Solutions to homework 11:

1. Determine if the following sets are countable, and prove your answers.

- (a) The set of all functions $f : \{0, 1\} \to \mathbb{N}$.
  - Let $G : S \to \mathbb{N} \times \mathbb{N}$ and be defined as $G(f) = \{f(0) = a, f(1) = b : (a, b) \in \mathbb{N} \times \mathbb{N}\}$.
  - Let $f, h \in S$ be functions such that $G(f) = G(h)$ then $G(f) = \{f(0) = a, f(1) = b : (a, b) \in \mathbb{N} \times \mathbb{N}\} = \{h(0) = a, h(1) = b : (a, b) \in \mathbb{N} \times \mathbb{N}\} = G(h)$.
  - This means that both f, h have the same inputs for outputs a, b hence $f = h$ due to the fact that the only possible inputs are 0 and 1.
  - Which shows that f is an injection.
  - Now let $(c, d) \in \mathbb{N} \times \mathbb{N}$, and let's define a function p so that $p(0) = c, p(1) = d$.
  - Therefore by definition, $G(p) = (c, d)$ showing us that $p \in S$ this a surjection.
  - Hence as G forms a bijection with $|S| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ and S is countable.
- (b) The set of all functions $f : \mathbb{N} \to \{0, 1\}$.
  - Let $G : S \to \mathcal{P}(\mathbb{N})$ and be defined as $G(f) = \{n : f(n) = 1\}$.
  - Let $f, h \in S$ be functions such that $G(f) = G(h)$.
  - This implies that $G(f) = \{n : f(n) = 1\} = \{n : h(n) = 1\} = G(h)$.
  - This means that both f, h have outputs 1 and 0 for the same inputs, therefore showing us that $f = h$.
  - Therefore G is an injection.
  - Now let $X \in \mathcal{P}(\mathbb{N})$ thus $X \subseteq \mathbb{N}$ and define a function $g : \mathbb{N} \to \{0, 1\}$ as $g(x) = 1$ when $x \in X$ otherwise $g(x) = 0$.
  - Hence by definition, $G(g) = X$ showing us that G is a surjection.
  - Therefore G a bijection showing us that $|S| = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$. Thus S is uncountable.

2. Prove the following statements

- (a) If A is countable but B is uncountable, then $B - A$ is uncountable.
  - Let's assume, to the contrary, that $B - A$ is countable.
  - Then $(B - A) \cup A = (A \cup B)$ is countable.
  - Therefore B is countable, however this contradicts with out assumption.
  - Hence, $B - A$ is uncountable.
- (b) Between any real numbers a, b such that $a < b$ there are uncountably many irrationals.
  - We can see that we can represent this set of numbers as $S = \{x : a < x < b, x \in \mathbb{I}\}$.
  - We can rephrase this expression to $S = \mathbb{R} - \{x : a < x < b, x \in \mathbb{Q}\}$.
  - We know that the set $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable.

  - Therefore, any subset of $\mathbb{Q}$ is countable hence $\{x : a < x < b, x \in \mathbb{Q}\}$ is therefore countable.
  - The original set $S = \mathbb{R} - \{x : a < x < b, x \in \mathbb{Q}\}$ is therefore uncountable.

3. Prove that $\mathbb{R}$ and $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ are equinumerous.

   - Let's consider a function $f : \mathbb{R} \to \mathbb{R}^+$.
   - We define the function $f(x) = e^x$ for $x \in \mathbb{R}$.
   - Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$. Then

$$e^a = e^b \tag{1}$$
$$e^a - e^b = 0 \tag{2}$$
$$e^b \cdot (e^{a-b} - 1) = 0 \tag{3}$$

   - We know that $\forall n \in \mathbb{R}$, $e^n \geq 0$. Hence $a - b = 0$ showing that $a = b$.
   - Thus f forms an injection and $|\mathbb{R}| \leq |\mathbb{R}^+|$.
   - Now let's consider a function $g : \mathbb{R}^+ \to \mathbb{R}$ and this function is defined as $g(x) = x$ for $x \in \mathbb{R}$.
   - Now let $c, d \in \mathbb{R}^+$ so that $g(c) = g(d)$.
   - Then, $g(c) = c = d = g(d)$ and thus shows us that g forms an injection.
   - Hence, $|\mathbb{R}^+| \leq |\mathbb{R}|$. And as we proved from earlier that $|\mathbb{R}| \leq |\mathbb{R}^+|$ by CSB $\mathbb{R}$ and $\mathbb{R}^+$ are equinumerous.

4. Let S, T be sets. Prove the following

   - (a) If $|S| \leq |T|$ then $|\mathcal{P}(S)| \leq |\mathcal{P}(T)|$.
     - Let $f : S \to T$ be a well-defined function that is an injection.
     - Now let $g : \mathcal{P}(S) \to \mathcal{P}(T)$ be defined as $g(f) = \{f(A) = B : A \subseteq S, B \subseteq T\}$.
     - Let $P, Q \in \mathcal{P}(S), X \in \mathcal{P}(T)$ such that $g(P) = g(Q) = X$.
     - Then this implies that $g(P) = \{f(P) = X : P \subseteq S, X \subseteq T\} = \{f(Q) = X : Q \subseteq S, X \subseteq T\} = g(Q)$.
     - Hence by definition $P = Q$ showing that g is an injection as we know that f is also an injection therefore $|\mathcal{P}(S)| \leq |\mathcal{P}(T)|$.
   - (b) If $|S| = |T|$ then $|\mathcal{P}(S)| = |\mathcal{P}(T)|$.
     - Let $h : S \to T$ form a bijection.
     - Now let $g : \mathcal{P}(S) \to \mathcal{P}(T)$ be defined as $g(h) = \{h(A) = B : A \subseteq S, B \subseteq T\}$.
     - As we proved in (a) we know that g forms an injection.
     - Now let's prove that g forms a surjection.
     - Let $X \subseteq S, Y \subseteq T$ such that $f(X) = Y$.
     - Then by definition $g(f) = \{f(X) = Y : X \subseteq S, Y \subseteq T\}$.

  - Hence is a surjection as required.
  - Therefore, g is a bijection hence $|\mathcal{P}(S)| = |\mathcal{P}(T)|$..

5. Show that there exist infinitely many pairs of distinct natural numbers a, b such that $17^a - 17^b$ is divisible by 2022.

   - Consider a sequence of 2023 numbers $17^1, 17^2, 17^3, \cdots, 17^{2023}$.
   - There are at most 2022 remainders when divided by 2022.
   - However there exists 2023 numbers in the sequence.
   - Thus there must exists two numbers with the same remainder when divided by 2022.
   - Then $\exists a, b \in \mathbb{N}$ where $17^a = 2022k + r, 17^b = 2022\ell + r$ where $r \in \mathbb{N}$ and $k, \ell \in \mathbb{Z}, k, \ell \geq 0$.

$$\begin{aligned} 17^b - 17^a &= (2022\ell + r) - (2022k + r) && (4) \\ &= 2022(\ell - k) && (5) \end{aligned}$$

   - Hence, $2022 \mid (17^b - 17^a)$.
   - In the previous example, we considered the sequence with the interval $[17^1, 17^{2023}]$.
   - However this relation will still be satisfied for other intervals too.
   - To generalize, there will always exist two natural numbers a, b where $2022 \mid (17^b - 17^a)$ for all intervals $[17^n, 17^{n+2022}], n \in \mathbb{N}$.
   - We know that $\mathbb{N}$ is an infinite set hence there are infinitely many pairs that exist.

6. Prove that $(-\infty, -\sqrt{29})$ and $\mathbb{R}$ are equinumerous by constructing an explicit bijection.

   - Let $f : (-\infty, -\sqrt{29}) \to \mathbb{R}$ be defined as $f(x) = \log(-x - \sqrt{29})$
   - And also let $g : \mathbb{R} \to (-\infty, -\sqrt{29})$ be defined as $g(y) = -e^y - \sqrt{29}$.
   - Then

$$\begin{aligned} f(g(y)) &= \log(-(-e^y - \sqrt{29}) - \sqrt{29}) && (6) \\ &= \log(e^y) && (7) \\ &= y && (8) \\ g(f(x)) &= -e^{\log(-x-\sqrt{29})} - \sqrt{29} && (9) \\ &= -(-x - \sqrt{29}) - \sqrt{29} && (10) \\ &= x && (11) \end{aligned}$$

   - We can see that $f \circ g$ and $g \circ f$ are both identity functions.
   - Thus f has a two side inverse which is g.
   - Therefore $|(-\infty, -\sqrt{29})| = |\mathbb{R}|$. Thus equinumerous as required.

7. Prove or disprove: for any non-empty sets A, B, C if $|A \times B| = |A \times C|$ then $|B| = |C|$.
   - First, let $f : |A \times B| \to |A \times C|$ be defined as $f((a_k, b_n)) = \{(a_k, c_n) : k, n \in \mathbb{N}\}$.
   - We know that $|A \times B| = |A \times C|$ hence we can see that f is a bijection.
   - Now let $g : B \to C$ be defined as $g(b_k) = \{c_k : f((a_1, b_k)) = (a_1, c_k)\}$.
   - Let's prove whether g is a bijection.
   - Let b_i, b_j such that $g(b_i) = g(b_j)$, then $g(b_i) = \{c_i : f((a_1, b_i)) = (a_1, c_i)\} = \{c_j : f((a_1, b_j)) = (a_1, c_j)\} = g(b_j)$.
   - We know that f is a bijection hence $c_i = c_j$, therefore showing us that g is an injection.
   - Now let $c_\ell \in C$, since f is a surjection $\exists (a_1, b_\ell) \in A \times B$ such that $f((a_1, b_\ell)) = (a_1, c_\ell)$.
   - Therefore, by the definition of g $\exists b_k \in B$ where $g(b_k) = c_k$.
   - Thus, g is also a surjection showing us that g is a bijection.
   - Since we proved that g is a bijection, $|B| = |C|$.

8. Let A be a finite set and $f : \mathbb{R} \to A$. Show that there exists some $a \in A$ such that $f^{-1}(\{a\})$ is uncountable.
   - Let's assume, to the contrary, that $f^{-1}(\{a\})$ is countable.
   - Then we $\mathbb{R}$ can be expressed with $f^{-1}(\{a\})$ as
   $$\mathbb{R} = \bigcup_{a \in A} f^{-1}(\{a\}), \forall a \in A.$$
   - As we know that A is a finite set, then $\mathbb{R}$ is the union of a finite amount of sets, therefore shows us that $\mathbb{R}$ is finite.
   - However, this contradicts with the fact that $\mathbb{R}$ is an infinite set.
   - Hence, by contradiction we can declare that $f^{-1}(\{a\})$ is uncountable.