Homework 2 - Report

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Problem 1

In the first exercise we are going to study some properties of the following directed graph:

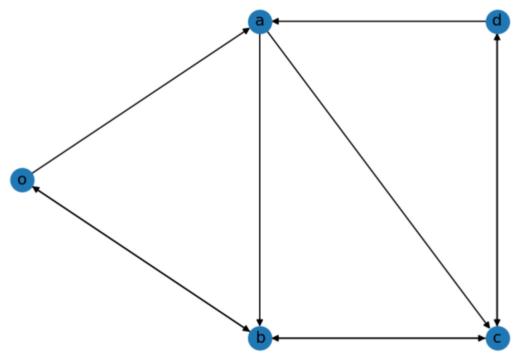


Figure 1

described by the following adjacency matrix:

$$\Lambda = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} o & a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 2/5 & 1/5 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 0 & 1/3 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ o \\ a \\ b \\ c \\ d \end{array}$$

First part

Estimation of return time

To estimate the return time of node a, we can simulate a random walk starting from a and, for each node v, waiting $X \sim exp\left(\text{outdeg}(v)\right)$ time and then moving to an edge u with probability $P_{v,u}$, ending when $u = a$. The estimation is $\mathbb{E}_a\left[T_a^+\right] = 6.74$.

Computation of return time

The retrun time is analytically computable as $\mathbb{E}_i[T_i^+] = \dfrac{1}{\omega_i \overline{\pi}_i}$,
where $\overline{\pi}$ is the normalized left dominant eigenvector of $\overline{P}$:

$$\overline{P}_{i,j} = \begin{cases} \dfrac{\Lambda_{i,j}}{\omega^*} & \Longleftarrow \; i \neq j \\ 1 - \sum_{j \in \mathcal{V}: j \neq i} \overline{P}_{i,j} & \Longleftarrow \; i = j \end{cases}, \quad \omega^* = \max_{v \in \mathcal{V}} \left(\text{outdeg}(v)\right)$$

$\mathbb{E}_a\left[T_a^+\right] = 6.75$

Estimation of hitting time

To estimeate the hitting time of node d starting from node o, we can use the same algorithm used to estimate the return time, starting from o and ending when $u = d$.
The estimation is $\mathbb{E}_o[T_d] = 8.82$.

Computation of hitting time

Hitting times satisfy the following relations:

$$\begin{cases} \mathbb{E}_i[T_\mathcal{S}] = 0 & \impliedby i \in \mathcal{S} \\ \mathbb{E}_i[T_\mathcal{S}] = \dfrac{1}{\omega_i} + \sum_{j\in\mathcal{V}} P_{i,j}\mathbb{E}_j[T_\mathcal{S}] & \impliedby i \notin \mathcal{S} \end{cases}$$

so, to compute $\mathbb{E}_o[T_d]$, we have to solve the linear system:

$$P := D^{-1}\Lambda$$

$$P_d : P_{d_{i,j}} = \begin{cases} P_{i,j} & \impliedby i \neq d \\ 0 & \impliedby i = d \quad \text{(because } \mathbb{E}_d[T_d] \text{ does not depend on other values)} \end{cases}$$

$$w\_inv_d = \begin{cases} \dfrac{1}{w_i} & \impliedby i \neq d \\ 0 & \impliedby i = d \quad \text{(because } \mathbb{E}_d[T_d] = 0\text{)} \end{cases}$$

$$\mathbb{E}[T_d] = w\_inv_d + P_d\mathbb{E}[T_d] \implies (I - P_d)\mathbb{E}[T_d] = w\_inv_d$$

In our problem, $\mathbb{E} = \begin{bmatrix} \mathbb{E}_o[T_d] \\ \mathbb{E}_a[T_d] \\ \mathbb{E}_b[T_d] \\ \mathbb{E}_c[T_d] \\ \mathbb{E}_d[T_d] \end{bmatrix}$
The result is $\mathbb{E}_o[T_d] = 8.79$.

Second part

French-DeGroot averaging dynamics

Considering the previous graph, where Λ is now the weight matrix, the dynamics converge because the condensation graph has only one sink and it is aperiodic in the original graph.
For example, starting with initial condition

$$x_0 = [0.38, 0.43, 0.29, 0.40, 0.64]$$

the dynamics converge to a consensus with value 0.42.
The dynamics converge because the condensation graph has only one sink and it is aperiodic in the original graph.

Consensus variance

We can compute the consensus value as

$$\alpha = \sum_{i\in\mathcal{V}} \pi_i x_i(0)$$

Assuming $x_i(0) = \xi_i,\ \{\xi_i\}_{i\in\mathcal{V}}\ iid,\ V[\xi_i] = \sigma^2$,

$$V[\alpha] = V\left[\sum_{i\in\mathcal{V}} \pi_i x_i(0)\right] \overset{\text{ind.}}{=} \sum_{i\in\mathcal{V}} V[\pi_i x_i(0)] = \sum_{i\in\mathcal{V}} \pi_i^2 V[x_i(0)] = \sum_{i\in\mathcal{V}} \pi_i^2\sigma^2$$

For example, assuming $\sigma^2 = 0.09$, $V[\alpha] = 0.019$
Simulating the dynamics, we obtain 0.018 as consensus variance.

Averaging dynamics on modified graph - 1

Removing edges (d, a) and (d, c) from the original graph, we obtain the following:

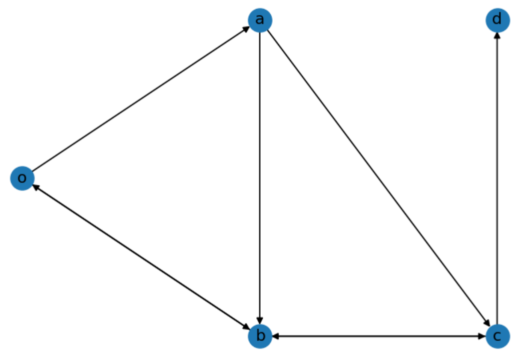


Figure 2

Removing the outgoing edges from d, the graph is not strongly connected anymore.
Now the asymptotic behaviour of the dynamics is determined only by the initial state of the nodes in the sinks of the condensation graph, because they are not influenced by the rest of the graph. Moreover, the sinks contain the support of the invariant distribution, which determines consensus value.
As in this case there is only one attracting component and it is aperiodic, a consensus will be reached, and its value will be the initial opinion of d.
Assuming $x_i(0) = \xi_i,\ \{\xi_i\}_{i\in\mathcal{V}}\ iid,\ V[\xi_i] = \sigma^2$,

$$V[\alpha] = V\left[\sum_{i\in\mathcal{V}} \pi_i x_i(0)\right] \overset{\text{ind.}}{=} \sum_{i\in\mathcal{V}} V[\pi_i x_i(0)] = V[\pi_d x_d(0)] = \pi_d^2\sigma^2 = \sigma^2$$

As d is the only node affecting the asymptotic dynamics, the variance of the consensus equals the variance of the opinion of d.

Averaging dynamics on modified graph - 2

Removing edges (c, b) and (d, a) from the original graph, we obtain the following:

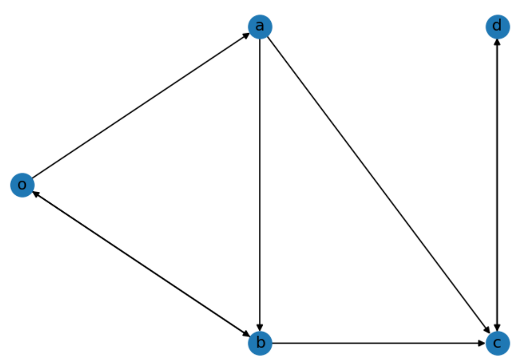


Figure 3

This case is similar to the previous one, but in this case we have two nodes (c and d) in the sink of the condensation graph (instead of only d).
Now the attracting component is not aperiodic, so a consensus will (generally) not be reached.
A consensus will be reached only in the cases where $x_c(0) = x_d(0)$.

Problem 2

In this problem we see how 100 particles behave in the same graph analyzed in Problem 1.

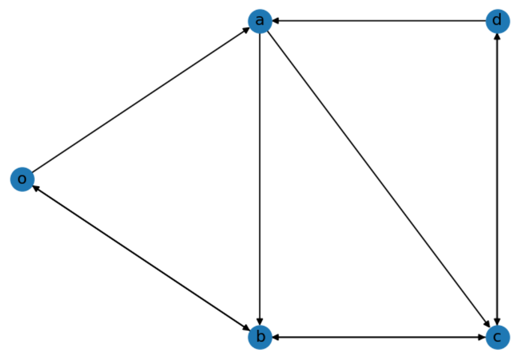


Figure 4

Particle perspective

As the particles do not influence each other while moving, they take on average the average time taken by only one particle to return to a starting from a, that is about 6.79 time units.

Node perspective

If we put 100 particles in node o and we wait 60 time units, the number of particles per node are, in function of time:

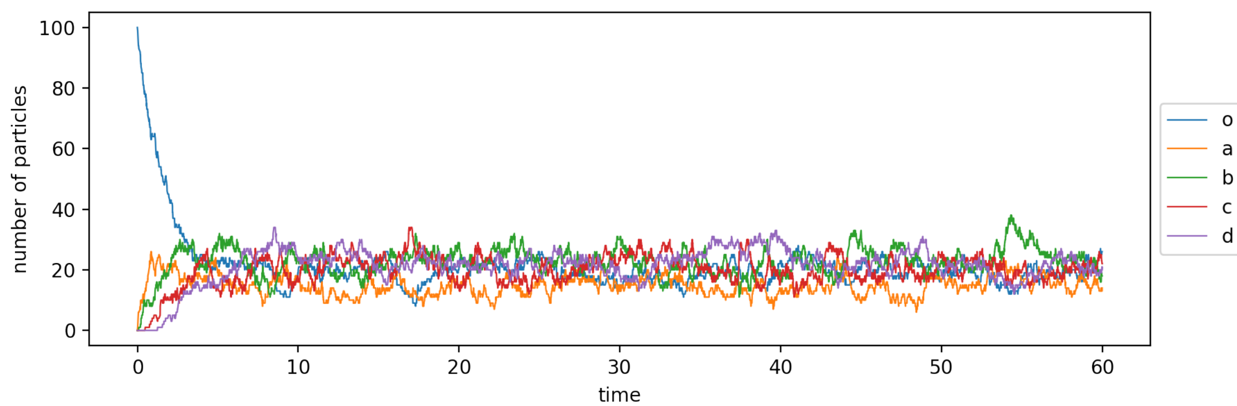
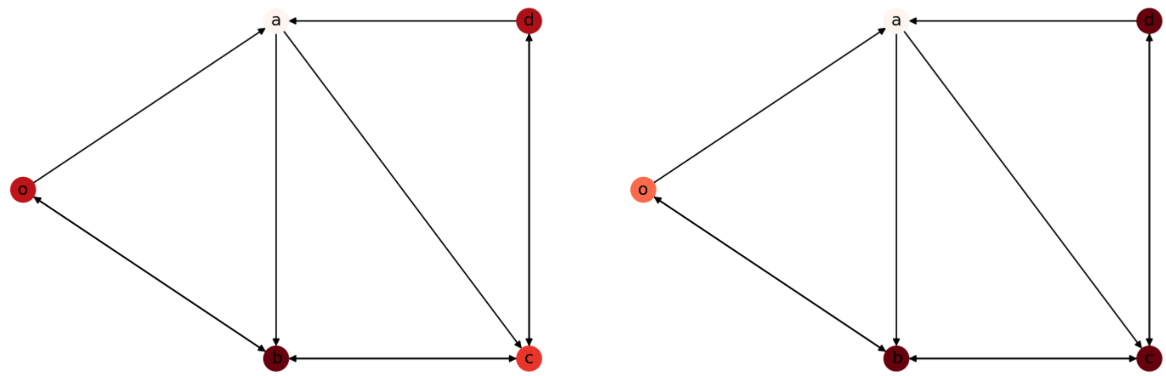


Figure 5

The average number of particles in the nodes are:

$$\{o : 21.09, a : 14.76, b : 22.73, c : 19.93, d : 21.49\}$$



On the left, the intensity of red is proportional to the average number of particles during the simulation in the nodes.
On the right, the intensity of red is proportional to the stationary distribution of the continuous time random walk followed by a single particle, that is
$\{o : 0.19, a : 0.15, b : 0.22, c : 0.22, d : 0.22\}$
We can see that they are very close to each other, in fact (normalizing the vector of average numbers of particles in nodes) their difference in norm-2 is 0.036.

Problem 3

Considering the following graph:

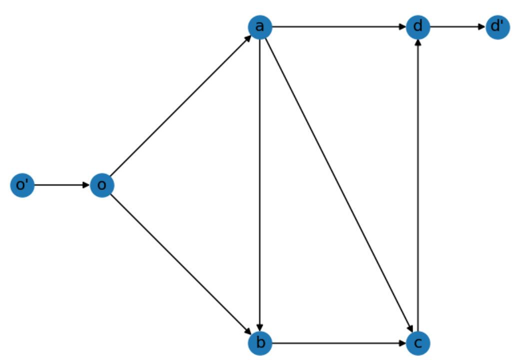


Figure 6

described by the transition rate matrix:

$$\Lambda = \begin{array}{c c} \begin{array}{ccccccc} o' & o & a & b & c & d & d' \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 1/4 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} o' \\ o \\ a \\ b \\ c \\ d \\ d' \end{array} \end{array}$$

Proportional rate

In this part we assume that each node is able to pass along a number of particles in a time unit proportional to the number of particles in it.
With an input rate of 1 particle per time unit, the distribution of particles among nodes during a simulation lasting 60 time units is like:

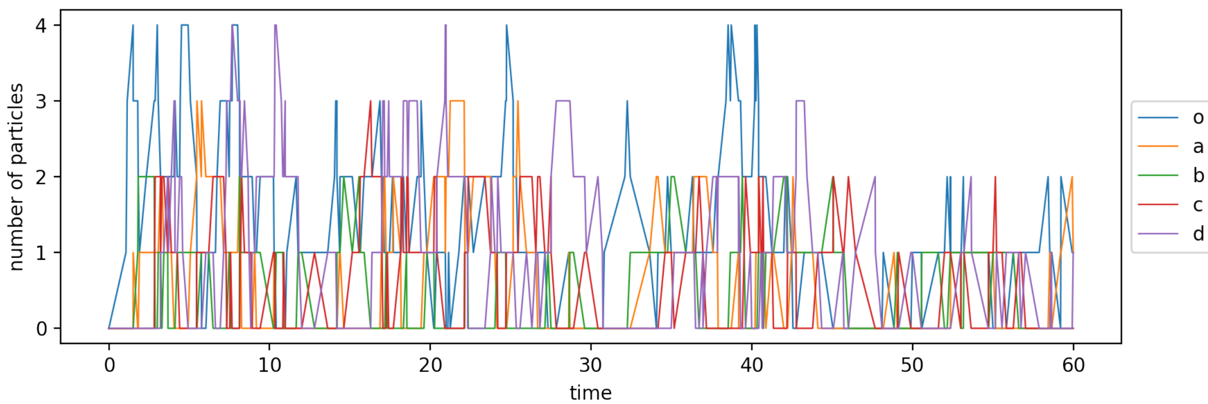


Figure 7

In this case, the system can handle any input rate without blowing up, because the rate a node passes along particles is proportional to the number of particles in the node, so when there will be a great number of particles in the system, a lot of them will move from a node inside the graph and not from the input.

More formally,

$$\lim_{n\to+\infty}\frac{r_i}{r_g}=0$$

where n is the number of particles in the system, r_i is the input rate, r_g is the sum of the rates of the nodes.

For example, we can see how the system behaves with an input rate of 1000 particles per time unit:

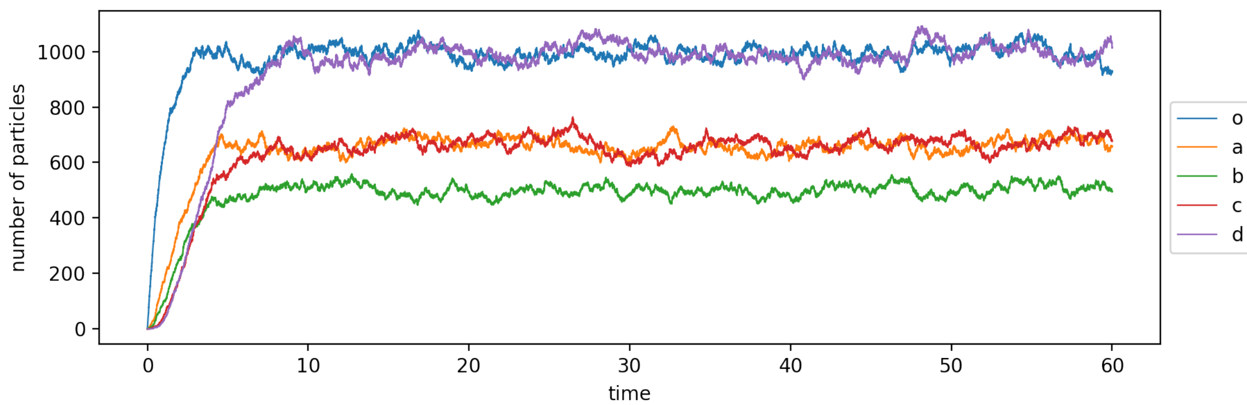


Figure 8

Fixed rate

Now we assume that each node is able to pass along a fixed number of particles in per time unit, in particular the waiting time of each node is a random variable exponentially distributed with a mean of 1 time unit.

With an input rate of 1 particle per time unit, the distribution of particles among nodes during a simulation lasting 60 time units is like:

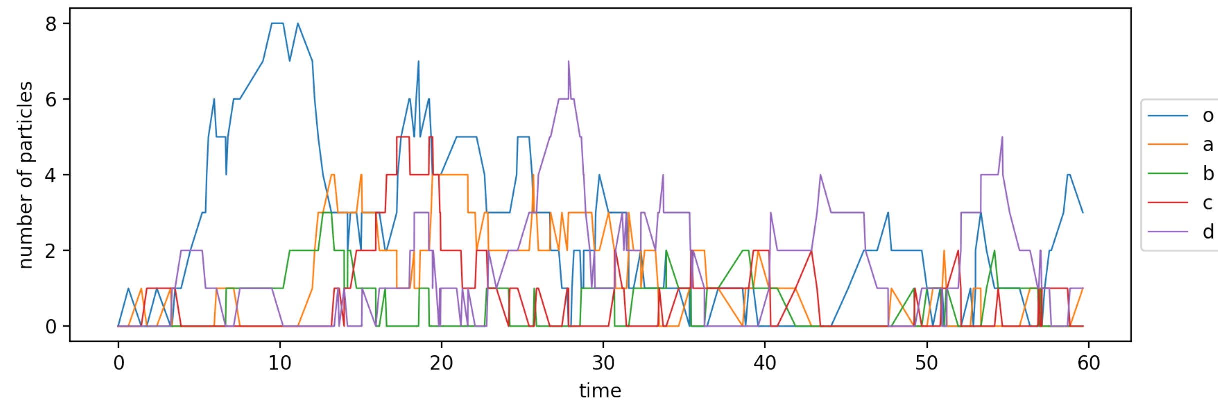


Figure 9

In this case, the system can handle an input rate of one particle per time unit, because the rate each node passes along particles is greater than or equal to the rate new particles are injected inside the graph, so particles won't stuck on any node.
The system blows up for each input rate greater than one, because the rate of node o would be smaller than the input rate, so particles will stuck on this node.

For example, we can see how the system behaves with an input rate of 2 particles per time unit:

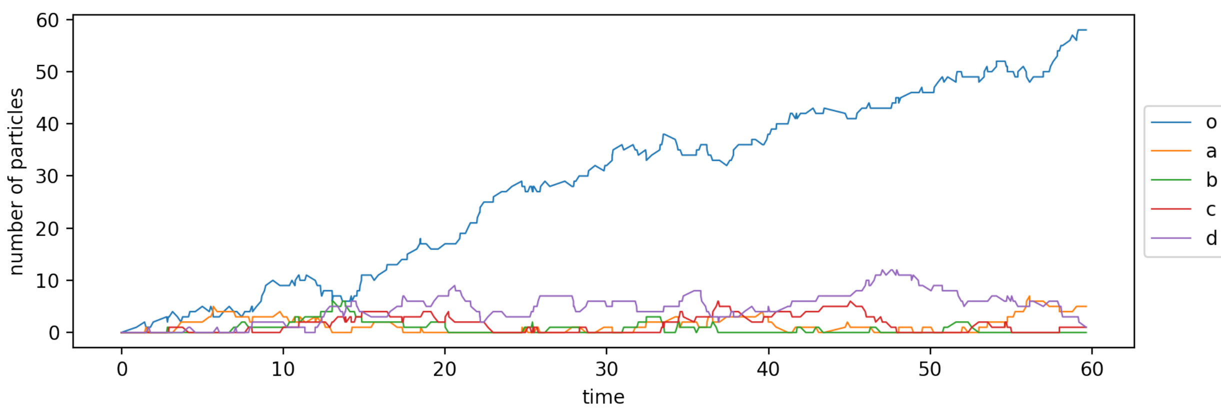


Figure 10