Solutions to homework 7:

1. We define a relation $\mathcal{R}$ on $\mathcal{P}(\{1,2\})$(the power set of $\{1,2\}$) by

$$S\mathcal{R}T \iff S \cap T = \emptyset.$$

   Write down all the elements in $\mathcal{R}$.

   - $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
   - Therefore, $\mathcal{R} = \{(\emptyset, \{1\}), (\{1\}, \emptyset), (\emptyset, \{2\}), (\{2\}, \emptyset), (\emptyset, \{1,2\}), (\{1,2\}, \emptyset), (\{1\}, \{2\})(\{2\}, \{1\})\}$

2. Let R be a relation on a nonempty set A. Then $\overline{R} = (A \times A) - R$ is also a relation on A. Prove or disprove each of the following statements.

   - If R is reflexive, then $\overline{R}$ is reflexive.
     - Let's assume that R is reflexive, then $\forall a \in A$, $a\ R\ a$.
     - Then $\forall a \in A$, $(a,a) \in R$.
     - Therefore, $(a,a) \notin \overline{R}$ and this shows that $\overline{R}$ is not reflexive.
   - If R is symmetric, then $\overline{R}$ is symmetric.
     - Let's prove by the contrapositive.
     - Then if $\overline{R}$ is not symmetric, then R is not symmetric.
     - Let's say for $a, b \in A$, $(a,b) \in \overline{R}$.
     - As $\overline{R}$ is not symmetric we can conclude that $(b,a) \notin \overline{R}$.
     - This is equivalent to saying that $(b,a) \in R$ however $(a,b) \notin R$.
     - Therefore, R is also not symmetric.
     - Thus as the contrapositive is true, the original statement is true.
   - If R is transitive, then $\overline{R}$ is transitive.
     - Let's prove by the contrapositive.
     - If $\overline{R}$ is not transitive, then R is not transitive.
     - Let's $(a,b),(b,c),(a,c) \notin \overline{R}$ for $a,b,c \in A$, thus proving that $\overline{R}$ is not transitive.
     - However, $(a,b),(b,c),(a,c) \notin \overline{R}$ shows us that $(a,b),(b,c),(a,c) \in R$.
     - Therefore, $a\ R\ b, b\ R\ c$ and $a\ R\ c$.
     - We can re-express this to if $(a\ R\ b) \wedge (b\ R\ c) \implies a\ R\ c$.
     - Thus, R is transitive.
     - As the contrapositive is false the original statement is false.

3. Let R be a relation on a set A. Suppose that R is reflexive and satisfy $(a\ R\ c \wedge b\ R\ c) \implies a\ R\ b$ for any $a,b,c \in A$. Prove that R is symmetric and transitive.

   - Let $a, b \in A$ such that $b\ R\ a$.

- As R is reflexive we know that $a\ R\ a$.
- By the given implication, $(a\ R\ a) \wedge (b\ R\ a) \implies a\ R\ b$.
- As $a\ R\ b$ and $b\ R\ a$, we can conclude that R is symmetric.
- Now let $a, b, c \in A$ such that $a\ R\ c, c\ R\ b$.
- As R is reflexive, $b\ R\ c$.
- By the given implication, $(a\ R\ c \wedge b\ R\ c) \implies a\ R\ b$.
- Thus, $a\ R\ b, b\ R\ c$ and $a\ R\ c$ co-exist. R is transitive.

4. Let R be a relation on set A and $f : A \to B$ a function. We define a relation $\mathcal{R}'$ on B as

$$\mathcal{R}' = \{(f(x), f(y)) : (x, y) \in \mathcal{R}\}$$

Determine (with proof) whether the following are true.

- (a) If $\mathcal{R}$ is reflexive then $\mathcal{R}'$ is reflexive.
  - Define $f(x) = x + 1$, and let $A = \{1\}, B = \{2, 3\}$.
  - And let $\mathcal{R} = \{1, 1\}$ such that $\mathcal{R}$ is reflexive.
  - Then $(2, 2) \in \mathcal{R}'$, however $(3, 3) \notin \mathcal{R}'$ therefore $\mathcal{R}'$ is not reflexive.
- (b) If $\mathcal{R}$ is symmetric then $\mathcal{R}'$ is symmetric.
  - Let $a, b \in A$, such that $a\ \mathcal{R}\ b$ and $b\ \mathcal{R}\ a$.
  - Therefore, $(a, b), (b, a) \in \mathcal{R}$.
  - When $(a, b), (b, a) \in \mathcal{R}$, $(f(a), f(b)), (f(b), f(a)) \in \mathcal{R}'$.
  - This shows that $f(a)\ \mathcal{R}'\ f(b)$ and $f(b)\ \mathcal{R}'\ f(a)$.
  - Thus, $\mathcal{R}'$ is symmetric.

5. We define a relation $\mathcal{R}$ on the real number as

$$\mathcal{R} = \{(x, x + n) : x \in \mathbb{R}, n \in \mathbb{N}\}.$$

Determine (with proof) whether the following holds:

- (a) If $x_1\ \mathcal{R}\ y_1$ and $x_2\ \mathcal{R}\ y_2$ then $(x_1 + x_2)\ \mathcal{R}\ (y_1 + y_2)$.
  - Let $x_1\ \mathcal{R}\ y_1$ and $x_2\ \mathcal{R}\ y_2$ then $y_1 = x_1 + n_1$ and $y_2 = x_2 + n_2$.
  - Then we can figure that $y_1 + y_2 = x_1 + x_2 + n_1 + n_2$.
  - As $n \in \mathbb{N}$, we can choose a new n_3 such that $n_3 = n_1 + n_2$.
  - Thus we can see that $y_1 + y_2 = x_1 + x_2 + n_3 = x_1 + x_2 + n_1 + n_2$.
  - Therefore, $(x_1 + x_2)\ \mathcal{R}\ (y_1 + y_2)$.
- (b) If $x_1\ \mathcal{R}\ y_1$ and $x_2\ \mathcal{R}\ y_2$ then $(x_1 \cdot y_1)\ \mathcal{R}\ (x_2 \cdot y_2)$.
  - Let $x_1\ \mathcal{R}\ y_1$ and $x_2\ \mathcal{R}\ y_2$ then $y_1 = x_1 + n$ and $y_2 = x_2 + n$.

- $y_2 \cdot x_2 = (x_2 + n) \cdot x_2$ and $y_1 \cdot x_1 = (x_1 + n) \cdot x_1$
- We can see that $x_2^2 + n \cdot x_2 \neq x_1^2 + n \cdot x_1 + n$.
- Hence, $(x_1 \cdot y_1) \not\mathcal{R} (x_2 \cdot y_2)$.

6. We define a relation T on $\mathbb{R} - \{0\}$ by

$$a\ T\ b \Longleftrightarrow \tfrac{a}{b} \in \mathbb{Q}$$

Show that T is symmetric, reflexive and transitive.

- Let $a \in \mathbb{R} - \{0\}$, then $\frac{a}{a} = 1 \in \mathbb{Q}$.
- Thus, we can see that $a\ T\ a$.
- Therefore T is reflexive.
- Now let $a, b \in \mathbb{R} - \{0\}$, such that $a\ T\ b$.
- Thus $\frac{a}{b} \in \mathbb{Q}$.
- As $a, b \in \mathbb{R} - \{0\}$, We can also conclude that $\frac{b}{a} \in \mathbb{Q}$.
- Which shows that $b\ T\ a$, thus T is symmetric.
- Lastly, let $a, b, c \in \mathbb{R} - \{0\}$, such that $a\ T\ b$ and $b\ T\ c$.
- Then, $\frac{a}{b}, \frac{b}{c} \in \mathbb{Q}$. $\frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c}$, and because $c \in \mathbb{R} - \{0\}$ $\frac{a}{c} \in \mathbb{Q}$.
- Therefore, $(a\ T\ b) \wedge (b\ T\ c) \implies a\ T\ c$. Making T transitive.

7. Let a relation $\mathcal{R}$ on $\{0, 1, 2, 3\}$ be such that $x\ \mathcal{R}\ y$ if $(x + y)$ is a multiple of 3.

- (a) Write out $\mathcal{R}$ as a set.
  - $\mathcal{R} = \{(0,0), (0,3), (1,2), (2,1), (3,0), (3,3)\}$.
- (b) Is this relation reflexive?
  - In order to be reflexive, $\forall a \in \{0, 1, 2, 3\}$, $a\ \mathcal{R}\ a$.
  - However, this is only satisfied when $a = 0$ or $a = 3$.
  - And because $(1,1), (2,2) \notin \mathcal{R}$, $\mathcal{R}$ is not reflexive.
- (c) Is it symmetric?
  - Let $a, b \in \{0, 1, 2, 3\}$, then if $a\ \mathcal{R}\ b$ then $b\ \mathcal{R}\ a$.
  - For $(a, b) = (0, 0)$, both $0\ \mathcal{R}\ 0$ and $0\ \mathcal{R}\ 0$ exist.
  - For $(a, b) = (1, 2)$, and as $(2, 1) \in \mathcal{R}$, both $1\ \mathcal{R}\ 2$ and $2\ \mathcal{R}\ 1$ exist.
  - For $(a, b) = (0, 3)$, and as $(3, 0) \in \mathcal{R}$, both $0\ \mathcal{R}\ 3$ and $3\ \mathcal{R}\ 0$ exist.
  - For $(a, b) = (3, 3)$, both $3\ \mathcal{R}\ 3$ and $3\ \mathcal{R}\ 3$ exist.
  - Therefore, $\mathcal{R}$ is symmetric.
- (d) What is the fewest number of elements that you need to add to $\mathcal{R}$ so as to obtain a transitive relation?
  - We only need to add $(1, 1)$ and $(2, 2)$ to satisfy a transitive relation.

  - Then $1 \mathcal{R} 1$ and $2 \mathcal{R} 2$ exist.
  - Thus when $(1 \mathcal{R} 2)$ and $(2 \mathcal{R} 1)$, $1 \mathcal{R} 1$.
  - Also when $(2 \mathcal{R} 1)$ and $(1 \mathcal{R} 2)$, $2 \mathcal{R} 2$.
  - Satisfying a transtive relation.

8. A relation on a set A is called **circular** if for all $a, b, c \in A$, $a\ R\ b$ and $b\ R\ c$ imply $c\ R\ a$. Prove that a relation is an equivalence relation if and only if it is reflexive and circular.

   - Let R be an equivalence relation.
   - Then R is automatically reflexive.
   - Now let $a, b, c \in A$ such that $a\ R\ b$ and $b\ R\ c$.
   - As R is an equivalence relation, R is transitive.
   - Thus $a\ R\ c$, also, R is transitive so $c\ R\ a$.
   - Showing us that R is circular.
   - Now let's assume that R is reflexive and circular.
   - Now let $a, b \in A$ such that $a\ R\ b$ and $b\ R\ b$ as R is reflexive.
   - As we know that R is circular $(a\ R\ b) \wedge (b\ R\ b) \implies b\ R\ a$.
   - Therefore, we can see that R is symmetric.
   - Also, by the circular implication we can see that when $a\ R\ b$ and bRc then $c\ R\ a$.
   - We know that R is reflexive, therefore $a\ R\ c$.
   - Thus, we can conclude that R is transitive.
   - To conclude, we know that R is reflexive, symmetric and transitive, thus R is an equivalence relation.