(3.46) **Definition. (Rank of a Matrix).** The **rank** of a matrix A is equal to the number of nonzero rows in its echelon (or reduced echelon) form or the order of I_r in the canonical form of A.

A more formal definition of the rank of a matrix is given in **(6.37)**.

To find the rank of A we just reduce A to its echelon (or reduced echelon) form or canonical form and count its nonzero rows.

Example 21. Find the rank of the matrix

$$A = \begin{bmatrix} 5 & 9 & 3 \\ -3 & 5 & 6 \\ -1 & -5 & -3 \end{bmatrix}$$

Solution. To find the rank of A, we reduce A to an echelon form. Thus:

$$A \underset{\sim}{R} \begin{bmatrix} -1 & -5 & -3 \\ -3 & 5 & 6 \\ 5 & 9 & 3 \end{bmatrix} \quad \text{by } R_{13}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 5 & 3 \\ -3 & 5 & 6 \\ 5 & 9 & 3 \end{bmatrix} \quad \text{by } (-1)\, R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 5 & 3 \\ 0 & 20 & 15 \\ 0 & -16 & -12 \end{bmatrix} \quad \text{by } R_2 + 3R_1 \text{ and } R_3 - 5R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & 3/4 \\ 0 & -16 & -12 \end{bmatrix} \quad \text{by } \frac{1}{20}R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & 3/4 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{by } R_3 + 16R_2.$$

This is an echelon form of A and the number of its nonzero rows is 2. Hence the rank of A is 2.

For an **Alternative Method** of finding the rank of a matrix, see **(6.42)**.

EXERCISE 3.2

1. (i) Show that the inverse of a diagonal matrix, with all diagonal elements nonzero, is a diagonal matrix.
   (ii) Show that the inverse of a scalar matrix is a scalar matrix.
2. For a nonsingular matrix A, show that
   (i) $(A^n)^{-1} = (A^{-1})^n$, here n is a positive integer.
   (ii) $(kA)^{-1} = k^{-1}A^{-1}$, k is any nonzero scalar.
   (iii) $(A^{-1})^T = (A^T)^{-1}$
   (iv) $(\bar{A})^{-1} = (\overline{A^{-1}})$
   (v) $\left(\overline{A^T}\right)^{-1} = (\overline{A^{-1}})^T$.
3. If A is invertible and $AB = \mathbf{0}$, then show that $B = \mathbf{0}$.
4. Let A and B be distinct $n \times n$ matrices with real entries. If $AB^2 = BA^2$ and $A^3 = B^3$, show that $A^2 + B^2$ is not invertible.
5. Find the inverse of each of the following matrices:

(i) $\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

(v) $\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}, \left(i=\sqrt{1}\right)$ (vi) $\begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}, \left(i=\sqrt{-1}\right)$

(vii) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 7 \end{bmatrix}$

6. Reduce each of the following matrices into the indicated form:

(i) $\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$ reduced echelon form

(ii) $\begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{bmatrix}$ reduced echelon form

(iii) $\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 3 \end{bmatrix}$ echelon form

(iv) $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$ reduced echelon form.

7. Show that

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \underset{\sim}{R} I_3.$$

8. Find the rank of each of the following matrices

(i) $\begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$

(v) $\begin{bmatrix} ar^{(1-1)n} & ar^{(1-1)n+1} & \cdots & ar^{(1-1)n+(n-1)} \\ ar^{(2-1)n} & ar^{(2-1)n+1} & \cdots & ar^{(2-1)n+(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ ar^{(n-1)n} & ar^{(n-1)n+1} & \cdots & ar^{(n-1)n+(n-1)} \end{bmatrix}$, ($a$ and r are nonzero)