3-(a) For natural angular frequency ω_0,

$Lq'' + \frac{q}{C} = \mathcal{E}_0 \cdot \sin(\omega_0 t)$ when $\mathcal{E}_0 = 0$

so $Lq'' + \frac{q}{C} = 0$ let's multiply both sides with C. $LC\,q'' + q = 0$. Let's assume $\tilde{q}(t) = e^{kt}$. $LC \cdot q'' + q = (LCk^2+1)e^{kt} = 0$ since $e^{kt} \neq 0$. $LCk^2+1=0$ thus $k = \pm\frac{1}{\sqrt{LC}}i$ making $\tilde{q}(t) = e^{\pm\frac{1}{\sqrt{LC}}it}$.

$\tilde{q}(t) = e^{\pm\frac{1}{\sqrt{LC}}it} = \cos(\frac{1}{\sqrt{LC}}t) \pm i\sin(\frac{1}{\sqrt{LC}}t)$ thus $q(t) = A\cdot\sin(\frac{1}{\sqrt{LC}}t) + B\cdot\cos(\frac{1}{\sqrt{LC}}t)$ where A, B are constants. Thus we can see that $\omega_0 = \frac{1}{\sqrt{LC}}$.

3-(b) when $\mathcal{E}_0 \neq 0$, $\omega \neq \omega_0$

$Lq'' + \frac{q}{C} = \mathcal{E}_0 \cdot \sin(\omega t)$

We found out at 3-(a) that $q_h(t) = A\sin(\frac{1}{\sqrt{LC}}t) + B\cos(\frac{1}{\sqrt{LC}}t)$. Now for $q_p'' \cdot L + \frac{q_p}{C} = \mathcal{E}_0 \cdot \sin(\omega t)$, let's assume $q_p(t) = D\sin\omega t + E\cos\omega t$. (D, E are constants)

Then $q_p'' \cdot L + \frac{q_p}{C} = L\omega^2(-D\sin\omega t - E\cos\omega t) + \frac{1}{C}(D\sin\omega t + E\cos\omega t)$

$= \sin\omega t(-L\omega^2 D + \frac{D}{C}) + \cos\omega t(-L\omega^2 E + \frac{E}{C})\cos\omega t = \mathcal{E}_0 \cdot \sin\omega t$. Since $\omega \neq \omega_0$, $E=0$. Making $\mathcal{E}_0 \cdot \sin\omega t = D(-L\omega^2 + \frac{1}{C})\sin\omega t$.

$\mathcal{E}_0 = D(-L\omega^2 + \frac{1}{C}) \Rightarrow D = \frac{\mathcal{E}_0}{-L\omega^2 + \frac{1}{C}} = -\frac{\mathcal{E}_0}{L}\frac{1}{\omega^2 - \frac{1}{LC}} = -\frac{\mathcal{E}_0}{L}\cdot\frac{1}{\omega^2-\omega_0^2}$

thus $q_p(t) = -\frac{\mathcal{E}_0}{L}\cdot\frac{1}{\omega^2-\omega_0^2}\cdot\sin\omega t$ therefore $q(t) = q_p(t) + q_h(t) = -\frac{\mathcal{E}_0}{L}\cdot\frac{1}{\omega^2-\omega_0^2}\cdot\sin\omega t + A\sin\omega_0 t + B\cos\omega_0 t$

$q'(t) = -\frac{\mathcal{E}_0\cdot\omega}{L}\cdot\frac{1}{\omega^2-\omega_0^2}\cdot\cos\omega t + A\omega_0\cos\omega_0 t - B\omega_0\sin\omega_0 t$

$\Rightarrow q(0) = B$, $q'(0) = \frac{\mathcal{E}_0\cdot\omega}{L}\cdot\frac{1}{\omega_0^2-\omega^2} + A\omega_0$

$\Rightarrow A = \frac{1}{\omega_0}(q'(0) - \frac{\mathcal{E}_0\omega}{L}\cdot\frac{1}{\omega_0^2-\omega^2})$

$\therefore q(t) = -\frac{\mathcal{E}_0}{L}\cdot\frac{1}{\omega^2-\omega_0^2}\cdot\sin\omega t + \frac{1}{\omega_0}(q'(0) - \frac{\mathcal{E}_0\omega}{L}\cdot\frac{1}{\omega_0^2-\omega^2})\sin\omega_0 t + q(0)\cdot\cos\omega_0 t$

3-(c) $\mathcal{E}_0 \neq 0$. $\omega = \omega_0$. $q_{\omega_0}(0) = q'_{\omega_0}(0) = 0$.

We solved from above that $q_h(t) = A\cos\omega_0 t + B\sin\omega_0 t$ (A, B are constants)

When $Lq'' + \frac{q}{L} = \mathcal{E}_0 \cdot \sin\omega_0 t$ $q_p(t) = Et\cos\omega_0 t + Ft\sin\omega_0 t$ (F, G are constants)

$q_p'(t) = E(\cos\omega_0 t - \omega_0 t\sin\omega_0 t) + F(\sin\omega_0 t + \omega_0 t\cos\omega_0 t)$

$q_p''(t) = E(-2\omega_0\sin\omega_0 t - \omega_0^2 t\cos\omega_0 t) + F(2\omega_0\cos\omega_0 t - \omega_0^2 t\sin\omega_0 t)$

So $Lq'' + \frac{q}{C} = \sin\omega_0 t(-2E\omega_0 L - F\omega_0^2 Lt + \frac{1}{C}\cdot Ft)$

$+ \cos\omega_0 t(2F\omega_0 L - E\omega_0^2 Lt + \frac{1}{C}Et)$

$= \mathcal{E}_0 \cdot \sin\omega_0 t$ $\Rightarrow \begin{cases} 2F\omega_0 L + Et(\frac{1}{C} - \omega_0^2 L) = 0 \\ -2E\omega_0 L + Ft(\frac{1}{C} - \omega_0^2 L) = \mathcal{E}_0 \end{cases}$ as $\frac{1}{C} - \omega_0^2 L = \frac{1}{C} - \frac{1}{C} = 0$ thus $2F\omega_0 L = 0$ and $-2E\omega_0 L = \mathcal{E}_0$ $\therefore F = 0, E = -\frac{\mathcal{E}_0}{2\omega_0 L}$

Hence, $q(t) = A\cos\omega_0 t + B\sin\omega_0 t - \frac{\mathcal{E}_0}{2\omega_0 L}\cdot t\cos\omega_0 t$. $q(0) = A = 0$

$q'(t) = -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t - \frac{\mathcal{E}_0}{2\omega_0 L}(\cos\omega_0 t - \omega_0 t\sin\omega_0 t)$. $q'(0) = B\omega_0 - \frac{\mathcal{E}_0}{2\omega_0 L} = 0 \Rightarrow B = \frac{\mathcal{E}_0}{2\omega_0^2 L}$

$\therefore q(t) = \frac{\mathcal{E}_0}{2\omega_0^2 L}\sin\omega_0 t - \frac{\mathcal{E}_0}{2\omega_0 L}\cdot t\cdot\cos\omega_0 t$

the charges does not decay and the amplitude increases

3 - (d)

Now $L \cdot q'' + R \cdot q' + \frac{q}{C} = \varepsilon_0$.

Let's first assume the particular solution.

$\tilde{q}_p(t) = A$, then $L \cdot q'' + R \cdot q' + \frac{1}{C} q = \frac{A}{C} = \varepsilon_0 \Rightarrow A = C \cdot \varepsilon_0$ hence $q_p(t) = C \cdot \varepsilon_0$

Now let's get the homogeneous solution. $L \cdot q'' + R \cdot q' + \frac{1}{C} q = 0$. Assume $\tilde{q}_h(t) = e^{kt}$, then $e^{kt}(Lk^2 + Rk + \frac{1}{C}) = 0$.

As $e^{kt} > 0$, $Lk^2 + Rk + \frac{1}{C} = 0$ has to be satisfied. Depending on R we can divide the general solution into three cases.

1) $R^2 > \frac{4L}{C}$, then $k = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$ as $R^2 - \frac{4L}{C} > 0$. $q(t) = A \cdot e^{\frac{-R + \sqrt{R^2 - \frac{4L}{C}}}{2L} t} + B \cdot e^{\frac{-R + \sqrt{R^2 - \frac{4L}{C}}}{2L} t} + C \cdot \varepsilon_0$

2) $R^2 = \frac{4L}{C}$, then $k = \frac{-R}{2L}$ thus $q_h(t) = C \cdot e^{-\frac{R}{2L} t} + D \cdot t \cdot e^{-\frac{R}{2L} t}$ making $q(t) = C \cdot e^{-\frac{R}{2L} t} + D \cdot t \cdot e^{-\frac{R}{2L} t} + C \cdot \varepsilon_0$

3) $R^2 < \frac{4L}{C}$, then $k = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$ but as $R^2 - \frac{4L}{C} < 0 \Rightarrow k = \frac{-R \pm \sqrt{\frac{4L}{C} - R^2} \cdot i}{2L}$ $(i = \sqrt{-1})$. which makes $q(t) = e^{-\frac{R}{2L}} \cdot \left(E \cdot \cos\left(\frac{\sqrt{\frac{4L}{C} - R^2}}{2L} t\right) + F \cdot \sin\left(\frac{\sqrt{\frac{4L}{C} - R^2}}{2L} t\right) \right) + C\varepsilon_0$.

We can see that $q(t)$ oscillates only when $R^2 < \frac{4L}{C}$.

$\therefore$ Hence if $R^2 \geq \frac{4L}{C}$ is satisfied the current will not oscillate. This is true as $I_{(current)} = \frac{dq}{dt}$.