Math 603

Homework 4

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1 Symmetry

Problem. 2

Answer.

(ii) Suppose $\|f_X(t)\| \leq b$ for some b. Then $\|f_X(t) - Id\| \leq \|f_X(t)\| + \|Id\| \leq b + b_0$ is bounded. Since $X \neq 0$, $\|X\| = c$ for some c. We then need to find an incredibly large a, so that

$$\begin{aligned}
\|f_X(a) - Id\| &= \left\| aX + \frac{1}{2}a^2X^2 + \cdots + \frac{1}{k!}a^kX^k \right\| \\
&\geq \left\| \frac{1}{k!}a^kX^k \right\| - \|aX\| - \left\| \frac{1}{2}a^2X^2 \right\| - \cdots - \left\| \frac{1}{(k-1)!}a^{k-1}X^{k-1} \right\| \\
&\geq \frac{1}{k!}(ac)^k - ac - \frac{1}{2}(ac)^2 - \cdots - \frac{1}{(k-1)!}(ac)^{k-1}
\end{aligned}$$

Since ac can be arbitrarily large and k is fixed for an strictly upper triangular matrix X, this is unbounded. The contradiction shows that $f_X(t)$ is not bounded.

(iii) We have that $\exp(tX) = diag(\exp tx_{11}, ..., \exp tx_{nn})$. Since whether a set is bounded or not does not depend on the norm we choose, we choose the norm of X to be $\left(\sum_{i,j} \|x_{ij}\|\right)^{\frac{1}{2}}$. Then apparently when $x_{11}, ..., x_{nn}$ are purely imaginary, $\exp(tX)$ is bounded by $\sqrt{n}$. Notice that $\|\exp(tX)\| = (\sum_i \exp(t\mathrm{Re}(x_{ii})))^{\frac{1}{2}}$. If any diagonal entry of X has real part, then this is certainly not bounded.

(vii) If $X = SYS^{-1}$ with Y being a diagonal matrix of purely imaginray numbers, then $\exp(tX) = S\exp(tY)S^{-1}$ is certainly bounded by the proof above. On the other hand, let $X = PJP^{-1}$ with J being its jordan canonical form. We have $J = D + N$ where D is diagonal and N is nilpotent and $DN = ND$. Then $\exp(tX) = S\exp(tD)p(tN)S^{-1}$ for some polynomial p. If this is bounded, then $\exp(tD)$ and $p(tN)$ is bounded. By question (ii), $p(tN)$ being bounded means $N = 0$. $\exp(tD)$ is bounded, so it has only purely imaginary eigenvalues. Then X is similar to a diagonal matrix that has only purely imaginary eigenvalues.

(viii) The closure of the image is compact if and only if it is bounded since a subspace of Euclidean space is compact if and only if it is closed and bounded. By criterion above, $f_X(t)$ is bounded only when X is similar to a diagonal matrix with only purely imaginary eigenvalues. Notice that such a matrix is skew-hermitian and all skew-hermitain matrix are diagonalizble and all eigenvalues are imaginary by spectral theorem (for skew-hermitian matrices). Therefore X must be similar to a skew-hermitian matrix.

2 Exercise Sheet 2

Problem. 1

Answer.

(a) First we show that $\exp(tX)$ actually lives in $GL_2(\mathbb{R})$. This is because $\det \exp(tX) = \exp(\mathrm{tr}(tX)) \neq 0$. Hence it is invertible. Also, since the exponential map is $\mathcal{C}^\infty$ with respect to t, the map keeps the $\mathcal{C}^\infty$ manifold structure. Now we just need to show that it is a homomorphism between groups. Since X commute with itself, this follows from:

$$r(s+t) = \exp(sX + tX) = \exp(sX)\exp(tX)$$

Therefore, r_X is a homomorphism of Lie group, $(\mathbb{R}, +)$.

(b)(i) Since $X^2 = 0$,

$$r(t) = \exp(tX) = 1 + tX = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

So there's a subrepresentation of r, $W = \mathrm{Span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$. It is obvious that for any t, $r(t)w \in W$ for any $w \in W$. Therefore it is not irreducible.

(b)(ii) Notice that $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ only has one eigenvalue associate with one eigenvector. Therefore, there's no other $\mathbb{R}$-stable subspace of V that is a subrepresentation. It follows that V can't be the direct sum of several subrepresentation. Hence r is not completely reducible.

(b)(iii) W as described above is $\mathbb{R} \times \{0\} \cong \mathbb{R}$, therefore $\mathbb{R}^2/W \cong \mathbb{R}$. Notice that $r(t)$ is the identity of W and the quotient representation is the identity $GL(\mathbb{R}^2/W)$. The direct sum of subrepresentation and the quotient representation is

$$r \oplus \bar{r}(t)(w, v) = (v, w) \in W \oplus \mathbb{R}^2/W \cong \mathbb{R}^2$$

.

Therefore the direct sum of two representations is isomorphic to $r'(t) = Id \in GL_2(\mathbb{R})$.

(b)(iv) Since $W \oplus \mathbb{R}^2/W$ is isomorphic to $\mathbb{R}^2$, we identiy $\rho(t) = r \oplus \bar{r}(t) = Id \in GL(W \oplus \mathbb{R}^2/W) \cong GL_2\mathbb{R}$. If ρ is isomorphic to r then there exists $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\rho(t) \circ T = T \circ r(t)$. Then $r(t)$ has to be diagonalizable. However it is not similar to any diagonal matrix as the eigenspace is not the who space. Therefore these two representation are not isomorphic.

(c)(i) Since X has two different eigenvalues, X is diagonalizable. Therefore let $X = SJS^{-1}$ with J being a diagonal matrix. Then

$$r(t) = \exp(tX) = \exp(tSJS^{-1}) = S\exp(tJ)S^{-1}$$

Since J is a diagonal matrix, $\exp(tJ)$ is also a diagonal matrix. Therefore $\exp(tX)$ has the same eigenvectors as $\exp(tJ)$ and thus it must have two eigenvectors. Suppose v is one eigenvector of $\exp(J)$, then since $t^k J^k v = t^k \lambda^k v$, it is also an eigenvector of $\exp(tJ)$. So the eigenvectors of $\exp(tX)$ doesn't depend on t. Since each eigenvector forms a $\mathbb{R}$-stable subspace, the representation is not irreducible.

(c)(ii) Suppose the eigenvectors for $r(t) = \exp(tX)$ are v and w. In, fact v and w are also eigenvectors of X. Then $V = \{sv, \forall s \in \mathbb{R}\}$ and $W = \{sw, \forall s \in \mathbb{R}\}$ are two stable subspaces with $W \oplus V = \mathbb{R}^2$. Therefore, r is completely reducible.

(c)(iii) Let $r_1 : \mathbb{R} \to GL(V)$ and $r_2 : \mathbb{R} \to GL(W)$ be two representations: $r_1(t)v = r(t)v = \lambda_1 v$ and $r_2(t)v = r(t)w = \lambda_2 w$. These are the irreducible representations since $\dim V = \dim W = 1$. Then let $\rho(t)(v, w) = r_1 \oplus r_2(t)(v, w) = (\lambda_1 v, \lambda_2 w)$ be the direct sum of r_1 and r_2. Let v' and w' be the eigenvectors of $r(t)$ which is basically v and w in $\mathbb{R}^2$. We use different notation since here we are interpreting v and w as basis for vector space V and W. Since v' and w' forms a basis for $\mathbb{R}^2$, There's a unique linear transformation $T : V \oplus W$ such that $T(v) = v'$

and $T(w) = w'$. Then we have $r(t)v' = T \circ \rho(t) \circ T^{-1}v'$ and $r(t)w' = T \circ \rho(t) \circ T^{-1}w'$. Therefore this explicitly constructs an isomorphism.

(d)(i) Suppose $\exp(tX)v = \lambda(t)v$ for all $t \in \mathbb{R}$. Then we have $\frac{d}{dt}\exp(tX)v = \lambda'(t)v$. However, since $\frac{d}{dt}\exp(tX) = X\exp(tX)$, we have $\frac{d}{dt}\exp(tX)v = \lambda(t)Xv = \lambda'(t)v$. But we know that X doesn't have any eigenvector. Therefore $\exp(tX)$ doesn't have a eigenvector that is not depend on t. So there won't be $\mathbb{R}$-stable subspace. Hence the representation is irreducible.

3 Tensor Products of Modules

Problem. 6.5

Answer.
(i) Let v be the basis for V. Then $v \wedge v = 0$. Therefore $\Lambda^k(V) = \{0\}$ for $k > 1$. Thus

$$\Lambda^\bullet(V) \cong \Lambda^0(V) \oplus \Lambda^1(V) \cong k \oplus V \cong V$$

(ii) Similarly, $\Lambda^3(V) = \{0\}$ as the wedge of three basis e_1 and e_2 results to 0. Thus Thus $\Lambda^0(V) = k$ has 1 which is the identity of field k, $\Lambda^1(V)$ has basis e_1 and e_2, and $\Lambda^2(V)$ has basis $e_1 \wedge e_2$.

(iii) The ideal generated by $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ where $e_1, e_2, ..., e_n$ form a basis of V

Problem. 6.6

Answer.
(ii) Let e_1, e_2 be the natural basis for V. Then $T^2(V)$ has basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$. Then the matrix of $f^{\otimes 2}$ is

$$\begin{pmatrix} a^2 & ab & ab & b^2 \\ 0 & ad & 0 & bd \\ 0 & 0 & ad & bd \\ 0 & 0 & 0 & d^2 \end{pmatrix}$$

(iii) For $Sym^2(V)$:

$$f^{\otimes 2} = \begin{pmatrix} a^2 & ab & b^2 \\ 0 & ad & 2bd \\ 0 & 0 & d^2 \end{pmatrix}$$

For $\Lambda^2(V)$

$$f^{\otimes 2} = bd$$

Problem. 6.7

Answer.
(ii) Since the wedge product of any $n+1$ basis vectors of $V = k^n$ is equal to 0, $\Lambda^k(V) = \{0\}$ for $k > n$. Then for $k \leq n$. $\Lambda^k(V)$ has $\binom{}{k}$ basis vectors since the wedge of any k basis vectors of V forms a basis vector of $\Lambda^k(V)$. Therefore the dimension is

$$\sum_{k=0}^{n} \binom{}{k} = 2^n$$