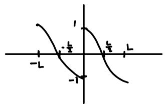
2.

(a) $f(x) = \cos\left(\frac{\pi x}{L}\right),\ 0 \le x \le L$



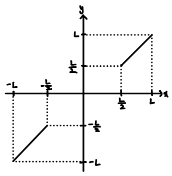
$S f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right),\ b_n = \frac{2}{L} \cdot \int_0^L \cos\left(\frac{\pi x}{L}\right) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$

$b_n = \frac{2}{L}\int_0^L \cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L \frac{\sin\left(\frac{\pi x}{L}(n+1)\right) + \sin\left(\frac{\pi x}{L}(n-1)\right)}{2} = \frac{1}{L}\left[\frac{\cos\left(\frac{\pi x}{L}(n+1)\right)}{-\frac{\pi}{L}(n+1)} - \frac{\cos\left(\frac{\pi x}{L}(n-1)\right)}{\frac{\pi}{L}(n-1)}\right]_0^L = \frac{1}{-\pi(n+1)} \cdot \left\{\cos(\pi(n+1)) - 1\right\} + \frac{1}{-\pi \cdot (n-1)} \cdot \left\{\cos(\pi(n-1)) - 1\right\}$

$= \frac{1}{-\pi(n+1)} \cdot (-\cos n\pi - 1) + \frac{1}{-\pi(n-1)} \cdot (-\cos n\pi - 1)$

$= (\cos n\pi + 1) \cdot \left(\frac{1}{\pi(n-1)} + \frac{1}{\pi(n+1)}\right) = \frac{2n}{\pi(n^2-1)} \cdot ((-1)^n + 1)$ $\therefore S f(x) = \sum_{n=1}^{\infty} \frac{2n}{\pi(n^2-1)} \cdot \left[(-1)^n + 1\right] \cdot \sin\left(\frac{n\pi x}{L}\right)$

(b) $f(x) = \begin{cases} 0, & 0 \le x < \frac{L}{2} \\ x, & \frac{L}{2} \le x \le L \end{cases}$



$S f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right),\ b_n = \frac{2}{L}\int_{\frac{L}{2}}^{L} x \cdot \sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L} \cdot \left[-\frac{L}{n\pi} \cdot x \cdot \cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)^2 \cdot \sin\left(\frac{n\pi x}{L}\right)\right]_{\frac{L}{2}}^{L}$

$= \frac{2}{L} \cdot \left[-\frac{L^2}{n\pi} \cdot (-1)^n - \left(\frac{L}{n\pi}\right)^2 \cdot (-1)^n\right] = 2(-1)^n \cdot \left[-\frac{L}{n\pi} - \frac{L}{(n\pi)^2}\right]$ $\therefore S f(x) = \sum_{n=0}^{\infty} \frac{2L \cdot (-1)^n}{n^2\pi^2}(-n\pi - 1) \cdot \sin\left(\frac{n\pi x}{L}\right)$

4. $u_t = u_{xx},\ t > 0,\ -2 \le x \le 2.\ u(-2,t) = u(2,t),\ u_x(-2,t) = u_x(2,t),\ u(x,0) = \cos\left(\frac{\pi x}{2}\right) + \sin(\pi x)$

Let $u = X \cdot T \longrightarrow X \cdot T' = X'' \cdot T \longrightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda \longrightarrow X'' + \lambda X = 0,\ T' = -\lambda T$

(1) $T = e^{-\lambda t}$

(2) $X'' + \lambda X = 0$, ① $\lambda = 0$, $X(x) = ax + b \longrightarrow X(-2) = X(2),\ -2a + b = 2a + b \longrightarrow a = 0,\ X'(-2) = X'(2),\ a = a \longrightarrow X_n = 1$

② $\lambda > 0,\ r^2 + \lambda = 0 \longrightarrow r = \pm\sqrt{\lambda}i \longrightarrow X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \longrightarrow A\sin(2\sqrt{\lambda}) + B\cos(2\sqrt{\lambda}) = -A\sin(2\sqrt{\lambda}) + B\cos(2\sqrt{\lambda}),\ 2A \cdot \sin(2\sqrt{\lambda}) = 0$

$A\sqrt{\lambda} \cdot \cos(2\sqrt{\lambda}) - B\sqrt{\lambda} \cdot \sin(2\sqrt{\lambda}) = A\sqrt{\lambda} \cdot \cos(2\sqrt{\lambda}) + B\sqrt{\lambda} \cdot \sin(2\sqrt{\lambda}),\ 2B \cdot \sin(2\sqrt{\lambda}) = 0$

if $A = 0 = B$, trivial sol., else, $2\sqrt{\lambda} = n\pi \longrightarrow \lambda = \left(\frac{n\pi}{2}\right)^2,\ n = 1, 2, 3 \ldots$

③ $\lambda < 0,\ \lambda = -\mu^2 \longrightarrow X'' - \mu^2 X = 0.\ r = \pm\mu \longrightarrow X(x) = Ae^{\mu x} + Be^{-\mu x},\ X(-2) = X(2) \longrightarrow Ae^{-2\mu} + Be^{2\mu} = Ae^{2\mu} + Be^{-2\mu} \longrightarrow A - B = 0$

$X'(-2) = X'(2) \longrightarrow \mu Ae^{-2\mu} - \mu Be^{2\mu} = \mu Ae^{2\mu} - \mu Be^{-2\mu} \longrightarrow A + B = 0$ $\therefore A = 0 = B$, trivial sol.

$\therefore u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cdot \cos\left(\frac{n\pi x}{2}\right) + b_n \cdot \sin\left(\frac{n\pi x}{2}\right)\right] \cdot e^{-\left(\frac{n\pi}{2}\right)^2 t}$

$u(x,0) = \cos\left(\frac{\pi x}{2}\right) + \sin(\pi x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cdot \cos\left(\frac{n\pi x}{2}\right) + b_n \cdot \sin\left(\frac{n\pi x}{2}\right)\right]$

$a_0 = \int_{-2}^{2} \cos\left(\frac{\pi x}{2}\right)dx = 0,\ a_1 = 1,\ b_2 = 1 \longrightarrow$ $\therefore u(x,t) = \cos\left(\frac{\pi x}{2}\right) \cdot e^{-\left(\frac{\pi}{2}\right)^2 t} + \sin(\pi x) \cdot e^{-\pi^2 t}$

3. $u_t = 3u_{xx},\ t > 0,\ 0 \le x \le \pi,\ u(0,t) = 0 = u_x(\pi,t),\ u(x,0) = \sin x + \sin\left(\frac{3x}{2}\right)$

Let $u = X \cdot T$. $X \cdot T' = 3 \cdot X'' T \longrightarrow \frac{T'}{3T} = \frac{X''}{X} = -\lambda \longrightarrow$ ① $T' = -3\lambda \cdot T,\ T(t) = e^{-3\lambda t}$, ② $X'' + \lambda X = 0$ (1) $\lambda = 0,\ X(x) = ax + b,\ X(0) = b = X'(\pi) = a = 0 \longrightarrow a = b = 0$, trivial sol.

(2) $\lambda > 0,\ r^2 + \lambda = 0 \longrightarrow r = \pm\sqrt{\lambda},\ X(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x \longrightarrow X(0) = B = 0 = X'(\pi) = A\sqrt{\lambda} \cdot \cos\sqrt{\lambda}\pi = 0 \longrightarrow \sqrt{\lambda}\pi = \frac{2n-1}{2}\pi \longrightarrow \lambda = \left(\frac{2n-1}{2}\right)^2$ if $A \ne 0$, $X_n = \sin\frac{2n-1}{2}\pi,\ n = 1, 2, 3 \ldots$

(3) $\lambda<0,\ \lambda=-\mu^2,\ X(x)=Ae^{\mu x}+Be^{-\mu x} \longrightarrow X(0)=A+B=0=X'(\pi)=\mu Ae^{\mu\pi}-\mu Be^{-\mu\pi}=0 \longrightarrow A=0=B$ (trivial sol.)

$$\therefore\ u(x,t)=\sum_{n=1}^{\infty}\left(b_n\cdot\sin\left(\frac{2n-1}{2}\right)x\right)\cdot e^{-3\left(\frac{2n-1}{2}\right)^2 t} \longrightarrow u(x,0)=\sum_{n=1}^{\infty}b_n\cdot\sin\left(\frac{2n-1}{2}\right)x=\sin x+\sin\frac{3x}{2},\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sin x+\sin\frac{3x}{2}\right)\cdot\sin\left(\frac{2n-1}{2}\right)x\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{-\cos\left(\frac{2n+1}{2}\right)x+\cos\left(\frac{2n-3}{2}\right)x-\cos\left(\frac{2n+2}{2}\right)x+\cos\left(\frac{2n-4}{2}\right)x}{2}\,dx$$

$$=\frac{1}{2\pi}\left[\frac{-2}{2n+1}\cdot\sin\left(\frac{2n+1}{2}\right)x+\frac{2}{2n-3}\cdot\sin\left(\frac{2n-3}{2}\right)x-\frac{2}{2n+2}\cdot\sin\left(\frac{2n+2}{2}\right)x+\frac{2}{2n-4}\cdot\sin\left(\frac{2n-4}{2}\right)x\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2\pi}\cdot\left\{\frac{-4}{2n+1}\cdot\sin\left(\frac{\pi}{2}+n\pi\right)+\frac{4}{2n-3}\cdot\sin\left(-\frac{3\pi}{2}+n\pi\right)\right\}=\frac{1}{\pi}\cdot\sin\left(n\pi+\frac{\pi}{2}\right)\cdot\left\{\frac{2}{2n-3}-\frac{2}{2n+1}\right\}=\frac{1}{\pi}\cdot(-1)^n\cdot\left\{\frac{8}{(2n+1)(2n-3)}\right\}$$

$$\therefore\ u(x,t)=\sum_{n=1}^{\infty}\frac{1}{\pi}\cdot(-1)^n\cdot\frac{8}{(2n+1)(2n-3)}\cdot\sin\left(\frac{2n-1}{2}\right)x\cdot e^{-3\left(\frac{2n-1}{2}\right)^2 t}$$