1.

(a)

Total probability equals $\sum_{i=1}^{3} P(\text{buy two same cakes at bakery } i)$. $P(\text{buy two cakes at bakery 1}) = P(\text{buy two same cakes} \mid \text{bakery 1}) \cdot P(\text{bakery 1}) = \frac{1}{3} \cdot \frac{{}_2C_1 \ {}_6C_2}{{}_{12}C_2} = \frac{1}{3} \cdot \frac{5}{11}$,

$P(\text{buy two same cakes at bakery 2}) = P(\text{buy same two cakes} \mid \text{bakery 2}) \cdot P(\text{bakery 2}) = \frac{1}{3} \cdot \frac{{}_3C_1 \cdot {}_4C_2}{{}_{12}C_2} = \frac{1}{3} \cdot \frac{3}{11}$. $P(\text{buy two same cakes at bakery 3}) = P(\text{buy two same cakes} \mid \text{bakery 3}) \cdot P(\text{bakery 3})$

$= \frac{1}{3} \cdot \frac{{}_4C_1 \cdot {}_3C_2}{{}_{12}C_2} = \frac{1}{3} \cdot \frac{2}{11}$. $\sum_{i=1}^{3} P(\text{buy two same cakes at bakery } i) = \frac{1}{3} \cdot \left(\frac{5}{11} + \frac{3}{11} + \frac{2}{11}\right) = \frac{10}{33}$ $\quad \therefore \frac{10}{33}$

(b)

$P(\text{bakery 2} \mid \text{two different types of cake}) = P(\text{bakery 2} \cap \text{two different types of cake}) / P(\text{two different types of cake}) = \dfrac{P(\text{two different types of cake} \mid \text{bakery 2}) \cdot P(\text{bakery 2})}{P(\text{two different types of cake})}$

$= \dfrac{P(\text{two different types of cake} \mid \text{bakery 2}) \cdot P(\text{bakery 2})}{1 - P(\text{same type cakes})} = \dfrac{\frac{1}{3} \cdot \frac{{}_3C_2 \cdot {}_4C_1 \cdot {}_4C_1}{{}_{12}C_2}}{1 - \frac{10}{33}} = \dfrac{\frac{1}{3} \cdot \frac{3 \cdot 16}{66}}{\frac{23}{33}} = \frac{8}{23}$ $\quad \therefore \frac{8}{23}$.

2.

(a) $P(\text{identify faulty}) = P(\text{test faulty} \cap \text{originally true}) + P(\text{test faulty} \cap \text{originally false}) = P(\text{test faulty} \mid \text{originally flawless})\, P(\text{originally flawless}) + P(\text{test faulty} \mid \text{originally faulty}) \cdot P(\text{originally faulty})$

$$= \frac{1}{100} \cdot \frac{98}{100} \cdot \frac{98}{100} + \frac{99}{100} \cdot \frac{5}{100} \cdot \frac{5}{100}$$

$P(\text{indeed faulty} \mid \text{test faulty}) = P(\text{indeed faulty} \cap \text{test faulty}) / P(\text{test faulty})$

$$= \frac{\frac{1}{100} \cdot \left(\frac{98}{100}\right)^2}{\frac{1}{100} \cdot \left(\frac{98}{100}\right)^2 + \frac{99}{100} \cdot \left(\frac{5}{100}\right)^2}$$

(b) If test is once, $P(\text{test faulty} \mid \text{originally flawless})\, P(\text{originally flawless}) = \frac{1}{100} \cdot \frac{98}{100}$ and $P(\text{test faulty} \mid \text{originally faulty}) \cdot P(\text{originally faulty}) = \frac{99}{100} \cdot \frac{5}{100}$.

Hence, $P(\text{indeed faulty} \mid \text{test faulty}) = \dfrac{\frac{1}{100} \cdot \left(\frac{98}{100}\right)}{\frac{1}{100} \cdot \left(\frac{98}{100}\right) + \frac{99}{100} \cdot \left(\frac{5}{100}\right)}$

3.

(a) $P(E) = P(\{U \text{ is even}\}) = \frac{1}{2}$, $P(F) = P(\{U \text{ is divisible by } 5\}) = \frac{1}{5}$.

$P(E \cap F) = P(\{U \text{ is even and divisible by } 5\}) = P(\{U \text{ is divisible by } 10\}) = \frac{1}{10} = \frac{1}{2} \cdot \frac{1}{5} = P(E) \cdot P(F)$ $\quad \therefore$ independent

(b) $P(A) = P(\{U \text{ is prime}\}) = \frac{1}{4}$ and $P(B) = P(\{\text{at least one digit of } U \text{ is } 2\}) = P(\{\text{first digit is } 2\}) + P(\{\text{second digit is } 2\}) - P(22) = \frac{19}{100}$

$P(A \cap B) = P(\{U \text{ is prime and at least one digit is } 2\}) = P(\{2, 23, 29\}) = \frac{3}{100} \neq \frac{1}{4} \cdot \frac{19}{100} = P(A) \cdot P(B)$ $\quad \therefore$ not independent

(c) Let's change it to 9. $P(E) = P(U \text{ is even}) = \frac{4}{9}$, $P(F) = P(U \text{ is divisible by } 5) = \frac{1}{9}$. Then $P(E \cap F) = \frac{0}{9} = 0 \neq \frac{1}{9} \cdot \frac{4}{9} = P(E)P(F)$ is now dependent.

4.

($\Rightarrow$)
$P(X = n+m \mid X > n) = P(X = n+m \cap X > n) / P(X > n)$. Since, $P(X = n+m \cap X > n) = P(X = n+m)$ then $P(X = n+m \mid X > n) = P(X = n+m) / P(X > n) = p(1-p)^{n+m-1} / (1-p)^n = p \cdot (1-p)^{m-1} = P(X = m)$ as required.

($\Leftarrow$)
Now, given that $P(X = n+m \mid X > n) = P(X = m)$, $\forall m, n \in \mathbb{N}$ and $p := P(X = 1)$, let's prove that $P(X = m) = p \cdot (1-p)^{m-1}$, $\forall m \in \mathbb{N}$

When $m = 1$, $P(X = 1) = p = p \cdot (1-p)^{1-1}$ as required. Now let's show $P(X = m \mid X > 1) = P(X = m-1)$. $P(X = m \mid X > 1) = P(X = (m-1)+1 \mid X > 1)$, using the memory less property $P(X = (m-1)+1 \mid X > 1) = P(X = m-1)$.

$P(X = m \mid X > 1) = \dfrac{P(X = m \cap X > 1)}{P(X > 1)}$ and given $m > 1$, $\dfrac{P(X = m \cap X = 1)}{P(X > 1)} = \dfrac{P(X = m)}{P(X > 1)} = P(X = m-1)$. Then for all integers k from 2 to m we get $P(k) = P(k-1) \cdot P(X > 1)$. Recursively proceeding from $X = m$ we can see that

$P(X = m) = P(X > 1) \cdot P(X = m-1) = P(X > 1)^2 \cdot P(X = m-2) = P(X > 1)^{m-1} \cdot P(X = 1) = (1 - P(X = 1))^{m-1} \cdot P(X = 1) = (1-p)^{m-1} \cdot p$ as required.

5.

(a) there are four suits with 13 cards each hence $p = \frac{{}_4C_1 \cdot {}_{13}C_{13}}{{}_{52}C_{13}} = \frac{{}_4C_1}{{}_{52}C_{13}}$

$P(X=k) = {}_{50}C_k \cdot \left(\frac{{}_4C_1}{{}_{52}C_{13}}\right)^k \cdot \left(1 - \frac{{}_4C_1}{{}_{52}C_{13}}\right)^{50-k}$, Binomial distribution

(b) P(number aces is at least one) = 1 − P(receive no aces) $= 1 - \frac{{}_{48}C_{13}}{{}_{52}C_{13}} = p$ and this is for each game.

$P(X=k)$ = P(k th game is first time with at least one ace) $= \left(1 - \frac{{}_{48}C_{13}}{{}_{52}C_{13}}\right) \cdot \left(\frac{{}_{48}C_{13}}{{}_{52}C_{13}}\right)^{k-1}$, geometric distribution

(c) P(getting exactly three aces) $= \frac{{}_4C_3 \cdot {}_{48}C_{10}}{{}_{52}C_{13}}$

$P(X=k)$ = P(k games out of 50 there are exactly 3 aces) $= {}_{50}C_k \cdot \left(\frac{{}_4C_3 \cdot {}_{48}C_{10}}{{}_{52}C_{13}}\right)^k \cdot \left(1 - \frac{{}_4C_3 \cdot {}_{48}C_{10}}{{}_{52}C_{13}}\right)^{50-k}$, Binomial distribution

7.

k	1	2	3	4	5
$\mathbb{P}(X=k)$	1/7	1/14	3/14	2/7	2/7

(a) $P(X \le 3) = P(X=1) + P(X=2) + P(X=3)$

$= \frac{1}{7} + \frac{1}{14} + \frac{3}{14} = \frac{3}{7} \quad \therefore \frac{3}{7}$

(b) $P(X<3) = P(X \le 3) - P(X=3)$

$= \frac{3}{7} - \frac{3}{14} = \frac{3}{14} \quad \therefore \frac{3}{14}$

(c) $P(X<4.12 \mid X>1.6)$

$= P(1.6 < X < 4.12) / P(X>1.6) = \frac{P(2 \le X \le 4)}{P(X \ge 2)} = \frac{P(2 \le X \le 4)}{1 - P(X<2)} = \frac{\frac{1}{14} + \frac{3}{14} + \frac{2}{7}}{1 - \left(\frac{1}{7}\right)} = \frac{8}{12} = \frac{2}{3} \quad \therefore \frac{2}{3}$