

Due Monday February 5, Submit online in a PDF document on Canvas by 11:59 pm of the due date

Problem 1 (Do not submit): Find all singular points of the followings equations and determine whether each of them is regular or irregular. For each regular singular point, determine the indicial equation and the exponents at the singularity.

1. $(x^2 - 4)^2 y'' + 8(x + 2)y' + y = 0$
2. $(x^2 - 2x - 3)y'' + (x - 3)y' + (x + 1)y = 0$
3. $xy'' + xy' + e^x y = 0$
4. $x^2 y'' + 2(x + \sin x)y' + y = 0$

Problem 2 (Submit): Consider the differential equation

$$3x^2(1+x)y'' + 4xy' - 4(1+x)y = 0 \quad (1)$$

- a. Classify the points $0 \leq x < \infty$ as ordinary points, regular singular points, or irregular singular points.
- b. Find the exponents at the singular point $x = 0$.
- c. Use the series expansion $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ to determine two linearly independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

Problem 3 (Submit): Consider the Bessel equation of order $1/2$

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \quad \nu = \frac{1}{2} \quad (2)$$

- a. Show that $x = 0$ is a regular singular point.
- b. Find the roots of the indicial equation for $x = 0$.
- c. Use a Frobenius series expansion about $x = 0$ to find two linearly independent solutions of equation (2). How does each of these solutions behave near $x = 0$?
- d. Find the radius of convergence of the series solutions in (c).

Problem 4 (Do not submit): Consider the Chebyshev equation

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0, \quad (3)$$

where α is a constant. The solutions to this equation are the famous Chebyshev polynomials that are used in the approximation theory and polynomial interpolation.

- a. Show that $x = 1$ and $x = -1$ are regular singular points of equation (3).
- b. Find the exponents at each of these singularities (the roots of the indicial equation).

- c. Find two linearly independent solutions about $x = 1$. (*Hint:* Write $(1 - x^2) = -(x - 1)(x + 1) = -(x - 1)(2 + (x - 1))$ and $x = 1 + (x - 1)$ or make the change of variable $x - 1 = t$.)
- d. Explain how you write a power series solution about $x = 0$ to find two linearly independent solutions. You must not try to compute the series solution itself.

Problem 5 (Do not submit): Consider the differential equation

$$(\ln x)y'' + \frac{1}{2}y' + y = 0 \tag{4}$$

- a. Show that equation 4 has a regular singular point at $x = 1$.
- b. Determine the roots of the indicial equation at $x = 1$.
- c. Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n(x - 1)^{n+r}$ corresponding to the largest root.