

Assignment 1 - AST4320

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Exercise 1

(1.1)

We start from the continuity equation:

$$\frac{d\bar{\rho}}{dt} + \bar{\rho} \nabla \cdot \mathbf{v} = \frac{\partial \bar{\rho}}{\partial t} + \mathbf{v} \cdot \nabla \bar{\rho} + \bar{\rho} \nabla \cdot \mathbf{v} = 0 \quad (1)$$

Now from the cosmological principle, we can assume there is no inherent density differences, and we get that $\nabla \rho_0 = 0$. Also, inserting for our assumption $\mathbf{v} = H\mathbf{r}$, the continuity equation reduces to

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\rho} \nabla \cdot H\mathbf{r} = 0 \quad (2)$$

Now $\nabla \cdot \mathbf{r} = 3$. We also insert $H = \frac{\dot{a}}{a}$ so we get

$$\frac{\partial \bar{\rho}}{\partial t} + 3\bar{\rho} \frac{\dot{a}}{a} = 0 \quad (3)$$

Now this is a separable differential equation (DE), solved simply by

$$\frac{1}{\bar{\rho}} d\bar{\rho} = -3 \frac{1}{a} da \quad (4)$$

Integrating from $t_0 \rightarrow t$, and naming $f(t) = f, f(t_0) = f_0$, gives us

$$\begin{aligned}
\ln(\bar{\rho}) - \ln(\bar{\rho}_0) &= -3[\ln(a) - \ln(a_0)] \\
\ln(\bar{\rho}) &= \ln(\bar{\rho}_0) + \ln\left[\frac{a}{a_0}\right]^{-3} \\
\bar{\rho} &= \bar{\rho}_0 \frac{a^{-3}}{a_0^{-3}} \\
\boxed{a_0 = 1} \\
\bar{\rho} &= \bar{\rho}_0 a^{-3}
\end{aligned}$$

(1.2)

We introduce perturbations in all relevant quantities. Velocity will be without vector notation for simplicity.

$$\rho = \rho_0 + \delta\rho, \quad \phi = \phi_0 + \delta\phi, \quad v = v_0 + \delta v, \quad p = p_0 + \delta p, \quad \delta = \frac{\delta\rho}{\bar{\rho}} \quad (5)$$

Euler equation

The Euler equation reads

$$\frac{dv}{dt} = -\frac{1}{\rho}\nabla p - \nabla\phi. \quad (6)$$

We introduce the perturbations into the equation. First, we look at the right hand side (RHS):

$$\begin{aligned}
-\frac{1}{\rho}\nabla p - \nabla\phi &= -\frac{1}{\rho_0 + \delta\rho}\nabla(p_0 + \delta p) - \nabla(\phi_0 + \delta\phi) \\
&= -\frac{1}{\rho_0} \left(\frac{1}{1 + \frac{\delta\rho}{\rho_0}} \right) (\nabla p_0 + \nabla\delta p) - \nabla\phi_0 - \nabla\delta\phi
\end{aligned}$$

Now for small perturbations, we have that $\frac{\delta\rho}{\rho_0} \ll 1$. We then get

$$-\frac{1}{\rho}\nabla p - \nabla\phi = -\frac{1}{\rho_0}\nabla p_0 - \frac{1}{\rho_0}\nabla\delta p - \nabla\phi_0 - \nabla\delta\phi. \quad (7)$$

Now the left hand side (LHS):

$$\begin{aligned}
\frac{dv}{dt} &= \frac{\partial v}{\partial t} + v \cdot \nabla v \\
&= \frac{\partial}{\partial t} (v_0 + \delta v) + (v_0 + \delta v) \cdot \nabla (v_0 + \delta v) \\
&= \frac{\partial v_0}{\partial t} + \frac{\partial \delta v}{\partial t} + v_0 \cdot \nabla v_0 + \delta v \cdot \nabla v_0 + v_0 \cdot \nabla \delta v + \delta v \cdot \nabla \delta v
\end{aligned}$$

Now applying the definition of total derivative, and noting that $v_0 \cdot \nabla \delta v = 0$ we get

$$\frac{dv}{dt} = \frac{d\delta v}{dt} + \frac{dv_0}{dt} + \delta v \cdot \nabla v_0 \quad (8)$$

We then get the full equation

$$\begin{aligned}
\frac{d\delta v}{dt} + \frac{dv_0}{dt} + \delta v \cdot \nabla v_0 &= -\frac{1}{\rho_0} \nabla p_0 - \frac{1}{\rho_0} \nabla \delta p - \nabla \phi_0 - \nabla \delta \phi \\
\frac{d\delta v}{dt} + \delta v \cdot \nabla v_0 + \left(\frac{dv_0}{dt} \right) &= -\frac{1}{\rho_0} \nabla \delta p - \nabla \delta \phi \left(-\frac{1}{\rho_0} \nabla p_0 - \nabla \phi_0 \right)
\end{aligned}$$

We see that the bracketed quantities are the unperturbed ones, and must obey the unperturbed Euler equation, thus canceling each other out. We get as a final equation

$$\frac{d\delta v}{dt} + \delta v \cdot \nabla v_0 = -\frac{1}{\rho_0} \nabla \delta p - \nabla \delta \phi. \quad (9)$$

Poisson equation

The original Poisson equation reads

$$\nabla^2 \phi = 4\pi G \rho. \quad (10)$$

Inserting the perturbed quantities, we get

$$\begin{aligned}
\nabla^2 (\phi_0 + \delta \phi) &= 4\pi G (\rho_0 + \delta \rho) \\
\nabla^2 \phi_0 + \nabla^2 \delta \phi &= 4\pi G \rho_0 + 4\pi G \delta \rho.
\end{aligned}$$

Again, the unperturbed quantities ϕ_0 and ρ_0 must follow the original Poisson equation, and we're left with

$$\nabla^2 \delta \phi = 4\pi G \delta \rho. \quad (11)$$

Exercise 2

(2.1)

We start from the relation

$$\frac{H^2}{H_0^2} = \sum_i \Omega_{i,0} \left(\frac{a}{a_0} \right)^{-3(1+\omega_i)} \quad (12)$$

We have that $\omega_m = 0$ and $\omega_\Lambda = -1$. With $a_0 = 1$, we get that

$$\begin{aligned} \frac{H^2}{H_0^2} &= [\Omega_m a^{-3} + \Omega_\Lambda] \\ H^2 &= H_0^2 [\Omega_m a^{-3} + \Omega_\Lambda] \\ \frac{\dot{a}^2}{a^2} &= H_0^2 [\Omega_m a^{-3} + \Omega_\Lambda] \\ H_0^2 &= \frac{\dot{a}^2}{[\Omega_m a^{-3} + \Omega_\Lambda] a^2} \\ H_0 &= \frac{1}{a \sqrt{[\Omega_m a^{-3} + \Omega_\Lambda]}} \frac{da}{dt} \\ H_0 dt &= \frac{1}{a \sqrt{[\Omega_m a^{-3} + \Omega_\Lambda]}} da \end{aligned}$$

Now we integrate this result from $0 \rightarrow t_0$, corresponding to $0 \rightarrow a_0$, and obtain

$$H_0 t = \int_0^{a_0} \frac{1}{a \sqrt{[\Omega_m a^{-3} + \Omega_\Lambda]}} da \quad (13)$$

Unfortunately, I could not solve this integral.

(2.2)

We start from the given differential equation, ignoring the pressure term because the perturbation is much larger than the Jeans length

$$\frac{d^2 \delta}{dt^2} + 2 \frac{\dot{a}}{a} \frac{d\delta}{dt} = 4\pi G \rho \quad (14)$$

From the Friedmann equations in a flat universe, we know that

$$\rho = \frac{3H^2}{8\pi G}, \quad (15)$$

such that we get

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} = \frac{3}{2}H^2\delta. \quad (16)$$

Now we know from the previous exercise that

$$H = \sqrt{H_0^2 [\Omega_m a^{-3} + \Omega_\Lambda]}, \quad (17)$$

and

$$\dot{a} = \sqrt{a^2 H_0^2 [\Omega_m a^{-3} + \Omega_\Lambda]} = \sqrt{H_0^2 [\Omega_m a^{-1} + \Omega_\Lambda a^2]} \quad (18)$$

We split the second order DE into two first order ones, by introducing $u = \frac{d\delta}{dt}$, which gives

$$\frac{du}{dt} + 2Hu = \frac{3}{2}H^2\delta \quad (19)$$

or

$$\frac{du}{dt} = \frac{3}{2}H^2\delta - 2Hu. \quad (20)$$

Using Euler-Cromer to numerically integrate, we get the integration scheme

$$u_{i+1} = u_i + \left(\frac{3}{2}H^2\delta - 2Hu_i \right) \cdot dt \quad (21)$$

$$\delta_{i+1} = \delta_i + u_{i+1} \cdot dt. \quad (22)$$

We change the integration variable from $t \rightarrow a$ by noting

$$\frac{da}{dt} = \dot{a} \Rightarrow dt = \frac{da}{\dot{a}} \quad (23)$$

We use the expressions in equations 17 and 18 to find H and \dot{a} . Since we assume that $\delta \propto a$ in the early universe, we also assume that $\dot{\delta} = \dot{a}$ as the boundary condition. The results are shown in figures 1 and 2.

We see that early on that we have the case where $\delta \propto a$. However, in the cases of $\Omega_\Lambda \neq 0$, the cosmological constant starts to dominate more and more, driving the expansion of the universe. Therefore δ will decrease, as matter is driven apart and "smeared" out over the growing universe.

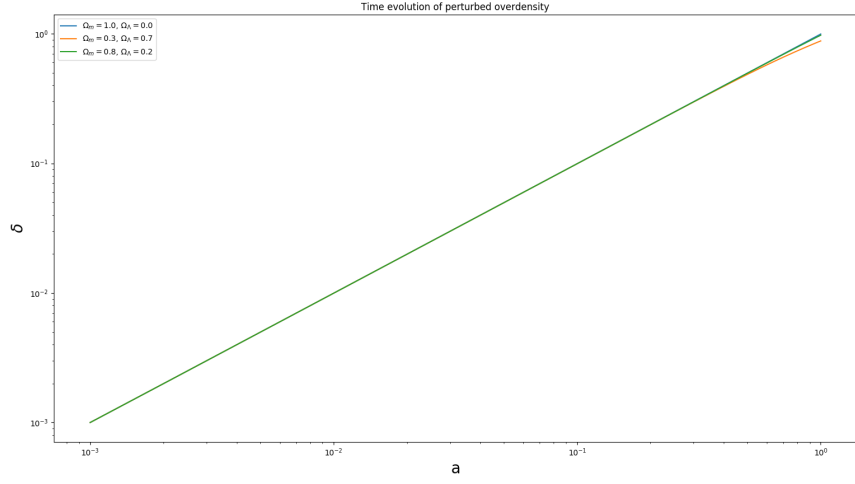


Figure 1: Time evolution of the perturbed overdensity, plotted as a function of the scale factor.

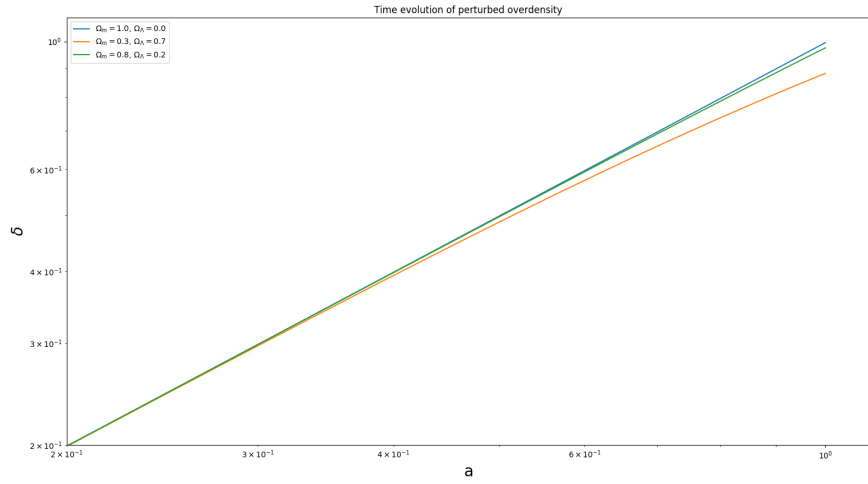


Figure 2: Close up of the last part of the time evolution of the perturbed overdensity.

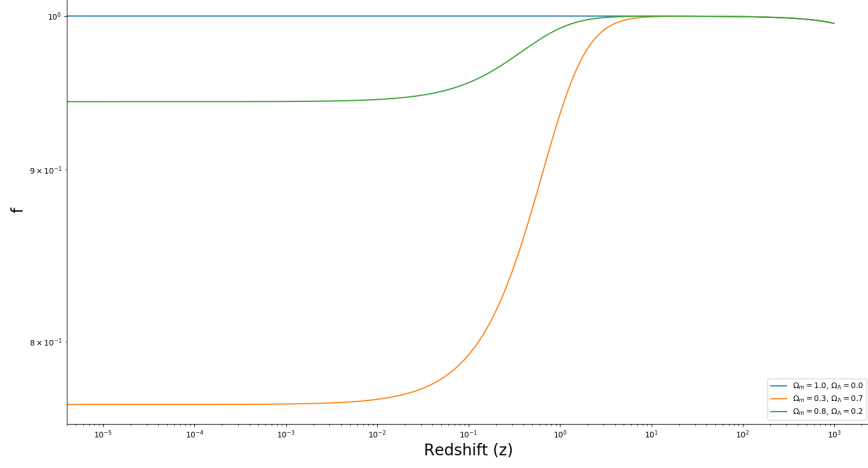


Figure 3: Growth factor f plotted as a function of redshift.

(2.3)

We solve this numerically. A numerical approach gives that

$$f_i = \frac{\Delta \ln(\delta)}{\Delta \ln(a)} = \frac{\ln(\delta_{i+1}) - \ln(\delta_i)}{\ln(a_{i+1}) - \ln(a_i)} = \frac{\ln\left(\frac{\delta_{i+1}}{\delta_i}\right)}{\ln\left(\frac{a_{i+1}}{a_i}\right)} \quad (24)$$

Also, from the relation

$$a = \frac{1}{1+z} \quad (25)$$

we get that

$$z = \frac{1}{a} - 1. \quad (26)$$

The results are shown in figure 3.

Exercise 3

(3.1)

For the gas, we use the expression for the adiabatic processes, namely

$$TV^{\gamma-1} = \text{constant}, \quad (27)$$

where

$$\gamma = \frac{f+2}{f} \quad (28)$$

and f is the degrees of freedom. For a mono-atomic gas, $f = 3$, and we get $\gamma = \frac{5}{3}$. The volume will expand along the universe, so we know that $V \propto a^3$ (each dimension scales with a). We then get

$$\begin{aligned} T_{gas} V^{\frac{5}{3}-1} &= C_1 \\ T_{gas} V^{\frac{2}{3}} &= C_1 \\ T_{gas} a^2 &= C_2 \\ T_{gas} &= \frac{C_2}{a^2} \end{aligned}$$

Now to find C_2 , we estimate that the temperature at recombination ($z = 1100$) was $T = 3000K$, we get

$$C_2 = T_{gas, recomb} a_{recomb}^2 = 3000K \cdot \left(\frac{1}{1+1100} \right)^2 = 2.47837 \cdot 10^{-3} K \approx 2.5 \cdot 10^{-3} K. \quad (29)$$

This means we get the final expression

$$T_{gas} = \frac{2.5 \cdot 10^{-3} K}{a^2} \quad (30)$$

Now for the radiation, we use start of with the photon energy

$$\begin{aligned} E_\gamma &= \frac{hc}{\lambda} \\ k_b T_\gamma &= \frac{hc}{\lambda} \\ T_\gamma &= \frac{hc}{k_b \lambda} \equiv \frac{C_1}{\lambda} \end{aligned}$$

Now for photons, we know their wavelength gets shifted along with the expansion of the universe, namely

$$\lambda \propto a. \quad (31)$$

We then get

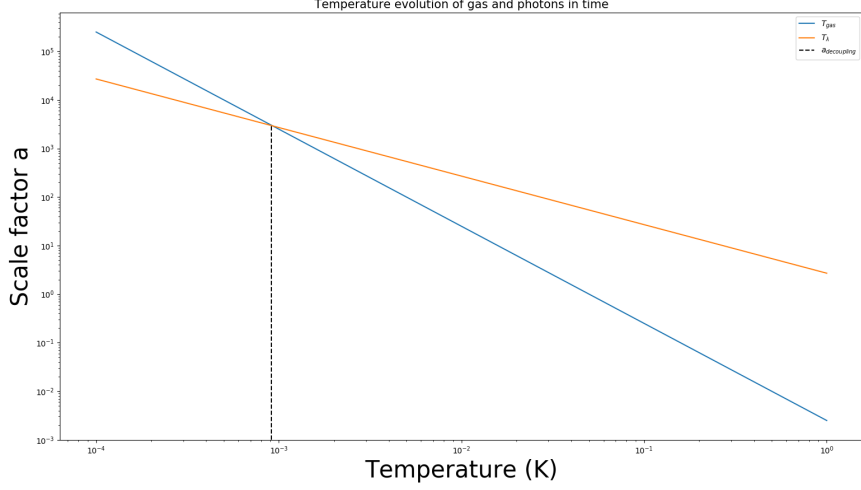


Figure 4: Time evolution of the temperature, plotted as a function of scale factor a .

$$T_\gamma = \frac{C_2}{a}. \quad (32)$$

Again, to find C_2 , we use that $T_\gamma = 3000K$ when $z = 1100$, and we get

$$C_2 = T_\gamma a = 3000K \cdot \left(\frac{1}{1 + 1100} \right) = 2.72480K \approx 2.7 \quad (33)$$

So we get

$$T_\gamma = \frac{2.7K}{a}. \quad (34)$$

Plotting this gives the result shown in figure 4. The model shows that the temperatures are decoupled before $z = 1100$, however this is not the case. We should have that $T_\gamma = T_{gas}$ before decoupling.

(3.1)

The Jeans length and -mass is given, as a function of the sound speed, as

$$\lambda_J = c_s \sqrt{\frac{\pi}{G\rho}}, \quad M_J = \frac{\pi^{5/2}}{6G^{3/2}\rho^{1/2}} \cdot c_s^3. \quad (35)$$

Before decoupling, the sound speed for both the gas and the photons is

$$c_s = \frac{c}{\sqrt{3}}. \quad (36)$$

We know that $\rho = \rho_0 a^{-3} = \rho_0 (1+z)^3$, so we get before decoupling that

$$\begin{aligned} \lambda_J &= \frac{c}{\sqrt{3}} \sqrt{\frac{\pi}{G\rho_0 (1+z)^3}} \\ \lambda_J &= (1+z)^{-3/2} \sqrt{\frac{c^2 \pi}{3G\rho_0}}. \end{aligned} \quad (37)$$

For the Jeans mass, we get

$$M_J = (1+z)^{-3/2} \frac{\pi^{5/2}}{6G^{3/2}\rho_0^{1/2}} \cdot \left(\frac{c}{\sqrt{3}}\right)^3$$

After decoupling, we have that the sound speed is generally given as

$$c_s = \sqrt{\frac{k_b T}{\mu m_p}}. \quad (38)$$

We use our previous relation for the temperature in the sound speed:

$$\begin{aligned} c_{s,gas} &= \sqrt{\frac{k_b T}{\mu m_p}} \\ c_{s,gas} &= \sqrt{\frac{k_b \left(\frac{2.5 \cdot 10^{-3} K}{a^2}\right)}{\mu m_p}} \\ c_{s,gas} &= \frac{0.05}{a} \sqrt{\frac{k_b \cdot K}{\mu m_p}} \\ c_{s,gas} &= 0.05 (1+z) \sqrt{\frac{k_b \cdot K}{\mu m_p}} \end{aligned}$$

Now inserting in the Jeans length, and noting that $\rho = \rho_0 a^{-3} = \rho_0 (1+z)^3$, we get

$$\lambda_J = 0.05 (1+z) \sqrt{\frac{k_b \cdot K}{\mu m_p}} \sqrt{\frac{\pi}{G\rho_0 (1+z)^3}}$$

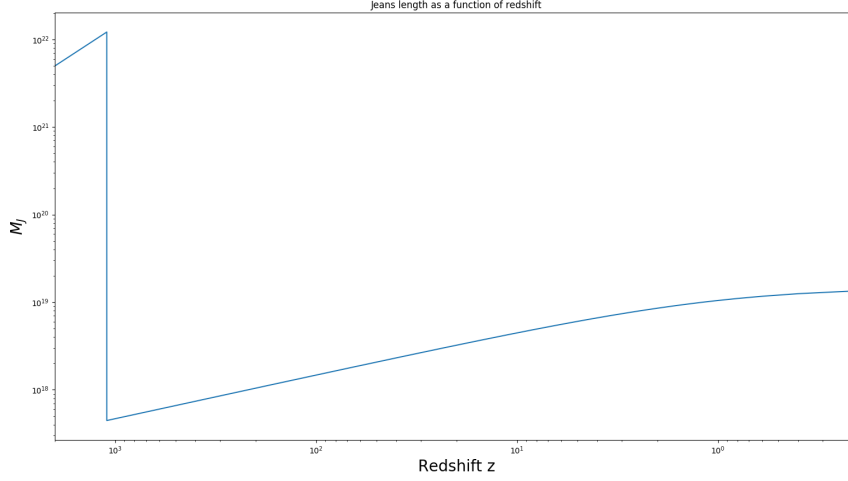


Figure 5: Jeans length plotted as a function of redshift.

$$\lambda_J = 0.05 (1+z)^{-1/2} \sqrt{\frac{k_b \cdot K}{\mu m_p} \frac{\pi}{G \rho_0}} \quad (39)$$

We do the same for the Jeans mass, and we get

$$M_J = \frac{\pi^{5/2}}{6G^{3/2} (\rho_0 (1+z)^3)^{1/2}} \cdot \left(0.05 (1+z) \sqrt{\frac{k_b \cdot K}{\mu m_p}} \right)^3$$

$$M_J = (1+z)^{3/2} \frac{1.25 \cdot 10^{-4} \pi^{5/2}}{6G^{3/2} \rho_0^{1/2}} \left(\frac{k_b \cdot K}{\mu m_p} \right)^{3/2}$$

The results from plotting the Jeans length and -mass is shown in figures 5 and 6.

We see a considerable drop in both the length and mass at decoupling, which is due to the drastic drop in the sound speed for the gas (5 orders of magnitude).

Exercise 4

We have the DE

$$\ddot{R} = -\frac{GM}{R^2} \quad (40)$$

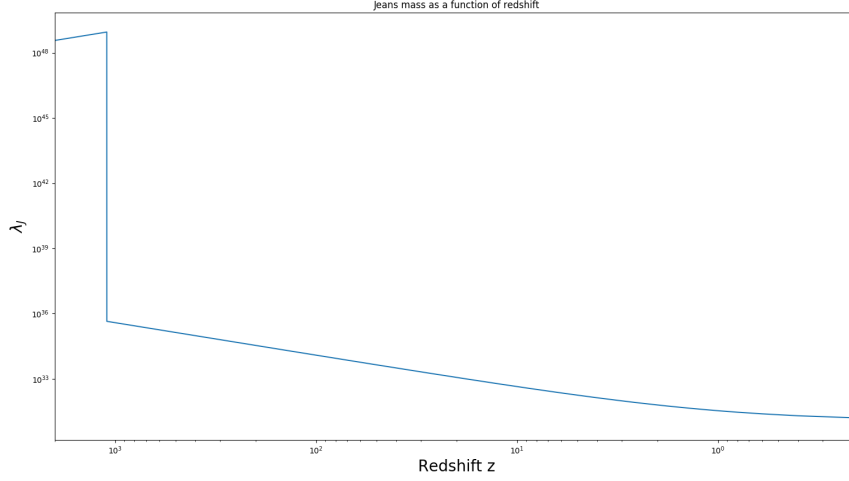


Figure 6: Jeans mass plotted as a function of redshift.

And the assumed parametric solutions

$$R = A(1 - \cos\theta), \quad t = B(\theta - \sin\theta), \quad A^3 = GMB^2. \quad (41)$$

We first note that

$$\frac{dt}{d\theta} = B(1 - \cos\theta) \quad \Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{B(1 - \cos\theta)}, \quad (42)$$

and that

$$\frac{dR}{dt} = \frac{dR}{d\theta} \frac{d\theta}{dt} = \frac{A \sin\theta}{B(1 - \cos\theta)}. \quad (43)$$

Now realizing that we can apply the chain rule on equation 40 if we multiply by $\frac{dR}{dt}$:

$$\begin{aligned} \frac{dR}{dt} \frac{d^2 R}{dt^2} &= -\frac{GM}{R^2} \frac{dR}{dt} \\ \frac{1}{2} \frac{d}{dt} \left(\frac{dR}{dt} \right)^2 &= \frac{d}{dt} \left(\frac{GM}{R} \right) \end{aligned}$$

Integrating both sides now gives us

$$\begin{aligned}\frac{1}{2} \left(\frac{dR}{dt} \right)^2 + C_1 &= \frac{GM}{R} + C_2 \\ \frac{1}{2} \left(\frac{dR}{dt} \right)^2 - \frac{GM}{R} &= C_2 - C_1.\end{aligned}$$

We see that the expression is a kind energy expression, energy per mass. We rename $C_2 - C_1 \equiv K$, and we get

$$\frac{1}{2} \left(\frac{dR}{dt} \right)^2 - \frac{GM}{R} = K. \quad (44)$$

Now inserting the parametric solutions for R and $\frac{dR}{dt}$, we get

$$\begin{aligned}K &= \frac{1}{2} \left[\frac{A \sin \theta}{B (1 - \cos \theta)} \right]^2 - \frac{GM}{A (1 - \cos \theta)} \\ K &= \frac{1}{2} \frac{A^2 \sin^2 \theta}{B^2 (1 - \cos \theta)^2} - \frac{GM}{A (1 - \cos \theta)}\end{aligned}$$

$$\boxed{\sin^2 \theta = 1 - \cos^2 \theta = (1 + \cos \theta) (1 - \cos \theta)}$$

$$\begin{aligned}K &= \frac{1}{2} \frac{A^2 (1 + \cos \theta)}{B^2 (1 - \cos \theta)} - \frac{GM}{A (1 - \cos \theta)} \\ K &= \frac{A^3 (1 + \cos \theta) - 2GM B^2}{2AB^2 (1 - \cos \theta)}\end{aligned} \quad (45)$$

Now since K is an integration constant, we know that it must be constant for all values of θ . Therefore, we can insert a value to get an expression for K. For simplicity, we use $\theta = \pi$:

$$K = -\frac{GM}{2A} \quad (46)$$

Now inserting this back into equation 45

$$\begin{aligned}
-\frac{GM}{2A} &= \frac{A^3(1 + \cos\theta) - 2GMB^2}{2AB^2(1 - \cos\theta)} \\
\frac{2GMB^2}{2AB^2(1 - \cos\theta)} - \frac{GM}{2A} &= \frac{A^3(1 + \cos\theta)}{2AB^2(1 - \cos\theta)} \\
GM \left(\frac{1}{A(1 - \cos\theta)} - \frac{1}{2A} \right) &= \frac{A^2(1 + \cos\theta)}{2B^2(1 - \cos\theta)} \\
GM \left(\frac{1 + \cos\theta}{2A(1 - \cos\theta)} \right) &= \frac{A^2(1 + \cos\theta)}{2B^2(1 - \cos\theta)} \\
\frac{A^2}{B^2} &= \frac{GM}{A}.
\end{aligned}$$

This reproduces the result

$$A^3 = GMB^2 \quad (47)$$

which shows the parametric solutions are correct.

Exercise 5

From the last result, we already know that

$$v = \frac{dR}{dt} = \frac{A \sin\theta}{B(1 - \cos\theta)}. \quad (48)$$

From the parametric solution for R , we also see that $R_{max} = 2A$ when $\theta = \pi$. We know that we have virialization at half of R_{max} , so we get

$$R_{vir} = \frac{1}{2}R_{max} = A \quad (49)$$

Now equating this with our parametric solution for R gives us

$$\begin{aligned}
R_{vir} &= R \\
A &= A(1 - \cos\theta) \\
\cos\theta &= 0
\end{aligned}$$

This gives the solutions $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. The first solution corresponds to halfway to R_{max} . But we know virialization happens on the way back, so the solution of interest here is $\theta = \frac{3\pi}{2}$. Inserting this into equation 48 yields

$$\begin{aligned}
v &= \frac{A \sin\left(\frac{3\pi}{2}\right)}{B \left(1 - \cos\left(\frac{3\pi}{2}\right)\right)} \\
&= -\frac{A}{B} \\
\Rightarrow v^2 &= \frac{A^2}{B^2}
\end{aligned}$$

Now using the relation found in equation 49 and that $\frac{A^2}{B^2} = \frac{GM}{A}$ from equation 41, we get

$$v^2 = \frac{GM}{R_{vir}} \quad (50)$$

$$v = \pm \sqrt{\frac{GM}{R_{vir}}} \quad (51)$$

Exercise 6

For finding the gravitational binding energy, we imagine a shell trying to escape the gravitational potential of the mass within the rest of the sphere. The energy required for a shell is given by

$$dU = -G \frac{m_{shell} m_{sphere}}{r}. \quad (52)$$

We have the mass for a shell with thickness dr given as

$$m_{shell} = 4\pi r^2 \rho dr, \quad (53)$$

and the mass of a general sphere with radius r as

$$m_{sphere} = \frac{4}{3}\pi r^3 \rho. \quad (54)$$

We insert into equation 52:

$$\begin{aligned}
dU &= -G \frac{4\pi r^2 \rho \cdot \frac{4}{3}\pi r^3 \rho}{r} dr \\
&= -G \left(\frac{4}{3}\pi \rho\right) (4\pi \rho) r^4 dr.
\end{aligned}$$

Now integrating from the center to the edge of the sphere, $0 \rightarrow R$, we get

$$\begin{aligned}
U(R) &= \int_0^R dU = -G \left(\frac{4}{3} \pi \rho \right) (4\pi \rho) \int_0^R r^4 dr \\
&= -G \left(\frac{4}{3} \pi \rho \right) (4\pi \rho) \frac{1}{5} R^5 \\
&= -\frac{3G}{5R} \left(\frac{4}{3} \pi \rho R^3 \right) \left(\frac{4}{3} \pi \rho R^3 \right) \\
U &= -\frac{3GM^2}{5R} \tag{55}
\end{aligned}$$