Chapter 5

DETERMINANTS

The reader may already be familiar with the concept of determinants. This concept played a basic role in the solution of systems of linear equations before the use of matrices and computers. The use of determinants is becoming less and less with every passing day. Neverthless determinants still play an important but minor role in finding solutions of linear problems.

This chapter includes concept of determinant of a square matrix, minors and cofactors, properties of determinants, Laplace expansion of a determinant, adjoint and inverse of a matrix.

DETERMINANT OF A SQUARE MATRIX

(5.1) **Definition.** In Chapter 4, methods for solving systems of homogeneous and nonhomogeneous linear equations were given. We shall make use of those methods to give formal definition of a **determinant.** Before taking up the general case, we consider the following special case of two linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

in two unknowns x_1, x_2. Here

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad A_b = \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \qquad (1)$$

are the **coefficient matrix** and the **augmented matrix** respectively with entries as real numbers. By the Gauss-Jordan procedure, the solution of these equations is

$$x_1 = \frac{(b_1a_{22} - b_2a_{12})}{(a_{11}a_{22} - a_{21}a_{12})}, \quad x_2 = \frac{(b_2a_{11} - b_1a_{21})}{(a_{11}a_{22} - a_{21}a_{12})}$$

provided that

$$a_{11}a_{22} - a_{21}a_{12} \neq 0.$$

The scalar $a_{11}a_{22} - a_{21}a_{12}$ is uniquely determined by the matrix A. It is called the **determinant of order 2** of the square matrix A given in (1) and is denoted by det A or $|A|$. Thus

$$|A| = \det A = a_{11}a_{22} - a_{21}a_{12} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{2}$$

Note the two vertical bars instead of square brackets used for matrices.

We note that det A of order 2 is a real number associated with a matrix A of order 2. So, in this case, we may regard det as a function whose domain is the set of all square matrices of order 2 and whose range is a subset of real numbers.

The following properties of determinants of order 2 follow directly from the above definition.

(i) For any real number k,

$$\begin{vmatrix} ka_{11} & a_{12} \\ ka_{21} & a_{22} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

(ii) If $a_{12} = b_{12} + c_{12}$, $a_{22} = b_{22} + c_{22}$, then

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & b_{12}+c_{12} \\ a_{21} & b_{22}+c_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & c_{12} \\ a_{21} & c_{22} \end{vmatrix}.$$

(iii) If the two columns of A are identical then det $A = 0$. That is

$$\begin{vmatrix} a_{11} & a_{11} \\ a_{21} & a_{21} \end{vmatrix} = 0.$$

(iv) The determinant of the unit matrix is 1. That is

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

It will be seen later that these properties are characteristics of a determinant of a square matrix of any order.

(5.2) **Definition. (Determinant of Order n).** Firstly we define the determinant of an $n \times n$ matrix inductively. That is, from our knowledge of a determinant of order 2, we define a determinant of order 3 and use this definition of a determinant of order 3 to describe a determinant of order 4 and so on.

Thus a determinant of order 3 is defined as follows:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Then } \det A = |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (1)$$

Note the minus sign before the second term on the right hand side of (1).

For example, if

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 3 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}, \text{ then}$$

$$\det A = \begin{vmatrix} 4 & 1 & 2 \\ 3 & 2 & 5 \\ 1 & 2 & 3 \end{vmatrix} = 4\begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 4(6-10) - 1(9-5) + 2(6-2)$$

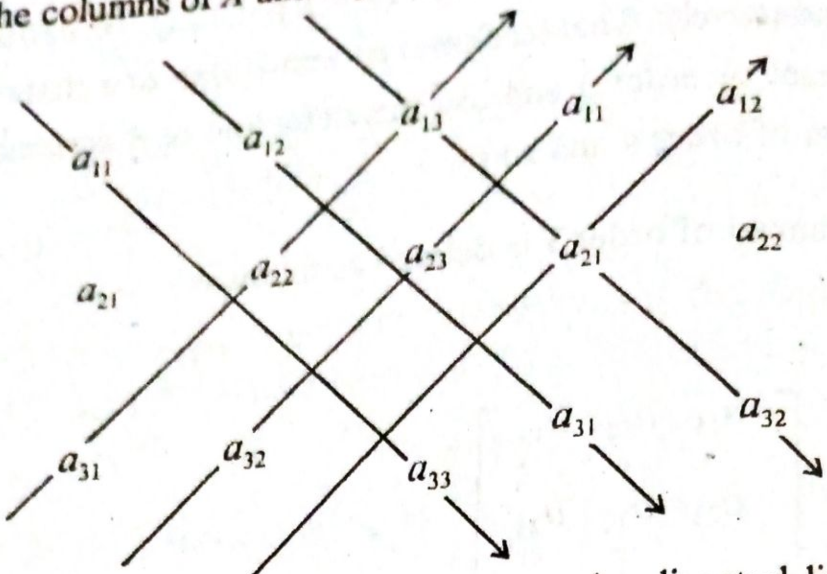
$$= -16 - 4 + 8 = -12.$$

There is a simplex method to calculate a determinant of order 3. For example, in (1) of (5.2).

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

We write the columns of A and adjoin to these the first two columns as below:



Now calculate the six products of numbers on the directed lines taking plus sign with those products on arrows pointing downwards and minus sign with products on arrows pointing upwards. Adding these products we get the value of det A. This is known as **Sarrus's rule.**

Note: The method of arrows given above works only for $n = 2, 3$, and not for $n \geq 4$.

(5.3) Definition. Let A be a square matrix of order n. The matrix obtained from A by deleting its ith row and jth column is again a matrix M_{ij} of order $n-1$. M_{ij} is called the ***ij* th minor of *A*.**[1]

(5.4) Definition. Let M_{ij} be the ijth minor of a square matrix A of order n. Then

$$A_{ij} = (-1)^{i+j} \det M_{ij}$$

is called the ijth **cofactor** of A.

Observe that the sign on the right hand side of the above equality is positive or negative according as $i + j$ is even or odd.

Example 1. Let

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 3 & 2 & 5 & -2 \\ 3 & 4 & 2 & 1 \\ -3 & 2 & 5 & 1 \end{bmatrix}.$$

Then

$$M_{23} = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 4 & 1 \\ -3 & 2 & 1 \end{bmatrix}, \quad M_{42} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

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[1] Some authors call det M_{ij} of the matrix

and $A_{23} = (-1)^{2+3} \det M_{23} = -82$

$A_{42} = (-1)^{4+2} \det M_{42} = -45.$

Consider the expansion of det A as given in (1) of (5.2). Using the fact that

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

we may write (1) of (5.2) as

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

We now define det A of a matrix A of order n as follows:

(5.5) **Definition.** For the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

of order n, we define det A by

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}. \qquad (1)$$

The expression on the right hand side of (1) is an **expansion** of det A by cofactors of the ith row of det A. This is called **Laplace's Expansion** of a determinant of order n. The definition (1) given above indicates that the determinant of a matrix can be evaluated by any row of the matrix. We only have to be careful about the sign of the cofactor of the corresponding element.

Example 6. Evaluate the determinant of the matrix

$$D = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

Solution. Expanding by the first row, we have

$$\det D = aA + hH + gG$$

where
$$A = \begin{vmatrix} b & f \\ f & c \end{vmatrix} = bc - f^2,$$

$$H = -\begin{vmatrix} h & f \\ g & c \end{vmatrix} = fg - ch,$$

$$G = \begin{vmatrix} h & b \\ g & f \end{vmatrix} = hf - bg$$

Hence
$$\begin{aligned} \det D &= a(bc - f^2) + h(fg - ch) + g(hf - bg) \\ &= abc - af^2 + fgh - ch^2 + fgh - bg^2 \\ &= abc + 2fgh - af^2 - bg^2 - ch^2. \end{aligned}$$

Example 7. Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 4 & 2 & 5 & 10 \\ 1 & 1 & 6 & 3 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{bmatrix}$$

Solution.

The solution given below makes use of Theorem 5.16 and the algorithm **(5.19)** by reducing the determinant into a determinant of smaller size. Thus

$$\det A = \begin{vmatrix} 4 & 2 & 5 & 10 \\ 1 & 1 & 6 & 3 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 6 & 3 \\ 4 & 2 & 5 & 10 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{vmatrix} \qquad \text{by } R_{12}\text{, changing the sign}$$

$$= -\begin{vmatrix} 1 & 1 & 6 & 3 \\ 0 & -2 & -19 & -2 \\ 0 & -4 & -42 & -16 \\ 0 & 2 & 5 & 8 \end{vmatrix} \qquad \text{by } R_2 - 4R_1,\ R_3 - 7R_1 \left(\begin{array}{c}\text{The value of } \det A \\ \text{does not change with} \\ \text{these row operations}\end{array}\right)$$

$$= -\begin{vmatrix} -2 & -19 & -2 \\ -4 & -42 & -16 \\ 2 & 5 & 8 \end{vmatrix}, \qquad \text{evaluating by the first column}$$

$$= -\begin{vmatrix} 2 & 19 & 2 \\ 4 & 42 & 16 \\ 2 & 5 & 8 \end{vmatrix} \qquad \begin{array}{l}\text{by } (-1)R_1,\ (-1)R_2 \\ \text{i.e. multiplying the} \\ \text{determinant by } (-1)^2.\end{array}$$

$$= -2\begin{vmatrix} 2 & 19 & 2 \\ 2 & 21 & 8 \\ 2 & 5 & 8 \end{vmatrix} \qquad \text{taking out 2 from } R_2$$

$$= -2\begin{vmatrix} 2 & 19 & 2 \\ 0 & 2 & 6 \\ 0 & -14 & 6 \end{vmatrix} \quad \text{by } R_2 - R_1 \text{ and } R_3 - R_1$$

$$= -4\begin{vmatrix} 2 & 6 \\ -14 & 6 \end{vmatrix} = -4(12 + 84) = -384$$

Example 8. Prove the identity from definition:

$$\begin{vmatrix} \beta\gamma\delta & \alpha & \alpha^2 & \alpha^3 \\ \gamma\delta\alpha & \beta & \beta^2 & \beta^3 \\ \delta\alpha\beta & \gamma & \gamma^2 & \gamma^3 \\ \alpha\beta\gamma & \delta & \delta^2 & \delta^3 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \beta^2 & \beta^3 & \beta^4 \\ 1 & \gamma^2 & \gamma^3 & \gamma^4 \\ 1 & \delta^2 & \delta^3 & \delta^4 \end{vmatrix}$$

Solution. Let

$$\Delta = \begin{vmatrix} \beta\gamma\delta & \alpha & \alpha^2 & \alpha^3 \\ \gamma\delta\alpha & \beta & \beta^2 & \beta^3 \\ \delta\alpha\beta & \gamma & \gamma^2 & \gamma^3 \\ \alpha\beta\gamma & \delta & \delta^2 & \delta^3 \end{vmatrix}$$

Multiply the first, second, third and fourth rows of Δ by α, β, γ and respectively. Since Δ is the multiplied by $\alpha\beta\gamma\delta$, we have

$$\Delta = \frac{1}{\alpha\beta\gamma\delta}\begin{vmatrix} \alpha\beta\gamma\delta & \alpha^2 & \alpha^3 & \alpha^4 \\ \alpha\beta\gamma\delta & \beta^2 & \beta^3 & \beta^4 \\ \alpha\beta\gamma\delta & \gamma^2 & \gamma^3 & \gamma^4 \\ \alpha\beta\gamma\delta & \delta^2 & \delta^3 & \delta^4 \end{vmatrix}$$

Take out $\alpha\beta\gamma\delta$ from the first column. Then

$$\Delta = \frac{\alpha\beta\gamma\delta}{\alpha\beta\gamma\delta}\begin{vmatrix} 1 & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \beta^2 & \beta^3 & \beta^4 \\ 1 & \gamma^2 & \gamma^3 & \gamma^4 \\ 1 & \delta^2 & \delta^3 & \delta^4 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \beta^2 & \beta^3 & \beta^4 \\ 1 & \gamma^2 & \gamma^3 & \gamma^4 \\ 1 & \delta^2 & \delta^3 & \delta^4 \end{vmatrix}$$

as required.

Example 9. Verify that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

Solution. Let

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Subtracting the first row from the second and third rows respectively, we have

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}.$$

Taking factors $(b-a)$, $(c-a)$ out of the second and third rows, we get

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix},$$ on expanding by the first column

$$= (b-a)(c-a)(c+a-b-a)$$

$$= (a-b)(b-c)(c-a).$$

DETERMINANTS AND INVERSES OF MATRICES

The section deals with another method of finding the inverse of a matrix. In this process we shall define the **adjugate or adjoint** of a matrix. We begin with the following simple result.

(5.20) Theorem. For an invertible matrix A, $\det A \neq 0$ and

$$\det(A^{-1}) = \frac{1}{\det A}$$

Proof. Let A be a nonsingular matrix with A^{-1} as its inverse.

Then

$$A^{-1}A = I.$$

By Theorem 5.18.

$$\det(A^{-1}A) = \det(A^{-1}) \cdot \det(A)$$

But $\quad \det(A^{-1}A) = \det(I) = 1$

Hence $\quad \det(A^{-1}) \cdot \det A = 1$

$\Rightarrow \quad \det(A^{-1}) \neq 0, \ \det A \neq 0$

and $\quad \det(A^{-1}) = \dfrac{1}{\det A}.$

Before we describe the other method of finding the inverse of a matrix A, we define the **adjugate or adjoint** of A as follows:

(5.21) Definition. Let $A = [\,a_{ij}\,]$ be an $n \times n$ matrix and A_{ij} be the cofactor of a_{ij} in A. Let

$$B = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$ be the matrix of cofactors.

Then the **adjugate or adjoint of A, written adj A, is the transpose of B.** Thus

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} = B^T.$$

Example 11. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \text{ Find adj } A.$$

Solution. Here, the cofactors are

$$A_{11} = d, \quad A_{12} = -c, \quad A_{21} = -b, \quad A_{22} = a.$$

Hence $B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

so that

$$\text{adj } A = B^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 12. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}. \text{ Compute adj } A.$$

Solution. Here, the cofactors are

$$A_{11} = 5, \quad A_{12} = -1, \quad A_{13} = -7, \quad A_{21} = -1, \quad A_{22} = -7$$
$$A_{23} = 5, \quad A_{31} = -7, \quad A_{32} = 5, \quad A_{33} = -1.$$

Hence, the matrix of cofactors is

$$B = \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}$$

and $\text{adj } A = B^T = \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}.$

Example 13. Find, by adjoint method, the inverse of

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$$

Solution. Here, the cofactors are

$$A_{11} = (-1)^{1+1} \det \begin{bmatrix} -1 & 8 \\ -2 & 7 \end{bmatrix} = 9, \quad A_{12} = (-1)^{1+2} \det \begin{bmatrix} 2 & 8 \\ 5 & 7 \end{bmatrix} = 26,$$

$$A_{13} = (-1)^{1+3} \det \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix} = 1, \quad A_{21} = (-1)^{2+1} \det \begin{bmatrix} 4 & 5 \\ -2 & 7 \end{bmatrix} = -38,$$

$$A_{22} = (-1)^{2+2} \det \begin{bmatrix} 3 & 5 \\ 5 & 7 \end{bmatrix} = -4, \quad A_{23} = (-1)^{2+3} \det \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} = 26,$$

$$A_{31} = (-1)^{3+1} \det \begin{bmatrix} 4 & 5 \\ -1 & 8 \end{bmatrix} = 37, \quad A_{32} = (-1)^{3+2} \det \begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix} = -14,$$

$$A_{33} = (-1)^{3+3} \det \begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix} = -11.$$

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 3 \times 9 + 4 \times 26 + 5 \times 1 = 136.$$

Therefore,

$$A^{-1} = \frac{1}{\det A} \text{ adj } A.$$

$$= \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

Example 14. Let

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}$$

Examine whether A is invertible and, if so, determine A^{-1}.

Solution. We have

$$\det A = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 0 \\ 3 & 1 & 5 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 4 \\ 3 & 1 & 5 \end{vmatrix} \text{ by } R_{12}$$

$$= -\begin{vmatrix} 1 & 2 & 0 \\ 0 & -1 & 4 \\ 0 & -5 & 5 \end{vmatrix} \text{ by } R_2 - R_1 \text{ and } R_3 - R_1$$

$$= -15 \neq 0.$$

Hence A is invertible.

Now, $A_{11} = 10$, $A_{12} = -5$, $A_{13} = -5$,
$A_{21} = -11$, $A_{22} = -2$, $A_{23} = 7$,
$A_{31} = -8$, $A_{32} = 4$, $A_{33} = 1$.

Therefore,

$$A^{-1} = \frac{1}{\det A} \text{ adj } A = -\frac{1}{15}\begin{bmatrix} 10 & -11 & -8 \\ -5 & -2 & 4 \\ -5 & 7 & 1 \end{bmatrix}$$

12. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha).$$

13. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma).$$

14. Show that

(i) $$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a-1)^3(a+3)$$

(ii) $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

(iii) $$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2.$$

15. Prove that

$$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6.$$