Homework 1 - Report

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Exercise 1

Disconnecting source and destination

The infimum of the total capacity that needs to be removed for no feasible unitary flows from origin o to destination d to exist is the minimum cut capacity, as stated by Menger's theorem, that is the minimum of the sums of the capacities of the edges that go from the subset containing the origin o to the subset containing the destination d, for every cut:

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}), \quad C^*_{ij} = \min_{\mathcal{U} \subseteq \mathcal{V} : o \in \mathcal{U} \wedge d \notin \mathcal{U}} \sum_{i \in \mathcal{U}} \sum_{j \in \mathcal{V} \setminus \mathcal{U}} C_{ij}$$

Here is the graph we are going to study:

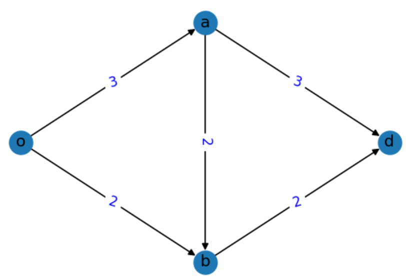


Figure 1

In this case, the minimum cut capacity is 5:

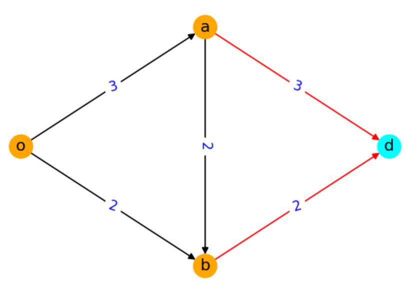


Figure 2: Node colors identify a minimal cut and the red edges are the ones that determine the minimum cut capacity.

For the Max-flow min-cut theorem, also the maximum flow is 5.

Allocate 1 unit of capacity

Additional units of capacity should be placed on edges that cover minimal cuts, in a way to cover the greatest possible number of them.
In this case we have edges (o, a) and (a, d) that cover two minumum cuts each (one of them in common).
Putting an additional unit of capacity on one edge however is not enough to improve the maximum flow of the graph, as none of them covers all the minimal cuts, e.g. incrementing (o, a)'s capacity:

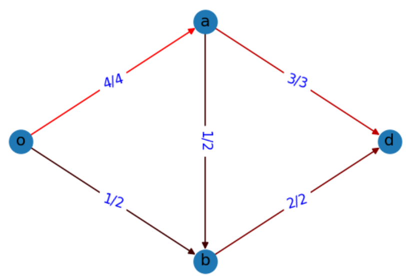


Figure 3: The maximum flow is still 5

Allocate 2 units of capacity

Two additional units of capacities can be used to improve the maximum flow of the graph from 5 to 6.

Throughput maximization

The assignments that allow this improvement are (the format is *edge*: *units of flow*):

$$(a, d) : 1, \ (o, a) : 1$$
$$(b, d) : 1, \ (o, a) : 1$$
$$(b, d) : 1, \ (o, b) : 1$$

Using the first assingment, we obtain:

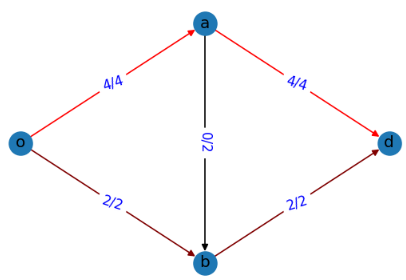


Figure 4: The maximum flow is 6.

Allocate 4 units of capacity

Four additional units of capacities can be used to improve the maximum flow of the graph from 5 to 7.

Throughput maximization

The assignments that allow this improvement are:

$$
\begin{aligned}
&(a,d):2,\ (o,a):2\\
&(a,d):1,\ (b,d):1,\ (o,a):2\\
&(b,d):2,\ (o,a):2\\
&(a,d):1,\ (b,d):1,\ (o,a):1,\ (o,b):1\\
&(b,d):2,\ (o,a):1,\ (o,b):1\\
&(b,d):2,\ (o,b):2
\end{aligned}
$$

Maximization of the sum of the capacities of cuts

One of the choices that maximizes the sum of the capacities of the cuts is
(o, a): 2, (a, d): 2
with a total increment of 8 units of capacity (summing the capacities of every cut).

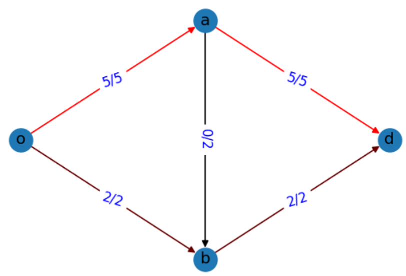


Figure 5: The maximum flow is 7.

Exercise 2

In this exercise we have a bipartite graph, where one set of nodes is the set of people and the other one is the set of books, and we study some matchings among them.

Graph representation

In the following plot, we can see the bipartite graph with undirected edges,
where left nodes represent the people and right ones the books:
an edge between a person and a book means that that person is interested in that book.

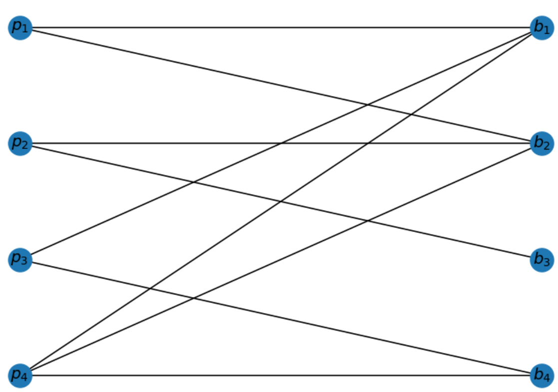


Figure 6

Perfect matching

Now we search a perfect matching in the previous graph.
To do that, we can make a directed graph, with edges from people to books with capacity 1 (each person can take only one copy of each book).
Then we add a source s with one edge of capacity 1 to each person (each person can take only one copy of only one book) and a sink t with one edge of capacity 1 incoming from each book (each book exists in only one copy).
The graph generated is the following:

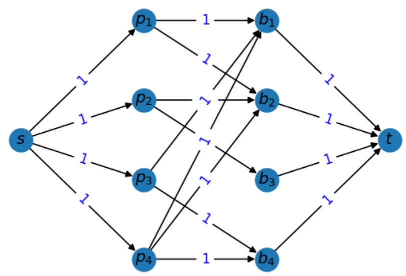


Figure 7

Running a maximum flow algorithm on this graph, from s to t, we can see how many assignments we can do. The following picture shows the results: an edge between a person p and a book b is red if and only if b is assigned to p.

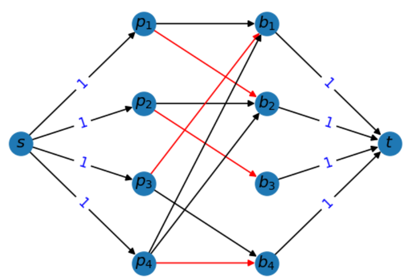


Figure 8

So a perfect matching is $\{(p_1, b_2), (p_2, b_3), (p_3, b_1), (p_4, b_4)\}$.

Matching with multiple copies of books

Assuming that each person can take only one copy of each book (but can take different books), we can set the capacities of the edges between s and peoples to ∞, and the ones between a book b and t to the number of copies of b. In this way, each person can take any number of books, but we limit the copies of books. The graph becomes the following:

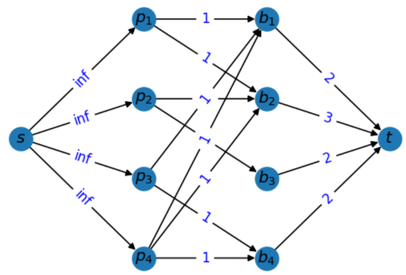


Figure 9

And the matching is

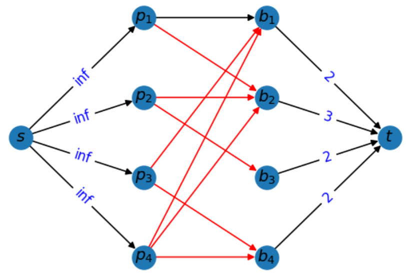


Figure 10

Exchange books to maximize the number of assignments

In Figure 10, we can see that there is an unused copy of b_3 and that p_1 would like to have a copy of b_1, but there aren't any left, so the library should sell one copy of b_3 and buy one of b_1.

Exercise 3

In this exercise we are going to study a simplified version of the highway network between Santa Monica and Santa Ana in Los Angeles. The network is modeled as a directed graph:

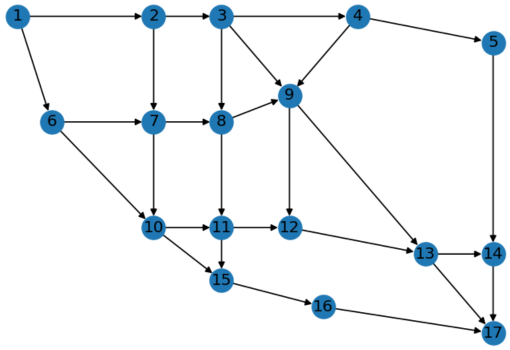


Figure 11

Shortest (fastest) path

The shortest (fastest) path can be computed using Dijkstra's algorithm.
The shortest path from node 1 to node 17 is (1, 2, 3, 9, 13, 17), and the time needed to travel on it is 0.53 hours.

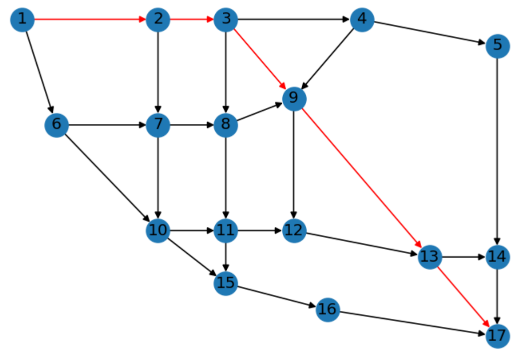


Figure 12: Red edges are the ones laying on the shortest path

Maximum flow

The maximum flow can be computed using Ford-Fulkerson's algorithm.
The maximum flow from node 1 to node 17 is 22448, the following plot shows how it is distributed:

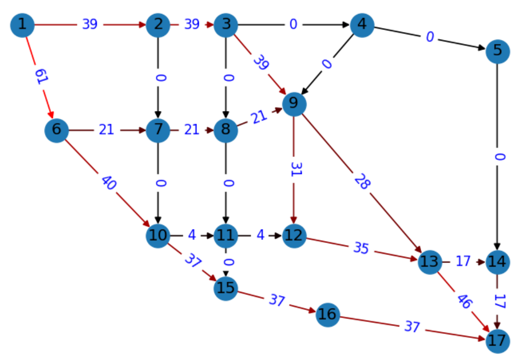


Figure 13: The intensity of red is proportional to the flow, an edge's label is the percentage of throughput that goes through that edge.

Exogenous net flow

Considering the flow vector f given in $flow.mat$, the exogenous net flow vector is
$\nu = (16806, 8570, 19448, 4957, -746, 4768, 413, -2, -5671, 1169, -5, -7131, -380, -7412, -7810, -3430, -23544)$

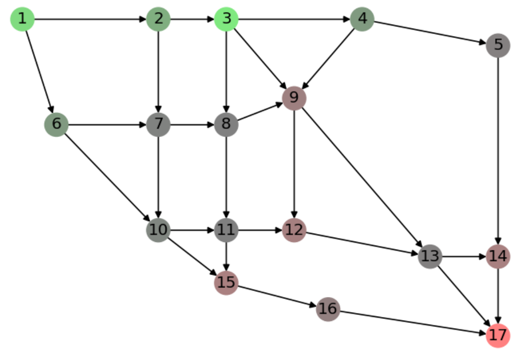


Figure 14: A green node has a positive exogenous net flow, a red node has a negative one.

In the following we will consider
$\nu = (16806,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -16806)$

Traffic Assignment Problem - 1

Consider the previous graph, with delay functions:

$$d_e(f_e) = \frac{l_e}{1 - \frac{f_e}{C_e}}, \quad 0 \le f_e < C_e$$

System Optimum Flow

The System Optimum Flow can be computed minimizing the sum of the products of flow and delay of the edges:

$$f^* = \underset{0 \le f_e \le C_e \wedge Bf = \nu}{\operatorname{argmin}} \sum_{e \in \mathcal{E}} f_e \cdot d_e(f_e)$$

The following plot shows how the System Optimum Flow is distributed among the edges:

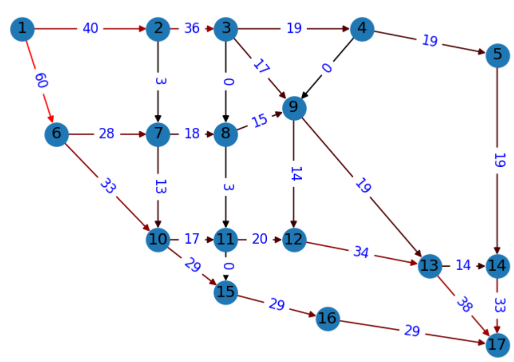


Figure 15: The intensity of red is proportional to the flow, an edge's label is the percentage of throughput that goes through that edge.

Wardrop equilibrium

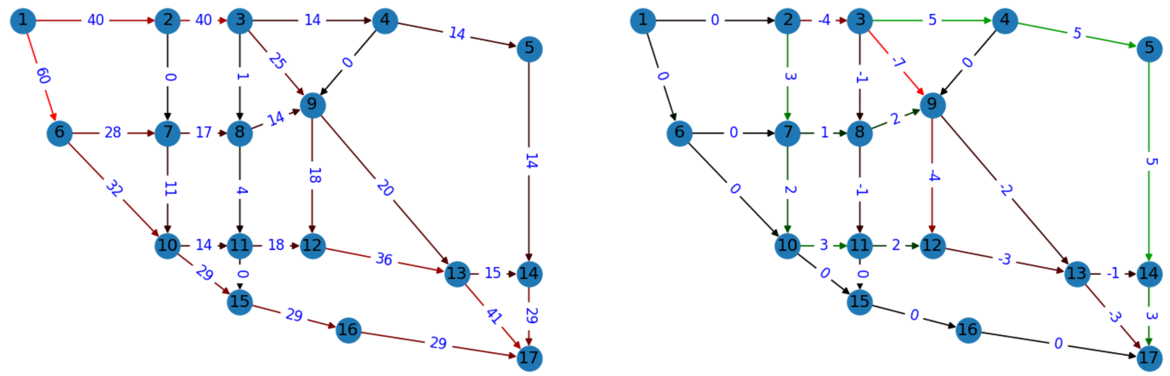
We have the Wardrop equilibrium when every path with a positive capacity has a total delay smaller than or equal to the delay of all other paths:

$$f^{(0)} = Az \quad : \quad p \in \Gamma^{(o,d)} \wedge z_p > 0 \implies \forall q \in \Gamma^{(o,p)}, \sum_{e \in \mathcal{E}} A_{ep}^{(o,d)} d_e\left(f_e^{(0)}\right) \le \sum_{e \in \mathcal{E}} A_{eq}^{(o,d)} d_e\left(f_e^{(0)}\right)$$

This equals the User Optimum Flow of the Traffic Assigment Problem:

$$f^{(0)} = \underset{f \geq 0 \wedge Bf = \tau\left(\delta^{(o)} - \delta^{(d)}\right)}{\operatorname{argmin}} \sum_{e \in \mathcal{E}} \int_0^{f_e} d_e(s) ds$$

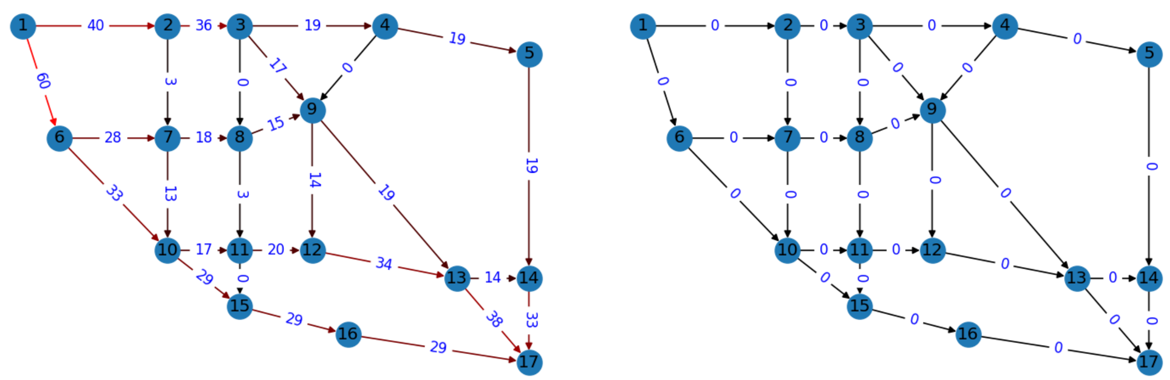
The following plot shows how the flow at Wardrop equilibrium is distributed among the edges:



On the left, the intensity of red is proportional to the User Optimum Flow.
On the right, on green edges $f_e^* > f_e^{(0)}$, on red edges $f_e^* < f_e^{(0)}$, color intensity is proportional to $\left|f_e^* - f_e^{(0)}\right|$.
On the right, an edge's label is $\dfrac{f_e^* - f_e^{(0)}}{\tau} \cdot 100$.
The maximum absolute difference in the flows is $\max\left(\left|f_e^* - f_e^{(0)}\right|\right) = 1245$.

Wardrop equilibrium with tolls

Let's now add a toll on every edge, $w_e = f_e^* d_e'(f_e^*)$.
The new delay function on the edge e is $d_e(f_e) + w_e$.
The following plot shows how the Flow at Wardrop equilibrium with tolls is distributed among the edges:



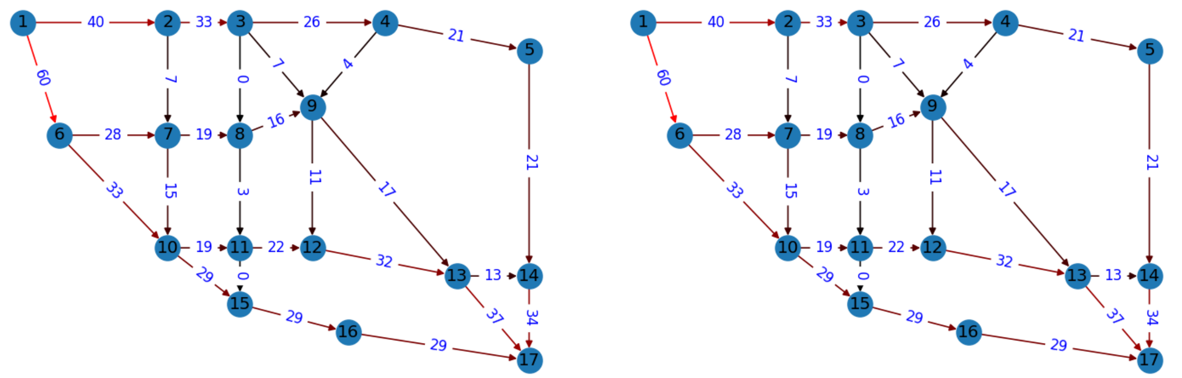
On the left, the intensity of red is proportional to the flow in Wardrop equilibrium with tolls condition.
On the right, on green edges $f_e^* > f_e^{(w)}$, on red edges $f_e^* < f_e^{(w)}$, color intensity is proportional to $\left|f_e^* - f_e^{(w)}\right|$ (same proportinality as in the difference with the Wardrop equilibrium without tolls).
The maximum absolute difference in the flows is 0.8, we can see that the flow in Wardrop equilibrium with tolls is very close to the System Optimum flow, the difference is given by computational and approximation errors.

Traffic Assignment Problem - 2

Let's now consider the delay function as the difference between the previous delay and the delay in free flow:

$$\tau_e(f_e) = d_e(f_e) - l_e$$

and let's recompute System Optmimum Flow f^* ad tolls $w : w_e = f_e^* \tau_e'(f_e^*)$.
The following plot shows how the flows are distributed among the edges:



On the left, the intensity of red is proportional to the System Optimum Flow.
On the right, the intensity of red is proportional to the flow in Wardrop equilibrium with tolls condition.
The maximum absolute difference in the flows is 1, we can see again that the flow in Wardrop equilibrium with tolls is very close to the System Optimum flow, the difference is given by computational and approximation errors.