INSTRUCTOR NOTES

This small class has recommended additional materials: one internet-enabled and Excel-capable laptop per team, and multiple colours of chalk or board markers.

For this small class, remember to focus on the *picture* of Newton's method. Students should leave with a strong visual representation of the method

Begin by getting students to sit in their teams. If possible, members should sit facing each other. Do not allow students to change teams.

If only 2 members show up for a team, they may work with another small team, but must submit their own worksheet. If only 1 member shows up for a team, they must work with another team, and may submit a blank worksheet with only their name on it for attendance-taking purposes.

Finally, pass out the handouts and announce the first question. Remember to have a routine to close questions (*e.g.* countdowns).

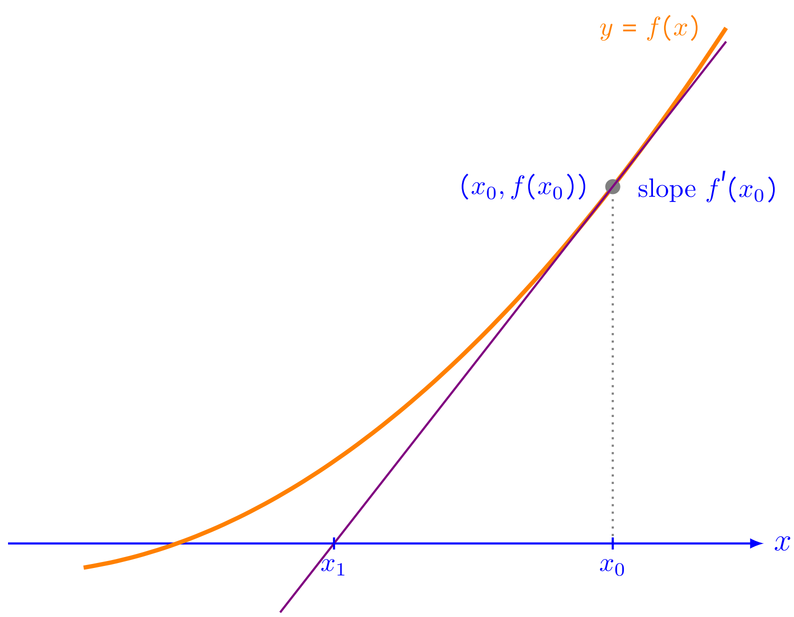
At the end of the class, remember to collect worksheets.

This week's tip: ***start with a question***. When you visit teams, especially for question 3, have a question in mind to ask. Students struggling with open-ended problems like question 3 often need a good, answerable question to focus their efforts.

NOTES ON QUESTIONS

The large lecture prior to this small class is on Euler's method, with an emphasized computational component using spreadsheets. This small class covers another computational tool, Newton's method.

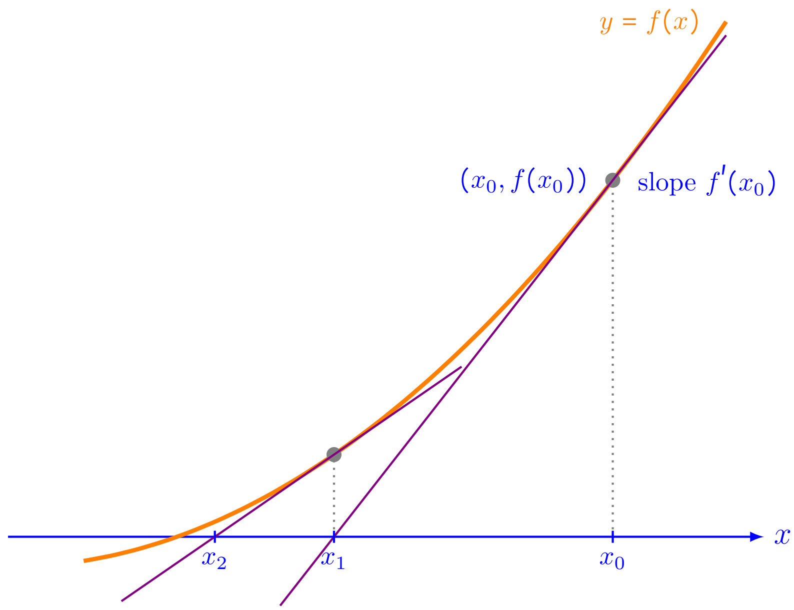
1. **10 minutes**.
   To set up the question, draw a very large version of the figure below for teams to copy.



Teams that are struggling may need the hint to calculate $f'(x_0)$.
To close the question, have a team with the correct answer explain it and write the conclusion on the board.
Since $f'(x_0) = \dfrac{f(x_0) - 0}{x_0 - x_1}$, we can solve for $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

2. **1 minute** of instructor-led class discussion.
   To close the question, have a student extend the function on the board (preferably using a different coloured chalk or marker) such that x_1 is a root.

3. **10 minutes**.
   Hints may need to be escalated if teams are making little progress.
   To close the question, once a team gets the idea, have them explain it to nearby teams. Once most of the teams have the idea, complete the picture on the board to look like the one below.



4. **5 minutes**.
   To close the question, once most teams have the idea, draw a few more iterations on the board, and then write down the answer below.
   This method of successively using the intercepts of linear approximations

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   to estimate a root is known as *Newton's method*.

5. **1 minute**. At this point (and not before), one laptop per team may be opened.
   To close the question, write down the answer on the board.
   $f'(x) = 5x^4 - 8x^3 + 6x$.

6. **5 minutes** for questions 6 and 7. Note that these questions are not strictly necessary; their goal is to indicate that $f(x)$ has a root, but it is not easy to calculate.

8. **Remaining time** for the remaining questions.
   See the spreadsheet "NewtonsMethod" for a sample. The initial guess is in red. Many teams will be

unfamiliar with the basics of programming in Excel, and may need help getting started. Some teams may not be able to complete questions 9 and 10 — this is fine.

To close the questions, as a majority of teams works through each initial guess, confirm their answers to the class.

The root is roughly -1.1079.

For $x = 0$, the tangent line at the first iteration is horizontal and fails to yield a value x_1. For $x = 1$, the tangent line at the second iteration is horizontal. For $x = 2$, various iterations yield values x_n close to 0 where $f'(x_n)$ is small, sending the next iteration far afield (but with 10 more iterations, stability is achieved).

SMALL CLASS: Newton's method

In this class, you will describe and apply Newton's method for estimating the root of an equation.

Contributing team members

Student number	Last name	First name

Small class questions

There are many circumstances in which you need to find a root (*i.e.* x-intercept) of a function $f(x)$, but you cannot easily "solve" the equation $f(x) = 0$. Newton's method allows you to estimate a root with only the expressions for $f(x)$ and $f'(x)$.

1. (★☆☆☆) Given the figure drawn by your instructor, find an expression for x_1 in terms of the other quantities labelled.

Answer:
Scribe:

2. Is it possible that x_1 is a root of $f(x)$?

Answer:
Scribe:

3. (Key concept) If x_1 is not a root, what is an idea to get closer to a root? Draw a picture to explain the idea.
Hint: You found x_1 using a linear approximation. Can you use the same idea again?

Answer:
Scribe:

4. (★☆☆☆) If you iterate the idea in question $n+1$ times, what is the formula you get for x_{n+1}?

Answer:
Scribe:

In the remaining questions, you apply Newton's method to the function

$$f(x) = x^5 - 2x^4 + 3x^2 + 1.$$

Your team will need one laptop open for these questions.

5. Find $f'(x)$.

Answer:
Scribe:

6. Use the Desmos graphing calculator to plot $f(x)$. By inspection, approximate the root of $f(x)$.

   Note: The Desmos graphing calculator, while helpful, is a "black box" — it is not clear how it implements.

Answer:
Scribe:

7. Use Wolfram Alpha to estimate the (real-valued) root of $f(x)$.

   Note: Wolfram Alpha is even more of a black box.

Answer:
Scribe:

8. Write an Excel program to estimate the root of $f(x)$ using Newton's method and 10 iterations. The columns of your Excel spreadsheet should be labelled: "Guess number", "Guess (x_n)", "Function at x_n ($f(x_n)$)", "Derivative at x_n ($f'(x_n)$)", and "Intercept of linear approximation about guess (x_{n+1})".

9. Use initial guesses of $x = -1, -2, -3$. What approximation(s) do you converge to?

Answer:
Scribe:

10. Something "goes wrong" for the guesses $x = 0, 1, 2$. In each case, describe what you observe, and give an explanation for what you think is happening.

Answer:
Scribe:

Practice questions

The questions below are for practice. They do not contribute to your grade, and it is not expected that you complete them during your small class. However, you are strongly encouraged to work through them.

11. (★★☆☆) Repeat questions 8 and 9 above for $f(x) = e^x - 3x^2$.

12. (★★☆☆) What would happen if you used Newton's method to estimate the root of $f(x) = ax + b$, where $a \neq 0$ and b are constants?

13. (★★★☆) Repeat questions 8 and 9 above for $f(x) = 2x^2 + x - 3$. In this case, $f(x)$ has two roots. Given an initial guess x_0, explain carefully how you can tell in advance which root Newton's method will converge to.