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Lense–Thirring precession in Plebański–Demiański Spacetimes

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Abstract An exact expression of Lense–Thirring precession rate is derived for non-extremal and extremal Plebański–Demiański spacetimes. This formula is used to find the exact Lense–Thirring precession rate in various axisymmetric spacetimes i.e., Kerr–Newman, Kerr–de Sitter etc. We also show that if the Kerr parameter vanishes in the Plebański–Demiański spacetime, the Lense–Thirring precession does not vanish due to the existence of NUT charge. To derive the Lense–Thirring precession rate in the extremal Plebański–Demiański spacetime, we first derive the *general extremal condition* for Plebański–Demiański spacetimes. This general result could be applied to obtain the extremal limit in any stationary and axisymmetric spacetimes.

1 Introduction

The axisymmetric vacuum solution of the Einstein equations are used to describe the various characteristics of different spacetimes. The most important and physical spacetime is Kerr spacetime [1], describing a rotating black hole which possesses a finite angular momentum J. The Kerr spacetime with a finite charge Q is expressed as a Kerr–Newman black hole. Actually inclusion of Cosmological constant may arises some complexity in calculations. Without this particular constant the spacetimes possess two horizons, namely event horizon and Cauchy horizon. But the presence of the cosmological constant leads to an extra horizon—the cosmological horizon. All of these spacetimes can be taken as the special cases of the most general axisymmetric spacetime of Petrov type D which was first given by Plebański and Demiański (PD) [2]. This spacetime contains seven parameters—acceleration, mass (M), Kerr parameter (a, angular momentum per unit mass), electric charge (Q_c), magnetic charge (Q_m), NUT parameter (n), and the Cosmological constant. At present, this is the most general known axially symmetric vacuum solution of the Einstein field equation. This solution is important at the present time because it is now being used in many ways. People working in semi-classical quantum gravity have used this type of metric to investigate the pair production of black holes in cosmological backgrounds ([3]). Some people are working to extend this type of solution to higher dimensions. But this spacetime is still not well-understood at the classical level of general relativity. In particular, the physical significance of the parameters employed in the original forms are only properly identified in the most simplified special cases and the most general PD metric covers all spacetimes which are well-known to us till now.

In this paper, we shall investigate an important effect of classical General Relativity in PD spacetime: the dragging of inertial frames which was discovered by Lense and Thirring [4]. It is well-known that any axisymmetric and stationary spacetime with angular momentum (rotation) is known to exhibit an effect called Lense–Thirring (LT) precession whereby locally inertial frames are dragged along the rotating spacetime, making any test gyroscope in such spacetimes *precess* with a certain frequency called the LT precession frequency [4].

More generally, we can say that frame-dragging effect is the property of all stationary spacetimes which may or may not be axisymmetric [5]. It has been discussed in detail about the LT effect in non-axisymmetric stationary spacetimes in a very recent paper by Chakraborty and Majumdar [6] in where the authors have shown that only Kerr parameter is not responsible for the LT precession, NUT parameter is equally important to continue the frame-dragging effect.

Hackmann and Lämmerzahl have shown that the LT precession vanishes (Eq. (45) of [7]) in PD spacetimes (with vanishing acceleration of the gravitating source), if the Kerr parameter $a = 0$. But it is not the actual scenario. We shall show in our present work, the LT precession does not vanish due to the presence of NUT charge n (*angular momentum*

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monopole [8]) in the PD spacetimes with *zero angular momentum* ($J = a = 0$).

It may be noted that the observations of LT precession due to locally inertial frame-dragging have so far focused on spacetimes where the curvatures are small enough; e.g., the LT precession in the earth's gravitational field which was probed recently by Gravity Probe B. There has been so far no attempt to measure LT precession effects due to frame-dragging in strong gravity regimes [9]. Our LT precession formulas for different spacetimes are valid in *strong as well as weak* gravitational field as we do not make any approximation to derive the LT precession rate. We also note that the earth's gravitational is described by Kerr geometry which is physically reliable. But the most general axisymmetric PD spacetime is not physically reliable till now because the existence of some parameters (e.g., NUT parameter (n), Q_e, Q_m) of PD spacetimes are not yet proved.

Lynden-Bell and Nouri-Zonos [10] studied a Newtonian analogue of monopole spacetimes and discussed the observational possibilities for (gravito)magnetic monopoles (n). In fact, the spectra of supernova, quasars, or active galactic nuclei might be good candidates to infer the existence of gravito-magnetic monopoles [11]. The Kerr–Taub-NUT solutions representing relativistic thin disks are of great astrophysical importance since they can be used as models of certain galaxies, accretion disks, the superposition of a black hole and a galaxy or an accretion disk as in the case of quasars.

In this manuscript, we will derive the exact LT precession rates in non-extremal PD spacetimes without invoking the weak field approximation. We will also find the exact LT precession rates for the other axisymmetric stationary spacetimes i.e. Kerr–Newman, Kerr–de Sitter etc. We will further show that the non-vanishing LT precession in 'zero angular momentum' PD spacetimes and find the extremal limit for PD spacetime including others axisymmetric and stationary spacetimes. Finally, we will find the exact LT precession rates in the extremal PD spacetime and others axisymmetric extremal spacetimes.

The structure of the paper as follows. In Sect. 2, we review the general Lense–Thirring precession formula in stationary and axisymmetric spacetimes and derive the exact Lense–Thirring precession rate in PD spacetimes with vanishing acceleration of the gravitating source and discuss the exact LT precession rates in some other stationary, axisymmetric spacetimes as special cases of PD spacetimes. If the Kerr parameter vanishes in PD spacetimes, the frame-dragging effect does not vanish due to the existence of NUT charge. It is shown in a subsections of Sect. 2, as a special case of LT precession in PD spacetimes. In Sect. 3, we derive the more general extremal condition for PD spacetimes and discuss the exact LT precession rates in PD spacetimes and also other various extremal axisymmetric spacetimes as the special cases of PD spacetimes. A short discussion closes the paper in Sect. 4.

2 Derivation of the Lense–Thirring Precession Frequency

The exact LT frequency of precession of test gyroscopes in strongly curved stationary spacetimes, analyzed within a Copernican frame, is expressed as a co-vector given in terms of the timelike Killing vector fields K of the stationary spacetime, as (in the notation of Eq. (1.159) of Ref. [5])

$$\tilde{\Omega} = \frac{1}{2K^2} * (\tilde{K} \wedge d\tilde{K}) \tag{1}$$

or

$$\Omega_\mu = \frac{1}{2K^2}\eta_\mu^{\nu\rho\sigma} K_\nu \partial_\rho K_\sigma, \tag{2}$$

where $\eta^{\mu\nu\rho\sigma}$ represent the components of the volume-form in spacetime and $\tilde{K}$ & $\tilde{\Omega}$ denote the one-form of K & Ω, respectively. $\tilde{\Omega}$ will vanish if and only if $(\tilde{K} \wedge d\tilde{K})$ does. This happens only in a static spacetime. Using the coordinate basis form of $K = \partial_0$, the co-vector components are easily seen to be $K_\mu = g_{\mu 0}$. This co-vector could also be written in the following form:

$$\tilde{K} = g_{00} dx^0 + g_{0i} dx^i \tag{3}$$

Now

$$d\tilde{K} = g_{00,k} dx^k \wedge dx^0 + g_{0i,k} dx^k \wedge dx^i, \tag{4}$$

$$\begin{aligned}(\tilde{K} \wedge d\tilde{K}) &= (g_{00} g_{0i,j} - g_{0i} g_{00,j}) dx^0 \wedge dx^j \wedge dx^i \\ &\quad + g_{0k} g_{0i,j} dx^k \wedge dx^j \wedge dx^i,\end{aligned} \tag{5}$$

$$\begin{aligned}(\tilde{K} \wedge d\tilde{K}) &= (g_{00} g_{0i,j} - g_{0i} g_{00,j}) * \left(dx^0 \wedge dx^j \wedge dx^i\right) \\ &\quad + g_{0k} g_{0i,j} * \left(dx^k \wedge dx^j \wedge dx^i\right)\end{aligned} \tag{6}$$

It can be shown that

$$\begin{aligned}&\left(dx^0 \wedge dx^j \wedge dx^i\right) = \eta^{0jil} g_{l\mu} dx^\mu = -\frac{1}{\sqrt{-g}}\epsilon_{jil} g_{l\mu} dx^\mu \\ &* \left(dx^k \wedge dx^j \wedge dx^i\right) = \eta^{kji0} g_{0\mu} dx^\mu = -\frac{1}{\sqrt{-g}}\epsilon_{kji} g_{0\mu} dx^\mu\end{aligned} \tag{7}$$

From Eq. (6) we get

$$\begin{aligned}&*(\tilde{K} \wedge d\tilde{K}) \\ &\quad = \frac{\epsilon_{ijl}}{\sqrt{-g}}\left[(g_{00} g_{0i,j} - g_{0i} g_{00,j})\left(g_{l0} dx^0 + g_{lk} dx^k\right) \right. \\ &\qquad \left. - g_{0l} g_{0i,j}\left(g_{00} dx^0 + g_{0k} dx^k\right)\right]\end{aligned} \tag{8}$$

Simplifying the above equation we find

$$*(\tilde{K} \wedge d\tilde{K}) = \frac{\epsilon_{ijl}}{\sqrt{-g}}\big[g_{0i,j}(g_{00}g_{kl} - g_{0k}g_{0l}) - g_{0i}g_{kl}g_{00,j}\big]dx^k \tag{9}$$

Now, using Eq. (2) we find that the spatial components of the precession rate (in the chosen frame) is

$$\Omega_k = \frac{1}{2}\frac{\epsilon_{ijl}}{g_{00}\sqrt{-g}}\big[g_{0i,j}(g_{00}g_{kl} - g_{0k}g_{0l}) - g_{0i}g_{kl}g_{00,j}\big] \tag{10}$$

[using $K^2 = g_{00}$].

The vector field corresponding to the LT precession co-vector in (10) can be expressed as

$$\begin{aligned}\Omega &= \Omega^\mu \partial_\mu \\ &= g^{\mu\nu}\Omega_\nu \partial_\mu \\ &= g^{\mu k}\Omega_k \partial_\mu \quad [\text{as } \Omega_0 = 0] \\ &= g^{0k}\Omega_k \partial_0 + g^{nk}\Omega_k \partial_n \\ &= \frac{1}{2}\frac{\epsilon_{ijl}}{\sqrt{-g}}\left[g_{0i,j}\left(\partial_l - \frac{g_{0l}}{g_{00}}\partial_0\right) - \frac{g_{0i}}{g_{00}}g_{00,j}\partial_l\right]\end{aligned} \tag{11}$$

For the axisymmetric spacetime, the only non-vanishing component is $g_{0i} = g_{0\phi}$, $i = \phi$ and $j, l = r, \theta$; substituting these in Eq. (11), the LT precession frequency vector is obtained as

$$\begin{aligned}\Omega_{\mathrm{LT}} = \frac{1}{2\sqrt{-g}}\Bigg[&\left(g_{0\phi,r} - \frac{g_{0\phi}}{g_{00}}g_{00,r}\right)\partial_\theta \\ &- \left(g_{0\phi,\theta} - \frac{g_{0\phi}}{g_{00}}g_{00,\theta}\right)\partial_r\Bigg]\end{aligned} \tag{12}$$

3 Non-extremal Case

3.1 Plebański–Demiański Spacetimes

The PD spacetime is the most general axially symmetric vacuum solution of the Einstein equation. The line element of six parameters (as we consider vanishing acceleration) PD spacetimes can be written as (taking $G = c = 1$) [7]

$$\begin{aligned}ds^2 = &-\frac{\Delta}{p^2}(dt - A\,d\phi)^2 + \frac{p^2}{\Delta}dr^2 + \frac{p^2}{\Xi}d\theta^2 \\ &+ \frac{\Xi}{p^2}\sin^2\theta(a\,dt - B\,d\phi)^2\end{aligned} \tag{13}$$

where

$$\begin{aligned}&p^2 = r^2 + (n - a\cos\theta)^2, \\ &A = a\sin^2\theta + 2n\cos\theta, \qquad B = r^2 + a^2 + n^2 \\ &\Delta = (r^2 + a^2 - n^2)\left(1 - \frac{1}{\ell^2}(r^2 + 3n^2)\right) \\ &\qquad - 2Mr + Q_c^2 + Q_m^2 - \frac{4n^2r^2}{\ell^2} \\ &\Xi = 1 + \frac{a^2\cos^2\theta}{\ell^2} - \frac{4an\cos\theta}{\ell^2}\end{aligned} \tag{14}$$

$\frac{1}{\ell^2} = \Lambda$ denotes the Cosmological constant divided by three (the Cosmological constant Λ' appears in the metric as $\frac{\Lambda'}{3}$ which we can take in a more simplified form as $\frac{\Lambda'}{3} = \Lambda$), represents the Plebański–Demiański(–de Sitter)[1] spacetimes and if ℓ^2 is replaced by $-\ell^2$, it represents the PD–AdS spacetimes. So, our metric $g_{\mu\nu}$ is thus following:

$$g_{\mu\nu} = \begin{pmatrix} -\frac{1}{p^2}(\Delta - a^2\Xi\sin^2\theta) & 0 & 0 & \frac{1}{p^2}(A\Delta - aB\Xi\sin^2\theta) \\ 0 & \frac{p^2}{\Delta} & 0 & 0 \\ 0 & 0 & \frac{p^2}{\Xi} & 0 \\ \frac{1}{p^2}(A\Delta - aB\Xi\sin^2\theta) & 0 & 0 & \frac{1}{p^2}(-A^2\Delta + B^2\Xi\sin^2\theta) \end{pmatrix} \tag{15}$$

The various metric components can be read off from the above metric (15). Likewise,

$$\sqrt{-g} = p^2\sin\theta \tag{16}$$

Substituting the metric components into Eq. (12) we can easily get the LT precession rate in PD spacetimes. But there is a problem in that formulation since the precession formula is in coordinate basis. So, we should transform the precession frequency formula from the coordinate basis to the orthonormal 'Copernican' basis: first note that

$$\Omega_{\mathrm{LT}} = \Omega^\theta \partial_\theta + \Omega^r \partial_r \tag{17}$$

$$\Omega^2_{\mathrm{LT}} = g_{rr}\left(\Omega^r\right)^2 + g_{\theta\theta}\left(\Omega^\theta\right)^2 \tag{18}$$

[1] In PD metric, Λ appears as either "positive" $(+\Lambda)$ or "negative" $(-\Lambda)$. In the whole paper, the PD metric with $+\Lambda$ represents the Plebański–Demiański–de Sitter or PD–dS spacetimes and the PD metric with $-\Lambda$ represents the Plebański–Demiański–anti-de Sitter or PD–AdS spacetimes.

Next, in the orthonormal 'Copernican' basis at rest in the rotating spacetime, with our choice of polar coordinates, Ω_{LT} can be written as

$$\boldsymbol{\Omega}_{\mathrm{LT}} = \sqrt{g_{rr}}\Omega^r \hat{r} + \sqrt{g_{\theta\theta}}\Omega^\theta \hat{\theta} \tag{19}$$

where $\hat{\theta}$ is the unit vector along the direction θ. Our final result of LT precession in non-accelerating PD spacetime is then

$$\begin{aligned}\boldsymbol{\Omega}^{\mathrm{PD}}_{\mathrm{LT}} = {}& \frac{\sqrt{\Delta}}{p}\biggl[\frac{a(\Xi\cos\theta + (2n - a\cos\theta)\frac{a}{\ell^2}\sin^2\theta)}{\Delta - a^2\Xi\sin^2\theta} \\ &- \frac{a\cos\theta - n}{p^2}\biggr]\hat{r} + \frac{\sqrt{\Xi}}{p}a\sin\theta \\ &\times \biggl[\frac{r - M - \frac{r}{\ell^2}(a^2 + 2r^2 + 6n^2)}{\Delta - a^2\Xi\sin^2\theta} - \frac{r}{p^2}\biggr]\hat{\theta}\end{aligned} \tag{20}$$

From the above expression we can easily derive the LT precession rates for various axisymmetric stationary spacetimes as special cases of PD spacetime.

3.2 Special Cases

(a) *Schwarzschild and Schwarzschild–de Sitter spacetimes*: As the Schwarzschild and Schwarzschild–de Sitter spacetimes both are static and $a = \Lambda = Q_c = Q_m = n = 0$, the inertial frames are not dragged along it. So, we cannot see any LT effect in these spacetimes. This is a very well-known feature of static spacetimes.

(b) *Kerr Spacetimes*: LT precession rate for non-extremal Kerr spacetimes is already discussed in detail in the paper by Chakraborty and Majumdar [6]. Setting $\Lambda = Q_c = Q_m = n = 0$ in Eq. (20) we can recover that result (Eq. (19) of [6]) which is applicable for Kerr spacetimes.

(c) *Kerr–Newman spacetime*: Rotating black hole spacetimes with electric charge Q_c and magnetic charge Q_m are described by Kerr–Newman metric, which is quite important in General Relativity. Setting $\Lambda = n = 0$ in Eq. (20), we can easily get the LT precession in Kerr–Newman spacetime. It is thus (taking $Q_c^2 + Q_m^2 = Q^2$),

$$\begin{aligned}\boldsymbol{\Omega}^{\mathrm{KN}}_{\mathrm{LT}} = {}& \frac{a}{\rho^3(\rho^2 - 2Mr + Q^2)}\bigl[\sqrt{\Delta}\bigl(2Mr - Q^2\bigr)\cos\theta\hat{r} \\ &+ \bigl(M\bigl(2r^2 - \rho^2\bigr) + rQ^2\bigr)\sin\theta\hat{\theta}\bigr]\end{aligned} \tag{21}$$

In the Kerr–Newman spacetime,

$$\Delta = r^2 - 2Mr + a^2 + Q^2 \quad \text{and} \quad \rho^2 = r^2 + a^2\cos^2\theta \tag{22}$$

From the expression (21) of LT precession in Kerr–Newman spacetime we can see that at the polar plane, the LT precession vanishes for the orbit $r = \frac{Q^2}{2M}$, though the spacetime is rotating ($a \neq 0$). So, if a gyroscope rotates in a polar orbit of radius $r = \frac{Q^2}{2M}$ in this spacetime, the gyroscope does not experience any frame-dragging effect. So, if any experiment is performed in future by which we cannot see any LT precession in that spacetime, it may be happened that the specified spacetime is a Kerr–Newman black hole and the gyroscope is rotating in a polar orbit whose radius is $r = \frac{Q^2}{2M}$. This is a very interesting feature of the Kerr–Newman geometry. Though the spacetime is rotating with the angular momentum J, the nearby frames are not dragged along it. Without this particular orbit the LT precession is continued in everywhere in this spacetimes.

(d) *Kerr–de Sitter spacetimes*: Kerr–de Sitter spacetime is more realistic, when we do not neglect the Cosmological constant parameter (though its value is very small, it may be very useful in some cases, where we need to very precise calculation). Setting $n = Q_c = Q_m = 0$, we get the following expression for Kerr–de Sitter spacetime:

$$\begin{aligned}\boldsymbol{\Omega}^{\mathrm{KdS}}_{\mathrm{LT}} = {}& \frac{a}{\rho^3(\rho^2 - 2Mr - \frac{1}{\ell^2}(a^4 + a^2r^2 - a^4\sin^2\theta\cos^2\theta))} \\ &\times \biggl[\sqrt{\Delta}\biggl(2Mr + \frac{1}{\ell^2}\rho^4\biggr)\cos\theta\hat{r} \\ &+ \sqrt{\Xi}\biggl[M\bigl(2r^2 - \rho^2\bigr) \\ &+ \frac{r}{\ell^2}\bigl\{a^4 + a^2r^2 - a^4\sin^2\theta\cos^2\theta \\ &- \rho^2\bigl(a^2 + 2r^2\bigr)\bigr\}\biggr]\sin\theta\hat{\theta}\biggr]\end{aligned} \tag{23}$$

where

$$\Delta = \bigl(r^2 + a^2\bigr)\biggl(1 - \frac{r^2}{\ell^2}\biggr) - 2Mr \quad \text{and} \quad \Xi = 1 + \frac{a^2}{\ell^2}\cos^2\theta \tag{24}$$

3.3 Non-vanishing Lense–Thirring Precession in 'Zero Angular Momentum' Plebański–Demiański Spacetimes

This subsection can be regarded as a special case of non-accelerating PD spacetime in where we take that PD spacetime is not rotating, we mean the Kerr parameter $a = 0$. In a very recent paper, Hackmann and Lämmerzahl show that LT effect vanishes (Eq. (45) of [7]) due to the vanishing Kerr parameter. But we can see easily from Eq. (20) that if a vanishes in PD spacetime, the LT precession rate will be

$$\boldsymbol{\Omega}^{\mathrm{PD}}_{\mathrm{LT}}|_{a=0} = \frac{n\sqrt{\Delta|_{a=0}}}{p^3}\hat{r} \tag{25}$$

where

$$\Delta|_{a=0} = \left(r^2 - n^2\right)\left(1 - \frac{1}{\ell^2}\left(r^2 + 3n^2\right)\right) - 2Mr + Q_c^2 + Q_m^2 - \frac{4n^2r^2}{\ell^2} \quad \text{and} \tag{26}$$

$$p^2 = r^2 + n^2$$

So, the LT precession does not vanish due to the vanishing Kerr parameter. The above expression reveals that NUT charge n is responsible for the LT precession in 'zero angular momentum' PD spacetimes. Here, M represents the "gravito-electric mass" or 'mass' and n represents the "gravito-magnetic mass" or 'dual' (or 'magnetic') mass [10] of this spacetime. It is obvious that the spacetime is not invariant under time reversal $t \to -t$, signifying that it must have a sort of 'rotational sense' which is analogous to a magnetic monopole in electrodynamics. One is thus led to the conclusion that the source of the non-vanishing LT precession in the "rotational sense" arising from a non-vanishing NUT charge. Without the NUT charge, the spacetime is clearly hypersurface orthogonal and frame-dragging effect vanishes. This *'dual' mass* has been investigated in detail in Ref. [12, 13] and it is also referred as an *'angular momentum monopole* [8] in Taub-NUT spacetime. This implies that the inertial frame dragging seen here in such a spacetime can be identified as a *gravito-magnetic* effect.

In [7] the authors have investigated that the timelike geodesic equations in the PD spacetimes. The *orbital plane precession frequency* ($\Omega_\phi - \Omega_\theta$) is computed, following the earlier work of Ref. [11, 14, 15], and a vanishing result ensues. This result is then interpreted as a signature for a null LT precession in the 'zero angular momentum' PD spacetime.

We would like to say that what we have focused in the present paper is quite different from the 'orbital plane precession' considered in [7]. Using a 'Copernican' frame, we calculate the precession of a gyroscope which is moving in an arbitrary integral curve (not necessarily geodesic). Within this frame, a torqueless gyro in a stationary but not static spacetime held fixed by a support force applied to its center of mass undergoes LT precession. Since the Copernican frame does not rotate (by construction) relative to the inertial frames at asymptotic infinity ("fixed stars"), the observed precession rate in the Copernican frame also gives the precession rate of the gyro relative to the fixed stars. It is thus, more an intrinsic property of the classical *spin* of the spacetime (as a torqueless gyro must necessarily possess), in the sense of a dual mass, rather than an *orbital* plane precession effect for timelike geodesics in a Taub-NUT spacetime.

In our case, we consider the gyroscope equation [5] in an arbitrary integral curve

$$\nabla_u S = \langle S, a \rangle u \tag{27}$$

where $a = \nabla_u u$ is the acceleration, u is the four velocity and S indicates the spacelike classical spin four vector $S^\alpha = (0, \mathbf{S})$ of the gyroscope. For geodesics $a = 0 \Rightarrow \nabla_u S = 0$.

In contrast, Hackmann and Lämmerzahl [7] consider the behavior of massive test particles with *vanishing spin* $S = 0$, and compute the orbital plane precession rate for such particles, obtaining a vanishing result. We are thus led to conclude that since two different situations are being considered, there is no inconsistency between our results and theirs.

We note that the detailed analyses on LT precession in Kerr–Taub-NUT, Taub-NUT [16–18] and massless Taub-NUT spacetimes has been done in [6].

4 Extremal Case

4.1 Extremal Plebański–Demiański Spacetime

In this section, we would like to describe the LT precession in extremal PD spacetime, whose non-extremal case is already described in the previous section. To get the extremal limit in PD spacetimes we should first determine the radius of the horizons r_h which can be determined by setting $\Delta|_{r=r_h} = 0$. We can make a comparison of coefficients in

$$\begin{aligned}\Delta &= -\frac{1}{\ell^2}r^4 + \left(1 - \frac{a^2}{\ell^2} - \frac{6n^2}{\ell^2}\right)r^2 - 2Mr \\ &\quad + \left[\left(a^2 - n^2\right)\left(1 - \frac{3n^2}{\ell^2}\right) + Q_c^2 + Q_m^2\right] \\ &= -\frac{1}{\ell^2}\left[r^4 + \left(a^2 + 6n^2 - \ell^2\right)r^2 + 2M\ell^2 r + b\right] \\ &= -\frac{1}{\ell^2}\Pi_{i=1}^4(r - r_{hi})\end{aligned} \tag{28}$$

where

$$b = \left(a^2 - n^2\right)\left(3n^2 - \ell^2\right) - \ell^2\left(Q_c^2 + Q_m^2\right) \tag{29}$$

and r_{hi} ($i = 1, 2, 3, 4$) denotes the zeros of Δ. From this comparison we can conclude that for the PD–AdS (when Λ is negative) black hole, there are two separated positive horizons at most, and Δ is positive outside the outer horizon of the PD black hole. In the same way, we can conclude for the PD–dS (when Λ is positive) black hole that there are three separated positive horizons at most, and Δ is negative outside the outer horizon of the black hole. Both in the above cases, when the two horizons of the PD black hole coincide, the black hole is extremal [19].

If we consider the extremal PD black hole, we have to make a comparison of coefficients in

$$\begin{aligned}\Delta &= (r - x)^2\left(a_2r^2 + a_1r + a_0\right) \\ &= -\frac{1}{\ell^2}\left[r^4 + \left(a^2 + 6n^2 - \ell^2\right)r^2 + 2Mr\ell^2 + b\right]\end{aligned} \tag{30}$$

with a_0, a_1, a_2 being real [7]. From this comparison we can get the following for the PD–AdS spacetime:

$$\frac{b_A}{x^2} - 3x^2 = a^2 + 6n^2 + \ell^2 \tag{31}$$

$$x^3 - \frac{b_A}{x} = -M\ell^2 \tag{32}$$

where b_A represents the value of b at PD–AdS spacetimes:

$$b_A = (a^2 - n^2)(3n^2 + \ell^2) + \ell^2(Q_c^2 + Q_m^2) \tag{33}$$

Solving Eq. (31) for x, we get

$$x = \sqrt{\frac{1}{6}\left[-(\ell^2 + a^2 + 6n^2) + \sqrt{(\ell^2 + a^2 + 6n^2)^2 + 12b_A}\right]} \tag{34}$$

Similarly, we can obtain for the PD–dS black hole

$$\frac{b}{x^2} - 3x^2 = a^2 + 6n^2 - \ell^2 \tag{35}$$

$$x^3 - \frac{b}{x} = M\ell^2 \tag{36}$$

In these equations x is positive and related to the coincided horizon of the extremal PD–dS black hole.

Solving Eq. (35) for x, we get

$$x_+ = \sqrt{\frac{1}{6}\left[\ell^2 - a^2 - 6n^2 + \sqrt{(\ell^2 - a^2 - 6n^2)^2 + 12b}\right]} \tag{37}$$

$$x_- = \sqrt{\frac{1}{6}\left[\ell^2 - a^2 - 6n^2 - \sqrt{(\ell^2 - a^2 - 6n^2)^2 + 12b}\right]} \tag{38}$$

where x_+ and x_- indicate the outer horizon and inner horizons, respectively.

This can be seen by calculating

$$\left.\frac{d^2\Delta}{dr^2}\right|_{r=x_+} = -\frac{2}{\ell^2}\sqrt{(\ell^2 - a^2 - 6n^2)^2 + 12b} \tag{39}$$

and

$$\left.\frac{d^2\Delta}{dr^2}\right|_{r=x_-} = \frac{2}{\ell^2}\sqrt{(\ell^2 - a^2 - 6n^2)^2 + 12b} \tag{40}$$

For the PD–dS black hole, on the outer extremal horizon, $\frac{d\Delta}{dr} = 0$ and $\frac{d^2\Delta}{dr^2} < 0$ and on the inner extremal horizon $\frac{d^2\Delta}{dr^2} > 0$. Now, we can solve M and a from the two Eqs. (35), (36):

$$M = \frac{x[x^4 + 2x^2(3n^2 - \ell^2) + (3n^2 - \ell^2)(7n^2 - \ell^2) + \ell^2(Q_c^2 + Q_m^2)]}{\ell^2(\ell^2 + x^2 - 3n^2)} \tag{41}$$

$$a_e^2 = \frac{3x^4 + (6n^2 - \ell^2)x^2 + n^2(3n^2 - \ell^2) + \ell^2(Q_c^2 + Q_m^2)}{(3n^2 - \ell^2 - x^2)} \tag{42}$$

From the above values of a_e^2 and M, we get

$$a_e^2 = -M\ell^2 \frac{[3x^4 + (6n^2 - \ell^2)x^2 + n^2(3n^2 - \ell^2) + \ell^2(Q_c^2 + Q_m^2)]}{x[x^4 + 2x^2(3n^2 - \ell^2) + (3n^2 - \ell^2)(7n^2 - \ell^2) + \ell^2(Q_c^2 + Q_m^2)]} \tag{43}$$

The ranges of x and a_e are determined from the following expressions:

$$x^2 < \left(\frac{\ell^2 + \ell\sqrt{\ell^2 - 12Q^2}}{6} - n^2\right) \tag{44}$$

$$0 < a_e^2 < \left[(7\ell^2 - 24n^2) - \sqrt{(7\ell^2 - 24n^2)^2 - (\ell^4 - 12\ell^2(Q_c^2 + Q_m^2))}\right] \tag{45}$$

Due to the presence of the Cosmological constant, there exist four roots of x in Eq. (42). When the Cosmological constant $\frac{1}{l^2} \to 0$, Eq. (41) and Eq. (42) reduces to

$$x = M \tag{46}$$

and,

$$a_e^2 = x^2 + n^2 - Q_c^2 - Q_m^2 \tag{47}$$

or,

$$a_e^2 = M^2 + n^2 - Q_c^2 - Q_m^2 \tag{48}$$

respectively.

Now, the line element of extremal PD spacetimes can be written as

$$ds^2 = -\frac{\Delta_e}{p_e^2}(dt - A_e\,d\phi)^2 + \frac{p_e^2}{\Delta_e}dr^2 + \frac{p_e^2}{\Xi_e}d\theta^2 + \frac{\Xi_e}{p_e^2}\sin^2\theta(a_e\,dt - B_e\,d\phi)^2 \tag{49}$$

and the final LT precession rate in the extremal PD spacetime is

$$\begin{aligned}\boldsymbol{\Omega}_{\mathrm{LT}}^{\mathrm{ePD}} = {}& \frac{\sqrt{\Delta_e}}{p_e}\left[\frac{a_e(\Xi_e\cos\theta + (2n - a_e\cos\theta)\frac{a_e}{\ell^2}\sin^2\theta)}{\Delta_e - a_e^2\Xi_e\sin^2\theta}\right.\\ &\left.-\frac{a_e\cos\theta - n}{p_e^2}\right]\hat{r}\\ &+\frac{\sqrt{\Xi_e}}{p_e}a_e\sin\theta\left[\frac{r - M - \frac{r}{\ell^2}(a_e^2 + 2r^2 + 6n^2)}{\Delta_e - a_e^2\Xi_e\sin^2\theta}\right.\\ &\left.-\frac{r}{p_e^2}\right]\hat{\theta}\end{aligned} \tag{50}$$

where

$$\begin{aligned}\Delta_e &= -\frac{1}{\ell^2}(r - x)^2\left(r^2 + 2rx + \frac{b}{x^2}\right),\\ \Xi_e &= 1 + \frac{a_e^2}{\ell^2}\cos^2\theta - \frac{4a_e n}{\ell^2}\cos\theta,\\ p_e &= r^2 + (n - a_e\cos\theta)^2,\\ A_e &= a_e\sin^2\theta + 2n\cos\theta,\\ B_e &= r^2 + a_e^2 + n^2\end{aligned} \tag{51}$$

and the value of a_e is determined from Eq. (42) and the range of x and a_e (‘e’ stands for the *extremal* case) are determined from Eq. (44) and (45), respectively. It could be noted that, for the extremal PD–dS spacetimes, *there are upper limiting values for angular momentum and extremal horizon of the black hole*. Substituting all the above mentioned values and ranges in Eq. (50), we get the exact LT precession rate in extremal PD spacetimes.

4.2 Extremal Kerr Spacetime

Substituting $a_e = M$ and $\Lambda = \frac{1}{\ell^2} = Q_c = Q_m = n = 0$ in Eq. (50) we can easily get the LT precession rate in extremal Kerr spacetime:

$$\boldsymbol{\Omega}_{\mathrm{LT}}^{\mathrm{eK}} = \frac{M^2[2r(r - M)\cos\theta\hat{r} + (r^2 - M^2\cos^2\theta)\sin\theta\hat{\theta}]}{(r^2 + M^2\cos^2\theta)^{\frac{3}{2}}(r^2 - 2Mr + M^2\cos^2\theta)} \tag{52}$$

The above result is also coming from Eq. (50), the general LT precession rate for extremal PD spacetime.

Case I: On the polar region, i.e. $\theta = 0$, the Ω_{LT} becomes

$$\Omega_{\mathrm{LT}}^{\mathrm{eK}} = \frac{2M^2r}{(r^2 + M^2)^{3/2}(r - M)} \tag{53}$$

Case II: On the equator, i.e. $\theta = \pi/2$, the Ω_{LT} becomes

$$\Omega_{\mathrm{LT}}^{\mathrm{eK}} = \frac{M^2}{r^2(r - 2M)} \tag{54}$$

It could be easily seen from Eq. (53) that LT precession diverges at $r = M$, since $r = M$ is the only direct ISCO in extremal Kerr geometry which coincides with the principal null geodesic generator of the horizon [20] and it is also the radius of the horizon which is a null surface. As the LT precession is not defined at the null surfaces and the spacelike surfaces, the general LT precession formula is derived only considering that the observer is at rest in a timelike Killing vector field. We have not also incorporated the LT effect for any null geodesics. So, our formula is valid only for $r > M$ (outside the horizon).

4.3 Extremal Kerr–Newman Spacetime

Substituting $a^2 = M^2 - (Q_c^2 + Q_m^2) = M^2 - Q^2$ and $\Lambda = n = 0$ in Eq. (21), we can obtain the following LT precession rate for extremal KN black hole:

$$\begin{aligned}\boldsymbol{\Omega}_{\mathrm{LT}}^{\mathrm{eKN}} = {}& \frac{\sqrt{M^2 - Q^2}}{\rho^3(\rho^2 - 2Mr + Q^2)}\big[(r - M)\big(2Mr - Q^2\big)\cos\theta\hat{r}\\ &+\big(M\big(2r^2 - \rho^2\big) + rQ^2\big)\sin\theta\hat{\theta}\big]\end{aligned} \tag{55}$$

where

$$\rho^2 = r^2 + \big(M^2 - Q^2\big)\cos^2\theta \tag{56}$$

From the above expression (Eq. (55)), we can make a similar comment again, like the extremal Kerr–Newman black hole that the gyroscope which is rotating in a polar orbit of radius $r = \frac{Q^2}{2M}$ cannot experience any frame-dragging effect. Apparently, it seems that this argument is also true for the gyroscope which is rotating at $r = M$ orbit. But this is not true, because this is the horizon of extremal Kerr–Newman spacetime. So, $r = M$ is a null surface. The general formula (Eq. (12)) which we have considered in our whole paper, is valid only in timelike spacetimes (outside the horizon), not in any null or spacelike regions.

4.4 Extremal Kerr–de Sitter Spacetime

The extremal Kerr–de Sitter spacetime is interesting because it involves the Cosmological constant. Setting $n =$

$Q_c = Q_m = 0$ in Eq. (20), we can find the following expression of LT precession at extremal Kerr–de Sitter spacetimes:

$$\begin{aligned}\boldsymbol{\Omega}_{\rm LT}^{\rm eKdS} &= \frac{a_e}{\rho^3(\rho^2 - 2Mr - \frac{1}{\ell^2}(a_e^4 + a_e^2 r^2 - a_e^4 \sin^2\theta\cos^2\theta))} \\ &\quad\times\left[\sqrt{\Delta_e}\left(2Mr + \frac{\rho^4}{\ell^2}\right)\cos\theta\hat{r}\right. \\ &\quad + \sqrt{\Xi_e}\left[M\left(2r^2 - \rho^2\right) + \frac{r}{\ell^2}\left\{a_e^4 + a_e^2 r^2\right.\right. \\ &\quad \left.\left.\left. - a_e^4\sin^2\theta\cos^2\theta - \rho^2\left(a_e^2 + 2r^2\right)\right\}\right]\sin\theta\hat{\theta}\right]\end{aligned} \tag{57}$$

where

$$\rho^2 = r^2 + a_e^2\cos^2\theta, \qquad \Xi_e = 1 + \frac{a_e^2}{\ell^2}\cos^2\theta \tag{58}$$

and

$$\Delta_e = -\frac{1}{\ell^2}(r - x)^2\left(r^2 + 2xr - \frac{a^2\ell^2}{x^2}\right)$$

Using Eqs. (42), (41), we see that the values for a and M are

$$\begin{aligned}a_e^2 &= \frac{(\ell^2 - 3x^2)x^2}{(\ell^2 + x^2)} \\ M &= \frac{x(\ell^2 - x^2)^2}{\ell^2(\ell^2 + x^2)}\end{aligned} \tag{59}$$

where the horizons are at

$$\begin{aligned}x_+ &= \sqrt{\frac{1}{6}\left[\ell^2 - a^2 + \sqrt{\left(\ell^2 - a^2\right)^2 - 12a^2\ell^2}\right]} \\ x_- &= \sqrt{\frac{1}{6}\left[\ell^2 - a^2 - \sqrt{\left(\ell^2 - a^2\right)^2 - 12a^2\ell^2}\right]}\end{aligned} \tag{60}$$

where x_+ and x_- indicate the outer horizon and inner horizon. The ranges of a_e^2 and x are the following:

$$0 < a_e^2 < (7 - 4\sqrt{3})\ell^2 \quad \text{and} \quad x^2 < \ell^2/3 \tag{61}$$

which is already discussed in [19]. Substituting all the above values in Eq. (57) and taking the ranges of a_e and x, we obtain the exact LT precession rate in extremal Kerr–dS spacetimes.

4.5 Extremal Kerr–Taub-NUT spacetime

To derive the extremal limit in Kerr–Taub-NUT spacetime we set

$$\Delta = r^2 - 2Mr + a^2 - n^2 = 0 \tag{62}$$

Solving for r, we get two horizons which are located at $r_\pm = M \pm \sqrt{M^2 + n^2 - a^2}$. So, we find that the extremal condition $(r_+ = r_-)$ for Kerr–Taub-NUT spacetimes is $a_e^2 = M^2 + n^2$. If we set $Q_c = Q_m = \Lambda = 0$ and $M^2 + n^2 = a_e^2$ in Eq. (20), we get the following exact LT precession rate at extremal Kerr–Taub-NUT spacetime:

$$\begin{aligned}\boldsymbol{\Omega}_{\rm LT}^{\rm eKTN} &= \frac{(r - M)}{p}\left[\frac{\sqrt{M^2 + n^2}\cos\theta}{p^2 - 2Mr - n^2}\right. \\ &\quad \left. - \frac{\sqrt{M^2 + n^2}\cos\theta - n}{p^2}\right]\hat{r} \\ &\quad + \frac{\sqrt{M^2 + n^2}\sin\theta}{p}\left[\frac{r - M}{p^2 - 2Mr - n^2} - \frac{r}{p^2}\right]\hat{\theta}\end{aligned} \tag{63}$$

where $p^2 = r^2 + (n \mp \sqrt{M^2 + n^2}\cos\theta)^2$.

5 Discussion

In this work we have explicitly derived the LT precession frequencies for extremal and non-extremal PD spacetime. The PD family of solutions are the solutions of Einstein field equations which contain a number of well known black hole solutions. Among them are included Schwarzschild, Schwarzschild–de Sitter, Kerr, Kerr–AdS, Kerr–Newman, Kerr–Taub-NUT etc. We observe that the LT precession frequency strongly depends upon the different parameters like mass M, spin a, Cosmological constant Λ, NUT charge n, electric charge Q_c, and magnetic charge Q_m.

An interesting point is that LT precession occurs solely due to the "dual mass". This "dual mass" is equivalent to angular momentum monopole (n) of NUT spacetime. For our completeness we have also deduced the LT precession for extremal PD spacetime and also for others axisymmetric PD-like spacetimes. To get the LT precession rate in extremal Kerr–Taub-NUT–de-Sitter spacetime, the basic procedure is the same as the PD spacetimes with the additional requirement $Q_c^2 + Q_m^2 = 0$ in Eqs. (41) and (42) and the range of $x^2 < (\frac{\ell^2}{3} - n^2)$ and a_e^2 has a range of $0 < a_e^2 < (7\ell^2 - 24n^2) - 4\sqrt{3(\ell^2 - 3n^2)(\ell^2 - 4n^2)}$.

In the case of Taub-NUT spacetime, the horizons are located at $r_\pm = M \pm \sqrt{M^2 + n^2}$. For extremal Taub-NUT spacetime, it has to satisfy the condition $r_+ = r_-$ or $M^2 + n^2 = 0$. The solution of this equation is either $M = in$ or $n = iM$ (where $i = \sqrt{-1}$) which is not possible as M and n are always real for Taub-NUT black hole. So, there is not any valid extremal condition at Taub-NUT spacetime (with M or without M) and we could not obtain any *real* LT precession rate due to the frame-dragging effect in *extremal* Taub-NUT spacetime. Since, the direct ISCO coincides with the principal null geodesic generator [20] in extremal Kerr spacetimes, we are unable to discuss the LT precession at that particular geodesic. So this formula is not valid for the domain of $r \leq M$ for the extremal Kerr spacetimes. Thus our formula is valid only for $r > M$. In general, the general formula for LT precession in stationary spacetime is valid only outside the horizon, as the observer is in timelike Killing vector field. The formula is not valid on the horizon and inside the horizon. We will discuss about this problem in the near future.

We also expect that the physically unobserved spacetimes will be found in future and the strong gravity LT precession will be measured in those physically *unknown* spacetimes.

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