Solutions to homework 5:

1. Prove that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n}(2k-1)\cdot 2^k = 6 + 2^n(4n-6).$$

- Let's prove by induction.
- Base case: $n = 1$,

$$\sum_{k=1}^{1}(2k-1) * 2^k = 6 + 2(4-6) \quad (1)$$
$$(2-1)\cdot 2 = 6 + 2(4-6) \quad (2)$$
$$2 = 2 \quad (3)$$

- Thus, the result holds.
- Inductive step: Let's assume that this case is satisfied for n.
- Then for $n+1$,

$$\sum_{k=1}^{n+1}(2k-1)\cdot 2^k = \sum_{k=1}^{n}(2k-1)\cdot 2^k + (2n+1)\cdot 2^{n+1} \quad (4)$$
$$= 6 + 2^n(4n-6) + (2n+1)\cdot 2^{n+1} \quad (5)$$
$$= (4n + 4n - 6 + 2)\cdot 2^n + 6 \quad (6)$$
$$= (8n-4)\cdot 2^n + 6 \quad (7)$$
$$= 6 + 2^{n+1}(4(n+1)-6) \quad (8)$$

- Therefore, by induction the result holds.

2. Let $n \in \mathbb{N}$. Prove that if $a_{n+2} = 5a_{n+1} - 6a_n$ and $a_1 = 1, a_2 = 5$, then $a_n = 3^n - 2^n$ for all $n \geq 3$.

- Let's prove by induction.
- Base case: $n = 1$,

$$a_3 = 5a_2 - 6a_1 \quad (9)$$
$$= 5\cdot 5 - 6\cdot 1 \quad (10)$$
$$= 19 \quad (11)$$
$$= 3^3 - 2^3 \quad (12)$$

- Inductive step: Let's assume that this case is true for $n = k$.

- Then,

$$a_k = 3^k - 2^k \text{ and } a_{k+2} = 5a_{k+1} - 6a_k \qquad \textit{bottom} \tag{13}$$
$$a_{k-1} = 3^{k-1} - 2^{k-1} \text{ and } a_{k+1} = 5a_k - 6a_{k-1} \qquad \textit{bottom} \tag{14}$$
$$a_{k+1} = 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}) \tag{15}$$
$$= 3^{k+1} - 2^{k+1} \tag{16}$$

- Then we can continue

$$a_{k+2} = 5a_{k+1} - 6a_k \tag{17}$$
$$= 5(3^{k+1} - 2^{k+1}) - 6(3^k - 2^k) \tag{18}$$
$$= 3^{k+2} - 2^{k+2} \tag{19}$$

- As this is true for some $n = 1, 2, 3, \cdots k$,

$$a_{k+3} = 5a_{k+2} - 6a_{k+1} \tag{20}$$
$$= 5(3^{k+2} - 2^{k+2}) - 6(3^{k+1} - 2^{k+1}) \tag{21}$$
$$= 3^{k+3} - 2^{k+3} \tag{22}$$

- Therefore, we can see that when that the conclusion is true by induction.

3. Let $n \in \mathbb{N}$ and suppose that $a_0 = 1, a_1 = 3, a_2 = 9$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$. Show that $a_n \leq 3^n$.

- Let's prove by induction
- Base case: $n = 3$

$$a_3 = a_2 + a_1 + a_0 \tag{23}$$
$$= 1 + 3 + 9 \tag{24}$$
$$= 13 \leq 27 = 3^3 \tag{25}$$

- Thus the result holds.
- Now let's assume that this satisfies for some $n = 1, 2, 3, \cdots k$.
- As $a_2, a_1, a_0 \geq 0$. We know that for any a_n, $a_n \geq 0$. Also $a_n \geq a_{n-1}, n \in \mathbb{N}$.
- As we assumed that $a_n \leq 3^n$, for a_{n+1} we can see that

$$a_{n+1} = a_n + a_{n-1} + a_{n-2} \tag{26}$$
$$a_n + a_{n-1} + a_{n-2} \leq a_n + a_n + a_n = 3a_n \tag{27}$$
$$\leq 3 \cdot 3^n = 3^{n+1} \tag{28}$$

- Thus we can see that $a_{n+1} \leq 3^{n+1}$, therefore the result holds.

4. Prove that for all integers $n > 1$,

$$\tfrac{1}{n+1} + \tfrac{1}{n+2} + \cdots + \tfrac{1}{2n} > \tfrac{13}{24}.$$

- Let's simplify the equation, $\sum_{k=1}^{n} \frac{1}{n+k}$.
- Now let's prove by induction.
- Base case: $n = 2$,

$$\sum_{k=1}^{2} \frac{1}{2+k} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{13}{24}. \tag{29}$$

- Thus the result holds.
- Inductive step: Let's assume that this relationship is satisfied for n.
- Then for $n+1$,

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{n+1+k} &= \sum_{k=2}^{n+2} \frac{1}{n+k} &(30)\\ &= \sum_{k=1}^{n} \frac{1}{n+k} - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2} &(31)\end{aligned}$$

- Then,

$$\begin{aligned}\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} &= \frac{1}{2n+1} - \frac{1}{2n+2} &(32)\\ &= \frac{1}{(2n+1)(2n+2)} > 0 &(33)\\ \sum_{k=1}^{n+1} \frac{1}{n+1+k} &= \sum_{k=1}^{n} \frac{1}{n+k} - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2} > \frac{13}{24} &(34)\end{aligned}$$

- Thus the result holds for $n+1$.
- Therefore the implication is true.

5. Prove that $7^{4n+3} + 2$ is a multiple of 5 for all non-negative integers n.

- Let's prove by induction.
- Base case: $n = 0$,

$$\begin{aligned}7^3 + 2 &= 345 &(35)\\ &= 5 \cdot 69 &(36)\end{aligned}$$

- Thus the result holds.

- Inductive step: Let's assume that $7^{4n+3}+2$ is a multiple of 5 for n.
- $7^{4n+3}+2 = 5k, k \in \mathbb{Z}$.

$$\begin{aligned} 7^{4(n+1)+3}+2 &= 7^4 \cdot (7^{4n+3}+2) - 2 \cdot 7^4 + 2 && (37)\\ &= 7^4 \cdot 5k - 14 \cdot 7^3 + 2 && (38)\\ &= 7^4 \cdot 5k - 14(7^3+2) + 30 && (39)\\ &= 7^4 \cdot 5k - 14 \cdot 245 + 30 && (40)\\ &= 5(7^4 k - 14 \cdot 49 + 6) && (41) \end{aligned}$$

- As we know that $7^4k - 14 \cdot 49 + 6 \in \mathbb{Z}$.
- Therefore, $7^{4(n+1)+3}+2$ is also a multiple of 5.

6. We define a sequence $(x_n)_{n\in\mathbb{N}}$ with $a_1 = 3$, and for every $n \geq 1, a_{n+1} = a_n^2 - a_n$. Show that (a_n) is increasing, which means that for all $n \in \mathbb{N}, a_n < a_{n+1}$.

- Let's prove by induction.
- Base case: when $n = 1$,

$$\begin{aligned} a_2 &= a_1^2 - a_1 && (42)\\ &= 3^2 - 3 && (43)\\ &= 6 && (44) \end{aligned}$$

- Therefore, we can see that it increases as $a_2 > a_1$.
- Inductive step: Let's assume for some n this relationship is satisfied.
- Also, for $n = 1, 2, 3, \cdots, k$ it is also satisfied.
- Thus, $a_{k+1} = a_k^2 - a_k$ and $a_{k+1} > a_k$.
- Through strong induction we know that $a_k > a_1 = 3$ as the sequence is increasing.

$$\begin{aligned} a_{k+2} &= a_{k+1}^2 - a_{k+1} && (45)\\ a_{k+2} - a_{k+1} &= a_{k+1}^2 - 2a_{k+1} && (46)\\ &= (a_k^2 - a_k)^2 - 2(a_k^2 - a_k) && (47)\\ &= (a_k^2 - a_k - 2)(a_k^2 - a_k) && (48) \end{aligned}$$

- We know that $a_k > 3$, therefore we can conclude that both $(a_k^2 - a_k - 2)$ and $(a_k^2 - a_k)$ are positive.
- Thus $a_{k+2} - a_{k+1} > 0$, therefore by mathematical induction the result holds.

7. Let $x \in \mathbb{R}$ with $x \neq 1$and let $N \in \mathbb{N}$. Use mathematical induction to show that

$$\textstyle\sum_{k=1}^{N} k \cdot x^{k-1} = \frac{1-x^N}{(1-x)^2} - \frac{Nx^N}{1-x}$$

- Let $N = 1$, then

$$1 = \sum_{k=1}^{1} k \cdot x^{k-1} = \frac{1-x}{(1-x)^2} - \frac{x}{1-x} \tag{49}$$
$$= \frac{(1-x)^2}{(1-x)^2} = 1 \tag{50}$$

- Thus, the result holds.
- Now let's assume the relationship is satisfied for some N.
- Then,

$$\sum_{k=1}^{N+1} k \cdot x^{k-1} = \sum_{k=1}^{N} k \cdot x^{k-1} + (N+1) \cdot x^N \tag{51}$$
$$= \frac{1-x^N}{(1-x)^2} - \frac{Nx^N}{1-x} + (N+1) \cdot x^N \tag{52}$$
$$= \frac{(1-x^N) - (1-x)(N \cdot x^N) + (N+1)x^N(1-x)^2}{(1-x)^2} \tag{53}$$
$$= \frac{(1-x^{N+1}) - (N+1)(1-x)(x^{N+1})}{(1-x)^2} \tag{54}$$

- We can further simplify this and obtain,

$$\frac{1-x^{N+1}}{(1-x)^2} - \frac{(N+1)x^{N+1}}{(1-x)} \tag{55}$$

- Therefore, by mathematical induction the result holds.

8. Find all positive integers n so that $n^3 > 2n^2 + n$. Prove your result using mathematical induction.

- $n = 3$,

$$3^3 > 2 \cdot 3^2 + 3 \tag{56}$$
$$27 > 21 \tag{57}$$

- Thus the result holds.
- Now let's assume that $n^3 > 2n^2 + n$ for all n such that $n \geq 3$.

$$(n^2-2)(n+1) > 0 (n \geq 3) \tag{58}$$
$$(n^2-2)(n+1) = n^3 + n^2 - 2n - 2 \tag{59}$$
$$= (n+1)^3 - 2(n+1)^2 - (n+1) > 0 \tag{60}$$
$$(n+1)^3 > 2(n+1)^2 + (n+1) \tag{61}$$

- Therefore, for all $n \geq 3$ we can see that $n^3 > 2n^2 + n$ by induction.