Some useful latex for you to use:

- For sets use the command we defined in the latex source

$$\left\{1, 2, 3\right\}, \left\{\emptyset, \left\{4, 5, 6\right\}\right\}, \left\{\frac{1}{2}, \frac{\alpha}{1+\beta}\right\}$$

  it will format the braces nicely.

- Sometimes it is nice to write $\ell$ instead of $l$ because it looks nice in formulas.

- For logic, latex defines the symbols we need:

$$\sim P \qquad P \vee Q \qquad P \wedge Q \qquad P \implies Q \qquad P \iff Q$$

  Unfortunately, we use $\sim$ for negation and not the default negation symbol $\neg$, so it is useful to redefine things in the header of your document (a bit like how we define the set command.)

- For a proof we can (and probably should) use the proof environment. It automatically puts the word "proof" at the start and the little square at the end:

  *Proof.* This is my proof. It is just missing a few details, but I'll put in an equation

$$a + b = c$$

  just because I can. □

  Sometimes we want to give the proof a title, and the proof environment helps us do that too. Here is a classic false-proof that $2 = 1$.

  *Not-quite-a-proof that two equals one.* Let $x, y$ be non-zero real numbers so that $x = y$. Then, multiplying by $x$ gives us

$$\begin{aligned}
x^2 &= xy && \text{now subtract } y^2 \\
x^2 - y^2 &= xy - y^2 && \text{now factor} \\
(x-y)(x+y) &= y(x-y) && \text{divide by common factor of } (x-y) \\
x + y &= y && \text{since } x = y \\
2y &= y && \text{now divide by y} \\
2 &= 1
\end{aligned}$$

  □

- For the truth tables you can use the following:

| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|---|---|---|---|---|
| $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $a_{16}$ |
| $a_{21}$ | $a_{22}$ | $a_{23}$ | $a_{24}$ | $a_{25}$ | $a_{26}$ |
| $a_{31}$ | $a_{32}$ | $a_{33}$ | $a_{34}$ | $a_{35}$ | $a_{36}$ |
| $a_{41}$ | $a_{42}$ | $a_{43}$ | $a_{44}$ | $a_{45}$ | $a_{46}$ |

- Remeember to cheque the speeling of your subbmisssion.
- Also remember that you should not include your scratchwork unless a question specifically asks for it.
- Finally, please try to make your work look nice and neat and use 12pt font — think about the reader!

Please do not include the above text in your homework solution — we have just included it here to help you write your homework.

## Solutions to homework 1:

1. Your solution to question 1.

   *Proof.* Let $n \in \mathbb{Z}$, Prove that if $3 \mid n+1$ then $3 \nmid n^2+5n+5$.

   - Assume for some $n \in \mathbb{Z}, 3 \mid n+1$
   - Then $(n+1) = 3\ell$ for $\ell \in \mathbb{Z}$
   - $n^2+5n+5 = (n+1)^2+3(n+1)+1 = (3\ell)^2+3(3\ell)+1$
   - By fact, $(3l)^2+(3l) \in \mathbb{Z}$ so $n^2+5n+5 = 3(3\ell^2+3\ell)+1$
   - This shows that $3 \nmid n^2+5n+5$

   □

2. Your solution to question 2.

   *Proof.* Let $a \in \mathbb{Z}$ . Prove that if $5a+11$ is odd then $9a+3$ is odd.

   - Let's assume that $5a+11$ is odd, then we can see that $5a+11 = 2\ell+1$ for $\ell \in \mathbb{Z}$.
   - $9a+3 = 10a-a+3 = 2(2\ell-10)-a+3 = 2(2l-8)-a-1$.
   - As $a \in \mathbb{Z}$, we know that $a$ is either an odd number or even number by fact.
   - Let's consider the case when a is an odd number. $a = 2k+1, k \in \mathbb{Z}$.
   - $9a+3 = 2(2\ell-8)-(2k+2) = 2*(2l-2k-10)$. And as $l, k \in \mathbb{Z}$ we know that $2l-2k-10$ is also an Integer.
   - However, in this case $5a+11 = 10k+16 = 2(5k+8)$ becomes an even number.
   - So, let's consider the case when $a = 2q, q \in \mathbb{Z}$.

- In this case, $5a + 11 = 10q + 11 = 2(5q + 5) + 1$. Thus satisfies that $5a + 11$ is an odd number due to the fact that $5q + 5 \in \mathbb{Z}$.
- And, $9a + 3 = 2(2\ell - 8) - 2q - 1 = 2(2\ell - q - 9) + 1$ shows us that 9a+3 is odd as $2\ell - q - 9 \in \mathbb{Z}$.
- Therfore, we can conclude that when $5a + 11$ is odd then $9a + 3$ is odd.

□

3. Your solution to question 3.

*Proof.* If $-1 < x < 2$, then $x^2 - x - 2 < 0$.

- Assume that $-1 < x < 2$.
- We change the expression $x^2 - x - 2$ to $(x - 2)(x + 1)$.
- For some $x \in (-1, 2)$, we know that $(x - 2) < 0$ and $(x + 1) > 0$.
- This shows that $(x - 2)(x + 1) < 0$ because we know that for some $a, b \in \mathbb{R}$ if $ab < 0$ then $a < 0, b > 0$ or $a > 0, b < 0$.
- Therefore, we can conclude that if $-1 < x < 2$ then $x^2 - x - 2 < 0$

□

4. Your solution to question 4.

*Proof.* Let $a, b, c, d$ be integers. Suppose that $a, c, b + d$ are all odd numbers. Prove $ab + cd$ is odd.

- Let's assume that $a, b, c, d$ be integers and $a, c, b + d$ are all odd numbers.
- Then we can see that $a = 2\ell + 1, c = 2k + 1, b + d = 2m + 1$. for $\ell, k, m \in \mathbb{Z}$.
- When $b + d$ is odd, we can express $b + d$ as a sum of an even number and an odd number.
- For instance, let's consider the case when $b$ is even and $d$ is odd.
- Then we can express these as $b = 2n$ and $d = 2p + 1$ for $n, p \in \mathbb{Z}$.
- We can see that $ab+cd = (2\ell+1)*(2n)+(2k+1)*(2p+1) = 2(2nl+n+2kp+k+p)+1$
- As we know that $n, l, k, p \in \mathbb{Z}$, $2nl + n + 2kp + k + p \in \mathbb{Z}$.
- Therfore, from the form $ab + cd = 2(2nl + n + 2kp + k + p) + 1$ we can conclude that $ab + cd$ is odd.

□

5. Your solution to question 5.

*Proof.* Let $x$ and $y$ be real numbers. Show that

$$xy \leq \frac{1}{2}(x^2 + y^2)$$

- Let's assume that $x, y \in \mathbb{R}$
- $\frac{1}{2}x^2 + \frac{1}{2}y^2 - xy = \frac{1}{2}(x^2 - 2xy - y^2) = \frac{1}{2}(x - y)^2$
- We know for some $n \in \mathbb{R}$ that $n^2 \geq 0$.
- As $x, y \in \mathbb{R}$ we also can find out that $(x - y) \in \mathbb{R}$ making $(x - y)^2 \geq 0$.
- If $(x - y)^2 \geq 0$, then $\frac{1}{2}(x - y)^2 = \frac{1}{2}x^2 + \frac{1}{2}y^2 - xy \geq 0$.
- Therefore, we can conclude that $\frac{1}{2}x^2 + \frac{1}{2}y^2 \geq xy$.

$\square$

6. Your solution to question 6.

*Proof.* Let $x$ and $y$ be real numbers. Suppose that $x < y$ and $y^2 < x^2$. Show that $x + y < 0$.

- Let's assume that $x < y$ and $y^2 < x^2$ for $x, y \in \mathbb{R}$.
- $x < y$ shows that $x - y < 0$. And $y^2 < x^2$ shows that $x^2 - y^2 > 0$.
- $x^2 - y^2 = (x - y)(x + y)$ and as we know that $x - y < 0$ then in order to satisfy $x^2 - y^2 > 0$ , $x + y < 0$ has to be satisfied.
- This is due to the fact that for $a, b \in \mathbb{R}$. If $a > 0$ then $b > 0$, and if $a < 0$ then $b < 0$ to make $ab > 0$.
- Therfore, $x + y < 0$ when $x < y$ and $y^2 < x^2$.

$\square$

7. Your solution to question 7.

*Proof.* For an integer $n$, prove that if $5 \mid (n + 7)$, then $5 \mid (n^2 + 1)$.

- Let's assume that for $n \in \mathbb{Z}$ , $5 \mid (n + 7)$.
- Then, $n + 7 = 5\ell$ for some $\ell \in \mathbb{Z}$.
- $n^2 + 1 = (5\ell - 7)^2 + 1 = 5(5\ell^2) - 5(14\ell) + 50 = 5(5\ell^2 - 14\ell + 10)$.
- We know that as $\ell \in \mathbb{Z}$ , $5\ell^2 - 14\ell + 10 \in \mathbb{Z}$.
- Then, $n^2 + 1 = 5(5\ell^2 - 14\ell + 10)$ shows that $5 \mid (n^2 + 1)$.
- Therefore, we can conclude that if $5 \mid (n + 7)$, then $5 \mid (n^2 + 1)$.

$\square$

8. Your solution to question 8.

*Proof.* Let $n, a, b, x, y \in \mathbb{Z}$ with $n > 0$. Prove that if $n \mid a$ and $n \mid b$ then $n \mid (ax + by)$.

- Let $n \mid a$ and $n \mid b$ for $n, a, b, x, y \in \mathbb{Z}$ and $n > 0$.
- Then $a = nl$ , $b = nk$ for $l, k \in \mathbb{Z}$.
- $ax = anl, by = nky$ we know that $ax + by = n(al + ky)$.
- We can see that $al + ky \in \mathbb{Z}$, because $a, l, k, y \in \mathbb{Z}$.
- Therefore, $ax + by = n(al + ky)$ shows that $n \mid ax + by$, when $n > 0$.

$\square$