SMALL CLASS: The Fundamental Theorem of Calculus Part 1

In this class, you will make explicit the relationship between derivatives and integrals using the area function.

Contributing team members

Student number	Last name	First name

Small class questions

1. The *scribe*'s role is to do the writing and record the answers.

   Nominate a team member to be the scribe. As you work through a worksheet, the scribe role *must* rotate from question to question, repeating only after every team member has taken a turn.

   Your team may also wish to have other roles. For example, the *manager*'s role is to keep the team on task ("Okay, what's next?"); the *skeptic*'s role is to question the team's answer to make sure it is sound ("How do we *know* that x^3 is small?"). Your team should always be working on the same problem together. The goal isn't speed, it's conversation and understanding.

2. **The area function**

   (a) (★☆☆☆) Consider the area function $A(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$. Suppose that $a = 0$ and $f(t) = \cos t$. Find $A(2\pi)$ by drawing a picture.

   Answer:

   Scribe:

(b) (★★☆☆☆) Again consider $A(x) = \int_0^x \cos t \, dt$. Let's get a rough sketch of the function $A(x)$. First, find $A(0)$ and $A(\pi)$. Next, determine the sign of $A(\pi/2)$ and $A(3\pi/2)$. Finally, think about the intervals $(0, \pi/2)$, $(\pi/2, 3\pi/2)$, and $(3\pi/2, 2\pi)$ and determine if $A(x)$ is increasing or decreasing on these intervals. What familiar function does $A(x)$ look like?

Answer:

Scribe:

(c) (★☆☆☆☆) With your guess for $A(x)$ in part (b) above, what do you think $A'(x)$ is?

Answer:

Scribe:

(d) (★☆☆☆☆) In general, if $A(x) = \int_0^x f(t) \, dt$ what do you think $A'(x)$ is? Remember, this is a function of x and not t.

Answer:

Scribe:

3. **The Fundamental Theorem of Calculus Part 1**

(a) (★★★★) The Theorem states that if $A(x) = \int_a^x f(t)\ \mathrm{d}t$, where f is a continuous function, then $A'(x) = f(x)$.

With the help of your small class instructor, understand the proof of the Fundamental Theorem of Calculus Part 1. You can find a helpful picture on page 47 of CLP-2.

Consider $A(x) = \int_a^x f(t)\ \mathrm{d}t$. We want to show that $A'(x) = f(x)$. Draw a picture showing both $A(x)$ and $A(x+h)$. Write $A(x+h) = A(x)+E(x)$ and indicate this difference $E(x) = A(x+h)-A(x)$ clearly. Write a sentence arguing that, based on your picture, $E(x) \approx h \cdot f(x)$ and that this becomes an equality when $h \to 0$. Rearrange everything to find that $f(x) \approx \dfrac{A(x+h)-A(x)}{h}$. What does this resemble? Take the limit as $h \to 0$ to achieve the desired result.

Answer:

Scribe:

4. **Using Part 1 of the Fundamental Theorem of Calculus.**

(a) (★☆☆☆)

Use the Fundamental Theorem to confirm our earlier suspicions and compute $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x \cos(t)\ \mathrm{d}t\right)$.

Answer: Scribe:

(b) (★☆☆☆)

Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^x \arctan t\ \mathrm{d}t\right)$.

Answer: Scribe:

(c) (★☆☆☆)

Compute $\frac{\mathrm{d}}{\mathrm{d}x}A(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^{x^2} \arctan t\ \mathrm{d}t\right)$. To do so, recognize $A(x) = f(g(x))$ as the composition of two functions. What are f and g here? Once you have them all you need is chain rule.

Answer: Scribe:

Practice questions

The questions below are for practice. They do not contribute to your grade, and it is not expected that you complete them during your small class. However, you are strongly encouraged to work through them.

5. (★★☆☆) Let $A(x) = \displaystyle\int_{e^x}^{x^2} \sin t \, \mathrm{d}t$. Find $A'(x)$. For this more challenging problem you will need the identity $\displaystyle\int_a^b f(t) \, \mathrm{d}t = -\int_b^a f(t) \, \mathrm{d}t$ which you will see later. Convince yourself it should be true and then solve the problem.