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1-(a)

First, $\frac{dT}{dt} = \lambda(T_a - T)$ we know $T_a = 25°C$, $T_0 = 77°C$ and $T_1 = 70°C$

$\frac{dT}{dt} = \lambda(25 - T) \longrightarrow \lambda T + \frac{dT}{dt} = 25\lambda$ we can multiply both sides with $e^{\lambda t}$ and get

$$e^{\lambda t} \cdot \lambda T + e^{\lambda t} \cdot \frac{dT}{dt} = 25\lambda e^{\lambda t} \Rightarrow \frac{d}{dt}(e^{\lambda t} \cdot T) = 25\lambda e^{\lambda t}$$

thus $e^{\lambda t} \cdot T = 25e^{\lambda t} + C$ (C is a constant) making $T = 25 + C \cdot e^{-\lambda t}$

at $t=0$, $T_0 = 77 = 25 + C \cdot e^{0} = 25 + C$ $\therefore C = 52$

at $\underset{\substack{\uparrow \\ T_1}}{t} = \tau = 100$ seconds, $70 = 25 + C \cdot e^{-100\lambda} = 25 + 52 \cdot e^{-100\lambda}$

$$\Rightarrow \frac{45}{52} = e^{-100\lambda} \Rightarrow \lambda = \frac{-1}{100} \ln\left(\frac{45}{52}\right) \approx 0.001446\, s^{-1}$$

1-(b) Now $T_a = 15°C$, therefore $\frac{dT}{dt} = \lambda(15 - T) \Rightarrow \frac{dT}{dt} + \lambda T = 15\lambda \Rightarrow e^{\lambda t}\left(\frac{dT}{dt} + \lambda T\right) = 15\lambda \cdot e^{\lambda t} \Rightarrow e^{\lambda t} \cdot T = 15e^{\lambda t} + D$ (D is a constant)

$$\Rightarrow T = 15 + D \cdot e^{-\lambda t}.$$

Now the new initial temperature is $T(0) = T_1 = 70$ hence $D = 55$.

$T_{sip} = T(360 - 100) = T(260) = 15 + 55 \cdot e^{-260\lambda} = 52.7665°C$

$\therefore 52.7665°C$

1-(c) $T_a = T_b(1 + at)$, $a = 0.0038462\, sec^{-1}$ and $T_b = 10°C$

$\frac{dT}{dt} = \lambda(T_a - T) = \lambda(10 + 10at - T) \Rightarrow \frac{dT}{dt} + \lambda T = \lambda(10 + 10at)$

$T_h(t) = E \cdot e^{-\lambda t}$, $\tilde{T_p}(t) = At + B \Rightarrow A + \lambda(At + B) = \lambda At + (A + \lambda B) = 10a\lambda t + 10\lambda \Rightarrow A = 10a$, $B = 10 - \frac{10}{\lambda}a$ hence $T_p(t) = 10at + 10\left(1 - \frac{a}{\lambda}\right)$

Hence, $T(t) = E \cdot e^{-\lambda t} + 10at + 10\left(1 - \frac{a}{\lambda}\right)$

As 1-(b), $T(0) = 70 = E + 10\left(1 - \frac{a}{\lambda}\right)$ $\therefore E = 60 + 10 \cdot \frac{a}{\lambda}$

Now $T_{sip} = T(260) = \left(60 + 10 \cdot \frac{a}{\lambda}\right)e^{-260\lambda} + 260 \cdot (10a) + 10\left(1 - \frac{a}{\lambda}\right) = 52.8644°C$ $\therefore 52.8644°C$

1-(d)

as both (b), (c) are in the bounds $44°C < T < 60°C$, both are uncomfortable situations.

2.

$F_d = \pi \cdot \rho_a \cdot R \cdot v \cdot \sqrt{a^2\nu^2 + c^2R^2v^2}$. When we hit the gowf ball drag force is opposite to motion, hence $\Sigma F = ma = -F_d$.

$m = \frac{4}{3}\pi R^3 \cdot \rho_b \Rightarrow \left(\frac{4}{3}\pi R^3 \rho_b\right) \cdot v \cdot \frac{dv}{dx} = \pi \cdot \rho_a \cdot R \cdot v \cdot \sqrt{a^2\nu^2 + c^2R^2v^2} \cdot (-1)$

$$\Rightarrow \frac{\frac{4}{3}R^2 \cdot \rho_b/\rho_a}{\sqrt{a^2\nu^2 + c^2R^2v^2}}\, dv = -dx \Rightarrow \frac{4R^2\rho_b}{3\rho_a \cdot a \cdot \nu} \cdot \frac{1}{\sqrt{1+\left(\frac{cR}{a\nu}v\right)^2}}\, dv = -dx \Rightarrow \int_{v_0}^{0} \frac{1}{\sqrt{1+\left(\frac{cR}{a\nu}v\right)^2}}\, dv \cdot \frac{4R^2\rho_b}{3\rho_a \cdot a \cdot \nu} = -x$$

let $\tan\theta = \frac{cR}{a\nu}v$, then $\sec^2\theta \cdot d\theta = \frac{cR}{a\nu}\, dv$. So $\int_{v_0}^{0} \frac{1}{\sqrt{1+\left(\frac{cR}{a\nu}v\right)^2}}\, dv = \underbrace{\int_{\alpha}^{0} \sec\theta\, d\theta}_{} \cdot \frac{a\nu}{cR}$ where $\alpha = \arctan\left(\frac{cR}{a\nu} \cdot v_0\right)$

$\left[\ln(\sec\theta + \tan\theta)\right]_{\alpha}^{0} = -\ln(\sec\alpha + \tan\alpha)$

Therefore, $\ln(\sec\alpha + \tan\alpha) \cdot \frac{4R^2 \cdot \rho_b}{3\rho_a \cdot a \cdot \nu} \cdot \frac{a\nu}{cR} = \ln(\sec\alpha + \tan\alpha) \cdot \frac{4R\rho_b}{3\rho_a \cdot c}$

$$= \ln\left(\frac{cR}{a\nu}v_0 + \sqrt{1+\left(\frac{cR}{a\nu}v_0\right)^2}\right) \cdot \frac{4R\rho_b}{3\rho_a c} = x$$

Let's call the original ball's travelling distance $x = \ln\left(\frac{cR}{a\nu}v_0 + \sqrt{1+\left(\frac{cR}{a\nu}v_0\right)^2}\right) \cdot \frac{4R\rho_b}{3\rho_a c}$

then professor dimple's ball is $x_{dimp} = f \cdot \frac{4R\rho_b}{3\rho_a c} \cdot \ln\left(\frac{1}{f^2} \cdot \frac{cR}{a\nu}v_0 + \sqrt{1+\left(\frac{1}{f^2}\frac{cR}{a\nu}v_0\right)^2}\right)$ since a increased to fa, and c decreased to $\frac{c}{f}$

Therefore professor dimple's ball is better if $f>1$ and $v_0 \gg \frac{a\nu}{cR}$.

3-(a) For natural angular frequency ω_0,

$Lq'' + \frac{q}{C} = \varepsilon_0 \cdot \sin(\omega_0 t)$ when $\varepsilon_0 = 0$

so $Lq'' + \frac{q}{C} = 0$ let's multiply both sides with C. $LC\, q'' + q = 0$. Let's assume $\tilde{q}(t) = e^{kt}$. $LC \cdot q'' + q = (LCk^2+1)e^{kt} = 0$ since $e^{kt} \neq 0$. $LCk^2+1=0$ thus $k = \pm\frac{1}{\sqrt{LC}} i$ making $\tilde{q}(t) = e^{\pm\frac{1}{\sqrt{LC}} it}$.

$\tilde{q}(t) = e^{\pm\frac{1}{\sqrt{LC}} it} = \cos(\frac{1}{\sqrt{LC}}t) \pm i\sin(\frac{1}{\sqrt{LC}}t)$ thus $q(t) = A\cdot\sin(\frac{1}{\sqrt{LC}}t) + B\cdot\cos(\frac{1}{\sqrt{LC}}t)$ where A, B are constants. Thus we can see that $\omega_0 = \frac{1}{\sqrt{LC}}$.

3-(b) when $\varepsilon_0 \neq 0$, $\omega \neq \omega_0$

$Lq'' + \frac{q}{C} = \varepsilon_0 \cdot \sin(\omega t)$

We found out at 3-(a) that $q_h(t) = A\sin(\frac{1}{\sqrt{LC}}t) + B\cos(\frac{1}{\sqrt{LC}}t)$. Now for $q_p'' \cdot L + \frac{q_p}{C} = \varepsilon_0 \cdot \sin(\omega t)$, let's assume $q_p(t) = D\sin\omega t + E\cos\omega t$. (D,E are constants)

Then $q_p'' \cdot L + \frac{q_p}{C} = L\omega^2(-D\sin\omega t - E\cos\omega t) + \frac{1}{C}(D\sin\omega t + E\cos\omega t)$

$= \sin\omega t(-L\omega^2 D + \frac{D}{C}) + \cos\omega t(-L\omega^2 E + \frac{E}{C}) = \varepsilon_0 \cdot \sin\omega t$. Since $\omega \neq \omega_0$, $E=0$. Making $\varepsilon_0 \cdot \sin\omega t = D(-L\omega^2 + \frac{1}{C})\sin\omega t$.

$\varepsilon_0 = D(-L\omega^2 + \frac{1}{C}) \Rightarrow D = \frac{\varepsilon_0}{-L\omega^2 + \frac{1}{C}} = -\frac{\varepsilon_0}{L}\frac{1}{\omega^2 - \frac{1}{LC}} = -\frac{\varepsilon_0}{L}\cdot\frac{1}{\omega^2 - \omega_0^2}$

thus $q_p(t) = -\frac{\varepsilon_0}{L}\cdot\frac{1}{\omega^2-\omega_0^2}\cdot\sin\omega t$ therefore $q(t) = q_p(t) + q_h(t) = -\frac{\varepsilon_0}{L}\cdot\frac{1}{\omega^2-\omega_0^2}\cdot\sin\omega t + A\sin\omega_0 t + B\cos\omega_0 t$

$q'(t) = -\frac{\varepsilon_0\cdot\omega}{L}\cdot\frac{1}{\omega^2-\omega_0^2}\cdot\cos\omega t + A\omega_0\cos\omega_0 t - B\omega_0\sin\omega_0 t$

$\Rightarrow q(0) = B$, $q'(0) = \frac{\varepsilon_0\cdot\omega}{L}\cdot\frac{1}{\omega_0^2-\omega^2} + A\omega_0$

$\Rightarrow A = \frac{1}{\omega_0}\left(q'(0) - \frac{\varepsilon_0\omega}{L}\cdot\frac{1}{\omega_0^2-\omega^2}\right)$

$\therefore q(t) = -\frac{\varepsilon_0}{L}\cdot\frac{1}{\omega^2-\omega_0^2}\cdot\sin\omega t + \frac{1}{\omega_0}\left(q'(0) - \frac{\varepsilon_0\omega}{L}\cdot\frac{1}{\omega_0^2-\omega^2}\right)\sin\omega_0 t + q(0)\cdot\cos\omega_0 t$

3-(c) $\varepsilon_0 \neq 0$. $\omega = \omega_0$. $q_{\omega_0}(0) = q'_{\omega_0}(0) = 0$.

We solved from above that $q_h(t) = A\cos\omega_0 t + B\sin\omega_0 t$ (A, B are constants)

When $Lq'' + \frac{q}{L} = \varepsilon_0 \cdot \sin\omega_0 t$ $q_p(t) = Et\cos\omega_0 t + Ft\sin\omega_0 t$ (F, G are constants)

$q_p'(t) = E(\cos\omega_0 t - \omega_0 t\sin\omega_0 t) + F(\sin\omega_0 t + \omega_0 t\cos\omega_0 t)$

$q_p''(t) = E(-2\omega_0\sin\omega_0 t - \omega_0^2 t\cos\omega_0 t) + F(2\omega_0\cos\omega_0 t - \omega_0^2 t\sin\omega_0 t)$

So $Lq'' + \frac{q}{C} = \sin\omega_0 t(-2E\omega_0 L - F\omega_0^2 Lt + \frac{1}{C}\cdot Ft)$

$+ \cos\omega_0 t(2F\omega_0 L - E\omega_0^2 Lt + \frac{1}{C}Et)$

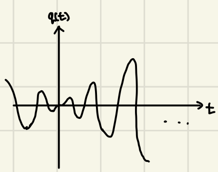
$= \varepsilon_0 \cdot \sin\omega_0 t$ $\Rightarrow \begin{cases} 2F\omega_0 L + Et(\frac{1}{C} - \omega_0^2 L) = 0 \\ -2E\omega_0 L + Ft(\frac{1}{C} - \omega_0^2 L) = \varepsilon_0 \end{cases}$ as $\frac{1}{C} - \omega_0^2 L = \frac{1}{C} - \frac{1}{C} = 0$ thus $2F\omega_0 L = 0$ and $-2E\omega_0 L = \varepsilon_0$ $\therefore F = 0, E = -\frac{\varepsilon_0}{2\omega_0 L}$

Hence, $q(t) = A\cos\omega_0 t + B\sin\omega_0 t - \frac{\varepsilon_0}{2\omega_0 L}\cdot t\cos\omega_0 t$. $q(0) = A = 0$

$q'(t) = -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t - \frac{\varepsilon_0}{2\omega_0 L}(\cos\omega_0 t - \omega_0 t\sin\omega_0 t)$. $q'(0) = B\omega_0 - \frac{\varepsilon_0}{2\omega_0 L} = 0 \Rightarrow B = \frac{\varepsilon_0}{2\omega_0^2 L}$

$\therefore q(t) = \frac{\varepsilon_0}{2\omega_0^2 L}\sin\omega_0 t - \frac{\varepsilon_0}{2\omega_0 L}\cdot t\cdot\cos\omega_0 t$

the charges does not decay and the amplitude increases



3 - (d)

Now $L\cdot q'' + R\cdot q' + \frac{q}{C} = \varepsilon_0$.

Let's first assume the particular solution.

$\tilde{q}_p(t) = A$, then $L\cdot q'' + R\cdot q' + \frac{1}{C}q = \frac{A}{C} = \varepsilon_0 \Rightarrow A = C\cdot\varepsilon_0$ hence $q_p(t) = C\cdot\varepsilon_0$

Now let's get the homogeneous solution. $L\cdot q'' + R\cdot q' + \frac{1}{C}q = 0$. Assume $\tilde{q}_h(t) = e^{kt}$, then $e^{kt}(Lk^2 + Rk + \frac{1}{C}) = 0$.

As $e^{kt} > 0$, $Lk^2 + Rk + \frac{1}{C} = 0$ has to be satisfied. Depending on R we can divide the general solution into three cases.

1) $R^2 > \frac{4L}{C}$, then $k = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$ as $R^2 - \frac{4L}{C} > 0$. $q(t) = A\cdot e^{\frac{-R+\sqrt{R^2-\frac{4L}{C}}}{2L}t} + B\cdot e^{-R+\frac{\sqrt{R^2-\frac{4L}{C}}t}{2L}} + C\cdot\varepsilon_0$

2) $R^2 = \frac{4L}{C}$, then $k = \frac{-R}{2L}$ thus $q_h(t) = C\cdot e^{-\frac{R}{2L}t} + D\cdot t\cdot e^{-\frac{R}{2L}t}$ making $q(t) = C\cdot e^{-\frac{R}{2L}t} + D\cdot t\cdot e^{-\frac{R}{2L}t} + C\cdot\varepsilon_0$

3) $R^2 < \frac{4L}{C}$, then $k = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$ but as $R^2 - \frac{4L}{C} < 0 \Rightarrow k = \frac{-R \pm \sqrt{\frac{4L}{C} - R^2}\cdot i}{2L}$ $(i = \sqrt{-1})$. which makes $q(t) = e^{-\frac{R}{2L}}\cdot\left(E\cdot\cos\left(\frac{\sqrt{\frac{4L}{C}-R^2}}{2L}t\right) + F\cdot\sin\left(\frac{\sqrt{\frac{4L}{C}-R^2}}{2L}t\right)\right) + C\varepsilon_0$.

We can see that $q(t)$ oscillates only when $R^2 < \frac{4L}{C}$.

$\therefore$ Hence if $R^2 \geq \frac{4L}{C}$ is satisfied the current will not oscillate. This is true as $I_{(current)} = \frac{dq}{dt}$.