**Week 4 Lecture Outline**
September 27 (2023)

**Topics:** The linearity of differentiation; the Power, Product and Quotient Rules

**Small Class:** Major focus on trignometric functions and their derivatives

Instructor notes:

- Problem 1 in the first section is inspired by an additional problem from the Week 1 lecture notes. It's purpose in this lecture is motivational, so feel free to choose other examples. Remember, we are not covering the Chain Rule this week, which is the reason for working with the so-called "AB form" of the potential. (They will see this terminology, which comes from chemistry, if they Google Lennard-Jones potential.)
- The Power Rule for negative integer exponents can simply be asserted; in that case, we need to introduce the Reciprocal Rule for the proof of the Quotient Rule.
- There are multiple approaches to Problem 5 in the fifth section: formal proof, "proof by picture", informal argument, *etc*. If we opt for something like "proof by example", problems 1 and 2 in that section should be skipped.
- In response to the question "Will this be tested on the final exam?", here are three answers: (a) it would be reasonable to ask students to prove, with guidance, something as complex as the Reciprocal Rule; and (b) it is good practice in mathematical thinking to try to explain why things are true.

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**Introduction**

The *Lennard-Jones potential*

$$V(r) = \varepsilon\left(\left(\frac{R}{r}\right)^{12} - 2\left(\frac{R}{r}\right)^{6}\right), \quad \varepsilon, R > 0$$

describes the potential energy of a diatomic molecule where the atoms are a distance $r > 0$ apart.

1. What function is a good approximation for $V(r)$ for small $r$? For large $r$?
2. Draw a rough sketch of $V(r)$.
3. Without differentiating, identify on the graph of $V(r)$ where $V'(r) = 0$. Where is $V'(r) < 0$? Where is $V'(r) > 0$?
4. **Note**: Our goal is to be able to differentiate a wide variety of functions, including $V(r)$.

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**Learning Objectives** (The linearity of differentiation):

- Demonstrate using the limit definition of derivative that differentiation is linear.
- Use linearity to "break down" derivatives of sums and constant multiples.

**Problems and takeaways** (The linearity of differentiation):

1. Write down the limit definition of $\frac{d}{dx}x$ and $\frac{d}{dx}x^2$.
2. Write down the limit definition of $\frac{d}{dx}\left(3x + x^2\right)$, and show that $\frac{d}{dx}\left(3x+x^2\right) = 3\frac{d}{dx}x + \frac{d}{dx}x^2$.
3. Form a conjecture: if $S(x) = \alpha f(x) + \beta g(x)$ where $\alpha, \beta$ are constants, then what is $S'(x)$?
4. **Takeaway**: If $S(x) = \alpha f(x) + \beta g(x)$ where $\alpha, \beta$ are constants, then $S'(x) = \alpha f'(x) + \beta g'(x)$ (provided the derivatives exist).

   **See CLP-1 Theorem 2.4.2**.
5. Use the limit definition of derivative to prove the previous Takeaway.

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**Learning Objectives** (The Power Rule):

- Demonstrate the Power Rule for integer exponents using the limit definition of derivative.
- Use the Power Rule for integer exponents.

**Problems and takeaways** (The Power Rule):

1. **Note**: By the property just shown, we have

$$V'(r) = \varepsilon R^{12} \frac{d}{dr}\frac{1}{r^{12}} - 2\varepsilon R^6 \frac{d}{dr}\frac{1}{r^6}.$$

   It remains to calculate $\frac{d}{dr}\frac{1}{r^{12}}$ and $\frac{d}{dr}\frac{1}{r^6}$
2. Without using the limit definition of derivative, find $\frac{d}{dx}x^0$ and $\frac{d}{dx}x^1$.
3. Use the limit definition of derivative to find $\frac{d}{dx}x^3$.
4. Recall from the Week 3 lecture that $\frac{d}{dx}x^2 = 2x$. Form a conjecture: what is $\frac{d}{dx}x^n$ where $n \geq 1$ is a positive integer?
5. **Takeaway**: $\frac{d}{dx}x^n = nx^{n-1}$ for positive integers $n$.

   **See CLP-1 Lemma 2.6.9.** This is the Power Rule for positive integer exponents
6. Use the limit definition of derivative to prove the previous Takeaway.
7. We can also calculate $\frac{d}{dx}x^{-n}$ for positive integers $n$ using the same technique. Another way is to use the more general *Reciprocal Rule*: $\frac{d}{dx}\frac{1}{g(x)} = -\frac{g'(x)}{g(x)^2}$ provided $g'(x) \neq 0$ and $g'(x)$ exists.

   Use the limit definition of derivative to prove the Reciprocal Rule.
8. Use the Reciprocal Rule to calculate $\frac{d}{dx}x^{-n}$ for positive integers $n$.
9. **Takeaway**: $\frac{d}{dx}x^n = nx^{n-1}$ for integers $n$.

   **See CLP-1 Example 2.6.15.** This is the Power Rule for integer exponents.
10. **Note**: By the Power Rule, we have

$$V'(r) = \frac{12\varepsilon R^6}{r^7} - \frac{12\varepsilon R^{12}}{r^{13}} = \frac{12\varepsilon R^6}{r^7}\left(1 - \frac{R^6}{r^6}\right).$$

   In other words, the derivative is negative when $r < R$, zero when $r = R$ and positive when $r > R$.

**Learning Objectives** (The Product and Quotient Rules):

- Use counterexamples to demonstrate that certain statements about derivatives are false.
- Explain why an example does not constitute a "proof".
- Use the Product and Quotient Rules to differentiate the product or quotient of functions.

**Problems and takeaways** (The Product and Quotient Rules):

1. In general, is $\frac{d}{dx}f(x)g(x) = f'(x)g'(x)$? Why or why not?
2. Can you think of two example functions $f(x)$ and $g(x)$ such that $\frac{d}{dx}f(x)g(x) = f'(x)g'(x)$?
3. It turns out that, in general:
   **Takeaway**: $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$.
   **See CLP-1 Theorem 2.4.3.** This is the Product Rule.
4. Confirm the Product Rule for two example functions $f(x)$ and $g(x)$ (whose product you can differentiate *without* using the Product Rule). Can you explain why this example does not constitute a "proof" of the Takeaway?
5. Demonstrate that $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$ in general.
6. Use the Product Rule and the Reciprocal Rule to calculate the derivative of the quotient $\frac{f(x)}{g(x)}$.
7. **Takeaway**: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$.
   **See CLP-1 Theorem 2.4.5**. This is the Quotient Rule.

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**Additional problems**:

- CLP-1 Problem Book Section 2.4: Q7-Q10, Q13, Q15, Q16.