Problems Proposed for the

Baltic Way 2016

Mathematical Team Contest

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BALTIC WAY 2016 OULU

November 3 to 7, 2016, Oulu, Finland

Algebra

A1. (p. 8) The set $\{1, 2, \ldots, 10\}$ is split to three parts. For each part the sum of its elements, the product of its elements and the sum of the digits of all its elements are calculated.

Is it possible that the first part has the largest sum of elements, the second part has the largest product of elements, and the third part has the largest sum of digits?

A2. (p. 8) Prove that

$$\sqrt{\frac{1}{3x+1}} + \sqrt{\frac{x}{x+3}} \geq 1$$

holds for all $x > 0$. For which values of x is there an equality?

A3. (p. 9) Find all combinations of four integers (a, b, c, d) satisfying the equations

$$\begin{cases} -a^2 + b^2 + c^2 + d^2 = 1 \\ 3a + b + c + d = 1. \end{cases}$$

A4. (p. 10) Find all positive integers n for which

$$3x^n + n(x+2) - 3 \geq nx^2$$

holds for all real numbers x.

A5. (p. 10) Is it true that for any real numbers a, b, c and d satisfying $a^2 + b^2 + (a-b)^2 = c^2 + d^2 + (c-d)^2$ also the equality

a) $a^3 + b^3 + (a-b)^3 = c^3 + d^3 + (c-d)^3$

b) $a^4 + b^4 + (a-b)^4 = c^4 + d^4 + (c-d)^4$

holds?

A6. (p. 11) Find all real numbers a for which there there exists a non-constant function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equations

1) $f(ax) = a^2 f(x)$

2) $f(f(x)) = a\ f(x)$.

A7. (p. 12) Find all real solutions of the equation

$$\frac{(x+y)(2-\sin(x+y))}{4\sin^2(x+y)} = \frac{xy}{x+y}.$$

A8. (p. 12) Find all quadruples (a, b, c, d) of real numbers that simultaneously satisfy the following equations:

$$\begin{cases} a^3 & + & c^3 & = & 2 \\ a^2b & + & c^2d & = & 0 \\ b^3 & + & d^3 & = & 1 \\ ab^2 & + & cd^2 & = & -6. \end{cases}$$

A9. (p. 13) Let $a_{0,1}, a_{0,2}, \ldots, a_{0,2016}$ be positive real numbers. For $n \geq 1$ and $1 \leq k < 2016$ set

$$a_{n+1,k} = a_{n,k} + \frac{1}{2a_{n,k+1}}$$

and

$$a_{n+1,2016} = a_{n,2016} + \frac{1}{2a_{n,1}}.$$

Let

$$m_n = \max_{1 \leq k \leq 2016} a_{n,k} \quad \text{for } n \geq 0.$$

Show that $m_{2016} > 44$.

Combinatorics

C1. (p. 13) Set A consists of 2016 natural numbers. All prime divisors of these numbers are smaller than 29. Prove that there are four distinct numbers a, b, c and d in A such that $abcd$ is a square.

C2. (p. 14) Does there exist a hexagon with side lengths 1, 2, 3, 4, 5, 6 (not necessarily in this order) that can be tiled with a) 31 b) 32 equilateral triangles with side length 1?

C3. (p. 14) Mobile operator has n clients and holds the following advertising campaign. In the beginning it deposits 1 € on account of each client. When two persons which have a € and b € in their accounts communicate by a phone the operator makes both accounts equal to $(a + b)$ €. It happens that after $h(m)$ phone calls all n clients' accounts become equal m €. Prove that $h(m) \leq \frac{1}{2} n \log_2 m$.

C4. (p. 15) A *magic octagon* is an octagon whose sides go along the grid lines of a square grid and side lengths are 1, 2, 3, 4, 5, 6, 7, 8 (in any order). What is the largest possible area of a magic octagon?

C5. (p. 16) In a computer game a $4 \times 4 \times 4$ cube is built using 4^3 unit cubes. At the beginning of the game each unit cube contains an integer. In each turn of the game, you choose a unit cube and increase by 1 all the integers in the cubes having a face in common with the chosen cube. You win the game if you reach a position in a finite number of turns where all the 4^3 integers are divisible by 3.

Is it possible to win the game no matter what the starting position is?

C6. (p. 17) Arbitrary integers are written in the cells of the 13×13 table. Prove that it is possible to choose 2 rows and 4 columns such that the sum of eight numbers in the intersecting cells is divisible by 8.

C7. (p. 17) All n-digit positive integers from 10^{n-1} to $10^n - 1$ are concatenated in the increasing order. What is the largest possible value of k for which it is possible to find in this sequence of digits the same k-digit substring in at least two different places?

C8. (p. 19) There are n students at a school. It is known that, for any two students A and B, either A loves B or B loves A (but not both). (A student may thus be in love with several other students.) A *love triangle* is a configuration of three students A, B, C such that A loves B, B loves C and C loves A. What is the maximal number of love triangles, given the number n of students?

C9. (p. 21) Let k and t be integers with $1 \leq k/2 < t < k$. Each square of a $k \times k$ checkerboard is coloured either red or blue. A move consists of choosing a row or a column with at most t red squares and switching the colour of these red squares to blue. Assume that it is possible to make all squares of the checkerboard blue with a sequence of moves, and let m be the least number of moves required to do so. Prove that if $m > k$, then the total number of initially red squares is at least $k + 2t$.

C10. (p. 22) Prove that

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{kn}{n} = (-n)^n$$

holds for every positive integer n.

C11. (p. 23) A graph has 2016 vertices. All the edges of the graph are coloured blue or red. It is known that the graph contains no blue path on 1062 vertices. Prove that it is possible to select two disjoint sets of 477 vertices each, such that all the edges between these sets are red.

Geometry

G1. (p. 23) In triangle ABC, the points D and E are the intersections of the angular bisectors from C and B with the sides AB and AC. Points F and G on the extensions of AB and AC beyond B and C satisfy $BF = CG = BC$. Prove $FG \parallel DE$.

G2. (p. 24) $ABCD$ is a convex quadrilateral such that $AB = AD$. T is a point on the diagonal AC such that $\sphericalangle ABT + \sphericalangle ADT = \sphericalangle BCD$. Prove that $AT + AC \geq AB + AD$.

G3. (p. 25) Let $ABCD$ be a parallelogram such that $\sphericalangle BAD = 60°$. Let K and L be the midpoints of BC and CD, respectively. Assuming that $ABKL$ is a cyclic quadrilateral, find $\sphericalangle ABD$.

G4. (p. 26) Let ABC be a triangle and D and E points on the lines CA and BA such that $CD = AB$, $BE = AC$ and A, D and E lie on the same side of BC. Let I be the incenter of ABC and let H be a point such that I I the orthocenter of BCI. Show that D, E and H are collinear.

G5. (p. 27) Consider triangles where each corner has integer coordinates. Such a triangle can be legally *transformed* by moving one corner parallell to the opposite side to a different point with integer coordiantes. Show that if two triangles with integer coordinates have the same area, then there exists a series of legal transforms that transforms one to the other.

G6. (p. 29) Let $ABCD$ be a convex quadrilateral. Let P be a point such that $\sphericalangle APC = \sphericalangle BPD = 30^\circ$. Prove that

$$2(AB + AC + AD + BC + BD + CD) \geq PA + PB + PC + PD.$$

G7. (p. 30) Let $ABCD$ be a cyclic quadrilateral. Let M be the midpoint of CD. Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD, and perpendicular bisector of MP are concurrent or parallel.

G8. (p. 31) Let ABC be a triangle and let P be a point such that AP is the angle bisector of $\sphericalangle BAC$ and segment BC bisects segment AP. Prove that perimeter of triangle ABC is greater than or equal to perimeter of triangle PBC.

G9. (p. 32) Starting with three points A, B, C in general position and the circumcircle k of $\triangle ABC$, a step consists of drawing a line ℓ and obtaining all points of intersection of ℓ with the lines already drawn and with k, where ℓ is

(1) the line through two distinct points that had been obtained before or

(2) the bisector of an angle $\sphericalangle XYZ$, where X, Y, Z are three previously obtained distinct points on k.

Is it always (i.e., for any choice of A, B, C) possible to obtain the orthocentre of $\triangle ABC$ in a finite number of steps?

G10. (p. 33) Let M and N be the midpoints of the sides AC and AB, respectively, of an acute triangle ABC. Let ω_B be the circle centered at M passing through B, and let ω_C be the circle centered at N passing through C. Let the point D be such that $ABCD$ is an isosceles trapezoid with AD parallel to BC. Assume that ω_B and ω_C intersect in two distinct points P and Q. Show that D lies on PQ.

Number Theory

N1. (p. 35) Prove that

$$\sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}}$$

is rational for every positive integer n.

N2. (p. 36) Do there exist positive integers a, b, c, such they have no common divisor and

$$ab + bc + ca = (a + b - c)(b + c - a)(c + a - b)?$$

N3. (p. 36) Determine all positive integers a and all primes p fulfilling the equation

$$(a - p)^3 = a + p.$$

N4. (p. 36) Find all pairs of primes (p, q) such that

$$p^3 - q^5 = (p + q)^2.$$

N5. (p. 37) A finite sequence $d_{n-1}, d_{n-2}, \ldots, d_1, d_0$ of digits is called a *stable final segment of length* n if it has the following property: If m is any positive integer such that the last n digits of m are $d_{n-1}d_{n-2}\ldots d_1d_0$ (in this order), then for every positive integer k the last n digits of m^k are $d_{n-1}d_{n-2}\ldots d_1d_0$ (in this order). Prove that for any positive integer n there are exactly four stable final segments of length n.

N6. (p. 38) Prove or disprove the following hypotheses.

1. Each sequence of at least two consecutive integers contains a number that is divisible by no prime number less than the amount of members in the sequence.
2. Each sequence of at least two consecutive integers contains a number that is relatively prime to all other members of the sequence.

N7. (p. 39) For which integers $n = 1, \ldots, 6$ does the equation

$$a^n + b^n = c^n + n$$

have a solution in integers?

N8. (p. 40) Let n be a positive integer and let a, b, c, d be integers such that $n | a+b+c+d$ and $n | a^2+b^2+c^2+d^2$. Show that

$$n | a^4+b^4+c^4+d^4+4abcd.$$

N9. (p. 41) Find all pairs p, q of distinct primes, sets $D \subseteq \mathbb{R}$ and functions $f\colon D \to D$ fulfilling

$$f^p(x) = x^p \qquad \text{and} \qquad f^q(x) = x^q$$

for all $x \in D$. (Here, f^n denotes the n'th iterate of f.)

N10. (p. 42) Let $a_1, \ldots, a_{2016}$ and $b_1, \ldots, b_{2016}$ be two reorderings of the numbers $1, \ldots, 2016$. Prove that

$$2017 \mid a_i b_i - a_j b_j$$

for some distinct indices i and j.

N11. (p. 42) Let $a_0, a_1, \ldots$ be a sequence of positive integers such that $a_n = a_{n-1}^{2^n}$ for all $n = 1, 2, \ldots$. Prove that for each prime p, $p > 3$, with residue 3 modulo 4 there exists a positive integer a_0 such that the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is not constant modulo p for any positive integer N.

Solutions

A1. *The set $\{1, 2, \ldots, 10\}$ is split to three parts. For each part the sum of its elements, the product of its elements and the sum of the digits of all its elements are calculated.*

Is it possible that the first part has the largest sum of elements, the second part has the largest product of elements, and the third part has the largest sum of digits?

Yes! For example

set	sum	digsum	prod
1,9,10	20✓	11	90
3,7,8	18	18✓	168
2,4,5,6	17	17	240✓

A2. *Prove that*

$$\sqrt{\frac{1}{3x+1}} + \sqrt{\frac{x}{x+3}} \geq 1$$

holds for all $x > 0$. For which values of x is there an equality?

As both sides of the inequality are positive by definition, one may equivalently replace both sides by their squares. The left side minus the right side then becomes

$$\begin{aligned} 0 &\leq \frac{1}{3x+1} + \frac{x}{x+3} + 2\sqrt{\frac{x}{(3x+1)(x+3)}} - 1 \\ &= \frac{-8x}{(3x+1)(x+3)} + 2\sqrt{\frac{x}{(3x+1)(x+3)}} = (-2y+1)y \end{aligned}$$

with

$$y = 2\sqrt{\frac{x}{(3x+1)(x+3)}}.$$

The inequality is thus seen to be equivalent to $0 \leq y \leq 1/2$. $x > 0$ implies $y > 0$ and $1/2 \geq y$ is eqivalent to

$$(3x+1)(x+3) \geq 16x,$$

which is equivalent to

$$3(x-1)^2 \geq 0,$$

which holds for every x.

As all the transformation were equivalences and all of them hold when inequality is replaced by equality, there is equality if and only if $x = 1$.

A3. *Find all combinations of four integers (a, b, c, d) satisfying the equations*

$$\begin{cases} -a^2 + b^2 + c^2 + d^2 = 1 \\ 3a + b + c + d = 1. \end{cases}$$

Answer: The solutions are: $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ and $(1,0,-1,-1)$, $(1,-1,0,-1)$, $(1,-1,-1,0)$.

Solution: We write the equations as:

$$\begin{aligned} b^2 + c^2 + d^2 &= 1 + a^2 \\ b + c + d &= 1 - 3a \end{aligned}$$

We now apply the root-mean square and the arithmetic mean inequality to the numbers $|b|, |c|$ and $|d|$. This yields

$$\begin{aligned} \sqrt{\frac{1+a^2}{3}} = \sqrt{\frac{b^2+c^2+d^2}{3}} &\geq \frac{|b|+|c|+|d|}{3} \\ &\geq \frac{|b+c+d|}{3} = \frac{|1-3a|}{3} \\ \Longrightarrow\ & 3 + 3a^2 \geq 1 - 6a + 9a^2 \\ \Longrightarrow\ & 6a^2 - 6a - 2 \leq 0 \\ \Longrightarrow\ & 3a^2 - 3a - 1 \leq 0 \\ \Longrightarrow\ & a \in \left[\frac{1}{2} - \frac{\sqrt{21}}{6}, \frac{1}{2} + \frac{\sqrt{21}}{6}\right] \\ \Longrightarrow\ & a \in \{0, 1\} \end{aligned}$$

We can now look two cases depending on the value of a. We obtain solutions: $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,0,-1,-1)$, $(1,-1,0,-1)$, $(1,-1,-1,0)$ (permutations of last three places of solutions $(0,1,0,0)$ and $(1,0,-1,-1)$).

A4. *Find all positive integers n for which*

$$3x^n + n(x+2) - 3 \geq nx^2$$

holds for all real numbers x.

Answer: The inequality holds if and only if n is even.

Solution: First suppose that n is odd. Setting $x=-1$ in the inequality, the left-hand side becomes $3\cdot(-1)^n + n - 3 = n-6$, while the right-hand side becomes $n\cdot(-1)^2 = n$, and this is a contradiction.

Then let n be even. Since $|x| \geq x$ for all x, it suffices to prove that $3x^n + 2n - 3 \geq nx^2 + n|x|$ for all x. Writing $y = |x| \geq 0$ and using the fact that n is even, it it enough to prove that

$$3y^n + (2n-3) \geq ny^2 + ny$$

for all $y \geq 0$. By the arithmetic-geometric inequality, we have

$$\begin{aligned} 2y^n + (n-2) &= y^n + y^n + 1 + \cdots + 1 && (1)\\ &\geq n\sqrt[n]{y^n \cdot y^n \cdot 1^{n-2}} = ny^2. && (2) \end{aligned}$$

Again by the arithmetic-geometric inequality, we obtain

$$y^n + (n-1) = y^n + 1 + \cdots + 1 \geq \sqrt[n]{y^n \cdot 1^{n-1}} = ny. \qquad (3)$$

Adding these two inequalities together yields the claim.

Remark. One could alternatively prove (1) and (3) by induction on n. For (3) this is straightforward, and for (1) the induction step would lead to proving

$$ny^3 - (n-2)y + (n-1) - (n+1)y^2 \geq 0,$$

which is true since the left-hand side equals $(y-1)^2(ny+n-1) \geq 0$.

A5. *Is it true that for any real numbers a, b, c and d satisfying $a^2 + b^2 + (a-b)^2 = c^2 + d^2 + (c-d)^2$ also the equality*

a) $a^3 + b^3 + (a-b)^3 = c^3 + d^3 + (c-d)^3$
b) $a^4 + b^4 + (a-b)^4 = c^4 + d^4 + (c-d)^4$

holds?

a) No, for example, if $a = b = 7$, $c = 8$ and $d = 3$ then

$$7^2 + 7^2 + 0^2 = 98 = 8^2 + 3^2 + 5^2,$$

but

$$7^3 + 7^3 + 0^3 = 686 \neq 664 = 8^3 + 3^3 + 5^3.$$

b) Yes, because

$$(a^2 + b^2 + (a-b)^2)^2 = 2(a^4 + b^4 + (a-b)^4)$$

(this is verified by simple algebra).

Remark: Part b is almost the same as problem 5 of BW 1992 contest.

A6. *Find all real numbers a for which there there exists a non-constant function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equations*

1) $f(ax) = a^2 f(x)$
2) $f(f(x)) = a\ f(x)$.

Answer: $a \in \{0, 1\}$

Solution: Examining $f(f(f(x)))$ we can write

$$\begin{aligned} a^2 f(x) &\overset{(2)}{=} af(f(x)) \overset{(2)}{=} f(f(f(x))) \\ &\overset{(2)}{=} f(af(x)) \overset{(1)}{=} a^2 f(f(x)) \overset{(2)}{=} a^3 f(x) \end{aligned}$$

which implies $a \in \{0, 1\}$ or $f(x) \equiv 0$.

If $a = 1$, then function $f(x) = x$ satisfies both conditions.

If $a = 0$, then function $f(x) = |x| - x$ satisfies both conditions. In general, every function which sends all negative numbers to non-negative numbers and all non-negative numbers to zero satisfies conditions. For example, function which sends -1 to 1 and everything else to zero is suitable.

Therefore the only suitable values of a are $a = 0$ and $a = 1$.

A7. *Find all real solutions of the equation*

$$\frac{(x+y)(2-\sin(x+y))}{4\sin^2(x+y)} = \frac{xy}{x+y}.$$

Answer: $x = y = \frac{\pi}{4} + n\pi$, for $n = 0, 1, \ldots$.

Solution: Under the condition $\sin(x+y) \neq 0$, the equation will be equivalent to

$$(x+y)^2(2-\sin(x+y)) = 4xy\sin^2(x+y).$$

The left-hand side is non-negative; hence $xy \geq 0$. We then have

$$(x+y)^2(2-\sin(x+y)) \geq (x+y)^2 \geq 4xy \geq 4xy\sin^2(x+y).$$

Equality holds in the second inequality if and only if $x = y$, and in the first and third inequalities if and only if $\sin(x+y) = 1$ (since $x+y = 0$ would imply $\sin(x+y) = 0$, violating the condition above). Together these equations yield the solution $x = y = \frac{\pi}{4} + n\pi$, for $n = 0, 1, \ldots$.

A8. *Find all quadruples (a, b, c, d) of real numbers that simultaneously satisfy the following equations:*

$$\begin{cases} a^3 & + & c^3 & = & 2 \\ a^2b & + & c^2d & = & 0 \\ b^3 & + & d^3 & = & 1 \\ ab^2 & + & cd^2 & = & -6. \end{cases}$$

Consider the polynomial $P(x) = (ax+b)^3 + (cx+d)^3 = 2x^3 - 18x + 1$. By $P(0) > 0$, $P(1) < 0$, $P(3) > 0$, it has two distinct real zeros x_1 and x_2. Since $P(x) = 0$ implies that $(a+c)x + (b+d) = 0$, it follows that $a + c = b + d = 0$. This contradicts the first equation $a^3 + c^3 = 2$. Hence, the system has no solution.

Remark: Asking to show that there is no solution would probably make the problem much easier.

The numbers on the RHS can of course be replaced by any other choice of numbers for which the corresponding polynomial $P(x)$ has distinct real roots.

A9. *Let $a_{0,1}, a_{0,2}, \ldots, a_{0,2016}$ be positive real numbers. For $n \geq 1$ and $1 \leq k < 2016$ set*

$$a_{n+1,k} = a_{n,k} + \frac{1}{2a_{n,k+1}}$$

and

$$a_{n+1,2016} = a_{n,2016} + \frac{1}{2a_{n,1}}.$$

Let

$$m_n = \max_{1\leq k\leq 2016} a_{n,k} \quad \text{for } n \geq 0.$$

Show that $m_{2016} > 44$.

We prove

$$m_n^2 \geq n \tag{4}$$

for all n. The claim then follows from $44^2 = 1936 < 2016$. To prove (4), first notice that the inequality certainly holds for $n = 0$.

Assume (4) is true for n. There is a k such that $a_{n,k} = m_n$. Also $a_{n,k+1} \leq m_n$ (or if $k = 2016$, $a_{n,1} \leq m_n$). Now (assuming $k < 2016$)

$$a_{n+1,k}^2 = \left(m_n + \frac{1}{2a_{n,k+1}}\right)^2 = m_n^2 + \frac{m_n}{a_{n,k+1}} + \frac{1}{4a_{n,k+1}^2} > n+1.$$

Since $m_{n+1}^2 \geq a_{n+1,k}^2$, we are done.

C1. *Set A consists of 2016 natural numbers. All prime divisors of these numbers are smaller than 29. Prove that there are four distinct numbers a, b, c and d in A such that $abcd$ is a square.*

There are nine prime numbers smaller than 29. Let us denote them as $p_1, p_2, \ldots, p_9$. To each number n from A we can assign a 9-element sequence $(n_1, n_2, \ldots, n_9)$ such that $n_i = 1$ when in factorization of n p_i has odd exponent, and $n_i = 0$ otherwise. There are only 512 different 9-element $\{0,1\}$-sequences, so there exist some four numbers a, b, c and d in A that have identical sequences assigned. It is easy to see that such numbers satisfy conditions of the problem.

Remark: We may raise 29 to 30 and allow 10 primes to be considered. Then there would be $2^{10} = 1024$ boxes, and if none of them contained 4 numbers, there would be two boxes containing pairs of given numbers. This suffices.

C2. *Does there exist a hexagon with side lengths 1, 2, 3, 4, 5, 6 (not necessarily in this order) that can be tiled with a) 31 b) 32 equilateral triangles with side length 1?*

a) Yes, for example, see Figure 1.

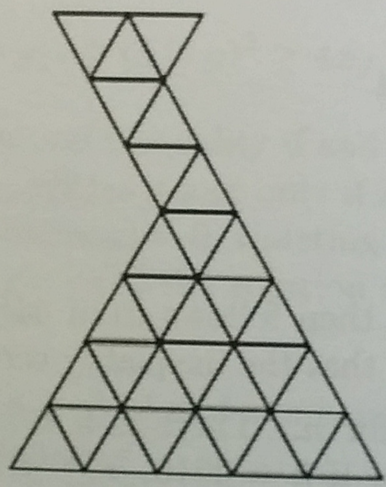


Figure 1:

b) No, the number of triangles cannot be an even number. Denote the number of triangles by x, then in total there are $3x$ sides. Some sides touch each other; let there be n such places. Others form the perimeter of the hexagon, whose length is $1+2+3+4+5+6 = 21$. From here we get an equation

$$3x = 2n + 1$$

where we see that the right hand side is odd, therefore the left hand side also should be odd, which implies that x should be odd.

C3. *Mobile operator has n clients and holds the following advertising campaign. In the beginning it deposits 1 € on account of each* *client. When two persons which have a € and b € in their accounts communicate by a phone the operator makes both accounts equal to $(a+b)$ €. It happens that after $h(m)$ phone calls all n clients' accounts become equal m €. Prove that $h(m) \leq \frac{1}{2} n \log_2 m$.*

Let the product of the clients' values after the k-th call be a_k. Suppose the values of two persons before a call were a and b. By the arithmetic-geometric mean inequality, $(a+b)(a+b) \geq 4ab$. Therefore, regardless of the choice of a call, $a_k \geq 4a_{k-1}$. Since the initial and final values of a_k are 1 and m^n, the number of calls is at most $\log_4(m^n)$.

C4. *A magic octagon is an octagon whose sides go along the grid lines of a square grid and side lengths are 1, 2, 3, 4, 5, 6, 7, 8 (in any order). What is the largest possible area of a magic octagon?*

Answer: 71.

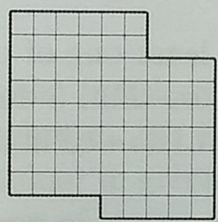


Figure 2:

Solution: Figure 2 shows an example of a magic octagon with area 71. Let us show that it cannot be larger. This octagon has some 90° angles and some 270° angles. From the equation

$$a \cdot 90^\circ + (8-a) \cdot 270^\circ = 6 \cdot 180^\circ$$

we get that is has exactly two 270° (and six 90°) angles.

First let us look at the case when these 270° angles are consecutive. Then the octagon looks like in figure 3. and its area is less than $m \cdot n$ which cannot exceed $7 \cdot 8 = 56$.

And now let us consider the case when these angles are not consecutive. Then they can be "pushed out" (see figure 4.) by increasing the size of the octagon by $m \cdot n$ and without changing its perimeter. If we do it with both 270° angles then we obtain a rectangle with

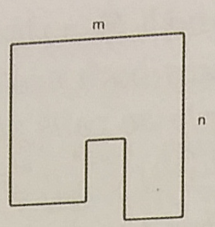


Figure 3:

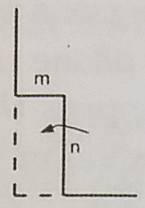


Figure 4:

the same perimeter as our original octagon (it is $1+2+\cdots+8=36$) and with area $S+ab+cd$ for some numbers a,b,c and d.

Maximum area of this rectangle with perimeter 36 is 81 (when it is a square) therefore we have an inequality

$$S+ab+cd \le 81.$$

It is easy to notice that at least one of these numbers is not less than 4, at least one is not less than 3 and at least one is not less than 2. The minimum value of $ab+cd$ in that case is $1\cdot 4+2\cdot 3=10$ from where we get $S \le 71$.

C5. *In a computer game a $4\times 4\times 4$ cube is built using 4^3 unit cubes. At the beginning of the game each unit cube contains an integer. In each turn of the game, you choose a unit cube and increase by 1 all the integers in the cubes having a face in common with the chosen cube. You win the game if you reach a position in a finite number of turns where all the 4^3 integers are divisible by 3.*

Is it possible to win the game no matter what the starting position is?

Answer: No.

Solution: Two unitcubes with a common face are called neighbours. Colour the cubes either black or white in such a way that two neighbours always have different colours. Notice that the integers in the white cubes only change when a black cube is chosen. Now recolour the white cubes that have exactly 4 neighbours and make them green. If we look at a random black cube it has either 0, 3 or 6 white neighbours. Hence if we look at the sum of the integers in the white cubes, it changes by 0, 3 or 6 in each turn. From this follows that if this sum is not divisible by 3 at the beginning, it will never be, and all the integers in the white cubes are not divisible by 3 at any state.

+ **C6.** *Arbitrary integers are written in the cells of the 13×13 table. Prove that it is possible to choose 2 rows and 4 columns such that the sum of eight numbers in the intersecting cells is divisible by 8.*

Lemma. For any seven integers we can choose four of them whose sum is divisible by 4.

Proof. Choose two numbers a_1 and b_1 of the same parity. The remaining five numbers contain two more numbers a_2 and b_2 of the same parity. The remaining three numbers contain two more numbers a_3 and b_3 of the same parity. Let $s_i = \frac{1}{2}(a_i + b_i)$. Then two of the numbers s_1, s_2, s_3 have the same parity. If these numbers are s_1 and s_2 then $a_1 + b_1 + a_2 + b_2 = 2(s_1 + s_2)$ is divisible by 4.

Now start the solution of the problem. Choose 7 numbers of the same parity in the first column. We have 7 rows in which these numbers are located, lets colour in green all the cells (and numbers) in these rows. Consider one of the remaining 12 columns. It has 7 green cells. If these cells contain a even and b odd numbers ($a+b=7$), then we have $a(a-1)/2+b(b-1)/2$ pairs of numbers of the same parity. The minimum of this expression equals $6+3=9$ (when $\{a,b\} = \{4,3\}$). Thus for each of 12 columns we have at least 9 pairs of green numbers, i.e. at least $12 \cdot 9 = 108$ green pairs of the same parity totally. Each of these (vertical!) 108 green pairs belongs to one of $7 \cdot 6/2 = 21$ pairs of green rows. Therefore we can find a pair of green rows that contains at least 6 vertical pairs. Taking into account the first column we found in these two rows 7 vertical pairs of numbers with even sums; denote the sums by $2t_1$, $2t_2$, $\ldots$, $2t_7$. By lemma, we can choose four numbers among t_1, $\ldots$, t_7 such that their sum is divisible by 4. Then the corresponding sum of the four numbers $2t_i$ is divisible by 8. This sum is a sum of four vertical pairs of numbers belonging to the two chosen rows.

+ **C7.** *All n-digit positive integers from 10^{n-1} to $10^n - 1$ are concatenated in the increasing order. What is the largest possible value of k*

for which it is possible to find in this sequence of digits the same k-digit substring in at least two different places?

Answer: $k = 2n - 1$.

Solution: Let us look at the sequence

$$\underbrace{33\ldots3}_{n-1}9\underbrace{33\ldots3}_{n-2}4$$

Its length is $2n-1$ and it is a substring for two consecutive integers

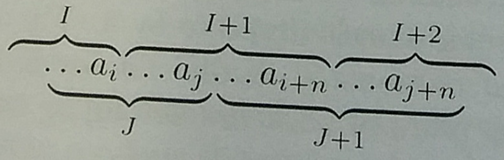
$$\underbrace{33\ldots3}_{n-1}9 \quad \text{and} \quad \underbrace{33\ldots3}_{n-2}40$$

as well as for

$$9\underbrace{33\ldots3}_{n-1} \quad \text{and} \quad 9\underbrace{33\ldots3}_{n-2}4.$$

It remains to prove that there is no such substring of length $2n$. We will use the following simple observation: if for two consecutive n-digit integers A and $A+1$ the $i-th$ digit differs and $i < n$ then the $(i+1)$-st digit of A is 9 and the $i+1$-st digit of $A+1$ is 0.

Assume the contrary that there is a $2n$ digit substring $a_1 \ldots a_{2n}$ that can be found twice in this sequence. In each of these occurrences there are two digits in our substring that are the last digit of the number, let's denote these digits a_i and a_{i+n} in the first occurrence and a_j and a_{j+n} in the second, wlog. assume that $a_i < a_j < a_{i+n} < a_{j+n}$. Let the number that ends with digit a_i in the first occurrence be I, then $I+1$ ends with a_{i+n} and $I+2$ follows. In the second occurrence let the number that ends with digit a_j be J, then $J+1$ ends with digita_{j+n}.



First we observe that $a_i \neq a_{i+n}$ and $a_j \neq a_{j+n}$ as these are the last digits of consecutive integers. As we see that for J and $J+1$ digits a_i and a_{i+n} differ then we conclude (using our observation) that digit $a_{i+1} = 9$ and $a_{i+n+1} = 0$. But a_{i+n+1} is the first digit for the number $I + 2$ what is a contradiction — a number cannot start with zero.

Note: Any n-digit number that ends with 9 (except $99\ldots9$) leads to a solution, for example if $n = 4$ we have both 2369 2370 and 9236 9237 having the same 7-digit substring 2369237.

+H **C8.** *There are n students at a school. It is known that, for any two students A and B, either A loves B or B loves A (but not both). (A student may thus be in love with several other students.) A love triangle is a configuration of three students A, B, C such that A loves B, B loves C and C loves A. What is the maximal number of love triangles, given the number n of students?*

Answer: $\dfrac{n(n^2-1)}{24}$ if n is odd and $\dfrac{n(n^2-4)}{24}$ if n is even.

Solution: Let a_i be the number of students that student i loves $(1 \le i \le n)$. Then the number of love triangles is

$$T = \binom{n}{3} - \binom{a_1}{2} - \cdots - \binom{a_n}{2}.$$

This is explained as follows. In a non-love triangle ABC, one student, say A, loves B and C, while B loves C. For each pair B, C of students that A loves, one triangle is thus deducted above from the total number of triangles.

Simplifying the above expression, we get

$$T = \binom{n}{3} + \frac{1}{2}(a_1 + \cdots + a_n) - \frac{1}{2}(a_1^2 + \cdots + a_n^2).$$

We have $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$ and $a_1 + \cdots + a_n = \binom{n}{2} = \frac{n(n-1)}{2}$, while the sum of squares is bounded by Chebyshev's Inequality:

$$\frac{a_1^2 + \cdots + a_n^2}{n} \ge \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 = \frac{1}{n^2}\binom{n}{2}^2 = \frac{(n-1)^2}{4},$$

yielding

$$a_1^2 + \cdots + a_n^2 \ge \frac{n(n-1)^2}{4}.$$

Hence an upper bound for the number of love triangles is given by

$$T \leq \frac{n(n-1)(n-2)}{6} + \frac{1}{2} \cdot \frac{n(n-1)}{2} - \frac{1}{2}\frac{n(n-1)^2}{4} = \frac{n(n^2-1)}{24}.$$

Equality holds if and only if $a_1 = \cdots = a_n = \frac{n-1}{2}$. This can easily be arranged when n is odd, by making student i fall in love with students $i+1, \ldots, i+\frac{n-1}{2}$ (counting cyclically).

When n is even, this bound is clearly impossible to attain, and we must proceed differently. We investigate, for n even, the minimal value of $a_1^2 + \cdots + a_n^2$, subject to the condition $a_1 + \cdots + a_n = \binom{n}{2}$.

Suppose first the sum of squares is at a minimum, but that $a_p - a_q \geq 2$ for some p and q. We may then replace a_p and a_q by $a_p - 1$ and $a_q + 1$, respectively, which will serve to decrease the sum of squares:

$$(a_p - 1)^2 + (a_q + 1)^2 = a_p^2 + a_q^2 + 2(1 + a_q - a_p) < a_p^2 + a_q^2.$$

Consequently, when the sum of squares is at a minimum, the maximal difference among the numbers a_i is at most 1.

Suppose next, without loss of generality, that $a_1 = \cdots = a_k = x$ and $a_{k+1} = \cdots = a_n = x + 1$. Then

$$\binom{n}{2} = a_1 + \cdots + a_n = kx + (n-k)(x+1) = nx + n - k$$

$$\Leftrightarrow \quad k = nx + n - \binom{n}{2},$$

transforming the inequality $0 \leq k \leq n$ into

$$0 \leq nx + n - \binom{n}{2} \leq n \quad \Leftrightarrow \quad \frac{n-1}{2} - 1 \leq x \leq \frac{n-1}{2}$$

$$\Leftrightarrow \quad x = \frac{n-2}{2}.$$

This corresponds to $k = \frac{n}{2}$, and so the minimum is attained when half of the a_i equal $\frac{n-2}{2}$ and the remaining half equal $\frac{n}{2}$. The minimal quadratic sum is

$$\frac{n}{2}\left(\frac{n-2}{2}\right)^2 + \frac{n}{2}\left(\frac{n}{2}\right)^2 = \frac{n(n^2-2n+2)}{4}.$$

We thus find

$$T \leq \frac{n(n-1)(n-2)}{6} + \frac{1}{2} \cdot \frac{n(n-1)}{2} - \frac{1}{2} \cdot \frac{n(n^2-2n+2)}{4} = \frac{n(n^2-4)}{24}.$$

The bound is attained when student i falls in love with students $i+1, \ldots, \frac{n}{2}-1$ (counting cyclically), as well as with student $i+\frac{n}{2}$ if $1 \leq i \leq \frac{n}{2}$.

Remark: This is of course a well-known result in the theory of tournaments in disguise, see, e.g., John Moon: *Topics of tournaments*, first Corollary in Chapter 5. We do not think this would reduce the potential of this problem as a contest problem.

+?

C9. *Let k and t be integers with $1 \leq k/2 < t < k$. Each square of a $k \times k$ checkerboard is coloured either red or blue. A move consists of choosing a row or a column with at most t red squares and switching the colour of these red squares to blue. Assume that it is possible to make all squares of the checkerboard blue with a sequence of moves, and let m be the least number of moves required to do so. Prove that if $m > k$, then the total number of initially red squares is at least $k + 2t$.*

Assume that $m \geq k+1$. Let T be the set of all initially red squares, and assume that $|T|$ is as small as possible. It follows that $m = k+1$ because otherwise after the first $m-(k+1)$ moves the number of red squares has decreased and still there are $k+1$ moves required, a contradiction.

If each row contained at most t red squares, then we could make the whole board blue in at most k moves. Hence, there is a row R with at least $t+1$ red squares. Similarly, there is a column C with at least $t+1$ red squares. Now we can make sure that there are two moves in each of which exactly t squares switch their colour from red to blue because as soon as there are exactly t red squares left on R or C, we can proceed with the move along R or C, respectively. (This may increase the total number of moves.) In each of the remaining at least $k-1$ moves at least one red square changes its colour to blue. Consequently, $|T| \geq 2t + k - 1$.

Finally, assume that $|T| = 2t + k - 1$. We can proceed with the move along R or C as soon as there are only t red squares left on R or C, respectively. As the total number of moves is at least

$k+1$ and with each move at least one square changes its colour, it follows that the total number of moves can only be $k+1$ and in each move exactly one square changes its colour. If there was a row $R' \neq R$ containing $r \in \{2, 3, \ldots, t\}$ red squares, then we could chose the first move to be the one along R', a contradiction. Hence, each row – and similarly, each column – contains at most one or at least $t+1$ red squares. Assume there is a column $C' \neq C$ containing at least $t+1$ red squares. By $t > k/2$, there must be a row $R'' \neq R$ which intersects both, C and C' in a red square. It follows that R'' must contain at least $t+1$ red squares. Now in total there are at least $4t$ red squares in R, R'', C, C'. This implies $2t + k - 1 = |T| \geq 4t$, a contradiction to $t > k/2$. Consequently, any row other than R contains at most one red square and, hence, $2k - 1 \geq |T| = 2t + k - 1$ contradicting $t > k/2$.

Remark: The bound can be attained. If the square in row i and column j is labelled (i, j), then choose the following $k + 2t$ squares to be red: $(i, 1)$ with $i = 1, 2, \ldots, t+1$, $(j, 1)$ with $j = 2, 3, \ldots, t+1$, and (i, i) with $i = 2, 3, \ldots, k$.

+ **C10.** *Prove that*

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k}\binom{kn}{n} = (-n)^n$$

holds for every positive integer n.

Consider an $n \times n$ checkerboard and count the number s of ways to color exactly one square in each column. On the one hand, $s = n^n$. On the other hand, if a_i denotes the number of ways to color exactly n squares such that some fixed i columns do not contain a colored square, then $s = \sum_{i=0}^{n-1}(-1)^i \binom{n}{i} a_i$ by inclusion-exclusion. By $a_i = \binom{(n-i)n}{n}$, we have

$$s = \sum_{i=0}^{n-1}(-1)^i \binom{n}{i}\binom{(n-i)n}{n} = n^n,$$

and setting $k = n - i$ gives the claim.

Variants: We think that asking to prove the more general identity

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{km}{n} = (-m)^n$$

makes the problem easier. Using an $m \times n$ checkerboard, the above solution can be adopted. Just the special case $m = 2$, i.e.

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = 2^n$$

is another option.

C11. *A graph has 2016 vertices. All the edges of the graph are coloured blue or red. It is known that the graph contains no blue path on 1062 vertices. Prove that it is possible to select two disjoint sets of 477 vertices each, such that all the edges between these sets are red.*

Denote the graph by G, let $n = 106\cancel{0}2$. Let P_n denote a path on n vertices. We perform the following algorithm on G and construct a blue path P. Let v_1 be an arbitrary vertex of G, let $P = (v_1)$, $U = V \setminus \{v_1\}$, and $W = \varnothing$. We investigate all edges from v_1 to U searching for a blue edge. If such an edge is found (say from v_1 to v_2), we extend the blue path as $P = (v_1, v_2)$ and remove v_2 from U. We continue extending the blue path P this way for as long as possible. Since there is no blue P_n, we must reach the point of the process in which P cannot be extended, that is, there is a blue path from v_1 to v_k ($k < n$) and there is no blue edge from v_k to U. This time, v_k is moved to W and we try to continue extending the path from v_{k-1}, reaching another critical point in which another vertex will be moved to W, etc. If P is reduced to a single vertex v_1 and no blue edge to U is found, we move v_1 to W and simply restart the process from another vertex from U, again arbitrarily chosen. During this algorithm there is never a blue edge between U and W. Moreover, in each step of the process, the size of U decreases by 1 or the size of W increases by 1. Finally, since there is no blue P_n, the number of vertices of the blue path P is always smaller than n. Hence, at some point of the process both U and W must have size at least $(2016 - n)/2$. After removing some vertices from U or W, if needed, both sets have sizes precisely $(2016 - 1062)/2 = 477$.

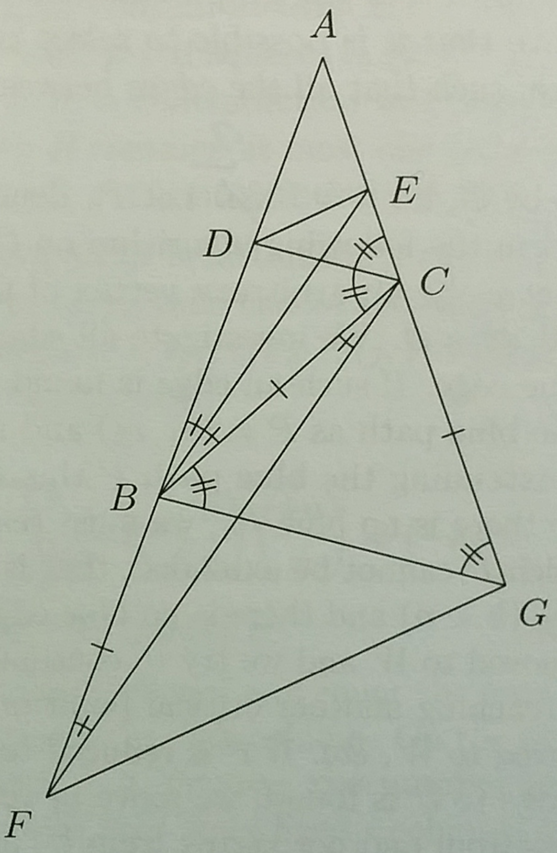
G1. *In triangle ABC, the points D and E are the intersections of the angular bisectors from C and B with the sides AB and AC. Points*

F and G on the extensions of AB and AC beyond B and C satisfy $BF = CG = BC$. Prove $FG \parallel DE$.

From $BC = BF$ follows $\sphericalangle BFC = \sphericalangle BCF$. From $\sphericalangle BFC + \sphericalangle BCF = \sphericalangle ABC$ then follows $\sphericalangle BFC = \sphericalangle ABE$. Hence $FC \parallel BE$. Analogously $GB \parallel CD$. Therefore

$$\frac{AF}{AG} = \frac{AF}{AC}\frac{AC}{AB}\frac{AB}{AG} = \frac{AB}{AE}\frac{AC}{AB}\frac{AD}{AC} = \frac{AD}{AE},$$

whence the assertion follows.



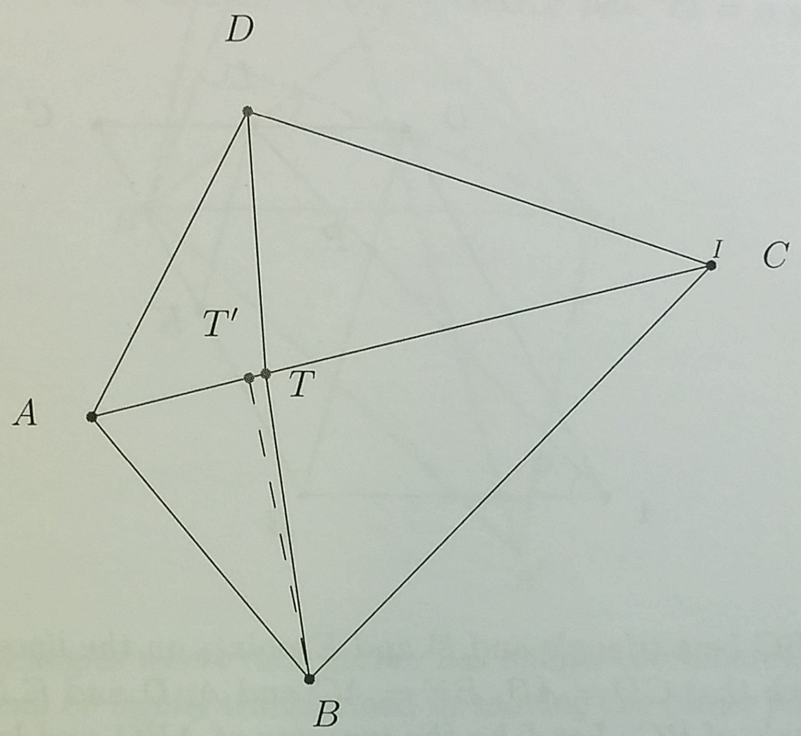
G2. *$ABCD$ is a convex quadrilateral such that $AB = AD$. T is a point on the diagonal AC such that $\sphericalangle ABT + \sphericalangle ADT = \sphericalangle BCD$. Prove that $AT + AC \geq AB + AD$.*

Let T' be a point on the ray AC such that $AT' \cdot AC = AB^2 = AD^2$. Then triangles ACB and ABT' are similar (by two sides and the angle between them), hence $\sphericalangle ACB = \sphericalangle ABT'$. The triangles ACD and ADT' are similar by analogous reasons, therefore $\sphericalangle ACD = \sphericalangle ADT'$. Thus,

$$\sphericalangle ABT + \sphericalangle ADT = \sphericalangle BCD = \sphericalangle ACB + \sphericalangle ACD = \sphericalangle ABT' + \sphericalangle ADT'.$$

But the sum $\sphericalangle ABT + \sphericalangle ADT$ changes monotonically when we move point T along the ray AC. Therefore the last equality implies $T = T'$. Then $AT \cdot AC = AB^2$ and

$$AT + AC \geq 2\sqrt{AT \cdot AC} = 2AB = AB + AD.$$



G3. *Let $ABCD$ be a parallelogram such that $\sphericalangle BAD = 60°$. Let K and L be the midpoints of BC and CD, respectively. Assuming that $ABKL$ is a cyclic quadrilateral, find $\sphericalangle ABD$.*

Answer: $75°$.

Solution: Let $\sphericalangle BAL = \alpha$. Observe that

$$\sphericalangle ADB = \sphericalangle CBD = \sphericalangle CKL = \sphericalangle BAL = \alpha.$$

The second equality holds since KL is a mid-segment of BCD, and the third equality holds since $ABKL$ is inscribed.

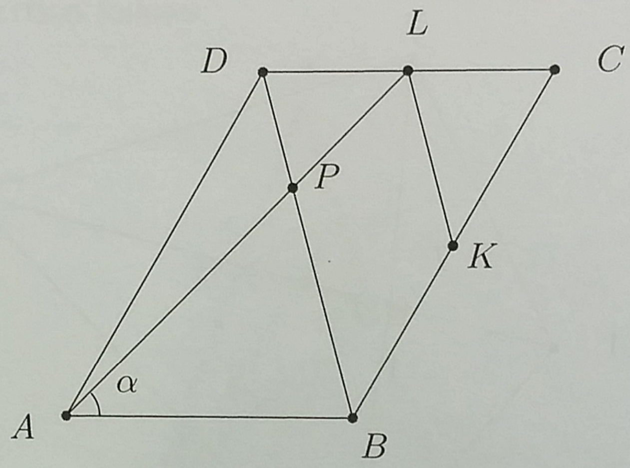
Let P be the intersection point of BD and AL. Triangles ABP and DBA are similar yielding

$$\frac{AB}{DB} = \frac{BP}{AB}.$$

Since triangles ABP and DLP are similar with the scale factor 2, we have $BP = \frac{2}{3}DB$. Therefore, $AB^2 = \frac{2}{3}DB^2$, or $AB = \sqrt{\frac{2}{3}}DB$. Now, law of sines in triangle ABD gives us

$$\frac{AB}{\sin\alpha} = \frac{BD}{\sin 60^\circ} \quad \Rightarrow \quad \sin\alpha = \frac{AB}{BD} \cdot \sin 60^\circ = \frac{\sqrt{2}}{2}$$

yielding $\alpha = 45^\circ$ and $\measuredangle ABD = 120^\circ - \measuredangle CBD = 75^\circ$.

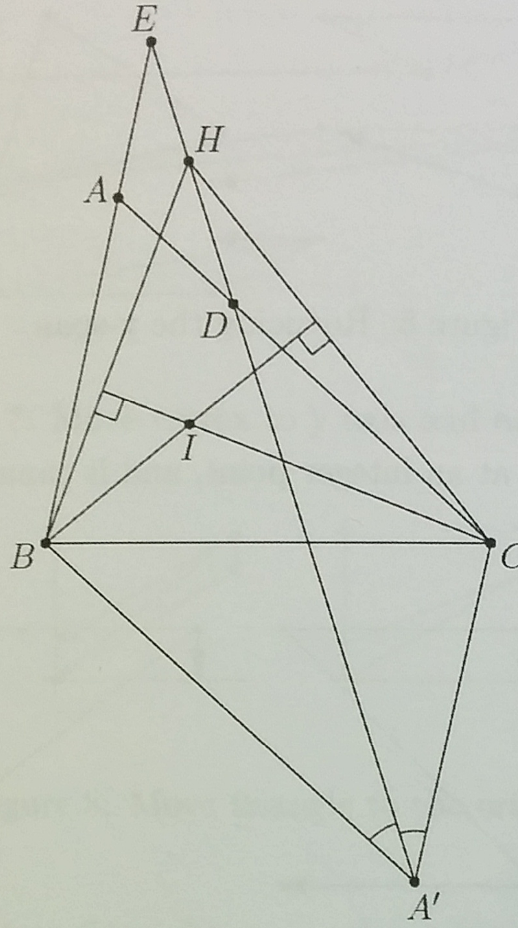


G4. *Let ABC be a triangle and D and E points on the lines CA and BA such that $CD = AB$, $BE = AC$ and A, D and E lie on the same side of BC. Let I be the incenter of ABC and let H be a point such that I I the orthocenter of BCI. Show that D, E and H are collinear.*

Let the point A' be such that $ABA'C$ is a parallelogram with $AB \parallel A'C$ and $AC \parallel A'B$. Denote $\alpha = \measuredangle BAC = \measuredangle CA'B$.

Since $CD = AB = CA'$, we find that CDA' is an isosceles triangle. As $\measuredangle DCA' = 180^\circ - \alpha$, we deduce that $\measuredangle A'DC = \measuredangle CA'D = \frac{\alpha}{2}$, so that D lies on the angle bisector ℓ of $\measuredangle CA'B$. Similarly E lies on ℓ.

Next notice that $\measuredangle CBH = 90^\circ - \measuredangle BCI = 90^\circ - \frac{1}{2}\measuredangle BCA = 90^\circ - \frac{1}{2}\measuredangle A'BC$, so BH is the exterior angle bisector of $\measuredangle A'BC$. Similarly CH is the exterior angle bisector of $\measuredangle A'CB$, so H is in fact the excenter of triangle $A'BC$ opposite A'. Therefore we conclude that D, E, and H all lie on ℓ.



G5. *Consider triangles where each corner has integer coordinates. Such a triangle can be legally transformed by moving one corner parallell to the opposite side to a different point with integer coordiantes. Show that if two triangles with integer coordinates have the same area, then there exists a series of legal transforms that transforms one to the other.*

We will first show that any such triangle can be transformed to a a *special* triangle whose corners are at $(0,0)$, $(0,1)$ and $(n,0)$. Since every transformation preserves the triangle's area, triangles with the same area will have the same value for n.

Define *y-span* of a triangle to be the difference between the largest and the smallest y value of its vertices. First we show that a triangle with a y-span greater than one can be transformed to a triangle with a strictly lower y-span.

If none of the vertices have the same y coordinate, move vertice with minimal y upwards, as in figure 5, reducing the y-span. The point is moved by a vector equal to the difference of the oppisite

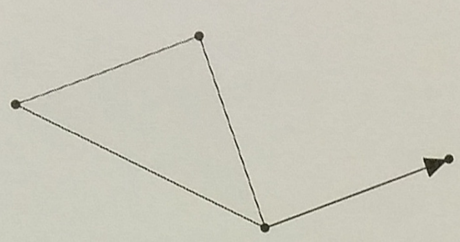


Figure 5: Reducing the y-span

side, so it ends up at an integer point, and it cannot pass the top of the old triangle.

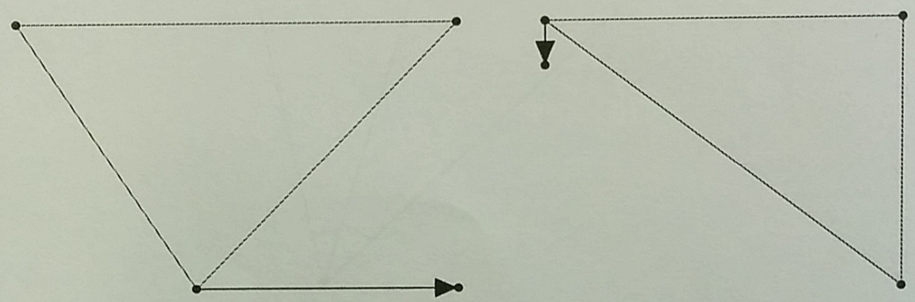


Figure 6: Prepare for y-span reduction

If two vertices have the same y coordinate, use figure 6 to move one of these between the others (which is possible since the y-span was at least two), and then do the previous transformation to reduce the y-span.

By induction, any triangle can be transformed to a triangle with y-span equal to one.

When the triangle has been transformed to an y-span of 1, use figure 7 to move one vertex to the y axis, and do two steps to make the opposite side vertical. It may be that the figure have to be flipped, but the next step removes this as a different case then the illustrated one.

Finally, use figure 8 to transform the triangle to the origin. Since the reverse of a legal transform is also a legal transform, any

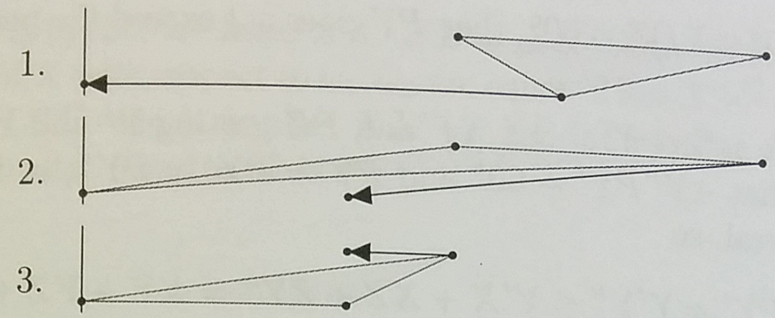


Figure 7: Move vertex to y axis and normalize

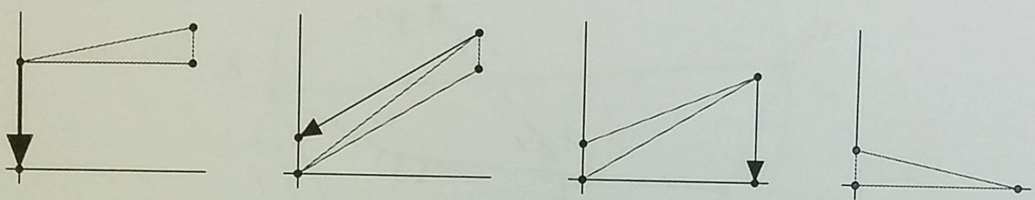


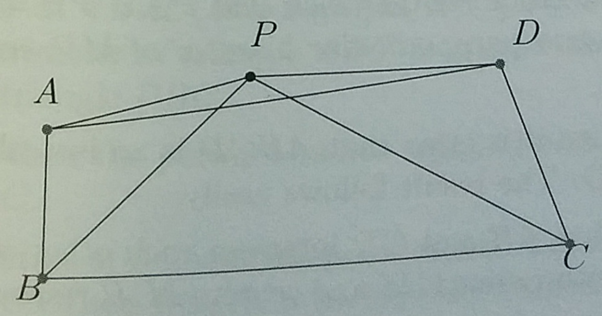
Figure 8: Move triangle to the origin

triangle can be transformed to any other triangle with the same area, via the special triangle.

Bonus question: Does there exist two triangles of the same area where you need more than 2016 steps to transform one to the other?

G6. *Let $ABCD$ be a convex quadrilateral. Let P be a point such that $\measuredangle APC = \measuredangle BPD = 30°$. Prove that*

$$2(AB + AC + AD + BC + BD + CD) \geq PA + PB + PC + PD.$$

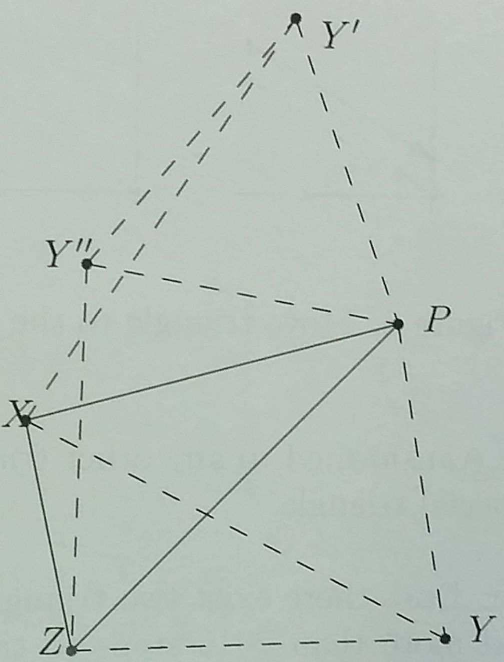


Lemma. If $\measuredangle XPZ = 30^\circ$, then PY does not exceed the perimeter of XYZ.

Proof: We reflect Y about XP and PZ getting Y' and Y''. We observe that $\measuredangle Y''PY' = 60^\circ$ and $PY = PY' = PY''$, so $PY'Y''$ is equilateral, so

$$PY = PY' = Y'Y'' \leq Y'X + XZ + ZY'' = XY + YZ + ZX$$

by triangle inequality. ■



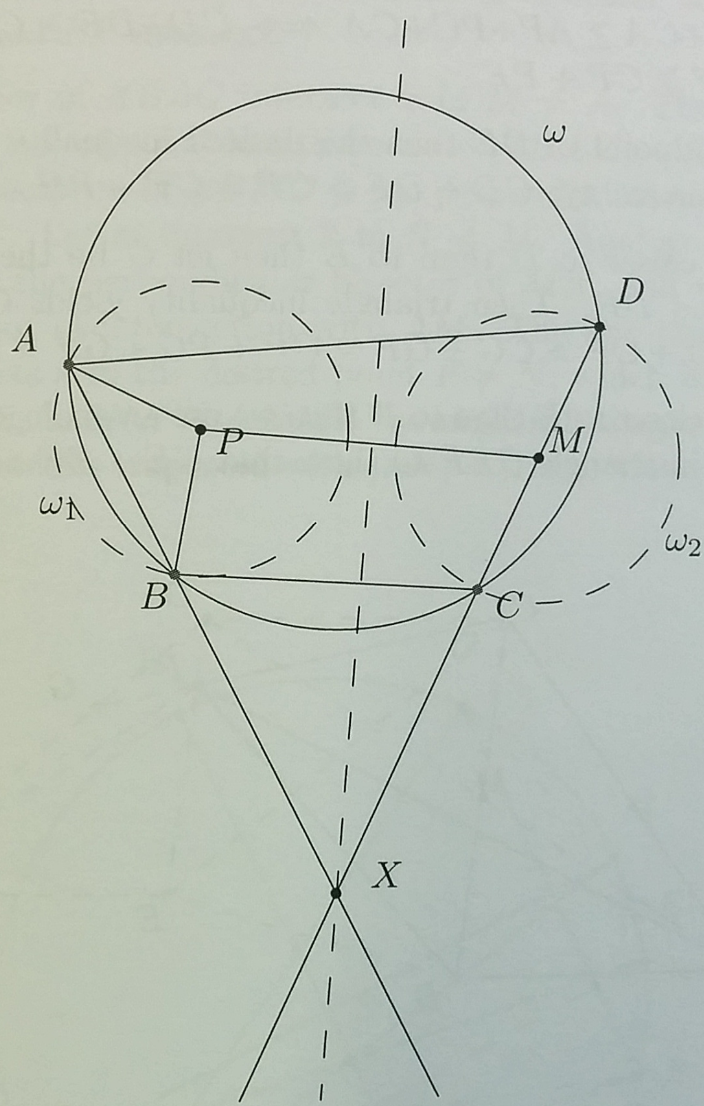
Now we use the lemma for $XYZ = ABC, BCD, CDA, DAB$ and sum up obtained inequalities.

G7. *Let $ABCD$ be a cyclic quadrilateral. Let M be the midpoint of CD. Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD, and perpendicular bisector of MP are concurrent or parallel.*

If $AB \parallel CD$ then it is clear that $ABCD$ is an isosceles trapezoid and $MP \perp CD$. The result follows easily.

Now assume that AB and CD intersect each other at X. Let ω_1, ω_2 be circles with radius CM and centers M, P respectively. Since

$ABCD$ is cyclic, we have $XA \cdot XB = XC \cdot XD$. Therefore the powers of X with respect to ω_1, ω_2 are equal. Thus $XM^2 - CM^2 = XP^2 - CM^2$ which implies $XM = XP$. Therefore X lies on perpendicular bisector of MP.



G8. *Let ABC be a triangle and let P be a point such that AP is the angle bisector of $\measuredangle BAC$ and segment BC bisects segment AP. Prove that perimeter of triangle ABC is greater than or equal to perimeter of triangle PBC.*

If $AB = AC$ then the perimeters are equal. Further we assume that $AB < AC$.

Let D be reflection of A about midpoint of BC. Let E be reflection of A about BC. Then $BCDE$ is an isosceles trapezoid. Moreover

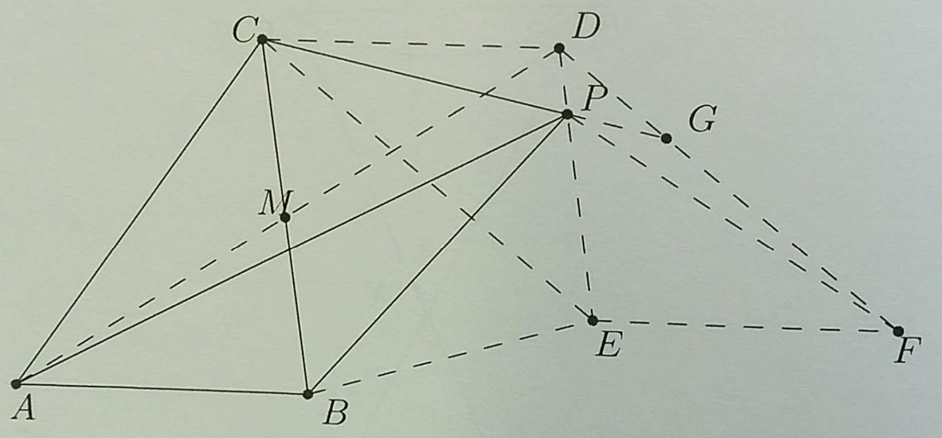
P lies on the segment DE because angle bisector lies between altitude and median.

Let F be a point such that $CDFE$ is a parallelogram, then F is symmetric to B with respect to DE. We have $EF = CD = AB$, $DF = CE = AC$, $PB = PF$. We need to prove that $AB+BC+CA \geq AP+PC+CA \iff CD+DF \geq CP+PF \iff CE+EF \geq CP+PF$.

If F is midpoint of DE then the desired inequality follows from triangle inequality: $CD + DF > CF = CP + PF$.

If P lies closer to D than to E then let G be the intersection of CP and DF. Then triangle inequality yields $CD + DF = CD + DG + GF > CG + GF = CP + PG + GF > CP + PF$.

IF P lies closer to E than to D then we use an analogous argument as above in triangle CEF to show that $CE + EF > CP + PF$.



G9. *Starting with three points A, B, C in general position and the circumcircle k of $\triangle ABC$, a step consists of drawing a line ℓ and obtaining all points of intersection of ℓ with the lines already drawn and with k, where ℓ is*

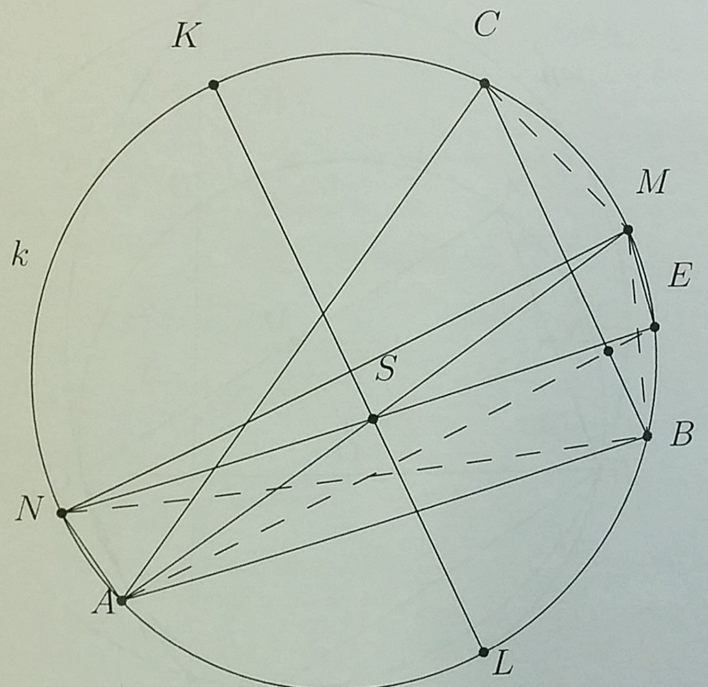
(1) the line through two distinct points that had been obtained before or

(2) the bisector of an angle $\sphericalangle XYZ$, where X, Y, Z are three previously obtained distinct points on k.

Is it always (i.e., for any choice of A, B, C) possible to obtain the orthocentre of $\triangle ABC$ in a finite number of steps?

The answer is yes. It suffices to describe how to obtain the point $E \neq A$ on k with $AE \perp BC$. We can then obtain the point $F \neq B$ on k with $BF \perp AC$ analogously, and the orthocentre of $\triangle ABC$ is where AE and BF intersect.

Let the bisector of $\sphericalangle BAC$ intersect k in $M \neq A$. The chords BM and CM are of equal length because $|\sphericalangle BAM| = |\sphericalangle CAM|$. Hence, the bisector m of $\sphericalangle BMC$ is the perpendicular bisector of the chord BC. Let m intersect k in $N \neq M$. Analogously, we can construct the perpendicular bisector of MN, and we let S denote its point of intersection with AM. The line through N and S intersects k in the desired point $E \neq N$, which is easy to verify. Indeed, $\triangle MNE \cong \triangle NMA$ by angle-side-angle, making $MNAE$ an isosceles trapezoid with $|AN| = |EM|$. Consequently, $AE \perp CB$.



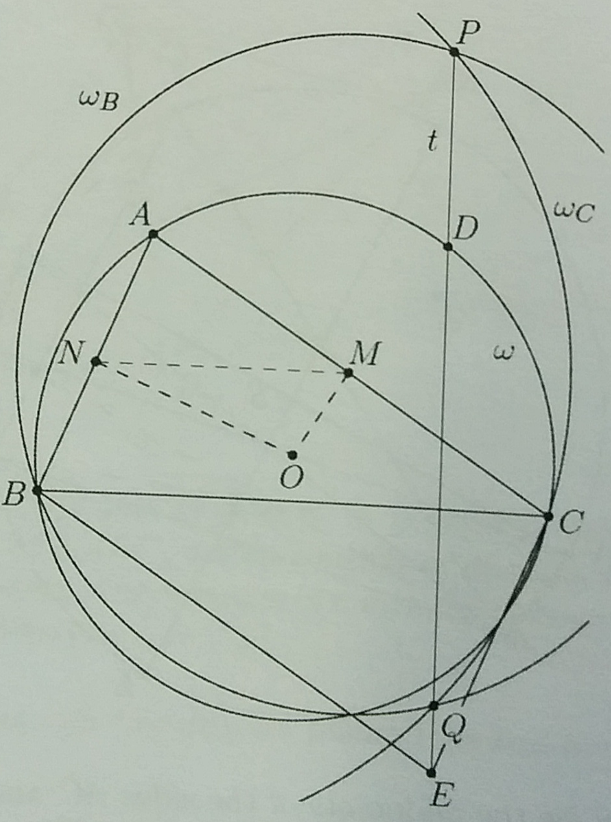
G10. *Let M and N be the midpoints of the sides AC and AB, respectively, of an acute triangle ABC. Let ω_B be the circle centered at M passing through B, and let ω_C be the circle centered at N*

passing through C. Let the point D be such that $ABCD$ is an isosceles trapezoid with AD parallel to BC. Assume that ω_B and ω_C intersect in two distinct points P and Q. Show that D lies on PQ.

Solution 1. Let E be such that $ABEC$ is a parallelogram with $AB \parallel CE$ and $AC \parallel BE$, and let ω be the circumcircle of ABC with center O.

It is known that the radical axis of two circles is perpendicular to the line connecting the two centers. Since $BE \perp MO$ and $CE \perp NO$, this means that BE and CE are the radical axes of ω and ω_B, and of ω and ω_C, respectively, so E is the radical center of ω, ω_B, and ω_C.

Now as $BE = AC = BD$ and $CE = AB = CD$ we find that BC is the perpendicular bisector of DE. Most importantly we have $DE \perp BC$. Denote by t the radical axis of ω_B and ω_C, ie. $t = PQ$. Then since $t \perp MN$ we find that t and DE are parallel. Therefore since E lies on t we get that D also lies on t.



Solution 2.

Reflect B across M to a point B' forming a parallelogram $ABCB'$. Then B' lies on ω_B diagonally opposite B, and since $AB' \parallel BC$ it lies on AD. Similarly reflect C across N to a point C', which satisfy analogous properties. Note that $CB' = AB = CD$, so we find that triangle CDB' and similarly triangle BDC' are isosceles.

Let B'' and C'' be the orthogonal projections of B and C onto AD. Since BB' is a diameter of ω_B we get that B'' lies on ω_B, and similarly C'' lies on ω_C. Moreover BB'' is an altitude of the isosceles triangle BDC' with $BD = BC'$, hence it coincides with the median from B, so B'' is in fact the midpoint of DC'. Similarly C'' is the midpoint of DB'. From this we get

$$2 = \frac{DC'}{DB''} = \frac{DB'}{DC''}$$

which rearranges as $DC' \cdot DC'' = DB' \cdot DB''$. This means that D has same the power wrt. ω_B and ω_C, hence it lies on their radicalaxis PQ.

N1. *Prove that*

$$\sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}}$$

is rational for every positive integer n.

As

$$\sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}} = \frac{\sqrt{n^2(n+1)^2 + (n+1)^2 + n^2}}{n(n+1)}$$

it suffices to show that $n^2(n+1)^2 + (n+1)^2 + n^2$ is a perfect square. This follows from

$$\begin{aligned} n^2(n+1)^2 + (n+1)^2 + n^2 &= n^2((n+1)^2 + 1) + (n+1)^2 \\ &= n^4 + 2n^2(n+1) + (n+1)^2 \\ &= (n^2 + n + 1)^2. \end{aligned}$$

Remark. We note that more generally if x, y and z are any positive numbers with $x + y = z$, then

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \left(\frac{x^2 + xy + y^2}{xy(x+y)}\right)^2.$$

N2. *Do there exist positive integers a, b, c, such they have no common divisor and*

$$ab + bc + ca = (a + b - c)(b + c - a)(c + a - b)?$$

We show that all of a, b and c have the number 3 as a common factor. First suppose that one of a, b, c is divisible by 3. By symmetry, we may assume that $a \equiv 0 \pmod 3$. Then the equation implies $bc \equiv (b - c)(b + c)(c - b) \pmod 3$. If neither of b and c is divisible by 3, this gives $bc \equiv 0 \pmod 3$, which is a contradiction. On the other hand, if either of b and c is divisible by 3, then from $bc \equiv (b-c)(b+c)(c-b) \pmod 3$ we see that they both are divisible by 3, which means that a, b and c are all divisible by 3. Now we are left with the case that a, b, c are each $\pm 1 \pmod 3$. If $a \equiv b \equiv c \pmod 3$, then clearly the left-hand side of the equation is divisible by 3, while the right-hand side is not, so we have a contradiction. In the opposite case that two of a, b and c are equal and the third one is distinct modulo 3, clearly the left-hand side of the equation is not divisible by 3, while the right-hand side is divisible. We conclude that a, b and c are all divisible by 3.

N3. *Determine all positive integers a and all primes p fulfilling the equation*

$$(a - p)^3 = a + p.$$

Answer: The only solution is $a = 5$ and $p = 3$.

Solution: Writing $n = a - p$ transforms the equation into

$$n^3 = a + p = n + 2p,$$

so that

$$2p = n^3 - n = n(n + 1)(n - 1),$$

which is divisible by 3. Therefore $p = 3$ and $n = 2$, whence $a = 5$.

N4. *Find all pairs of primes (p, q) such that*

$$p^3 - q^5 = (p + q)^2.$$

Answer: The only solution is $p = 7$ and $q = 3$.

Solution: First let us assume that neither of the numbers equals 3. Then if $p \equiv q \pmod 3$ then the left hand side is divisible by 3, but the right hand side is not. But if $p \equiv -q \mod 3$ then the left hand side is not divisible by 3, but the right hand side is divisible. So this is not possible.

If $p = 3$, then $q^5 < 27$, which is impossible. Therefore $q = 3$ and the equation turns into $p^3 - 243 = (p+3)^2$ or

$$p(p^2 - p - 6) = 252.$$

As $p > 3$ then $p^2 - p - 6$ is positive and increasing, so the equation has at most one solution. It is easy to see that $p = 7$ is the one and that $(7, 3)$ is a solution to the given equation.

N5. *A finite sequence $d_{n-1}, d_{n-2}, \ldots, d_1, d_0$ of digits is called a* stable final segment of length n *if it has the following property: If m is any positive integer such that the last n digits of m are $d_{n-1}d_{n-2}\ldots d_1 d_0$ (in this order), then for every positive integer k the last n digits of m^k are $d_{n-1}d_{n-2}\ldots d_1 d_0$ (in this order). Prove that for any positive integer n there are exactly four stable final segments of length n.*

Let $a = d_{n-1}\ldots d_1 d_0$ (where initial zeros are ignored if there are any). The sequence $d_{n-1}, d_{n-2}, \ldots, d_1, d_0$ is a stable final segment if and only if $a^k - a \equiv 0 \pmod{10^n}$ for all integers $k \geq 2$. This again is equivalent to $a^2 - a \equiv 0 \pmod{10^n}$ because $a^2 - a = a(a-1)$ is a factor of $a^k - a = a(a^{k-1} - 1)$. Since a and $a - 1$ cannot both be even or both divisible by 5, the congruence $a(a-1) \equiv 0 \pmod{10^n}$ holds if and only if:

1. $a \equiv 0 \pmod{10^n}$, i.e. $a = 0 = d_{n-1} = \cdots = d_1 = d_0$, or
2. $a \equiv 1 \pmod{10^n}$, i.e. $a = 1 = d_0$, $d_{n-1} = \cdots = d_1 = 0$, or
3. $a \equiv 0 \pmod{2^n}$, $a \equiv 1 \pmod{5^n}$, or
4. $a \equiv 1 \pmod{2^n}$, $a \equiv 0 \pmod{5^n}$.

Also in Cases 3 and 4 there are unique solutions modulo 10^n by the Chinese Remainder Theorem. (Of course, for these simple systems of congruences this can also be argued without the theorem.)

Remark. The following is an alternative argument for solving the problem. As above, the problem reduces to showing that the congruence $a^2 \equiv a \pmod{10^n}$ has exactly 4 non-congruent solutions. We show this by induction on n. For $n = 1$, the only solutions are clearly $a \equiv 0, 1, 5, 6 \pmod{10}$. Suppose the case n has been proved and consider the case $n + 1$. Let a_1, a_2, a_3 and a_4 be the solutions to $a^2 \equiv a \pmod{10^n}$. If b is any number satisfying the same congruence with n replaced by $n + 1$, we must have $b \equiv a_i \pmod{10^n}$ for some $i \leq 4$. Hence $b \equiv a_i + k_i \cdot 10^n \pmod{10^{n+1}}$ for some $i \leq 4$ and some integer $0 \leq k_i \leq 9$. What remains to be shown is that each k_i can be chosen in a unique way. If $b \equiv a_i + k_i \cdot 10^n \pmod{10^{n+1}}$, $b^2 - b \equiv 0 \pmod{10^{n+1}}$ simplifies to $\frac{a_i^2 - a_i}{10^n} \equiv (1 - 2a_i) \cdot k_i \pmod{10}$, since $\frac{a_i^2 - a_i}{10^n}$ is an integer. Since a_i is $0, 1, 5$ or $6 \pmod{10}$, the number $1 - 2a_i$ is coprime to 10, so the congruence for k_i has a unique solution.

N6. *Prove or disprove the following hypotheses.*

1. *Each sequence of at least two consecutive integers contains a number that is divisible by no prime number less than the amount of members in the sequence.*

2. *Each sequence of at least two consecutive integers contains a number that is relatively prime to all other members of the sequence.*

Answer: Neither hypothesis is true.

Solution:

1. The sequence $(2, 3, 4, 5, 6, 7, 8, 9)$ contains 8 consecutive integers which all are divisible by some prime less than 8.

2. By Chinese Remainder Theorem, there exists an integer x that satisfies the following conditions:

   - x is divisible by 2, 5 and 11;
   - $x + 16$ is divisible by 3, 7 and 13.

Then the sequence $(x, x+1, \ldots, x+16)$ contains 17 consecutive integers, each of which has a common prime factor with some other:

Number	Factors common to some other
x	$2, 5, 11$
$x+1$	3
$x+2$	$2, 7$
$x+3$	13
$x+4$	$2, 3$
$x+5$	5
$x+6$	2
$x+7$	3
$x+8$	2

Number	Factors common to some other
$x+9$	7
$x+10$	$2, 3, 5$
$x+11$	11
$x+12$	2
$x+13$	3
$x+14$	2
$x+15$	5
$x+16$	$2, 3, 7, 13$

Remark 1: The counterexample given to either hypothesis is shortest possible.

Remark 2: The only counterexamples of length 8 to the first hypothesis are those where numbers give remainders $2, 3, \ldots, 9$ or $3, 4, \ldots, 10$ or $-2, -3, \ldots, -9$ or $-3, -4, \ldots, -10$ modulo 210. The only counterexamples of length 17 to the second hypothesis are those where the numbers give remainders $2184, 2185, \ldots, 2200$ or $-2184, -2185, \ldots, -2200$ modulo 30030

N7. *For which integers $n = 1, \ldots, 6$ does the equation*

$$a^n + b^n = c^n + n$$

have a solution in integers?

Answer: Solutions exist for $n = 1, 2, 3$.

Solution: For $n = 6$, we consider the eqation $a^6 + b^6 = c^6 + 6$ modulo 13. We always have $x^6 \equiv 0, 1$ or $-1 \pmod{13}$ (by Fermat's little theorem or a direct computation). However, then it is evident that $a^6 + b^6 - c^6$ cannot be $6 \pmod{13}$.

For $n = 5$, we consider the eqation $a^5 + b^5 = c^5 + 5$ modulo 11. We always have $x^5 \equiv 0, 1$ or $-1 \pmod{11}$ (by Fermat's little theorem or a direct computation). This is a contradiction just as in the previous case.

For $n = 4$, we consider the equation $a^4 + b^4 = c^4 + 4$ modulo 8. Since always $x^4 \equiv 0, 1 \mod 8$, we clearly have a contradiction as in the previous cases. For $n = 1, 2, 3$, there are solutions

$$1^1 + 0^1 = 0^1 + 1, \qquad 1^2 + 1^2 = 0^2 + 2, \qquad 1^3 + 1^3 = (-1)^3 + 3.$$

Remark: Using the same arguement as in the cases $n = 5$ and $n = 6$, one can show that $a^n + b^n = c^n + n$ has no solutions whenever $n \geq 5$ and $2n + 1$ is a prime. Indeed, then by Fermat's little theorem $x^{2n} \equiv 1 \pmod{2n+1}$ or $x \equiv 0 \pmod{2n+1}$. Since $x^{2n} - 1 = (x^n - 1)(x^n + 1) \equiv 0 \pmod{2n+1}$ or $x \equiv 0 \pmod{2n+1}$, we see that $x^n \equiv 0, 1$ or $-1 \pmod{2n+1}$. From this it follows that the expression $a^n + b^n - c^n$ can never be congruent to n modulo $2n + 1$, hence the equation $a^n + b^n = c^n + n$ has no solutions.

N8. *Let n be a positive integer and let a, b, c, d be integers such that $n | a + b + c + d$ and $n | a^2 + b^2 + c^2 + d^2$. Show that*

$$n | a^4 + b^4 + c^4 + d^4 + 4abcd.$$

Consider the polynomial

$$w(x) = (x - a)(x - b)(x - c)(x - d) = x^4 + Ax^3 + Bx^2 + Cx + D.$$

It is clear that $w(a) = w(b) = w(c) = w(d) = 0$. By adding these values we get

$$\begin{aligned} w(a) + w(b) + w(c) + w(d) &= a^4 + b^4 + c^4 + d^4 \\ &+ A(a^3 + b^3 + c^3 + d^3) + B(a^2 + b^2 + c^2 + d^2) \\ &+ C(a + b + c + d) + 4D = 0. \end{aligned}$$

Hence

$$\begin{aligned} a^4 + b^4 + c^4 + d^4 + 4D = &-A(a^3 + b^3 + c^3 + d^3) \\ &- B(a^2 + b^2 + c^2 + d^2) \qquad (5) \\ &- C(a + b + c + d). \end{aligned}$$

Using Vieta's formulas we can see that $D = abcd$ and $-A = a + b + c + d$, and therefore right side of (5) is divisible by n, and so is left side.

N9. *Find all pairs p, q of distinct primes, sets $D \subseteq \mathbb{R}$ and functions $f\colon D \to D$ fulfilling*

$$f^p(x) = x^p \qquad \text{and} \qquad f^q(x) = x^q$$

for all $x \in D$. (Here, f^n denotes the n'th iterate of f.)

Answer: For odd p, q, the possibilities are $D \subseteq \{0, \pm 1\}$ and $f(x) = x$. When one of p, q is even, the possibilities are $D \subseteq \{0, 1\}$ and $f(x) = x$.

Solution: From

$$x^{p^q} = f^{pq}(x) = f^{qp}(x) = x^{q^p}$$

we have $x^{p^q}(x^{q^p - p^q} - 1) = 0$ or $x^{q^p}(x^{p^q - q^p} - 1) = 0$. We infer that $x = 0, \pm 1$, hence $D \subseteq \{0, \pm 1\}$. If either of p and q is even, we are led to $x = 0, 1$ and the sharper inclusion $D \subseteq \{0, 1\}$.

From the restricted form of D it is evident that

$$f^p(x) = x^p = x \qquad \text{and} \qquad f^q(x) = x^q = x$$

for $x \in D$. Further, using Bezout's Identity we may write $ap = bq + 1$ (or $bq = ap + 1$) for positive integers a, b. We then have

$$x = f^{ap}(x) = f^{bq+1}(x) = f(f^{bq}(x)) = f(x).$$

Conversely, one easily verifies that the identity function $f\colon D \to D$ on domains D of the above form has the required property.

Remark: One can also solve the problem without Bezout's identity as follows. As above, we conclude that $D \subseteq \{-1, 0, 1\}$ if p and q are odd and $D \subseteq \{0, 1\}$ if one of p and q is even. In the latter case, there are only $2^2 = 4$ functions from the set $\{0, 1\}$ to itself, and all of them are easy to describe, so one sees that the identity function is the only solution. In the former case where p and q are odd, the function $x \mapsto x^p$ is a bijection of $\{-1, 0, 1\}$. Hence $x \mapsto f^p(x)$ is also a bijection of $\{-1, 0, 1\}$, so f is a surjection. Since D is a finite set, surjections are also bijections, so f is a bijection. The number of bijections of $\{-1, 0, 1\}$ is $3! = 6$, and all of these are easy to describe, so one sees again that there are no other solutions than the identity function.

N10. *Let $a_1, \ldots, a_{2016}$ and $b_1, \ldots, b_{2016}$ be two reorderings of the numbers $1, \ldots, 2016$. Prove that*

$$2017 \mid a_i b_i - a_j b_j$$

for some distinct indices i and j.

The number 2017 is prime. Clearly, all $a_n b_n \not\equiv 0 \mod 2017$. If all $a_n b_n$ were non-congruent modulo 2017, then

$$a_1 b_1, a_2 b_2, \ldots, a_{2016} b_{2016}$$

would be another reordering of $1, \ldots, 2016$ modulo 2017, and so

$$\prod_{n=1}^{2016} a_n b_n \equiv 2016! \equiv -1 \pmod{2017}$$

by Wilson's theorem; yet

$$\prod_{n=1}^{2016} a_n \prod_{n=1}^{2016} b_n = 2016!^2 \equiv (-1)^2 = 1 \pmod{2017},$$

which is a contradiction. Hence there exist distinct i and j such that $a_i b_i \equiv a_j b_j \pmod{2017}$, as wanted.

Remark: This problem has appeared earlier[1] in the following form:

If p is an odd prime and $\{a_1, a_2, \ldots, a_p\}$ and $\{b_1, b_2 \ldots, b_p\}$ are two reorderings of the residue classes $\pmod{p}$, then $\{a_1 b_1, a_2 b_2, \ldots, a_p b_p\}$ is not a reordering of the residue classes $\pmod{p}$.

N11. *Let $a_0, a_1, \ldots$ be a sequence of positive integers such that $a_n = a_{n-1}^{2^n}$ for all $n = 1, 2, \ldots$. Prove that for each prime p, $p > 3$, with residue 3 modulo 4 there exists a positive integer a_0 such that the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is not constant modulo p for any positive integer N.*

Let p be a prime with residue 3 modulo 4 and $p > 3$. Then $p - 1 = u \cdot 2$ where $u > 1$ is odd. Choose $a_0 = 2$. The order of 2 modulo p (that is, the smallest positive integer t such that

[1] See for instance `https://math.berkeley.edu/ mcivor/math115su12/HW/HW2soln.pdf`

$2^t \equiv 1 \pmod p$) is a divisor of $p-1 = u \cdot 2$, but not a divisor of 2 since $2^2 \not\equiv 1 \pmod p$. Hence the order of 2 modulo p is not a power of 2. From the definition we see that $a_n = a_0^{2^{1+2+\cdots+n}}$. Since the order of $a_0 = 2$ modulo p is not a power of 2, we know that $a_n \not\equiv 1 \pmod p$ for all $n = 1, 2, 3, \ldots$. We prove the statement by contradiction. Assume there exists a positive integer N such that $a_n \equiv a_N \pmod p$ for all $n \geq N$. Let $d > 1$ be the order of a_N modulo p (that is, the smallest positive integer s such that $a_N^s \equiv 1 \pmod p$). Then $a_N \equiv a_n \equiv a_{n+1} = a_n^{2^{n+1}} \equiv a_N^{2^{n+1}} \pmod p$, and hence $a_N^{2^{n+1}-1} \equiv 1 \pmod p$ for all $n \geq N$. Now d divides $2^{n+1}-1$ for all $n \geq N$, but this is a contradiction since

$$\begin{aligned}\gcd(2^{n+1}-1, 2^{n+2}-1) &= \gcd(2^{n+1}-1, 2^{n+2}-1-2(2^{n+1}-1)) \\ &= \gcd(2^{n+1}-1, 1) = 1.\end{aligned}$$

Hence there does not exist such an N.

Remark. The following is an alternative solution to the problem, without using the concept of orders of integers. We choose $a_0 = 4$. From the definition we see that $a_n = 4^{2^{n(n+1)/2}}$ for $n = 1, 2, \ldots$. Suppose that there exists an integer N for which $a_n \equiv a_N \pmod p$ whenever $n \geq N$. Pick an integer $n \geq N$ such that $n \equiv 0 \pmod{2\varphi(\frac{p-1}{2})}$ (where $\varphi(\cdot)$ stands for Euler's function). Then $\frac{n(n+1)}{2} \equiv 0 \pmod{\varphi(\frac{p-1}{2})}$. Since $\frac{p-1}{2}$ is odd by assumption, Euler's theorem implies $2^{n(n+1)/2} \equiv 1 \pmod{\frac{p-1}{2}}$. Writing $2^{n(n+1)/2} = 1 + k \cdot \frac{p-1}{2}$ for some integer $k \geq 0$, we see that $a_n = 4^{1+k\cdot(p-1)/2} = 4 \cdot 2^{k(p-1)} \equiv 4 \pmod p$ by Fermat's little theorem. After this, pick another integer $n \geq N$, this time satisfying $n \equiv 1 \pmod{2\varphi(\frac{p-1}{2})}$. Then $\frac{n(n+1)}{2} \equiv 1 \pmod{\varphi(\frac{p-1}{2})}$. By Euler's theorem, we get $2^{n(n+1)/2} \equiv 2^1 \pmod{\frac{p-1}{2}}$. Again writing $2^{n(n+1)/2} = 2 + k \cdot \frac{p-1}{2}$ for some integer $k \geq 0$, we see that $a_n \equiv 4^2 \pmod p$. Since $a_n \pmod p$ was constant for $n \geq N$, we deduce $4^2 \equiv 4 \pmod p$, which gives the desired contradiction.