作业十：

29、(1) $\int_0^1 x^2(x^2-2)^2 dx$

$= \int_0^1 x^6-4x^4+4x^2 dx$

$= \frac{x^7}{7} - \frac{4x^5}{5} + \frac{4}{3}x^3 \Big|_0^1 = \frac{1}{7} - \frac{4}{5} + \frac{4}{3} = \frac{15-84+140}{105}$

$= \frac{71}{105}$

(5) $\int_0^{\frac{\pi}{2}} |\sin x - \cos x| dx$

$= \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x \, dx$

$= (\sin x + \cos x)\Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= (\sqrt{2}-1) - (1-\sqrt{2}) = 2\sqrt{2}-2$

30、(1) $\int_{-5}^{2} \frac{dx}{\sqrt[3]{(x-3)^2}} = \int_{-5}^{2} \frac{d(x-3)}{(x-3)^{\frac{2}{3}}} = 3(x-3)^{\frac{1}{3}} \Big|_{-5}^{2}$

$= -3-(-6) = 3$

(5) $\int_0^{\frac{\pi}{4}} \frac{\sin x}{1+\sin x} dx = \int_0^{\frac{\pi}{4}} \frac{\sin x - \sin^2 x}{\cos^2 x} dx$

$= -\int_0^{\frac{\pi}{4}} \frac{d\cos x}{\cos^2 x} - \int_0^{\frac{\pi}{4}} \tan^2 x \, dx$

$= \frac{1}{\cos x}\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^2 x - 1 \, dx$

$= (\sqrt{2}-1) - \left[\int_0^{\frac{\pi}{4}} \sec^2 x dx - \int_0^{\frac{\pi}{4}} 1 \, dx\right]$

$= (\sqrt{2}-1) - \left[\tan x\Big|_0^{\frac{\pi}{4}} - x\Big|_0^{\frac{\pi}{4}}\right]$

$= (\sqrt{2}-1) - (1 - \frac{\pi}{4})$

$= \sqrt{2} - 2 + \frac{\pi}{4}$

31.(1) $\int_0^1 \frac{x dx}{1+\sqrt{x}} \xlongequal{u=\sqrt{x}} \int_0^1 \frac{u^2}{1+u} \cdot 2u \, du$

$= 2\int_0^1 \frac{u^3}{1+u} du$

$= 2\int_0^1 u^2 - \frac{u^2}{1+u} du$

$= \frac{2}{3}u^3\Big|_0^1 - 2\int_0^1 \frac{u^2}{1+u} du$

$= \frac{2}{3} - 2\int_0^1 (u - \frac{u}{1+u}) du$

$= \frac{2}{3} - 2 \times \frac{1}{2}u^2\Big|_0^1 + 2\int_0^1 \frac{u}{1+u} du$

$= \frac{2}{3} - 1 + 2\int_0^1 (1 - \frac{1}{1+u}) du$

$= -\frac{1}{3} + 2u\Big|_0^1 - 2\int_0^1 \frac{1}{1+u} d(u+1)$

$$= -\frac{1}{3} + 2 - 2\ln|1+u|\Big|_0^1$$

$$= \frac{5}{3} - 2\ln 2$$

(5) $$\int_0^2 \frac{dx}{2+\sqrt{4+x^2}} = \int_0^2 \frac{2-\sqrt{4+x^2}}{-x^2}\,dx$$

$$= \int_0^2 \left(2-\sqrt{4+x^2}\right)\,d\left(\frac{1}{x}\right)$$

$$= \frac{2-\sqrt{4+x^2}}{x}\Big|_0^2 + \int_0^2 \frac{1}{x}\cdot\frac{x}{\sqrt{x^2+4}}\,dx$$

$$= \frac{2-2\sqrt{2}}{2} + \ln\left(x+\sqrt{x^2+4}\right)\Big|_0^2$$

$$= 1-\sqrt{2} + \ln(2+2\sqrt{2}) - \ln 2$$

$$= 1-\sqrt{2} + \ln(1+\sqrt{2})$$

$\lim\limits_{x\to 0^+}$ ——

$= \lim\limits_{x\to 0^+} -\frac{x}{\sqrt{4+x^2}}$

$= \lim\limits_{x\to 0}\left(-\frac{1}{\sqrt{\frac{4}{x^2}+1}}\right)$

$= 0$

(9) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+e^x}\,dx = \underbrace{\int_{-\frac{\pi}{2}}^{0} \frac{\sin^2 x}{1+e^x}\,dx}_{I} + \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+e^x}\,dx$$

*定积分对称范围，可考虑拆

$$I \xlongequal{x=-t} \int_0^{\frac{\pi}{2}} \frac{e^t\sin^2 t}{1+e^t}\,dt = \int_0^{\frac{\pi}{2}} \frac{e^x\sin^2 x}{1+e^x}\,dx$$

$$\text{原式} = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2x\,dx$$

$$= \frac{\pi}{4} - \frac{1}{2}\cdot\frac{1}{2}\sin 2x\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

32. (1) $$\int_0^{\frac{1}{2}} \arcsin x\,dx = x\cdot\arcsin x\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x\,d\arcsin x$$

$$= \frac{1}{2}\cdot\frac{\pi}{6} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{12} + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,d(1-x^2)$$

$$= \frac{\pi}{12} + \frac{1}{2}\cdot 2\sqrt{1-x^2}\Big|_0^{\frac{1}{2}}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

(5) $$\int_0^1 e^{\sqrt{x}}\,dx \xlongequal{x=t^2} \int_0^1 e^t\,dt^2$$

$$= 2\int_0^1 t\cdot e^t\,dt$$

$$= 2\int_0^1 t\cdot d(e^t)$$

$$= 2\left(t\cdot e^t\Big|_0^1 - \int_0^1 e^t\,dt\right)$$

$$= 2(\cancel{1}^{e} - e^t|_0^1) = 2(1-e+1)$$

$$e-1 \qquad = 4-2e \times 2$$

$$=2$$

(9) $\int_0^4 x^2\sqrt{4x-x^2}\,dx \xlongequal{x=4\sin^2 t} 4\int_0^{\frac{\pi}{2}} 16\sin^4 t \cdot 4\sin t\cdot\cos t\, d\sin^2 t$

$$= 32\times 16\int_0^{\frac{\pi}{2}} \sin^6 t\cos^2 t\,dt$$

$$= 32\times 16\left[\int_0^{\frac{\pi}{2}} \sin^6 t\,dt - \int_0^{\frac{\pi}{2}} \sin^8 t\,dt\right]$$

$$= 32\times 16\left[\frac{1\times 3\times 5}{6\times 4\times 2} - \frac{7\times 5\times 3\times 1}{8\times 6\times 4\times 2}\right]\cdot\frac{\pi}{2}$$

$$= 20\cdot\frac{\pi}{2} \times \quad 10\pi.$$

33、(1) $f(x)\in C(0,+\infty)$，$f(x)=\sin x+\int_0^{\pi} f(x)\,dx$，求$f(x)$的表达式.

解：$\int_0^{\pi} f(x)\,dx = \int_0^{\pi}\sin x\,dx + \pi\int_0^{\pi} f(x)\,dx$

$$= 2+\pi\int_0^{\pi} f(x)\,dx$$

$$\Rightarrow \int_0^{\pi} f(x)\,dx = \frac{2}{1-\pi}$$

$$\Rightarrow f(x) = \sin x + \frac{2}{1-\pi}$$

34、求定积分：

(1) $\int_1^3 f(x-2)\,dx$，其中 $f(x) = \begin{cases} 1+x^2, & x \le 0 \\ \frac{1}{e^x}, & x > 0 \end{cases}$

解：令 $u = x-2$，$x = u+2$

$$\int_1^3 f(x-2)\,dx = \int_{-1}^1 f(u)\,du$$

$$= \int_{-1}^0 f(u)\,du + \int_0^1 f(u)\,du$$

$$= \int_0^1 e^{-u}\,du + \int_{-1}^0 (1+u^2)\,du$$

$$= -e^{-u}\Big|_0^1 + \left(u+\frac{1}{3}u^3\right)\Big|_{-1}^0$$

$$= (-e^{-1}+1) + 1 + \frac{1}{3} = \frac{7}{3} - \frac{1}{e}$$

(2) $\int_1^4 f(x-2)\,dx$，其中 $f(x)=\begin{cases} xe^{-x^2}, & x\geq 0 \\ \dfrac{1}{1+e^x}, & x<0 \end{cases}$

解：设 $u=x-2$， $x=u+2$

$$\int_1^4 f(x-2)\,dx = \int_{-1}^2 f(u)\,du$$

$$= \int_{-1}^0 f(u)\,du + \int_0^2 f(u)\,du$$

$$= \int_{-1}^0 \frac{1}{1+e^u}\,du + \int_0^2 u e^{-u^2}\,du$$

$$= \int_{-1}^0 1\,du - \int_{-1}^0 \frac{e^u}{1+e^u}\,du + \frac{1}{2}\int_0^2 e^{-u^2}\,du^2$$

$$= 1 - \int_{-1}^0 \frac{1}{1+e^u}\,d(e^u+1) + \frac{1}{2}\cdot\left(-e^{-u^2}\right)\Big|_0^2$$

$$= 1 - \ln(e^u+1)\Big|_{-1}^0 + \frac{1}{2}\left(-e^{-4}+1\right)$$

$$= 1 - \left(\ln 2 - \ln\left(\frac{1}{e}+1\right)\right) - \frac{1}{2}e^{-4} + \frac{1}{2}$$

$$\ln \frac{2}{\frac{e+1}{e}} = \ln \frac{2e}{e+1}$$

$$\ln e - \ln \frac{2e}{e+1} = \ln \frac{e+1}{2}$$

36、已知函数 $f(x)$ 在 $[0,+\infty)$ 上具有二阶连续导数：

(1) 设 $f(2)=1$，$f'(2)=0$，且 $\int_0^2 f(x)dx=4$，求 $\int_0^1 x^2 f''(2x)dx$.

解：$\int_0^1 x^2 f''(2x)dx = \frac{1}{2}\int_0^1 x^2\, d f'(2x)$

$= \frac{1}{2}\left( x^2 f'(2x)\Big|_0^1 - \int_0^1 f'(2x)dx^2 \right)$

$= -\frac{1}{4}\int_0^1 2x f'(2x)dx$

$= -\frac{1}{4}\int_0^1 2x\, d f(2x)$

$= -\frac{1}{4}\left(2x\cdot f(2x)\Big|_0^1 - \int_0^1 f(2x)\, d2x\right)$

$= -\frac{1}{4}\left(2 - \int_0^2 f(x)dx\right) = \frac{1}{2}$

(2) 设 $f(0)=2$，$f(\pi)=1$，求 $\int_0^\pi [f(x)+f''(x)]\sin x\, dx$.

解：原式 $= -\int_0^\pi f(x)\, d\cos x + \underbrace{\int_0^\pi \sin x\, d f'(x)}_{I}$

$I = \sin x f'(x)\Big|_0^\pi - \int_0^\pi \cos x\, d f(x)$

$= -\left(\cos x\cdot f(x)\Big|_0^\pi - \int_0^\pi f(x)\, d\cos x\right) = \int_0^\pi f(x) d\cos x - (-1-2)$

$\Rightarrow$ 原式 $= 3$

37. 已知函数 $f(x)$ 连续，且分别满足下列条件：

(1) 设 $\int_0^x t f(x-t)\,dt = 1-\cos x$，求 $\int_0^{\frac{\pi}{2}} f(x)\,dx$

解：两边同时求导，$\times$ 因为被积函数中含 $x$.

$x f(0) = \sin x$

$u = x-t$，$\int_0^x t f(x-t)\,dt = \int_0^x -(x-u) f(u)\,du$

$= \int_0^x u f(u)\,du - x\int_0^x f(u)\,du$

同求导，$\int_0^x f(u)\,du = \sin x$，$\int_0^{\frac{\pi}{2}} f(u)\,du = 1$.

(2) 设 $f(1)=1$，且 $\int_0^x t f(2x-t)\,dt = \frac{\arctan x^2}{2}$，求 $\int_1^2 f(x)\,dx$.

解，令 $u = 2x-t$，$t = 2x-u$

左 $= -\int_{2x}^{x} (2x-u) f(u)\,du$

$$= \int_x^{2x} (2x-u) f(u)\, du$$

$$= 2x\int_x^{2x} f(u)\,du - \int_x^{2x} u f(u)\,du = \frac{\arctan x^2}{2}$$

$\frac{1}{2} \times \frac{1}{1+x^4} \times 2x$

两边求导：

$$2\int_x^{2x} f(u)\,du + 2x\cdot\left[\cancel{f(2x)\cdot 2} - f(x)\right] - \left[\cancel{4x f(2x)} - x f(x)\right]$$

$$= \frac{x}{1+x^4}$$

$$2\int_x^{2x} f(u)\,du - x f(x) = \frac{x}{1+x^4}$$

代入 $x=1$，$\int_1^2 f(u)\,du = \frac{1}{2} \times \left( 1 \times 1 + \frac{1}{2} \right)$

$$= \frac{3}{4}$$

38. 设函数 $f(x)$ 在 $U(0)$ 可导，且 $f(0)=0$．求极限：

$$\lim_{x\to 0} \frac{\int_0^x t^{n-1} f(x^n - t^n)\, dt}{x^{2n}} \quad (n \in \mathbb{N}_+)$$

$\int_{t=0}^{t=x}$

解：令 $u = x^n - t^n$，$t^n = x^n - u$

$$原式 = \lim_{x\to 0} \frac{\int_0^x f(x^n - t^n)\,dt^n}{n\cdot x^{2n}}$$

$$= \lim_{x\to 0} \frac{-\int_{x^n}^0 f(u)\,du}{n\cdot x^{2n}}$$

$$= \lim_{x\to 0} \frac{\int_0^{x^n} f(u)\,du}{n\cdot x^{2n}}$$

$$= \lim_{x\to 0} \frac{f(x^n)\cdot \cancel{n x^{n-1}}}{2n^2\cdot \cancel{x^{2n-1}}\, x^n}$$

$$= \lim_{x\to 0} \frac{f'(x^n)\cdot \cancel{n x^{n-1}}}{2n\cdot \cancel{n x^{n-1}}} = \frac{f'(0)}{2n}$$

40. 求下列曲线所围成的图形的面积：

(1) $y=\frac{1}{x}$ 与直线 $y=x$ 及 $x=2$；

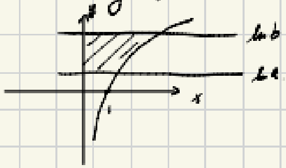
解：$S = \int_1^2 \left(x - \frac{1}{x}\right) dx$

$$= \int_1^2 x\,dx - \int_1^2 \frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2\Big|_1^2 - \ln|x|\Big|_1^2$$

$= 2 - \frac{1}{2} - \ln 2 = \frac{3}{2} - \ln 2$

(5) $y = \ln x$ 与直线 $y = \ln a$, $y = \ln b$ $(b > a > 0)$ 及 $y$ 轴




解: $S = \int_{\ln a}^{\ln b} e^{x}\, dx$

$= e^{\ln b} - e^{\ln a} = b - a$

43、求下列曲线围成的图形的面积.

(1) 星形线 $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$

解: $x \in [-a, a]$, $\frac{1}{2} S = \int_{-a}^{a} a\sin^3 t\, dx$

$= \int_{\pi}^{0} a\sin^3 t\, d(a\cos^3 t)$

$= \int_{\pi}^{0} 3a^2 \sin^3 t \cos^2 t \cdot (-\sin t)\, dt$

$= 3a^2 \int_{0}^{\pi} \sin^4 t \cos^2 t\, dt$

$= 6a^2 \left[ \int_{0}^{\frac{\pi}{2}} \sin^4 t\, dt - \int_{0}^{\frac{\pi}{2}} \sin^6 t\, dt \right]$

$$= 6a^2\left[\frac{3\times1}{4\times2} - \frac{5\times3\times1}{6\times4\times2}\right]\times\frac{\pi}{2}$$

$$= 3a^2\pi\cdot\frac{5}{2\cancel{5}\times4\times2}$$

$$= \frac{3a^2\pi}{16}$$

$$\Rightarrow S = \frac{3a^2\pi}{8}$$

45. 求下列曲线所围成的图形的面积：

(1) 心脏线 $r = 2a(1-\cos\theta)$ $(a>0)$

解：参 $\begin{cases} x = 2a(1-\cos\theta)\cos\theta = 2a\cos\theta - 2a\cos^2\theta \\ y = 2a(1-\cos\theta)\sin\theta = 2a\sin\theta - a\sin2\theta \end{cases}$

$$S = -\int_0^{2\pi} 2a\sin\theta - a\sin2\theta \ d(2a\cos\theta - 2a\cos^2\theta)$$

$$= 2a^2\int_0^{2\pi}(2\sin\theta - \sin2\theta)(-\sin\theta + \sin2\theta)\,d\theta$$

$$= -2a^2\int_0^{2\pi}(-2\sin^2\theta + 2\sin\theta\sin2\theta + \sin\theta\sin2\theta - \sin^2 2\theta)\,d\theta$$

$$= -2a^2\left[\int_0^{2\pi}(\cos2\theta - 1)\,d\theta - \frac{1}{2}\int_0^{2\pi}(1-\cos2\theta)\,d\theta - 6\int_0^{2\pi}\overset{(1-\cos^2\theta)}{\sin^2\theta}\,d(\cos\theta)\right]$$

$(1-u^2)\ du$

$$= -2a^2\left[\left(\frac{1}{2}\sin 2\theta-\theta\right)\Big|_0^{2\pi}-\frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}-6\left(\cos\theta-\frac{1}{3}\cos^3\theta\right)\Big|_0^{2\pi}\right]$$

$$= -2a^2\left[-2\pi-\pi-6\left(1-\frac{1}{3}-1+\frac{1}{3}\right)\right]$$

$$= 6a^2\pi$$

(2) 双纽线 $r^2=a^2\cos 2\theta$

解：参：$\begin{cases} x=a\sqrt{\cos 2\theta}\cos\theta \\ y=a\sqrt{\cos 2\theta}\sin\theta \end{cases}$ （第一象限）

$$\frac{1}{4}S=-\int_0^{\frac{\pi}{4}} a\sqrt{\cos 2\theta}\sin\theta\, d\left(a\sqrt{\cos 2\theta}\cos\theta\right)$$

$$= -\int_0^{\frac{\pi}{4}} a^2\sqrt{\cos 2\theta}\sin\theta\cdot\left[\frac{-\sin 2\theta\cos\theta}{\sqrt{\cos 2\theta}}-\sqrt{\cos 2\theta}\sin\theta\right]d\theta$$

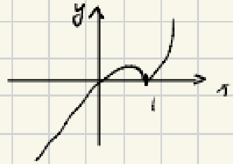
$$= -a^2\int_0^{\frac{\pi}{4}}\left(-\sin 2\theta\sin\theta\cos\theta-\cos 2\theta\sin^2\theta\right)d\theta$$

$$= a^2\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}\sin^2 2\theta+\frac{\cos 2\theta-\cos^2 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{4}}\left(\cos 2\theta-\cos 4\theta\right)d\theta$$

50、用"薄壳法"求下列各旋转体体积：

(1) 由曲线 $y=x(x-1)^2$ 与 $x$ 轴所围成的图形绕 $y$ 轴旋转所得的旋转体



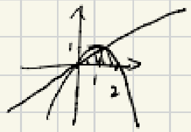

解：$V = 2\pi\int_0^1 x^2(x-1)^2\,dx$

$$= 2\pi\int_0^1 x^4-2x^3+x^2\,dx$$

$$= 2\pi\left(\frac{1}{5}x^5-\frac{1}{2}x^4+\frac{1}{3}x^3\right)\Bigg|_0^1$$

$$= 2\pi\cdot\frac{6-15+10}{30} = \frac{\pi}{15}$$

(2) 由抛物线 $y=2x-x^2$ 与直线 $y=x$ 及 $x$ 轴所围成的图形绕 $y$ 轴旋转所得的旋转体




$x(2-x)$

解：

$2x-x^2=x$

$x=0$ or $1$

$$V = 2\pi \int_0^1 x^2\,dx + 2\pi \int_1^2 2x^2 - x^3\,dx$$

$$= 2\pi \cdot \frac{1}{3}x^3 \Big|_0^1 + 2\pi \left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_1^2$$

$$= \frac{2\pi}{3} + 2\pi\left(\frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{2\pi}{3} + 2\pi \cdot \frac{11}{12} = \frac{5}{2}\pi$$

64、讨论下列反常积分的敛散性，如果收敛求出它的值.

(1) $$\int_0^{+\infty} e^{-ax}\,dx \ (a>0) = \lim_{m\to+\infty} \int_0^m e^{-ax}\,dx$$

$$= \lim_{m\to+\infty} -\frac{1}{a}e^{-ax}\Big|_0^m$$

$$= -\frac{1}{a}\lim_{m\to+\infty}\left(e^{-am} - 1\right)$$

$$= \frac{1}{a}$$ 收敛

(5) $\int_1^{+\infty} \frac{\arctan x}{x^2} dx = -\int_1^{+\infty} \arctan x \, d\frac{1}{x}$

$= \lim\limits_{m\to+\infty} -\int_1^m \arctan x \, d\frac{1}{x}$

$= \lim\limits_{m\to+\infty} -\left( \frac{\arctan x}{x}\Big|_1^m - \int_1^m \frac{1}{x^3+x} dx \right)$

$= -\left(1-\frac{\pi}{4}\right) + \lim\limits_{m\to+\infty} \int_1^m \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$

$= \frac{\pi}{4} - 1 + \lim\limits_{m\to+\infty} \left( \ln|x| \Big|_1^m - \frac{1}{2}\ln(x^2+1)\Big|_1^m \right)$

$\frac{1}{\sqrt{1+\frac{1}{m^2}}}$

$= \frac{\pi}{4} - 1 + \lim\limits_{m\to+\infty} \ln\frac{x}{\sqrt{x^2+1}} \Big|_1^m$

$= \frac{\pi}{4} - 1 + \left(1 - \ln\frac{\sqrt{2}}{2}\right)$

$= \frac{\pi}{4} - \ln\frac{\sqrt{2}}{2}$

(9) $\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} = 2\int_0^{+\infty} \frac{dx}{(1+x^2)^2}$ $\qquad -\frac{2x^2}{(1+x^2)^2}$

$= 2\left[ \int_0^{+\infty} \frac{1}{1+x^2} dx - \int_0^{+\infty} \frac{x^2}{(1+x^2)^2} dx \right]$

$= 2\lim\limits_{m\to+\infty} \arctan x \Big|_0^m + \int_0^{+\infty} x \, d\left(\frac{1}{1+x^2}\right)$

$$= 2\times\frac{\pi}{2} + \lim_{m\to+\infty}\left(\frac{x}{1+x^2}\Big|_0^m - \int_0^m \frac{1}{1+x^2}dx\right)$$

$$= 2\times\frac{\pi}{2} + \lim_{m\to+\infty}\left(\frac{1}{\frac{1}{x}+x}\Big|_0^m - \arctan x\Big|_0^m\right)$$

$$= \cancel{\frac{\pi}{2}}\,\pi + 0 - \frac{\pi}{2} = \cancel{0}\,\frac{\pi}{2}.\ \text{收敛}$$

67、求由笛卡尔叶形线 $x^3+y^3-3axy=0\ (a>0)$

所围图形的面积.

解：令 $y=tx$，$x^3+t^3x^3-3atx^2=0$

$(t^3+1)x^3-3atx^2=0\quad (x\neq 0)$

$$x=\frac{3at}{t^3+1}\le\frac{3a}{2}$$

$$y=\frac{3at^2}{t^3+1}$$

$$S=\int_{t=0}^{t=\frac{3a}{2}}\frac{3at^2}{t^3+1}\,d\frac{3at}{t^3+1}$$

$$=\int_0^1\frac{3at}{t^3+1}\cdot\frac{3at^3+3a-6at^3}{(t^3+1)^2}\,dt$$

$$= \int_0^1 \frac{9a^2 t(1-t^2)}{(t^2+1)^3} \, dt \qquad ?$$

# 作业十一

3、判别下列级数的敛散性，并求出其中收敛级数的和.

(1) $\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{6^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} + (-1)^n \sum_{n=1}^{\infty} \frac{1}{3^n}$

$= 1 + (-1)^n \sum_{n=1}^{\infty} \frac{1}{3^n}$

由于 $0 < \frac{1}{3^{n+1}} \leq \frac{1}{3^n}$，$\lim_{n\to\infty} \frac{1}{3^n} = 0$

故原级数收敛，和为 $\frac{3}{4}$

(3) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

$S_n = \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$，故 $S_n$ 存在上界 $\frac{3}{2}$

$\Rightarrow$ 原级数收敛，和为 $\frac{3}{4}$

(5) $\sum_{n=1}^{\infty} \frac{n}{2n+1}$，由于 $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$

发散

(7) $\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)} = \sum_{n=1}^{\infty}$ $= \frac{1}{4}$

5、(1) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ ，记为 $a_n$

令 $b_n = \frac{1}{n}$ ，则 $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln(n+1)} = +\infty$

又 $\sum_{n=1}^{\infty} b_n$ 发散，故原级数发散

(3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+1)}}$ ，令 $b_n = \frac{1}{\sqrt{n^3}}$

由 $\sum_{n=1}^{\infty} b_n$ 收敛，$\frac{1}{\sqrt{n(n^2+1)}} < \frac{1}{\sqrt{n^3}}$ ，故 $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+1)}}$ 收敛

(5) $\sum_{n=1}^{\infty} \frac{1}{n \cdot \sqrt[n]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$

$\Rightarrow$ 相同敛散性

由于 $1+\frac{1}{n} > 1$ ，故级数收敛 × 发散

(7) $\sum_{n=1}^{\infty} \frac{a^n}{1+a^{2n}} \ (a>0) = \sum_{n=1}^{\infty} \frac{1}{a^n + \frac{1}{a^n}}$

$> \sum_{n=1}^{\infty} \frac{1}{(a^{\frac{n}{2}} + \frac{1}{a^{\frac{n}{2}}})^2}$ ×

$$= \frac{a^2}{2} \cdot \left( \frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta \right) \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{a^2}{2} \cdot \left( \frac{1}{2} - 0 \right) = \frac{a^2}{4} \quad \Rightarrow \quad S = a^2$$

46、求下列曲线所围成的图形的公共部分的面积

(2) $r = \sqrt{2}\sin\theta$ 及 $r^2 = \cos 2\theta$

$$\begin{cases} x = \sqrt{2}\sin\theta\cos\theta = \frac{\sqrt{2}}{2}\sin 2\theta \\ y = \sqrt{2}\sin^2\theta = \frac{\sqrt{2}}{2}(1-\cos 2\theta) \end{cases}$$

解：$\begin{cases} r = \sqrt{2}\sin\theta \\ r^2 = \cos 2\theta \end{cases} \Rightarrow 2\sin^2\theta = \cos 2\theta$

$$1 - \cos 2\theta = \cos 2\theta$$

$$\cos 2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ 或 } \frac{2}{3}\pi$$

$$\Rightarrow S = 2\int_0^{\frac{\pi}{6}} \frac{1}{2}(\sqrt{2}\sin\theta)^2 d\theta + 2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2}\cos 2\theta \, d\theta = \frac{\pi}{6} + \frac{1}{2} - \frac{\sqrt{3}}{2}$$

48. 求下列各立体的体积：

(1) 以椭圆域 $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ $(a>b>0)$ 为底面，且垂直于长轴的截面都是等边三角形的立体.

$$y^2 = b^2 - \frac{b^2}{a^2}x^2$$

解： 

$$V = \int_{-a}^{a} \frac{\sqrt{3}}{4} \cdot 4\left(b^2 - \frac{b^2}{a^2}x^2\right) dx$$

$$= \sqrt{3}\left[\int_{-a}^{a} b^2 dx - \frac{b^2}{a^2}\int_{-a}^{a} x^2 dx\right]$$

$$= \sqrt{3}\left[2ab^2 - \frac{b^2}{a^2} \cdot \frac{1}{3}x^3 \Big|_{-a}^{a}\right]$$

$$= 2\sqrt{3}ab^2 - \frac{2\sqrt{3}b^2}{3\cancel{a^2}} a^{\cancel{3}} = \frac{4\sqrt{3}}{3}ab^2$$

(2) 求曲面 $y^2+z^2 = e^{-2x}$ 与平面 $x=0, x=1$ 所围成的体积.

解： $$V = \int_0^1 \pi e^{-2x} dx$$

$$= \pi \cdot \left(-\frac{1}{2}\right) e^{-2x} \Big|_0^1$$

$$= -\frac{1}{2}\pi\left(e^{-2} - 1\right)$$

13、(1) $\sum_{n=1}^{\infty}\left(\frac{1-3n}{3+4n}\right)^n$

$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{3n-1}{4n+3}\right)^n$ ，令 $a_n=\left(\frac{3n-1}{4n+3}\right)^n>0$

$\lim_{n\to\infty}\sqrt[n]{a_n}=\frac{3}{4}$ ，$\Rightarrow\sum_{n=1}^{\infty}a_n$ 收敛

$\Rightarrow$ 原级数绝对收敛

(5) $\sum_{n=1}^{\infty}\sin(\pi\sqrt{n^2+1})$

$a<1$ 时，$\sum_{n=1}^{\infty}\frac{a^n}{1+a^{2n}} < \sum_{n=1}^{\infty} a^n$（收敛），收敛

$a=1$ 时，$\lim_{n\to+\infty}\frac{a^n}{1+a^{2n}} \neq 0$，发散

$a>1$ 时，$\sum_{n=1}^{\infty}\frac{a^n}{1+a^{2n}} < \sum_{n=1}^{\infty}\frac{a^n}{a^{2n}}$（收敛），收敛

6.（1）$\sum_{n=1}^{\infty}\frac{n^3}{3^n}$，$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}}$

$$= \lim_{n\to\infty}\frac{1}{3}\left(1+\frac{1}{n}\right)^3$$

$$< \frac{e}{3} < 1$$

$\Rightarrow$ 收敛

（3）$\sum_{n=1}^{\infty}\frac{n^{n+1}}{(n+1)!}$，$\lim_{n\to\infty}\frac{n^{n+1}}{(n+1)!} > 1$，故发散

（5）$\sum_{n=1}^{\infty}\frac{a^n}{\ln(n+1)}$ $(a>0)$，$\lim_{n\to\infty}\frac{\frac{a^{n+1}}{\ln(n+2)}}{\frac{a^n}{\ln(n+1)}} = \lim_{n\to\infty} a\cdot\frac{\ln(n+1)}{\ln(n+2)}$

$= a$

$\Rightarrow$ ① $a\in(0,1)$，收敛 ② $a\in(1,+\infty)$，发散 ③ $a=1$，发散

（7）$\sum_{n=1}^{\infty}(\sqrt[n]{3}-1)^n$，$\lim_{n\to\infty}\sqrt[n]{(\sqrt[n]{3}-1)^n} = \lim_{n\to\infty}(\sqrt[n]{3}-1) = 0$

$\Rightarrow$ 收敛

(9) $\sum\limits_{n=1}^{\infty}\left(2n\arcsin\frac{1}{n}\right)^{\frac{n}{2}}$

$\lim\limits_{n\to\infty}\sqrt[n]{\left(2n\arcsin\frac{1}{n}\right)^{\frac{n}{2}}}=\lim\limits_{n\to\infty}\left(2n\cdot\frac{1}{n}\right)^{\frac{1}{2}}=\sqrt{2}>1$

$\Rightarrow$ 发散

7. (1) $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n}$

$\int_{1}^{+\infty}\frac{\ln x}{x}\,dx=\frac{1}{2}(\ln x)^2\Big|_{1}^{+\infty}=+\infty \quad \left(\frac{\ln x}{x}\text{ 在 }(0,+\infty)\searrow\right)$

$\Rightarrow$ 发散

(3) $\sum\limits_{n=1}^{\infty}\frac{\arctan n}{n^2+1}$

$\int_{1}^{+\infty}\frac{\arctan x}{x^2+1}\,dx=\frac{1}{2}(\arctan x)^2\Big|_{1}^{+\infty}=\frac{\pi^2}{8}-\frac{\pi^2}{32}$

$\Rightarrow$ 收敛

9. (1) $\sum\limits_{n=1}^{\infty}\frac{1+2^n}{1+3^n}$, $\lim\limits_{n\to\infty}\sqrt[n]{\frac{1+2^n}{1+3^n}}=\frac{2}{3}\in[0,1)$

$\Rightarrow$ 收敛

(3) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(n+1)\,n^2}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^2}{(n+2)(n+1)^2} \times \frac{(n+1)\,n^2}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,n^2}{n+2} = +\infty$$

$\Rightarrow$ 发散

(5) $\sum_{n=1}^{\infty} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n}$，由于 $\lim_{n\to\infty} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n}$

$$= \lim_{n\to\infty} n^{\frac{1}{n}} \cdot \lim_{n\to\infty} \frac{n^n}{(n+\frac{1}{n})^n}$$

$$= 1$$

$\Rightarrow$ 发散

(7) $\sum_{n=1}^{\infty} \frac{1}{2^{\sqrt{n}}}$，$\lim_{n\to\infty} \frac{\frac{1}{2^{\sqrt{n}}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{2^{\sqrt{n}}}$

$$= \lim_{n\to\infty} \frac{2n}{2^{\sqrt{n}} \cdot \ln 2}$$

$$= \lim_{n\to\infty} \frac{2}{2^{\sqrt{n}} \cdot (\ln 2)^2} = 0$$

由 $\sum_{n=1}^{\infty} \frac{1}{n^2}$ 收敛，知原级数收敛

10、证明：

(1) 若 $a_n \geq 0$，且 $\sum_{n=1}^{\infty} a_n$ 收敛，则 $\sum_{n=1}^{\infty} a_n^2$ 也收敛.

证：$\lim_{n\to\infty} \frac{a_n^2}{a_n} = \lim_{n\to\infty} a_n$，由 $\sum_{n=1}^{\infty} a_n$ 收敛，知

$$\lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n^2 \text{ 与 } \sum_{n=1}^{\infty} a_n \text{ 具有相同敛散性}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n^2 \text{ 收敛.} \quad \square$$

(2) 若 $a_n \geq 0$，且数列 $\{na_n\}$ 收敛，则 $\sum_{n=1}^{\infty} a_n^2$ 收敛.

证

(3) 若 $a_n \geq 0$，$b_n \geq 0$，且 $\sum_{n=1}^{\infty} a_n$ 和 $\sum_{n=1}^{\infty} b_n$ 都收敛，则 $\sum_{n=1}^{\infty} a_n b_n$ 和 $\sum_{n=1}^{\infty} (a_n+b_n)^2$ 收敛

证：$\lim_{n\to\infty} \frac{a_n b_n}{a_n} = \lim_{n\to\infty} b_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n b_n$ 与 $\sum_{n=1}^{\infty} a_n$ 敛散性相同，

故收敛

$\sum_{n=1}^{\infty} a_n$，$\sum_{n=1}^{\infty} b_n$ 收敛，故 $\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n$ 收敛

由(1)得，$\sum_{n=1}^{\infty}(a_n+b_n)^2$ 收敛 □

(4) 若 $a_n \geq 0$，且 $\sum_{n=1}^{\infty} a_n^2$ 收敛，则 $\sum_{n=1}^{\infty}\frac{a_n}{n}$ 也收敛

证：由 $\sum_{n=1}^{\infty} a_n^2$ 收敛，得 $\lim_{n\to\infty}\sqrt[n]{a_n^2} \in [0,1)$

$$\Rightarrow \lim_{n\to\infty}\sqrt[n]{a_n} \in [0,1)$$

$$\Rightarrow \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n}} = \lim_{n\to\infty}\sqrt[n]{a_n} \in [0,1) \Rightarrow \sum_{n=1}^{\infty}\frac{a_n}{n} \text{ 收敛} \quad □$$

(5) 若数列 $\{na_n\}$ 收敛，且级数 $\sum_{n=1}^{\infty} n(a_n-a_{n-1})$ 收敛 $(a_0=0)$，则级数 $\sum_{n=1}^{\infty} a_n$ 也收敛

11、利用不等式 $\frac{1}{2\sqrt{n}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{2n+1}}$，证明：级数 $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$ 发散而级数 $\sum_{n=2}^{\infty} \frac{(2n-3)!!}{(2n)!!}$ 收敛

证：由 $\frac{(2n-1)!!}{(2n)!!} > \frac{1}{2\sqrt{n}}$，且 $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ 发散

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \text{ 发散}$$

由 $\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{2n+1}}$，得 $\frac{(2n-3)!!}{(2n)!!} < \frac{1}{\sqrt{2n+1}\,(2n-1)}$

$(2n-1>0)$

取 $b_n = \frac{1}{n^{\frac{3}{2}}}$，$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{2n+1}\,(2n-1)}}{\frac{1}{n^{\frac{3}{2}}}} = \frac{1}{2\sqrt{2}}$，且 $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ 收敛

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}\,(2n-1)}$ 收敛，又 $\frac{(2n-3)!!}{(2n)!!} < \frac{1}{\sqrt{2n+1}\,(2n-1)}$

$\Rightarrow \sum_{n=2}^{\infty} \frac{(2n-3)!!}{(2n)!!}$ 收敛　　证

12、(1) $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}\ln n}{n}$

由于 $0 < \frac{\ln(n+1)}{n+1} \leq \frac{\ln n}{n}$，且 $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

$\Rightarrow$ 原级数收敛，且是~~绝~~对收敛

条件　　$\frac{\ln n}{n} > \frac{1}{n}$

(5) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2n-1)!!}{(2n)!!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)!!}{(2n+2)!!} \times \frac{(2n)!!}{(2n-1)!!}$$

$$= \lim_{n\to\infty} \frac{2n+1}{2n+2} = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \text{ 发散}$$

$\Rightarrow$ 原级数发散 ✗

$$a_{n+1} = \frac{(2n+1)!!}{(2n+2)!!} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{2n+1}{2n+2} < a_n$$

下证 $\lim_{n\to\infty} a_n = 0$

$$a_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{2n-1}{2n}$$

$$\le \frac{2}{3} \cdot \frac{4}{5} \cdot \cdots \cdot \frac{2n}{2n+1} = \frac{1}{a_n(2n+1)}$$

$\Rightarrow 0 \le a_n \le \frac{1}{a_n(2n+1)}$，$n \to \infty$，$a_n = 0 \Rightarrow$ 收敛

条件收敛

作业 P2

14. 求下列函数项级数的收敛域

(1) $\sum_{n=1}^{\infty} \frac{\sin^n x}{n^2}$，令 $y=\sin x$，原级数 $= \sum_{n=1}^{\infty} \frac{y^n}{n^2}$

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{1}{\sqrt[n]{n}}\right)^2 = 1 \Rightarrow R=1$

当 $y=\pm1$ 时，收敛

$\Rightarrow y\in[-1,1] \Leftrightarrow x\in\mathbb{R}$

收敛域

(2) $\sum_{n=1}^{\infty} (\ln x)^n$，令 $y=\ln x$，$\Rightarrow R=1$

$y=\pm1$ 时，不收敛 $\Rightarrow y\in(-1,1) \Leftrightarrow I=\left(\frac{1}{e}, e\right)$

15. 求下列幂级数的收敛域

(1) $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$，$a_n = \frac{1}{n(n+1)}$ $\Rightarrow \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n(n+1)}{(n+1)(n+2)}$
$= 1$

$\Rightarrow R=1$

$x=1$ 时，$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$，故收敛

$x=-1$ 时，由莱布尼兹判别法，收敛

$\Rightarrow I=[-1,1]$

(3) $\sum_{n=1}^{\infty}\frac{2^n}{n^2+1}x^n$ ，$a_n=\frac{2^n}{n^2+1}$ .

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\frac{2^{n+1}}{(n+1)^2+1}}{\frac{2^n}{n^2+1}}=\lim_{n\to\infty}2\cdot\frac{n^2+1}{(n+1)^2+1}=2$$

$\Rightarrow R=\frac{1}{2}$

$x=\frac{1}{2}$ 时，原级数 $=\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ ，收敛

$x=-\frac{1}{2}$ 时，由莱布尼兹判别法，收敛

$\Rightarrow I=[-\frac{1}{2},\frac{1}{2}]$

(5) $\sum_{n=1}^{\infty}\frac{x^n}{(n+1)^p}$ ，$a_n=\frac{1}{(n+1)^p}$ .

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{(n+1)^p}{(n+2)^p}=1\Rightarrow R=1$$

$x=1$ 时，$\begin{cases} p>1 & 收敛 \\ p\le 1 & 发散 \end{cases}$ $x=-1$ 时 $\begin{cases} p>0 & 收敛 \\ p\le 0 & 发散 \end{cases}$

$$\Rightarrow I=\begin{cases}[-1,1] & p>1\\ [-1,1) & 0<p\le 1\\ (-1,1) & p\le 0\end{cases}$$

16、设 $\sum_{n=0}^{\infty} a_n x^n$ 与 $\sum_{n=0}^{\infty} b_n x^n$ 的收敛半径分别为 $R_1$ 和 $R_2$，且 $R_1 \neq R_2$，证明 $\sum_{n=0}^{\infty}(a_n+b_n)x^n$ 的收敛半径 $R=\min\{R_1, R_2\}$，若 $R_1=R_2$，则上结论是否成立？

证：$\sum_{n=0}^{\infty}(a_n+b_n)x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n$

当 $|x|<\min\{R_1,R_2\}$ 时，$\sum a_n x^n$，$\sum b_n x^n$ 收敛 $\Rightarrow \sum(a_n+b_n)x^n$ 收敛

当 $|x|>\min\{R_1,R_2\}$ 时，不妨令 $R_1<R_2$，则 $\sum b_n x^n$ 发散

$\Downarrow$

$\sum(a_n+b_n)x^n$ 发散

$\Rightarrow$ 收敛半径为 $\min\{R_1, R_2\}$

$R_1=R_2$ 时，结论成立，因为 $\sum a_n x^n$，$\sum b_n x^n$ 收敛

$\Downarrow$

$\sum(a_n+b_n)x^n$ 收敛

17、利用幂级数的和函数的分析性质，求下列级数在各自收敛域上的和函数。

(1) $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\int_0^x t^{2n}\,dt$

$= \int_0^x \sum_{n=0}^{\infty} t^{2n}\,dt$ $\qquad \sum_{n=0}^{\infty} t^{2n} = 1^2 + t^2 + \cdots$

$$= \int_0^x \left(\frac{1}{1-t}\right)^2 dt \qquad \int_0^x \frac{1}{1-t^2}\,dt$$

$$= \int_0^x \frac{1}{(t-1)^2}\, d(t-1) \qquad = \frac{1}{2}\ln\frac{1+x}{1-x} \quad (-1<x<1)$$

$$= -\frac{1}{x-1}$$

(3) $\sum_{n=1}^{\infty}(-1)^{n-1}n^2x^n$ 收敛域为 $(-1,1)$

$$= x\sum_{n=1}^{\infty}(-1)^{n-1}n^2x^{n-1}$$

$$= x\left(\sum_{n=1}^{\infty}(-1)^{n-1}n x^n\right)' = x\left(x\cdot\sum_{n=1}^{\infty}(-1)^{n-1}n x^{n-1}\right)'$$

$$= x\left[x\cdot\left(\sum_{n=1}^{\infty}(-1)^{n-1}x^n\right)'\right]'$$

$$= x\cdot\left[x\cdot\left(\frac{x}{x+1}\right)'\right]'$$

$$= x\cdot\left(x\cdot\frac{x+1-x}{(x+1)^2}\right)'$$

$$= x\cdot\left(\frac{x}{(x+1)^2}\right)'$$

$$= x\cdot\frac{(x+1)^2-x(2x+2)}{(x+1)^4}$$

$$= x\cdot\frac{x^2+2x+1-2x^2-2x}{(x+1)^4} = \frac{x-x^2}{(x+1)^3}$$

$$x\cdot\frac{-x^2+1}{(x+1)^4}$$

$$-x\cdot\frac{(x-1)\cancel{(x+1)}}{(x+1)^{\cancel{4}\,3}}$$

(5) $\sum_{n=0}^{\infty} \frac{2n+1}{n!} x^{2n}$, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2n+3}{(n+1)!} \cdot \frac{n!}{2n+1}$

$= \lim_{n\to\infty} \frac{2n+3}{(n+1)(2n+1)} = \cancel{1}$ 0

收敛域 $[-1,1]$ ✗ $(-\infty, +\infty)$

$= \left( \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!} \right)'$

$= \left( x \cdot \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} \right)'$

$= \left( x \cdot e^{x^2} \right)' = e^{x^2} + 2x^2 e^{x^2}$ ✓

18、利用幂级数的性质求下列级数的和：

(1) $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$, 令 $S(x) = \sum_{n=1}^{\infty} n \cdot x^{n-1}$

$= \left( \sum_{n=1}^{\infty} x^n \right)'$

$= \left( \frac{1}{1-x} - 1 \right)'$

$= \frac{1}{(1-x)^2}$

$\Rightarrow$ 原式 $= S\left(\frac{1}{2}\right) = 4$ ✓

(2) $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \left( \frac{\pi}{4} \right)^{2n+1}$

令 $S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \cdot x^{2n+1}$

$$= \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} \, dt$$

$$= \int_0^x \left(1 - t^2 + (t^2)^2 - \cdots\right) dt$$

$$= \int_0^x \frac{1}{1+t^2} \, dt$$

$$= \arctan t \Big|_0^x$$

$$\Rightarrow S\left(\frac{\pi}{4}\right) = 1 \times \quad \arctan\frac{\pi}{4}$$

(3) $$\sum_{n=1}^{\infty} \frac{2n+1}{9^n} = \sum_{n=1}^{\infty} \frac{2n+1}{3^{2n}}$$

$$\frac{1}{3} S(x) = \sum_{n=1}^{\infty} (2n+1) x^{2n}$$

$$= \left(\sum_{n=1}^{\infty} x^{2n+1}\right)'$$

$$= \left(x \cdot \frac{x^2}{1-x^2}\right)'$$

$$= \frac{3x^2(1-x^2) + x^3 \cdot 2x}{(1-x^2)^2}, \quad S\left(\frac{1}{3}\right) = \frac{13}{32}$$

(4) $\sum_{n=2}^{\infty} \frac{1}{(n^2-1)\, 2^n}$ , 令 $S(x) = \sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)} x^n$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n-1)(n+1)}{(n+2)\,n} = 1$$

$$= \frac{1}{x} \int_0^x \sum_{n=2}^{\infty} \frac{1}{n-1} \cdot t^n \, dt$$

$$= \frac{1}{x} \int_0^x \left( t \cdot \int_0^x \sum_{n=2}^{\infty} t^{n-2} dt \right) dt$$

$$= \frac{1}{x} \int_0^x \left( t \cdot \int_0^x \frac{1}{1-t} dt \right) dt$$

$$= \frac{1}{x} \int_0^x \left( t \cdot \ln \frac{1}{1-x} \right) dt$$

$$= \frac{-\ln(1-x)}{x} \cdot \frac{1}{2} x^2$$

$$= \frac{-x \cdot \ln(1-x)}{2}$$

$$\left( \frac{x^n}{n-1} - \frac{x^n}{n+1} \right) \times \frac{1}{2}$$

$$S\left(\tfrac{1}{2}\right) = \frac{-\ln \frac{1}{2}}{4} = \frac{1}{4} \ln 2$$

19. 将下列函数在给定点 $x_0$ 展开成 $(x-x_0)$ 的幂级数，并指出展开式成立的区间.

(1) $x^2 e^{x^2}$ , $x_0 = 0$

在 $x_0 = 0$ 处展开 $e^{x^2}$ , $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

$\Rightarrow x^2 e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2(n+1)}}{n!}$ , $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0 \Rightarrow I = (-\infty, +\infty)$