1.

Table 1: The joint p.m.f. of X, Y

$X \downarrow Y \rightarrow$	0	1	2	3
1/2	1/12	1/8	1/8	1/12
1	0	1/12	1/9	1/9
6	1/12	1/12	0	1/9

(a) $p_X(\frac{1}{2}) = \sum_{y=0}^{3} p_{XY}(\frac{1}{2}, y) = \frac{5}{12}$

$p_X(1) = \sum_{y=0}^{3} p_{XY}(1, y) = \frac{11}{36}$

$p_X(6) = \sum_{y=0}^{3} p_{XY}(6, y) = \frac{5}{18}$

$$p_X(k) = \begin{cases} \frac{5}{12}, & k=\frac{1}{2} \\ \frac{11}{36}, & k=1 \\ \frac{5}{18}, & k=6 \end{cases},$$

$A = \{\frac{1}{2}, 1, 6\}$, $p_Y(0) = \sum_{x\in A} p_{XY}(x,0) = \frac{1}{6}$

$p_Y(1) = \sum_{x\in A} p_{XY}(x,1) = \frac{7}{24}$

$p_Y(2) = \sum_{x\in A} p_{XY}(x,2) = \frac{17}{72}$

$p_Y(3) = \sum_{x\in A} p_{XY}(x,3) = \frac{11}{36}$

$$p_Y(k) = \begin{cases} \frac{1}{6}, & k=0 \\ \frac{7}{24}, & k=1 \\ \frac{17}{72}, & k=2 \\ \frac{11}{36}, & k=3 \end{cases}$$

(b) $p_{XY}(\frac{1}{2}, 0) = \frac{1}{12} \neq \frac{5}{12}\cdot\frac{1}{6} = p_X(\frac{1}{2})\cdot p_Y(0)$ $\quad \therefore X, Y$ are not independent

(c) $Cov(X,Y) = E[XY] - \mu_X\mu_Y$.

$\mu_X = \frac{5}{24} + \frac{11}{36} + \frac{5}{3} = \frac{15+22+120}{72} = \frac{157}{72}$, $\mu_Y = \frac{7}{24} + \frac{17}{36} + \frac{11}{12} = \frac{21+34+66}{72} = \frac{121}{72}$

$$E[XY] = \sum_{y=0}^{3}\sum_{x\in A} x\cdot y\cdot p_{XY}(x,y) = \sum_{y=0}^{3} y\left(\tfrac{1}{2}p_{XY}(\tfrac{1}{2},y) + p_{XY}(1,y) + 6p_{XY}(6,y)\right)$$

$$= \tfrac{1}{2}\left(p_{XY}(\tfrac{1}{2},1) + 2p_{XY}(\tfrac{1}{2},2) + 3p_{XY}(\tfrac{1}{2},3)\right) + \left(p_{XY}(1,1) + 2p_{XY}(1,2) + 3p_{XY}(1,3)\right) + 6\left(p_{XY}(6,1) + 2p_{XY}(6,2) + 3p_{XY}(6,3)\right)$$

$$= \tfrac{1}{2}\left(\tfrac{1}{8} + \tfrac{2}{8} + \tfrac{3}{12}\right) + \left(\tfrac{1}{12} + \tfrac{2}{9} + \tfrac{3}{9}\right) + 6\left(\tfrac{1}{12} + \tfrac{3}{9}\right) = \tfrac{5}{16} + \tfrac{23}{36} + \tfrac{5}{2}$$

$\therefore \frac{5}{16} + \frac{23}{36} + \frac{5}{2} - \frac{157}{72}\cdot\frac{121}{72} = -\frac{1105}{5184}$

2.

$E(d(Z_1,Z_2)^2) = E[(X_1-X_2)^2 + (Y_1-Y_2)^2]$

$= E[X_1^2 + X_2^2 - 2X_1X_2 + Y_1^2 + Y_2^2 - 2Y_1Y_2]$. Since D is symmetric and X_1, X_2 and Y_1, Y_2 are independent, $E[d(Z_1,Z_2)^2] = E[X_1^2] + E[X_2^2] + E[Y_1^2] + E[Y_2^2]$

$E[X_1^2+Y_1^2] = \iint_D (x^2+y^2)\cdot\frac{1}{\pi}\, dA = \int_0^{2\pi}\int_0^1 \frac{r^2}{\pi} r\, dr\, d\theta = \frac{1}{2}$ similarly $E[X_2^2+Y_2^2] = \frac{1}{2}$, $E[d(Z_1,Z_2)^2] = 1$ $\quad \therefore 1$

4.

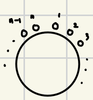
Exercise 8.16. Let $E[X] = 1$, $E[X^2] = 3$, $E[XY] = -4$ and $E[Y] = 2$. Find $\mathrm{Cov}(X, 2X + Y - 3)$.

$Cov(X, 2X+Y-3) = 2Cov(X,X) + Cov(X,Y)$

$= 2(E[X^2] - E[X]^2) + (E[XY] - E[X]E[Y])$

$= 2(3 - 1^2) + (-4 - 1\cdot 2) = 4 - 6 = -2$ $\quad \therefore -2$

6.



$n \geq 5$

$X :=$ number of shirts that a color with at least one of the two shirts adjacent to it

$X = \sum_{i=1}^{n} \xi_i$, ξ_i is an indicator such that $\xi_i = \begin{cases} 1, & \text{color with at least one of the two shirts adjacent to it} \\ 0, & \text{o.w.} \end{cases}$ and $P(\xi_i = k) = \begin{cases} \frac{5}{9}, & \xi_i = 1 \\ \frac{4}{9}, & \xi_i = 0 \end{cases}$

· $E[X] = E[\sum_{i=1}^{n}\xi_i] = \sum_{i=1}^{n} E[\xi_i] = \sum_{i=1}^{n} P(\xi_i = 1) = n\cdot\frac{5}{9}$ $\quad \therefore E[X] = \frac{5}{9}n$

· $Var(X) = Var(\sum_{i=1}^{n}\xi_i) = \sum_{i=1}^{n} Var(\xi_i) + 2\sum_{1\leq j<k\leq n} Cov(\xi_j, \xi_k)$

let's denote $Y_i := 1 - \xi_i$ so that Y_i is the complement of ξ_i. Then $Cov(\xi_j, \xi_k) = Cov(Y_j, Y_k)$, now let's consider the cases ① $|k \bmod n - j \bmod n| = 1$ ② $|k \bmod n - j \bmod n| \geq 3$ ③ $|k \bmod n - j \bmod n| = 2$

For case ①, we know adjacent t-shirts are dependent.

For case ②, let's consider the sets of t-shirts, let $T_i :=$ T-shirt at position i, (T_{j-1}, T_j, T_{j+1}), $\overbrace{(T_{k-1}, T_k, T_{k+1})}^{\text{all mod } n}$ where $|k \bmod n - j \bmod n| \geq 3$ then every t-shirt from the first set is at least 3 positions different from the second set, hence they are independent.

For case ③, let's consider the set of t-shirts $\underbrace{(T_{j-1}, T_j, T_{j+1}, T_{j+2}, T_{j+3})}_{\text{mod } n}$ < Continue next page >

For the set $(T_{j-1}, T_j, T_{j+1}, T_{j+2}, T_{j+3})$, let's see if positions j and $j+2$ are independent.

Since we're calculating $Cov(t_j, t_{j+2})$ we see if P(t-shirt j doesn't share same color j-1, j+1 and t-shirt j+2 is different color from j+1, j+3) = P(t-shirt j doesn't share same color j-1, j+1) · P(t-shirt j+2 is different color from j+1, j+3)

P(t-shirt j doesn't share same color j-1, j+1 and t-shirt j+2 is different color from j+1, j+3) ⇒ let's say T_{j+1} is G then T_j, T_{j+2} are not G and T_{j-1}, T_{j+3} are not the same as T_j, T_{j+2} respectively.

then $p = 3 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^2 \times \left(\frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)^4$

P(t-shirt j doesn't share same color j-1, j+1) $= 3 \cdot \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right) \cdot \left(\frac{2}{3}\right) =$ P(t-shirt j+2 is different color from j+1, j+3)

So since both LHS and RHS have probability $\left(\frac{2}{3}\right)^4$ they are independent.

Therefore, $Var(X) = \sum_{i=1}^{n} Var(S_i) + 2\sum_{i=1}^{n} Cov(r_{i \bmod n}, r_{i+1 \bmod n})$

$= n \cdot Var(S_1) + 2n \cdot Cov(t_1, t_2)$

$= n \cdot P(S_1=1)(1-P(S_1=1)) + 2n \cdot (P(t_1=1 \cap t_2=1) - P(t_1=1)P(t_2=1))$

$= n \cdot \frac{5}{9} \cdot \frac{4}{9} + 2n \cdot \left(\frac{24}{81} - \frac{16}{81}\right)$

$= \frac{20}{81}n + \frac{16}{81}n = \frac{36}{81}n = \frac{4}{9}n$

$\therefore \frac{4}{9}n$

7. $r \geq 1$, $t_1 \cdots t_n$ be i.i.d each having pdf $f(y) = \begin{cases} \frac{1}{y^2}, & y \geq 1 \\ 0, & y < 1 \end{cases}$

$E[N] = E\left[\sum_{i=1}^{n} \mathbb{1}_{A_i}\right] = \sum_{i=1}^{n} E[\mathbb{1}_{A_i}] = n \cdot P(t_2 \geq r \cdot t_1)$, for $B := \{(y_1, y_2) : y_2 \geq r y_1\}$, $\iint_B f_{t_1 t_2}(x, y)\, dy_1 dy_2 = \int_1^\infty \int_{rx}^\infty \frac{1}{x^2} \cdot \frac{1}{y^2}\, dy dx = \int_1^\infty \frac{1}{x^2} \cdot \left[-\frac{1}{y}\right]_{rx}^\infty dx = \int_1^\infty \frac{1}{rx^3} dx = \left[-\frac{1}{2rx^2}\right]_1^\infty = \frac{1}{2r}$

$\therefore E[N] = n\left(\frac{1}{2r}\right)$

$Var(N) = Var\left(\sum_{i=1}^{n} \mathbb{1}_{A_i}\right) = \sum_{i=1}^{n} Var(\mathbb{1}_{A_i}) + 2\sum_{1 \leq j < k \leq n} Cov(\mathbb{1}_{A_j}, \mathbb{1}_{A_k})$ ⟶ if $|k-j| \geq 2$, then events $\mathbb{1}_{A_j}$ and $\mathbb{1}_{A_k}$ are independent ex) $P(\mathbb{1}_{A_1} \cap \mathbb{1}_{A_3}) = P(t_2 \geq r t_1 \cap t_4 \geq 3 \cdot t_3) = P(t_2 \geq r \cdot t_1)P(t_4 \geq r \cdot t_3)$

$= n \cdot Var(\mathbb{1}_{A_1}) + 2 \cdot \sum_{i=1}^{n-1} Cov(\mathbb{1}_{A_i}, \mathbb{1}_{A_{i+1}})$

$= n \cdot Var(\mathbb{1}_{A_1}) + 2(n-1) \cdot Cov(\mathbb{1}_{A_1}, \mathbb{1}_{A_2})$

$= n\, Var(\mathbb{1}_{A_1}) + 2 \cdot (n-1) \{P(A_1 \cap A_2) - P(A_1)P(A_2)\}$

$= n \{E[\mathbb{1}_{A_1}^2] - E[\mathbb{1}_{A_1}]E[\mathbb{1}_{A_1}]\} + 2 \cdot (n-1) \{P(A_1 \cap A_2) - P(A_1)P(A_2)\}$

$= n \{E[\mathbb{1}_{A_1}](1 - E[\mathbb{1}_{A_1}])\} + 2 \cdot (n-1) \{P(A_1 \cap A_2) - P(A_1)P(A_2)\}$

$= n \left(\frac{1}{2r} \cdot \left(1 - \frac{1}{2r}\right)\right) + 2(n-1) \{P(A_1 \cap A_2) - P(A_1)P(A_2)\}$

$P(A_1 \cap A_2) = P(t_3 \geq r t_2 \geq r^2 \cdot t_1) = \int_1^\infty \int_{rx}^\infty \int_{ry}^\infty \frac{1}{x^2} \cdot \frac{1}{y^2} \cdot \frac{1}{z^2}\, dz dy dx = \int_1^\infty \int_{rx}^\infty \frac{1}{x^2 y^2} \frac{1}{ry}\, dy dx = \int_1^\infty \frac{1}{rx^2} \cdot \left[-\frac{1}{2y^2}\right]_{rx}^\infty dx = \int_1^\infty \frac{1}{2r^3 x^4} dx = \left[-\frac{1}{6r^3 x^3}\right]_1^\infty = \frac{1}{6r^3}$

$= n \cdot \left(\frac{1}{2r} \cdot \left(1 - \frac{1}{2r}\right)\right) + 2 \cdot (n-1) \cdot \left(\frac{1}{6r^3} - \frac{1}{4r^2}\right)$

$\therefore E[X] = n \cdot \frac{1}{2r}$, $Var(X) = n \cdot \frac{1}{2r} \cdot \left(1 - \frac{1}{2r}\right) + 2 \cdot (n-1) \cdot \left(\frac{1}{6r^3} - \frac{1}{4r^2}\right)$