INSTRUCTOR NOTES

Begin by getting students to sit in their teams. If possible, members should sit facing each other. Do not allow students to change teams.

If only 2 members show up for a team, they may work with another small team, but must submit their own worksheet. If only 1 member shows up for a team, they must work with another team, and may submit a blank worksheet with only their name on it for attendance-taking purposes.

Finally, pass out the handouts and announce the first question. Remember to have a routine to close questions (*e.g.* countdowns).

At the end of the class, remember to collect worksheets.

*This week's tip: **honour student contributions**.* It is not always easy for students to volunteer an answer. Encourage volunteers by respecting all contributions that you get, right or wrong.

NOTES ON QUESTIONS

The large lecture prior to this small class has a section on the definition of the derivative. That definition is used to define the centrepiece of this class, the exponential function e^x.

1. **3 minutes**.
   To close the question, ask a team with the correct answer to put it on the board.
   $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$.

2. **2 minutes**.
   To close the question, ask a team with a sensible answer to put it on the board. Label the y-intercept if it is not already labelled.

3. **3 minutes**.
   Teams may struggle to know when they're "done", so make sure to visit many teams quickly.
   To close the question, ask a team with the correct answer to put it on the board.
   $f'(0) = \lim_{h\to 0} \frac{b^h - 1}{h}$.

4. **6 minutes**.
   Prompt teams that are stuck to begin with their calculation from question 4, but with e instead of b.
   To close the question, write the conclusion ($\frac{d}{dx}e^x = e^x$) on the board, and draw a box around it to indicate its importance.
   $f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x = f(x)$.

5. **3 minutes**. 1 minute for teams to write down answers, 2 minutes for class discussion.
   There are many acceptable answers: ideal population growth, continuous compound interest, a chain reaction of uranium atoms undergoing fission, *etc.* Encourage students to come up with answers relevant to their intended specialization.

6. **2 minutes**.
   To close the question, ask a team that got the answer below to volunteer it. Then write down the answer on the board.
   $y(t) = e^t$.

7. **2 minutes**.
   To close the question, ask for teams to volunteer answers. If they came up with specific answers, write them down before writing down the general answer.
   e.g. $y(t) = 2e^t$, $y(t) = 3e^t$, $y(t) = -5e^t$, $y(t) = Ce^t$ for any constant C.

8. **5 minutes**.
   This will be a tricky problem for many teams. Prompt them along the following steps: identify the general solution $y(t) = Ce^t$; what is $y(0)$? what does C represent?
   $y(t) = 200e^t$

9. **Remaining time** for this question.
   When $200e^t = 500$, $300e^t = 750$.

SMALL CLASS: The exponential function

In this class, you will describe the exponential function e^x *in terms of its derivative, and explore some applications of the function to differential equations.*

Contributing team members

Student number	Last name	First name

Small class questions

1. Write down the limit definition of the derivative $f'(x_0)$.

Answer:
Scribe:

2. Imagine a constant $b > 1$. Sketch the graph of the function $f(x) = b^x$.

Answer:
Scribe:

3. (★☆☆☆) Write down the limit definition of $f'(0)$ for the function $f(x) = b^x$. You should simplify your answer, but your answer should be a limit.

Answer:
Scribe:

4. (★☆☆☆, key concept) One definition of *Euler's number* e is that it is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Use this definition along with the limit definition of derivative to find the derivative of $f(x) = e^x$.

Answer:
Scribe:

5. (Key concept) A *differential equation* is an equation involving an unknown function and its derivatives. A *solution* to a differential equation is a function that satisfies that equation.

   List two or more real-life phenomena that could be described using the differential equation $y'(t) = y(t)$.

   Answer:

   Scribe:

6. What is one solution to the differential equation $y'(t) = y(t)$?

   Answer:

   Scribe:

7. Propose another solution to the differential equation $y'(t) = y(t)$. Can you demonstrate why your proposal is correct?

   Answer:

   Scribe:

8. (★☆☆☆, key concept) Imagine a quantity A satisfying the differential equation $y'(t) = y(t)$. At $t = 0$, A is equal to 200. What is the solution to the differential equation that describes A?

   (Solving a differential equation given a point on the solution, as you are asked to do here, is an example of an *initial value problem*.)

Answer:
Scribe:

9. (★☆☆☆) Imagine a quantity B satisfying the same differential equation $y'(t) = y(t)$. At $t = 0$, B is equal to 300. What is B when A is equal to 500?

Answer:
Scribe:

Practice questions

The questions below are for practice. They do not contribute to your grade, and it is not expected that you complete them during your small class. However, you are strongly encouraged to work through them.

10. (★★★☆) Propose a nontrivial (not 0) solution to the differential equation $y'(t) = -y(t)$. Then demonstrate that your proposal is correct using the limit definition of derivative.

11. As question 4 implies, there are multiple definitions of Euler's number e. One definition has to do with compound interest.

    Using an example, explain carefully how the quantity

    $$\left(1 + \frac{1}{n}\right)^n$$

    is equal to the total accumulated value after 1 year given an annual compound interest rate of 100% and a compounding frequency of n times per year.

12. (★★★★) Let e be defined to be $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. Show that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

    Hint: Let $m = e^h - 1$. You may need to use $\log(x)$, the inverse function to e^x, to simplify your limit.

    Note: This question shows that the compound interest definition given in this question is consistent with the definition given in question 4.