Due date: September 27, 2016

Problem 1: Let P be a convex polyhedron in $\mathbb{R}^n$ defined as the intersection of m halfspaces, i.e.,

$$P = \{x \in \mathbb{R}^n \mid a_i^t x \leq b_i,\ i = 1, \ldots, m\}.$$

Formulate the problem of computing the largest ball inscribed in P (i.e., a ball that lies inside P) as an instance of LP. (**Hint:** *What is the condition that a ball of radius r centered at x lies inside a halfspace?*)

Problem 2: In many applications the LP constraints are not known exactly. More precisely, for $1 \leq i \leq m$, let $\bar{a}_i \in \mathbb{R}^n$ and $\rho_i \geq 0$, and set $A_i = \{a \mid \|a - \bar{a}_i\| \leq \rho_i\}$. Consider the following optimization problem:

$$\max c^t x \quad s.t. \quad a_i^t x \leq b_i \ \forall a_i \in A_i, \quad x \geq 0.$$

Show that the above problem can be rewritten as an LP with polynomially number of variables and constraints.

Problem 3: (i) Consider an LP $\max c^T x$ such that $Ax \leq b$ and $x \geq 0$. A vertex x of the feasible set F is called *degenerate* if more than $n - m$ variables are zero. Is the optimal solution to the LP is unique if there are no degenerate vertices in F? Justify your answer.

(ii) Can a pivot of the simplex algorithm move the feasible point a positive distance in $\mathbb{R}^n$ while leaving the value of the objective function unchanged? Justify your answer.

Problem 4: Let $G = (V, E)$ be a directed weighted graph, with $w : E \to \mathbb{R}^+$ being the edge weights, and let s, t be two vertices in G.

(i) Show that the problem of computing the shortest path from s to t is equivalent to finding an st-flow f that minimizes $\sum_e w(e) f_e$ subject to the constraint $|f| = 1$. There are no capacity constraints

(ii) Write the shortest-path problem as a linear program.

(iii) Write the dual of this LP.

Problem 5: Let $G = (A \cup B, E)$ with $|A| = |B|$ and $E \subseteq A \times B$ be a bipartite graph, and let $w : E \to \mathbb{R}_{\geq 0}$ be the edge-cost function. A subset $M \subseteq E$ is a *perfect matching* if every vertex is incident on exactly one edge of M. The minimum-weight pefect matching problem asks for computing a pefect matching of the minimum weight. Formulate this as an LP problem and write its dual.

Show that if G has a perfect matching, then there is an optimal solution for the LP of the minimum-weight bipartite matching that is integral. (**Hint:** *Show that if there is a fractional optimal solution, then there exists a cycle in G such that every edge in the cycle has a fractional value. Now compute another optimal solution with fewer fractional values.*)