Assignment 2 - AST4320

Markus Bjørklund

October 2018

Exercise 1

We consider the top hat smoothing function $W(x)$, which is defined as follows

$$W(x) = \begin{cases} 1, & \text{if } |x| < R \\ 0, & \text{if } |x| \geq R \end{cases} \tag{1}$$

Finding the Fourier transform of this is done by computing

$$\widetilde{W}(k) = \int_{-\infty}^{\infty} W(x) e^{-ikx} dx \tag{2}$$

Since our smoothing function is zero everywhere but inside the range of R where it is one, we get

$$\begin{aligned}
\widetilde{W}(k) &= \int_{-R}^{R} e^{-ikx} dx \\
&= -\frac{1}{ik} \left[e^{-ikx} \right]_{-R}^{R} \\
&= \frac{1}{k} \frac{e^{ikR} - e^{-ikR}}{i} \\
&\boxed{sinz = \frac{1}{2i} \left(e^{iz} - e^{-iz} \right)} \\
&= \frac{2}{k} sin\left(kR \right)
\end{aligned}$$

To avoid numerical problems, we note that

$$
\begin{aligned}
&\lim_{k\to 0}\frac{2}{k}sin\left(kR\right)\\
\boxed{L'Hopital} &= \lim_{k\to 0} 2\frac{Rcos\left(kR\right)}{1}\\
&= 2R.
\end{aligned}
$$

We also want to compute the Full Width Half Maximum (FWHM) of $\widetilde{W}\left(k\right)$. We know that the function has its maximum at $k=0$, where $\widetilde{W}\left(0\right) = 2R$. We must therefore have that

$$\frac{2}{k}sin\left(kR\right) = R. \tag{3}$$

We solve this numerically, to find that $FWHM = 3.80$. Both results are plotted in figure 1

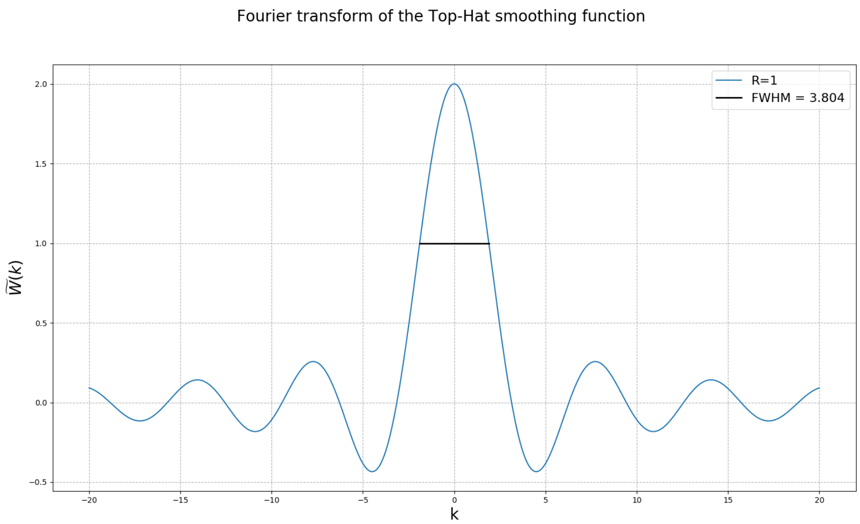

Figure 1: The Fourier transform of the Top-Hat smoothing function, plotted against k.

Exercise 2.

We have the scale defined as

$$S_c = \frac{2\pi}{k}, \tag{4}$$

and the variance defined as

$$\sigma^2(S_c) = \frac{\pi}{S_c^4}. \tag{5}$$

Equations 4 and 5 together, gives us that

$$\sigma^2 = \frac{\pi}{\left(\frac{2\pi}{k}\right)^4}$$
$$\frac{\sigma^2}{\pi} = \frac{k^4}{(2\pi)^4}$$

$$k = 2\pi \left(\frac{\sigma^2}{\pi}\right)^{\frac{1}{4}} \tag{6}$$

We insert this result into equation 4, to get

$$S(\sigma) = \frac{2\pi}{2\pi \left(\frac{\sigma^2}{\pi}\right)^{\frac{1}{4}}} = \left(\frac{\sigma^2}{\pi}\right)^{-\frac{1}{4}} \tag{7}$$

We require that we have a radius such that $\sigma^2(S_c) = \sigma^2(S_1) < 10^{-4}$. So simply insert some $\sigma^2 < 10^{-4}$ to aquire S_1.

The analytic PDF for the overdensity is given by

$$P(\delta|M) = \frac{1}{\sqrt{2\pi}\sigma(M)\, exp\left[-\frac{\delta^2}{2\sigma^2(M)}\right]}. \tag{8}$$

For the case where δ never is larger than δ_{crit}, we have the probability given by

$$P_{\text{nc}}(\delta \mid M) = P(\delta|M) - P([2\delta_{\text{crit}} - \delta|M])$$

$$P_{\text{nc}}(\delta \mid M) = \frac{1}{\sqrt{2\pi}\sigma(M)} \left(\exp\left[-\frac{\delta^2}{2\sigma^2(M)}\right] - \exp\left[-\frac{[2\delta_{\text{crit}} - \delta]^2}{2\sigma^2(M)}\right] \right). \tag{9}$$

These are plotted with their respective distributions in figures 2 and 3.

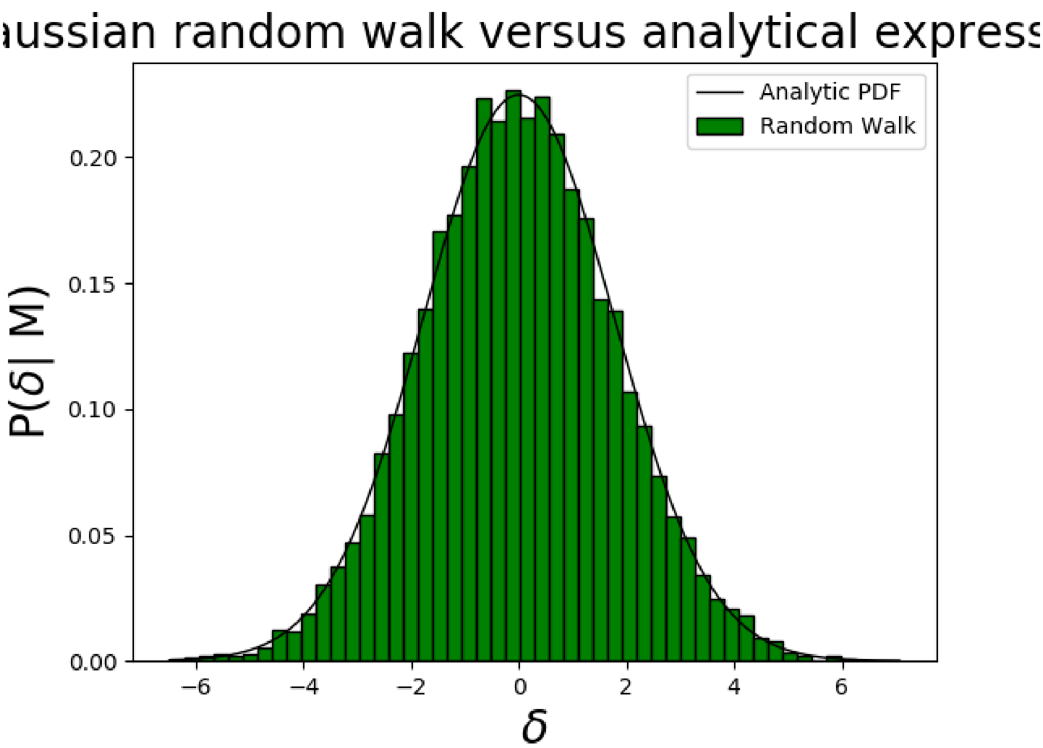

Figure 2: Random walk simulation of the PDF, as a function of δ.

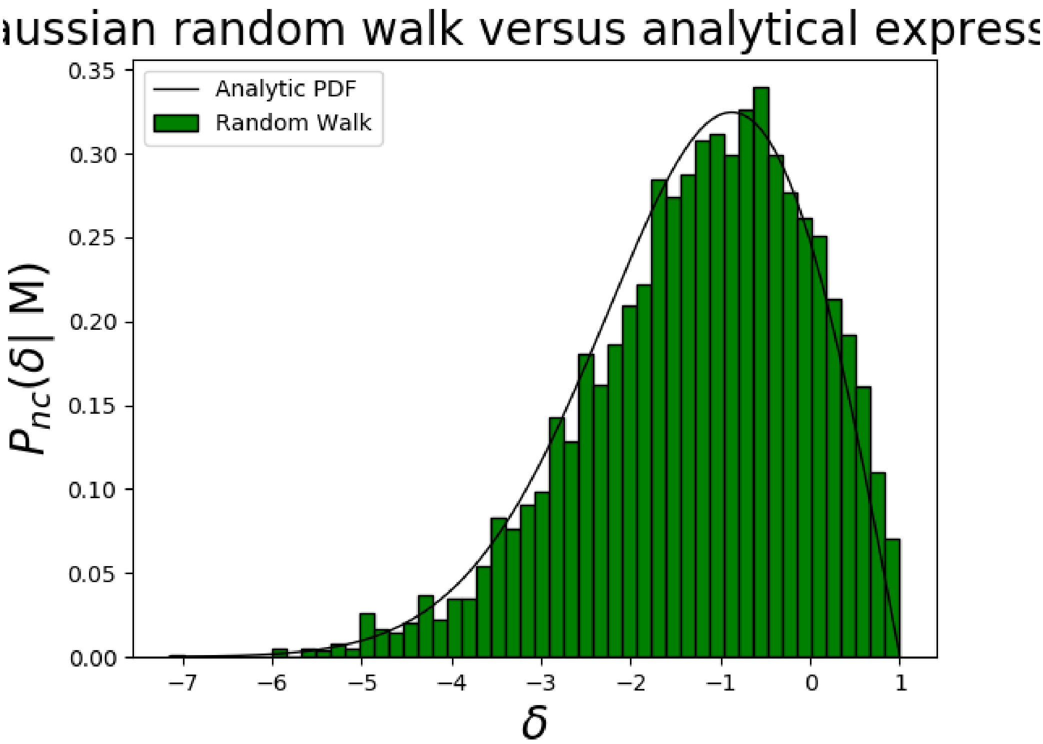

Figure 3: Random walk simulation of the PDF, as a function of δ. Only chains which do not cross the critical density.

Exercise 3.

(1)

We know the total probability must equal to one, I.E.

$$
\begin{aligned}
1 &= P\,(< M) + P\,(> M) \\
P\,(> M) &= 1 - P\,(< M)
\end{aligned}
$$

Now for a mass smaller than M, we have $\delta < \delta_{crit}$, and therefore no collapse, and the probability is given by equation 9. We then get

$$
P\,(> M) = 1 - \int_{-\infty}^{\delta_{crit}} P_{\mathrm{nc}}(\delta \mid M) d\delta \tag{10}
$$

(2)

We want to solve the integral in equation 10. We insert equation 9 into equation 10, and obtain

$$P\left(>M\right)=1-\int_{-\infty}^{\delta_{crit}}\frac{1}{\sqrt{2\pi}\sigma\left(M\right)}\left(\exp\left[-\frac{\delta^2}{2\sigma^2\left(M\right)}\right]-\exp\left[-\frac{\left[2\delta_{\mathrm{crit}}-\delta\right]^2}{2\sigma^2\left(M\right)}\right]\right)d\delta$$

$$=1-\left(\underbrace{\frac{1}{\sqrt{2\pi}\sigma\left(M\right)}\int_{-\infty}^{\delta_{crit}}\exp\left[-\frac{\delta^2}{2\sigma^2\left(M\right)}\right]d\delta}_{I_1}-\underbrace{\frac{1}{\sqrt{2\pi}\sigma\left(M\right)}\int_{-\infty}^{\delta_{crit}}\exp\left[-\frac{\left[2\delta_{\mathrm{crit}}-\delta\right]^2}{2\sigma^2\left(M\right)}\right]d\delta}_{I_2}\right)$$

We solve these separately.

I_1:

We introduce

$$u=\frac{\delta}{\sqrt{2}\sigma}. \tag{11}$$

We get the differential

$$\frac{du}{d\delta}=\frac{1}{\sqrt{2}\sigma}\Rightarrow d\delta=\sqrt{2}\sigma du. \tag{12}$$

u also goes to minus infinity as δ goes to minus infinity, but for the upper limit we get

$$\frac{\delta_{crit}}{\sqrt{2}\sigma}\equiv\frac{\nu}{\sqrt{2}}. \tag{13}$$

We get the integral

$$\begin{aligned}I_1&=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{\nu}{\sqrt{2}}}e^{-u^2}du\\&=\frac{1}{\sqrt{\pi}}\left[\int_{-\infty}^{0}e^{-u^2}du\int_{0}^{\frac{\nu}{\sqrt{2}}}e^{-u^2}du\right]\\&=\frac{1}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}+\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right]\end{aligned}$$

which gives us

$$I_1=\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right] \tag{14}$$

I_2**:**

Now for the second integral, we again do a substitution

$$u = \frac{[2\delta_{crit} - \delta]}{\sqrt{2}\sigma}. \tag{15}$$

$$\frac{du}{d\delta} = -\frac{1}{\sqrt{2}\sigma} \Rightarrow d\delta = -\sqrt{2}\sigma du \tag{16}$$

We now see that for the lower limit, $u \to \infty$ as $\delta \to -\infty$. For the upper limit we get $\frac{\delta_{crit}}{\sqrt{2}\sigma} = \frac{\nu}{\sqrt{2}}$. We get the integral

$$\begin{aligned} I_2 &= -\frac{1}{\sqrt{\pi}} \int_{\infty}^{\frac{\nu}{\sqrt{2}}} e^{-u^2} du \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{\nu}{\sqrt{2}}}^{\infty} e^{-u^2} du \\ &= \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \text{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \end{aligned}$$

which finally gives us

$$I_2 = \frac{1}{2}\left[1 - \text{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right] \tag{17}$$

So, everything put together, we get

$$\begin{aligned} P(> M) &= 1 - (I_1 - I_2) \\ &= 1 - \left(\frac{1}{2}\left[1 + \text{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right] - \frac{1}{2}\left[1 - \text{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right]\right) \\ &= 1 - \text{erf}\left(\frac{\nu}{\sqrt{2}}\right) \\ &= 2P\left(\delta > \delta_{\text{crit}} | M\right) \end{aligned}$$