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Research Statement

My dissertation is on generalizations of Descartes' Rule of Signs to multivariate polynomial systems. It focuses on identifying conditions on the signs of negatively coupled Pham systems that gurantee that a positive real solution exists. Future work will be in contributing to the growing body of knowledge about sums of nonnegative circuit (SONC) polynomials. What follows is an introduction to Descartes' Rule of Signs, my contributions in extending one aspect of this, and an introdution to SONC polynomials.

DESCARTES' RULE OF SIGN

Descartes' Rule of Sign is a theorem that uses the signs of the coefficients of a real univariate polynomial to answer questions about the number of positive roots the polynomial has. To do this it uses the number of sign changes in a polynomial's coefficients.

A univariate real polynomial $f(x)$ can be expressed as

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where a_i are real numbers. The number of sign changes in f can be calculated using the list

$$a_n, a_{n-1}, \ldots, a_0$$

removing any coefficients that equal zero, and counting the number of times a positive number is succeeded by a negative number or vice versa. Letting $s(f)$ represent the number of sign changes in f and letting $n(f)$ represent the number of positive roots of f, then Descartes' Rule of Signs can be succinctly stated as

$$n(f) = s(f) - 2k$$

where k is a positive integer.

One corollary of this theorem is that if the sign of the highest degree term and the sign of the lowest degree term are opposites, then for all instantiations of coefficients the polynomial is guaranteed to have at least one positive real solution. Additionally, if the signs of the highest degree term and the lowest degree terms are the same, then there exist coefficients for which the polynomial has no positive real solutions. It is this corollary of Descartes' Rule of Sign that my dissertation works to generalize and in it the term uniform consistency is used to describe polynomial systems that for all choices of coefficients will have at least one positive real root.

SIGN CONDITIONS FOR UNIFORM CONSISTENCY

A Pham system is a multivariate polynomial system $f \in \mathbb{R}[x_1, \ldots, x_n]^n$ with the following form

$$\begin{aligned} f_1 &= x_1^{d_1} + \sum_{|\alpha|<d_1} c_{\alpha 1}\mathbf{x}^{\alpha} \\ &\vdots \\ f_n &= x_n^{d_n} + \sum_{|\alpha|<d_n} c_{\alpha n}\mathbf{x}^{\alpha} \end{aligned}$$

where $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_n})$.

It is known that for Pham systems, the number of complex roots when counting multiplicities is exactly equal to Bezout's bound, $d_1 d_2 \cdots d_{n-1} d_n$. Statements about the number of real solutions are less known.

In my dissertation I introduce negatively coupled Pham systems. If $f = (f_1, \ldots, f_n) \in \mathbb{R}[x_1, \ldots, x_n]^n$ is a Pham system then each f_i can be partitioned into g_i and h_i such that

1. $g_i \in \mathbb{R}[x_i]$ is maximal
2. $h_i = f_i - g_i$

where maximal means that g_i contains all terms in f_i that can be expressed as cx_i^k. Here f_i is defined as negatively coupled when the coefficients of every term in h_i is negative.

It is shown in my dissertation that if a Pham system f is negatively coupled and each g_i has a negative constant, then f will have at least one positive solution. The importance of this result lies with weakly coupled dynamical systems where the existence of a positive real solution is a stabilization point of the dynamical system.

SUMS OF NONNEGATIVE CIRCUIT POLYNOMIALS

Certifying that a polynomial $f \in \mathbb{R}[\mathbf{x}]$ is nonnegative over all possible inputs can be done by finding square polynomials g_i^2 and decomposing f into a sum of them. However, the Motzkin polynomial

$$M(x, y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$$

is a nonnegative polynomial that cannot be expressed as a sum of square polynomials and is an example that using sums of squares as the only method of nonnegativity certification is insufficient.

A circuit polynomial is a polynomial whose support defines a simplex and a single interior point. The Motzkin polynomial is such a polynomial. The coefficient of monomial corresponding to the interior point in relation to the other coefficients determines if the circuit polynomial is nonnegativity. Thus the certification of nonnegativity of a polynomial f can be accomplished by decomposing f into a sum of nonnegative circuit polynomials.

The first abstract on arxiv.org that included SONC polynmials appeared in 2017. At this point of time papers outlining the dual cone of SONC polynomials, a positivstellensatz for SONC polynomials, and using SONC polynomials in polynomial optimization have all been published. Despite this, much more is known about the cone of SOS polynomials and it's intersection with the cone of nonnegative polynomials than is known about the cone of SONC polynomials. Combined this shows that study of SONC polynomials is an important area and is still a promising area of research.