1. $P(E\setminus F)=\frac{2}{5}$, $P(E\cup F)=\frac{1}{2}$, $P((E\cap F)^c)=\frac{3}{4}$.

$P(F\setminus E) = P(F) - P(F\cap E) = P(E\cup F) - P(E\setminus F) - P(F\cap E)$

$= P(E\cup F) - P(E\setminus F) - (1 - P((E\cap F)^c)) = \frac{1}{2} - \frac{2}{5} - (1-\frac{3}{4}) = \frac{1}{2} - \frac{2}{5} - \frac{1}{4} = \frac{10-8-5}{20} = -\frac{3}{20} < 0$. Since probability P needs to satisfy $0\le P\le 1$ within the sample space, the events E, F can't exist.

2 Assume E_1, E_2 are independent.

a) $P(E_1^c\cap E_2^c) = P((E_1\cup E_2)^c) = 1 - P(E_1\cup E_2) = 1 - (P(E_1) + P(E_2) - P(E_1\cap E_2))$

$= 1 - P(E_1) - P(E_2) + P(E_1\cap E_2) = 1 - P(E_1) - P(E_2) + P(E_1)P(E_2) = (1-P(E_1))(1-P(E_2)) = P(E_1^c)\cdot P(E_2^c)$. Thus, events E_1^c and E_2^c are independent.

b) $P(E_1)=\frac{1}{2}$, $P(E_2)=\frac{1}{3}$, prove $P(E_1\cup E_2)=\frac{2}{3}$.

$P(E_1\cup E_2) = P(E_1) + P(E_2) - P(E_1\cap E_2) = P(E_1) + P(E_2) - P(E_1)P(E_2)$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{2}\frac{1}{3} = \frac{5}{6} - \frac{1}{6} = \frac{2}{3}$. $\therefore P(E_1\cup E_2) = \frac{2}{3}$.

c) When $P(E_3)=\frac{1}{4}$, show $\frac{17}{24} \le P(E_1\cup E_2\cup E_3) \le \frac{19}{24}$.

$P(E_1\cup E_2\cup E_3) = P(E_1)+P(E_2)+P(E_3) - P(E_1\cap E_2) - P(E_2\cap E_3) - P(E_1\cap E_3) + P(E_1+E_2+E_3)$. Since $E_1\cdot E_2$, $E_2\cdot E_3$, $E_1\cdot E_3$ are indepent to each other, $P(E_1\cup E_2\cup E_3) = P(E_1)+P(E_2)+P(E_3) - P(E_1)\cdot P(E_2) - P(E_2)\,P(E_3) - P(E_1)\cdot P(E_3) + P(E_1\cap E_2\cap E_3)$

$= \frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{6}-\frac{1}{12}-\frac{1}{8} + P(E_1\cap E_2\cap E_3) = \frac{12+8+6-4-2-3}{24} + P(E_1\cap E_2\cap E_3) = \frac{17}{24} + P(E_1\cap E_2\cap E_3)$. We know for a given event, in this case $E_1\cap E_2\cap E_3$, $P(E_1\cap E_2\cap E_3)\ge 0$. And it is also true that $P(E_1\cap E_2\cap E_3) \le P(E_1\cap E_2)$, $P(E_2\cap E_3)$, $P(E_1\cap E_3)$.

Hence, $0 \le P(E_1\cap E_2\cap E_3) \le P(E_2\cap E_3) = \frac{1}{12} \le P(E_1\cap E_3) = \frac{1}{8} \le P(E_1\cap E_2) = \frac{1}{6}$. Therefore, $\frac{17}{24} \le P(E_1\cup E_2\cup E_3) \le \frac{1}{12} + \frac{17}{24} = \frac{19}{24}$.

3.

For the first row, we place a rook in one column, ${}_8C_1$. On the second row, we can't place a rook on the column of the previous row, ${}_7C_1$. So on and so on, we get ${}_8C_1\cdot{}_7C_1\cdot{}_6C_1\cdots{}_1C_1 = 8!$. Since the total odds of placing 8 rooks is ${}_{64}C_8$.

Hence, $P(A) = \frac{8!}{{}_{64}C_8}$.

4.

First, let's get the probability of each events. $P(E) = P(\text{first odd second odd}) + P(\text{first even second even}) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$. $P(F) = P(\text{At least one outcome is 6.}) = 1 - P(\text{no out come is 6}) = 1 - \frac{5}{6}\cdot\frac{5}{6} = \frac{11}{36}$.

$P(E\cap F) = P(\text{sum is even and at least one outcome is 6.}) = P(\text{one even and one six}) = P(\text{first even second 6}) + P(\text{first 6 second even}) - P(\text{both six}) = \frac{1}{6}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{6} - \frac{1}{36} = \frac{5}{36}$.

$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{\frac{5}{36}}{\frac{11}{36}} = \frac{5}{11}$. $P(F|E) = \frac{P(E\cap F)}{P(E)} = \frac{\frac{5}{36}}{\frac{1}{2}} = \frac{5}{18}$.

5.

(a) We can get the probability by adding the three cases.

① four doesn't appear in first fiue rolls. ${}_5C_0\cdot(\frac{5}{6})^5\cdot(\frac{1}{6})^0$

② four appears once in first five rolls. ${}_5C_1\cdot(\frac{5}{6})^4\cdot(\frac{1}{6})$

③ four appears twice in first five rolls. ${}_5C_2\cdot(\frac{5}{6})^3\cdot(\frac{1}{6})^2$

$\therefore \sum_{j=0}^{2} {}_5C_j\cdot(\frac{1}{6})^j\cdot(\frac{5}{6})^{5-j}$

(b)

In order to not have a two before the fifth roll, we simply get $P(A) = {}_4C_0 \cdot (\frac{1}{6})^0 \cdot (\frac{5}{6})^4$.

(c)

We need to get the probability where the first six appears between the fourth and twentieth roll.

$P(A) = P($ no six before fifth and first six from 5th to 19th $)$, and since these events are independent

$P($ no six before fifth and first six from 5th to 19th$) = P($no six before fifth$) \cdot P($first six from 5th to 19th$) = P($no six before fifth$) \cdot (1 - P($no six between fourth and twentieth$))$

$$= {}_4C_0 \cdot (\tfrac{5}{6})^4 \cdot (1 - {}_{15}C_0 \cdot (\tfrac{5}{6})^{15}) = (\tfrac{5}{6})^4 \cdot (1 - (\tfrac{5}{6})^{15})$$

$$\therefore (\tfrac{5}{6})^4 \cdot (1 - (\tfrac{5}{6})^{15})$$

6.

In the statement " some days are snowy", the length of the words are 4, 4, 3 and 5 hence X can be 3, 4 or 5.

There are two words of length 4, and each for length 3 and 5. We can express the probability mass function as

$$P(X=k) = \begin{cases} \frac{3}{16} & (k=3) \\ \frac{8}{16} & (k=4) \\ \frac{5}{16} & (k=5) \end{cases}$$

7.

Let's prove by induction. when $n=1$, $P(E_1) = \frac{1}{1+1} = \frac{1}{2}$ hence the base case holds.

Now for all k s.t. $1 \le k \le n$ let's assume the assumption holds. Then since the events are jointly independent, $P(E_1 \cup E_2 \cdots \cup E_{n+1}) = P((E_1 \cup E_2 \cdots \cup E_n) \cup E_{n+1}) = P(E_1 \cup E_2 \cdots \cup E_n) + P(E_{n+1}) - P((E_1 \cup \cdots \cup E_n) \cap E_{n+1})$

$$= P(E_1 \cup E_2 \cdots \cup E_n) + P(E_{n+1}) - P(E_1 \cup E_2 \cdots \cup E_n) \cdot P(E_{n+1})$$

$$= \frac{n}{n+1} + \frac{1}{n+2} - \frac{n}{n+1} \cdot \frac{1}{n+2}$$

$$= \frac{n(n+2) + (n+1) - n}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{n+1}{n+2} = \frac{n+1}{(n+1)+1}$$ as required.