Week 2 Lecture Outline
September 13 (2023)

Topics: Limits; horizontal and vertical asymptotes

Instructor notes:

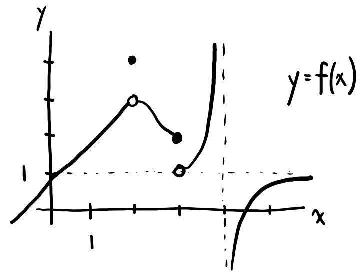
- The notion of *limit* is approached with an "intuitive" definition in MATH 100. Drawing some pictures and using some explicit (including graphically) examples helps students build their understanding of this core concept. There is a balance of dealing with interesting cases and also having them comfortable that many functions (e.g. continuous ones – an as-yet undefined concept in the course) behave as they expect.
- We will introduce vertical and horizontal asymptotes this week. This naturally follows on from the Week 1 conversation about functions. In particular, rational functions provide some rich and manageable examples for students to become comfortable with these ideas.
- Consider some examples that bust myths: e.g. many students believe graphs of functions cannot cross asymptotes. The damped oscillator makes is a good example for such a discussion as students can link the mathematical behavour to a physical behaviour they can observe.
- It's good to make explicit links to sections in the CLP Notes so students get used to the idea the text is an important source for them and we expect them to work with it. Mention the additional problems at the end in the CLP Notes as encouragement for students to do problems as part of developing their understanding of the mathematics.

Learning Objectives (Limits):

- Explain using both words and pictures what $\lim_{x\to a} f(x) = L$, $\lim_{x\to a^-} f(x) = L$, and $\lim_{x\to a^+} f(x) = L$ mean (including the case where L is equal to ∞ or $-\infty$).
- Explain using using both words and pictures what $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = L$ mean (including the case where L is equal to ∞ of $-\infty$).
- Find the limit of a function at a point given the graph of the function.
- Evaluate limits of rational, trigonometric, exponential, and logarithmic functions.

Problems and takeaways (Limits):

1. Consider the function $f(x)$ whose graph is below.



What value (if any) is $f(x)$ "arbitrarily close to" if x is "sufficiently close to" 1? 2? 3? 4?

2. **Definition**: $\lim_{x\to a} f(x) = L$ (read "the limit of $f(x)$ as x approaches a is equal to L") means $f(x)$ is arbitrarily close to L provided x is sufficiently close to a (but not equal to a).

   See CLP-1 Definition 1.3.3.

3. What value (if any) is $f(x)$ arbitrarily close to if x is sufficiently close to 3 and $x < 3$? What if x is sufficiently close to 3 and $x > 3$?

4. **Definition**: One-sided limits are defined analogously.

   See CLP-1 Definition 1.3.7.

5. What value (if any) is $f(x)$ arbitrarily close to if x is sufficiently close to 4 and $x < 4$? What if x is sufficiently close to 4 and $x > 4$?

6. **Definition**: $\lim_{x\to a} f(x) = \infty$ means $f(x)$ is arbitrarily large and positive provided x is sufficiently close to a (but not equal to a). $\lim_{x\to a} f(x) = -\infty$, as well as one-sided versions, are definited analogously.

   See CLP-1 Definitions 1.3.10 and 1.3.11.

7. Find $\lim_{x\to 5} \left(x^3 - x\right)$.

8. **Takeaway**: Polynomials like $x^3 - x$ are "nice" functions: their limits can be computed simply by "plugging in" values of x. But...

9. Find $\lim_{x\to 2} \dfrac{x-2}{x^2+x-6}$.

10. **Takeaway**: ...not all functions are "nice"!

11. Recall the Hill function $f(x) = \dfrac{Ax^n}{B+x^n}$. Find $\lim_{x\to\infty} f(x)$ by first determining which term in the denominator is "dominant" for large values of x.

12. **Takeaway**: There is a more mechanical method of computing the limit, starting by dividing the numerator and denominator of $f(x)$ by x^n — but we will favour *asymptotic reasoning* , where we investigate the behaviour of functions by determining which terms are "dominant" for particular values of x.

Learning Objectives (Asymptotes):

- Explain using both informal language and the language of limits what it means for a function to have a horizontal or vertical asymptote.
- Given a simple function, find its vertical and horizontal asymptotes by asymptotic reasoning or by taking limits.

Problems and takeaways (Asymptotes):

1. (a) Find the following limits: $\lim_{x\to-\infty} \frac{x-1}{x+3}$, $\lim_{x\to\infty} \frac{x-1}{x+3}$.

   (b) Find the following limits: $\lim_{x\to-3^+} \frac{x-1}{x+3}$, $\lim_{x\to-3^-} \frac{x-1}{x+3}$, $\lim_{x\to-3} \frac{x-1}{x+3}$.

   (c) What are the intercepts of $f(x) = \frac{x-1}{x+3}$?

   (d) Sketch the graph of $f(x) = \frac{x-1}{x+3}$.

2. **Takeaway**: Computing limits gives us information about the shape of a function.
3. **Definition**: A function has a *horizontal asymptote* $y = L$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$.

   A function has a *vertical asymptote* $x = a$ if either $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

   See CLP-1 Section 3.6.1.
4. Can a function cross its horizontal asymptote?
5. Can you come up with a function that crosses its horizontal asymptote once? Twice? Many times? Infinitely many times?
6. Sketch the graph of the "damped oscillator" $f(x) = e^{-x}\sin x$ for $x \geq 0$.
7. **Takeaway**: It is *not true* that a function cannot cross its horizontal asymptote.

Additional Problems:

1. CLP-1 Problem Book Section 1.3: Q1-Q17.
2. CLP-1 Problem Book Section 1.5: Q1-Q7, Q27.
3. CLP-1 Problem Book Section 1.5: Q8, Q13-Q15, Q17-Q19 — but solve these problems using *asymptotic reasoning* (investigating the behaviour of functions by determining which terms are "dominant").
4. CLP-1 Problem Book Section 3.6.1: Q1, Q4, Q5.