Solutions to homework 1:

1. Your solution to question 1.

   Proof. Prove that if $a \in \mathbb{Z}$, then $4 \nmid (a^2+1)$.

   - Let's assume that $a \in \mathbb{Z}$.
   - By euclidean division $a = 4k, a = 4\ell+1, a = 4l+2, a = 4m+3$ when $\ell, k, m, n \in \mathbb{Z}$.
   - Case 1: $a = 4k(k \in \mathbb{Z})$. Then, $a^2 + 1 = 4(4k^2) + 1$.
   - Notice that when $k \in \mathbb{Z}$, then $4k^2 \in \mathbb{Z}$ thus we can conclude $4 \nmid (a^2+1)$.
   - Case 2: $a = 4\ell + 1(\ell \in \mathbb{Z})$. Then, $a^2 + 1 = 4(4\ell^2 + 2\ell) + 2$.
   - As $(4\ell^2 + 2\ell) \in \mathbb{Z}$, then $4 \nmid (a^2+1)$.
   - Case 3: $a = 4m + 2(m \in \mathbb{Z})$. Then, $a^2 + 1 = 4(4m^2 + 4m) + 4$.
   - We can conclude that $(4m^2 + 4m) \in \mathbb{Z}$, so $4 \nmid (a^2+1)$.
   - Case 4: $a = 4n + 3(n \in \mathbb{Z})$. Then, $a^2 + 1 = 4(4n^2 + 6n + 2) + 1$.
   - We can conclude that $4 \nmid (a^2 + 1)$ as we know that $(4n^2 + 6n + 2)$.

   $\square$

2. Your solution to question 2.

   Proof. Let x be a positive number. Prove that if $2x - \frac{1}{x} > 1$, then $x > 1$.

   - Let's assume $x > 0, 2x - \frac{1}{x} > 1$ when $x \in \mathbb{R}$.
   - $2x - \frac{1}{x} - 1 = \frac{1}{x}(2x^2 - x - 1)$ for $x \neq 0$.
   - And $\frac{1}{x} - 1 = \frac{1}{x}(2x^2 - x - 1) = \frac{1}{x}(2x+1)(x-1)$.
   - When $x > 0$, $\frac{1}{x} > 0$ and $(2x + 1) > 0$.
   - Therefore, we can conclude that in order to satisfy $\frac{1}{x}(2x+1)(x-1) > 0$. $(x - 1) > 0$ needs to be satisfied.
   - To conclude, $x > 1$.

   $\square$

3. Your solution to question 3.

   Proof. Prove that if $k \in \mathbb{Z}$ then $3 \mid \big(k(2k+1)(4k+1)\big)$.

   $\square$

4. Your solution to question 4.

   Proof. $\square$

5. Your solution to question 5.

   Proof. □

6. Your solution to question 6.

   Proof. Let $x \in \mathbb{R}$. Then, prove that $x^2 + |x - 6| > 5$.

   - Let's assume that $x \in \mathbb{R}$.
   - Case 1: When $x \geq 6$, then

   $$x^2 + |x - 6| - 5 \qquad bottom \tag{1}$$

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   □