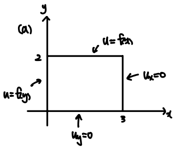
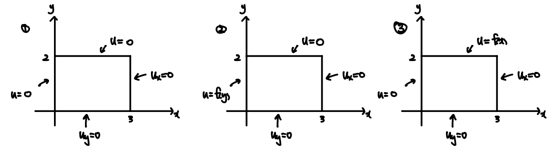
1. $u_{xx}+u_{yy}=0$, $0\le x\le 3$, $0\le y\le 2$. B.C's $u_x(3,y)=0$, $u_y(x,0)=0$, $u(x,2)=f(x)=\begin{cases} x , & 0\le x<1 \\ 2-x , & 1\le x<3 \end{cases}$, $u(0,y)=f(y)=\begin{cases} y , & 0\le y<1 \\ 2-y , & 1\le y<2 \end{cases}$



We only need to consider the three cases



Let $u(x,y)=X(x)\cdot Y(y) \longrightarrow \frac{X''}{X}=-\frac{Y''}{Y}=\lambda$

Case ①

$u(0,y)=X(0)\cdot Y(y)=0 \rightarrow X(0)=0$, $u_x(3,y)=X'(3)\cdot Y(y)=0 \rightarrow X'(3)=0$, $u_y(x,0)=X(x)\cdot Y'(0)=0 \rightarrow Y'(0)=0$, $u(x,2)=X(x)\cdot Y(2)=0 \longrightarrow Y(2)=0$

1. $\lambda=0$. $X''=0 \longrightarrow X(x)=Ax+B$ $X(0)=B=0$. $X'(3)=A=0$ so we get the trivial solution.

2. $\lambda=\mu^2$. $X''-\mu^2X=0 \longrightarrow X(x)=A\cosh(\mu x)+B\sinh(\mu x)$, $X(0)=A=0$. $X'(3)=B\mu\cdot\cosh(3\mu)=0 \rightarrow B=0$ so we get the trivial solution.

3. $\lambda=-\mu^2$. $Y''-\mu^2Y=0 \longrightarrow Y(y)=A\cosh(\mu y)+B\sinh(\mu y)$, $Y'(0)=B\cdot\mu=0 \longrightarrow B=0$, $Y(2)=A\cdot\cosh(2\mu)=0 \longrightarrow A=0$ so we get the trivial solution.

Case ②

$u(0,y)=X(0)\cdot Y(y)=f(y)$, $u_x(3,y)=X'(3)\cdot Y(y)=0 \rightarrow X'(3)=0$, $u_y(x,0)=X(x)\cdot Y'(0)=0 \rightarrow Y'(0)=0$, $u(x,2)=X(x)\cdot Y(2)=0 \longrightarrow Y(2)=0$

1. $\lambda=0$. $Y''=0$. $Y(y)=Ay+B$. Since $Y'(0)=0=Y(2)$ we get the trivial solution.

2. $\lambda=-\mu^2$. $Y''-\mu^2Y=0$, $Y(y)=A\cdot\cosh(\mu y)+B\cdot\sinh(\mu y)$. Since $Y'(0)=0=Y(2)$ we get the trivial solution.

3. $\lambda=\mu^2$. $Y''+\mu^2Y=0$. $Y(y)=A\cos(\mu y)+B\sin(\mu y)$. $Y'(0)=B\mu=0 \rightarrow B=0$. $Y(2)=A\cos(2\mu)=0$. When $A\neq 0$, $\mu_n=\frac{2n-1}{4}\pi$. $Y_n=\cos(\mu_n y)$ and $\lambda_n=-\left(\frac{2n-1}{4}\pi\right)^2=-\mu_n^2$ $(n=1,2,\cdots)$

$X''-\mu^2X=0$. $X(x)=C\cdot\cosh(\mu x)+D\cdot\sinh(\mu x)$ since $X'(3)=0=\mu C\cdot\sinh(3\mu)+\mu D\cdot\cosh(3\mu)=0 \longrightarrow D=-\frac{\sinh(3\mu)}{\cosh(3\mu)}\cdot C$

therefore $X(x)=C\cdot\left(\cosh(\mu x)-\frac{\sinh(3\mu)}{\cosh(3\mu)}\cdot\sinh(3x)\right)=C\cdot\frac{\cosh(\mu x)\cdot\cosh(3\mu)-\sinh(\mu x)\cdot\sinh(3\mu)}{\cosh(3\mu)}=C\cdot\frac{\cosh(\mu(x-3))}{\cosh(3\mu)}$

$\therefore u(x,y)=\sum_{n=1}^{\infty}C_n\cdot\frac{\cosh(\mu_n(x-3))}{\cosh(3\mu_n)}\cdot\cos(\mu_n y)$ where $\mu_n=\frac{2n-1}{4}\pi$

$u(0,y)=f(y)=\sum_{n=1}^{\infty}C_n\cdot\cos(\mu_n y) \longrightarrow C_n=\frac{2}{2}\cdot\int_0^2 f(y)\cdot\cos(\mu_n y)\,dy=\int_0^1 y\cdot\cos(\mu_n y)\,dy+\int_1^2(2-y)\cdot\cos(\mu_n y)\,dy=\frac{-1}{\mu_n^2}+\frac{2\cos(\mu_n)}{\mu_n^2}-\frac{\cos(2\mu_n)}{\mu_n^2}=\frac{2\cos(\mu_n)-1}{\mu_n^2}$

Thus, $u(x,y)=\sum_{n=1}^{\infty}\left(\frac{2\cos(\mu_n)-1}{\mu_n^2}\right)\cdot\frac{\cosh(\mu_n(x-3))}{\cosh(3\mu_n)}\cdot\cos(\mu_n y)$ $\left(\mu_n=\frac{2n-1}{4}\pi\right)$

case ③

$u(0,y)=X(0)\cdot Y(y)=0 \to X(0)=0$, $u_x(3,y)=X'(3)\cdot Y(y)=0 \to X'(3)=0$, $u_y(x,0)=X(x)\cdot Y'(0)=0 \to Y'(0)=0$, $u(x,2)=X(x)\cdot Y(2)=f(x)$

1. $\lambda=0.$ $X''=0 \longrightarrow X(x)=Ax+B$, since $X(0)=0=X'(3)$ we get the trivial solution.

2. $\lambda=\mu^2.$ $X''-\mu^2X=0 \longrightarrow X(x)=A\cdot\cosh(\mu x)+B\cdot\sinh(\mu x)$, since $X(0)=0=X'(3)$ we get the trivial solution.

3. $\lambda=-\mu^2.$ $X''+\mu^2X=0 \longrightarrow X(x)=A\cdot\cos(\mu x)+B\cdot\sin(\mu x)$. $X(0)=A=0$, $X'(3)=B\mu\cdot\cos(3\mu)$. If $B\neq0$, then $\mu=\frac{2n-1}{6}\pi$. therefore $X_n(x)=\sin(\mu_n x)$. $\lambda_n=-\mu_n^2=-\left(\frac{2n-1}{6}\pi\right)^2$ $(n=1,2,\cdots)$

$Y''-\mu^2Y=0 \longrightarrow Y(y)=A\cdot\cosh(\mu y)+B\cdot\sinh(\mu y)$. $Y'(0)=\mu B=0$ so $B=0$.

$u(x,y)=\sum_{n=1}^{\infty}A_n\cdot\cosh(\mu_n y)\cdot\sin(\mu_n x).$

$u(x,2)=\sum_{n=1}^{\infty}A_n\cdot\cosh(2\mu_n)\cdot\sin(\mu_n x)=f(x) \longrightarrow A_n\cdot\cosh(2\mu_n)=\frac{2}{3}\cdot\int_0^3 f(x)\cdot\sin(\mu_n x)\,dx=\frac{2}{3}\cdot\left\{\int_0^1 x\cdot\sin(\mu_n x)\,dx+\int_1^3(2-x)\cdot\sin(\mu_n x)\,dx\right\}=\frac{2(2\sin(\mu_n)-\sin(3\mu_n)+\mu_n\cos(3\mu_n))}{3\mu_n^2}=\frac{4\sin(\mu_n)+2\cdot(-1)^n}{3\mu_n^2}$

$\therefore u(x,y)=\sum_{n=1}^{\infty}\frac{4\sin(\mu_n)+2\cdot(-1)^n}{3\mu_n^2}\cdot\frac{1}{\cosh(2\mu_n)}\cdot\cosh(\mu_n y)\cdot\sin(\mu_n x)$ $\left(\mu_n=\frac{2n-1}{6}\pi\right)$

$\therefore$ Therefore, $u(x,y)=\sum_{n=1}^{\infty}\left(\frac{2\cos(\mu_n)-1}{\mu_n^2}\right)\cdot\frac{\cosh(\mu_n(x-3))}{\cosh(3\mu_n)}\cdot\cos(\mu_n y)+\sum_{n=1}^{\infty}\frac{4\sin(\mu_n')+2\cdot(-1)^n}{3\mu_n'^2}\cdot\frac{1}{\cosh(2\mu_n')}\cdot\cosh(\mu_n' y)\cdot\sin(\mu_n' x)$ $\left(\mu_n=\frac{2n-1}{4}\pi,\ \mu_n'=\frac{2n-1}{6}\pi\right)$

(b)

$u_x(x,y)=\sum_{n=1}^{\infty}\left(\frac{2\cos(\mu_n)-1}{\mu_n^2}\right)\cdot\frac{\mu_n\cdot\sinh(\mu_n(x-3))}{\cosh(3\mu_n)}\cdot\cos(\mu_n y)+\sum_{n=1}^{\infty}\frac{4\sin(\mu_n')+2\cdot(-1)^n}{3\mu_n'^2}\cdot\frac{1}{\cosh(2\mu_n')}\cdot\cosh(\mu_n' y)\cdot\mu_n'\cdot\cos(\mu_n' x)$

$u_y(x,y)=\sum_{n=1}^{\infty}\left(\frac{2\cos(\mu_n)-1}{\mu_n^2}\right)\cdot\frac{\cosh(\mu_n(x-3))}{\cosh(3\mu_n)}\cdot(-\mu_n\cdot\sin\mu_n y)+\sum_{n=1}^{\infty}\frac{4\sin(\mu_n')+2\cdot(-1)^n}{3\mu_n'^2}\cdot\frac{\mu_n'}{\cosh(2\mu_n')}\cdot\sinh(\mu_n' y)\cdot\sin(\mu_n' x)$

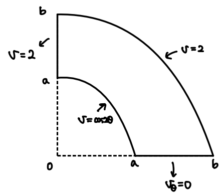
$u_x(3,y)=\sum_{n=1}^{\infty}\left(\frac{2\cos(\mu_n)-1}{\mu_n^2}\right)\cdot\frac{\mu_n\cdot 0}{\cosh(3\mu_n)}\cdot\cos(\mu_n y)+\sum_{n=1}^{\infty}\frac{4\sin(\mu_n')+2\cdot(-1)^n}{3\mu_n'^2}\cdot\frac{1}{\cosh(2\mu_n')}\cdot\cosh(\mu_n' y)\cdot\mu_n'\cdot 0=0$

$u_y(x,0)=\sum_{n=1}^{\infty}\left(\frac{2\cos(\mu_n)-1}{\mu_n^2}\right)\cdot\frac{\cosh(\mu_n(x-3))}{\cosh(3\mu_n)}\cdot(-\mu_n\cdot 0)+\sum_{n=1}^{\infty}\frac{4\sin(\mu_n')+2\cdot(-1)^n}{3\mu_n'^2}\cdot\frac{\mu_n'}{\cosh(2\mu_n')}\cdot 0\cdot\sin(\mu_n' x)=0$

$u(x,2)=\sum_{n=1}^{\infty}\left(\frac{2\cos(\mu_n)-1}{\mu_n^2}\right)\cdot\frac{\cosh(\mu_n(x-3))}{\cosh(3\mu_n)}\cdot 0+\sum_{n=1}^{\infty}\frac{4\sin(\mu_n')+2\cdot(-1)^n}{3\mu_n'^2}\cdot\frac{1}{\cosh(2\mu_n')}\cdot\cosh(\mu_n' 2)\cdot\sin(\mu_n' x)=\sum_{n=1}^{\infty}\frac{4\sin(\mu_n')+2\cdot(-1)^n}{3\mu_n'^2}\cdot\frac{1}{\cosh(2\mu_n')}\cdot\cosh(\mu_n' 2)\cdot\sin(\mu_n' x)=f(x)$ from (A)

$u(0,y)=\sum_{n=1}^{\infty}\left(\frac{2\cos(\mu_n)-1}{\mu_n^2}\right)\cdot\frac{\cosh(\mu_n(-3))}{\cosh(3\mu_n)}\cdot\cos(\mu_n y)+\sum_{n=1}^{\infty}\frac{4\sin(\mu_n')+2\cdot(-1)^n}{3\mu_n'^2}\cdot\frac{1}{\cosh(2\mu_n')}\cdot\cosh(\mu_n' y)\cdot 0=\sum_{n=1}^{\infty}\left(\frac{2\cos(\mu_n)-1}{\mu_n^2}\right)\cdot\cos(\mu_n y)=f(y)$ from (A)

2. $v_{rr} + \frac{1}{r}\cdot v_r + \frac{1}{r^2}\cdot v_{\theta\theta} = 0$ in Ω. $v_\theta(r,0)=0$ for $a<r<b$, $v(r,\frac{\pi}{2})=2$ for $a<r<b$



$v(a,\theta)=\cos 2\theta$ for $0<\theta<\frac{\pi}{2}$, $v(b,\theta)=2$ for $0<\theta<\frac{\pi}{2}$

First. let $v(r,\theta)=w(\theta)+v(r,\theta)$ where $w'(0)=0$, $w(\frac{\pi}{2})=2 \longrightarrow$ Then $w(r,\theta)=2$. and $u(a,\theta)=\cos 2\theta-2$, $u(b,\theta)=0$, $u_\theta(r,0)=0$, $u(r,\frac{\pi}{2})=0$.

$(u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta})+(w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\theta\theta})=0 \longrightarrow u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$

Let $u(r,\theta)=R(r)\cdot\Theta(\theta)$ then $r^2\cdot\frac{R''}{R}+r\cdot\frac{R'}{R}=-\frac{\Theta''}{\Theta}=\lambda$.

① $\lambda=0$, $\Theta''=0 \rightarrow \Theta(\theta)=A\theta+B$, $\Theta'(0)=0=\Theta(\frac{\pi}{2})$ hence $\Theta=0$ (trivial solution)

② $\lambda=-\mu^2$. $\Theta''-\mu^2\Theta=0 \rightarrow \Theta(\theta)=A\cdot\cosh(\mu\theta)+B\cdot\sinh(\mu\theta) \longrightarrow \Theta'(0)=B\mu=0, B=0$. $\Theta(\frac{\pi}{2})=A\cdot\cosh(\mu\cdot\frac{\pi}{2})$, $A=0$ (trivial solution)

③ $\lambda=\mu^2$, $\Theta''+\mu^2\Theta=0 \longrightarrow \Theta(\theta)=A\cdot\cos(\mu\theta)+B\cdot\sin(\mu\theta) \longrightarrow \Theta'(0)=B\mu=0, B=0$. $\Theta(\frac{\pi}{2})=A\cdot\cos(\frac{\pi}{2}\theta)=0 \longrightarrow$ if $A\neq0$. $\theta=2n-1$ $\therefore \Theta_n(\theta)=\cos(\mu_n\theta)$ $(n=1,2,\cdots)$ where $\lambda_n=-(\mu_n)^2=-(2n-1)^2$

$r^2\cdot R''+r\cdot R'-\mu^2R=0$. $R(r)=r^\alpha$, $r^\alpha\cdot\{\alpha(\alpha-1)+\alpha-\mu^2\}=0$ so $\alpha=\pm\mu$ thus $R(r)=A\cdot r^{\mu}+B\cdot r^{-\mu}$

Making $u(r,\theta)=\sum_{n=1}^{\infty}\{A_n\cdot r^{\mu_n}+B_n\cdot r^{-\mu_n}\}\cdot\cos(\mu_n\theta)$. $(\mu_n=2n-1)$.

$u(b,\theta)=\sum_{n=1}^{\infty}\{A_n\cdot b^{\mu_n}+B_n\cdot b^{-\mu_n}\}=0 \longrightarrow B_n=-A_n\cdot b^{2\mu_n} \longrightarrow u(r,\theta)=\sum_{n=1}^{\infty}A_n\cdot b^{\mu_n}\cdot\{(\frac{r}{b})^{\mu_n}-(\frac{b}{r})^{\mu_n}\}\cdot\cos(\mu_n\theta)$

$u(a,\theta)=\cos 2\theta-2=\sum_{n=1}^{\infty}A_n\cdot b^{\mu_n}\cdot\{(\frac{a}{b})^{\mu_n}-(\frac{b}{a})^{\mu_n}\}\cdot\cos(\mu_n\theta)$

$$A_n\cdot b^{\mu_n}\cdot\{(\frac{a}{b})^{\mu_n}-(\frac{b}{a})^{\mu_n}\}=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}(\cos(2\theta)-2)\cdot\cos(\mu_n\theta)\,d\theta=\frac{4(12n^2-12n-5)\cdot\cos(\pi n)}{\pi(8n^3-12n^2-2n+3)} \longrightarrow A_n\cdot b^{\mu_n}=\frac{\frac{4(12n^2-12n-5)\cdot\cos(\pi n)}{\pi(8n^3-12n^2-2n+3)}}{\{(\frac{a}{b})^{\mu_n}-(\frac{b}{a})^{\mu_n}\}}$$

$$\therefore u(r,\theta)=\frac{\frac{4(12n^2-12n-5)\cdot\cos(\pi n)}{\pi(8n^3-12n^2-2n+3)}}{\{(\frac{a}{b})^{\mu_n}-(\frac{b}{a})^{\mu_n}\}}\{(\frac{r}{b})^{\mu_n}-(\frac{b}{r})^{\mu_n}\}\cdot\cos(\mu_n\theta)$$

$$\therefore v(r,\theta)=2+\frac{\frac{4(12n^2-12n-5)\cdot\cos(\pi n)}{\pi(8n^3-12n^2-2n+3)}}{\{(\frac{a}{b})^{\mu_n}-(\frac{b}{a})^{\mu_n}\}}\{(\frac{r}{b})^{\mu_n}-(\frac{b}{r})^{\mu_n}\}\cdot\cos(\mu_n\theta)$$