## Solutions to homework 4:

1. Prove that for every integer $n \geq 0$, the sum $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

- By Euclidean division $n = 3k, n = 3p+1, n = 3q+2$ when $k, p, q \in \mathbb{Z}$. And let $k, q, p \geq 0$.
- Case 1: $n = 3k$,

$$\begin{aligned} n^3 + (n+1)^3 + (n+2)^3 &= (3k)^3 + (3k+1)^3 + (3k+2)^3 && (1)\\ &= 81k^3 + 81k^2 + 45k + 9 && (2)\\ &= 9(9k^3 + 9k^2 + 5k + 1) && (3)\end{aligned}$$

- We know that $9k^3 + 9k^2 + 5k + 1 \in \mathbb{Z}$, thus $9 \mid \big(n^3 + (n+1)^3 + (n+2)^3\big)$.
- Case 2: $n = 3p+1$,

$$\begin{aligned} n^3 + (n+1)^3 + (n+2)^3 &= (3p+1)^3 + (3p+2)^3 + (3p+3)^3 && (4)\\ &= 81p^3 + 162p^2 + 126p + 36 && (5)\\ &= 9(9p^3 + 18p^2 + 14p + 4) && (6)\end{aligned}$$

- $9p^3 + 18p^2 + 14p + 4 \in \mathbb{Z}$, therefore $9 \mid \big(n^3 + (n+1)^3 + (n+2)^3\big)$.
- Case 3: $n = 3q+2$,

$$\begin{aligned} n^3 + (n+1)^3 + (n+2)^3 &= (3q+2)^3 + (3q+3)^3 + (3q+4)^3 && (7)\\ &= 81q^3 + 243q^2 + 261q + 99 && (8)\\ &= 9(9q^3 + 27q^2 + 29q + 11) && (9)\end{aligned}$$

- $9q^3 + 27q^2 + 29q + 11 \in \mathbb{Z}$, therefore $9 \mid \big(n^3 + (n+1)^3 + (n+2)^3\big)$.

2. Recall Bézout's identity: Let $a, b \in \mathbb{Z}$ such that $a$ and $b$ are not both zero. Then there exists $x, y \in \mathbb{Z}$ such that $ax + by =\gcd(a, b)$.
Use this result to prove the following result.
Let $a, b, c \in \mathbb{Z}$ such that $\gcd(a, b) = 1$. Then

$$(a \mid bc) \implies (a \mid c)$$

- Let's assume the hypothesis is true.
- Then we know that $\exists x, y \in \mathbb{Z}$ such that $ax + by = 1$.
- Also we know that $bc = ak, \in \mathbb{Z}$.

$$\begin{aligned} ax + by &= 1 && (10)\\ acx + bcy &= c && (11)\\ axc + aky &= c && (12)\\ a(cx + yk) &= c && (13)\end{aligned}$$

- We can notice that $cx + yk \in \mathbb{Z}$, therefore we can figure that $(a \mid c)$.
- Thus the result holds.

3. Let $P \subset \mathbb{N}$ be the set of prime numbers $P = \{2, 3, 5, 7, 11, \ldots\}$. Determine whether the following statements are true or false. Prove your answers (”true” or ”false” is not sufficient.)

   - (a) $\forall x \in P, \forall y \ \in P, x + y \in P$.
     - Negation: $\exists x \in P, \exists y \in P, x + y \notin P$
     - If we choose $x = 2, y = 5$ then $x + y = 7 \in P$
     - Thus, the negation is false which shows that the original statement is true.
   - (b) $\forall x \in P, \exists y \in P$ such that $x + y \in P$.
     - Negation: $\exists x \in P, \forall y \in P$ such that $x + y \notin P$.
     - Case 1: $x = 2$
     - In this case, if we say $y \equiv 1(\mod 3)$, then $y = 3g + 1, g \in \mathbb{Z}$
     - Then $x + y = 3(g + 1)$, and we know that $g + 1 \in \mathbb{Z}$.
     - Thus we can conclude that $x + y \notin P$.
     - Case 2: $x \neq 2$
     - Case 2-(a): We can choose $y \in P$ such that $y \equiv 1(\mod 3)$, $y = 3q + 1, q \in \mathbb{Z}$.
     - If we choose $x$ to be $x = 3\ell + 2$ s.t. $x \in P$.
     - Then $x + y = 3(q + \ell + 1)$ and $q + \ell + 1 \in \mathbb{Z}$.
     - Therefore we can conclude that $x + y \notin P$.
     - Case 2-(b): Let's choose $y \in P$ s.t. $y \equiv 2(\mod 3)$, $y = 3m + 2, m \in \mathbb{Z}$.
     - Then we can choose $x \in P$ s.t. $x = 3n + 1, n \in \mathbb{Z}$.
     - Thus $x + y = 3(m + n + 1), m + n + 1 \in \mathbb{Z}$.
     - Thus $x + y \notin P$.
     - As the negation is true, the original statement is false.
   - (c) $\exists x \in P$ such that $\forall y \in P, x + y \in P$. Again, we leverage the fact that every number is odd *except* 2.
     - Let's negate the statement, $\forall x \in P$ s.t. $\exists y \in P, x + y \notin P$.
     - Case 1: $x \neq 2$
     - Case 1-(a): $x = 3\ell + 1$
     - Then we can choose $y = 2$, so that $x + y = 3\ell + 3 = 3(\ell + 1)$.
     - As $\ell + 1 \in \mathbb{Z}$, thus $3 \mid (x + y)$ therefore $x + y \notin P$.
     - Case 1-(b): $x = 3p + 2$
     - Then we can choose $y = 1$, so that $x + y = 3p + 3 = 3(p + 1)$.
     - We know that $p + 1 \in \mathbb{Z}$, thus $x + y \notin P$.
     - Case 2: $x = 2$

- Then we can just choose $y = 7$, $x + y = 9$.
- As 9 is not a prime number, $x + y \notin P$.
- Therefore as the negation is true the original statement is false.

- (d) $\exists x \in p$ such that $\exists y \in P, x + y \in P$.
  - Let's choose $x = 2$ and $y = 5$.
  - Then $x + y = 7$
  - And $7 \in P$, so the statement is true.

4. Prove the following statement: For every positive number $\epsilon$ there is a positive number $M$ such that

$$|\tfrac{2x^2}{x^2+1} - 2| < \epsilon$$

Whenever $x \geq M$.

- Let's rephrase the statement. Then we get $\forall \epsilon > 0, \exists M > 0$ such that $(x \geq M) \implies |\frac{2x^2}{x^2+1} - 2| < \epsilon$.
- Let's then negate the statement. Then we will obtain that $\exists \epsilon > 0, \forall M > 0$ such that $(x \geq M) \wedge (|\frac{2x^2}{x^2+1} - 2| \geq \epsilon)$.

$$|\frac{2x^2}{x^2+1} - 2| = |\frac{-2}{x^2+1}| \tag{14}$$

$$= \frac{2}{x^2+1} \leq 2 \tag{15}$$

$$= \frac{2}{x^2+1} > 0 \tag{16}$$

- Therefore, if we choose an $\epsilon$ such that $\epsilon > 2$ the negated statement becomes false.
- Thus, the original statement is true.

5. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if $\lim_{x \to a} f(x) = f(a)$. Let $f(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$.

Is $f$ continuous a $x = 0$?

- Let $\epsilon > 0$ be given, and let's choose $\delta = \sqrt{\epsilon}$ such that $0 < |x - 0| < \delta = \sqrt{\epsilon}$.
- Then $|x| < \delta$, therefore $|x^2| < \delta^2 = (\sqrt{\epsilon})^2 = \epsilon$.
- Also, we can notice that $|x^2 \sin(\frac{1}{x}) - 0| < \delta^2 = \epsilon$.
- Then we can conclude that $\lim_{x \to 0} f(x) = 0 = f(0)$.
- Making $f(x)$ continuous.

6. We say that the sequence $(x_n)$ is bounded if

$$\exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M.$$

Prove that if a sequence $(x_n)$ converges to 0, then $(x_n)$ is bounded.

- Let's assume that $(x_n)$ converges to 0.
- Then we know that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and $\forall n \geq N$ then $|x_n - 0| < \epsilon$.
- Case 1: consider the case where $n \geq N$,
- Then we know that $|x_n| < \epsilon$, thus $x_n$ is bounded by $\epsilon$.
- Case 2: when $n < N$,
- As we know that $n, N \in \mathbb{N}$, $x_n$ will always have a minimum or a maximum in that boundary.
- Let's then say that $A = max\{x_n\}$ and $B = min\{x_n\}$, Therefore, we can choose a $M$ such that $M = max\{|A|, |B|\}$ and thus $M$ bounds $x_n$.

7. A function is said to be unbounded on the interval $(a, b)$ if

$$\forall M \in \mathbb{R}, \exists t \in (a, b) \text{ s.t. } |f(t)| > M.$$

Prove that $\log x$ is unbounded on (0,1).

- Let's negate the statement.
- Then a function is bounded when $\exists M \in R, \forall t \in (a, b)$ s.t. $|f(t)| \leq M$.
- As the given interval is (0,1), let's first choose $M$ to be 1.
- Then we can find a counter example when $t = e^{-2}$, then $f(t) = \log(e^{-2}) = -2$.
- Then we can notice that $|f(t)| = 2 > 1$.
- As the negation is false, the original statement it true.

8. We say that a sequence $(x_n)_{n \in \mathbb{N}}$ converges to $L$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \epsilon.$$

Using the definition, prove that the sequence $(x_n)$ with $x_n = (-1)^n + \frac{1}{n}$ does not converge to any $L \in \mathbb{R}$.

- Let's negate the statement, $\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N, |x_n - L| \geq \epsilon$.
- Case 1: $L \geq 0$.
- Let's choose $\epsilon$ s.t $\epsilon = |L|$.
- When $n$ is odd s.t. $n = 2N + 1$,

$$x_n - L = (-1)^n + \frac{1}{n} - L \tag{17}$$

$$= -1 + \frac{1}{n} - L \leq -L \tag{18}$$

$$|-1 + \frac{1}{n} - L| \geq |-L| = L \tag{19}$$

- Then for all $L$, $|x_n - L| \geq L = \epsilon$, for some $n > N$.
- Case 2: $L < 0$.
- Let's choose $\epsilon < 1$.
- When $n$ is even s.t. $n = 2N$,

$$x_n - L = (-1)^n + \frac{1}{n} - L \tag{20}$$
$$= 1 + \frac{1}{n} - L > 1 \tag{21}$$
$$|x_n - L| > 1 \tag{22}$$

- For some $n > N$ and for all $L$, $|x_n - L| > 1 > \epsilon$.
- Thus it doesn't converge for any $L$.