Consistency Criterion

(Method for solving a consistent system of equations)

A general system
$$Ax=b$$
of m linear equations in n unknowns $x_1, x_2, \dots, x_n$ may or may not have a solution. If the system $Ax=b$ has a solution $y=(y_1, y_2, \dots, y_n)$ then it is called consistent system. If the system has $Ax=b$ has no solution, (If y does not exist) then the system is termed as inconsistent.

→ If $m=n$, that the number of equations equals to number of unknowns and matrix A of coefficient is non singular then system is consistent.

Theorem:

The system $Ax=b$, with $m=n$ and matrix A non-singular has a unique solution $x=A^{-1}b$.

Corollary:

A system $Ax=0$ of n homogenous linear equations in n unknowns has a unique solution $x=0$ if and only if A is non singular matrix. $x=0$ is a trivial solution.

Theorem (Consistency criterion)

Let $Ax=b$ be a system of m linear equation in n unknowns. Then the equation have a solution iff the rank of A is equals to the rank of augmented matrix A_b.

Example:

Examine the following homogenous system for non trivial solution

$$x_1 - x_2 + 2x_3 + x_4 = 0$$
$$3x_1 + 2x_2 + x_4 = 0$$
$$4x_1 + x_2 + 2x_3 + 2x_4 = 0$$

The matrix of coefficients is

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix}$$

Bring A into echelon form.

$$A \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -2 \\ 0 & 5 & -6 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 4R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 - R_2$$

$$\sim R \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & \frac{-6}{5} & \frac{-2}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{5} R_2$$

$$\sim R \begin{bmatrix} 1 & 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & 1 & \frac{-6}{5} & \frac{-2}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 + R_2$$

The rank of matrix is $2 < 4$ thus equation has non trivial solution

The first two rows of above matrix give the following relations

$$x_1 + \frac{4}{5}x_3 + \frac{3}{5}x_4 = 0$$

$$x_2 - \frac{6}{5}x_3 - \frac{2}{5}x_4 = 0$$

i.e $x_1 = -\frac{4}{5}x_3 - \frac{3}{5}x_4$

$$x_2 = \frac{6}{5}x_3 + \frac{2}{5}x_4$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5}x_3 - \frac{3}{5}x_4 \\ \frac{6}{5}x_3 + \frac{2}{5}x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

Assigning arbitrary values to x_3 & x_4.

let $x_3 = a$, $x_4 = b$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5}a - \frac{3}{5}b \\ \frac{6}{5}a + \frac{2}{5}b \\ a \\ b \end{bmatrix} = a\begin{bmatrix} -\frac{4}{5} \\ \frac{6}{5} \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3/5 \\ 2/5 \\ 0 \\ 1 \end{bmatrix}$$

which gives the solution for arbitrary values of a & b.

Example.

For what value of λ the equations

$$(5-\lambda)x_1 + 4x_2 + 2x_3 = 0$$

$$4x_1 + (5-\lambda)x_2 + 2x_3 = 0$$

$$2x_1 + 2x_2 + (2-\lambda)x_3 = 0$$

have non trivial solution. For these equations.

The matrix of coefficients is

$$A = \begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

$$A \underset{\sim}{R} \begin{bmatrix} 2 & 2 & 2-\lambda \\ 4 & 5-\lambda & 2 \\ 5-\lambda & 4 & 2 \end{bmatrix} \text{ by } R_{13}$$

$$\sim R \begin{bmatrix} 1 & 1 & 1-\lambda/2 \\ 4 & 5-\lambda & 2 \\ 5-\lambda & 4 & 2 \end{bmatrix} \text{ by } \tfrac{1}{2} R_1$$

$$\sim R \begin{bmatrix} 1 & 1 & 1-\lambda/2 \\ 0 & 1-\lambda & -2+2\lambda \\ 5-\lambda & 4 & 2 \end{bmatrix} R_2 - 4R_1$$

$$\sim R \begin{bmatrix} 1 & 1 & 1-\lambda/2 \\ 0 & 1-\lambda & -2+2\lambda \\ 0 & -1+\lambda & \frac{-6+7\lambda-\lambda^2}{2} \end{bmatrix} R_3 - (5-\lambda)R_1$$

$$\sim R \begin{bmatrix} 1 & 1 & 1-\lambda/2 \\ 0 & 1-\lambda & -2(1-\lambda) \\ 0 & -1(1-\lambda) & -\frac{(1-\lambda)(6-\lambda)}{2} \end{bmatrix} \longrightarrow \text{eq } 1$$

$$\sim R \begin{bmatrix} 1 & 1 & 1-\lambda/2 \\ 0 & 1 & -2 \\ 0 & -1 & -\frac{(6-\lambda)}{2} \end{bmatrix} \begin{matrix} \frac{1}{1-\lambda} R_2 \\ \\ \frac{1}{1-\lambda} R_3 \end{matrix} \rightarrow$$

$$\sim R \begin{bmatrix} 1 & 0 & 3-\lambda/2 \\ 0 & 1 & -2 \\ 0 & 0 & \frac{\lambda-10}{2} \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ R_3 + R_2 \end{matrix} \rightarrow \text{eq } 2$$

If $\lambda = 10$

$$A \underset{\sim}{R} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

R.w

$2-(5-\lambda)(1-\lambda/2)$

$2-(5-\frac{5\lambda}{2}-\lambda+\frac{\lambda^2}{2})$

$2-5+\frac{5\lambda}{2}+\lambda-\frac{\lambda^2}{2}$

$-3+\frac{7\lambda}{2}-\frac{\lambda^2}{2}$

$\frac{-6+7\lambda-\lambda^2}{2}$

$-(\lambda^2-7\lambda+6)$

$-(\lambda^2-6\lambda-\lambda+6)$

$-(\lambda(\lambda-6)-1(\lambda-6))$

$-(\lambda-1)(\lambda-6)$

$-(1-\lambda)(6-\lambda)$

$1-\frac{\lambda}{2}+2$

$3-\frac{\lambda}{2}$

$\frac{-6+\lambda}{2}-2$

$\frac{-6+\lambda-4}{2} = \frac{\lambda-10}{2}$

given system reduces to

$$x_1 - 2x_3 = 0$$
$$x_2 - 2x_3 = 0$$
$$x_1 = x_2 = 2x_3$$

Take $x_3 = a$.

$$x_1 = 2a, \ x_2 = 2a, \ x_3 = a.$$

So $(x_1, x_2, x_3) = (2a, 2a, a) = a(2,2,1)$ gives all solutions for $a \in \mathbb{R}$.

If $\lambda \neq 10$ with $1-\lambda = 0$ within eq ① gives

$$A \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \frac{2-\lambda}{2} = \frac{1+1-\lambda}{2} = \frac{1}{2} + \frac{1-\lambda}{2} = \frac{1}{2} + 0. \right.$$

in eq ①.

The given system reduced to the single equation.

$$x_1 + x_2 + \frac{1}{2}x_3 = 0.$$

Take $x_3 = 2a$

$x_2 = b$

Then

$$x_1 = -x_2 - \frac{1}{2}x_3$$
$$= -b - a$$

Hence the solution vector is

$$(x_1, x_2, x_3) = (-b-a, b, 2a).$$
$$= b(-1,1,0) + a(-1,0,2) \quad a, b \in \mathbb{R}.$$

For $\lambda \neq 1$, $\lambda \neq 10$ the given system has only the trivial solution $(0,0,0)$.