Math 603

Homework 6

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An Introduction to Manifolds

Problem. 14.2

Answer.
To show that it is a tangent vector at every point of the sphere, we need the canonical map of $T_p(\mathbb{R}^n) \cong T_p^*(\mathbb{R}^n) \cong \mathbb{R}^n$. Let $f = \sum(x^i)^2 + (y^i)^2 - 1$. Then the normal vector of the hypersurface, sphere in this case, is identified in $\mathbb{R}^2n$ as $\text{grad}f = \sum 2x^i dx^i + 2y^i dy^i \sim (2x^1, ..., 2x^n, 2y^1, ..., 2y^n)$. Since X is identified as $(-y^1, ..., -y^n, x_1, ..., x_n)$, the inner product of X and $\text{grad}f$ at each point is zero. Therefore X is tangent to the sphere at each point. Hence it is a vector field. Notice that each coefficient is certainly a $\mathcal{C}^\infty$ function and this vector field does not vanish at each point on the sphere since $(0, ..., 0)$ is not a point on the sphere. Therefore it is a nowhere vanishing smooth vector field.

Problem. 14.10

Answer.
For any $h \in \mathcal{C}^\infty(M)$, we have that

$$\begin{aligned}
[fX, gY]h &= (fX(gY) - gY(fX))h \\
&= fX(gYh) - gY(fXh) \\
&= f(Xg)(Yh) + fgX(Yh) - g(Yf)(Xh) - gfY(Xh) \\
&= (fg[X, Y] + f(Xg)Y - g(Yf)X)h
\end{aligned}$$

Therefore,

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

Problem. 14.12

Answer.
For any $h \in \mathcal{C}^\infty(M)$, we have that

$$[X, Y]h = \sum_{i,j} a^i \frac{\partial b^j}{\partial x^i} \frac{\partial h}{\partial x^j} - \sum_{i,j} b^j \frac{\partial a^i}{\partial x^j} \frac{\partial h}{\partial x^i} = \sum_{i,j} a^i \frac{\partial b^j}{\partial x^i} \frac{\partial h}{\partial x^j} - \sum_{i,j} b^i \frac{\partial a^j}{\partial x^i} \frac{\partial h}{\partial x^j}$$

Therefore

$$[X, Y] = \sum_{i,j} \left(a^i \frac{\partial b^j}{\partial x^i} \frac{\partial}{\partial x^j} - b^i \frac{\partial a^j}{\partial x^i} \frac{\partial}{\partial x^j} \right) = \sum_k \sum_i \left(a^i \frac{\partial b^k}{\partial x^i} \frac{\partial}{\partial x^k} - b^i \frac{\partial a^k}{\partial x^i} \frac{\partial}{\partial x^k} \right)$$

Hence

$$c^k = \sum_i \left(a^i \frac{b^k}{x^i} - b^i \frac{a^k}{x^i} \right)$$

Problem. 16.5

Answer.

The left translation ℓ_g is $\ell_g(x) = g + x$ for $x, g \in \mathbb{R}^n$. Let $\partial_i = \frac{\partial}{\partial x^i}$. Suppose X is a left invariant vector field. Assume $X_0 = \sum_i x_0^i \partial_i|_0$ and $X_g = \sum_i x_g^i \partial_i|_g$. Then since it is left invariant, the induced map satisfies $\ell_g^*(X_0) = X_g$. Apply $f_i(x) = x^i$ on each side for $i = 1, 2, ..., n$, then we have

$$x_g^i = X_g(f_i) = \ell_g^*(X_0)(f_i) = X_0(f_i \circ \ell_g) = x_i^0$$

Therefore $X = \sum_i c^i \partial_i$ for constants c^i. Hence, the left invariant vector fields are constant vector fields in the sense that it is constant under the canonical map $T_p(\mathbb{R}^n)$ to $\mathbb{R}^n$.

Problem. 16.6

Answer 1.

We use the fact that $S^1 \cong \mathbb{R}/\mathbb{Z}$. Let S^1 be the unit circle in complex plane, i.e. $S^1 = \{e^{2\pi i x} | \, x \in \mathbb{R}/\mathbb{Z}\}$. The left translation is $\ell_g(x) = x + g$. Suppose $X = a\partial_x$ is a left invariant vector, then $X_0 = a_0 \partial_x|_0$ and $X_g = a_g \partial_x|_g$. The induced map on vector fields satisfies $\ell_g^*(X_0) = X_g$. Apply $f(x) = x$ on both side of the equation, then we have

$$a_g = X_g(x) = \ell_0^*(X_0)(x) = X_0(x \circ \ell_g) = a_0$$

Therefore the left invariant vector fields on S^1 is a multiple of $\partial_x = \frac{d}{dx}$.