CPSC 320 2023S2: Tutorial 1 Solutions

1 Generating permutations lexicographically

This tutorial question will help you to get started one of the problems in the first assignment.

Let $\pi[1..n]$ and $\pi'[1..n]$ be two permutations over a set of n integers. We say that $\pi < \pi'$ if and only if for some i, $1 \leq i < n$,

$$\pi[1..i-1] = \pi'[1..i-1] \text{ and } \pi[i] < \pi'[i].$$

For example, if the set of integers is $\{1, 5, 6, 8\}$, then $5168 < 5618$ (the conditions hold with $i = 2$), and also $5168 < 5186$ (the conditions hold with $i = 3$).

Let $\pi_1, ..., \pi_{n!}$ be an ordering of all $n!$ permutations of a set of n integers. This is a *lexicographic ordering* if $\pi_k < \pi_{k+1}$ for all $k, 1 \leq k < n!$.

1. Write down the first four permutations in the lexicographic ordering of all permutations over $\{1, 5, 6, 8\}$.

   SOLUTION: The first four permutations, in order, are 1568, 1586, 1658, 1685.

2. How can you tell if a permutation $\pi[1..n]$ is the *first* permutation in lexicographic order?

   SOLUTION: The integers stored in π are sorted in increasing order.

3. How can you tell if a permutation $\pi[1..n]$ is the *last* permutation in lexicographic order?

   SOLUTION: The integers stored in π are sorted in decreasing order.

4. How can you tell if permutation $\pi'[1..n]$ is the lexicographically next permutation after permutation $\pi[1..n]$, that is, π' immediately follows π in the lexicographic ordering? Write down necessary and sufficient criteria that can be checked efficiently. (Play around with some examples to build your intuition.)

   SOLUTION: Find the smallest value, say i, in the range $[0..n-1]$ such that $\pi[i+1..n]$ is sorted in descending order. If $i = 0$, π must be the last permutation in lexicographic order, so π' cannot follow π.

   Otherwise, if $i \geq 1$, find j, where $\pi[j]$ is the smallest value in $\pi[i+1..n]$ that is larger than $\pi[i]$. Then π' follows π in lexicographic order if and only if

   (a) $\pi'[1..i-1] = \pi[1..i-1]$[1],
   (b) $\pi'[i] = \pi[j]$, and
   (c) $\pi'[i+1..n]$ is sorted in ascending order.

[1]When $i = 1$, the left and right sides here are empty, and so trivially equal

5. Suppose that all $n!$ permutations of a set of n integers are listed lexicographically. How many times does one permutation $\pi[1..n]$ differ from its successor at exactly *two* positions? Which positions are these?

   SOLUTION: This happens exactly $n!/2$ times, at the rightmost two positions. In this case, the leftmost change is at position $n-1$.

6. Suppose that all $n!$ permutations of a set of n integers are listed lexicographically. How many times is the leftmost change between a permutation $\pi[1..n]$ and its successor at position $n-2$?

   SOLUTION: Suppose that the leftmost change is at position $n-2$. There are $\binom{n}{3}$ ways to choose the integers in the last three positions. For each such choice there are $(n-3)!$ ways that the integers in positions $1, 2, ..., n-3$ can be permuted.

   In order for the leftmost change to be at position $n-2$, it must be the case that $\pi[n-1]$ is the largest of the three integers in the rightmost three positions. The other two integers can be in either of two orders, so there are two possible orderings for these two integers. So overall, $\binom{n}{3}(n-3)!2$ permutations differ from their successor at exactly three positions. We can simplify this expression a little:

$$\binom{n}{3}(n-3)!2 = \frac{n(n-1)(n-2)}{1.2.3}(n-3)!2 = n!/3.$$

2 Review proof by contradiction

Let a, b and n be non-negative real numbers. Suppose that $a \cdot b = n$. Prove that the statement

$$\text{either} \quad a \leq \sqrt{n} \tag{1}$$
$$\text{or} \quad b \leq \sqrt{n} \tag{2}$$

is true.

SOLUTION: We use proof by contradiction. Suppose not, such that the statement is false. This means that "(1) or (2)" is false. By De Morgan's law, that means that (1) is false *and* (2) is false. In other words, $a > \sqrt{n}$ **and** $b > \sqrt{n}$. Multiplying these two inequalities, we obtain that $a \cdot b > n$. This contradicts our hypothesis that $a \cdot b = n$. Therefore we were incorrect when we assumed the statement is false, so the statement is true.