19th April-24 Friday

Chapter# 5 ...Iqra
Linear Algebra

Question:- 12, 13, 14, 15

12- $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

Solution:-

L-H-S

$$\begin{array}{ccc} C_1 & C_2 & C_3 \end{array}$$
$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix}$$

Here, $C_2 - C_1$, $C_3 - C_1$ $\therefore 1-1=0$ $\therefore 1-1=0$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \alpha & \beta-\alpha & \gamma-\alpha \\ \beta\gamma & \gamma\alpha-\beta\gamma & \alpha\beta-\beta\gamma \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \alpha & \beta-\alpha & \gamma-\alpha \\ \beta\gamma & -\gamma(\beta-\alpha) & -\beta(\gamma-\alpha) \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha) \begin{vmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta\gamma & -\gamma & -\beta \end{vmatrix}$$

Now,

$$= (\beta-\alpha)(\gamma-\alpha)(-\beta \times 1 - (-\gamma \times 1))$$

$= (\beta - \alpha)(\gamma - \alpha)(-\beta + \gamma)$

$= -(\alpha - \beta)(\gamma - \alpha)[-(\beta - \gamma)]$

$= (-)(-)(\alpha - \beta)(\gamma - \alpha)(\beta - \gamma)$

$= (\alpha - \beta)(\gamma - \alpha)(\beta - \gamma)$

Ans:- proved. $\therefore (-)(-) =$

13- $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} = (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(\alpha + \beta + \gamma)$

Solution:-

L-H-S

$$= \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} \end{array}$$

Here, $C_2 - C_1$

$C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \alpha & \beta - \alpha & \gamma - \alpha \\ \alpha^3 & \beta^3 - \alpha^3 & \gamma^3 - \alpha^3 \end{vmatrix}$$

Here,

$\therefore \beta^3 - \alpha^3 = (\beta - \alpha)(\beta^2 + \alpha^2 + \beta\alpha)$

$\therefore \gamma^3 - \alpha^3 = (\gamma - \alpha)(\gamma^2 + \alpha^2 + \gamma\alpha)$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \alpha^3 & \beta^2+\alpha^2+\beta\alpha & \gamma^2+\alpha^2+\gamma\alpha \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha)[\gamma^2+\cancel{\alpha^2}+\gamma\alpha - \beta^2 - \cancel{\alpha^2} - \beta\alpha]$$

$$= (\beta-\alpha)(\gamma-\alpha)[(\gamma^2-\beta^2)+(\gamma\alpha-\beta\alpha)]$$

$$\because \gamma^2-\beta^2 = (\gamma+\beta)(\gamma-\beta)$$

$$= (\beta-\alpha)(\gamma-\alpha)[(\gamma+\beta)(\gamma-\beta)+\alpha(\gamma-\beta)]$$

Taking $(\gamma-\beta)$ common,

$$= (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)(\alpha+\gamma+\beta)$$

$$= -(\alpha-\beta)(\gamma-\alpha) - (\beta-\gamma)(\alpha+\gamma+\beta)$$

$$= +(\alpha-\beta)(\gamma-\alpha)(\beta-\gamma)(\alpha+\gamma+\beta)$$

Ans:- proved.

14- $\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a-1)^3(a+3)$

Solution:-

$$\begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array}$$
$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} \Rightarrow R_1+(R_2+R_3+R_4)$$

$$\begin{vmatrix} a+3 & a+3 & a+3 & a+3 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}$$

$$= a+3 \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array}$$
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}$$

Here, $C_2 - C_1$, $C_3 - C_1$, $C_4 - C_1$

$$=(a+3)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a-1 & 0 & 0 \\ 1 & 0 & a-1 & 0 \\ 1 & 0 & 0 & a-1 \end{vmatrix}$$

As above determinant is lower traingular.

$$= (a+3)(1)(a-1)(a-1)(a-1)$$
$$= (a+3)(a-1)^3 \text{ Ans:-}$$

ii- $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\right.$$

Solution:-

L-H-S

$$= \begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

Taking a, b, c, d common,

$$= abcd \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d}+1 \end{vmatrix}$$

Now, $R_1 + (R_2 + R_3 + R_4)$

$$abcd \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d}+1 \end{vmatrix}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d}+1 \end{vmatrix} \end{array}$$

Then, $C_2 - C_1,\ C_3 - C_1,\ C_4 - C_1$

$$abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 \\ \frac{1}{c} & 0 & 1 & 0 \\ \frac{1}{d} & 0 & 0 & 1 \end{vmatrix}$$

As above determinant is lower triangular.

$$= 1\,(abcd)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

$$= (abcd)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Hence proved.

iii- $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2$

Solution:-

L-H-S

$$= \begin{array}{c} \begin{matrix} C_1 & C_2 & C_3 & C_4 \end{matrix} \\ \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} \end{array}$$

Now, $C_1 - (1+x)C_4$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ -x & -x & 0 & 1 \\ -x & 0 & y & 1 \\ -x+y+xy & y & y & 1-y \end{vmatrix} \begin{matrix} C_2 - C_4 \\ C_3 - C_4 \end{matrix}$$

$$= -1 \begin{vmatrix} -x & -x & 0 \\ -x & 0 & y \\ -x+y+xy & y & y \end{vmatrix} \; C_2 - C_1$$

$$= - \begin{vmatrix} 0 & -x & 0 \\ -x & 0 & y \\ -x+xy & y & y \end{vmatrix}$$

$= \{-(+x)[-x(y) - y(-x+xy)]\}$

$= -x(-xy + xy - xy^2)$

$= x^2y^2$ Ans:-

proved.

15- $\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+1 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6$

Solution:-

L-H-S

$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+1 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$

Now, $R_1 - R_3$, $R_2 - R_4$, $R_3 - R_4$

$= \begin{vmatrix} a^3-1 & 3a^2-3 & 3a-3 & 0 \\ a^2-1 & a^2+2a-3 & 2a-2 & 0 \\ a-1 & 2a-2 & a-1 & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix}$

$= 1 \begin{vmatrix} (a-1)(a^2+a+1) & 3(a-1)(a+1) & 3(a-1) \\ (a-1)(a+1) & (a-1)(a+3) & 2(a-1) \\ (a-1) & 2(a-1) & (a-1) \end{vmatrix}$

$$= (a-1)^3 \begin{array}{c} \begin{matrix} C_1 & C_2 & C_3 \end{matrix} \\ \begin{vmatrix} a^2+a+1 & 3(a+1) & 3 \\ (a+1) & (a+3) & 2 \\ 1 & 2 & 1 \end{vmatrix} \end{array}$$

Now, $C_1 - C_3$, $C_2 - 2C_3$

$$= (a-1)^3 \begin{vmatrix} a^2+a-2 & 3(a-1) & 3 \\ a-1 & a-1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-1)^3 \begin{vmatrix} (a-1)(a+2) & 3(a-1) & 3 \\ a-1 & a-1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-1)^3 (a-1)^2 \begin{vmatrix} a+2 & 3 \\ 1 & 1 \end{vmatrix}$$

Then, $(a+2)-3 = a-1$

$\Rightarrow$ $1-1=0$

$= (a-1)^5 (a-1) = (a-1)^6$ Ans.