Solutions to homework 1:

1. Your solution to question 1.

   Proof. If $n \in \mathbb{Z}$, $3 \mid n+1$ can be re-written as $n+1 = 3k$ for some $k \in \mathbb{Z}$. The expression $3 \nmid n^2+5n+5$ can be also re-written as:

$$\begin{aligned} n^2+5n+5 &= (n+1)((n+1)+3)+1 \\ &= (n+1)^2+3(n+1)+1 \\ &= (3k)^2+3(3k)+1 \\ &= 3(3k^2+3k)+1 \end{aligned}$$

   Since $(3k^2+3k) \in \mathbb{Z}$, it shows that $3 \nmid n^2+5n+5$.

   □

2. Your solution to question 2.

   Proof. Let $a \in \mathbb{Z}$ and $5a+11$ is odd. If $5a+11 = 2k+1$ for some $k \in \mathbb{Z}$, we can express the statement as

$$\begin{aligned} 9a+3 &= (5a+11)+(4a-8) \\ &= 2k+1+4a-8 \\ &= 4a+2k-7 \\ &= 2(2a+k-4)+1 \end{aligned}$$

   Since $(2a+k-4) \in \mathbb{Z}$, we can conclude that $9a+3$ is odd.

   □

3. Your solution to question 3.

   Proof. Assume $x \in \mathbb{R}$. For $x^2-x-2 = (x-2)(x+1)$, the term $(x-2)$ is negative for any $x < 2$ and $(x+1)$ is positive for any $x > -1$. Since $ab < 0$ for $a < 0$ and $b > 0$ where $a, b \in \mathbb{R}$, we can conclude that $(x^2-x-2) < 0$ for the given interval $-1 < x < 2$.

   □

4. Your solution to question 4.

Proof. Let $a, b, c, d \in \mathbb{Z}$ and $a, c, b+d$ be odd numbers. Then we can express a, b, c, d in terms of k, ℓ, m, n where $k, \ell, m, n \in \mathbb{Z}$:

$$a = 2k+1 \qquad c = 2m+1 \qquad b+d = 2(\ell+n)+1 \text{ where } b = 2\ell,\ d = 2n+1$$

Then,

$$\begin{aligned} ab + cd &= (2k+1)(2\ell) + (2m+1)(2n+1) \\ &= 2(2k\ell + 2mn + \ell + m + n) + 1 \end{aligned}$$

Since $(2kl + 2mn + \ell + m + n) \in \mathbb{Z}$, we can conclude that $ab + cd$ is odd.

$\square$

5. Your solution to question 5.

Proof. Let $x, y \in \mathbb{R}$. By multiplying the inequality

$$xy \leq \frac{1}{2}\left(x^2 + y^2\right)$$

by 2 and rearranging, we obtain

$$\begin{aligned} 0 &\leq x^2 - 2xy + y^2 \\ 0 &\leq (x+y)^2 \end{aligned}$$

Since $(x+y) \in \mathbb{R}$ and a square of a real number is positive, we can conclude that the expression holds true. Therefore, $xy \leq \frac{1}{2}\left(x^2 + y^2\right)$ is true.

$\square$

6. Your solution to question 6.

Proof. Let $x, y \in \mathbb{R}$. $x < y$ is equivalent to $x - y < 0$ and $y^2 < x^2$ can be re-written as

$$\begin{aligned} y^2 &< x^2 \\ 0 &< x^2 - y^2 \\ 0 &< (x+y)(x-y) \end{aligned}$$

For $ab > 0$ where $a, b \in \mathbb{R}$, conditions $a > 0$, $b > 0$ or $a < 0$, $b < 0$ must be true for the statement to be valid. Since $(x-y) < 0$, $(x+y) < 0$ must hold true in order to satisfy the inequality $y^2 < x^2$. Therefore, we can conclude that $(x+y) < 0$ when given $(x-y) < 0$ and $y^2 < x^2$.

$\square$

7. Your solution to question 7.

Proof. Let $n \in \mathbb{Z}$ and $5 \mid (n+7)$. To check n^2+1 is divisible by 5, we can re-write (n^2+1) and assume $(n+7) = 5k$ for $k \in \mathbb{Z}$.

$$\begin{aligned}(n^2+1) &= (n+1)(n-1)+2 \\ &= (5k-6)(5k-8)+2 && \text{derived from } (n+7)=5k \\ &= 25k^2-70k+50 \\ &= 5(5k^2-14k+10)\end{aligned}$$

Since $(5k^2-14k+10) \in \mathbb{Z}$, it shows that $5 \mid (n^2+1)$ holds true.

$\square$

8. Your solution to question 8.

Proof. Let $n, a, b, x, y \in \mathbb{Z}$ with $n > 0$. If $a = nk$ and $b = n\ell$ for $k, \ell \in \mathbb{Z}$, we can express $(ax+by)$ as

$$\begin{aligned}(ax+by) &= nkx + n\ell y \\ &= n(kx+\ell y)\end{aligned}$$

Since $(kx+\ell y) \in \mathbb{Z}$, we can conclude that $n \mid (ax+by)$ is true.

$\square$