1.

(a) $f(x) = \begin{cases} 3x - b, & x \in [0,1] \\ 0, & \text{o.w.} \end{cases}$

$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 (3x-b)\,dx = \frac{3}{2} - b = 1$ so we should have $b = \frac{1}{2}$. But let's see if $f(1) = \lim_{\varepsilon \to 0} \frac{P(X \le a+\varepsilon) - P(X \le a)}{\varepsilon}$ the right derivate of the cdf.

$F'(1)$ of the right is $0 \neq f(1) = 3 - b = \frac{5}{2}$. So no value of b exists to make f a pdf of some r.v.

(b) $f(x) = \begin{cases} \frac{1}{2}\cos x & x \in [-b, b] \\ 0 & \text{o.w.} \end{cases}$

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-b}^{b} \frac{1}{2}\cos x\,dx = \left[\frac{1}{2}\sin x\right]_{-b}^{b} = \frac{1}{2}(\sin b - \sin(-b)) = \sin b$, in order to have $\sin b = 1$, $b = \frac{\pi}{2} + n\pi$ for some even n.

Again, let's check the right derivative. $F'(x) = 0$ on the right and $f(b) = 0$.

However, if f is the pdf of some r.v. for some $a \in [-b, b]$, $\int_{-\infty}^{a} f(x)\,dx \ge 0$ needs to be satisfied. If $n > 0$, then there becomes at point where $\int_{-\infty}^{a} f(x)\,dx < 0$ hence $n = 0$ and $b = \frac{\pi}{2}$ would be the only value.

2

Let $c > 0$ and $X \sim \text{Unif}[0, c]$ and $Y := c - X$.

$F_Y(t) = P(Y \le t) = P(c - X \le t) = P(c - t \le X) = 1 - P(X < c - t)$. If $c - t < 0 \to c < t$ then $1 - P(X < c - t) = 1$. Now if $c - t < c$ $\to$ then $0 < t$ hence $P(Y \le t) = 0$

Now if $0 \le c - t < c$ then $0 < t \le c$, $P(Y \le t) = P(c - t \le X) = 1 - P(X < c - t) = 1 - P(X \in [0, c-t)) = 1 - \frac{c-t}{c} = \frac{t}{c} = P(X \le t)$ for $0 \le t < c$ showing us that X and Y share the same pdf.

$F_X(t), F_Y(t) = \begin{cases} 0, & t < 0 \\ \frac{t}{c}, & 0 \le t < c \\ 1, & c \le t \end{cases}$ thus $f_X(t), f_Y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{c}, & 0 \le t < c \\ 0, & c \le t \end{cases}$ X and Y have the same pdf.

3.

Let X be a r.v. with the pdf $f(x) = \begin{cases} cx^{-3} & x > 2 \\ 0 & \text{o.w.} \end{cases}$

(a) $1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_2^{\infty} cx^{-3}\,dx = \left[-\frac{c}{2}x^{-2}\right]_2^{\infty} = \frac{c}{2}\frac{1}{4} \Rightarrow c = 8.$

(b) $x > 2$, $F_X(x) = \int_{-\infty}^{x} \frac{c}{t^3}\,dt = \int_2^{x} \frac{8}{t^3}\,dt = \left[\frac{-4}{t^2}\right]_2^{x} = -4\left(\frac{1}{x^2} - \frac{1}{4}\right) = 1 - \frac{4}{x^2}$.

$x \le 2$, $F_X(x) = \int_{-\infty}^{x} 0\,dt = 0.$

$\therefore F_X(x) = \begin{cases} 1 - \frac{4}{x^2}, & x > 2 \\ 0, & \text{o.w} \end{cases}$

(c) $P(X > 3 \mid X < 5) = \frac{P(3 < X < 5)}{P(X < 5)} = \frac{F_X(5) - F_X(3)}{F_X(5)} = \frac{\frac{4}{9} - \frac{4}{25}}{\frac{21}{25}} = \frac{100 - 36}{189} = \frac{64}{189}$

(d) $P(X > m) = P(X \le m) = 1 - P(X \le m)$ so $P(X \le m) = \frac{1}{2} = 1 - \frac{4}{x^2}$ $\therefore x = 2\sqrt{2}$ $(x > 2)$

(e) $E[\sqrt{X}] = \int_{-\infty}^{\infty} \sqrt{x} \cdot f(x)\,dx = \int_2^{\infty} 8x^{-3} \cdot \sqrt{x}\,dx = \int_2^{\infty} 8 \cdot x^{-\frac{5}{2}}\,dx = \left[-\frac{16}{3}x^{-\frac{3}{2}}\right]_2^{\infty} = \frac{16}{3} \cdot \frac{1}{2\sqrt{2}} = \frac{4}{3}\sqrt{2}$ $\therefore \frac{4}{3}\sqrt{2}$

4.

$Y := \min(X, \ell - X)$

(a) $P(Y \le b) = P(\min(X, \ell - X) \le b) = 1 - P(\min(X, \ell - X) > b) = 1 - P(X > b \;\cap\; \ell - b > X)$

(b) if $b < 0$ then $P(Y \le b) = 1 - P(\ell - b > X > b) = 1 - P(\ell > X > 0) = 0$

if $b \ge \frac{\ell}{2}$ then $P(Y \le b) = 1 - P(X > b \cap \ell - b > X) = 1 - 0 = 1$

if $0 \le b < \frac{\ell}{2}$ then $P(Y \le b) = 1 - P(X > b \cap \ell - b > X) = 1 - P(\ell - b > X > b) = 1 - \dfrac{\ell - 2b}{\ell} = \dfrac{2b}{\ell} = \dfrac{b - 0}{\frac{\ell}{2} - 0}$

then $F_Y(b) = \begin{cases} 0 & b < 0 \\ \dfrac{b}{\frac{\ell}{2}} & 0 \le b < \frac{\ell}{2} \\ 1 & \frac{\ell}{2} \le b \end{cases}$

then $f_Y(b) = \begin{cases} 0, & b < 0 \\ \frac{1}{\frac{\ell}{2}}, & 0 \le b < \frac{\ell}{2} \\ 0, & \frac{\ell}{2} \le b \end{cases}$

so $Y \sim \mathrm{Unif}[0, \frac{\ell}{2}]$

5.

$P(X \in [0,1]) = 1 - e^{-2}$

$P(X \in [a,2]) = \begin{cases} a \le 0, & 1 - e^{-4} \\ a > 0, & e^{-2a} - e^{-4} \end{cases}$

$P(X \in [0,1] \cap X \in [a,2]) = \begin{cases} 1 < a, & 0 \\ 0 < a \le 1, & P(X \in [a,1]) = e^{-2a} - e^{-2} \\ a \le 0, & P(X \in [0,1]) = 1 - e^{-2} \end{cases}$

$a \le 0$, $(1 - e^{-2})(1 - e^{-4}) = 1 - e^{-2}$ such a DNE

$0 < a < 1$, $(e^{-2a} - e^{-4})(1 - e^{-2}) = e^{-2a} - e^{-2} \Rightarrow e^{-2a} - e^{-2a-2} - e^{-4} + e^{-6} = e^{-2a} - e^{-2} \Rightarrow e^{-2a-2} = e^{-2} - e^{-4} + e^{-6}$

$e^{-2a} = 1 - e^{-2} + e^{-4}$ $\therefore$ $a = -\frac{1}{2}\ln(1 - e^{-2} + e^{-4})$

$a \ge 1$, $(1 - e^{-2})(e^{-2a} - e^{-4}) = 0$ such $a < 2$ DNE

7. Let $X \sim N(2,4)$

(a) $P(X < 6) = P(\frac{X-2}{2} < 2) = 1 - P(\frac{X-2}{2} \ge 2) = 1 - \Phi(-2) = 1 - 0.02275 = 0.97725$

(b) $P(X \le 6) = P(\frac{X-2}{2} \le 2) = \Phi(2) = 0.97725$

(c) $P(X < 1 \mid X > -1) = \dfrac{P(-1 < X < 1)}{P(X > -1)} = \dfrac{P(\frac{-3}{2} < \frac{X-2}{2} < \frac{-1}{2})}{P(\frac{X-2}{2} > -\frac{3}{2})} = \dfrac{1 - (P(\frac{X-2}{2} \ge -\frac{1}{2}) + P(\frac{X-2}{2} \le -\frac{3}{2}))}{1 - P(\frac{X-2}{2} \le -\frac{3}{2})} = \dfrac{1 - (\Phi(\frac{1}{2}) + \Phi(-\frac{3}{2}))}{1 - \Phi(-\frac{3}{2})} = \dfrac{1 - (0.69146 + 0.06681)}{1 - 0.06681} = 0.2561$

(d) $E[X^2] = E[4 \cdot (\frac{X-2}{2})^2 + 8 \cdot (\frac{X-2}{2}) + 4] = 4E[(\frac{X-2}{2})^2] + 8E[\frac{X-2}{2}] + 4 = 4(E[(\frac{X-2}{2})^2] - E[(\frac{X-2}{2})]^2) + 4E[(\frac{X-2}{2})]^2 + 8 \cdot E[\frac{X-2}{2}] + 4 = 8$ $\therefore 8$

(e) $P(X > c) = P(\frac{X-2}{2} > \frac{c-2}{2}) = \Phi(\frac{2-c}{2}) = \frac{1}{3}$ we can approximate $\frac{c-2}{2} = -0.43$ so $c = 1.14$

10

$S_n \sim \mathrm{Bin}(10000, \frac{1}{36}) \approx N(\frac{2500}{9}, \frac{21875}{81})$

$P(280 \le S_n \le 300) = P(280 \le \frac{\sqrt{21875}}{9} Z + \frac{2500}{9} \le 300) = P(\frac{20}{\sqrt{21875}} \le Z \le \frac{200}{\sqrt{21875}}) = P(Z \le \frac{200}{\sqrt{21875}}) - P(Z < \frac{20}{\sqrt{21875}})$

$= \Phi(\frac{200}{\sqrt{21875}}) - \Phi(\frac{20}{\sqrt{21875}})$

$\approx \Phi(1.35) - \Phi(0.135) \approx \Phi(1.35) - \Phi(0.14) = 0.91149 - 0.55567 = 0.35582$