_2) = (0,2), (1,2), (2,1), (2,0), (2,2)$

$Y_2 = 3$ when $(X_1, X_2) = (0,3), (1,3), (2,3)$

$Y_1 \backslash Y_2$	0	1	2	3
0	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$
1	0	$\frac{1}{12}$	0	0
2	0	0	$\frac{2}{12}$	0
3	0	0	0	$\frac{1}{12}$
4	0	0	$\frac{1}{12}$	0
5	0	0	0	0
6	0	0	0	$\frac{1}{12}$

(c) Are Y_1 and Y_2 independent

$P(Y_1=0, Y_2=0) = \frac{1}{12}$

$P(Y_1=0) = \frac{1}{2}$, $P(Y_2=0) = \frac{1}{12}$ $\quad \frac{1}{2}\cdot\frac{1}{12} \neq \frac{1}{12}$ so $P(Y_1=0 \cap Y_2=0) \neq P(Y_1=0)\ P(Y_2=0)$ so Y_1, Y_2 are not independent.

4. $X \sim Geom(p)$

$$M_X(t) = E[e^{Xt}] = \sum_{k=1}^{\infty} p(1-p)^{k-1}\cdot e^{kt} = p\cdot e^{t}\cdot\sum_{k=1}^{\infty}(1-p)^{k-1}\cdot e^{(k-1)t} = p\cdot e^{t}\cdot\sum_{k=1}^{\infty}((1-p)e^{t})^{k-1} = \frac{pe^{t}}{1-(1-p)e^{t}}, \quad |(1-p)e^{t}| < 1.$$

5.

(a) $f(X) \in B, g(Y) \in B' \longrightarrow X \in \{x : f(x) \in B\}, Y \in \{y : g(y) \in B'\}$

$P(f(X) \in B, g(Y) \in B') = P(X \in \{x : f(x) \in B\}, Y \in \{y : g(y) \in B'\}) = P(X \in \{x : f(x) \in B\}) \cdot P(Y \in \{y : g(y) \in B'\}) = P(f(X) \in B) \cdot P(g(Y) \in B')$ as required.

(b) $M_{X+Y}(t) = E[e^{(X+Y)t}]$ since X, Y are independent, e^{Xt}, e^{Yt} are also independent from part a. $E[e^{(X+Y)t}] = E[e^{Xt}\cdot e^{Yt}] = E[e^{Xt}]\cdot E[e^{Yt}] = M_X(t)\cdot M_Y(t) = M_{X+Y}(t)$

(c) $X \sim Pois(\mu)$. $M_X(t) = E[e^{Xt}] = \sum_{k=0}^{\infty} e^{kt}\cdot\frac{e^{-\mu}\cdot\mu^k}{k!} = e^{-\mu}\cdot\sum_{k=0}^{\infty}\frac{(\mu\cdot e^{t})^k}{k!} = e^{-\mu}\cdot e^{\mu\cdot e^{t}} = e^{\mu(e^{t}-1)}$ and similarly when $Y \sim pois(\lambda)$, $M_Y(t) = e^{\lambda(e^{t}-1)}$

$M_{X+Y}(t) = M_X(t)\cdot M_Y(t)$ (from part b) $= e^{(\mu+\lambda)(e^{t}-1)}$. $W \sim Pois(\mu+\lambda) \to M_W(t) = \sum_{k=0}^{\infty}\frac{e^{-(\mu+\lambda)}\cdot(\mu+\lambda)^k\cdot e^{kt}}{k!} = e^{-(\mu+\lambda)}\cdot\sum_{k=0}^{\infty}\frac{\{(\mu+\lambda)\cdot e^{t}\}^k}{k!} = e^{-(\mu+\lambda)}\cdot e^{(\mu+\lambda)e^{t}} = e^{(\mu+\lambda)(e^{t}-1)} = e^{\mu(e^{t}-1)}\cdot e^{\lambda(e^{t}-1)} = M_X(t)\cdot M_Y(t) = M_{X+Y}(t)$

(d) $X \sim N(a,b)$, $Y \sim N(c,d)$

$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$, $M_X(t) = E[e^{Xt}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi b}} \cdot e^{\frac{-(x-a)^2}{2b}} \cdot e^{xt}\, dx = \frac{1}{\sqrt{2\pi \cdot b}} \cdot \int_{-\infty}^{\infty} e^{\frac{-x^2+2ax-a^2+2bxt}{2b}} dx = \frac{1}{\sqrt{2\pi \cdot b}} \cdot \int_{-\infty}^{\infty} e^{\frac{-(x^2-2(a+bt)+a^2)}{2b^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \cdot b}} \cdot e^{\frac{-(x-(a+bt))^2+(a+bt)^2-a^2}{2b}} dx = e^{\frac{(a+bt)^2-a^2}{2b}} = e^{\frac{2abt+b^2t^2}{2b}} = e^{at+\frac{1}{2}bt^2}$

Similarly, $M_Y(t) = e^{ct+\frac{1}{2}dt^2}$. Hence, $M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = e^{(a+c)t+\frac{1}{2}(b+d)t^2}$

$W \sim N(a+c, b+d)$. $M_W(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \cdot (b+d)}} \cdot e^{\frac{-(x-(a+c))^2}{2(b+d)}} \cdot e^{xt}\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \cdot (b+d)}} \cdot e^{\frac{-x^2+\{2(a+c)+2(b+d)t\}x-(a+c)^2}{2(b+d)}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \cdot (b+d)}}\, e^{\frac{-[x-\{(a+c)+(b+d)t\}]^2+(a+c)^2+2(a+c)(b+d)t+(b+d)^2t^2-(a+c)^2}{2(b+d)}} = e^{(a+c)t+\frac{1}{2}(b+d)t}$

$= M_{X+Y}(t)$

6. $X \sim Exp(\mu)$. p.d.f of $Y = \ln(X)$.

$P(X>t) = \int_t^{\infty} \mu \cdot e^{-\mu x}\, dx = [-e^{-\mu x}]_t^{\infty} = e^{-\mu t}$

c.d.f
$P(X \le t) = 1 - e^{-\mu t}$. $P(Y \le t) = P(\ln(X) \le t) = P(X \le e^t) = 1 - e^{-\mu e^t}$. $f_Y(t) = \frac{d}{dt}\{1 - e^{-\mu e^t}\} = -e^{-\mu e^t} \cdot (-\mu e^t) = \mu e^{t-\mu e^t}$ $\therefore f_Y(t) = \mu e^{t-\mu e^t}$

7. $X \sim Bin(n,p)$

Let $X = \sum_{i=1}^{n} \xi_i$ where ξ_i is i.i.d Bern(p). $M_X(t) = M_{\sum_{i=1}^{n} \xi_i}(t) = \prod_{i=1}^{n} M_{\xi_i}(t) = \{M_{\xi_1}(t)\}^n$. $M_{\xi_1}(t) = E[e^{\xi_1 t}] = (1-p) \cdot e^0 + p \cdot e^t = p \cdot e^t + (1-p) = p(e^t-1)+1$.

$M_X(t) = \{p(e^t-1)+1\}^n$. $E[X] = M_X'(0)$, $M_X'(t) = n \cdot \{p(e^t-1)+1\}^{n-1} \cdot pe^t \longrightarrow M_X'(0) = np$. $E[X^2] = M_X''(0)$, $M_X''(t) = np \cdot e^t \cdot \{p(e^t-1)+1\}^{n-1} + n(n-1)(pe^t)^2\{p(e^t-1)+1\}^{n-2} \longrightarrow M_X''(0) = np + n(n-1)p^2$

$Var(X) = E[X^2] - E[X]^2 = np + n(n-1)p^2 - (np)^2 = np - np^2 = np(1-p)$

$\therefore Var(X) = np(1-p)$

$E[X] = np$

13. X_1, X_2 are independent, $Exp(\lambda)$ r.v.. Let $W_1 := \min\{X_1, X_2\}$, $W_2 := \max\{X_1,X_2\} - \min\{X_1,X_2\}$. Show $W_1 \sim exp(2\lambda)$, $W_2 \sim exp(\lambda)$ and W_1, W_2 are independent

W_1: $P(W_1>t) = P(\min\{X_1,X_2\}>t) = P(X_1>t, X_2>t) = P(X_1>t) \cdot P(X_2>t)$ · when $t<0$, $P(W_1>t) = 1 = P(exp(2\lambda)>t) = 1$.

· when $t \ge 0$, $P(X_1>t) \cdot P(X_2>t) = e^{-\lambda t} \cdot e^{-\lambda t} = e^{-2\lambda t} = P(Exp(2\lambda)>t) \longrightarrow \therefore W_1 \sim Exp(2\lambda)$

W_2: $P(W_2=0) = P(X_1=X_2) = 0$, $P(W_2<0) = 0$ since $W_2 := \max\{X_1,X_2\} - \min\{X_1,X_2\}$.

Let $X_1<X_2$ and $X_1=t$ so $X_1=t<X_2$. Then we are getting the distribution of X_2-t given that $X_1=t$ by memoryless property $W_2 = X_2 - t \sim Exp(\lambda)$ and by symmetry $W_2 = \max\{X_1,X_2\} - \min\{X_1,X_2\} \sim Exp(\lambda)$

$P[W_2>t] = P[\max\{X_1,X_2\} - \min\{X_1,X_2\} > t] = P[|X_1-X_2|>t]$. Let $Z := |X_2-X_1| \ge 0$, so for $t<0$, $P[Z>t]=1$.

For $t \ge 0$, $P[W_2>t] = P[X_1>X_2+t] + P[X_2>X_1+t]$

$P[X_1>X_2+t] = \int_0^{\infty}\int_{y+t}^{\infty} f_{X_1,X_2}(x,y)\, dx\, dy = \int_0^{\infty}\int_{y+t}^{\infty} \lambda^2 e^{-\lambda x} \cdot e^{-\lambda y}\, dx\, dy = \int_0^{\infty} \lambda \cdot e^{-\lambda y}\, [-e^{-\lambda x}]_{y+t}^{\infty} dy = \int_0^{\infty} \lambda \cdot e^{-\lambda y} \cdot e^{-\lambda(y+t)} dy = e^{-\lambda t} \cdot \int_0^{\infty} \lambda\, e^{-2\lambda y} dy = e^{-\lambda t} \cdot [-\frac{1}{2} e^{-2\lambda y}]_0^{\infty} = \frac{1}{2} e^{-\lambda t} = P[X_2>X_1+t]$ by symmetry.

$P[W_2>t] = 2 \cdot \frac{1}{2} e^{-\lambda t} = e^{-\lambda t} \longrightarrow f_{W_2}(t) = \frac{d}{dt}\{1 - e^{-\lambda t}\} = \lambda e^{-\lambda t}$ so $W_2 \sim Exp(\lambda)$.

$P[W_1>s, W_2>t] = P[X_1>s, X_2>t+X_1] + P[X_2>s, X_1>t+X_2]$

$P[X_1>s, X_2>t+X_1] = \int_s^{\infty}\int_{x+t}^{\infty} f_{X_1X_2}(x,y)\, dy\, dx = \int_s^{\infty}\int_{x+t}^{\infty} \lambda^2 e^{-\lambda x} e^{-\lambda y}\, dy\, dx = \int_s^{\infty} \lambda \cdot e^{-\lambda x} \cdot e^{-\lambda(x+t)} dx = e^{-\lambda t} \cdot \int_s^{\infty} \lambda \cdot e^{-2\lambda x} dx = \frac{1}{2} \cdot e^{-\lambda t} \cdot e^{-2\lambda s}$

$P[X_2>s, X_1>t+X_2] = \int_s^{\infty}\int_{y+t}^{\infty} f_{X_1X_2}(x,y)\, dx\, dy = \int_s^{\infty}\int_{y+t}^{\infty} \lambda^2 e^{-\lambda x} e^{-\lambda y}\, dx\, dy = \int_s^{\infty} \lambda \cdot e^{-\lambda y} \cdot e^{-\lambda(y+t)} dy = e^{-\lambda t} \cdot \int_s^{\infty} \lambda \cdot e^{-2\lambda y} dy = \frac{1}{2} \cdot e^{-\lambda t} \cdot e^{-2\lambda s}$

Hence, $P[W_1>s, W_2>t] = e^{-\lambda t} \cdot e^{-2\lambda s} = P[W_1>s] \cdot P[W_2>t]$ hence independent.