Effects of Couple-stresses in Linear Elasticity

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Communicated by C. TRUESDELL

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Introduction

In a theory of deformation of continua, originated by VOIGT [*1*] and amplified by E. & F. COSSERAT [*2*], the couple per unit area, acting across a surface within a material volume or on its boundary, was taken into account in addition to the usual force per unit area. Some typical effects of such "couple-stresses" are exhibited, in the present paper, by means of solutions of problems of wave-propagation, vibration, stress-concentration and nuclei of strain — all within the framework of a linearized form of the couple-stress theory for perfectly elastic, centrosymmetric-isotropic materials.

A modern derivation of the Cosserat equations has been given by TRUESDELL & TOUPIN [*3*]. More recently, TOUPIN [*4*] has derived the associated constitutive equations for finite deformation of perfectly elastic materials. Upon linearization, TOUPIN's equations become identical with those which are obtained, for example by AERO & KUVSHINSKII [*5*], without first establishing constitutive equations for finite deformation. On the other hand, GRIOLI [*6*] has also obtained constitutive equations for finite deformation — equally correct, though of different form — but, upon linearization he obtains results at variance with TOUPIN'S. The source of the discrepancy is not readily apparent owing to the different procedures employed in the two treatments. In view of this situation, it seems advisable, before proceeding with solutions, to review, briefly, derivations of the equations of both authors, retaining the essentials of their reasoning but following an intermediate procedure. In this way the point of divergence is identified and it is shown how GRIOLI'S results may be brought into conformity with TOUPIN'S.

The discussion of the derivations of the finite and linearized equations occupies the first four sections of the present paper. This is followed, in Section 5, by a theorem of uniqueness of solution of the linear equations. The theorem is based on the assumption of positive definiteness of the internal energy-density. Although no proof of positive definiteness is available, some justification is given on intuitive grounds and by an example, later, of failure of uniqueness when the internal energy-density is not positive-definite. In the course of the formulation of the uniqueness theorem, it appears, at first, that six boundary conditions are required; but it is shown that five are sufficient. The reduction from six to five is analogous to the reduction from three to two conditions in the classical theory of flexure of thin plates.

In the linear theory that takes into account couple-stresses in a centro-symmetric-isotropic, elastic material, there is an additional modulus of elasticity (the ratio of couple-stress to curvature or twist, *i.e.*, a modulus of bending and twisting) with the dimensions of force. The square root of the ratio of the bending-twisting modulus to the usual shear modulus has the dimension of length. This length, l, is a material property which carries with it all of the difference between analogous equations or solutions with and without couple-stresses. The larger l may be, the greater is the difference. Presumably l is small, in comparison with bodily dimensions and wave-lengths normally encountered, as there appears to be no conclusive experimental evidence of its existence. However, even though small, its influence might become important as dimensions of a body or wave-lengths diminish to the order of the length l. The assumption of positive-definiteness of the internal energy-density requires the bending-twisting modulus to be positive. In the contrary case, l^2 would be replaced by its negative, in the equations; and the forms of solutions would be drastically different.

The propagation of plane waves is considered in Section 6. There is a non-dispersive dilatational wave, as in the usual theory, propagating at the usual velocity; but there are two rotational waves: one propagating and the other non-propagating. Both rotational waves are dispersive. The group velocity of the propagating rotational wave increases with increasing wave number and increasing l. The non-propagating wave produces a boundary layer effect.

In Section 7, the displacement equations of motion are reduced to equations corresponding to Lamé's. The equation on the potential of the dilatation is the usual one; but, in the equation on the potential of the rotation, Laplace's operator ∇^2 is replaced by $\nabla^2 - l^2\nabla^4$. In the case of steady vibration, the latter equation is separated into two Helmholtz equations.

The solution of the problem of antisymmetric, thickness-shear vibrations of an infinite plate is given in Section 8. The natural frequencies exceed the integral multiples of the fundamental which are found without couple-stresses. Since the excess increases with the order of the mode and with the magnitude of the material constant l, a laboratory experiment aimed at the measurement of l might be based on a solution of this type. The lowest mode is affected very little by the couple-stress, regardless of the value of l. However, this is due to an accident of mode shape in the case of thickness-shear vibrations. It does not occur, for example, in the solution of the problem of torsional vibrations of a cylinder, which is given in Section 9.

In preparation for solutions of problems of equilibrium involving stress concentrations and singularities, GREEN's formulas are derived, in Section 10, for functions whose governing equations contain the operators $1 - l^2\nabla^2$ and $\nabla^2 - l^2\nabla^4$. Following this, in Section 11, stress functions analogous to those of PAPKOVITCH and GALERKIN are introduced.

Solutions of the problems of stresses around spherical and cylindrical cavities in an infinite body under simple tension are given in Sections 12 and 14. Unlike in the classical solutions, the maximum tension at the surface depends on the radius of the cavity — appearing in the form of the ratio of the radius to the material constant l. In the limiting cases of vanishingly small cavities, the maximum tensions at the surfaces are found to be thirty to forty percent less than they are if couple-stresses are neglected.

The problem of the spherical cavity in a field of simple tension is considered again in Section 13; but this time with the bending-twisting modulus negative. Two solutions are found: exhibiting failure of uniqueness with non-positive internal energy-density.

In the last two sections, the singularities appropriate to the concentrated force, the concentrated couple and a variety of nuclei of strain are established. The center of dilatation is the same as it is without couple-stresses and the center of rotation is identical with the concentrated body-couple.

1. The Cosserat equations

Let S' be a surface which separates a portion of a material volume, V, from the remainder. At each point of S', let $\boldsymbol{n}$ be the unit vector normal to S' in a specified sense. The material on the side of S' toward which $\boldsymbol{n}$ is directed exerts loads, on the material on the other side, which are assumed to consist, at each point of S', of a force per unit area, $\boldsymbol{t}_n$, and a couple per unit area, $\boldsymbol{m}_n$. Also, at each point in V, let $\boldsymbol{f}$ be an extrinsic force per unit mass and $\boldsymbol{c}$ be an extrinsic couple per unit mass. The force-stress vector $\boldsymbol{t}_n$ and the body-force vector $\boldsymbol{f}$ are polar vectors; whereas the couple-stress vector $\boldsymbol{m}_n$ and the body-couple vector $\boldsymbol{c}$ are axial vectors. Axial vectors are taken as positive in the direction of advance of a right handed screw.

We now consider the motion of a portion V, of a material volume, bounded by a surface S with outward normal $\boldsymbol{n}$. Across S there act force-stress and couple-stress vectors, $\boldsymbol{t}_n$ and $\boldsymbol{m}_n$, and within V there act body-force and body-couple vectors $\boldsymbol{f}$ and $\boldsymbol{c}$. The equations of conservation of mass, balance of momentum and moment of momentum and conservation of mechanical energy are taken to be, respectively,

$$\frac{d}{dt}\int_V \varrho\, dV = 0, \tag{1.1}$$

$$\frac{d}{dt}\int_V \boldsymbol{v}\varrho\, dV = \int_S \boldsymbol{t_n}\, dS + \int_V \boldsymbol{f}\varrho\, dV, \tag{1.2}$$

$$\frac{d}{dt}\int_V \boldsymbol{r}\times\boldsymbol{v}\varrho\, dV = \int_S (\boldsymbol{r}\times\boldsymbol{t_n} + \boldsymbol{m_n})\, dS + \int_V (\boldsymbol{r}\times\boldsymbol{f} + \boldsymbol{c})\,\varrho\, dV, \tag{1.3}$$

$$\begin{aligned}\frac{d}{dt}\int_V \left(\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + U\right)\varrho\, dV = \int_S \left(\boldsymbol{t_n}\cdot\boldsymbol{v} + \frac{1}{2}\boldsymbol{m_n}\cdot\boldsymbol{\nabla}\times\boldsymbol{v}\right) dS + \\ + \int_V \left(\boldsymbol{f}\cdot\boldsymbol{v} + \frac{1}{2}\boldsymbol{c}\cdot\boldsymbol{\nabla}\times\boldsymbol{v}\right)\varrho\, dV,\end{aligned} \tag{1.4}$$

where d/dt is the material time-derivative, ϱ is the mass density, $\boldsymbol{r}$ is the spatial position vector from a fixed origin, $\boldsymbol{v}$ is the material velocity $d\boldsymbol{r}/dt$, U is the internal energy per unit mass, $\boldsymbol{\nabla}$ is the spatial gradient $\partial/\partial\boldsymbol{r}$, and $\frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{v}$ is the vorticity.

Consideration of the equilibrium of forces acting on an elementary tetrahedron, as the volume of the tetrahedron shrinks to zero, leads to the definition of the usual force-stress dyadic* $\boldsymbol{\tau}$:

$$\boldsymbol{t_n} = \boldsymbol{n}\cdot\boldsymbol{\tau}. \tag{1.5}$$

An analogous treatment of the equilibrium of moments acting on the tetrahedron yields the definition of the couple-stress dyadic $\boldsymbol{\mu}$:

$$\boldsymbol{m_n} = \boldsymbol{n}\cdot\boldsymbol{\mu}. \tag{1.6}$$

Now, in (1.2),

$$\int_S \boldsymbol{t_n}\, dS = \int_S \boldsymbol{n}\cdot\boldsymbol{\tau}\, dS = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{\tau}\, dV, \tag{1.7}$$

by (1.5) and the divergence theorem. With (1.7) and (1.1), (1.2) becomes

$$\int_V (\boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \varrho\boldsymbol{f} - \varrho\dot{\boldsymbol{v}})\, dV = 0,$$

where $\dot{\boldsymbol{v}} = d\boldsymbol{v}/dt$. Hence

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \varrho\boldsymbol{f} = \varrho\dot{\boldsymbol{v}}, \tag{1.8}$$

which is the usual force-stress equation of motion.

In (1.3),

$$\frac{d}{dt}\int_V \boldsymbol{r}\times\boldsymbol{v}\varrho\, dV = \int_V \frac{d(\boldsymbol{r}\times\boldsymbol{v})}{dt}\varrho\, dV = \int_V (\boldsymbol{v}\times\boldsymbol{v} + \boldsymbol{r}\times\dot{\boldsymbol{v}})\,\varrho\, dV = \int_V \boldsymbol{r}\times\dot{\boldsymbol{v}}\varrho\, dV,$$

$$\begin{aligned}\int_S \boldsymbol{r}\times\boldsymbol{t_n}\, dS &= \int_S \boldsymbol{r}\times(\boldsymbol{n}\cdot\boldsymbol{\tau})\, dS = -\int_S \boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{r}\, dS \\ &= -\int_V \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}\times\boldsymbol{r})\, dV = \int_V [\boldsymbol{r}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \boldsymbol{\tau}\underset{\cdot}{\times}\mathbf{I}]\, dV,\end{aligned}$$

$$\int_S \boldsymbol{m_n}\, dS = \int_S \boldsymbol{n}\cdot\boldsymbol{\mu}\, dS = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{\mu}\, dV,$$

* See WEATHERBURN [7] or WILSON [8] for the dyadic notation.

where $\mathbf{I}$ ($\equiv \nabla \boldsymbol{r}$) is the unit spatial dyadic. Hence (1.3) becomes

$$\int_V \boldsymbol{r} \times (\nabla \cdot \boldsymbol{\tau} + \varrho \boldsymbol{f} - \varrho \dot{\boldsymbol{v}})\, dV + \int_V (\nabla \cdot \boldsymbol{\mu} + \varrho \boldsymbol{c} + \boldsymbol{\tau} \overset{\times}{\cdot} \mathbf{I})\, dV = 0;$$

from which, with (1.8), we have the couple-stress equation of motion:

$$\nabla \cdot \boldsymbol{\mu} + \varrho \boldsymbol{c} + \boldsymbol{\tau} \overset{\times}{\cdot} \mathbf{I} = 0. \tag{1.9}$$

The antisymmetric part of $\boldsymbol{\tau}$ is $\boldsymbol{\tau}^A = -\frac{1}{2} \mathbf{I} \times (\boldsymbol{\tau} \overset{\times}{\cdot} \mathbf{I})$. Hence

$$\boldsymbol{\tau}^A = \tfrac{1}{2} \mathbf{I} \times (\nabla \cdot \boldsymbol{\mu} + \varrho \boldsymbol{c}). \tag{1.10}$$

If the couple-stress $\boldsymbol{\mu}$ and the body couple $\boldsymbol{c}$ were absent, $\boldsymbol{\tau}^A$ would vanish and $\boldsymbol{\tau}$ would be a symmetric dyadic.

Writing

$$\boldsymbol{\tau} = \boldsymbol{\tau}^S + \boldsymbol{\tau}^A, \tag{1.11}$$

where $\boldsymbol{\tau}^S$ is the symmetric part of $\boldsymbol{\tau}$, substituting (1.10) in (1.11) and the latter in (1.8), we find an alternative form of the equation of motion:

$$\nabla \cdot \boldsymbol{\tau}^S + \tfrac{1}{2} \nabla \times \nabla \cdot \boldsymbol{\mu} + \varrho \boldsymbol{f} + \tfrac{1}{2} \nabla \times \varrho \boldsymbol{c} = \varrho \dot{\boldsymbol{v}}. \tag{1.12}$$

In (1.4), with $\dot{U} = dU/dt$,

$$\frac{d}{dt} \int_V \left(\frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v} + U \right) \varrho\, dV = \int_V (\boldsymbol{v} \cdot \dot{\boldsymbol{v}} + \dot{U})\, \varrho\, dV,$$

$$\int_S \boldsymbol{t}_n \cdot \boldsymbol{v}\, dS = \int_S \boldsymbol{n} \cdot \boldsymbol{\tau} \cdot \boldsymbol{v}\, dS = \int_V \nabla \cdot (\boldsymbol{\tau} \cdot \boldsymbol{v})\, dV = \int_V [\boldsymbol{\tau} : \nabla \boldsymbol{v} + (\nabla \cdot \boldsymbol{\tau}) \cdot \boldsymbol{v}]\, dV,$$

$$\begin{aligned} \int_S \boldsymbol{m}_n \cdot \nabla \times \boldsymbol{v}\, dS &= \int_S \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \nabla \times \boldsymbol{v}\, dS = \int_V \nabla \cdot (\boldsymbol{\mu} \cdot \nabla \times \boldsymbol{v})\, dV \\ &= \int_V [\boldsymbol{\mu} : \nabla \nabla \times \boldsymbol{v} + (\nabla \cdot \boldsymbol{\mu}) \cdot \nabla \times \boldsymbol{v}]\, dV. \end{aligned}$$

Hence, (1.4) may be written as

$$\int_V \varrho \dot{U}\, dV = \int_V [(\nabla \cdot \boldsymbol{\tau} + \varrho \boldsymbol{f} - \varrho \dot{\boldsymbol{v}}) \cdot \boldsymbol{v} + \tfrac{1}{2} (\nabla \cdot \boldsymbol{\mu} + \varrho \boldsymbol{c}) \cdot \nabla \times \boldsymbol{v} + \boldsymbol{\tau} : \nabla \boldsymbol{v} + \tfrac{1}{2} \boldsymbol{\mu} : \nabla \nabla \times \boldsymbol{v}]\, dV,$$

which, by (1.8) and (1.9), becomes

$$\int_V \varrho \dot{U}\, dV = \int_V [\boldsymbol{\tau} : \nabla \boldsymbol{v} - \tfrac{1}{2} (\boldsymbol{\tau} \overset{\times}{\cdot} \mathbf{I}) \cdot \nabla \times \boldsymbol{v} + \tfrac{1}{2} \boldsymbol{\mu} : \nabla \nabla \times \boldsymbol{v}]\, dV.$$

Now, $\frac{1}{2} (\boldsymbol{\tau} \overset{\times}{\cdot} \mathbf{I}) \cdot \nabla \times \boldsymbol{v} = \boldsymbol{\tau}^A : \nabla \boldsymbol{v}$. Hence

$$\int_V \varrho \dot{U}\, dV = \int_V (\boldsymbol{\tau}^S : \nabla \boldsymbol{v} + \tfrac{1}{2} \boldsymbol{\mu} : \nabla \nabla \times \boldsymbol{v})\, dV;$$

or

$$\varrho \dot{U} = \boldsymbol{\tau}^S : \nabla \boldsymbol{v} + \tfrac{1}{2} \boldsymbol{\mu} : \nabla \nabla \times \boldsymbol{v}. \tag{1.13}$$

Equations (1.8), (1.9), (1.12) and (1.13) comprise the couple-stress theory essentially as it was left by the COSSERATS. The four equations may be identified with equations (205.2), (205.10), (205.17) and (241.4), respectively, in the article by TRUESDELL & TOUPIN [3], except for the omisssion, here, of non-mechanical terms.

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We note, in (1.13), that the antisymmetric part of the force-stress does not contribute to the internal energy. Also, since the scalar of $\boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{v}$ is identically zero ($\mathbf{I}:\boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{v}=0$), the scalar of $\boldsymbol{\mu}$ does not contribute to the internal energy. Thus, the right hand side of (1.13) is the sum of $(9-3)+(9-1)=14$ products and (1.13) may be written as

$$\varrho\dot{U}=\boldsymbol{\tau}^S:\boldsymbol{\nabla}\boldsymbol{v}+\tfrac{1}{2}\boldsymbol{\mu}^D:\boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{v}, \tag{1.14}$$

where $\boldsymbol{\mu}^D$ is the deviator of $\boldsymbol{\mu}$:

$$\boldsymbol{\mu}^D=\boldsymbol{\mu}-\tfrac{1}{3}\boldsymbol{\mu}:\mathbf{II}. \tag{1.15}$$

Furthermore, since $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\cdot(\mathbf{II}:\boldsymbol{\mu})=0$, the scalar of $\boldsymbol{\mu}$ does not appear in the equation of motion (1.12). That is, (1.12) may be written as

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau}^S+\tfrac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\cdot\boldsymbol{\mu}^D+\varrho\boldsymbol{f}+\tfrac{1}{2}\boldsymbol{\nabla}\times\varrho\boldsymbol{c}=\varrho\dot{\boldsymbol{v}}. \tag{1.16}$$

Thus, a peculiarity of the Cosserat equations is that the scalar of the couple-stress and, by (1.10), the antisymmetric part of the force-stress are left indeterminate.

2. Toupin's constitutive equations

In formulating constitutive equations, we shall have to employ the material position vector, $\mathring{\boldsymbol{r}}$, the material gradient, $\mathring{\boldsymbol{\nabla}}\equiv\partial/\partial\mathring{\boldsymbol{r}}$, and the material unit dyadic $\mathring{\mathbf{I}}\equiv\mathring{\boldsymbol{\nabla}}\mathring{\boldsymbol{r}}$. To convert between material and spatial gradients, $\mathring{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$, we note that, if $\boldsymbol{\Phi}$ is any polyadic,

$$\mathring{\boldsymbol{\nabla}}\boldsymbol{\Phi}=\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\cdot\boldsymbol{\nabla}\boldsymbol{\Phi},\qquad \boldsymbol{\Phi}\mathring{\boldsymbol{\nabla}}=\boldsymbol{\Phi}\boldsymbol{\nabla}\cdot\boldsymbol{r}\mathring{\boldsymbol{\nabla}},\qquad \boldsymbol{\nabla}\boldsymbol{\Phi}=\boldsymbol{\nabla}\mathring{\boldsymbol{r}}\cdot\mathring{\boldsymbol{\nabla}}\boldsymbol{\Phi},\qquad \boldsymbol{\Phi}\boldsymbol{\nabla}=\boldsymbol{\Phi}\mathring{\boldsymbol{\nabla}}\cdot\mathring{\boldsymbol{r}}\boldsymbol{\nabla}. \tag{2.1}$$

In particular, if $\boldsymbol{\Phi}$ is $\mathring{\boldsymbol{r}}$ or $\boldsymbol{r}$, we have

$$\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\cdot\boldsymbol{\nabla}\mathring{\boldsymbol{r}}=\mathring{\boldsymbol{r}}\boldsymbol{\nabla}\cdot\boldsymbol{r}\mathring{\boldsymbol{\nabla}}=\mathring{\mathbf{I}},\qquad \boldsymbol{\nabla}\mathring{\boldsymbol{r}}\cdot\mathring{\boldsymbol{\nabla}}\boldsymbol{r}=\boldsymbol{r}\mathring{\boldsymbol{\nabla}}\cdot\mathring{\boldsymbol{r}}\boldsymbol{\nabla}=\mathbf{I}. \tag{2.2}$$

When couple-stresses are not taken into account, the specific energy of an elastic material may be expressed as a function of the material strain-dyadic

$$\mathbf{E}\equiv\tfrac{1}{2}(\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\cdot\boldsymbol{r}\mathring{\boldsymbol{\nabla}}-\mathring{\mathbf{I}}),\qquad \mathbf{E}=\mathbf{E}_C, \tag{2.3}$$

where $\mathbf{E}_C$ designates the conjugate of $\mathbf{E}$. R. A. TOUPIN [4] has shown that, when couple-stresses are considered, an appropriate second variable is

$$\mathbf{K}\equiv-\mathbf{E}\times\mathring{\boldsymbol{\nabla}}=\tfrac{1}{2}\mathring{\boldsymbol{\nabla}}\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\mathbin{\dot\times}\mathring{\boldsymbol{\nabla}}\boldsymbol{r},\qquad \mathring{\mathbf{I}}:\mathbf{K}=0. \tag{2.4}$$

$\mathbf{E}$ and $\mathbf{K}$ are "appropriate" as they have the correct number, $6+8=14$, of independent components; $U(\mathbf{E},\mathbf{K})$ is invariant in a rigid rotation of the deformed body, since $\boldsymbol{r}$ appears only as $\boldsymbol{r}\cdot\boldsymbol{r}$ in (2.3) and (2.4); and the right hand side of (1.13) may be expressed as a sum of terms with $\dot{\mathbf{E}}$ and $\dot{\mathbf{K}}$ as double scalar factors, as will be shown.

Thus, we assume

$$U=U(\mathbf{E},\mathbf{K}), \tag{2.5}$$

whence

$$\varrho\dot{U}=\varrho\frac{\partial U}{\partial\mathbf{E}}:\dot{\mathbf{E}}+\varrho\frac{\partial U}{\partial\mathbf{K}}:\dot{\mathbf{K}},\qquad \mathring{\mathbf{I}}:\dot{\mathbf{K}}=0, \tag{2.6}$$

where it is understood that $\partial U/\partial\mathbf{E}=\partial U/\partial\mathbf{E}_C$, $\mathring{\mathbf{I}}:\partial U/\partial\mathbf{K}=0$.

We have now to put the energy equation

$$\varrho \dot{U} = \boldsymbol{\tau}^S : \nabla \boldsymbol{v} + \tfrac{1}{2} \boldsymbol{\mu}^D : \nabla\nabla \times \boldsymbol{v}, \tag{2.7}$$

into the form $\varrho \dot{U} = \boldsymbol{\Phi} : \dot{\mathbf{E}} + \boldsymbol{\Psi} : \dot{\mathbf{K}}$, where $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are dyadics to be determined, so that we may equate coefficients of $\dot{\mathbf{E}}$ and of $\dot{\mathbf{K}}$ in (2.6) and (2.7).

First we have, from (2.3) and (2.1),

$$\dot{\mathbf{E}} = \tfrac{1}{2}(\mathring{\nabla}\boldsymbol{r} \cdot \boldsymbol{v}\mathring{\nabla} + \mathring{\nabla}\boldsymbol{v} \cdot \boldsymbol{r}\mathring{\nabla}) = \tfrac{1}{2}(\mathring{\nabla}\boldsymbol{r} \cdot \boldsymbol{v}\nabla \cdot \boldsymbol{r}\mathring{\nabla} + \mathring{\nabla}\boldsymbol{r} \cdot \nabla\boldsymbol{v} \cdot \boldsymbol{r}\mathring{\nabla}),$$

whence

$$\dot{\mathbf{E}} = \mathring{\nabla}\boldsymbol{r} \cdot (\nabla\boldsymbol{v})^S \cdot \boldsymbol{r}\mathring{\nabla}, \tag{2.8}$$

where $(\nabla\boldsymbol{v})^S = \frac{1}{2}(\nabla\boldsymbol{v} + \boldsymbol{v}\nabla)$, *i.e.*, the symmetric part of $\nabla\boldsymbol{v}$. Then, from (2.8) and (2.2),

$$(\nabla\boldsymbol{v})^S = \nabla\mathring{\boldsymbol{r}} \cdot \dot{\mathbf{E}} \cdot \mathring{\boldsymbol{r}}\nabla.$$

Hence

$$\boldsymbol{\tau}^S : \nabla\boldsymbol{v} = \boldsymbol{\tau}^S : (\nabla\boldsymbol{v})^S = \boldsymbol{\tau}^S : (\nabla\mathring{\boldsymbol{r}} \cdot \dot{\mathbf{E}} \cdot \mathring{\boldsymbol{r}}\nabla) = (\mathring{\boldsymbol{r}}\nabla \cdot \boldsymbol{\tau}^S \cdot \nabla\mathring{\boldsymbol{r}}) : \dot{\mathbf{E}}, \tag{2.9}$$

which last has the required form.

Second, we have

$$\begin{aligned} \dot{\mathbf{K}} &= -\dot{\mathbf{E}} \times \mathring{\nabla} = -[\mathring{\nabla}\boldsymbol{r} \cdot (\nabla\boldsymbol{v})^S \cdot \boldsymbol{r}\mathring{\nabla}] \times \mathring{\nabla}, \\ &= \mathring{\nabla}\boldsymbol{r} \cdot [(\nabla\boldsymbol{v})^S \mathring{\nabla}] \dot{\times} \boldsymbol{r}\mathring{\nabla} + \mathring{\nabla}\mathring{\nabla}\boldsymbol{r} \underset{\cdot}{\times} [\mathring{\nabla}\boldsymbol{r} \cdot (\nabla\boldsymbol{v})^S], \\ &= \mathring{\nabla}\boldsymbol{r} \cdot [(\nabla\boldsymbol{v})^S \mathring{\nabla}] \dot{\times} \boldsymbol{r}\mathring{\nabla} + \mathring{\nabla}\mathring{\nabla}\boldsymbol{r} \underset{\cdot}{\times} [\mathring{\nabla}\boldsymbol{r} \cdot \nabla\mathring{\boldsymbol{r}} \cdot \dot{\mathbf{E}} \cdot \mathring{\boldsymbol{r}}\nabla], \\ &= \mathring{\nabla}\boldsymbol{r} \cdot [(\nabla\boldsymbol{v})^S \mathring{\nabla}] \dot{\times} \boldsymbol{r}\mathring{\nabla} + \mathring{\nabla}\mathring{\nabla}\boldsymbol{r} \underset{\cdot}{\times} (\dot{\mathbf{E}} \cdot \mathring{\boldsymbol{r}}\nabla). \end{aligned} \tag{2.10}$$

Then, noting that [8, p. 317]

$$\mathring{\nabla}\boldsymbol{r} = (\mathring{\boldsymbol{r}}\nabla)_C^{-1} = \tfrac{1}{2}(\nabla\mathring{\boldsymbol{r}})_d^{-1} \mathring{\boldsymbol{r}}\nabla \overset{\times}{\times} \mathring{\boldsymbol{r}}\nabla, \tag{2.11}$$

where $(\;)^{-1}$ and $(\;)_d$ denote reciprocal and determinant, we form the product, from (2.10) and (2.11),

$$\begin{aligned} \dot{\mathbf{K}} \cdot (\mathring{\boldsymbol{r}}\nabla\mathring{\boldsymbol{r}} \overset{\times}{\times} \nabla) = {} & \{\mathring{\nabla}\boldsymbol{r} \cdot [(\nabla\boldsymbol{v})^S \mathring{\nabla}] \dot{\times} \boldsymbol{r}\mathring{\nabla}\} \cdot (\mathring{\boldsymbol{r}}\nabla \overset{\times}{\times} \mathring{\boldsymbol{r}}\nabla) + \\ & + 2(\nabla\mathring{\boldsymbol{r}})_d [\mathring{\nabla}\mathring{\nabla}\boldsymbol{r} \dot{\times} (\dot{\mathbf{E}} \cdot \mathring{\boldsymbol{r}}\nabla)] \cdot \mathring{\nabla}\boldsymbol{r}. \end{aligned} \tag{2.12}$$

The first term of (2.12) contains the factor

$$\begin{aligned} \{[(\nabla\boldsymbol{v})^S \mathring{\nabla}] \dot{\times} \boldsymbol{r}\mathring{\nabla}\} \cdot (\mathring{\boldsymbol{r}}\nabla \overset{\times}{\times} \mathring{\boldsymbol{r}}\nabla) &= 2\{[(\nabla\boldsymbol{v})^S \mathring{\nabla}] \cdot \mathring{\mathbf{I}} \cdot \mathring{\boldsymbol{r}}\nabla\} \dot{\times} (\boldsymbol{r}\mathring{\nabla} \cdot \mathring{\mathbf{I}} \cdot \mathring{\boldsymbol{r}}\nabla), \\ &= 2(\nabla\boldsymbol{v})^S \nabla \dot{\times} \mathbf{I} = -2(\nabla\boldsymbol{v})^S \times \nabla, \\ &= \nabla\nabla \times \boldsymbol{v}. \end{aligned}$$

Hence

$$\dot{\mathbf{K}} \cdot (\mathring{\boldsymbol{r}}\nabla \overset{\times}{\times} \mathring{\boldsymbol{r}}\nabla) = \mathring{\nabla}\boldsymbol{r} \cdot \nabla\nabla \times \boldsymbol{v} + 2(\nabla\mathring{\boldsymbol{r}})_d [\mathring{\nabla}\mathring{\nabla}\boldsymbol{r} \underset{\cdot}{\times} (\dot{\mathbf{E}} \cdot \mathring{\boldsymbol{r}}\nabla)] \cdot \mathring{\nabla}\boldsymbol{r}, \tag{2.13}$$

from which

$$\nabla\nabla \times \boldsymbol{v} = 2(\nabla\mathring{\boldsymbol{r}})_d \nabla\mathring{\boldsymbol{r}} \cdot [\dot{\mathbf{K}} - \mathring{\nabla}\mathring{\nabla}\boldsymbol{r} \underset{\cdot}{\times} (\dot{\mathbf{E}} \cdot \mathring{\boldsymbol{r}}\nabla)] \cdot \mathring{\nabla}\boldsymbol{r}. \tag{2.14}$$

Then

$$\tfrac{1}{2}\boldsymbol{\mu}^D : \nabla\nabla \times \boldsymbol{v} = (\nabla\mathring{\boldsymbol{r}})_d \{(\mathring{\boldsymbol{r}}\nabla \cdot \boldsymbol{\mu}^D \cdot \boldsymbol{r}\mathring{\nabla}) : \dot{\mathbf{K}} - [(\mathring{\boldsymbol{r}}\nabla \cdot \boldsymbol{\mu}^D \cdot \boldsymbol{r}\mathring{\nabla}) \dot{\times} \mathring{\nabla}\mathring{\nabla}\boldsymbol{r} \cdot \nabla\mathring{\boldsymbol{r}}]^S : \dot{\mathbf{E}}\}. \tag{2.15}$$

Thus, with (2.9) and (2.15), (2.7) may be written in the form

$$\varrho \dot{U} = \boldsymbol{\Phi} : \dot{\mathbf{E}} + \boldsymbol{\Psi} : \dot{\mathbf{K}}, \tag{2.16}$$

where

$$\boldsymbol{\Phi} = \mathring{\boldsymbol{r}} \boldsymbol{\nabla} \cdot \boldsymbol{\tau}^S \cdot \boldsymbol{\nabla} \mathring{\boldsymbol{r}} - (\boldsymbol{\nabla} \mathring{\boldsymbol{r}})_d [(\mathring{\boldsymbol{r}} \boldsymbol{\nabla} \cdot \boldsymbol{\mu}^D \cdot \boldsymbol{r} \mathring{\boldsymbol{\nabla}}) \dot{\times} \mathring{\boldsymbol{\nabla}} \mathring{\boldsymbol{\nabla}} \boldsymbol{r} \cdot \boldsymbol{\nabla} \mathring{\boldsymbol{r}}]^S, \tag{2.17}$$

$$\boldsymbol{\Psi} = (\boldsymbol{\nabla} \mathring{\boldsymbol{r}})_d \mathring{\boldsymbol{r}} \boldsymbol{\nabla} \cdot \boldsymbol{\mu}^D \cdot \boldsymbol{r} \mathring{\boldsymbol{\nabla}}. \tag{2.18}$$

From (2.6),

$$\varrho \dot{U} = \varrho \frac{\partial U}{\partial \mathbf{E}} : \dot{\mathbf{E}} + \varrho \frac{\partial U}{\partial \mathbf{K}} : \dot{\mathbf{K}}, \tag{2.19}$$

$$0 = A \mathring{\mathbf{I}} : \dot{\mathbf{K}}, \tag{2.20}$$

where A is an arbitrary multiplier, introduced because the nine components of $\mathbf{K}$ are not independent. Then, subtracting the sum of (2.19) and (2.20) from (2.16), we have

$$\left(\boldsymbol{\Phi} - \varrho \frac{\partial U}{\partial \mathbf{E}}\right) : \dot{\mathbf{E}} + \left(\boldsymbol{\Psi} - \varrho \frac{\partial U}{\partial \mathbf{K}} - A \mathring{\mathbf{I}}\right) : \dot{\mathbf{K}} = 0. \tag{2.21}$$

We now assume that $\boldsymbol{\tau}^S$, $\boldsymbol{\mu}^D$ and U are independent of $\dot{\mathbf{K}}$ and $\dot{\mathbf{E}}$. Hence, from (2.21),

$$\boldsymbol{\Phi} = \varrho \frac{\partial U}{\partial \mathbf{E}}, \tag{2.22}$$

$$\boldsymbol{\Psi} = \varrho \frac{\partial U}{\partial \mathbf{K}} + A \mathring{\mathbf{I}}. \tag{2.23}$$

But $\mathring{\mathbf{I}} : \boldsymbol{\Psi} = 0$ and $\mathring{\mathbf{I}} : \partial U / \partial \mathbf{K} = 0$; hence $A = 0$. Then solving (2.22) and (2.23) for $\boldsymbol{\tau}^S$ and $\boldsymbol{\mu}^D$, we find

$$\begin{aligned} \boldsymbol{\tau}^S &= \varrho \boldsymbol{r} \mathring{\boldsymbol{\nabla}} \cdot \frac{\partial U}{\partial \mathbf{E}} \cdot \mathring{\boldsymbol{\nabla}} \boldsymbol{r} + \varrho \left[\boldsymbol{r} \mathring{\boldsymbol{\nabla}} \cdot \left(\frac{\partial U}{\partial \mathbf{K}} \dot{\times} \mathring{\boldsymbol{\nabla}} \mathring{\boldsymbol{\nabla}} \boldsymbol{r}\right)\right]^S, \\ \boldsymbol{\mu}^D &= \varrho (\mathring{\boldsymbol{\nabla}} \boldsymbol{r})_d \boldsymbol{r} \mathring{\boldsymbol{\nabla}} \cdot \frac{\partial U}{\partial \mathbf{K}} \cdot \mathring{\boldsymbol{r}} \boldsymbol{\nabla}. \end{aligned} \tag{2.24}$$

Equations (2.24) are one form of Toupin's constitutive equations [4].

3. Linearization

Let us assume that the specific internal energy can be expressed as a polynomial in $\mathbf{E}$ and $\mathbf{K}$:

$$\varrho_0 U = \tfrac{1}{2} \mathbf{K} : \boldsymbol{a} : \mathbf{K} + \mathbf{E} : \boldsymbol{b} : \mathbf{K} + \tfrac{1}{2} \mathbf{E} : \boldsymbol{c} : \mathbf{E} + \cdots, \tag{3.1}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are constant, material tetradics and the coefficients of the terms of higher degree in $\mathbf{E}$ and $\mathbf{K}$ are polyadics of higher even order. The terms linear in $\mathbf{E}$ and $\mathbf{K}$ are omitted so as to make the undeformed state ($\mathbf{E} = 0$), with mass density ϱ_0, one of zero stress. Further, with the material displacement defined as $\boldsymbol{u} = \boldsymbol{r} - \mathring{\boldsymbol{r}}$, we restrict the absolute values of the components of the displacement gradient to be small in comparison with unity:

$$|\mathring{\boldsymbol{\nabla}} \boldsymbol{u}| \ll 1. \tag{3.2}$$

Then

$$\boldsymbol{\nabla} \approx \mathring{\boldsymbol{\nabla}}, \qquad \varrho \approx \varrho_0, \qquad \mathbf{I} \approx \mathring{\mathbf{I}}, \tag{3.3}$$

$$\mathbf{E} \approx \tfrac{1}{2} (\boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{u} \boldsymbol{\nabla}) \equiv \boldsymbol{\epsilon}, \qquad \mathbf{K} \approx -\boldsymbol{\epsilon} \times \boldsymbol{\nabla} = \tfrac{1}{2} \boldsymbol{\nabla} \boldsymbol{\nabla} \times \boldsymbol{u} \equiv \boldsymbol{\varkappa}, \tag{3.4}$$

and hence

$$\varrho_0 U \equiv W \approx \tfrac{1}{2}\boldsymbol{\varkappa}:\boldsymbol{a}:\boldsymbol{\varkappa} + \boldsymbol{\epsilon}:\boldsymbol{b}:\boldsymbol{\varkappa} + \tfrac{1}{2}\boldsymbol{\epsilon}:\boldsymbol{c}:\boldsymbol{\epsilon} + \cdots, \tag{3.5}$$

$$\boldsymbol{\tau}^S \approx \frac{\partial W}{\partial \boldsymbol{\epsilon}} + \left(\frac{\partial W}{\partial \boldsymbol{\varkappa}} \dot{\times} \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{u}\right)^S, \qquad \boldsymbol{\mu}^D \approx \frac{\partial W}{\partial \boldsymbol{\varkappa}}. \tag{3.6}$$

In (3.5), we note that $\boldsymbol{\epsilon}$ is dimensionless and $\boldsymbol{\varkappa}$ has the dimension of reciprocal of length. Hence, the dimensions of $|\boldsymbol{a}/\boldsymbol{c}|$ are the square of length and those of $|\boldsymbol{b}/\boldsymbol{c}|$ are length. Let C be one of the components of $\boldsymbol{c}$ and let L^2 be one of the positive components of $\boldsymbol{a}/C$; *i.e.*, L is a material constant with the dimension of length. Then (3.5) may be written as

$$W \approx C\left(\tfrac{1}{2}L^2\boldsymbol{\varkappa}:\boldsymbol{a}^*:\boldsymbol{\varkappa} + L\boldsymbol{\epsilon}:\boldsymbol{b}^*:\boldsymbol{\varkappa} + \tfrac{1}{2}\boldsymbol{\epsilon}:\boldsymbol{c}^*:\boldsymbol{\epsilon} + \cdots\right), \tag{3.7}$$

where $\boldsymbol{a}^*$, $\boldsymbol{b}^*$, $\boldsymbol{c}^*, \ldots$ are dimensionless material constants. We now assume that $\boldsymbol{a}^*$, $\boldsymbol{b}^*$, $\boldsymbol{c}^*, \ldots$ are of order of magnitude unity and that, in addition to (3.2),

$$|L\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{u}| \ll 1. \tag{3.8}$$

Then, in (3.7), terms of degree higher than the second, in $\boldsymbol{\epsilon}$ and $L\boldsymbol{\varkappa}$, are negligible in comparison with the quadratic terms; and also, in $(3.6)_1$,

$$\left|\frac{\partial W}{\partial \boldsymbol{\varkappa}} \dot{\times} \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{u}\right| \ll \left|\frac{\partial W}{\partial \boldsymbol{\epsilon}}\right|.$$

Thus, to the approximations (3.2) and (3.8),

$$W = \tfrac{1}{2}\boldsymbol{\varkappa}:\boldsymbol{a}:\boldsymbol{\varkappa} + \boldsymbol{\epsilon}:\boldsymbol{b}:\boldsymbol{\varkappa} + \tfrac{1}{2}\boldsymbol{\epsilon}:\boldsymbol{c}:\boldsymbol{\epsilon}, \tag{3.9}$$

$$\boldsymbol{\tau}^S = \frac{\partial W}{\partial \boldsymbol{\epsilon}} = \boldsymbol{c}:\boldsymbol{\epsilon} + \boldsymbol{b}:\boldsymbol{\varkappa}, \tag{3.10}$$

$$\boldsymbol{\mu}^D = \frac{\partial W}{\partial \boldsymbol{\varkappa}} = \boldsymbol{\epsilon}:\boldsymbol{b} + \boldsymbol{\varkappa}:\boldsymbol{a}. \tag{3.11}$$

These are the linearized forms of TOUPIN's constitutive equations. They are identical with those of AERO & KUVSHINSKII [5].

The dyadic $\boldsymbol{\epsilon}$ is the usual small-strain dyadic and $\boldsymbol{\varkappa}$ is the gradient of the small-rotation vector:

$$\boldsymbol{\varkappa} = \boldsymbol{\nabla}\boldsymbol{w}, \qquad \boldsymbol{w} = \tfrac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{u}; \tag{3.12}$$

i.e., $\boldsymbol{w}$ is the vector of the small-rotation dyadic

$$\boldsymbol{\omega} = \tfrac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{u}\boldsymbol{\nabla}). \tag{3.13}$$

Hence $\boldsymbol{\varkappa}$ may be termed a curvature-twist dyadic.

The existence of a displacement field, $\boldsymbol{u}$, requires the usual compatibility condition

$$\boldsymbol{\nabla}\times\boldsymbol{\epsilon}\times\boldsymbol{\nabla} = 0. \tag{3.14}$$

Hence we have, on $\boldsymbol{\varkappa}$, the compatibility conditions

$$\boldsymbol{\nabla}\times\boldsymbol{\varkappa} = 0, \qquad \boldsymbol{\varkappa}:\mathbf{I} = 0. \tag{3.15}$$

If $\boldsymbol{u}_0$ and $\boldsymbol{\omega}_0$ are the displacement and rotation at a point M_0, with position vector $\boldsymbol{r}_0$; $\boldsymbol{u}_1$ is the displacement at a point M_1, with position vector $\boldsymbol{r}_1$; and $\boldsymbol{r}$ is the position vector of the running point along a path connecting M_0 and

M_1, then Cesàro's theorem [9, p. 223, Eq. (2)] may be written as

$$u_1 = u_0 + (r_1 - r_0)\cdot\omega_0 + \int_{M_0}^{M_1} dr\cdot[\epsilon + \varkappa\times(r_1 - r)]. \tag{3.16}$$

Returning to (3.9), we note that, owing to the symmetry of the products $\varkappa : a : \varkappa$ and $\epsilon : c : \epsilon$,

$$\Phi : a = a : \Phi, \qquad \Phi : c = c : \Phi, \tag{3.17}$$

where Φ is an arbitrary dyadic, giving 36 relations among the 81 components of a and 36 relations among the 81 components of c. Owing to the symmetry $\epsilon = \epsilon_C$,

$$\Phi : b = \Phi_C : b, \qquad \Phi : c = \Phi_C : c; \tag{3.18}$$

giving 27 relations among the 81 components of b and 24 additional relations among the remaining components of c. Finally, owing to the property $\varkappa : \mathbf{I} = 0$, we have

$$a : \mathbf{I} = 0, \qquad b : \mathbf{I} = 0, \tag{3.19}$$

giving 9 additional relations among the components of a and 6 additional relations among the components of b. Thus, there are $81 - 36 - 9 = 36$ independent components of a; $81 - 27 - 6 = 48$ independent components of b and the usual $81 - 36 - 24 = 21$ independent components of c.

We also note that ϵ is a polar dyadic and $\varkappa$ is an axial dyadic. Since W, in (3.9), is a scalar, a and c must be polar tetradics and b must be an axial tetradic. We shall consider only centrosymmetric-isotropic materials; in which case the material constants a, b, c must be invariant in a central inversion and an arbitrary rotation of coordinate axes. Hence $b = 0$, since it is axial, while a and c can be only linear combinations of the three isotropic tetradics $\mathbf{II}$, $\mathbf{I}\times\mathbf{I}\times\mathbf{I}$ and

$$(\mathbf{I}\times\mathbf{I})\overset{\times}{\times}(\mathbf{I}\times\mathbf{I}).$$

Thus, for a centrosymmetric-isotropic material, we may take

$$W = \sum_{i=1}^{3} a_i\,\varkappa : J_i : \varkappa + \sum_{i=1}^{3} c_i\,\epsilon : J_i : \epsilon, \tag{3.20}$$

where the a_i and c_i are scalar constants and

$$J_1 = \mathbf{II}, \qquad J_2 = (\mathbf{I}\times\mathbf{I})\overset{\times}{\times}(\mathbf{I}\times\mathbf{I}) - J_1 - J_3, \qquad J_3 = \tfrac{1}{2}\mathbf{I}\times\mathbf{I}\times\mathbf{I}. \tag{3.21}$$

Now

$$\begin{aligned} \varkappa : J_1 : \varkappa &= \varkappa_S^2 = 0, & \epsilon : J_1 : \epsilon &= \epsilon_S^2, \\ \varkappa : J_2 : \varkappa &= \varkappa : \varkappa^S, & \epsilon : J_2 : \epsilon &= \epsilon : \epsilon, \\ \varkappa : J_3 : \varkappa &= \varkappa : \varkappa^A, & \epsilon : J_3 : \epsilon &= 0, \end{aligned}$$

where $\varkappa^S$ and $\varkappa^A$ are the symmetric and antisymmetric parts of $\varkappa$; and $\epsilon_S = \epsilon : \mathbf{I}$. Then (3.20) becomes

$$\begin{aligned} W &= a_2\,\varkappa : \varkappa^S + a_3\,\varkappa : \varkappa^A + c_1\epsilon_S^2 + c_2\,\epsilon : \epsilon, \\ &= \tfrac{1}{2}(a_2 + a_3)\,\varkappa : \varkappa + \tfrac{1}{2}(a_2 - a_3)\,\varkappa : \varkappa_C + c_1\epsilon_S^2 + c_2\,\epsilon : \epsilon. \end{aligned}$$

Finally, setting

$$4\eta = a_2 + a_3, \qquad 4\eta' = a_2 - a_3, \qquad \lambda = 2c_1, \qquad \mu = c_2, \tag{3.22}$$

the internal energy-density of a centrosymmetric-isotropic material becomes

$$W = 2\eta \boldsymbol{\kappa}:\boldsymbol{\kappa} + 2\eta' \boldsymbol{\kappa}:\boldsymbol{\kappa}_C + \tfrac{1}{2}\lambda \boldsymbol{\epsilon}_S^2 + \mu \boldsymbol{\epsilon}:\boldsymbol{\epsilon}, \tag{3.23}$$

from which

$$\begin{aligned} \boldsymbol{\tau}^S &= \frac{\partial W}{\partial \boldsymbol{\epsilon}} = \lambda \boldsymbol{\epsilon}_S \mathbf{I} + 2\mu \boldsymbol{\epsilon}, \\ \boldsymbol{\mu}^D &= \frac{\partial W}{\partial \boldsymbol{\kappa}} = 4\eta \boldsymbol{\kappa} + 4\eta' \boldsymbol{\kappa}_C. \end{aligned} \tag{3.24}$$

In the analogous expressions found by GRIOLI [6], the constant η' does not appear. This is discussed in the next section.

If we replace $\boldsymbol{\epsilon}$ by $\frac{1}{2}(\nabla \boldsymbol{u} + \boldsymbol{u}\nabla)$ and $\boldsymbol{\kappa}$ by $\frac{1}{2}\nabla\nabla\times\boldsymbol{u}$, in (3.24), the latter become

$$\begin{aligned} \boldsymbol{\tau}^S &= \lambda \nabla\cdot\boldsymbol{u}\mathbf{I} + \mu(\nabla\boldsymbol{u} + \boldsymbol{u}\nabla), \\ \boldsymbol{\mu}^D &= 2\eta \nabla\nabla\times\boldsymbol{u} + 2\eta' \nabla\times\boldsymbol{u}\nabla. \end{aligned} \tag{3.25}$$

When these are inserted in the linearized stress-equation of motion (1.16):

$$\nabla\cdot\boldsymbol{\tau}^S + \tfrac{1}{2}\nabla\times\nabla\cdot\boldsymbol{\mu}^D + \varrho \boldsymbol{f} + \tfrac{1}{2}\varrho\nabla\times\boldsymbol{c} = \varrho\ddot{\boldsymbol{u}} \tag{3.26}$$

(where $\ddot{\boldsymbol{u}}$ now designates $\partial^2\boldsymbol{u}/\partial t^2$), we find the displacement-equation of motion

$$\mu \nabla^2 \boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \eta\nabla^2\nabla\times\nabla\times\boldsymbol{u} + \varrho\boldsymbol{f} + \tfrac{1}{2}\varrho\nabla\times\boldsymbol{c} = \varrho\ddot{\boldsymbol{u}}. \tag{3.27}$$

It will be observed that η' does not appear in (3.27), so that (3.27) is identical with the displacement-equation of motion that would be obtained with GRIOLI's constitutive equations.

The sign of the bending-twisting modulus, η, is of great importance as solutions of (3.27) are different in character for positive and negative η. If we require the energy density W, in (3.23), to be positive definite (implying that energy is stored, rather than produced, as a result of deformation) we find, as necessary and sufficient conditions,

$$\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad \eta > 0, \qquad -1 < \eta'/\eta < 1. \tag{3.28}$$

In this case we can establish a theorem of uniqueness of solution, of the Neumann type, as shown in Section 5. It seems reasonable to adopt $\mu > 0$, $3\lambda + 2\mu > 0$, as these are acceptable in the absence of couple-stresses. Positiveness of η conforms to the intuitive expectation that the sign of the curvature, say $\varkappa_{12}$, produced by a bending couple μ_{12}, will be the same as the sign of μ_{12}. Furthermore, an example of failure of uniqueness, for $\eta < 0$, is exhibited in Section 13. Finally, η'/η is the ratio of the transverse curvature (probably anticlastic) to the principal curvature and it seems unlikely that this ratio would lie outside the range given in $(3.28)_4$. Thus, it is plausible to assume that W is positive definite when couple stresses are considered.

With $\mu > 0$ and $\eta > 0$, we may adopt, as the material length L, in (3.8),

$$l = \sqrt{\eta/\mu}. \tag{3.29}$$

If $l=0$, there are no effects of couple-stress. Inasmuch as the ordinary theory of elasticity has been verified experimentally in great detail, l is probably very small in comparison with bodily dimensions and wave-lengths that are commonly encountered.

The indeterminacy of $\boldsymbol{\mu}:\mathbf{I}$ and $\boldsymbol{\tau}^A$, mentioned in Section 1, carries over, of course, to the linear theory. From a solution, $\boldsymbol{u}$, of (3.27), we can compute only $\boldsymbol{\tau}^S$ and $\boldsymbol{\mu}^D$ from (3.25). The remaining parts of the stress, namely $\boldsymbol{\mu}:\mathbf{I}$ and $\boldsymbol{\tau}^A$, are related through (1.10), which may be written as

$$\boldsymbol{\tau}^A=\tfrac{1}{2}\mathbf{I}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\mu}+\varrho\boldsymbol{c})=\tfrac{1}{2}\mathbf{I}\times(2\eta\nabla^2\boldsymbol{\nabla}\times\boldsymbol{u}+\tfrac{1}{3}\boldsymbol{\nabla}\boldsymbol{\mu}:\mathbf{I}+\varrho\boldsymbol{c}). \tag{3.30}$$

Thus, with $\boldsymbol{u}$ and $\varrho\mathbf{c}$ known, there are only three scalar equations relating the scalar $\boldsymbol{\mu}:\mathbf{I}$ and the three components of $\boldsymbol{\tau}^A$.

4. Remarks on Grioli's constitutive equations

In place of (2.5), GRIOLI [6] assumed

$$U=U(\boldsymbol{\Delta},\boldsymbol{\Lambda}), \tag{4.1}$$

where

$$\boldsymbol{\Delta}=\mathring{\boldsymbol{\nabla}}\boldsymbol{r}, \tag{4.2}$$

$$\boldsymbol{\Lambda}=-\tfrac{1}{2}\boldsymbol{\Delta}\times\boldsymbol{\nabla}=-\tfrac{1}{2}(\mathring{\boldsymbol{\nabla}}\boldsymbol{r})\times\mathring{\boldsymbol{\nabla}}\cdot(\mathring{\boldsymbol{r}}\boldsymbol{\nabla})=-\tfrac{1}{2}\mathring{\boldsymbol{\nabla}}\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\dot{\times}\mathring{\boldsymbol{r}}\boldsymbol{\nabla}, \tag{4.3}$$

and he noted that $\boldsymbol{\Delta}$ and $\boldsymbol{\Lambda}$ are connected by the single scalar relation

$$\boldsymbol{\Lambda}:\boldsymbol{\Delta}_C^{-1}=-\tfrac{1}{2}(\mathring{\boldsymbol{\nabla}}\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\dot{\times}\mathring{\boldsymbol{r}}\boldsymbol{\nabla}):\mathring{\boldsymbol{r}}\boldsymbol{\nabla}=-\tfrac{1}{2}\mathbf{I}\times\mathbf{I}:\boldsymbol{\nabla}\mathring{\boldsymbol{r}}\cdot(\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\mathring{\boldsymbol{\nabla}})\cdot\mathring{\boldsymbol{r}}\boldsymbol{\nabla}=0. \tag{4.4}$$

Thus, U is a function of $9+9-1=17$ variables.

From (4.1), we have

$$\varrho\dot{U}=\varrho\frac{\partial U}{\partial\boldsymbol{\Delta}}:\dot{\boldsymbol{\Delta}}+\varrho\frac{\partial U}{\partial\boldsymbol{\Lambda}}:\dot{\boldsymbol{\Lambda}}, \tag{4.5}$$

and, from (4.4),

$$\dot{\overline{\boldsymbol{\Lambda}:\boldsymbol{\Delta}_C^{-1}}}=\boldsymbol{\Lambda}:\dot{\boldsymbol{\Delta}}_C^{-1}+\boldsymbol{\Delta}_C^{-1}:\dot{\boldsymbol{\Lambda}}=0. \tag{4.6}$$

Now, $\boldsymbol{\Delta}_C^{-1}\cdot\boldsymbol{\Delta}_C=\mathring{\mathbf{I}}$, whence

$$\dot{\overline{\boldsymbol{\Delta}_C^{-1}\cdot\boldsymbol{\Delta}_C}}=\boldsymbol{\Delta}_C^{-1}\cdot\dot{\boldsymbol{\Delta}}_C+\dot{\boldsymbol{\Delta}}_C^{-1}\cdot\boldsymbol{\Delta}_C=0. \tag{4.7}$$

Upon multiplying $(4.7)_2$ by $\cdot\,\boldsymbol{\Delta}_C^{-1}$, we have

$$\dot{\boldsymbol{\Delta}}_C^{-1}=-\boldsymbol{\Delta}_C^{-1}\cdot\dot{\boldsymbol{\Delta}}_C\cdot\boldsymbol{\Delta}_C^{-1}. \tag{4.8}$$

Then

$$\boldsymbol{\Lambda}:\dot{\boldsymbol{\Delta}}_C^{-1}=-\boldsymbol{\Lambda}:(\boldsymbol{\Delta}_C^{-1}\cdot\dot{\boldsymbol{\Delta}}_C\cdot\boldsymbol{\Delta}_C^{-1})=-\boldsymbol{\Delta}_C^{-1}\cdot\boldsymbol{\Lambda}_C\cdot\boldsymbol{\Delta}_C^{-1}:\dot{\boldsymbol{\Delta}}.$$

Thus, $(4.6)_2$ becomes

$$-\boldsymbol{\Delta}_C^{-1}\cdot\boldsymbol{\Lambda}_C\cdot\boldsymbol{\Delta}_C^{-1}:\dot{\boldsymbol{\Delta}}+\boldsymbol{\Delta}_C^{-1}:\dot{\boldsymbol{\Lambda}}=0, \tag{4.9}$$

which is the companion equation to (4.5), taking into account the relation $\boldsymbol{\Lambda}:\boldsymbol{\Delta}_C^{-1}=0$.

We have now to put the energy equation,

$$\varrho\dot{U}=\boldsymbol{\tau}^S:\boldsymbol{\nabla}\boldsymbol{v}+\tfrac{1}{2}\boldsymbol{\mu}^D:\boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{v}, \tag{4.10}$$

into the form $\varrho\dot{U}=\boldsymbol{\Phi}:\dot{\boldsymbol{\Delta}}+\boldsymbol{\Psi}:\dot{\boldsymbol{\Lambda}}$.

First, noting that

$$\nabla(\ )=\nabla\mathring{r}\cdot\mathring{\nabla}(\ ),\qquad \overset{\cdot}{\overline{\mathring{\nabla}r}}=\mathring{\nabla}\dot{r}=\mathring{\nabla}v, \tag{4.11}$$

we have

$$\tau^S:\nabla v=\tau^S:\nabla\dot{r}=\tau^S:(\nabla\mathring{r}\cdot\mathring{\nabla}\dot{r})=\tau^S:(\Delta^{-1}\cdot\dot{\Delta})=\Delta_C^{-1}\cdot\tau^S:\dot{\Delta}. \tag{4.12}$$

Next,

$$\begin{aligned}\tfrac{1}{2}\mu^D:\nabla\nabla\times v&=\tfrac{1}{2}\mu^D:\nabla\mathring{r}\cdot\mathring{\nabla}[(\nabla\mathring{r}\cdot\mathring{\nabla})\times v],\\&=\tfrac{1}{2}\mu^D:\nabla\mathring{r}\cdot[\mathring{\nabla}(\mathring{r}\nabla)\dot{\times}\mathring{\nabla}v-\mathring{\nabla}\mathring{\nabla}v\dot{\times}\mathring{r}\nabla],\\&=\tfrac{1}{2}\mu^D:\Delta^{-1}\cdot[\mathring{\nabla}\Delta_C^{-1}\dot{\times}\dot{\Delta}-\mathring{\nabla}\dot{\Delta}\dot{\times}\Delta_C^{-1}].\end{aligned}$$

Also,

$$\begin{aligned}\mathring{\nabla}\dot{\Delta}\dot{\times}\Delta_C^{-1}&=\overset{\cdot}{\overline{\mathring{\nabla}\Delta\dot{\times}\Delta_C^{-1}}}-\mathring{\nabla}\Delta\dot{\times}\dot{\Delta}_C^{-1}=-2\dot{\Lambda}+\mathring{\nabla}\Delta\dot{\times}(\Delta_C^{-1}\cdot\dot{\Delta}_C\cdot\Delta_C^{-1})\\&=-2\dot{\Lambda}+(\Delta\nabla)\dot{\times}(\Delta^{-1}\cdot\dot{\Delta}),\end{aligned}$$

by (4.3) and (4.8). Hence

$$\begin{aligned}\tfrac{1}{2}\mu^D:\nabla\nabla\times v&=\tfrac{1}{2}\mu^D:\Delta^{-1}\cdot[\mathring{\nabla}\Delta_C^{-1}\dot{\times}\dot{\Delta}+2\dot{\Lambda}-(\Delta\nabla)\dot{\times}(\Delta^{-1}\cdot\dot{\Delta})],\\&=\tfrac{1}{2}\{[\Delta_C^{-1}\cdot\mu^D\dot{\times}(\mathring{\nabla}\Delta^{-1})]_C-\Delta_C^{-1}\cdot[\Delta_C^{-1}\cdot\mu^D\dot{\times}(\Delta\nabla)]\}:\dot{\Delta}+\Delta_C^{-1}\cdot\mu^D:\dot{\Lambda}.\end{aligned} \tag{4.13}$$

Substituting (4.13) and (4.12) into (4.10), we obtain

$$\varrho\dot{U}=\Phi:\dot{\Delta}+\Psi:\dot{\Lambda}, \tag{4.14}$$

where

$$\Phi=\Delta_C^{-1}\cdot\tau^S+\tfrac{1}{2}[(\Delta_C^{-1}\cdot\mu^D)\dot{\times}(\mathring{\nabla}\Delta^{-1})]_C-\tfrac{1}{2}\Delta_C^{-1}\cdot[(\Delta_C^{-1}\cdot\mu^D)\dot{\times}(\Delta\nabla)], \tag{4.15}$$

$$\Psi=\Delta_C^{-1}\cdot\mu^D. \tag{4.16}$$

Then, multiplying (4.9) by an arbitrary scalar, A, and adding the result to the difference between (4.14) and (4.5), we have

$$\left(\Phi-\varrho\frac{\partial U}{\partial\Delta}-A\,\Delta_C^{-1}\cdot\Lambda_C\cdot\Delta_C^{-1}\right):\dot{\Delta}+\left(\Psi-\varrho\frac{\partial U}{\partial\Lambda}+A\,\Delta_C^{-1}\right):\dot{\Lambda}=0. \tag{4.17}$$

We now assume that τ^S, μ^D and U are independent of $\dot{\Delta}$ and $\dot{\Lambda}$, whence

$$\Phi-\varrho\frac{\partial U}{\partial\Delta}-A\,\Delta_C^{-1}\cdot\Lambda_C\cdot\Delta_C^{-1}=0, \tag{4.18}$$

$$\Psi-\varrho\frac{\partial U}{\partial\Lambda}+A\,\Delta_C^{-1}=0. \tag{4.19}$$

Then, multiplying (4.19) by $\Delta_C\cdot$ and using (4.16), we find

$$\mu^D=\varrho\,\Delta_C\cdot\frac{\partial U}{\partial\Lambda}-A\,\mathbf{I}; \tag{4.20}$$

and, noting that $\mu^D:\mathbf{I}=0$,

$$A=\frac{1}{3}\varrho\,\mathbf{I}:\left(\Delta_C\cdot\frac{\partial U}{\partial\Lambda}\right);$$

so that

$$\mu^D=\varrho\left(\Delta_C\cdot\frac{\partial U}{\partial\Lambda}\right)^D. \tag{4.21}$$

Now, substitute (4.20) into (4.15) and the latter into (4.18) to obtain

$$\begin{aligned}\mathbf{\Delta}_C^{-1}\cdot\boldsymbol{\tau}^S + \frac{1}{2}\varrho\left[\frac{\partial U}{\partial\mathbf{\Lambda}}\dot{\times}(\mathring{\mathbf{\nabla}}\mathbf{\Delta}^{-1})\right]_C - \frac{1}{2}\varrho\,\mathbf{\Delta}_C^{-1}\cdot\left[\frac{\partial U}{\partial\mathbf{\Lambda}}\dot{\times}(\mathbf{\Delta}\,\mathbf{\nabla})\right] - \varrho\frac{\partial U}{\partial\mathbf{\Delta}} - \\ - A\left\{\mathbf{\Delta}_C^{-1}\cdot\mathbf{\Lambda}_C\cdot\mathbf{\Delta}_C^{-1} + \frac{1}{2}[\mathbf{\Delta}_C^{-1}\dot{\times}(\mathring{\mathbf{\nabla}}\mathbf{\Delta}^{-1})]_C - \frac{1}{2}\mathbf{\Delta}_C^{-1}\cdot[\mathbf{\Delta}_C^{-1}\dot{\times}(\mathbf{\Delta}\,\mathbf{\nabla})]\right\} = 0.\end{aligned}\tag{4.22}$$

But

$$\mathbf{\Delta}_C^{-1}\dot{\times}(\mathring{\mathbf{\nabla}}\mathbf{\Delta}^{-1}) = \mathbf{\nabla}\times\mathbf{\Delta}^{-1} = 0$$

and

$$\tfrac{1}{2}\mathbf{\Delta}_C^{-1}\dot{\times}(\mathbf{\Delta}\,\mathbf{\nabla}) = -\tfrac{1}{2}[(\mathring{\mathbf{\nabla}}\boldsymbol{r})\times\mathbf{\nabla}]_C\cdot\mathring{\boldsymbol{r}}\mathbf{\nabla} = \mathbf{\Lambda}_C\cdot\mathbf{\Delta}_C^{-1}.$$

Hence, the coefficient of A, in (4.22), vanishes. Then, multiplying (4.22) by $\mathbf{\Delta}_C\,\cdot$, we have

$$\boldsymbol{\tau}^S = \varrho\,\mathbf{\Delta}_C\cdot\frac{\partial U}{\partial\mathbf{\Delta}} + \frac{1}{2}\varrho\frac{\partial U}{\partial\mathbf{\Lambda}}\dot{\times}(\mathbf{\Delta}\,\mathbf{\nabla}) - \frac{1}{2}\varrho(\mathbf{\Delta}_C\mathbf{\nabla})\dot{\times}\frac{\partial U}{\partial\mathbf{\Lambda}}.\tag{4.23}$$

Since $\boldsymbol{\tau}^S$ is symmetric, so also must be the right hand side of (4.23); *i.e.*,

$$\left[\mathbf{\Delta}_C\cdot\frac{\partial U}{\partial\mathbf{\Delta}} + \frac{1}{2}\frac{\partial U}{\partial\mathbf{\Lambda}}\dot{\times}(\mathbf{\Delta}\,\mathbf{\nabla}) - \frac{1}{2}(\mathbf{\Delta}_C\mathbf{\nabla})\dot{\times}\frac{\partial U}{\partial\mathbf{\Lambda}}\right]\dot{\times}\mathbf{I} = 0.\tag{4.24}$$

The first term in (4.24) may be written as

$$\mathbf{\Delta}_C\cdot\frac{\partial U}{\partial\mathbf{\Delta}}\dot{\times}\mathbf{I} = \mathbf{\Delta}\dot{\times}\frac{\partial U}{\partial\mathbf{\Delta}}\tag{4.25}$$

and the remaining terms are

$$\frac{1}{2}\left[\frac{\partial U}{\partial\mathbf{\Lambda}}\dot{\times}(\mathbf{\Delta}\,\mathbf{\nabla})\right]\dot{\times}\mathbf{I} = \frac{1}{2}(\mathbf{\Delta}_C\mathbf{\nabla}):\frac{\partial U}{\partial\mathbf{\Lambda}} - \frac{1}{2}(\mathbf{\nabla}\cdot\mathbf{\Delta}_C)\cdot\frac{\partial U}{\partial\mathbf{\Lambda}},\tag{4.26}$$

$$-\frac{1}{2}\left[(\mathbf{\Delta}_C\mathbf{\nabla})\dot{\times}\frac{\partial U}{\partial\mathbf{\Lambda}}\right]\dot{\times}\mathbf{I} = \frac{1}{2}(\mathbf{\nabla}\cdot\mathbf{\Delta}_C)\cdot\frac{\partial U}{\partial\mathbf{\Lambda}} - \frac{1}{2}(\mathbf{\nabla}\mathbf{\Delta}):\frac{\partial U}{\partial\mathbf{\Lambda}}.\tag{4.27}$$

The sum of (4.26) and (4.27) is

$$\frac{1}{2}(\mathbf{\Delta}_C\mathbf{\nabla} - \mathbf{\nabla}\mathbf{\Delta}):\frac{\partial U}{\partial\mathbf{\Lambda}} = \frac{1}{2}\frac{\partial U}{\partial\mathbf{\Lambda}}\dot{\times}(\mathbf{\Delta}\times\mathbf{\nabla}) = \mathbf{\Lambda}\dot{\times}\frac{\partial U}{\partial\mathbf{\Lambda}}.\tag{4.28}$$

Hence, (4.24) may be written as

$$\mathbf{\Delta}\dot{\times}\frac{\partial U}{\partial\mathbf{\Delta}} + \mathbf{\Lambda}\dot{\times}\frac{\partial U}{\partial\mathbf{\Lambda}} = 0.\tag{4.29}$$

TOUPIN [*10*, p. 885] had noted, earlier, that equations such as (4.29) are necessary and sufficient conditions for the invariance of U in a rigid rotation of the deformed material.

To summarize, GRIOLI's constitutive equations for the fourteen components of $\boldsymbol{\tau}^S$ and $\boldsymbol{\mu}^D$, in terms of a function, U, of the eighteen components of $\mathbf{\Delta}$ and $\mathbf{\Lambda}$, are [*6*, (56), (57)]

$$\begin{aligned}\boldsymbol{\tau}^S &= \varrho\,\mathbf{\Delta}_C\cdot\frac{\partial U}{\partial\mathbf{\Delta}} + \frac{1}{2}\varrho\frac{\partial U}{\partial\mathbf{\Lambda}}\dot{\times}(\mathbf{\Delta}\,\mathbf{\nabla}) - \frac{1}{2}\varrho(\mathbf{\Delta}_C\mathbf{\nabla})\dot{\times}\frac{\partial U}{\partial\mathbf{\Lambda}},\\ \boldsymbol{\mu}^D &= \varrho\left(\mathbf{\Delta}_C\cdot\frac{\partial U}{\partial\mathbf{\Lambda}}\right)^D,\end{aligned}\tag{4.30}$$

subject to the four conditions [6, (54), (71)]

$$\boldsymbol{\Lambda} : \boldsymbol{\Delta}_C^{-1} = 0, \tag{4.31}$$

$$\boldsymbol{\Delta} \dot{\times} \frac{\partial U}{\partial \boldsymbol{\Delta}} + \boldsymbol{\Lambda} \dot{\times} \frac{\partial U}{\partial \boldsymbol{\Lambda}} = 0. \tag{4.32}$$

To obtain the linearized form of (4.30)—(4.32), we assume, first, that U can be expressed as a polynomial in $\boldsymbol{\Delta}$ and $\boldsymbol{\Lambda}$:

$$\varrho_0 U = \tfrac{1}{2} \boldsymbol{\Lambda} : a : \boldsymbol{\Lambda} + \boldsymbol{\Delta} : b : \boldsymbol{\Lambda} + \tfrac{1}{2} \boldsymbol{\Delta} : c : \boldsymbol{\Delta} + \cdots . \tag{4.33}$$

Then, with the assumption (3.2), we have

$$\boldsymbol{\Delta} \approx \mathbf{I} + \nabla \boldsymbol{u}, \qquad \boldsymbol{\Lambda} \approx \boldsymbol{\varkappa}. \tag{4.34}$$

Hence, dropping terms linear in $\nabla \boldsymbol{u}$ and $\boldsymbol{\varkappa}$, (4.33) becomes

$$\varrho_0 U \equiv W \approx \tfrac{1}{2} \boldsymbol{\varkappa} : a : \boldsymbol{\varkappa} + \nabla \boldsymbol{u} : b : \boldsymbol{\varkappa} + \tfrac{1}{2} \nabla \boldsymbol{u} : c : \nabla \boldsymbol{u} + \cdots . \tag{4.35}$$

Also, (4.30) reduce to

$$\boldsymbol{\tau}^S = \frac{\partial W}{\partial \nabla \boldsymbol{u}} + \frac{1}{2} \frac{\partial W}{\partial \boldsymbol{\varkappa}} \dot{\times} (\nabla \boldsymbol{u} \nabla) - \frac{1}{2} (\boldsymbol{u} \nabla \nabla) \dot{\times} \frac{\partial W}{\partial \boldsymbol{\varkappa}}, \qquad \boldsymbol{\mu}^D = \frac{\partial W}{\partial \boldsymbol{\varkappa}}. \tag{4.36}$$

The condition (4.31) reduces to $\boldsymbol{\varkappa} : \mathbf{I} = 0$, which is satisfied identically; while (4.32) becomes

$$\mathbf{I} \dot{\times} \frac{\partial W}{\partial \nabla \boldsymbol{u}} + \boldsymbol{\varkappa} \dot{\times} \frac{\partial W}{\partial \boldsymbol{\varkappa}} = 0. \tag{4.37}$$

To complete the linearization, we must again assume (3.8). Then (4.35) to (4.37) further reduce to

$$W = \tfrac{1}{2} \boldsymbol{\varkappa} : a : \boldsymbol{\varkappa} + \nabla \boldsymbol{u} : b : \boldsymbol{\varkappa} + \tfrac{1}{2} \nabla \boldsymbol{u} : c : \nabla \boldsymbol{u}, \tag{4.38}$$

$$\boldsymbol{\tau}^S = \frac{\partial W}{\partial \nabla \boldsymbol{u}}, \qquad \boldsymbol{\mu}^D = \frac{\partial W}{\partial \boldsymbol{\varkappa}}, \tag{4.39}$$

$$\mathbf{I} \dot{\times} \frac{\partial W}{\partial \nabla \boldsymbol{u}} = 0. \tag{4.40}$$

(4.38) and (4.39) are equivalent to GRIOLI's forms; but, in effect, he retained (4.37) in place of (4.40). However, to the order of approximation employed in the passage from (4.35) and (4.36) to (4.38) and (4.39), the second vector in (4.37) is negligible in comparison with the first.

In an isotropic material,

$$W = \sum_{i=1}^{3} a_i \boldsymbol{\varkappa} : J_i : \boldsymbol{\varkappa} + \sum_{i=1}^{3} b_i \nabla \boldsymbol{u} : J_i : \boldsymbol{\varkappa} + \sum_{i=1}^{3} c_i \nabla \boldsymbol{u} : J_i : \nabla \boldsymbol{u}, \tag{4.41}$$

where the J_i are again the isotropic tetradics given in (3.21). Then

$$W = a_2 \boldsymbol{\varkappa} : \boldsymbol{\varkappa}^S + a_3 \boldsymbol{\varkappa} : \boldsymbol{\varkappa}^A + b_2 \boldsymbol{\epsilon} : \boldsymbol{\varkappa} + b_3 \boldsymbol{\omega} : \boldsymbol{\varkappa} + c_1 \boldsymbol{\epsilon}_S^2 + c_2 \boldsymbol{\epsilon} : \boldsymbol{\epsilon} + c_3 \boldsymbol{\omega} : \boldsymbol{\omega} \tag{4.42}$$

If, in addition, the material is centrosymmetric, we must have $b_2 = b_3 = 0$: since $\boldsymbol{\epsilon}$ and $\boldsymbol{\omega}$ are polar but $\boldsymbol{\varkappa}$ is axial. Then, rearranging terms for comparison with (3.23), we write (4.42) as

$$W = 2\eta\,\boldsymbol{\varkappa}:\boldsymbol{\varkappa} + 2\eta'\boldsymbol{\varkappa}:\boldsymbol{\varkappa}_C + \tfrac{1}{2}\lambda\boldsymbol{\epsilon}_S^2 + \mu\boldsymbol{\epsilon}:\boldsymbol{\epsilon} + c_3\,\boldsymbol{\omega}:\boldsymbol{\omega}, \tag{4.43}$$

where the definitions (3.22) are employed again.

We now substitute (4.43) into the rigid rotation condition (4.40) to obtain

$$c_3\mathbf{I}\overset{\cdot}{\times}\boldsymbol{\omega} = 0,$$

whence $c_3 = 0$. The rôle of the condition of invariance of the energy function in a rigid rotation is, thus, to exclude dependence of W on the rotation $\boldsymbol{\omega}$.

On the other hand, at the stage analogous to (4.43), GRIOLI first eliminated dependence of W on $\boldsymbol{\omega}$, on the grounds that $\boldsymbol{\tau}^S$ is symmetric, and then redundantly applied the condition of invariance of W in a rigid rotation. However, he applied (4.37) instead of (4.40) and, with c_3 already eliminated, he obtained

$$\eta'\boldsymbol{\varkappa}\overset{\cdot}{\times}\boldsymbol{\varkappa}_C = 0$$

from which he concluded that $\eta' = 0$.

As remarked earlier, following (3.27), the constant η', even when properly included in the constitutive equations, does not appear in the displacement-equations of motion. Nevertheless, it may enter the displacement field through certain of the boundary conditions.

GRIOLI's correct finite constitutive equations, (4.30)–(4.32), may be transformed to TOUPIN's by means of a theorem by CAUCHY, previously employed by TOUPIN in another connection [*10*, p. 888], to the effect that a function of any number of vectors, which is invariant in a rigid rotation of the system of vectors, must reduce to a function of the lengths and scalar products of the vectors and the determinants of their components taken three at a time. In the case of GRIOLI's $U(\boldsymbol{\Delta}, \boldsymbol{\Lambda})$, Cauchy's theorem requires that U must reduce to a function of the three products

$$\boldsymbol{\Delta}\cdot\boldsymbol{\Delta}_C, \qquad \boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda}_C, \qquad \boldsymbol{\Delta}\cdot\boldsymbol{\Lambda}_C; \tag{4.44}$$

the two scalar determinants

$$(\boldsymbol{\Delta})_d = \tfrac{1}{6}\boldsymbol{\Delta}\overset{\times}{\times}\boldsymbol{\Delta}:\boldsymbol{\Delta}, \qquad (\boldsymbol{\Lambda})_d = \tfrac{1}{6}\boldsymbol{\Lambda}\overset{\times}{\times}\boldsymbol{\Lambda}:\boldsymbol{\Lambda}; \tag{4.45}$$

and the two determinants

$$\tfrac{1}{2}\boldsymbol{\Delta}\overset{\times}{\times}\boldsymbol{\Delta}\cdot\boldsymbol{\Lambda}_C, \qquad \tfrac{1}{2}\boldsymbol{\Lambda}\overset{\times}{\times}\boldsymbol{\Lambda}\cdot\boldsymbol{\Delta}_C. \tag{4.46}$$

However, all of these can be expressed in terms of the material strain dyadic $\mathbf{E}$ and TOUPIN's variable $\mathbf{K}$. We have

$$\boldsymbol{\Delta}\cdot\boldsymbol{\Delta}_C = 2\mathbf{E} + \mathring{\mathbf{I}}, \tag{4.47}$$

$$\tfrac{1}{2}\boldsymbol{\Delta}\overset{\times}{\times}\boldsymbol{\Delta}\cdot\boldsymbol{\Lambda}_C = -\tfrac{1}{4}\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\overset{\times}{\times}\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\cdot(\mathring{\boldsymbol{\nabla}}\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\overset{\cdot}{\times}\mathring{\boldsymbol{r}}\boldsymbol{\nabla})_C = \tfrac{1}{2}(\mathring{\boldsymbol{\nabla}}\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\underset{\cdot}{\times}\mathring{\boldsymbol{\nabla}}\boldsymbol{r})_C = \mathbf{K}_C; \tag{4.48}$$

and the remaining variables in (4.44)—(4.46) can be expressed in terms of $\boldsymbol{\Delta}\cdot\boldsymbol{\Delta}_C$ and $\mathbf{K}$ as follows:

$$(\boldsymbol{\Delta})_d = \pm\left[(\boldsymbol{\Delta}\cdot\boldsymbol{\Delta}_C)_d\right]^{\frac{1}{2}}; \tag{4.49}$$

$$\begin{aligned}\boldsymbol{\Delta}\cdot\boldsymbol{\Lambda}_C &= \boldsymbol{\Delta}\cdot(\boldsymbol{\Delta}_C\cdot\boldsymbol{\Delta}_C^{-1})\cdot\boldsymbol{\Lambda}_C,\\ &= \boldsymbol{\Delta}\cdot\boldsymbol{\Delta}_C\cdot\left[\tfrac{1}{2}(\boldsymbol{\Delta})_d^{-}\ \boldsymbol{\Delta}\overset{\times}{\times}\boldsymbol{\Delta}\right]\cdot\boldsymbol{\Lambda}_C,\\ &= (\boldsymbol{\Delta})_d^{-1}(\boldsymbol{\Delta}\cdot\boldsymbol{\Delta}_C)\cdot\mathbf{K}_C;\end{aligned} \tag{4.50}$$

$$\begin{aligned}\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda}_C &= \boldsymbol{\Lambda}\cdot(\boldsymbol{\Delta}_C\cdot\boldsymbol{\Delta}_C^{-1})\cdot(\boldsymbol{\Delta}^{-1}\cdot\boldsymbol{\Delta})\cdot\boldsymbol{\Lambda}_C,\\ &= (\boldsymbol{\Lambda}\cdot\boldsymbol{\Delta}_C)\cdot(\boldsymbol{\Delta}_C^{-1}\cdot\boldsymbol{\Delta}^{-1})\cdot(\boldsymbol{\Delta}\cdot\boldsymbol{\Lambda}_C),\\ &= (\boldsymbol{\Delta}\cdot\boldsymbol{\Lambda}_C)_C\cdot(\boldsymbol{\Delta}\cdot\boldsymbol{\Delta}_C)^{-1}\cdot(\boldsymbol{\Delta}\cdot\boldsymbol{\Lambda}_C);\end{aligned} \tag{4.51}$$

$$(\boldsymbol{\Lambda})_d = \pm\left[(\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda}_C)_d\right]^{\frac{1}{2}}; \tag{4.52}$$

$$\begin{aligned}\tfrac{1}{2}\boldsymbol{\Lambda}\overset{\times}{\times}\boldsymbol{\Lambda}\cdot\boldsymbol{\Delta}_C &= (\boldsymbol{\Lambda})_d\boldsymbol{\Lambda}_C^{-1}\cdot\boldsymbol{\Delta}_C,\\ &= (\boldsymbol{\Lambda})_d\boldsymbol{\Lambda}_C^{-1}\cdot(\boldsymbol{\Lambda}^{-1}\cdot\boldsymbol{\Lambda})\cdot\boldsymbol{\Delta}_C,\\ &= (\boldsymbol{\Lambda})_d(\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda}_C)^{-1}\cdot(\boldsymbol{\Delta}\cdot\boldsymbol{\Lambda}_C)_C.\end{aligned} \tag{4.53}$$

Thus, taking only the positive signs in (4.49) and (4.52), each of the variables (4.44)—(4.46) can be expressed in terms of $\mathbf{E}$ and $\mathbf{K}$ and, hence, GRIOLI's $U(\boldsymbol{\Delta},\boldsymbol{\Lambda})$ can be replaced by TOUPIN's $U(\mathbf{E},\mathbf{K})$.

5. Uniqueness of solution

We consider solutions of the thirty-seven equations

{3} $$\nabla\cdot\boldsymbol{\tau}+\varrho\boldsymbol{f}=\varrho\ddot{\boldsymbol{u}}, \tag*{(5.1)$_1$}$$

{3} $$\nabla\cdot\boldsymbol{\mu}+\varrho\boldsymbol{c}+\boldsymbol{\tau}\overset{\times}{\cdot}\mathbf{I}=0, \tag*{(5.1)$_2$}$$

{6} $$\boldsymbol{\epsilon}=\tfrac{1}{2}(\nabla\boldsymbol{u}+\boldsymbol{u}\nabla), \tag*{(5.2)$_1$}$$

{3} $$\boldsymbol{w}=\tfrac{1}{2}\nabla\times\boldsymbol{u}, \tag*{(5.2)$_2$}$$

{8} $$\boldsymbol{\varkappa}=\nabla\boldsymbol{w}, \tag*{(5.2)$_3$}$$

{6} $$\boldsymbol{\tau}^S=\lambda\boldsymbol{\epsilon}_S\mathbf{I}+2\mu\boldsymbol{\epsilon}, \tag*{(5.3)$_1$}$$

{8} $$\boldsymbol{\mu}^D=4\eta\boldsymbol{\varkappa}+4\eta'\boldsymbol{\varkappa}_C, \tag*{(5.3)$_2$}$$

in thirty-eight dependent variables (9 components of $\boldsymbol{\tau}$, 9 of $\boldsymbol{\mu}$, 3 of $\boldsymbol{u}$, 6 of $\boldsymbol{\epsilon}$, 3 of $\boldsymbol{w}$ and 8 of $\boldsymbol{\varkappa}$).

As noted previously, the scalar of $\boldsymbol{\mu}$ cannot be determined, so that we shall have to be content with lack of uniqueness to the extent of an arbitrary value of $\boldsymbol{\mu}:\mathbf{I}$. With this reservation, we proceed in the usual manner to consider two solutions $\boldsymbol{u}', \boldsymbol{\epsilon}', \boldsymbol{w}', \boldsymbol{\varkappa}', \boldsymbol{\tau}', \boldsymbol{\mu}'$ and $\boldsymbol{u}'', \boldsymbol{\epsilon}'', \boldsymbol{w}'', \boldsymbol{\varkappa}'', \boldsymbol{\tau}'', \boldsymbol{\mu}''$, of (5.1)—(5.3), and their differences, $\boldsymbol{u}=\boldsymbol{u}'-\boldsymbol{u}''$, *etc.*, which are also solutions, since the equations are linear. Similarly, body force and body couple differences are defined by $\boldsymbol{f}=\boldsymbol{f}'-\boldsymbol{f}''$ and $\boldsymbol{c}=\boldsymbol{c}'-\boldsymbol{c}''$. From $(5.1)_1$, we form the equation

$$\int_{t_0}^{t} dt \int_V (\nabla\cdot\boldsymbol{\tau}+\varrho\boldsymbol{f}-\varrho\ddot{\boldsymbol{u}})\cdot\dot{\boldsymbol{u}}\,dV=0, \tag{5.4}$$

on the difference variables. Now,

$$(\nabla\cdot\boldsymbol{\tau})\cdot\dot{\boldsymbol{u}}=\nabla\cdot(\boldsymbol{\tau}\cdot\dot{\boldsymbol{u}})-\boldsymbol{\tau}:\nabla\dot{\boldsymbol{u}}=\nabla\cdot(\boldsymbol{\tau}\cdot\dot{\boldsymbol{u}})-\boldsymbol{\tau}^S:\nabla\dot{\boldsymbol{u}}-\boldsymbol{\tau}^A:\nabla\dot{\boldsymbol{u}}. \tag{5.5}$$

But $\boldsymbol{\tau}^A : \nabla \dot{\boldsymbol{u}} = \frac{1}{2}(\boldsymbol{\tau} \dot{\times} \mathbf{I}) \cdot \nabla \times \dot{\boldsymbol{u}}$. Hence, using $(5.1)_2$,

$$\begin{aligned} \boldsymbol{\tau}^A : \nabla \dot{\boldsymbol{u}} &= -\tfrac{1}{2}(\nabla \cdot \boldsymbol{\mu} + \varrho \boldsymbol{c}) \cdot \nabla \times \dot{\boldsymbol{u}} \\ &= -\tfrac{1}{2}[\nabla \cdot (\boldsymbol{\mu} \cdot \nabla \times \dot{\boldsymbol{u}}) - \boldsymbol{\mu} : \nabla \nabla \times \dot{\boldsymbol{u}}] - \tfrac{1}{2}\varrho \boldsymbol{c} \cdot \nabla \times \dot{\boldsymbol{u}}. \end{aligned} \tag{5.6}$$

Inserting (5.6) in $(5.5)_2$ and using $(5.2)_1$, $(5.2)_2$ and $(5.2)_3$, we have

$$(\nabla \cdot \boldsymbol{\tau}) \cdot \dot{\boldsymbol{u}} = \nabla \cdot (\boldsymbol{\tau} \cdot \dot{\boldsymbol{u}} + \boldsymbol{\mu} \cdot \dot{\boldsymbol{w}}) - \boldsymbol{\tau}^S : \dot{\boldsymbol{\epsilon}} - \boldsymbol{\mu} : \dot{\boldsymbol{\varkappa}} + \varrho \boldsymbol{c} \cdot \dot{\boldsymbol{w}}. \tag{5.7}$$

Then, noting that $\boldsymbol{\mu} : \dot{\boldsymbol{\varkappa}} = \boldsymbol{\mu}^D : \dot{\boldsymbol{\varkappa}}$, since $\boldsymbol{\varkappa} : \mathbf{I} = 0$, and using $(5.3)_1$ and $(5.3)_2$,

$$\begin{aligned} (\nabla \cdot \boldsymbol{\tau}) \cdot \dot{\boldsymbol{u}} &= \nabla \cdot (\boldsymbol{\tau} \cdot \dot{\boldsymbol{u}} + \boldsymbol{\mu} \cdot \dot{\boldsymbol{w}}) - (\lambda \epsilon_S \dot{\epsilon}_S + 2\mu \boldsymbol{\epsilon} : \dot{\boldsymbol{\epsilon}} + 4\eta \boldsymbol{\varkappa} : \dot{\boldsymbol{\varkappa}} + 4\eta' \boldsymbol{\varkappa} : \dot{\boldsymbol{\varkappa}}_C) + \varrho \boldsymbol{c} \cdot \dot{\boldsymbol{w}}, \\ &= \nabla \cdot (\boldsymbol{\tau} \cdot \dot{\boldsymbol{u}} + \boldsymbol{\mu} \cdot \dot{\boldsymbol{w}}) - \dot{W} + \varrho \boldsymbol{c} \cdot \dot{\boldsymbol{w}}, \end{aligned} \tag{5.8}$$

where W is defined as in (3.23), but in terms of the difference variables. Inserting (5.8) in (5.4) and applying the divergence theorem and an integration with respect to time, we find

$$\begin{aligned} \int_V [\tfrac{1}{2}\varrho \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} + W]_{t_0}^t dV &= \int_{t_0}^t dt \int_V \varrho (\boldsymbol{f} \cdot \dot{\boldsymbol{u}} + \boldsymbol{c} \cdot \dot{\boldsymbol{w}}) dV + \\ &\quad + \int_{t_0}^t dt \int_S (\boldsymbol{n} \cdot \boldsymbol{\tau} \cdot \dot{\boldsymbol{u}} + \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \dot{\boldsymbol{w}}) dS. \end{aligned} \tag{5.9}$$

From the integrand of the surface integral, it would appear that the equations require six boundary conditions. However, five are sufficient. We note that

$$\begin{aligned} \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \dot{\boldsymbol{w}} &= \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \boldsymbol{n}\boldsymbol{n} \cdot \dot{\boldsymbol{w}} + \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot (\mathbf{I} - \boldsymbol{n}\boldsymbol{n}) \cdot \dot{\boldsymbol{w}} \\ &= \tfrac{1}{2}\mu_{nn} \boldsymbol{n} \cdot \nabla \times \dot{\boldsymbol{u}} + \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot (\mathbf{I} - \boldsymbol{n}\boldsymbol{n}) \cdot \dot{\boldsymbol{w}}, \end{aligned} \tag{5.10}$$

where $\mu_{nn} = \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \boldsymbol{n}$. Also,

$$\tfrac{1}{2}\mu_{nn} \boldsymbol{n} \cdot \nabla \times \dot{\boldsymbol{u}} = \tfrac{1}{2}\boldsymbol{n} \cdot \nabla \times (\mu_{nn} \dot{\boldsymbol{u}}) - \tfrac{1}{2}\boldsymbol{n} \times \nabla \mu_{nn} \cdot \dot{\boldsymbol{u}}. \tag{5.11}$$

By Stokes' theorem, if the surface S is smooth,

$$\int_S \boldsymbol{n} \cdot \nabla \times (\mu_{nn} \dot{\boldsymbol{u}}) dS = 0. \tag{5.12}$$

It may be observed, from (5.11) and (5.12), that a uniform distribution of μ_{nn}, on a smooth surface, does no work.

Inserting (5.10)–(5.12) in (5.9), we find

$$\begin{aligned} \int_V [\tfrac{1}{2}\varrho \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} + W]_{t_0}^t dV &= \int_{t_0}^t dt \int_V \varrho (\boldsymbol{f} \cdot \dot{\boldsymbol{u}} + \boldsymbol{c} \cdot \dot{\boldsymbol{w}}) dV + \\ &\quad + \int_{t_0}^t dt \int_S [(\boldsymbol{n} \cdot \boldsymbol{\tau} - \tfrac{1}{2}\boldsymbol{n} \times \nabla \mu_{nn}) \cdot \dot{\boldsymbol{u}} + \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot (\mathbf{I} - \boldsymbol{n}\boldsymbol{n}) \cdot \dot{\boldsymbol{w}}] dS. \end{aligned} \tag{5.13}$$

Thus, the normal component of the couple-stress vector on S enters only in the combination with the force-stress vector shown in the coefficient of $\dot{\boldsymbol{u}}$ in the surface integral in (5.13). Accordingly, specification of the boundary values of the three components of the force-stress vector and the three components of the couple-stress vector results in only five boundary conditions rather than six. Alternatively, the three components of displacement and only the tangential components of rotation, $(\mathbf{I} - \boldsymbol{n}\boldsymbol{n}) \cdot \boldsymbol{w}$, may be specified on the boundary.

To apply the conditions, implied in (5.13), to problems involving force-stress and couple-stress boundary values, it is necessary to express the coefficient of

$\dot{\boldsymbol{u}}$, in the surface integral, in terms of $\boldsymbol{\tau}^S$ and $\boldsymbol{\mu}^D$ since only these portions of $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ appear in the constitutive equations. To this end, we write $\boldsymbol{\tau}=\boldsymbol{\tau}^S+\boldsymbol{\tau}^A$ in (5.13). Then, from (1.10), which is an alternative form of $(5.1)_2$, we may write

$$\boldsymbol{n}\cdot\boldsymbol{\tau}^A\cdot\dot{\boldsymbol{u}}=\tfrac{1}{2}\boldsymbol{n}\cdot\mathbf{I}\times(\nabla\cdot\boldsymbol{\mu}+\varrho\boldsymbol{c})\cdot\dot{\boldsymbol{u}}=\tfrac{1}{2}\boldsymbol{n}\times(\nabla\cdot\boldsymbol{\mu})\cdot\dot{\boldsymbol{u}}+\tfrac{1}{2}\varrho\boldsymbol{n}\times\boldsymbol{c}\cdot\dot{\boldsymbol{u}}. \tag{5.14}$$

Noting that $\nabla\cdot\boldsymbol{\mu}-\nabla\mu_{nn}=\nabla\cdot\boldsymbol{\mu}^D-\nabla\mu^D_{nn}$, we have, then,

$$\int_S(\boldsymbol{n}\cdot\boldsymbol{\tau}-\tfrac{1}{2}\boldsymbol{n}\times\nabla\mu_{nn})\cdot\dot{\boldsymbol{u}}\,dS=\int_S\boldsymbol{p}\cdot\dot{\boldsymbol{u}}\,dS+\tfrac{1}{2}\varrho\int_S\boldsymbol{n}\times\boldsymbol{c}\cdot\dot{\boldsymbol{u}}\,dS, \tag{5.15}$$

where

$$\boldsymbol{p}=\boldsymbol{n}\cdot\boldsymbol{\tau}^S+\tfrac{1}{2}\boldsymbol{n}\times(\nabla\cdot\boldsymbol{\mu}^D-\nabla\mu^D_{nn}). \tag{5.16}$$

Also, in (5.13),

$$\begin{aligned}\varrho\int_V\boldsymbol{c}\cdot\dot{\boldsymbol{w}}\,dV&=\tfrac{1}{2}\varrho\int_V\nabla\times\boldsymbol{c}\cdot\dot{\boldsymbol{u}}\,dV-\tfrac{1}{2}\varrho\int_V\nabla\cdot(\boldsymbol{c}\times\dot{\boldsymbol{u}})\,dV,\\&=\tfrac{1}{2}\varrho\int_V\nabla\times\boldsymbol{c}\cdot\dot{\boldsymbol{u}}\,dV-\tfrac{1}{2}\varrho\int_S\boldsymbol{n}\times\boldsymbol{c}\cdot\dot{\boldsymbol{u}}\,dS.\end{aligned} \tag{5.17}$$

From these results, we may write (5.13) in the final form:

$$\begin{aligned}\int_V[\tfrac{1}{2}\varrho\,\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}}+W]^t_{t_0}dV&=\int_{t_0}^t dt\int_V\varrho(\boldsymbol{f}+\tfrac{1}{2}\nabla\times\boldsymbol{c})\cdot\dot{\boldsymbol{u}}\,dV+\\&+\int_{t_0}^t dt\int_S[\boldsymbol{p}\cdot\dot{\boldsymbol{u}}+\boldsymbol{n}\cdot\boldsymbol{\mu}\cdot(\mathbf{I}-\boldsymbol{n}\boldsymbol{n})\cdot\dot{\boldsymbol{w}}]\,dS.\end{aligned} \tag{5.18}$$

Consider, for example, orthogonal curvilinear or rectilinear coordinates α,β,γ and unit vectors $\boldsymbol{e}_\alpha,\boldsymbol{e}_\beta,\boldsymbol{e}_\gamma$ in the directions α,β,γ increasing. Let $\gamma=\gamma_0$ be the bounding surface S, whence $\boldsymbol{n}=\boldsymbol{e}_\gamma$. Then the integrand of the surface integral in (5.18) is

$$p_\alpha\dot{u}_\alpha+p_\beta\dot{u}_\beta+\tau_{\gamma\gamma}\dot{u}_\gamma+\mu_{\gamma\alpha}\dot{w}_\alpha+\mu_{\gamma\beta}\dot{w}_\beta, \tag{5.19}$$

where

$$p_\alpha=\tau^S_{\gamma\alpha}-\tfrac{1}{2}\boldsymbol{e}_\beta\cdot(\nabla\cdot\boldsymbol{\mu}^D-\nabla\mu^D_{\gamma\gamma}),\qquad p_\beta=\tau^S_{\gamma\beta}+\tfrac{1}{2}\boldsymbol{e}_\alpha\cdot(\nabla\cdot\boldsymbol{\mu}^D-\nabla\mu^D_{\gamma\gamma}), \tag{5.20}$$

and $\tau^S_{\gamma\alpha}=\boldsymbol{e}_\gamma\cdot\boldsymbol{\tau}^S\cdot\boldsymbol{e}_\alpha$, *etc.*; $\mu_{\gamma\alpha}=\boldsymbol{e}_\gamma\cdot\boldsymbol{\mu}\cdot\boldsymbol{e}_\alpha$, *etc.*; $w_\alpha=\boldsymbol{e}_\alpha\cdot\boldsymbol{w}$, *etc.*

If S has an edge, C, formed by the intersection of two coordinate surfaces, say $\gamma=\gamma_0$ and $\alpha=\alpha_0$, the surface integral in (5.12) is equal to $\int_C\mu_{nn}\dot{\boldsymbol{u}}\cdot d\boldsymbol{r}$, instead of zero, for each surface; and this contributes to (5.18) the additional integral

$$\int_C\{[\mu_{\gamma\gamma}]_{\gamma_0}-[\mu_{\alpha\alpha}]_{\alpha_0}\}\dot{\boldsymbol{u}}\cdot d\boldsymbol{r}=\int_C\{[\mu^D_{\gamma\gamma}]_{\gamma_0}-[\mu^D_{\alpha\alpha}]_{\alpha_0}\}\dot{\boldsymbol{u}}\cdot d\boldsymbol{r}, \tag{5.21}$$

where the integration is along the edge. The quantity in braces, in (5.21), is a force, per unit length of edge, tangent to the edge.

Conditions sufficient for a unique solution are now obtained from (5.18) in the usual manner [9, p. 176], based on the positive definiteness of W and $\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}}$. Returning to a single solution, $\boldsymbol{u},\boldsymbol{\epsilon},\boldsymbol{w},\boldsymbol{\varkappa},\boldsymbol{\tau},\boldsymbol{\mu}$, we see that sufficient conditions for uniqueness are

1. In V: at each point, $\boldsymbol{f}$, $\nabla\times\boldsymbol{c}$ and initial values of $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$;

2. On S: at each point, a component of $\boldsymbol{u}$ (or $\boldsymbol{p}$), in any direction, and the resultant of $\boldsymbol{p}$ (or $\boldsymbol{u}$) in the plane at right angles; and, in the tangent plane, a component of $\boldsymbol{w}$ (or $\boldsymbol{m}_n$) and the component of $\boldsymbol{m}_n$ (or $\boldsymbol{w}$) at right angles. In orthogonal coordinates, with γ increasing outward, normal to S, these components are one member of each of the five products

$$p_\alpha u_\alpha,\ p_\beta u_\beta,\ \tau_{\gamma\gamma} u_\gamma,\ \mu_{\gamma\alpha} w_\alpha,\ \mu_{\gamma\beta} w_\beta, \tag{5.22}$$

where

$$p_\alpha = \tau^S_{\gamma\alpha} - \tfrac{1}{2} \boldsymbol{e}_\beta \cdot (\nabla \cdot \boldsymbol{\mu}^D - \nabla \mu^D_{\gamma\gamma}), \qquad p_\beta = \tau^S_{\gamma\beta} + \tfrac{1}{2} \boldsymbol{e}_\alpha \cdot (\nabla \cdot \boldsymbol{\mu}^D - \nabla \mu^D_{\gamma\gamma}); \tag{5.23}$$

3. On an edge: at each point, the force per unit length or the displacement — both tangent to the edge. If the edge is the intersection of orthogonal surfaces $\gamma=\gamma_0$ and $\alpha=\alpha_0$, the quantities are

$$[\mu^D_{\gamma\gamma}]_{\gamma_0} - [\mu^D_{\alpha\alpha}]_{\alpha_0} \quad \text{or} \quad u_\beta. \tag{5.24}$$

The reduction of the number of boundary conditions from six to five is analogous to the reduction from three to two conditions in the classical theory of flexure of thin plates.

6. Wave motion

Consider the displacement-equation of motion, (3.27), in the absence of body force and body couple:

$$\mu \nabla^2 \boldsymbol{u} + (\lambda+\mu) \nabla \nabla \cdot \boldsymbol{u} + \eta \nabla^2 \nabla \times \nabla \times \boldsymbol{u} = \varrho \ddot{\boldsymbol{u}}. \tag{6.1}$$

Taking the divergence and curl of (6.1), we obtain, respectively,

$$c_1^2 \nabla^2 \nabla \cdot \boldsymbol{u} = \nabla \cdot \ddot{\boldsymbol{u}}, \tag{6.2}$$

$$c_2^2 (1 - l^2 \nabla^2) \nabla^2 \nabla \times \boldsymbol{u} = \nabla \times \ddot{\boldsymbol{u}}, \tag{6.3}$$

where

$$l^2 = \eta/\mu, \qquad c_1^2 = (\lambda + 2\mu)/\varrho, \qquad c_2^2 = \mu/\varrho. \tag{6.4}$$

Thus, the dilatation is propagated non-dispersively, with velocity c_1, as it is in the absence of couple-stress; but the propagation of the rotation is influenced by couple-stress.

Consider, for example, the plane wave

$$\nabla \times \boldsymbol{u} = \boldsymbol{d} A \exp[i\xi(\boldsymbol{n}\cdot\boldsymbol{r} - ct)] = \boldsymbol{d} A \exp[i(\xi \boldsymbol{n}\cdot\boldsymbol{r} - \omega t)], \tag{6.5}$$

where $\boldsymbol{d}$ is a unit vector, A is the amplitude, ξ is the wave number, $\boldsymbol{n}$ is the unit wave normal, c is the phase velocity and ω is the circular frequency. Upon substituting (6.5) in (6.3), we find

$$c^2 = c_2^2 (1 + l^2 \xi^2), \qquad \omega^2 = \xi^2 c_2^2 (1 + l^2 \xi^2); \tag{6.6}$$

and, solving $(6.6)_2$ for ξ^2, we find the two roots

$$\xi_1^2 = \tfrac{1}{2} l^{-2} [(1 + 4 l^2 \omega^2 / c_2^2)^{\frac{1}{2}} - 1], \qquad \xi_2^2 = -\tfrac{1}{2} l^{-2} [(1 + 4 l^2 \omega^2 / c_2^2)^{\frac{1}{2}} + 1]; \tag{6.7}$$

so that one wave number, ξ_1, is real and the other is a pure imaginary, since, by (3.28) and (3.29), l is real. Thus, there are two rotational waves: one propagating, the other non-propagating and both dispersive.

The group velocity of the real wave,

$$d\omega/d\xi_1 = c_2 (1 + 2 l^2 \xi_1^2)/(1 + l^2 \xi_1^2)^{\frac{1}{2}}, \tag{6.8}$$

increases monotonically with increasing $\xi_1 l$. This increase might produce a detectable effect of couple-stress in high frequency vibrations. If $\xi_1 l$ is small, *i.e.*, if the wave length is large in comparison with the material constant l, the group velocity is approximately the same (c_2) as without couple-stress. The phase velocity of the real wave increases monotonically with increasing $\xi_1 l$ and reaches the velocity, c_1, of the dilatational wave at

$$\xi_1 l = [(\lambda+\mu)/\mu]^{\frac{1}{2}} = (1-2\sigma)^{-\frac{1}{2}}, \tag{6.9}$$

in which

$$\sigma = \lambda/2(\lambda+\mu) \tag{6.10}$$

is Poisson's ratio.

The imaginary branch has a wave number cut-off at $\xi_2 l = \pm i$ and zero frequency. This branch is a second major source of the difference between elastic fields with and without couple-stresses. In both dynamic and static fields, it contributes local effects near boundaries and singularities.

7. Potentials of dilatation and rotation

Solutions of the equation of motion may be expressed as solutions of equations which reduce to Lamé's in the absence of couple-stress. Consider the resolution of the displacement into its lamellar and solenoidal components:

$$\boldsymbol{u} = \nabla\varphi + \nabla\times\boldsymbol{H}, \qquad \nabla\cdot\boldsymbol{H} = 0. \tag{7.1}$$

Upon substituting (7.1) in (6.1), we obtain

$$\nabla(c_1^2 \nabla^2\varphi - \ddot{\varphi}) + \nabla\times[c_2^2(1-l^2\nabla^2)\nabla^2\boldsymbol{H} - \ddot{\boldsymbol{H}}] = 0, \tag{7.2}$$

which is satisfied by

$$c_1^2\nabla^2\varphi = \ddot{\varphi}, \qquad c_2^2(1-l^2\nabla^2)\nabla^2\boldsymbol{H} = \ddot{\boldsymbol{H}}. \tag{7.3}$$

That (7.1) and (7.3) comprise the complete solution of the equation of motion (6.1) may be proved by following exactly the steps in Sternberg's [*11*] form of the proof for the case $l=0$. It is only necessary to replace the operator

$$\Box_2^2 = \nabla^2 - c_2^{-2}\,\partial^2/\partial t^2,$$

in Sternberg's proof, by

$$\Box_2^2 = (1-l^2\nabla^2)\nabla^2 - c_2^{-2}\,\partial^2/\partial t^2.$$

Also, as in the case $l=0$, the condition $\nabla\cdot\boldsymbol{H}=0$ is not necessary.

For steady vibrations ($\varphi = \overline{\varphi} e^{i\omega t}$, $\boldsymbol{H} = \overline{\boldsymbol{H}} e^{i\omega t}$), (7.3) become

$$(\nabla^2+\alpha^2)\overline{\varphi} = 0, \qquad (\nabla^2+\beta_1^2)(\nabla^2-\beta_2^2)\overline{\boldsymbol{H}} = 0, \tag{7.4}$$

where $\alpha = \omega/c_1$ and

$$\beta_1 = 2^{-\frac{1}{2}} l^{-1}[(1+4l^2\omega^2/c_2^2)^{\frac{1}{2}} - 1]^{\frac{1}{2}}, \qquad \beta_2 = 2^{-\frac{1}{2}} l^{-1}[(1+4l^2\omega^2/c_2^2)^{\frac{1}{2}} + 1]^{\frac{1}{2}}. \tag{7.5}$$

Then, with $\overline{\boldsymbol{H}} = \overline{\boldsymbol{H}}' + \overline{\boldsymbol{H}}''$, the complete solution [*11*] is

$$\overline{\boldsymbol{u}} = \nabla\overline{\varphi} + \nabla\times\overline{\boldsymbol{H}}' + \nabla\times\overline{\boldsymbol{H}}'', \tag{7.6}$$

where $\overline{\varphi}$, $\overline{\boldsymbol{H}}'$ and $\overline{\boldsymbol{H}}''$ are governed by the Helmholtz equations:

$$(\nabla^2+\alpha^2)\overline{\varphi} = 0, \qquad (\nabla^2+\beta_1^2)\overline{\boldsymbol{H}}' = 0, \qquad (\nabla^2-\beta_2^2)\overline{\boldsymbol{H}}'' = 0. \tag{7.7}$$

8. Thickness-shear vibrations of a plate

We consider an infinite plate bounded by free surfaces $y=\pm b$ in a rectangular coordinate system x, y, z. From (5.22), the boundary conditions on $y=\pm b$ are

$$p_\alpha = p_\beta = \tau_{\gamma\gamma} = \mu_{\gamma\alpha} = \mu_{\gamma\beta} = 0, \tag{8.1}$$

where α, β, γ are z, x, y respectively. Thus,

$$\begin{gathered} p_\alpha = \tau^S_{yz} - \tfrac{1}{2} e_x \cdot (\nabla \cdot \boldsymbol{\mu}^D - \nabla \mu^D_{yy}), \qquad p_\beta = \tau^S_{yx} + \tfrac{1}{2} e_z \cdot (\nabla \cdot \boldsymbol{\mu}^D - \nabla \mu^D_{yy}), \\ \tau_{\gamma\gamma} = \tau_{yy}, \qquad \mu_{\gamma\alpha} = \mu_{yz}, \qquad \mu_{\gamma\beta} = \mu_{yx}. \end{gathered} \tag{8.2}$$

Hence, on $y=\pm b$,

$$\begin{gathered} \tau^S_{yz} - \frac{1}{2}\left(\frac{\partial \mu^D_{xx}}{\partial x} + \frac{\partial \mu_{yx}}{\partial y} + \frac{\partial \mu_{zx}}{\partial z} - \frac{\partial \mu^D_{yy}}{\partial x}\right) = 0, \\ \tau^S_{yx} + \frac{1}{2}\left(\frac{\partial \mu_{xz}}{\partial x} + \frac{\partial \mu_{yz}}{\partial y} + \frac{\partial \mu^D_{zz}}{\partial z} - \frac{\partial \mu^D_{yy}}{\partial z}\right) = 0, \\ \tau_{yy} = \mu_{yz} = \mu_{yx} = 0. \end{gathered} \tag{8.3}$$

Of main interest are the antisymmetric, thickness-shear modes of vibration of the plate. For these, we take, as solutions of (7.7),

$$\bar{\varphi} = \bar{H}_x = \bar{H}_y = 0, \qquad \bar{H}'_z = A_1 \cos\beta_1 y, \qquad \bar{H}''_z = A_2 \cosh\beta_2 y, \tag{8.4}$$

where β_1 and β_2 are defined in (7.5). Then, from (8.4) and (7.6),

$$\bar{u}_x = -A_1\beta_1 \sin\beta_1 y + A_2\beta_2 \sinh\beta_2 y, \qquad \bar{u}_y = \bar{u}_z = 0. \tag{8.5}$$

From (8.5) and (5.2), all the components of strain and curvature are zero except ε_{xy} and $\varkappa_{yz}$. Then, from (5.3), the only non-zero components of $\boldsymbol{\tau}^S$ and $\boldsymbol{\mu}^D$, across $y=$ constant, are

$$\tau^S_{yx} = -\mu\,(A_1\beta_1^2 \cos\beta_1 y - A_2\beta_2^2 \cosh\beta_2 y), \tag{8.6}$$

$$\mu_{yz} = -2\eta\,(A_1\beta_1^3 \sin\beta_1 y + A_2\beta_2^3 \sinh\beta_2 y). \tag{8.7}$$

As a result, the first, third and fifth of the boundary conditions (8.3) are satisfied identically and the remaining two reduce to

$$\tau^S_{yx} + \tfrac{1}{2}\partial\mu_{yz}/\partial y = 0, \qquad \mu_{yz} = 0, \quad \text{on } y = \pm b. \tag{8.8}$$

Substituting (8.6) and (8.7) in (8.8), we obtain

$$A_1\beta_1^2(1+l^2\beta_1^2)\cos\beta_1 b - A_2\beta_2^2(1-l^2\beta_2^2)\cosh\beta_2 b = 0, \tag{8.9}$$

$$A_1\beta_1^3 \sin\beta_1 b + A_2\beta_2^3 \sinh\beta_2 b = 0. \tag{8.10}$$

Upon eliminating A_1 and A_2 from (8.9) and (8.10), we find, after some manipulation, the frequency equation

$$\tan\gamma = f^3 \tanh f\gamma, \tag{8.11}$$

where

$$\gamma = \beta_1 b, \qquad f = [(1+g^2)^{\frac{1}{2}} + 1]/g, \qquad g = \pi\Omega l/b; \tag{8.12}$$

and

$$\Omega = \omega/(\pi c_2/2b) = \gamma\pi^{-1}\{2[(1+g^2)^{\frac{1}{2}}+1]\}^{\frac{1}{2}} = 2\gamma\pi^{-1}(1+\gamma^2 l^2/b^2)^{\frac{1}{2}}. \tag{8.13}$$

In the limit as l/b goes to zero (absence of couple stress), g goes to zero and f goes to infinity, so that the frequency equation reduces to $\cot\gamma=0$, or $\gamma=n\pi/2$, n odd; and $(8.13)_3$ reduces to $\Omega=n$. Thus, Ω is the ratio of the frequency with couple-stress to the frequency of the lowest mode without couple-stress.

To compute the frequency spectrum, Ω vs. b/l, a value of g is chosen and f is computed from $(8.12)_2$. Then the roots, γ, of (8.11) are computed for that value of f. For each root γ, Ω is computed from $(8.13)_2$ and b/l is computed from $(8.12)_3$. This process is then repeated for successive values of g. The spectrum is illustrated in Fig. 1.

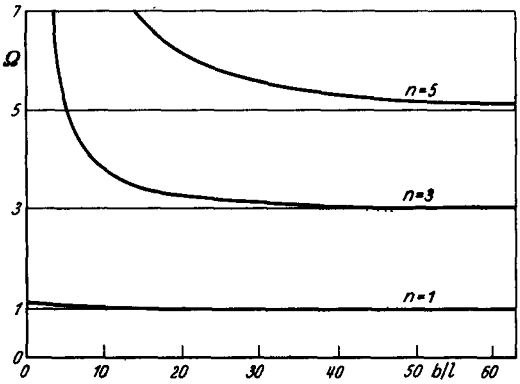

Fig. 1

As l/b or Ω becomes large, g increases and f approaches unity, in accordance with $(8.12)_3$ and $(8.12)_2$. Except for the lowest root, the roots of (8.11) are then approximately

$$\gamma=(2n-1)\pi/4, \quad n=3,5,7,\ldots. \tag{8.14}$$

Then, from $(8.13)_3$, for large Ω and large n,

$$\Omega\approx n[1+(n\pi l/2b)^2]^{\frac{1}{2}}. \tag{8.15}$$

The lowest root of (8.11) approaches zero as g increases. An expansion in power series yields

$$\gamma^2\approx 6/g \tag{8.16}$$

for large g. Then, from $(8.13)_2$, the limiting frequency of the lowest mode, as l/b approaches infinity, is

$$\Omega=(12)^{\frac{1}{2}}/\pi. \tag{8.17}$$

This is an increase of only about 10% over the frequency of the first mode without couple-stress. Thus, it would not be advisable to look for the effect of couple-stress by measuring the frequencies of the first mode of successively thinner plates.

On the other hand, the frequencies of higher and higher modes of the same plate eventually tend toward (8.15) with couple-stress, as opposed to $\Omega=n$ without. For detecting the effect of couple-stress, then, the most favorable experiment of this type would be made with as high modes and as thin a plate as possible. As is well known [*12*], end effects in a finite plate raise the frequencies of the lower modes above $\Omega=n$; but this effect diminishes with increasing n. Accordingly, as n increases, experimental measurements of Ω/n should first drop to a minimum, due to diminishing end effects, and then begin to increase in accordance with (8.15) if n can be made large enough in comparison with b/l. Following this, Ω/n would again drop: owing to the influence of rotatory inertia, which is not taken into account here.

The insensitivity of the lowest mode to varying b/l is due to an accident of mode shape. The displacement distribution contributed by the imaginary branch is a hyperbolic sinusoid, while that of the real branch is a sinusoid with one node. As b/l approaches zero, the displacement becomes linear, corresponding to infinite wave-length. The discrepancy between the solutions with and without couple-stress is then solely that due to the difference between a linear and a sinusoidal distribution of displacement. This is known [13] to produce a ratio $(12)^{\frac{1}{2}}/\pi$ in the frequencies. For bodies of other shape, insensitivity of the frequency of the lowest mode, to the ratio of the pertinent dimension to l, need not occur — as is illustrated in the following section.

9. Torsional vibrations of a circular cylinder

We consider an infinite, circular cylinder with free surface $r=a$ in a cylindrical coordinate system r, ϑ, z. From (5.22), the boundary conditions on $r=a$ are

$$p_\alpha = p_\beta = \tau_{\gamma\gamma} = \mu_{\gamma\alpha} = \mu_{\gamma\beta} = 0, \tag{9.1}$$

where α, β, γ are ϑ, z, r, respectively, and

$$\begin{gathered} p_\alpha = \tau^S_{r\vartheta} - \tfrac{1}{2} \mathbf{e}_z \cdot (\nabla \cdot \boldsymbol{\mu}^D - \nabla \mu^D_{rr}), \qquad p_\beta = \tau^S_{rz} + \tfrac{1}{2} \mathbf{e}_\vartheta \cdot (\nabla \cdot \boldsymbol{\mu}^D - \nabla \mu^D_{rr}), \\ \tau_{\gamma\gamma} = \tau_{rr}, \qquad \mu_{\gamma\alpha} = \mu_{r\vartheta}, \qquad \mu_{\gamma\beta} = \mu_{rz}. \end{gathered} \tag{9.2}$$

Hence, on $r=a$, the boundary conditions are

$$\begin{gathered} \tau^S_{r\vartheta} - \frac{1}{2}\left(\frac{\partial \mu_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \mu_{\vartheta z}}{\partial \vartheta} + \frac{\partial \mu^D_{zz}}{\partial z} + \frac{\mu_{rz}}{r} - \frac{\partial \mu^D_{rr}}{\partial z}\right) = 0, \\ \tau^S_{rz} + \frac{1}{2}\left(\frac{\partial \mu_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial \mu^D_{\vartheta\vartheta}}{\partial \vartheta} + \frac{\partial \mu_{z\vartheta}}{\partial z} + \frac{\mu_{r\vartheta} + \mu_{\vartheta r}}{r} - \frac{1}{r}\frac{\partial \mu^D_{rr}}{\partial \vartheta}\right) = 0, \\ \tau_{rr} = \mu_{r\vartheta} = \mu_{rz} = 0. \end{gathered} \tag{9.3}$$

With the boundary conditions (9.3), a solution of (7.7) that is analogous to the solution for antisymmetric thickness-shear vibrations of an infinite plate, given in the preceding section, is one independent of ϑ and z: corresponding to torsional vibrations of the cylinder in a state of plane strain. To this end, we take, as solutions of (7.7),

$$\bar{\varphi} = \bar{H}_r = \bar{H}_\vartheta = 0, \qquad \bar{H}'_z = A_1 J_0(\beta_1 r), \qquad \bar{H}''_z = A_2 I_0(\beta_2 r), \tag{9.4}$$

where J_0 and I_0 are Bessel's function and Bessel's modified function of the first kind and order zero. Then, from (7.6),

$$\bar{u}_r = \bar{u}_z = 0, \qquad \bar{u}_\vartheta = A_1 \beta_1 J_1(\beta_1 r) - A_2 \beta_2 I_1(\beta_2 r). \tag{9.5}$$

Substituting (9.5) in (5.2), we find that all the components of strain and curvature are zero except $\varepsilon_{r\vartheta}$ and $\varkappa_{rz}$. Then, from (5.3), the only non-zero components of $\boldsymbol{\tau}^S$ and $\boldsymbol{\mu}^D$, across $r=$ constant, are

$$\tau^S_{r\vartheta} = \mu \{A_1 \beta_1 [\beta_1 J_0(\beta_1 r) - 2r^{-1} J_1(\beta_1 r)] - A_2 \beta_2 [\beta_2 I_0(\beta_2 r) - 2r^{-1} I_1(\beta_2 r)]\}, \tag{9.6}$$

$$\mu_{rz} = -2\eta [A_1 \beta_1^3 J_1(\beta_1 r) + A_2 \beta_2^3 I_1(\beta_2 r)]. \tag{9.7}$$

Hence, the boundary conditions on $r=a$ reduce to

$$\tau^S_{r\vartheta} - \tfrac{1}{2} r^{-1} \partial(r \mu_{rz})/\partial r = 0, \qquad \mu_{rz} = 0. \tag{9.8}$$

When (9.6) and (9.7) are substituted in (9.8), there result

$$A_1[\beta_1^2(1+l^2\beta_1^2)J_0(\beta_1 a)-2\beta_1 a^{-1}J_1(\beta_1 a)]- \\ -A_2[\beta_2^2(1-l^2\beta_2^2)I_0(\beta_2 a)-2\beta_2 a^{-1}I_1(\beta_2 a)]=0, \tag{9.9}$$

$$A_1\beta_1^3 J_1(\beta_1 a)+A_2\beta_2^3 I_1(\beta_2 a)=0. \tag{9.10}$$

The frequency equation is obtained by equating to zero the determinant of the coefficients of A_1 and A_2 in (9.9) and (9.10). It may be written in the form

$$\tfrac{1}{2}p\,\mathscr{I}_1(\gamma)-2-f^{-2}[\tfrac{1}{2}q\,\mathscr{I}_1(f\gamma)+2]=0, \tag{9.11}$$

where

$$\mathscr{I}_1(\gamma)=\gamma J_0(\gamma)/J_1(\gamma), \qquad \mathscr{I}_1(f\gamma)=f\gamma I_0(f\gamma)/I_1(f\gamma), \tag{9.12}$$

$$\gamma=\beta_1 a, \qquad f=[(1+g^2)^{\frac{1}{2}}+1]/g, \qquad g=2\bar{\gamma}\Omega l/a, \tag{9.13}$$

$$p=(1+g^2)^{\frac{1}{2}}+1, \qquad q=(1+g^2)^{\frac{1}{2}}-1, \tag{9.14}$$

$$\Omega=\omega/(\bar{\gamma}c_2/a)=(\gamma/\bar{\gamma})(p/2)^{\frac{1}{2}}=(\gamma/\bar{\gamma})(1+\gamma^2 l^2/a^2)^{\frac{1}{2}}, \tag{9.15}$$

and $\bar{\gamma}$ is the lowest root of

$$\mathscr{I}_1(\gamma)=2. \tag{9.16}$$

In the absence of couple-stress, l/a goes to zero, g goes to zero, f goes to infinity and $p=2$. Hence, the frequency equation (9.11) reduces to (9.16) and $(9.15)_2$ reduces to $\Omega=\gamma/\bar{\gamma}$. Thus Ω is the ratio of the frequency with couple-stress to the frequency of the lowest mode without couple-stress.

The frequency spectrum, Ω vs. a/l, may be computed, with the aid of ONOE's tables [*14*], by the same procedure as that described for (8.11). The results are similar to those shown in Fig. 1 but, in this case, the frequency ratios of all the modes approach infinity as a/l approaches zero; with the exception, of course, of the mode of rigid rotation, which has zero frequency for all a/l.

10. Green's formulas

In the study of problems of equilibrium, we shall make use of GREEN'S formulas for a function, say ψ, for which

$$\nabla^2\psi \quad \text{or} \quad (1-l^2\nabla^2)\psi \quad \text{or} \quad (1-l^2\nabla^2)\nabla^2\psi$$

is known. The formulas may be obtained in the usual way [7, p. 31] from GREEN'S identity

$$\int_S \boldsymbol{n}\cdot(\varphi\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\varphi)\,dS=\int_V(\varphi\nabla^2\psi-\psi\nabla^2\varphi)\,dV. \tag{10.1}$$

When $\nabla^2\psi$ is known, we have the well known result [7, p. 32]

$$4\pi\psi=\int_S \boldsymbol{n}\cdot(r_1^{-1}\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}r_1^{-1})\,dS-\int_V r_1^{-1}\nabla^2\psi\,dV, \tag{10.2}$$

where

$$r_1=[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{\frac{1}{2}} \tag{10.3}$$

is the distance from the field point $P(x,y,z)$ to the source point $Q(\xi,\eta,\zeta)$.

To obtain the formula applicable if $(1-l^2\nabla^2)\psi$ is known, write (10.1) in the form

$$\int_S \boldsymbol{n}\cdot(\varphi_1\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\varphi_1)\,dS=l^{-2}\int_V[\psi(1-l^2\nabla^2)\varphi_1-\varphi_1(1-l^2\nabla^2)\psi]\,dV, \qquad (10.4)$$

and let

$$\varphi_1=-r_1^{-1}e^{-r_1/l}. \qquad (10.5)$$

We note that

$$(1-l^2\nabla^2)\varphi_1=0 \qquad (10.6)$$

except at P. Then, by the usual limit process, (10.4) reduces to

$$4\pi\psi=\int_S \boldsymbol{n}\cdot(\psi\boldsymbol{\nabla}\varphi_1-\varphi_1\boldsymbol{\nabla}\psi)\,dS-l^{-2}\int_V\varphi_1(1-l^2\nabla^2)\,\psi\,dV \qquad (10.7)$$

For the case of $(1-l^2\nabla^2)\nabla^2\psi$, first replace φ and ψ, in (10.1), by

$$(1-l^2\nabla^2)\varphi_2, \qquad (1-l^2\nabla^2)\psi,$$

respectively, to obtain

$$\begin{aligned}&\int_S \boldsymbol{n}\cdot[(\varphi_2-l^2\nabla^2\varphi_2)\boldsymbol{\nabla}(1-l^2\nabla^2)\psi-(\psi-l^2\nabla^2\psi)\boldsymbol{\nabla}(1-l^2\nabla^2)\varphi_2]\,dS\\&\quad=\int_V[\varphi_2(1-l^2\nabla^2)\nabla^2\psi-\psi(1-l^2\nabla^2)\nabla^2\varphi_2+l^4\nabla^2\varphi_2\nabla^4\psi-l^4\nabla^2\psi\nabla^4\varphi_2]\,dV.\end{aligned} \qquad (10.8)$$

Then replace φ by $l^2\nabla^2\varphi_2$ and ψ by $l^2\nabla^2\psi$ in (10.1):

$$l^4\int_S \boldsymbol{n}\cdot(\nabla^2\varphi_2\boldsymbol{\nabla}\nabla^2\psi-\nabla^2\psi\boldsymbol{\nabla}\nabla^2\varphi_2)\,dS=l^4\int_V(\nabla^2\varphi_2\nabla^4\psi-\nabla^2\psi\nabla^4\varphi_2)\,dV. \qquad (10.9)$$

Subtracting (10.9) from (10.8), we have

$$\begin{aligned}&\int_S \boldsymbol{n}\cdot[(\varphi_2-l^2\nabla^2\varphi_2)\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}(\varphi_2-l^2\nabla^2\varphi_2)+l^2\nabla^2\psi\boldsymbol{\nabla}\varphi_2-l^2\varphi_2\boldsymbol{\nabla}\nabla^2\psi]\,dS\\&\quad=\int_V[\varphi_2(1-l^2\nabla^2)\nabla^2\psi-\psi(1-l^2\nabla^2)\nabla^2\varphi_2]\,dV.\end{aligned} \qquad (10.10)$$

In (10.10) let

$$\varphi_2=r_1^{-1}(1-e^{-r_1/l}) \qquad (10.11)$$

and note that

$$(1-l^2\nabla^2)\nabla^2\varphi_2=0$$

except at P. Again, by the usual limit process,

$$\begin{aligned}4\pi\psi=&\int_S \boldsymbol{n}\cdot[r_1^{-1}\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}r_1^{-1}+l^2\nabla^2\psi\boldsymbol{\nabla}\varphi_2-l^2\varphi_2\boldsymbol{\nabla}\nabla^2\psi]\,dS-\\&-\int_V\varphi_2(1-l^2\nabla^2)\nabla^2\psi\,dV.\end{aligned} \qquad (10.12)$$

11. Stress functions

We may find a complete solution of the displacement-equation of equilibrium,

$$\mu\nabla^2\boldsymbol{u}+(\lambda+\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u}+\eta\nabla^2\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u}+\varrho\boldsymbol{f}+\tfrac{1}{2}\varrho\boldsymbol{\nabla}\times\boldsymbol{c}=0, \qquad (11.1)$$

in terms of stress functions which reduce to the PAPKOVITCH functions [*15*] when $l=0$. The procedure follows that in [*16*] and [*17*].

Substitute

$$\boldsymbol{u}=\boldsymbol{\nabla}\varphi+\boldsymbol{\nabla}\times\boldsymbol{H}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{H}=0, \qquad (11.2)$$

in (11.1) and obtain

$$\mu\nabla^2[\alpha\boldsymbol{\nabla}\varphi+(1-l^2\nabla^2)\boldsymbol{\nabla}\times\boldsymbol{H}]+\varrho\boldsymbol{f}+\tfrac{1}{2}\varrho\boldsymbol{\nabla}\times\boldsymbol{c}=0, \qquad (11.3)$$

where

$$\alpha = (\lambda + 2\mu)/\mu. \tag{11.4}$$

Now define a vector function $\boldsymbol{B}$ by

$$4\pi l^2 \boldsymbol{B} = \int_V r_1 e^{-r_1/l} [\alpha \boldsymbol{\nabla} \varphi + (1 - l^2 \nabla^2) \boldsymbol{\nabla} \times \boldsymbol{H}] \, dV. \tag{11.5}$$

Then, by (10.7),

$$(1 - l^2 \nabla^2) \boldsymbol{B} = \alpha \boldsymbol{\nabla} \varphi + (1 - l^2 \nabla^2) \boldsymbol{\nabla} \times \boldsymbol{H}; \tag{11.6}$$

and, from (11.6) and (11.3),

$$\mu (1 - l^2 \nabla^2) \nabla^2 \boldsymbol{B} = -\varrho \boldsymbol{f} - \tfrac{1}{2} \varrho \boldsymbol{\nabla} \times \boldsymbol{c}. \tag{11.7}$$

We must now solve (11.6) for $\boldsymbol{\nabla}\varphi$ and $\boldsymbol{\nabla} \times \boldsymbol{H}$. First take the divergence of (11.6) and obtain

$$\alpha \nabla^2 \varphi = (1 - l^2 \nabla^2) \boldsymbol{\nabla} \cdot \boldsymbol{B}, \tag{11.8}$$

the solution of which is

$$2\alpha \varphi = \boldsymbol{r} \cdot (1 - l^2 \nabla^2) \boldsymbol{B} + B_0, \tag{11.9}$$

where

$$\mu \nabla^2 B_0 = \boldsymbol{r} \cdot (\varrho \boldsymbol{f} + \tfrac{1}{2} \varrho \boldsymbol{\nabla} \times \boldsymbol{c}). \tag{11.10}$$

Then, substituting (11.9) in (11.6), we obtain

$$(1 - l^2 \nabla^2) \boldsymbol{\nabla} \times \boldsymbol{H} = (1 - l^2 \nabla^2) \boldsymbol{B} - \tfrac{1}{2} \boldsymbol{\nabla} [\boldsymbol{r} \cdot (1 - l^2 \nabla^2) \boldsymbol{B} + \boldsymbol{B}_0], \tag{11.11}$$

the solution of which is

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{B} - l^2 \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{B} - \tfrac{1}{2} \boldsymbol{\nabla} [\boldsymbol{r} \cdot (1 - l^2 \nabla^2) \boldsymbol{B} + B_0] + \boldsymbol{B}', \tag{11.12}$$

where

$$(1 - l^2 \nabla^2) \boldsymbol{B}' = 0. \tag{11.13}$$

We may now substitute (11.9) and (11.12) in (11.2) to obtain

$$\boldsymbol{u} = \boldsymbol{B} - l^2 \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{B} - \alpha' \boldsymbol{\nabla} [\boldsymbol{r} \cdot (1 - l^2 \nabla^2) \boldsymbol{B} + B_0] + \boldsymbol{B}', \tag{11.14}$$

where

$$\alpha' = (\lambda + \mu)/2(\lambda + 2\mu) = 1/4(1 - \sigma). \tag{11.15}$$

Next, we insert (11.14) in (11.1) and find, with the aid of (11.7), (11.10) and (11.13),

$$\boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{B}' = 0. \tag{11.16}$$

In view of (11.13), (11.16), (11.7) and the form of (11.14), $\boldsymbol{B}'$ may be absorbed in $\boldsymbol{B}$. Thus we have, finally,

$$\boldsymbol{u} = \boldsymbol{B} - l^2 \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{B} - \alpha' \boldsymbol{\nabla} [\boldsymbol{r} \cdot (1 - l^2 \nabla^2) \boldsymbol{B} + B_0], \tag{11.17}$$

$$\mu (1 - l^2 \nabla^2) \nabla^2 \boldsymbol{B} = -\varrho \boldsymbol{f} - \tfrac{1}{2} \varrho \boldsymbol{\nabla} \times \boldsymbol{c}, \tag{11.18}$$

$$\mu \nabla^2 B_0 = \boldsymbol{r} \cdot (\varrho \boldsymbol{f} + \tfrac{1}{2} \varrho \boldsymbol{\nabla} \times \boldsymbol{c}). \tag{11.19}$$

The functions $\boldsymbol{B}$ and B_0 reduce to PAPKOVITCH's functions when $l = 0$. It may be observed that the property $\boldsymbol{\nabla} \cdot \boldsymbol{H} = 0$ was not used in the derivation.

For later use, we note that

$$\alpha \boldsymbol{\nabla} \cdot \boldsymbol{u} = (1 - l^2 \nabla^2) \boldsymbol{\nabla} \cdot \boldsymbol{B}, \qquad \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{\nabla} \times \boldsymbol{B}, \qquad \boldsymbol{\varkappa} = \tfrac{1}{2} \boldsymbol{\nabla} \boldsymbol{\nabla} \times \boldsymbol{B}. \tag{11.20}$$

Further, in spherical coordinates r, ϑ, φ, if ψ is a function of r only, the complete solution of

$$(1-l^2\nabla^2)\nabla^2\psi=0 \tag{11.21}$$

is

$$\psi=a_0+b_0 r^{-1}+c_0 r^{-1}e^{-r/l}+d_0 r^{-1}e^{r/l}. \tag{11.22}$$

Also, in cylindrical coordinates r, ϑ, z, if ψ is a function or r only, the complete solution of (11.21) is

$$\psi=a_0+b_0\log r+c_0 K_0(r/l)+d_0 I_0(r/l), \tag{11.23}$$

where K_0 and I_0 are the modified Bessel functions of order zero.

The displacement may also be expressed in terms of a vector function, $\boldsymbol{G}$, which reduces to the Galerkin stress function [18] when $l=0$:

$$(\lambda+2\mu)\boldsymbol{u}=(\lambda+2\mu)\nabla^2\boldsymbol{G}-(\lambda+\mu)\boldsymbol{\nabla\nabla}\cdot\boldsymbol{G}-\eta\nabla^2\boldsymbol{\nabla\nabla}\cdot\boldsymbol{G}, \tag{11.24}$$

where $\boldsymbol{G}$ satisfies

$$\mu(1-l^2\nabla^2)\nabla^4\boldsymbol{G}=-\varrho\boldsymbol{f}-\tfrac{1}{2}\varrho\boldsymbol{\nabla}\times\boldsymbol{c}. \tag{11.25}$$

The relations between $\boldsymbol{G}$ and $\boldsymbol{B}$ and B_0 are

$$\boldsymbol{B}=\nabla^2\boldsymbol{G},\qquad B_0=2(1-l^2\nabla^2)\boldsymbol{\nabla}\cdot\boldsymbol{G}-\boldsymbol{r}\cdot(1-l^2\nabla^2)\nabla^2\boldsymbol{G}. \tag{11.26}$$

12. Spherical cavity in a field of simple tension

In spherical coordinates r, ϑ, φ, a stress-field of simple tension, τ_0, is given by

$$\tau_{rr}=\tfrac{1}{2}\tau_0(1+\cos 2\vartheta),\quad \tau_{\vartheta\vartheta}=\tfrac{1}{2}\tau_0(1-\cos 2\vartheta),\quad \tau_{r\vartheta}=\tau_{\vartheta r}=-\tfrac{1}{2}\tau_0\sin 2\vartheta, \tag{12.1}$$

$$\tau_{\varphi r}=\tau_{r\varphi}=\tau_{\varphi\vartheta}=\tau_{\vartheta\varphi}=\tau_{\varphi\varphi}=0,\qquad \boldsymbol{\mu}=0. \tag{12.2}$$

We wish to add a stress field $(\boldsymbol{\tau},\boldsymbol{\mu})$ which will produce a free surface at $r=a$ and vanish at infinity. From (5.22), (5.23), (12.1) and (12.2), the conditions which the additional field must satisfy on $r=a$ are

$$\begin{aligned}
p_\alpha&=\tau^S_{r\vartheta}-\tfrac{1}{2}\boldsymbol{e}_\varphi\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\mu}^D-\boldsymbol{\nabla}\mu^D_{rr})=\tfrac{1}{2}\tau_0\sin 2\vartheta,\\
p_\beta&=\tau^S_{r\varphi}+\tfrac{1}{2}\boldsymbol{e}_\vartheta\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\mu}^D-\boldsymbol{\nabla}\mu^D_{rr})=0,\\
\tau_{\gamma\gamma}&=\tau_{rr}=-\tfrac{1}{2}\tau_0(1+\cos 2\vartheta),\\
\mu_{\gamma\alpha}&=\mu_{r\vartheta}=0,\qquad \mu_{\gamma\beta}=\mu_{r\varphi}=0.
\end{aligned} \tag{12.3}$$

We take the stress functions $\boldsymbol{B}$ and B_0, for the additional field, to be of the form

$$B_x=B_y=0,\qquad B_z=B(r,\vartheta),\qquad B_0=B_0(r,\vartheta), \tag{12.4}$$

$$(1-l^2\nabla^2)\nabla^2 B=0,\qquad \nabla^2 B_0=0. \tag{12.5}$$

Then, from (11.17) and (3.25), the terms that appear in (12.3) are

$$\begin{aligned}
\tau^S_{r\vartheta}=\mu\Big\{&\cos\vartheta\frac{1}{r}\frac{\partial B}{\partial\vartheta}-2l^2\frac{\partial}{\partial r}\Big[\frac{1}{r}\frac{\partial}{\partial\vartheta}\Big(\cos\vartheta\frac{\partial B}{\partial r}-\sin\vartheta\frac{1}{r}\frac{\partial B}{\partial\vartheta}\Big)\Big]-\\
&-\sin\vartheta\frac{\partial B}{\partial r}-\frac{\lambda+\mu}{\lambda+2\mu}\frac{\partial^2}{\partial r\,\partial\vartheta}\Big[\cos\vartheta(1-l^2\nabla^2)B+\frac{B_0}{r}\Big]\Big\},
\end{aligned} \tag{12.6}$$

$$\tau_{rr} = \frac{\mu\lambda}{\lambda+2\mu}\left(\cos\vartheta\frac{\partial}{\partial r} - \sin\vartheta\frac{1}{r}\frac{\partial}{\partial\vartheta}\right)(1-l^2\nabla^2)\,B +$$
$$+ 2\mu\left[\cos\vartheta\frac{\partial B}{\partial r} - l^2\frac{\partial^2}{\partial r^2}\left(\cos\vartheta\frac{\partial B}{\partial r} - \sin\vartheta\frac{1}{r}\frac{\partial B}{\partial\vartheta}\right)\right] -$$
$$- \frac{\mu(\lambda+\mu)}{\lambda+2\mu}\frac{\partial^2}{\partial r^2}\left[r\cos\vartheta(1-l^2\nabla^2)\,B+B_0\right], \tag{12.7}$$

$$\mu_{r\varphi} = -2\left(\eta\frac{\partial}{\partial r} - \eta'\frac{1}{r}\right)\left(\sin\vartheta\frac{\partial B}{\partial r} + \cos\vartheta\frac{1}{r}\frac{\partial B}{\partial\vartheta}\right), \tag{12.8}$$

$$\nabla\cdot\boldsymbol{\mu}^D = 2\eta\nabla\times\boldsymbol{e}_z\nabla^2 B = -2\eta\boldsymbol{e}_\varphi\left(\sin\vartheta\frac{\partial}{\partial r} + \cos\vartheta\frac{1}{r}\frac{\partial}{\partial\vartheta}\right)\nabla^2 B, \tag{12.9}$$

$$\tau^S_{r\varphi} = \mu^D_{rr} = \mu_{r\vartheta} = 0. \tag{12.10}$$

For B and B_0 we take

$$B = a_1 r^{-2}\cos\vartheta + a_2 e^{-r/l}(r^{-1}l^{-1} + r^{-2})\cos\vartheta, \tag{12.11}$$

$$B_0 = a_3 r^{-1} + a_4 r^{-3}(1+3\cos 2\vartheta). \tag{12.12}$$

It may be verified that these functions satisfy (12.5) and give displacements and stresses that vanish at infinity.

Substituting (12.11) and (12.12) in (12.6)–(12.10), the result in (12.3) and equating coefficients of like functions of ϑ, we obtain a set of four equations in the constants a_1, a_2, a_3, a_4, the solution of which is

$$\begin{aligned} &a_1 = 5T, \qquad a_2 = -15e^k k' T,\\ &a_3 = -[6-5\sigma-6k'(1-\sigma)(1+k)]\,T,\\ &a_4 = -\tfrac{1}{2}a^2\{1-4(1-\sigma)[5k^{-2} - k'(15k^{-2}+15k^{-1}+6+k)]\}\,T,\end{aligned} \tag{12.13}$$

where

$$k = a/l, \qquad k' = (3+\sigma')/[9+9k+4k^2+k^3+\sigma'(3+3k+k^2)], \tag{12.14}$$

$$\sigma' = \eta'/\eta, \qquad T = (1-\sigma)\,a^3\tau_0/\mu\,[7-5\sigma+18k'(1-\sigma)(1+k)]. \tag{12.15}$$

This completes the solution of the problem.

The total force-stress component $\tau_{\vartheta\vartheta}$ is

$$\tau_{\vartheta\vartheta} = \tfrac{1}{2}\tau_0(1-\cos 2\vartheta) + \frac{\mu\lambda}{\lambda+2\mu}\left(\cos\vartheta\frac{\partial}{\partial r} - \sin\vartheta\frac{1}{r}\frac{\partial}{\partial\vartheta}\right)(1-l^2\nabla^2)\,B -$$
$$- 2\mu\left[\sin\vartheta\frac{1}{r}\frac{\partial B}{\partial\vartheta} + \left(\frac{l^2}{r}\frac{\partial}{\partial r} + \frac{l^2}{r^2}\frac{\partial^2}{\partial\vartheta^2}\right)\left(\cos\vartheta\frac{\partial B}{\partial r} - \sin\vartheta\frac{1}{r}\frac{\partial B}{\partial\vartheta}\right)\right] -$$
$$- \frac{\mu(\lambda+\mu)}{\lambda+2\mu}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2}\right)\left[r\cos\vartheta(1-l^2\nabla^2)\,B + B_0\right], \tag{12.16}$$

which yields a maximum tension at the surface:

$$[\tau_{\vartheta\vartheta}]_{r=a,\,\vartheta=\pi/2} = \frac{3\,[9-5\sigma+6k'(1-\sigma)(1+k)]\,\tau_0}{2\,[7-5\sigma+18k'(1-\sigma)(1+k)]}. \tag{12.17}$$

Thus, the maximum tension at the spherical surface depends on the ratio, k, of the radius of the cavity to the material constant l. In the absence of couple-stress, (12.17) becomes

$$\lim_{k\to\infty}[\tau_{\vartheta\vartheta}]_{r=a,\,\vartheta=\pi/2} = \frac{3(9-5\sigma)\,\tau_0}{2(7-5\sigma)}, \tag{12.18}$$

which coincides with SOUTHWELL & GOUGH's result [19, p. 361] for the case $l=0$. For a cavity with a radius vanishingly small in comparison with l, we find

$$\lim_{k\to 0}\,[\tau_{\vartheta\vartheta}]_{r=a,\,\vartheta=\pi/2}=\frac{3(11-7\sigma)\,\tau_0}{2(13-11\sigma)}. \tag{12.19}$$

In the range $0\leqq\sigma\leqq\frac{1}{2}$, (12.19) is about 30% less than (12.18).

13. Example of failure of uniqueness

In this section we take $\eta<0$ and find two solutions of the boundary value problem of the preceding section.

Let

$$\bar{\eta}=-\eta,\qquad \bar{\eta}'=-\eta',\qquad \bar{l}=il=(\bar{\eta}/\mu)^{\frac{1}{2}}, \tag{13.1}$$

where $\bar{\eta}>0$ and $\bar{l}>0$. Then the stress functions satisfy

$$(1+\bar{l}^2\nabla^2)\,\nabla^2 B=0,\qquad \nabla^2 B_0=0, \tag{13.2}$$

instead of (12.5).

For B and B_0 we consider the two sets of functions

$$\begin{aligned} B &= \frac{a_1'}{r^2}\cos\vartheta + a_2'\frac{\partial}{\partial z}\left(\frac{1}{r}\cos\frac{r}{\bar{l}}\right),\\ B_0 &= a_3' r^{-1} + a_4' r^{-3}(1+3\cos 2\vartheta), \end{aligned} \tag{13.3}$$

and

$$\begin{aligned} B &= \frac{a_1''}{r^2}\cos\vartheta + a_2''\frac{\partial}{\partial z}\left(\frac{1}{r}\sin\frac{r}{\bar{l}}\right),\\ B_0 &= a_3'' r^{-1} + a_4'' r^{-3}(1+3\cos 2\vartheta). \end{aligned} \tag{13.4}$$

It may be verified that (13.3) and (13.4) satisfy (13.2) and yield displacements and stresses that vanish at infinity.

Upon substituting the functions (13.3) in (12.6)–(12.10), the results in the boundary conditions (12.3) and solving for the a_i', we find:

$$a_1'=\bar{T}',\qquad a_2'=-3\bar{k}'\bar{T}',$$

$$a_3'=-\frac{\bar{T}'[7\lambda+12\mu+6(\lambda+2\mu)\,\bar{k}'(\cos\bar{k}+\bar{k}\sin\bar{k})]}{10(\lambda+\mu)},$$

$$a_4'=-\frac{a^2\bar{T}'\{\lambda+\mu+2(\lambda+2\mu)\,[5\bar{k}^{-2}-3\bar{k}'(2-5\bar{k}^{-2})\cos\bar{k}-\bar{k}'(\bar{k}-15\bar{k}^{-1})\sin\bar{k}]\}}{10(\lambda+\mu)},$$

where $\bar{k}=a/\bar{l}$ and

$$\bar{T}'=\frac{5\tau_0 a^3(\lambda+2\mu)}{\mu[9\lambda+14\mu-18(\lambda+2\mu)\,\bar{k}'(\cos\bar{k}+\bar{k}\sin\bar{k})]},$$

$$\bar{k}'=\frac{3\bar{\eta}+\bar{\eta}'}{\bar{\eta}[(4\bar{k}^2-9)\cos\bar{k}+(\bar{k}^3-9\bar{k})\sin\bar{k}]-\bar{\eta}'[3\bar{k}\sin\bar{k}+(3-\bar{k}^2)\cos\bar{k}]}.$$

Similarly, for the stress functions (13.4), we are also able to satisfy the same boundary conditions with constants a_i'' which differ from the a_i' in that $\cos\bar{k}$ is replaced by $\sin\bar{k}$ and $\sin\bar{k}$ is replaced by $-\cos\bar{k}$.

Thus, with $\eta<0$, we have two solutions of the same boundary value problem — illustrating failure of uniqueness when the internal energy density is not positive definite.

14. Cylindrical cavity in a field of simple tension

The statement of the problem of the cylindrical cavity in a field of simple tension is exactly the same as in (12.1)–(12.10) except with η' set equal to zero, with φ replaced by z and with r and ϑ interpreted as the r and ϑ of cylindrical coordinates.

In place of (12.11) and (12.12), we take, as stress functions for the present problem,

$$B = a_1 r^{-1} \cos\vartheta + a_2 K_1(r/l)\cos\vartheta, \tag{14.1}$$

$$B_0 = a_3 \log r + a_4 r^{-2}\cos 2\vartheta. \tag{14.2}$$

Then, substituting in the boundary conditions and solving for the constants, we find

$$\begin{aligned} a_1 &= T', \\ a_2 &= -k'' T'/a K_1(a/l), \\ a_3 &= \tfrac{1}{2} T'[3 - 4\sigma - 2k''(1-\sigma)], \\ a_4 &= -\tfrac{1}{4} T' a^2 [1 - 2k''(1-\sigma)], \end{aligned} \tag{14.3}$$

where

$$T' = \frac{2(1-\sigma)\tau_0 a^2}{\mu[1+2k''(1-\sigma)]}, \qquad k'' = \frac{4}{4+k^2+2kK_0/K_1}, \tag{14.4}$$

and $k = a/l$.

The stress $\tau_{\vartheta\vartheta}$ is again given by (12.16) and we find, for the maximum tension at the surface of the cavity,

$$[\tau_{\vartheta\vartheta}]_{r=a,\,\vartheta=\pi/2} = \frac{[3+2(1-\sigma)k'']\tau_0}{1+2(1-\sigma)k''}. \tag{14.5}$$

Thus, the maximum tension at the surface depends on both Poisson's ratio and the ratio of the radius of the cavity to the material constant l.

We note that

$$\lim_{k\to 0} kK_0(k)/K_1(k) = 0, \qquad kK_0(k)/K_1(k) \approx k \quad \text{for} \quad k \gg 1; \tag{14.6}$$

and hence

$$\lim_{k\to 0} k'' = 1, \qquad \lim_{k\to\infty} k'' = 0. \tag{14.7}$$

Thus, in the absence of couple-stress ($k\to\infty$), the maximum tension at the surface reduces to $3\tau_0$, which coincides with the result from the Kirsch solution [*19*, p. 80]. For the vanishingly small cavity, we have

$$\lim_{k\to 0} [\tau_{\vartheta\vartheta}]_{r=a,\,\vartheta=\pi/2} = \frac{(5-2\sigma)\tau_0}{3-2\sigma}. \tag{14.8}$$

This is less than $3\tau_0$ by about thirty to forty percent over the range $0 \leqq \sigma \leqq \frac{1}{2}$.

15. Concentrated force and couple

In an infinite medium acted upon statically by body forces and body couples, we have, from (10.11), (10.12) and (11.18),

$$\boldsymbol{B} = \frac{\varrho}{4\pi\mu} \int\limits_V \left(\frac{1}{r_1} - \frac{e^{-r_1/l}}{r_1}\right)\left(\boldsymbol{f} + \frac{1}{2}\nabla\times\boldsymbol{c}\right) dV. \tag{15.1}$$

Also, from (10.2), (10.3) and (11.19),

$$B_0 = -\frac{\varrho}{4\pi\mu} \int_V \frac{\boldsymbol{r}' \cdot (\boldsymbol{f} + \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{c})\, dV}{r_1}, \tag{15.2}$$

where $\boldsymbol{r}' \cdot \boldsymbol{r}' = \xi^2 + \eta^2 + \zeta^2$.

For the concentrated force, we let $\boldsymbol{c} = 0$ everywhere and $\boldsymbol{f} = 0$ outside a region V' which contains the origin and a non-vanishing field of parallel forces $\boldsymbol{f}$. We define a concentrated force by

$$\boldsymbol{P} = \lim_{V' \to 0} \int_V \varrho \boldsymbol{f}\, dV. \tag{15.3}$$

Then, since

$$\lim_{V' \to 0} r_1 = r, \qquad \lim_{V' \to 0} \boldsymbol{r}' = 0, \tag{15.4}$$

(15.1) and (15.2) reduce to

$$\boldsymbol{B} = \frac{\boldsymbol{P}}{4\pi\mu r}(1 - e^{-r/l}), \qquad B_0 = 0, \tag{15.5}$$

which are the stress functions for the concentrated force.

For the concentrated couple, we let $\boldsymbol{f} = 0$ everywhere and $\boldsymbol{c} = 0$ except in V', where $\boldsymbol{c}$ is a field of parallel couples. The concentrated couple is defined by

$$\boldsymbol{C} = \lim_{V' \to 0} \int_V \varrho \boldsymbol{c}\, dV. \tag{15.6}$$

Now, we note that

$$r_1^{-1}(1 - e^{-r_1/l})\boldsymbol{\nabla}\times\boldsymbol{c} = \boldsymbol{\nabla}\times[r_1^{-1}(1 - e^{-r_1/l})\boldsymbol{c}] + \boldsymbol{c}\times\boldsymbol{\nabla}[r_1^{-1}(1 - e^{-r_1/l})], \tag{15.7}$$

and

$$\int_V \boldsymbol{\nabla}\times[r_1^{-1}(1 - e^{-r_1/l})\boldsymbol{c}]\, dV = \int_S \boldsymbol{n}\times[r_1^{-1}(1 - e^{-r_1/l})\boldsymbol{c}]\, dS = 0. \tag{15.8}$$

Hence, (15.1) becomes

$$\boldsymbol{B} = \frac{1}{8\pi\mu} \int_V \varrho \boldsymbol{c}\times\boldsymbol{\nabla}[r_1^{-1}(1 - e^{-r_1/l})]\, dV. \tag{15.9}$$

Then, with (15.4) and (15.6), and noting that the gradients with respect to the coordinates of the source point and field point have opposite signs, (15.9) and (15.2) become

$$\boldsymbol{B} = -\frac{1}{8\pi\mu} \boldsymbol{C}\times\boldsymbol{\nabla}\left(\frac{1}{r} - \frac{e^{-r/l}}{r}\right), \qquad B_0 = 0, \tag{15.10}$$

which are the stress functions for the concentrated couple.

16. Nuclei of strain

Because of the presence of the factor $\boldsymbol{r}$ in (11.17), it is more convenient to derive formulas for nuclei of strain in terms of the stress function $\boldsymbol{G}$ than in terms of $\boldsymbol{B}$.

For the concentrated force and couple we have, respectively,

$$\boldsymbol{G} = \boldsymbol{P}\chi/4\pi\mu, \qquad \boldsymbol{G} = -\boldsymbol{C}\times\boldsymbol{\nabla}\chi/8\pi\mu, \tag{16.1}$$

where

$$\chi = \tfrac{1}{2}r + l^2 r^{-1}(1 - e^{-r/l}). \tag{16.2}$$

It may be verified that (16.1) yield the stress functions $\boldsymbol{B}$ and B_0 in (15.5) and (15.10) through the relations (11.26).

The stress functions, $\boldsymbol{G}$, for various nuclei of strain may now be obtained in the usual way [9, p. 186]. Thus, for double forces in the directions of x, y, z, of strength p, we have, respectively,

$$\boldsymbol{G}=\boldsymbol{e}_x p\,\partial\chi/\partial x, \qquad \boldsymbol{G}=\boldsymbol{e}_y p\,\partial\chi/\partial y, \qquad \boldsymbol{G}=\boldsymbol{e}_z p\,\partial\chi/\partial z. \tag{16.3}$$

The sum of these is the stress function for a center of dilatation:

$$\boldsymbol{G}=p\,\boldsymbol{\nabla}\chi, \tag{16.4}$$

or, from (11.26),

$$\boldsymbol{B}=p\,\nabla^2\boldsymbol{\nabla}\chi=p\,\boldsymbol{\nabla}(r^{-1}-r^{-1}e^{-r/l}), \tag{16.5}$$

$$B_0=p\,[2(1-l^2\nabla^2)\nabla^2\chi-\boldsymbol{r}\cdot(1-l^2\nabla^2)\nabla^2\boldsymbol{\nabla}\chi]=3p\,r^{-1}. \tag{16.6}$$

By (11.17), both (16.5) and (16.6) give displacements proportional to $\boldsymbol{\nabla}r^{-1}$. Thus, the center of dilatation is the same with or without couple-stress.

For a double force in the x-direction with positive moment about the z-axis,

$$\boldsymbol{G}=q\,\boldsymbol{e}_x\,\partial\chi/\partial y, \tag{16.7}$$

where q is the strength of the double force with moment. For a double force in the y-direction with positive moment about the z-axis,

$$\boldsymbol{G}=-q\,\boldsymbol{e}_y\,\partial\chi/\partial x. \tag{16.8}$$

The sum of (16.7) and (16.8) is the stress function for a center of rotation about the z-axis:

$$\boldsymbol{G}=q\,(\boldsymbol{e}_x\,\partial\chi/\partial y-\boldsymbol{e}_y\,\partial\chi/\partial x). \tag{16.9}$$

This may be compared with the stress function $(16.1)_2$, specialized to a couple, of magnitude C, about the z-axis:

$$\boldsymbol{G}=-C\,\boldsymbol{e}_z\times\boldsymbol{\nabla}\chi/8\pi\mu=(C/8\pi\mu)\,(\boldsymbol{e}_x\,\partial\chi/\partial y-\boldsymbol{e}_y\,\partial\chi/\partial x). \tag{16.10}$$

Thus, with $q=C/8\pi\mu$, the center of rotation is identical with the concentrated body-couple; a conclusion which holds with or without couple-stress [9, p. 187].

Acknowledgements. We wish to thank Dr. R. A. Toupin for valuable discussions of the constitutive equations. The senior author also wishes to thank the Office of Naval Research for support through a contract with Columbia University.

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(Received July 27, 1962)