Exercise sheet 3

The Lie groups $SU(2)$, $SO_{\mathbb{R}}(3)$, and the compact symplectic group Sp_2.

1. The Hamiltonian quaternions. Define $\mathbb{H} = \{\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} : w, z \in \mathbb{C}\}$.

(i) Check privately, but do not write up, that $\mathbb{H}$ is a subring of $M_2(\mathbb{C})$ (i.e. $\mathbb{H}$ is closed under $+$, $\times$, $-$ and $1 \in \mathbb{H}$).

(ii) What is the image of the map, $det : \mathbb{H} \to \mathbb{C}$? (We will call this map the reduced norm and we will denote it by n.)

(iii) Suppose that $h \in \mathbb{H} - \{0\}$. Is h a unit in $\mathbb{H}$? If so give an explicity decription of its inverse.

(iv) Consider the elements, 1, (respectively $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) of $\mathbb{H}$ defined by taking $(z, w) = (1, 0)$ (respectively $(i, 0)$, $(0, 1)$, $(0, i)$). Compute $\mathbf{i}^2$, $\mathbf{j}^2$, $\mathbf{k}^2$, $\mathbf{ij}$, $\mathbf{ji}$.

(v) Describe the action of matrix transpose on the matrices $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Short answer. (Matrix transpose is called quaternionic conjugation.)

(vi) Find a relationship between $det(h)$ and $h^T h$ for all $h \in \mathbb{H}$.

(vii) Give without justification an $\mathbb{R}$-basis for $\mathbb{H}$.

2. The norm 1 Hamiltonian quaternions.

(i) Take $z = a + bi$ and $w = c + di$ and compute the reduced norm of the associated quaternion.

(ii) To what familiar $\mathcal{C}^\infty$ manifold is the space of reduced norm 1 quaternions, $\mathbb{H}_1$, diffeomorphic?

(iii) Suppose $h \in \mathbb{H}^\times$ has norm equal to 1. Is there a formula for h^{-1} involving quaternionic conjugation?

(iv) Do the reduced norm 1 quaternions form a Lie group? (Don't allow your written answer to be too long.)

(v) Recall from exercise 3.5 on the handout "computation of tangent spaces of some closed subgroups" the compact symplectic group $Sp(2)$. What relationship exists between that group and this exercise?

3. The unitary and special unitary groups. Let $U(2) = \{A \in GL_2(\mathbb{C}) : \overline{A}^T A = Id\}$, where $\overline{A}$ is obtained from A by taking the complex conjugate of each entry. Let $SU(2) = \{A \in U(2) : det(A) = 1\}$.

(i) Relate $SU(2)$ to Lie groups in the previous exercise.

(ii) To what familiar $\mathcal{C}^\infty$ manifold is $SU(2)$ isomorphic?

(iii) Is $U(2)$ isomorphic to $\mathbb{H}^\times$ in the category of Lie groups?

4. The special orthogonal group $SO_{\mathbb{R}}(3)$.

Let $\mathbb{I} \subset \mathbb{H}$ denote the $\mathbb{R}$-vector space of trace 0 quaternions.

(i) Explain briefly why the trace of the product of two matrices gives a map $\mathbb{I} \times \mathbb{I} \xrightarrow{(\ ,\ )} \mathbb{R}$ and why this map is a symmetric bilinear form.

(ii) Compute $tr(AB)$ where $A = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $B = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$.

(iii) Define $SO_{\mathbb{R}}(3) = \{A \in GL_3(\mathbb{R}) : A^T A = Id\}$. Is $SO_{\mathbb{R}}(3)$ isomorphic to $G = \{A \in GL(\mathbb{I}) : (Ah_1, Ah_2) = (h_1, h_2)\ \forall\ h_1, h_2 \in \mathbb{I}\}$?

(iv) Show that $H^\times$ acts on $\mathbb{I}$ by conjugation: $h * \ell = h\ell h^{-1}$.
(v) Use (iv) the produce a Lie group homomorphism, $\xi : \mathbb{H}_1 \to G$.
(vi) What are the dimensions of the Lie groups H_1 and G?
(vii) Show that the map ξ is not an isomorphism.
(viii) Show that the map ξ is a covering space.