Combinations and Permutations

Permutations

How many ways are there to line up 5 students?

Step 1: choose 1st student 5 options
Step 2: choose 2nd 4 options
Step 3: choose 3rd 3 options
Step 4: choose 4th 2 options
Step 5: choose 5th 1 options

$5 * 4 * 3 * 2 * 1 = 240$ ways to line up 5 students

How does this change as the number of students changes? What if we were lining up 10 students?

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$$

"10 factorial"

- The number ways to arrange n objects is $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$

$0! = 1$

Each way of arranging the students is called a <u>permutations</u>. The number of <u>n</u>-long permutations of <u>n</u> elements is $n!$.

- How many ways are there to line up 5 students from a group of 10?

$$\underbrace{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}_{\text{5 \#s being multiplied}}$$

started at 10 and counded down

- If there are n elements, the number of k-long permutations

$$P(n,k) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdots (n-(k-1))}_{\text{k \#s being multiplied}}$$

$$= \frac{n!}{(n-k)!}$$

Suppose an album contained 13 tracks and you wanted to make a playlist that contained 8 songs from it without repeating songs. How many playlists can be made?

$$P(13,8) = 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$$

$$= \frac{13!}{(13-8)!} = \frac{13!}{5!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

$= 51,891,840$ different playlists

Combinations

If we think of permutations as counting ordered lists then combinations count unordered sets.

Let's suppose the order of pizza toppings doesn't affect the kind of pizza it is. Imagine there are 5 toppings Pe, On, Ha, Bp, Sp. How many two topping pizzas are there?

Pe On
On Pe

Pe Ha
Ha Pe

Pe Bp
Bp Pe

Pe Sp
Sp Pe

On Ha
Ha On

On Bp Bp On

On Sp Sp On

Ha Bp Bp Ha

Ha Sp Sp Ha

Bp Sp Sp Bp

$P(5,2) = 20$

$C(5,2) = 10$

$$C(5,2) = \frac{P(5,2)}{2}$$

$2! = 2$

In general if there are n objects to choose from, the number of k-sized unordered sets is

$$C(n,k) = \frac{P(n,k)}{k!}$$

$$= \frac{\overbrace{n\cdot(n-1)\cdot \cdots \cdot (n-(k-1))}^{k \text{ terms being multiplied}}}{k!}$$

$$C(n,k) = \frac{n!}{(n-k)!\,k!}$$

There are 15 students. How many ways are there to choose 4 students to form a group?

$$C(15,4) = \frac{15\cdot \cancel{14}^{7}\cdot 13\cdot \cancel{12}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 15\cdot 7\cdot 13 = 1365$$

$$= \frac{15!}{(15-4)!\,4!} = \frac{15!}{11!\,4!} = \frac{15\cdot 14\cdot 13\cdot 12\cdot \cancel{11}\cdot\cancel{10}\cdot \cancel{\cdots}}{(\cancel{11}\cdot\cancel{10}\cdot\cancel{\cdots})\,4\cdot 3\cdot 2\cdot 1}$$

There are 1365 ways to form a 4 person group out of 15 people.

Combination & Permutation Word Problems

There are 18 runners in a race. (a) How many ways for first, second, and third place runners to finish? (b) How many ways to hand out 4 good sportsmanship awards?

(a) Because a first place finish is different than a second place finish this is a permutations question.

$$P(18,3) = 18 \cdot 17 \cdot 16 = 4{,}896$$

$$= \frac{18!}{(18-3)!} = \frac{18!}{15!}$$

(b) This question is asking about 'who' gets awards and not 'in what order' do people get awards, so this a combinations question.

$$C(18,4) = \frac{\cancel{18}^{9} \cdot 17 \cdot \cancel{16}^{4} \cdot \cancel{15}^{5}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 9 \cdot 17 \cdot 4 \cdot 5 = 3060$$

$$= \frac{18!}{(18-4)!\,4!} = \frac{18!}{14!\;4!}$$

A pizza shop offers 12 toppings. (a) If each week they make one 2 topping pizza half off, how long would it take to cover every 2 topping pizza? (b) If they wanted to do this, how many ways could they arrange the 2 topping pizza discounts?

(a) How many 2 topping pizzas are there?

$$C(12,2) = \frac{12 \cdot 11}{2} = 6 \cdot 11 = 66$$

$$= \frac{12!}{(12-2)!\,2!} = \frac{12!}{10!\,2!}$$

66 weeks to cover all 2 topping pizzas

(b) $P(66,66) = 66!$

$$= \frac{66!}{(66-66)!} = \frac{66!}{0!}$$

≈ too big to write out (more than # atoms in known observable universe)

A collector owns 3 red marbles, 2 green marbles, 4 yellow marbles, and 4 orange marbles. (a) If you blindly choose 4 marbles, who many ways could you end up with exactly 1 red and 2 yellow marbles? (b) If you blindly choose 4 marbles, how many ways could you end up with at most 1 orange marble?

a)

red

yellow

yellow

not red nor yellow

Step 1: choose 1 red, $C(3,1) = 3$

Step 2: choose 2 yellow, $C(4,2) = \frac{4 \cdot 3}{2} = 6$

Step 3: choose 1 not yellow not red, $C(6,1) = 6$

$$3 \cdot 6 \cdot 6 = 108$$

b)

alt 1

orange

not orange

not orange

not orange

or

alt 2

not oranges

Step 1: choose 4 not orange

$C(9,4)$

$= \frac{9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6}^{2}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 126$

Step 1: choose 1 orange $C(4,1) = 4$

Step 2: choose 3 not orange $C(9,3) = \frac{\cancel{9}^{3} \cdot \cancel{8}^{4} \cdot 7}{\cancel{3} \cdot \cancel{2}} = 84$

$$4(84) + 126 = 462$$