Math 603 - Representation Theory

Homework 5

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Exercise 3

Problem. 1

Answer.
(i) Checked.
(ii) The norm is $z\bar{z} + w\bar{w}$, which is a non-negative real number. So the image is $\mathbb{R}^+ \cup \{0\}$.
(iii) If $det(h) = z\bar{z} + w\bar{w} \neq 0$. Then let the inverse of $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ is

$$\frac{1}{z\bar{z} + w\bar{w}} \begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix}$$

(iv) $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = (-1, 0) = -1$ and $\mathbf{ij} = \mathbf{k}, \mathbf{ji} = (0, -i) = -\mathbf{k}$.
(v) The conjugate of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are $-\mathbf{i}$, $-\mathbf{j}$, $-\mathbf{k}$ respectively.
(vi) $h^*h = det(h)I$ where I is the identity of $GL_2(\mathbb{C})$.
(vii) $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

Problem. 2

Answer.
(i) Reduced norm $n(z, w) = z\bar{z} + w\bar{w} = a^2 + b^2 + c^2 + d^2$.
(ii) According to (i), the reduced norm equals to 1 means $a^2 + b^2 + c^2 + d^2 = 1$. So it is diffeomorphic to unit S^3 in $\mathbb{R}^4$. That also means it is diffeomorphic to S^3 in general.
(iii) Yes. $h^* = h^{-1}$ in this case.
(iv) Yes. It is a closed subgroup of $GL_2(\mathbb{C})$. It is closed topologically since 1 is closed in $\mathbb{R}^\times$ and det is a continuous map. Therefore the $\mathbb{H}_1$ forms a Lie group.
(v) They are the same. Both are the unitary group of $GL_1(\mathbb{H})$.

Problem. 3

Answer.
(i) Suppose we have $A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$. Then we can use the equation $\bar{A}^T A = 1$ and $det(A) = 1$ to solve z_4 and z_3 in terms of z_1 and z_2. We have that $z_4 = \bar{z_1}$ and $z_3 = -\bar{z_2}$. This shows that $SU(2)$ is actually the Hamiltonian quaternions of reduced norm 1, i.e. $SU(2) = \mathbb{H}_1$.
(ii) As shown in problem 2, since $SU(2) = \mathbb{H}_1$, it is diffeomorphic to S^3.
(iii) No. Because the image det map of $U(2)$ is $U(1)$ thich is not isomorphic to $\mathbb{R}^\times$ which is the image of det map of $\mathbb{H}$.

Problem. 4

Answer.

(i) Clearly trace maps $h = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ to a real number $z + \bar{z} = 2\mathbf{Re}(z)$. It is symmetric since $\text{tr}(AB) = \sum_{i,j} a_{ij}bji = \sum_{i,j} b_{ji}a_{ij} = \text{tr}(BA)$. It is also bilinear since $\text{tr}(aA) = a\text{tr}(A)$ and $A(aB) = (aA)B$ for a real number a. Therefore it is a symmetric bilinear form.

(ii) $\text{tr}(AB) = -x_1y_1 - x_2y_2 - x_3y_3$.

(iii) Yes. Notice that $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis for $\mathbb{H}$ and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ has 0 trace. So $\mathbb{I}$ is spanned by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Since it is a $\mathbb{R}$ vector space. Therefore $GL(\mathbb{I})$ has entries in $\mathbb{R}$ and it is isomorphic to $GL_3(\mathbb{R})$. Also notice that tr respect to the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is just the natural inner product in $\mathbb{R}^3$ with a negative sign. So $\{A \in GL(\mathbb{I}) | (Ah_1, Ah_2) = (h_1, h_2)\}$ is just the orthonormoal group in $GL(\mathbb{I}) \cong GL_3(\mathbb{R})$. And if we require $det(A) = 1$, it is just the special orthogonal group of $GL_(\mathbb{I})$. Therefore $SO(3) \cong G$.

(iv) Check two condition of group action: (a) $e * l = ele^{-1} = ele = l$, where e is the identity of $H^\times \subset GL_2(\mathbb{C})$. (b) $h_1 * (h_2 * l) = h_1 * (h_2lh_2^{-1}) = h_1(h_2lh_2^{-1})h_1^{-1} = (h_1h_2)l(h_1h_2)^{-1} = (h_1h_2) * l$

Therefore, it is an action.

(v) Note that for $h \in \mathbb{H}_1$ and $l_1, l_2 \in \mathbb{I}$, we have that $(h * l_1, h * l_2) = \text{tr}(hl_1h^{-1}hl_2h^{-1}) = \text{tr}(hl_1l_2h^{-1}) = (hl_1l_2, h^{-1}) = (h^{-1}, hl_1l_2) = \text{tr}(l_1l_2) = (l_1, l_2)$. We also know that $det(h) = 1$ for any $h \in \mathbb{H}_1$. Therefore for any h, we have a unique $A \in G$ such that $h * l = Al$ for any $l \in \mathbb{I}$. The mapping h to A is a Lie group homomorphism. We can also see this as conjugation homomophism $H^\times \to Aut(\mathbb{H})$ and it fixes $\mathbb{I}$.

(vi) $\mathbb{H}_1$ has dimension 3 and $G \cong SO(3)$ has dimension 3 as well.

(vii) Since both 1 and -1 in $\mathbb{H}^\times$ maps to $Id \in SO(3)$, the kernel is not trivial. So it is not a isomorphism.

(viii) Notice that the kernel of the homomorphism is actually $\{1, -1\} \subset \mathbb{H}^\times$. Since this is discrete, the preimage of any open set $V \in SO(3)$ is disjoint in $\mathbb{H}^\times \cong SU(2)$. Therefore it is a covering map. In fact $SU(2)$ being diffeomorphic to S^3 and the kernel being isomorphic to $\mathbb{Z}/2\mathbb{Z}$ implies $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$.

Lie Algebras

Problem. 4.4

Answer.

(i) Let $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$ where a_i and b_j are real numbers. Then for $h \in \mathcal{C}^\infty(\mathbb{R}^n)$,

$$[X, Y]h = \sum_{i,j} (a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \frac{\partial}{\partial x_i})(h) = 0$$

Therefore it is a Lie subalgebra.

(ii) By (i), it is abelian.

(iii) Yes. For $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^n f_j \frac{\partial}{\partial x_j}$ where $a_i \in IR$ and $f_j \in \mathcal{C}^\infty(\mathbb{R}^n)$. Then,

$$[X, Y] = \sum_{i,j} a_i \frac{\partial f_j}{\partial x_i} \frac{\partial}{\partial x_j} \in Der(\mathcal{C}^\infty(\mathbb{R}^n))$$