3. $u_t = u_{xx}$, $0<x<1$, $t>0$ B.C's: $u_x(0,t)=1$, $u(1,t)=0$ I.C: $u(x,0)=0$

(a) Determine $u_\infty(x)$

Let $u_\infty(x) = w(x)$, $w_t = 0$ hence $w_{xx}=0 \longrightarrow w(x)= Ax+B$, $w_x(0)= A=1$, $w(1) = A+B=0 \longrightarrow w(x) = x-1$ $\therefore u_\infty(x)=x-1$

(b)

Let $u(x,t) = w(x) + v(x,t) = (x-1) + v(x,t)$, plug it into the original PDE. $u_t = v_t = u_{xx} = w_{xx}+v_{xx}=v_{xx}$ $\therefore$ PDE: $v_t=v_{xx}$

For B.C's, $u_x(0,t) = w_x(0)+v_x(0,t) = 1+v_x(0,t)=1$ $\therefore v_x(0,t)=0$, $u(1,t)=0 = w(1)+v(1,t) = v(1,t)$ $\therefore v(1,t)=0$

For I.C, $u(x,0)=w(x)+v(x,0) \longrightarrow v(x,0) = u(x,0)-w(x) = 0-(x-1) = 1-x$ $\therefore v(x,0)=1-x$

(c)

Given $v_t=v_{xx}$, let $v=X\cdot T$, $X\cdot T' = X''\cdot T \longrightarrow \frac{X''}{X}=\frac{T'}{T}=\lambda$ then $T(t)= e^{\lambda t}$. $X''-\lambda X=0$, 1) $\lambda=0$, then $X''=0 \longrightarrow X= Ax+B$, $X'(0)=0=X(1)$ so $X=0$ (trivial solution)

2) $\lambda>0$, $\lambda=\mu^2 \rightarrow X= A\sinh(\mu x)+B\cosh(\mu x)$, $X'(0)= A\mu=0$ $\therefore A=0$, $X(1)= B\cdot\cosh(\mu)$ $\therefore B=0$, $X=0$ (trivial solution)

3) $\lambda<0$, $\lambda=-\mu^2\rightarrow X''+\mu^2X=0 \longrightarrow X= A\cos(\mu x)+B\sin(\mu x)$, $X'(0)= B\mu=0$ $\therefore B=0$, $X(1)= A\cos(\mu)=0$ if $A=0$, $X=0$ (trivial solution)

if $A\neq0$, $\mu=\frac{2n-1}{2}\pi$, $n=1,2,3,\cdots$ $\therefore X_n= \cos(\frac{2n-1}{2}\pi x)$

Thus, $v(x,t)=\sum_{n=1}^{\infty}\cos(\frac{2n-1}{2}\pi x)\cdot e^{-(\frac{2n-1}{2}\pi)^2 t}$ and hence, $u(x,t)= x-1+\sum_{n=1}^{\infty}\cos(\frac{2n-1}{2}\pi x)\cdot e^{-(\frac{2n-1}{2}\pi)^2 t}$

4. $u_t= u_{xx}-u+x$, $0<x<1$, $t>0$ B.C's: $u_x(0,t)=1$, $u_x(1,t)=2$ I.C: $u(x,0)=x$

Let $u(x,t)= w(x)+v(x,t)$, and also let $w_t= w_{xx}-w+x$ and $w'(0)=1$, $w'(1)=2$. Given $w_t= w_{xx}-w+x \longrightarrow 0=w_{xx}-w+x$, let's first get the homogeneous solution w_h for $w_{xx}-w=0$. Let $w= e^{rx}$, then $r^2-1=0$ hence $r=\pm1$. $\therefore w(x) = A\sinh(x)+B\cosh(x)$.

Now let's get the particular solution, w_p, for $w_{xx}-w+x=0$. Let $w=Ax+B$ then $0=0-(Ax+B)+x=0$, hence $w_p=x$. Knowing $w(x) = x+ A\sinh(x)+B\cosh(x)$ and $w'(0)=1$, $w'(1)=2 \longrightarrow w'(x)= 1+A\cosh(x)+B\sinh(x)$. $w'(0)= 1+A=1$, $w'(1) = 1+A\cosh(1)+B\sinh(1)=2$

$\therefore w(x)= x+\frac{1}{\sinh(1)}\cdot\cosh(x)$

With the $w(x)$ solved above the PDE, B.C. for $v(x,t)$ becomes PDE: $v_t= v_{xx}-v$, B.C.: $v_x(0,t)=0$, $v_x(1,t)=0$ and the I.C for $v(x,t)$ becomes $v(x,0) = u(x,0)-w(x) = -\frac{1}{\sinh(1)}\cdot\cosh(x) = f(x)$.

Let $v(x,t)=X\cdot T \longrightarrow X\cdot T' = X''\cdot T-X\cdot T \longrightarrow \frac{X''}{X}=\frac{T+T'}{T}=\frac{T'}{T}+1=\lambda$. Then, $T(t)= e^{(\lambda-1)t}$ and $X''-\lambda X=0$.

① $\lambda=0$, then $X''=0$. $X=Ax+B$, since $X'(0)=0=X'(1)$ $A=0$ hence $X_0=1$

② $\lambda<0$, then $X''-\lambda X=0$ shows $X(x)= A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$. $X'(0)= B\sqrt{\lambda}=0$, $B=0$. $X'(1)= A\sqrt{\lambda}\cdot\sin\sqrt{\lambda}=0$. $\longrightarrow$ if $A=0$, trivial solution. else, $\lambda=(n\pi)^2$ $(n=1,2,3,\cdots)$ hence $X_n= \cos(n\pi x)$.

③ $\lambda > 0$, then $X(x) = A\sinh(\sqrt{\lambda}x) + B\cosh(\sqrt{\lambda}x)$. $X'(0) = A\sqrt{\lambda}\cdot\cosh(0) = 0$, $A = 0$. $X'(1) = \sqrt{\lambda}B\cdot\sinh(\sqrt{\lambda}) = 0$, $B = 0$. (trivial solution).

Thus, $v(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(n\pi x)\cdot e^{((n\pi)^2-1)t}$. $a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{-1}{\sinh(1)}\cdot\cosh(x)\,dx = \frac{-1}{\sinh(1)}\left[\sinh(1) - \sinh(-1)\right] = -2$.

$$a_n = \frac{-2}{\sinh(1)}\cdot\int_0^1 \cos(n\pi x)\cdot\cosh(x)\,dx = \frac{-2}{\sinh(1)}\cdot\left[\frac{\cos(n\pi x)\cdot\sinh(x) + (n\pi)\cdot\sin(n\pi x)\cosh(x)}{(n\pi)^2+1}\right]_0^1 = \frac{-2}{\sinh(1)}\cdot\frac{\cos(n\pi)\sinh(1)}{(n\pi)^2+1} = \frac{2(-1)^{n+1}}{(n\pi)^2+1}$$

$\therefore v(x,t) = -2 + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{(n\pi)^2+1}\cdot\cos(n\pi x)\cdot e^{((n\pi)^2-1)t}$

And hence $u(x,t) = x + \frac{1}{\sinh(1)}\cdot\cosh(x) - 2 + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{(n\pi)^2+1}\cdot\cos(n\pi x)\cdot e^{((n\pi)^2-1)t}$