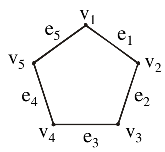
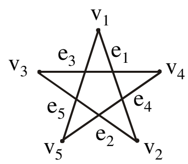
ISOMORPHISM OF GRAPHS

Here we have a graph



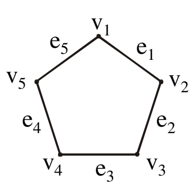
Which can also be defined as



Its vertices and edges can be written as:
$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$
Edge endpoint function is:

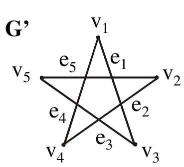
Edge	**Endpoints**
E_1	$\{v_1,v_2\}$
E_2	$\{v_2,v_3\}$
E_3	$\{v_3,v_4\}$
E_4	$\{v_4,v_5\}$
E_5	$\{v_5,v_1\}$

G



Another graph $G^'$ is

G'



Edge endpoint function of G is:

Edge	**Endpoints**
e_1	$\{v_1,v_2\}$
e_2	$\{v_2,v_3\}$
e_3	$\{v_3,v_4\}$
e_4	$\{v_4,v_5\}$
e_5	$\{v_5,v_1\}$

Edge endpoint function of G' is:

Edge	**Endpoints**
e_1	$\{v_1,v_3\}$
e_2	$\{v_2,v_4\}$
e_3	$\{v_3,v_5\}$
e_4	$\{v_1,v_4\}$
e_5	$\{v_2,v_5\}$

Two graphs (G and G') that are the same except for the labeling of their vertices are not considered different.

GRAPHS OF EDGE POINT FUNCTIONS

Edge point function of G is:

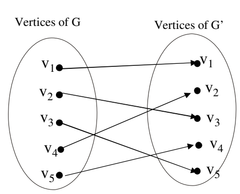
Edge	Endpoints
e_1	$\{v_1,v_2\}$
e_2	$\{v_2,v_3\}$
e_3	$\{v_3,v_4\}$
e_4	$\{v_4,v_5\}$
e_5	$\{v_5,v_1\}$

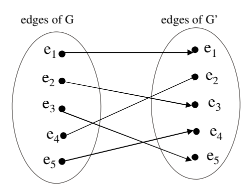
Edge point function of G' is:

Edge	Endpoints
e_1	$\{v_1,v_3\}$
e_2	$\{v_2,v_4\}$
e_3	$\{v_3,v_5\}$
e_4	$\{v_1,v_4\}$
e_5	$\{v_2,v_5\}$

Note it that the graphs G and G' are looking different because in G the end points of e_1 are v_1,v_2 but in G' are v_1, v_3 etc.

Buts G' is very similar to G ,if the vertices and edges of G' are relabeled by the function shown below, then G' becomes same as G:





It shows that if there is one-one correspondence between the vertices of G and G', then also one-one correspondence between the edges of G and G'.

ISOMORPHIC GRAPHS:

Let G and G' be graphs with vertex sets V(G) and V(G') and edge sets E(G) and E(G'), respectively.

G is isomorphic to G' if, and only if, there exist one-to-one correspondences g: $V(G) \rightarrow V(G')$ and h: $E(G) \rightarrow E(G')$ that preserve the edge-endpoint functions of G and G' in the sense that for all $v \in V(G)$ and $e \in E(G)$.

v is an endpoint of e $\Leftrightarrow$ g(v) is an endpoint of h(e).

EQUIVALENCE RELATION:

Graph isomorphism is an equivalence relation on the set of graphs.

1. Graphs isomorphism is Reflexive (It means that the graph should be isomorphic to itself).
2. Graphs isomorphism is Symmetric (It means that if G is isomorphic to G' then G' is also isomorphic to G).
3. Graphs isomorphism is Transitive (It means that if G is isomorphic to G' and G' is isomorphic to G'', then G is isomorphic to G'').

ISOMORPHIC INVARIANT:

A property P is called an isomorphic invariant if, and only if, given any graphs G and G', if G has property P and G' is isomorphic to G, then G' has property P.

THEOREM OF ISOMORPHIC INVARIANT:

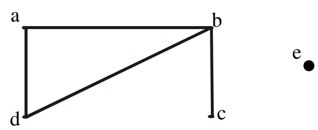
Each of the following properties is an invariant for graph isomorphism, where n, m and k are all non-negative integers, if the graph:
1. has n vertices.
2. has m edges.
3. has a vertex of degree k.
4. has m vertices of degree k.
5. has a circuit of length k.
6. has a simple circuit of length k.
7. has m simple circuits of length k.
8. is connected.
9. has an Euler circuit.
10. has a Hamiltonian circuit.

DEGREE SEQUENCE:

The degree sequence of a graph is the list of the degrees of its vertices in non-increasing order.

EXAMPLE:

Find the degree sequence of the following graph.



SOLUTION:

Degree of a = 2, Degree of b = 3, Degree of c = 1,
Degree of d = 2, Degree of e = 0

By definition, degree of the vertices of a given graph should be in decreasing (non-increasing) order.

Therefore Degree sequence is: 3, 2, 2, 1, 0

GRAPH ISOMORPHISM FOR SIMPLE GRAPHS:

If G and G' are simple graphs (means the "graphs which have no loops or parallel edges") then G is isomorphic to G' if, and only if, there exists a one-to-one correspondence (1-1 and onto function) g from the vertex set V (G) of G to the vertex set V (G') of G' that preserves the edge-endpoint functions of G and G' in the sense that for all vertices u and v of G,

{u, v} is an edge in G ⇔ {g(u), g(v)} is an edge in G'.

OR

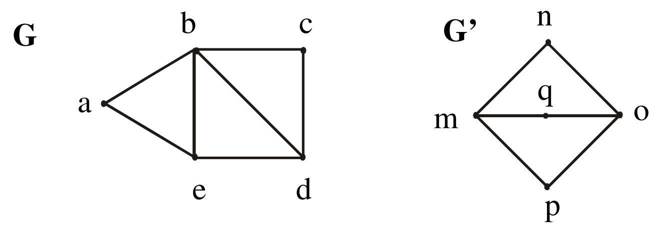
You can say that with the property of one-one correspondence, u and v are adjacent in graph G ⇔ if g (u) and g (v) are adjacent in G' .

Note:

It should be noted that unfortunately, there is no efficient method for checking that whether two graphs are isomorphic(methods are there but take so much time in calculations).Despite that there is a simple condition. Two graphs are isomorphic if they have the same number of vertices(as there is a 1-1 correspondence between the vertices of both the graphs) and the same number of edges(also vertices should have the same degree.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.



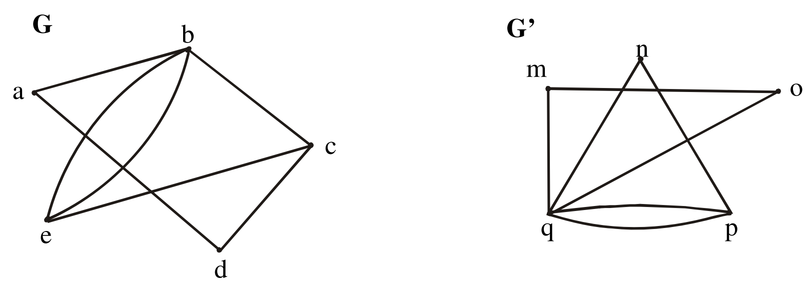
SOLUTION:

As both the graphs have the same number of vertices. But the graph G has 7 edges and the graph G' has only 6 edges. Therefore the two graphs are not isomorphic.

Note: As the edges of both the graphs G and G' are not same then how the one-one correspondence is possible ,that the reason the graphs G and G' are not isomorphic.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.

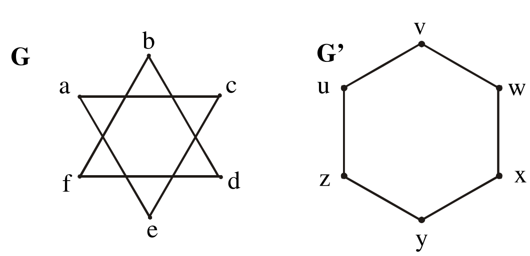


SOLUTION:

Both the graphs have 5 vertices and 7 edges. The vertex q of G' has degree 5. However G does not have any vertex of degree 5 (so one-one correspondence is not possible). Hence, the two graphs are not isomorphic.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.

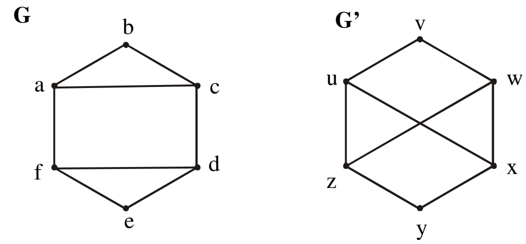


SOLUTION:

Clearly the vertices of both the graphs G and G' have the same degree (i.e "2") and having the same number of vertices and edges but isomorphism is not possible. As the graph G' is a connected graph but the graph G is not connected due to have two components (eca and bdf). Therefore the two graphs are non isomorphic.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.



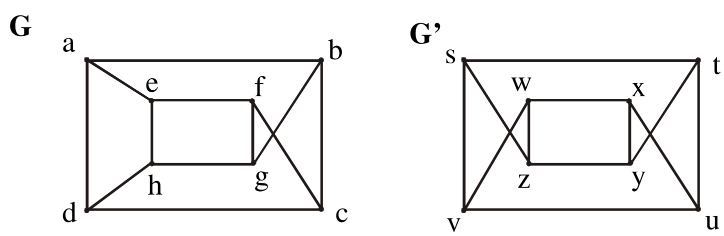
SOLUTION:

Clearly G has six vertices,G' also has six vertices. And the graph G has two simple circuits of length 3; one is abca and the other is defd.But G' does not have any simple circuit of length 3(as one simple circuit in G' is uxwv of length 4).Therefore the two graphs are non-isomorphic.

Note: A simple circuit is a circuit that does not have any other repeated vertex except the first and last.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.

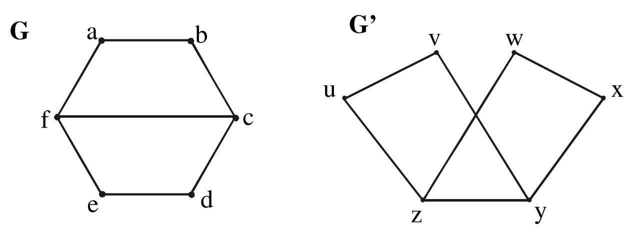


SOLUTION:

Both the graph G and G' have 8 vertices and 12 edges and both are also called regular graph(as each vertex has degree 3).The graph G has two simple circuits of length 5; abcfea(i.e starts and ends at a) and cdhgfc(i.e starts and ends at c). But G' does not have any simple circuit of length 5 (it has simple circuit tyxut,vwxuv of length 4 etc). Therefore the two graphs are non-isomorphic.

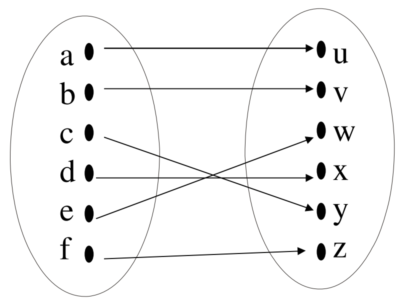
EXERCISE:

Determine whether the graph G and G' given below are isomorphic.



SOLUTION:

We note that all the isomorphism invariants seems to be true.
We shall prove that the graphs G and G' are isomorphic.
Here G has four vertices of degree "2" and two vertices of degree"3". Similar case in G'. Also G and G' have circuits of length 4.As a is adjacent to b and f in graph G.In graph G' u is adjacent to v and z. And as a and u has degree 2 so both are mapped. And b mapped with v, f mapped with z(as both have the same degree also a is adjacent to f and u is to z), and as we moves further we get the 1-1 correspondence.
Define a function f: V(G) →V(G') as follows.

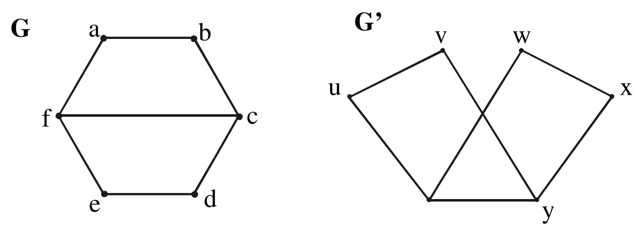


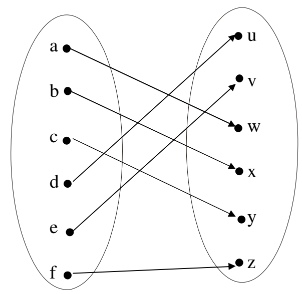
Clearly the above function is one and onto that is a bijective mapping. Note that I write the above mapping by keeping in mind the invariants of isomorphism as well as the fact that the mapping should preserve edge end point function. Also you should note that the mapping is not unique.
f is clearly a bijective function. The fact that f preserves the edge endpoint functions of G and G' is shown below.

Edges of G	Edges of G'
{a, b}	{u, v} = {g(a), g(b)}
{b, c}	{v, y} = {g(b), g(c)}
{c, d}	{y, x} = {g(c), g(d)}
{d, e}	{x, w} = {g(d), g(e)}
{e, f}	{w, z} = {g(e), g(f)}
{a, f}	{u, z} = {g(a), g(f)}
{c, f}	{y, z} = {g(c), g(f)}

ALTERNATIVE SOLUTION:

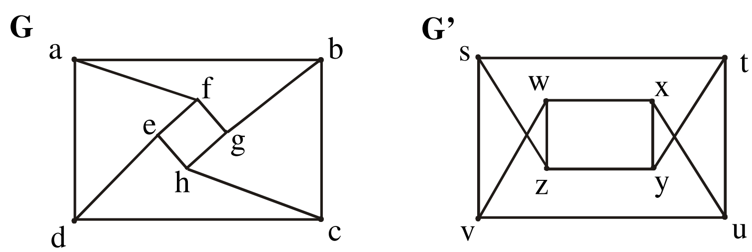
We shall prove that the graphs G and G' are isomorphic.
Define a function f: V(G) →V(G') as follows.





EXERCISE:

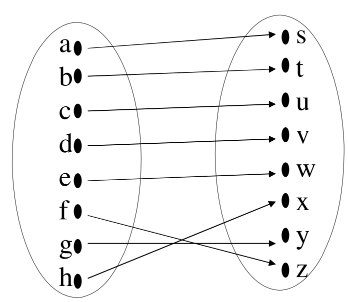
Determine whether the graph G and G' given below are isomorphic.



SOLUTION:

We shall prove that the graphs G and G' are isomorphic.
Clearly the isomorphism invariants seems to be true between G and G'.

Define a function f: V(G) →V(G') as follows.



f is clearly a bijective function(as it satisfies conditions the one-one and onto function clearly). The fact that f preserves the edge endpoint functions of G and G' is shown below.

Edges of G	**Edges of G'**

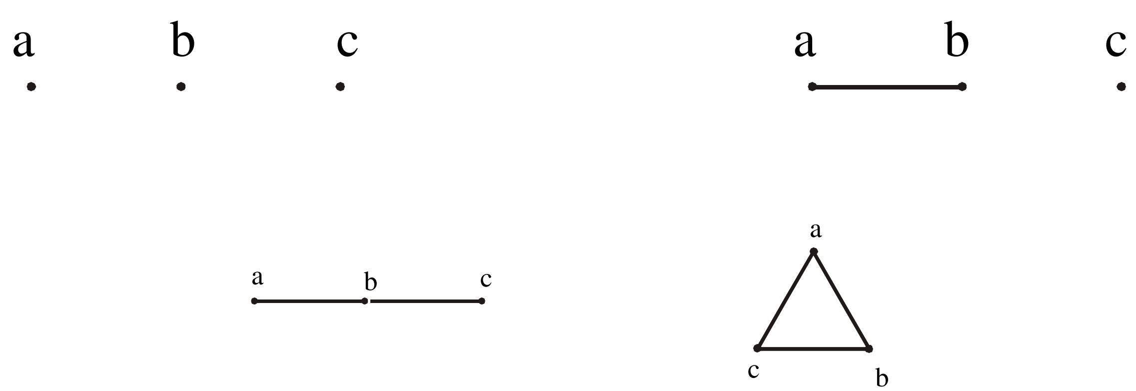
{a, b}	{s, t} = {f(a), f(b)}
{b, c}	{t, u} = {f(b), f(c)}
{c, d}	{u, v} = {f(c), f(d)}
{a,d}	{s, v} = {f(a), f(d)}
{a, f}	{s, z} = {f(a), f(f)}
{b, g}	{t, y} = {f(b), f(g)}
{c, h}	{u, x} = {f(c), f(h)}
{d, e}	{v, w} = {f(d), f(e)}
{e, f}	{w, z} = {f(e), f(f)}
{f, g}	{z, y} = {f(f), f(g)}
{g, h}	{y, x} = {f(g), f(h)}
{h, e}	{x, w} = {f(h), f(e)}

EXERCISE:

Find all non isomorphic simple graphs with three vertices.

SOLUTION:

There are four simple graphs with three vertices as given below(which are non-isomorphic simple graphs).

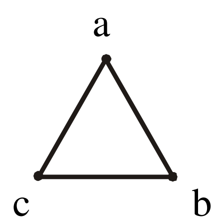


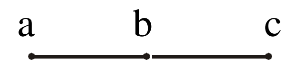
EXERCISE:

Find all non isomorphic simple connected graphs with three vertices.

SOLUTION:

There are two simple connected graphs with three vertices as given below(which are non-isomorphic connected simple graphs).





<u>EXERCISE:</u>

Find all non isomorphic simple connected graphs with four vertices.

<u>SOLUTION:</u>

There are six simple connected graphs with four vertices as given below.