k	1	2	3	4	5
$\mathbb{P}(X=k)$	1/7	1/14	3/14	2/7	2/7

1

(a) $E[X] = \sum_{k=1}^{5} k \cdot p_X(k) = 1 \cdot 1/7 + 2 \cdot 1/14 + 3 \cdot 3/14 + 4 \cdot 2/7 + 5 \cdot 2/7 = \frac{2+2+9+16+20}{14} = \frac{49}{14} = \frac{7}{2}$ $\quad \therefore \frac{7}{2}$

(b) $E[|X-2|] = \sum_{k=1}^{5} |k-2| p_X(k) = p_X(1) + 0 + p_X(3) + 2p_X(4) + 3 \cdot p_X(5)$

$= 1/7 + 3/14 + 2 \cdot 2/7 + 3 \cdot 2/7 = \frac{2+3+8+12}{14} = \frac{25}{14}$ $\quad \therefore \frac{25}{14}$

2.

(a) Let's denote k as the cost of tickets until you draw the first win. $\longrightarrow$ $P(X=k) = \begin{cases} \frac{5}{10} & (k=1) \\ \frac{5}{10}\cdot\frac{5}{9} & (k=2) \\ \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{5}{8} & (k=3) \\ \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{5}{7} & (k=4) \\ \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{5}{6} & (k=5) \\ \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{1}{6}\cdot 1 & (k=6) \end{cases}$

And let $k \in A$ where $A \in \{1,2,3,4,5,6\}$

(b) Since we're getting the expectation of a p.m.f, $E[X] = \sum_{k \in A} k \cdot p_X(k)$

$= 1\cdot\frac{5}{10} + 2\cdot\frac{5}{10}\cdot\frac{5}{9} + 3\cdot\frac{5}{10}\cdot\frac{4}{9}\cdot\frac{5}{8} + 4\cdot\frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{5}{7} + 5\cdot\frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{5}{6} + 6\cdot\frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{1}{6}$

$= \frac{11}{6}$

(c) Since the expected winnings will be \$8 (winning ticket) − cost of ticket drawing

the expected will be $E[-2^X+8] = \sum_{k \in A} (-2^k+8) p_X(k) = 8 + \sum_{k \in A} (-2^k \cdot p(k)) = 8 - \frac{319}{63} = \frac{185}{63}$

$\therefore \frac{185}{63}$

3.

(a) $E[aX+b] = \sum_k (ak+b) p_X(k) = a \cdot \sum_k k \cdot p_X(k) + b \sum_k p_X(k) = aE[X]+b$

(b) $E[(aX+b-E[aX+b])^2] = E[(aX+b-aE[X]-b)^2] = E[(aX-aE[X])^2] = a^2 E[(X-E[X])^2]$

(c) $E[(X-E[X])^2]$, let's denote $E[X]$ as μ, $E[(X-\mu)^2] = E[X^2-2\mu X+\mu^2] = E[X^2] - 2\mu \cdot E[X] + \mu^2 = E[X^2] - 2E[X]^2 + E[X]^2 = E[X^2] - E[X]^2$

(d) $E[X+Y]$, let $p(x,y) = P(X=x, Y=y)$ then we know $p_X(x) = P(X=x) = \sum_y P(x,y)$ and $p_Y(y) = \sum_x P(x,y)$. Then $E[X+Y] = \sum_x \sum_y (x+y)P(x,y) = \sum_x \sum_y x\, p(x,y) + y \cdot p(x,y) = \sum_x x \cdot p_X(x) + \sum_y y \cdot p_Y(y) = E[X]+E[Y]$

4.

$$F_X(t) := \mathbb{P}[X \le t] = \begin{cases} 0 & t<1 \\ 1/4 & 1 \le t < 2 \\ 1/3 & 2 \le t < 5 \\ 1/2 & 5 \le t < 100 \\ 1 & t \ge 100 \end{cases}.$$

(A) Since F_X is a piece-wise function then X is a descrete random variable and then $P(X=x)$ equals the magnitude of the jump of F at x.

Then $p_X(k) = P(X=k) = \begin{cases} \frac{1}{4} & (k=1) \\ \frac{1}{12} & (k=2) \\ \frac{1}{6} & (k=5) \\ \frac{1}{2} & (k=100) \end{cases}$

5.

$$F_X(t) := \mathbb{P}[X \le t] = \begin{cases} 0 & t<\sqrt{3} \\ t^2-3 & \sqrt{3} \le t < 2 \\ 1 & t \ge 2 \end{cases}.$$

probability distribution function

(a) $P[X=1.9] = P[X \le 1.9] - P[X<1.9] = \int_{1.9}^{1.9} f(x)\,dx = 0$ or $P[X=1.9] = F_X(1.9) - \lim_{t \to 1.9^+} F_X(t) = (1.9)^2 - 3 - (1.9)^2 + 3 = 0$.

(b) $P[1 \le X \le 1.9] = P[1 < X \le 1.9] = P[X \le 1.9] - P[X \le 1] = F_X(1.9) - F_X(1) = 1.9^2 - 3 - 0 = 0.61$

(c) For the probability distribution function f, $f(t) \approx \frac{P[X \le t+\varepsilon] - P[X \le t]}{\varepsilon} = \frac{F_X(t+\varepsilon) - F_X(t)}{\varepsilon}$ hence $f(t) = \lim_{\varepsilon \to 0} \frac{F_X(t+\varepsilon) - F_X(t)}{\varepsilon} = F_X'(t) = \begin{cases} 0 & t<\sqrt{3} \\ 2t & \sqrt{3} \le t < 2 \\ 0 & t \ge 2 \end{cases} = \begin{cases} 0 & t<\sqrt{3} \text{ or } t \ge 2 \\ 2t & \sqrt{3} \le t < 2 \end{cases}$

6. Let $I_{\{i,j,k\}}$ be the indicator of a triangle involving i, j and k with having an edge with each other. Thus, $I_{\{i,j,k\}} = \begin{cases} 1 & , \text{ i,j,k form a triangle} \\ 0 & , \text{ other wise} \end{cases}$

Then $E[I_{\{i,j,k\}}] = P(E_{\{i,j,k\}} = 1) \cdot 1 + P(E_{\{i,j,k\}} = 0) \cdot 0 = (\frac{1}{2})^3$ for all $\{i,j,k\} \subset [n]$ where $i \neq j$ and $j \neq k$ and $i \neq k$.

Therefore, $E[X] = E[\sum I_{\{i,j,k\}}] = \binom{n}{3} \cdot I_{\{1,2,3\}} = \binom{n}{3} \cdot (\frac{1}{2})^3$ $\quad \therefore \binom{n}{3} \cdot (\frac{1}{2})^3$