Solutions to homework 1:

1. Your solution to question 1.

   Proof. Let $n \in \mathbb{Z}$, Prove that if $3 \mid n+1$ then $3 \nmid n^2+5n+5$.

   - Assume for some $n \in \mathbb{Z}, 3 \mid n+1$
   - Then $(n+1) = 3\ell$ for $\ell \in \mathbb{Z}$
   - $n^2+5n+5 = (n+1)^2 + 3(n+1) + 1 = (3\ell)^2 + 3(3\ell) + 1$
   - By fact, $(3l)^2 + (3l) \in \mathbb{Z}$ so $n^2 + 5n + 5 = 3(3\ell^2 + 3\ell) + 1$
   - This shows that $3 \nmid n^2 + 5n + 5$

   □

2. Your solution to question 2.

   Proof. Let $a \in \mathbb{Z}$. Prove that if $5a + 11$ is odd then $9a + 3$ is odd.

   - Let's assume that $5a + 11$ is odd, then we can see that $5a + 11 = 2\ell + 1$ for $\ell \in \mathbb{Z}$.
   - $9a + 3 = (5a + 11) + (4a - 8) = (2\ell + 1) + (4a - 8) = 2(\ell + 2a - 4) + 1$
   - As it is known by fact that $2\ell + 2a - 4 \in \mathbb{Z}$, we can conclude that 9a+3 is odd.

   □

3. Your solution to question 3.

   Proof. If $-1 < x < 2$, then $x^2 - x - 2 < 0$.

   - Assume that $-1 < x < 2$.
   - We change the expression $x^2 - x - 2$ to $(x-2)(x+1)$.
   - For some $x \in (-1, 2)$, we know that $(x-2) < 0$ and $(x+1) > 0$.
   - This shows that $(x-2)(x+1) < 0$ because we know that for some $a, b \in \mathbb{R}$ if $ab < 0$ then $a < 0, b > 0$ or $a > 0, b < 0$.
   - Therefore, we can conclude that if $-1 < x < 2$ then $x^2 - x - 2 < 0$

   □

4. Your solution to question 4.

   Proof. Let a, b, c, d be integers. Suppose that $a, c, b + d$ are all odd numbers. Prove $ab + cd$ is odd.

   - Let's assume that a, b, c, d be integers and $a, c, b + d$ are all odd numbers.

- Then we can see that $a = 2\ell + 1, c = 2k + 1, b + d = 2m + 1$. for $\ell, k, m \in \mathbb{Z}$.
- When $b + d$ is odd, we can express $b + d$ as a sum of an even number and an odd number.
- For instance, let's consider the case when b is even and d is odd.
- Then we can express these as $b = 2n$ and $d = 2p + 1$ for $n, p \in \mathbb{Z}$.
- We can see that $ab+cd = (2\ell+1)*(2n)+(2k+1)*(2p+1) = 2(2nl+n+2kp+k+p)+1$
- As we know that $n, l, k, p \in \mathbb{Z}$, $2nl + n + 2kp + k + p \in \mathbb{Z}$.
- Therfore, from the form $ab + cd = 2(2nl + n + 2kp + k + p) + 1$ we can conclude that $ab + cd$ is odd.
- And this also is satisfied even for the case when b is an odd and d is an even number.

$\square$

5. Your solution to question 5.

Proof. Let x and y be real numbers. Show that

$$xy \leq \tfrac{1}{2}(x^2 + y^2)$$

- Let's assume that $x, y \in \mathbb{R}$
- $\frac{1}{2}x^2 + \frac{1}{2}y^2 - xy = \frac{1}{2}(x^2 - 2xy - y^2) = \frac{1}{2}(x - y)^2$
- We know for some $n \in \mathbb{R}$ that $n^2 \geq 0$.
- As $x, y \in \mathbb{R}$ we also can find out that $(x - y) \in \mathbb{R}$, thus $(x - y)^2 \geq 0$.
- If $(x - y)^2 \geq 0$, then $\frac{1}{2}(x - y)^2 = \frac{1}{2}x^2 + \frac{1}{2}y^2 - xy \geq 0$.
- Therefore, we can conclude that $\frac{1}{2}x^2 + \frac{1}{2}y^2 \geq xy$.

$\square$

6. Your solution to question 6.

Proof. Let x and y be real numbers. Suppose that $x < y$ and $y^2 < x^2$. Show that $x + y < 0$.

- Let's assume that $x < y$ and $y^2 < x^2$ for $x, y \in \mathbb{R}$.
- $x < y$ shows that $x - y < 0$. And $y^2 < x^2$ shows that $x^2 - y^2 > 0$.
- $x^2 - y^2 = (x - y)(x + y)$ and as we know that $x - y < 0$ then in order to satisfy $x^2 - y^2 > 0$, $x + y < 0$ has to be satisfied.
- This is due to the fact that for $a, b \in \mathbb{R}$. If $a > 0$ then $b > 0$, and if $a < 0$ then $b < 0$ to make $ab > 0$.

- Notice that if $x, y \in \mathbb{R}$ then both $x - y, x + y \in \mathbb{R}$
- Therefore, $x + y < 0$ when $x < y$ and $y^2 < x^2$.

$\square$

7. Your solution to question 7.

Proof. For an integer n, prove that if $5 \mid (n+7)$, then $5 \mid (n^2+1)$.

- Let's assume that for $n \in \mathbb{Z}$, $5 \mid (n+7)$.
- Then, $n + 7 = 5\ell$ for some $\ell \in \mathbb{Z}$.
- $n^2 + 1 = (5\ell - 7)^2 + 1 = 5(5\ell^2) - 5(14\ell) + 50 = 5(5\ell^2 - 14\ell + 10)$.
- We know that as $\ell \in \mathbb{Z}$, $5\ell^2 - 14\ell + 10 \in \mathbb{Z}$.
- Then, $n^2 + 1 = 5(5\ell^2 - 14\ell + 10)$ showds that $5 \mid (n^2 + 1)$.
- Therefore, we can conclude that if $5 \mid (n+7)$, then $5 \mid (n^2+1)$.

$\square$

8. Your solution to question 8.

Proof. Let $n, a, b, x, y \in \mathbb{Z}$ with $n > 0$. Prove that if $n \mid a$ and $n \mid b$ then $n \mid (ax + by)$.

- Let $n \mid a$ and $n \mid b$ for $n, a, b, x, y \in \mathbb{Z}$ and $n > 0$.
- Then $a = nl$, $b = nk$ for $l, k \in \mathbb{Z}$.
- $ax = nlx, by = nky$ we know that $ax + by = n(lx + ky)$.
- We can see that $lx + ky \in \mathbb{Z}$, because $l, k, x, y \in \mathbb{Z}$.
- Therefore, $ax + by = n(lx + ky)$ shows that $n \mid ax + by$, when $n > 0$.

$\square$