Solutions to homework 8:

1. Prove or disprove. If a relation $\mathcal{R}$ on a set A is symmetric and transitive, then it is also reflexive.

   - Let $x, y \in A$ such that $x \,\mathcal{R}\, y$.
   - As $\mathcal{R}$ is symmetric $y \,\mathcal{R}\, x$.
   - Also, as $\mathcal{R}$ is transitive, $(x \,\mathcal{R}\, y) \wedge (y \,\mathcal{R}\, x) \implies x \,\mathcal{R}\, x$.
   - Therefore, $\mathcal{R}$ is reflexive.

2. Define a relation on $\mathbb{Z}$ as $a \; R \; b$ if $3 \mid (5a - 8b)$. Is R an equivalence relation? Justify your answer.

   - Let $a \in \mathbb{Z}$, then $5a - 8a = -3a = 3(-a)$ and $-a \in \mathbb{Z}$.
   - Thus, $3 \mid (5a - 8a)$ making $a \; R \; a$ showing that R is reflexive.
   - Let $a, b \in \mathbb{Z}$ such that $a \; R \; b$, then $(5a - 8b) = 3k$ $(k \in \mathbb{Z})$.

$$5a - 8b = 3k \tag{1}$$
$$8a - 5b = 3k + 3a + 3b \tag{2}$$
$$5b - 8a = 3(-k - a - b) \tag{3}$$

   - $5b - 8a = 3(-k - a - b)$, $-k - a - b \in \mathbb{Z}$ therefore $3 \mid (5b - 8a)$.
   - Hence, $b \; R \; a$ and R is reflexive.
   - Now let $a, b, c \in \mathbb{Z}$ such that $a \; R \; b$ and $b \; R \; c$.
   - Then $5a - 8b = 3k$ and $5b - 8c = 3\ell$ for $k, \ell \in \mathbb{Z}$.

$$(5a - 8b) + (5b - 8c) = 3(k + \ell). \tag{4}$$
$$5a - 8c = 3(k + \ell + b) \tag{5}$$

   - As we know that $k + \ell + b \in \mathbb{Z}$, then $3 \mid (5a - 8c)$.
   - Therefore, $(a \; R \; b) \wedge (b \; R \; c) \implies a \; R \; c.$ and R is transitive.

3. Determine whether the following relations are reflexive, symmetric and transitive.

   - (a) On the set X of all functions $\mathbb{R} \to \mathbb{R}$, we define the relation

$$f \,\mathcal{R}\, g \text{ if there exists } x \in \mathbb{R} \text{ such that } f(x) = g(x).$$

     - Let $f(x) = x$, then $x = x, \forall x \in \mathbb{R}$
     - Hence, $f \,\mathcal{R}\, f$ making $\mathcal{R}$ reflexive.
     - Now assume $g(x), f(x)$ exist such that $f(\alpha) = g(\alpha)$ for some $\alpha \in \mathbb{R}$.
     - Then for some $\alpha \in \mathbb{R}$, $g(\alpha) = f(\alpha)$.
     - Therefore, $g \,\mathcal{R}\, f$ hence $\mathcal{R}$ is symmetric.

  - Let $f(x) = x^2, g(x) = x + 1, h(x) = x^2 + 1$.
  - Then there exists some $p, q \in \mathbb{R}$ such that $f(p) = g(p), g(q) = h(q)$.
  - So $f \mathcal{R} g, g \mathcal{R} h$.
  - However there exists no real number that makes $f(x) = x^2 = x^2 + 1 = h(x)$.
  - Hence, $f \not\mathcal{R} h$ making $\mathcal{R}$ not transitive.

- (b) Let R be a relation on $\mathbb{Z}$ defined by:

$$x \; R \; y \text{ if } xy \equiv 0 (\mod 4).$$

  - Let $x = 1$ then $x \cdot x = 1 \cdot 1 = 1$.
  - And $1 \not\equiv 0 (\mod 4)$. Hence, $1 \not R 1$ so R is not reflexive.
  - Let's take $x, y \in \mathbb{Z}$ such that $xy = 4k$ $(k \in \mathbb{Z})$.
  - Then $yx = 4k$ $(k \in \mathbb{Z})$ showing that $y \; R \; x$.
  - Therefore, R is symmetric.
  - Let $x, y, z \in \mathbb{Z}$ such that $x = 1, y = 4, z = 1$.
  - As $xy = 4$ and $yz = 4$ showing that $x \; R \; y$ and $y \; R \; z$.
  - However, $xz = 1$ showing that $x \not R z$.
  - Hence R is not transitive.

4. Let A be a non-empty set and $P \subseteq \mathcal{P}(A)$ and $Q \subseteq \mathcal{P}(A)$ be partitions of A. Show that R defined as

$$R = \{S \cap T : S \in P, T \in Q\} - \{\emptyset\}$$

is a partition of A.

- In order for R to be a partition of A, we need to show that $\bigcup_{X \in R} X = A$ and $\bigcap_{X \in R} X = \emptyset$.
- First, let's choose an element $a \in A$. Then there exists a P and T such that $a \in S \cap T$. And because $S \cap T \subseteq R$, therefore $A \subseteq \bigcup_{X \in R} X$.
- Next, as $X \in R$ and due to the definition of R, $X \subseteq A$. Hence, $\bigcup_{X \in R} X \subseteq A$.
- Therefore, $\bigcup_{X \in R} X = A$.
- Now, let $U_1, U_2 \in R$. And $U_1 = S_1 \cap T_1, U_2 = S_2 \cap T_2$.
- We have to prove that either $U_1 \cap U_2 = \emptyset$ or $U_1 = U_2$.
- Let's first assume that $U_1 \cap U_2 = \emptyset$, then we're done.
- Now let's assume that $U_1 \cap U_2 \neq \emptyset$. Hence, $\exists x \in U_1 \cap U_2$. Then $x \in S_1, S_2, T_1, T_2$.
- As $S_1, S_2 \in P, S_1$ must be equals to S_2.
- Similarly to T_1, T_2, $T_1 = T_2$ must be satisfied.
- Hence, $U_1 \cap U_2 \neq \emptyset$ therefore $U_1 = U_2$.
- Thus, we can conclude that $\bigcap_{X \in R} X = \emptyset$. Making R a partition of A.

5. Let E be a non-empty set and $x \in E$ be a fixed element of E. Consider the relation $\mathcal{R}$ on $\mathcal{P}(E)$ defined as

$$A \mathcal{R} B \Longleftrightarrow (x \in A \cap B) \vee (x \in \overline{A} \cap \overline{B}),$$

where for any set $S \subseteq E$, we write $\overline{S} = E - S$ for the compliment of S in E. Prove or disprove that $\mathcal{R}$ is an equivalence relation.

   - Let $X \in \mathcal{P}(E)$, and $x \in X$. Then $x \in X \cap X$ showing us that $(x \in X \cap X) \vee (x \in \overline{X} \cap \overline{X})$ is true. Hence, $X \mathcal{R} X$ therefore $\mathcal{R}$ is reflexive.
   - Let $X, Y \in \mathcal{P}(E)$ such that $X \mathcal{R} Y$. Then $(x \in X \cap Y) \vee (x \in \overline{X} \cap \overline{Y})$ is true.
   - This is equivalent to $(x \in Y \cap X) \vee (x \in \overline{Y} \cap \overline{X})$, hence $Y \mathcal{R} X$. Therefore, $\mathcal{R}$ is symmetric.
   - Let $X, Y, Z \in \mathcal{P}(E)$ such that $X \mathcal{R} Y$ and $Y \mathcal{R} Z$.
   - Then $(x \in X \cap Y) \vee (x \in \overline{X} \cap \overline{Y})$ and $(x \in Y \cap Z) \vee (x \in \overline{Y} \cap \overline{Z})$ are both true statements.
   - There are only two possible cases that can exists.
   - Case 1: $(x \in X \cap Y)$ and $(x \in Y \cap Z)$.
   - In this case, $x \in X \cap Z$.
   - Case 2: $(x \in \overline{X} \cap \overline{Y})$ and $(x \in \overline{Y} \cap \overline{Z})$.
   - In other words, $x \in E - (X \cup Y)$ and $x \in E - (Y \cup Z)$.
   - This clarifies that $x \in E - (X \cup Z)$, hence $x \in (\overline{X} \cap \overline{Z})$.
   - By these two cases we can prove that $(x \in X \cap Z) \vee (x \in \overline{X} \cap \overline{Z})$ is true.
   - Hence, $X \mathcal{R} Z$ thus showing that $\mathcal{R}$ is transitive.

6. Let $n \geq 2$ and i be integers. For $0 \leq i \leq n-1$, define

$$X_i = \{x \in \mathbb{Z} | x = nk + i, \text{for some } k \in \mathbb{Z}\} \text{ and}$$
$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | a, b \in X_i \text{ for some } i\}.$$

Show that

   - (a) $S = \{X_0, \ldots, X_{n-1}\}$ forms a partition of $\mathbb{Z}$.
     - We have to prove that $\bigcap_{i=0}^{n-1} X_i = \emptyset$ and $\bigcup_{i=0}^{n-1} X_i = \mathbb{Z}$.
     - First let's prove that $\bigcap_{i=0}^{n-1} X_i = \emptyset$.
     - Let's $X_p, X_q \in S$. If we assume that $X_p \cap X_q = \emptyset$ we're done.
     - Now let's assume that $X_p \cap X_q \neq \emptyset$.
     - Let's assume that an element α exists such that $\alpha \in X_p, X_q$.
     - Then $\alpha \equiv p(\mod n) \equiv q(\mod n)$. And $p = q$ must be satisfied since $0 \leq p, q \leq n-1$.

- Hence, $X_p \cap X_q \neq \emptyset$ therefore $X_p = X_q$.
- Now let's prove that $\bigcup_{i=0}^{n-1} X_i = \mathbb{Z}$.
- Let's choose an element $x \in \bigcup_{i=0}^{n-1} X_i$, then $x \in \mathbb{Z}$.
- Hence, $\bigcup_{i=0}^{n-1} X_i \subseteq \mathbb{Z}$.
- Now let's choose an element $y \in \mathbb{Z}$.
- From Euclidean division, we can express any integer(including y) in the form of $y = a \cdot n + b$ where $a, b, n \in \mathbb{Z}$ and $b \in [0, n-1]$.
- This is equivalent to some X_b then $y \in X_b$.
- Hence, $\mathbb{Z} \subseteq \bigcup_{i=0}^{n-1} X_i$.
- Therefore, S is a partition of $\mathbb{Z}$.

- (b) R is an equivalence relation on $\mathbb{Z}$.
  - Let $a \in \mathbb{Z}$, then $a \in X_i$ for some $i \in [0, n-1]$.
  - If $a \in X_i$ then $a \in X_i$ therefore for $(a, a) \in \mathbb{Z} \times \mathbb{Z}$, $a, a \in X_i$ hence $a\ R\ a$ so R is reflexive.
  - Now let $a, b \in \mathbb{Z}$ such that $a, b \in X_i$ for some $i \in [0, n-1]$.
  - Then $b, a \in \mathbb{Z}, X_i$ thus $b\ R\ a$ thus R is symmetric.
  - Let $a, b, c \in \mathbb{Z}$ such that $a\ R\ b$ and $b\ R\ c$.
  - Then we can say $a, b \in X_i$ and $b, c \in X_j$ for $i, j \in [0, n-1]$.
  - Because S is a partition of $\mathbb{Z}$, $X_i \cap X_j = \emptyset$ or $X_i = X_j$.
  - And we can see that $X_i \cap X_j = b \neq \emptyset$ hence $X_i = X_j$.
  - Thus, we can conclude that $c \in X_i = X_j$. Therefore $a, c \in X_i, X_j$.
  - Then we can conclude that $a\ R\ c$ thus R is transitive.
  - As R is reflexive, symmetric and transitive, R is an equivalence relation.

- (c) S equals the set of the equivalence classes of R.
  - Let's choose $X_m \in S$, we know that S is a partition hence X_m is a non-empty set.
  - So let's choose an element $x \in X_m$, we know that R is reflexive. Hence, $x \in [x]$. Then $X_m \subseteq [x]$.
  - Now we choose $x \in [x]$, then $x \in X_n$. As we proved above then $X_n = X_m$. Hence, $[x] \subseteq X_m$.
  - Therefore, $X_m = [x]$ which shows that an element of S is an equivalence class.
  - Now let's choose an equivalence class from R, $[y]$,
  - Since R is reflexive, $y \in [y]$ then $y \in X_f$. Therefore, $[y] \subseteq X_f$.
  - Let's choose an element $y \in X_g$, we know by fact that $y \in [y]$.
  - As $X_g, X_f \in S$ which is a partition of $\mathbb{Z}$. Therefore, $X_g = X_f$.
  - Thus, $X_g = X_f \subseteq [y]$. Therefore, $[y] = X_f$.
  - Therefore S is the set of equivalence classes of R.

7. Suppose that $n \in \mathbb{N}$ and $\mathbb{Z}_n$ is the set of equivalence classes of congruence modulo n on $\mathbb{Z}$. In this question we will call an element $[u]_n$ if there is another class $[v]_n$ so that

$$[u]_n \cdot [v]_n = [u \cdot v]_n = [1]_n.$$

That is, $[u]_n$ has a multiplicative inverse. For example, since $[3]_7 \cdot [5]_7 = [15]_7 = [1]_7$, we say that $[3]_7$ is invertible. However, $[2]_4$ is not invertible (you can check!).
Now, define a relation R on $\mathbb{Z}_n$ by $x\ R\ y$ iff $xu = y$ for some invertible $[u]_n \in \mathbb{Z}_n$.

- (a) Show that R is an equivalence relation.
  - First, let $[a]_n \in \mathbb{Z}_n$ then $[a]_n \cdot [1]_n = [a \cdot 1]_n = [a]_n$.
  - Now let's prove that $[1]_n$ is an invertible class.

$$\begin{aligned} [1]_n \cdot [1]_n &= [1 \cdot 1]_n && (6) \\ &= [1]_n. && (7) \end{aligned}$$

  - Hence, as $[1]_n$ is an invertible class $[a]_n\ R\ [a]_n$ so R is reflexive.
  - Let $[a]_n, [b]_n \in \mathbb{Z}_n$ such that $[a]_n\ R\ [b]_n$.
  - Then there exists some invertible class $[u]_n \in \mathbb{Z}_n$ so that $[a]_n \cdot [u]_n = [b]_n$.
  - We know that $[u]_n$ is invertible, hence there exists some class $[v]_n \in \mathbb{Z}_n$ such that $[u]_n \cdot [v]_n = [1]_n$.
  - Then

$$\begin{aligned} [a]_n \cdot [u]_n \cdot [v]_n &= [b]_n \cdot [v]_n && (8) \\ [a]_n &= [b]_n \cdot [v]_n && (9) \end{aligned}$$

  - We know that as $[u]_n$ is invertible hence $[v]_n$ is invertible.
  - Therefore, $[b]_n\ R\ [a]_n$ showing that R is symmetric.
  - Finally, let $[a]_n, [b]_n, [c]_n \in \mathbb{Z}_n$ such that $[a]_n\ R\ [b]_n$ and $[b]_n R [c]_n$.
  - Then there exists invertible $[p]_n, [q]_n \in \mathbb{Z}$ such that $[a]_n \cdot [p]_n = [b]_n, [b]_n \cdot [q]_n = [c]_n$.

$$\begin{aligned} [a]_n \cdot [p]_n &= [b]_n && (10) \\ [a]_n \cdot [p]_n \cdot [q]_n &= [b]_n \cdot [q]_n && (11) \\ &= [c]_n && (12) \end{aligned}$$

  - We know that $[p]_n, [q]_n$ are invertible, thus we can see that $[p]_n \cdot [q]_n$ is also invertible. Thus $[a]_n\ R\ [c]_n$ so R is transitive.
  - Therefore, R is an equivalence relation.
- (b) Compute the equivalence classes of this relation for $n = 6$.
  - The only possible invertible classes are $[1]_n, [5]_n$.
  - If we compute each equivalence class for $\mathbb{Z}_6$.
  - $[[0]_6] = \{[0]_6\}$,$[[1]_6] = \{[1]_6, [5]_6\}$,$[[2]_6] = \{[2]_6, [4]_6\}$ and $[[3]_6] = \{[3]_6\}$.
  - These for equivalent classes cover all elements in $\mathbb{Z}_6$, hence these classes are the equivalence classes for $\mathbb{Z}_6$.