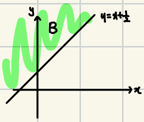
1. Let $X \sim \text{Exp}(2)$, $Y \sim \text{Unif}[1,3]$, and assume that X, Y are independent. $P(Y-X \ge \frac{1}{2})$?

$B := \{(x,y): y-x \ge \frac{1}{2}\}$, $P(Y-X \ge \frac{1}{2}) = \iint_B f(x,y)\, dB$ when $f(x,y)$ is the joint pdf of x, y.



so $P(Y-X\ge\frac{1}{2}) = \int_{-\infty}^{\infty}\int_{-\infty}^{y-\frac{1}{2}} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{y-\frac{1}{2}} f_X(x) f_Y(y)\,dx\,dy$ since $f_X(x) = \begin{cases} 2e^{-2x} & (x>0) \\ 0 & (x\le 0)\end{cases}$ we can consider the case $y \ge \frac{1}{2}$ and $y < \frac{1}{2}$

but since $-\infty < y < \infty$ and $Y \sim \text{Unif}[1,3] = \begin{cases} \frac{1}{2}, & 1\le y\le 3 \\ 0, & \text{o.w.}\end{cases}$ $\rightarrow P(Y-X\ge\frac{1}{2}) = \int_1^3\int_0^{y-\frac{1}{2}} f_X(x)\cdot f_Y(y)\,dx\,dy = \int_1^3\int_0^{y-\frac{1}{2}} 2e^{-2x}\cdot\frac{1}{2}\,dx\,dy = \int_1^3\left[-\frac{1}{2}e^{-2x}\right]_0^{y-\frac{1}{2}} = \int_1^3 \frac{1}{2}(1-e^{-2y+1})\,dy = \frac{1}{2}\left[y+\frac{1}{2}e^{-2y+1}\right]_1^3 = \frac{1}{2}\left(2+\frac{1}{2}e^{-5}-\frac{1}{2}e^{-1}\right)$

$= 1 + \frac{e^{-5}-e^{-1}}{4}$

2. The random variables X, Y have the joint pdf $f(x,y) = \begin{cases} c\cdot\frac{e^{-x}-e^{-x-2y}}{e^{y}-1}, & \text{if } x>0 \text{ and } y>0 \\ 0, & \text{o.w.}\end{cases}$

(a) $\int_0^{\infty}\int_0^{\infty} c\cdot\frac{e^{-x}-e^{-x-2y}}{e^{y}-1}\,dx\,dy = \int_0^{\infty}\left[c\cdot\frac{-e^{-x}+e^{-x-2y}}{e^{y}-1}\right]_0^{\infty} dy$

$= \int_0^{\infty} c\cdot\left(0-\frac{-1+e^{-2y}}{e^{y}-1}\right) dy = \int_0^{\infty} c\cdot\frac{1-e^{-2y}}{e^{y}-1}\,dy = \int_0^{\infty} c\cdot\frac{e^{-2y}(e^{2y}-1)}{e^{y}-1}\,dy = \int_0^{\infty} c\cdot e^{-2y}(e^{y}+1)\,dy = \int_0^{\infty} c\cdot(e^{-y}+e^{-2y})\,dy = c\cdot\left[-e^{-y}-\frac{1}{2}e^{-2y}\right]_0^{\infty} = c\cdot\left(0-\left(-1-\frac{1}{2}\right)\right) = \frac{3}{2}c = 1$, $\therefore c = \frac{2}{3}$

(b) Are X and Y independent

$f_X(x) = \int_0^{\infty} f(x,y)\,dy = \int_0^{\infty}\frac{2}{3}\cdot\frac{e^{-x}-e^{-x-2y}}{e^{y}-1}\,dy = \int_0^{\infty}\frac{2}{3}\cdot e^{-x}\cdot\frac{1-e^{-2y}}{e^{y}-1}\,dy = \int_0^{\infty}\frac{2}{3}\cdot e^{-x}\cdot e^{-2y}\cdot(e^{y}+1)\,dy = \frac{2}{3}\cdot e^{-x}\int_0^{\infty} e^{-y}+e^{-2y}\,dy = \frac{2}{3}e^{-x}\left[-e^{-y}-\frac{1}{2}e^{-2y}\right]_0^{\infty} = e^{-x}$

$f_Y(y) = \int_0^{\infty} f(x,y)\,dx = \int_0^{\infty}\frac{2}{3}\cdot\frac{e^{-x}(1-e^{-2y})}{e^{y}-1}\,dy = \frac{2}{3}\cdot\frac{1-e^{-2y}}{e^{y}-1}\cdot\left[-e^{-x}\right]_0^{\infty} = \frac{2}{3}\cdot\frac{1-e^{-2y}}{e^{y}-1}\cdot 1 = \frac{2}{3}\cdot\frac{1-e^{-2y}}{e^{y}-1}$

$f_X(x)\cdot f_Y(y) = e^{-x}\cdot\frac{2}{3}\cdot\frac{1-e^{-2y}}{e^{y}-1} = \frac{2}{3}\cdot\frac{e^{-x}-e^{-x-2y}}{e^{y}-1} = f(x,y)$ $\therefore$ X and Y are independent.

(c) Find $P(X<Y)$

$P(X<Y) = \iint_B f(x,y)\,dB$ where $B = \{(x,y): x<y\}$. $P(X<Y) = \int_0^{\infty}\int_x^{\infty}\frac{2}{3}\cdot\frac{e^{-x}-e^{-x-2y}}{e^{y}-1}\,dy\,dx = \int_0^{\infty}\frac{2}{3}e^{-x}\int_x^{\infty} e^{-y}+e^{-2y}\,dy\,dx = \int_0^{\infty}\frac{2}{3}e^{-x}\left[-e^{-y}-\frac{1}{2}e^{-2y}\right]_x^{\infty} dx = \int_0^{\infty}\frac{2}{3}e^{-x}\left(e^{-x}+\frac{1}{2}e^{-2x}\right) dx = \int_0^{\infty}\frac{2}{3}e^{-2x}+\frac{1}{3}e^{-3x}\,dx = \left[-\frac{1}{3}e^{-2x}-\frac{1}{9}e^{-3x}\right] = \frac{4}{9}$ $\therefore \frac{4}{9}$

3. $X \sim \text{Exp}(\mu)$, $Y \sim \text{Exp}(\lambda)$ are independent

$f_{X+Y}(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(x)\cdot f_Y(z-x)\,dx$

if) $0 \ge x$ or $z \le x$ then $f_X(x)\cdot f_Y(z-x)\,dx$

thus $(f_X * f_Y)(z) = \int_0^z f_X(x)\cdot f_Y(z-x)\,dx = \int_0^z \mu e^{-\mu x}\cdot\lambda e^{-\lambda(z-x)}\,dx = \mu\lambda e^{-\lambda z}\cdot\int_0^z e^{(\lambda-\mu)x}\,dx$

if) $\lambda=\mu$, $f_{X+Y}(z) = \mu\lambda e^{-\lambda z}\cdot\int_0^z 1\,dx = \mu\lambda e^{-\lambda z}\cdot z = \lambda^2 e^{-\lambda z}\cdot z = \mu^2\cdot e^{-\mu z}\cdot z$

if) $\lambda\ne\mu$, $f_{X+Y}(z) = \mu\lambda e^{-\lambda z}\cdot\int_0^z e^{(\lambda-\mu)x}\,dx = \mu\lambda e^{-\lambda z}\cdot\left[\frac{1}{\lambda-\mu}\cdot e^{(\lambda-\mu)x}\right]_0^z$

$= \mu\lambda e^{-\lambda z}\cdot\frac{1}{\lambda-\mu}\left\{e^{(\lambda-\mu)z}-1\right\} = \frac{\mu\lambda}{\lambda-\mu}\cdot\left(e^{-\mu z}-e^{-\lambda z}\right)$

4.

Exercise 6.57. Let $X_1, \ldots, X_n$ be independent random variables, each one distributed uniformly on $[0,1]$. Let Z be the minimum and W the maximum of these numbers. Find the joint density function of Z and W.

Hint. Try to find the joint cumulative distribution function first.

$Z = \min\{X_1,\cdots,X_n\}$, $W = \max\{X_1,\cdots,X_n\}$

$F_{Z,W}(z,w) = P(Z\le z, W\le w) = P(W\le w) - P(Z>z, W\le w)$

$= P(X_1\le w,\cdots,X_n\le w) - P(z<X_1\le w,\cdots,z<X_n\le w)$

if) $z\ge w$, then $F_{Z,W}(z,w)=1 \Rightarrow f_{Z,W}(z,w) = \partial_z\partial_w F_{Z,W}(z,w) = 0$

if) $z<w$, then $F_{Z,W}(z,w) = P(X_1\le w,\cdots,X_n\le w) - P(z<X_1\le w,\cdots,z<X_n\le w) = \prod_{i=1}^{n} P(X_i\le w) - \prod_{j=1}^{n} P(z<X_j\le w)$

$= P(X_1\le w)^n - P(z<X_1\le w)^n = (F_X(w))^n - (F_X(w)-F_X(z))^n$

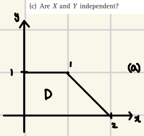
$F_{Z,W}(z,w) = \begin{cases} 1, & \text{if } z<w \text{ and } (w\ge 1, z\ge 1) \\ 1-(1-z)^n, & \text{if } z<w \text{ and } w\ge 1 \text{ and } 1>z\ge 0 \\ w^n-(w-z)^n, & \text{if } z<w \text{ and } 1>w\ge 0 \text{ and } 1>z\ge 0 \\ 0, & \text{o.w.}\end{cases}$

hence the joint pdf is $f_{Z,W}(z,w) = \begin{cases} n(n-1)(w-z)^{n-2}, & \text{if } z<w \text{ and } 0\le w<1,\ 0\le z<1 \\ 0, & \text{o.w.}\end{cases}$

(6.34)

Exercise 6.34. Let (X, Y) be a uniformly distributed random point on the quadrilateral D with vertices $(0,0)$, $(2,0)$, $(1,1)$ and $(0,1)$.

(a) Find the joint density function of (X, Y) and the marginal density functions of X and Y.
(b) Find $E[X]$ and $E[Y]$.
(c) Are X and Y independent?



(a) $f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{Area}(D)} = \frac{2}{3}, & (x,y) \in D \\ 0, & \text{o.w.} \end{cases}$, $f_X(x)$: if $x \notin [0,2]$ then $f_X(x) = 0$

if $x \in [0,2]$ then if $x \in [0,1]$, $f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 \frac{2}{3}\,dy = \frac{2}{3}$

$x \in [1,2]$, $f_X(x) = \int_0^{2-x} f(x,y)\,dy = \int_0^{2-x} \frac{2}{3}\,dy = \frac{2}{3}(2-x)$

, $f_Y(y)$ · if $y \notin [0,1]$ then $f_Y(y) = 0$

if $y \in [0,1]$ then $f_Y(y) = \int_0^{2-y} \frac{2}{3}\,dy = \frac{2}{3}(2-y)$

b) $E[X] = \int_0^2 x f_X(x)\,dx = \int_0^1 x \cdot \frac{2}{3}\,dx + \int_1^2 x \cdot \left(\frac{2}{3}\right)(2-x)\,dx$

$= \frac{1}{3} + \frac{2}{3}\left[x^2 - \frac{1}{3}x^3\right]_1^2 = \frac{1}{3} + \frac{2}{3}\left(3 - \frac{7}{3}\right) = \frac{7}{9}$

$E[Y] = \int_0^1 y f_Y(y)\,dy = \int_0^1 \frac{2}{3} y(2-y)\,dy$

$= \left[\frac{2}{3}\left(y^2 - \frac{1}{3}y^3\right)\right]_0^1 = \frac{4}{9}$ $\quad \therefore \frac{4}{9}$

c) for $0 \le x \le 1$, $0 \le y \le 1$ $\Rightarrow$ $f_{X,Y}(x,y) = \frac{2}{3}$

but $f_X(x) f_Y(y) = \frac{4}{9}(2-y)$ $\quad \therefore$ not independent

(6.36)

Exercise 6.36. Suppose that X, Y are jointly continuous with joint probability density function

$$f(x,y) = ce^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}, \quad x, y \in (-\infty, \infty),$$

for some constant c.

(a) Find the value of the constant c.
(b) Find the marginal density functions of X and Y.
(c) Determine whether X and Y are independent.

(a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c \cdot e^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}\,dx\,dy$

$= \int_{-\infty}^{\infty} c \cdot e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{2}}\,dy\,dx = \int_{-\infty}^{\infty} \sqrt{2\pi}\, c \cdot e^{-\frac{x^2}{2}} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(y-x)^2}{2}}\,dy\,dx$

$= \int_{-\infty}^{\infty} \sqrt{2\pi}\, c \cdot e^{-\frac{x^2}{2}}\,dx = \int_{-\infty}^{\infty} 2\pi \cdot c \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\,dx = 2\pi c = 1$ $\quad \therefore c = \frac{1}{2\pi}$

(b) $f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot e^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}\,dy = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-y)^2}{2}}\,dy = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$, $f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot e^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}\,dx = \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot e^{-x^2 + xy - \frac{y^2}{2}}\,dx = \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot e^{-(x-\frac{y}{2})^2 - \frac{y^2}{4}}\,dx = \frac{1}{2\sqrt{\pi}} \cdot e^{-\frac{y^2}{4}} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \cdot e^{-(x-\frac{y}{2})^2}\,dx = \frac{1}{2\sqrt{\pi}} \cdot e^{-\frac{y^2}{4}}$

(c) $f_X(x) \cdot f_Y(y) = \frac{1}{2\sqrt{2}\,\pi} \cdot e^{-\frac{x^2}{2} - \frac{y^2}{4}} \neq \frac{1}{2\pi} \cdot e^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}} = f_{X,Y}(x,y)$ $\quad \therefore$ Not independent

(7.18)

Exercise 7.18. Suppose that X and Y are independent random variables with density functions

$$f_X(x) = \begin{cases} 2e^{-2x}, & x \ge 0 \\ 0, & x < 0, \end{cases} \quad \text{and} \quad f_Y(x) = \begin{cases} 4xe^{-2x}, & x \ge 0 \\ 0, & x < 0. \end{cases}$$

Find the density function of $X + Y$.

if) $z < 0$, $f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x) \cdot f(z-x)\,dx = \int_0^z 0\,dx = 0$

if) $z \ge 0$, $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\,dx$

$= \int_0^z 2e^{-2x}\, 4(z-x)e^{-2(z-x)}\,dx = \int_0^z 8(z-x)e^{-2z}\,dx$

$= 8e^{-2z}\int_0^z (z-x)\,dx = 8e^{-2z}\left[zx - \frac{1}{2}x^2\right]_0^z = 4z^2e^{-2z}$ $\quad \therefore f_{X+Y}(z) = 4z^2e^{-2z}$

$\therefore \quad f_{X+Y}(z) = \begin{cases} 4z^2e^{-2z}, & z \ge 0 \\ 0, & z \end{cases}$