Solutions to homework 3:

1. Negate the following statement. For every positive number ϵ there is a positive number M for which

$$|1 - \frac{x^2}{x^2+1}| < \epsilon,$$

whenever $x \geq M$.

- There exists some positive number ϵ s.t. there exists some positive number M that $x \geq M$ and $|1 - \frac{x^2}{x^2+1}| \geq \epsilon$.

2. Write down the negation of the statement

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{R}, \big((x \geq y) \Rightarrow (\frac{x}{y} = 1)\big)$$

and determine whether the original is true or false.

- Let's first negate the statement.
- $\exists x \in \mathbb{Z}, \forall y \in \mathbb{R}$ s.t. $(x \geq y) \wedge (\frac{x}{y} \neq 1)$.
- If we choose an x where $x = 0$, there isn't always a y that satisfies $x \geq y$, even though whether $y = 0$ or $y \neq 0$, $\frac{x}{y} \neq 1$ is always satisfied.
- Thus the negated statement is false, concluding that the original statement is true.

3. Let $A = \{n \in \mathbb{N} : 3 \mid n \text{ or } 4 \mid n\} \subset \mathbb{N}$. Note that all numbers in A are positive. Determine whether the following four statements are true or false — explain your answers ("true" or "false" is not sufficient.)

- (a) $\exists x \in A$ s.t. $\exists y \in A$ s.t. $x + y \in A$.
  - Let's choose the case where $x = 3$ and $y = 6$. Both are divisible by three and positive thus there are in A.
  - Then $x + y = 3 + 6 = 9 = 3 * 3$, so $x + y$ is also positive and in A. So this statement is true.
- (b) $\forall x \in A, \forall y \in A, x + y \in A$.
  - Let's negate the original statement, then we get $\exists x \in A, \exists y \in A$ s.t. $x+y \notin A$.
  - Let's choose $x = 3$ and $y = 4$, then

$$\begin{aligned} x + y &= 3 + 4 && (1) \\ &= 7 && (2) \\ &= 2 * 3 + 1 && (3) \\ &= 1 * 4 + 3 && (4) \end{aligned}$$

  - Thus, we can see for this case $x + y \notin A$.

- Therefore, as the negation is false the original statement is true.

- (c) $\exists x \in A$ s.t. $\forall y \in A$, $x + y \in A$.
  - Let's say that $x = 24$, then consider three cases.
  - Case 1: $3 \mid y$, $y = 3k \in \mathbb{Z}$.
  - Then

$$x + y = 24 + 3k \tag{5}$$
$$= 3(8 + k) \tag{6}$$

  - We can notice that $8 + k \in \mathbb{Z}$, therefore $x + y \in \mathbb{Z}$.
  - Case 2: $4 \mid y$, $y = 4\ell \in \mathbb{Z}$.
  - Then

$$x + y = 24 + 4\ell \tag{7}$$
$$= 4(6 + \ell) \tag{8}$$

  - We know by fact that $6 + \ell \in \mathbb{Z}$, $x + y \in A$.
  - Case 3: $3 \mid y$ and $4 \mid y$, We know from the previous project that $12 \mid y$.
  - Then we can express y as $y = 12m, m \in \mathbb{Z}$.
  - Therefore,

$$x + y = 24 + 12m \tag{9}$$
$$= 12(2 + m) \tag{10}$$

  - As $2 + m \in \mathbb{Z}$, therefore $x + y \in A$.
  - To conclude, this statement is true.

4. Negate the following statements and determine whether the original statements are true or false. Justify your answer.

- (a) $\forall n \in \mathbb{Z}, \exists y \in \mathbb{R} - \{0\}$ such that $y^n \leq y$.
  - Let's negate the statement then we get $\exists n \in \mathbb{Z}, \forall y \in \mathbb{R} - \{0\}$ such that $y^n > y$.
  - Let's choose the case where $n = 0$, then $y^n = y^0 = 1$.
  - Then in order for this statement to be true for then any $y \in \mathbb{R} - \{0\}$, then $0 > y$.
  - However in this domain y is either a positive or negative number. Thus for cases when y is positive. This statement becomes false.
  - Thus, as the negation is false the original statement is true.

- (b) $\exists y \in \mathbb{R} - \{0\}$ such that $\forall n \in \mathbb{Z}$, $y^n \leq y$.
  - Let's choose the instance when $y = 1$.

- Then

$$y^n \leq y \tag{11}$$
$$1^n \leq 1 \tag{12}$$

- In this case, no matter what n is the relationship is always true.
- This is because for any $k \in \mathbb{R}$, $1^k = 1$.
- Therefore, the statement is true.

- (c) $\forall x \in \mathbb{R}$ where $x \neq 0$, we have $x \leq 1$ or $\frac{1}{x} \leq 1$.
  - Let's negate this statement, then $\exists x \in \mathbb{R}$ where $x \neq 0$, $x > 1$ and $\frac{1}{x} > 1$.
  - Choose $x = 2$, then $2 > 1$ but $\frac{1}{2} < 1$. therefore the original is false.
  - Thus, the original statement is true.

5. After cleaning you basement, you find a set of keys K and a set of locks L. For every one of the following statements (a), (b) and (c),

   - "At least one of the keys unlocks one of the locks."
     - $\exists k \in K$, $\exists l \in L$ k unlocks l.
     - negation : $\forall k \in K, \forall l \in L$. k doesn't unlock l.
     - all keys don't unlock all locks.
   - "Some key unlocks all the locks."
     - $\exists k \in K, \forall l \in L$, k unlocks l.
     - negation : $\forall k \in K, \exists l \in L$, k doesn't unlock l.
     - all keys don't unlock some lock.
   - "Some lock is not unlocked by any key."
     - $\forall k \in K, \exists l \in L$, k doesn't unlock l.
     - negation : $\exists k \in K, \forall l \in L$, k unlocks l.
     - at least one key unlocks all locks.

6. Prove that $\forall a \in \mathbb{Z}$, $\exists b \in \mathbb{Z}$, $a^2 + b^2 \equiv 1 \mod 3$.

   - Let's choose some a for $a \in \mathbb{Z}$, and let's choose some $b \in \mathbb{Z}$.
   - This b satisfies the equation $b^2 = 2a^2 + 1$, as $a \in \mathbb{Z}$ therefore we can conclude that $2a^2 + 1 \in \mathbb{Z}$.
   - Thus we can notice that

$$a^2 + b^2 = a^2 + (2a^2 + 1) \tag{13}$$
$$= 3a^2 + 1 \tag{14}$$

   - Thus, we can see that $a^2 + b^2 \equiv 1 \mod 3$.

7. Prove or disprove:

$$\forall x, y, z \in \{3, 6\}, \left(x = y = z \text{ or } \tfrac{x+y+z}{3} > \tfrac{x}{y} + \tfrac{y}{z} + \tfrac{z}{x}\right).$$

- Let's think of two big cases. When x, y, z are all equal and when x, y, z are not all equal.
- Case 1: When x, y, z are equal, we can choose the two options.
- Either when $x = y = z = 3$ or $x = y = z = 6$. In this case, the condition $x = y = z$ is fulfilled.
- Case 2: For this case, we can divide this instance into two cases. One case when there are two 3's and one 6, and the other where there are two 6's and one 3.
- Case 2-1: Let's see the case when we have two 3's and one 6.
- No matter what combination of x, y, z fulfills this, the result maintains the same. (This is also the same for the other case with two 6's and one 3.)
- Therefore,

$$\frac{x+y+z}{3} = \frac{3+3+6}{3} \tag{15}$$
$$= 4 > \frac{3}{3} + \frac{3}{6} + \frac{6}{3} = \frac{7}{2} \tag{16}$$

- Case 2-2: When there are two 6's and one 3.
- This case,

$$\frac{x+y+z}{3} = \frac{3+6+6}{3} \tag{17}$$
$$= 5 > \frac{6}{6} + \frac{6}{3} + \frac{3}{6} = \frac{7}{2} \tag{18}$$

- Thus we can see that all cases satisfy the statement.
- Therefore, the statement holds true.

8. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is increasing if

$$\forall a, b \in \mathbb{R}, (a < b \Rightarrow f(a) < f(b))$$

Show that

- (a) $f(x) = x^3 + 3x + 4$ is increasing.
  - Let's choose an arbitrary $a, b \in \mathbb{R}$, such that $a < b$.
  - Then $f(a) = a^3 + 3a + 4$ and $f(b) = b^3 + 3b + 4$.

$$f(b) - f(a) = b^3 + 3b - a^3 - 3a \tag{19}$$
$$= (b-a)(b^2 + ab + a^2) + 3(b-a) \tag{20}$$
$$= (b-a)\big((b-a)^2 + 3ab + 3\big) \tag{21}$$
$$= (b-a)\big((b+a)^2 - ab + 3\big) \tag{22}$$

  - Let's consider the two cases, $ab \geq 0$ and $ab < 0$.
  - Case 1: $ab \geq 0$.
  - We already know that $b-a > 0$, therefore we can conclude $(b-a)^2+3ab+3 > 0$.
  - Thus, $f(b) - f(a) > 0$ for $a, b \in \mathbb{R}$ that satisfy $b > a$.
  - Case 2: $ab < 0$.
  - We know that $b - a > 0$, and we know by axiom that $(b + a)^2 \geq 0$ so $(b + a)^2 + 3 > 0$.
  - As $ab < 0$ it is noticeable that $-ab > 0$.
  - Thus $(b + a)^2 - ab + 3 > 0$, which shows that $f(b) - f(a) > 0$ for this case.
  - To conclude, we can see that when $b > a$, $f(b) > f(a)$ is satisfied for the function $f(x) = x^3 + 3x + 4$. Thus $f(x)$ is increasing.

- (b) $g(x) = \sin x$ is not increasing.
  - Let's negate the statement, then we get $\exists a, b \in \mathbb{R}, (f(a) < f(b) \Rightarrow a < b)$.
  - Let's choose an instance when $f(a) = 0$ and $f(b) = 1$.
  - This fulfills $f(a) < f(b)$.
  - However, when $f(a) = 0$ then $a = n\pi, n \in \mathbb{Z}$. Also, for $f(b) = 1$ then $b = \frac{\pi}{2} + (2k)\pi, k \in \mathbb{Z}$.
  - So, if we grab the instance when $a = 0$ and $b = -\frac{3\pi}{2}$. Then $f(a) < f(b)$, but $a > b$.
  - Therefore, by this counter example the negation is false thus $\sin x$ is not increasing.