Week 3 Lecture Outline
September 20 (2023)

Topics: The derivative; tangent lines; linear approximations

Small Class: Limit Definition of the Derivative; Derivatives of Exponential Functions from Definition; first ODE with exponential function solution

Instructor notes:

- In Flavours B and C, Questions 4 and 5 in the second section are optional, but I think we should introduce the idea of linear approximation in this lecture – it's at the heart of what we are doing. Question 3 and the introduction of the idea of antiderivative in the third section are optional at this point in the course.
- Students know a lot about lines and the equations describing them. Reminding them they have this knowledge and have been using it for many years may help them see they have important knowledge to help them build their understanding of the derivative and its applications. That said, many students will really want to use $y = mx + b$ as the only way they think about lines and we should encourage them see the other ways they've dealt with lines in the past are useful to them in this course.
- One approach to motivating the derivative is to begin by asking students to think about how we might describe quantitatively the way a function changes as its input changes; link the ideas to geometric ones in terms of the graphs of functions.. The conversation starts with lines, but some simple functions and their graphs will help them visualize the concepts as they consider other functions. The average rate of change of a function is a good way point in this conversation.
- A useful perspective is ask students to think of approximating the graph of a function by a finite set of lines generated by choosing a set of points on the curve. The set of slopes of these approximating lines encode some useful information about the way the function is changing. Of course, choosing a different set of points gives a different set of slopes describing the way the given function is changing as its input changes. At this point, we focus on trying to understand what happens when we look at one of these approximating segments and under the condition we choose the two points close together. (I suppose there's a way to carry on considering the whole set of approximating points and refining the approximation appropriately and building a story from there, but....)
- Be careful to eventually tease apart the need to consider finding a way to think about the way a function changes "at a point". The students will need to build their understanding of thinking about the local nature of the derivative and also thinking about putting all of this information together to describe the way the function is changing anywhere. As you know, many will be confused by this relationship. (This is developed in parts 2 and 3 of the outline below, so consider the flow of your story as you build out the ideas with the students.)
- The rules of differentiation will be developed staring in Week 4. Telling the students the plan will be to assume and use the derivatives of elementary functions along with a set of rules for differentiation that allow us to calculate derivatives of functions built up from the elementary functions.
- The material on antiderivatives is in Chapter 4 in the CLP Notes. It's been labelled as optional for Week 3, but it might be good to at introduce it this week if it doesn't rush the rest of your presentation. Better to be sure to nail down the early concepts than to rush to try to tell too much of the story at once.

Learning Objectives (Definition of the derivative at a point):

- Explain what a derivative is and state its formal definition.

  Build the story in a way that adds the technical elements in layers – start with visualization of functions the students know and then progress to having students use their existing understanding the slope of a line to see that using secant lines leads to understanding how a more general function is changing. Eventually, limits are needed to define the derivative.

Problems and takeaways (Definition of the derivative at a point):

1. How would you describe a *line*, formally or informally? Recall what you have learned about lines and the equations describing them.

   What is the slope of a line and why is it important?

   What functions have lines as their graphs?

2. How can we describe the line of a given slope m through a given point (x_0, y_0)?

3. How can we describe the line through two given points — say $(-1, 2)$ and $(3, 1)$?

4. Given a function $f(x)$ and suppose the graph of $y = f(x)$ is not a line, How might you use what you know about slopes of lines to help you describe how the function $f(x)$ is changing as x changes?

   Image choosing two points on the graph of $y = f(x)$. What is the slope of the *secant line* through these two given points — say $(x_0, f(x_0))$ and $(x, f(x))$? What relationship does the slope of this secant line have to the way the function $f(x)$ is changing near these given points?

5. A function can change a lot between two given points. How does this affect the usefulness of the slope of a secant line connecting two points as a description of how a function is changing near these points? What happens if the two points you look at are very close to each other? That is, what if we set $x = x_0 + h$ for some small value h? In that case, what is the slope of the secant line through $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ and how well might it describe the way the function is changing near x_0, say?

   At this point, it is best to press forward with defining the derivative at a point. The first derivative calculation I choose to present is for $f(x) = kx$ to show them it is sometimes good to demonstrate a new definition produces the expected result to build one's understanding of a definition.

   Later in the lecture you could come back to the idea of approximating the graph of a function by a set of line segments as a way of motivating the idea that a set of numbers can be used to describe the way this function changes. This ultimately leads to the idea of the derivative as a function. There are other approaches, of course, and you are free to approach this as you wish.

6. **Definition**: The *derivative of $f(x)$ at $x = x_0$*, labelled $f'(x_0)$, is equal to $\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ provided the limit exists. Equivalently, it is equal to $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$ provided the limit exists. Informally, it is the "slope of $f(x)$ at x_0". Formally, $f'(x_0)$ is the slope of the tangent line to $f(x)$ at $x = x_0$.

   See CLP-1 Definition 2.2.1.

Learning Objectives (Tangent lines and linear approximations):

- Use the definition of derivative to define and find the tangent line to a function at a given point.

  There is a geometric notion of tangent line that will coincide with the analytic version in this course when the derivative exists at a point. There are curves associated with functions the students know that *have tangent lines to their graphs at points where the derivative does not exist. For example, the graph of $y = f(x) = \sqrt[3]{x}$ has a tangent line at the point $(0,0)$, but $f'(0)$ does not exist. An example like this is useful once you feel your've developed the link between derivatives and tangent lines sufficiently. There's a coordinate system story here, but perhaps it should wait until we discuss implicit differentiation.*

- Describe the tangent line as an approximation to a function at a given point.

Problems and takeaways (Tangent lines and linear approximations):

1. If $f(x) = 3$, what is the derivative of $f(x)$ at $x = 1$? Can you justify the answer in more than one way?
2. Let $f(x) = x^2$.
   (a) What is the derivative of $f(x)$ at $x = 1$? What is the tangent line to $f(x)$ at $x = 1$?
   (b) Write down the equation of another line that passes through the point $(1, 1)$.
   (c) Which line — the tangent line from part (a), or the other line from part (b) — is a "better approximation" to $f(x)$ near $x = 1$? Why? (Hint: imagine "zooming in" to the point $(1, 1)$.)
3. **Definition**: The *linear approximation* to $f(x)$ at $x = 1$ is $L(x) = f(1) + f'(1)(x - 1)$. This is simply the tangent line to $f(x)$ at $x = 1$.

   In general, the *linear approximation* to $f(x)$ at $x = a$ is $f(a) + f'(a)(x - a)$. This is simply the tangent line to $f(x)$ at $x = a$.

   See CLP-1 Equation 3.4.3.
4. Find the linear approximation to $f(x) = e^x$ at $x = 0$.

   In the small classes, the students will be working with a more detailed calculation of the derivative of e^x, making use of
   $$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$
   as a stated fact, so it is sufficient to state the final result and to let the students know they'll work through more of the details later.
5. The linear approximation to $f(x)$ at a is a better approximation to $f(x)$ near $x = a$ than other lines through $(a, f(a))$. What kind of polynomial approximation might be a better approximation than the linear approximation? Why?

Learning Objectives (Definition of the derivative as a function):

- Describe the derivative of a function as a function itself.
- Given the graph of a function, sketch the graph of its derivative.

  Highlight the relationship between the idea of the derivative at a point and the slope of the secant line connecting this given point to a nearby point as a staring point. On one hand, we took limits to get the definition of the derivative at a given point. On the other hand, we could have chosen any other point to consider and carried out the same limiting process to obtain the derivative of the function at that other point.

Problems and takeaways (Definition of the derivative as a function):

1. Let $f(x) = x^2$.
   (a) What is the derivative of $f(x)$ at $x = 2$? What about at $x = 3$? At $x = -3$?

(b) What is the derivative of $f(x)$ at a "generic x-value" x?

(c) Sketch the graph of $f(x)$. On another set of axes immediately beneath that, sketch the graph of your answer from part (b).

As with all of these lecture outlines, this is just an outline: Feel free to present other examples.

2. **Definition**: The *derivative of* $f(x)$, labelled $f'(x)$ or $\frac{d}{dx}f(x)$, is the function $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ provided the limit exists. Informally, it is the "slope of $f(x)$". If the derivative exists on an interval, $f(x)$ is *differentiable* on that interval.

   See CLP-1 Definition 2.2.6.

   The rest of this outline is optional. The suggested optional in-class learning exercise would likely take about 10 minutes to run, so take that into consideration if you plan to use it.

3. (a) Form a cycle of 3 with students sitting near you.

   (b) Draw 3 sets of axes on a single sheet of paper: one at the top, one in the middle, and one at the bottom. On the top axes, sketch the graph of a function. (Your function should not be too complicated. You don't have to have an algebraic expression for your function.) Pass your paper to the left.

   (c) On the middle axes of the paper you received, sketch the graph of the derivative of the top function. Then fold the paper so that the top function is hidden. Pass your paper to the left.

   (d) On the bottom axes of the paper you received, sketch the graph of a function whose derivative is the middle function. Pass your paper to the left.

   (e) Unfold your paper. Are the top and bottom graphs similar? Are they identical? Why or why not?

4. **Definition**: A function $F(x)$ whose derivative is $f(x)$ is an *antiderivative* of $f(x)$.

   See CLP-1 Definition 4.1.1.

5. **Takeaway**: If $F(x)$ is an antiderivative of $f(x)$, then so is $F(x)+c$ where c is any constant.

   See CLP-1 Lemma 4.1.1.

Additional problems

- CLP-1 Problem Book Section 2.2: Q1-Q5, Q9, Q10, Q12, Q18, Q26.
- CLP-1 Problem Book Section 3.4.2: Q1, Q5.
- CLP-1 Problem Book Section 2.3: Q1-Q7.