Assignment 1 - AST4320

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Exercise 1

(1.1)

We start from the continuuity equation:

$$\frac{d\bar{\rho}}{dt} + \bar{\rho}\nabla \cdot \mathbf{v} = \frac{\partial \bar{\rho}}{\partial t} + \mathbf{v} \cdot \nabla \bar{\rho} + \bar{\rho}\nabla \cdot \mathbf{v} = 0 \tag{1}$$

Now from the cosmological principle, we can assume there is no inherent density differences (homogeniouty and isotropy), and we get that $\nabla \bar{\rho} = 0$. Also, inserting for our assumption $\mathbf{v} = H\mathbf{r}$, the continuuity equation reduces to

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\rho}\nabla \cdot H\mathbf{r} = 0 \tag{2}$$

Now $\nabla \cdot \mathbf{r} = 3$. We also insert $H = \frac{\dot{a}}{a}$ so we get

$$\frac{\partial \bar{\rho}}{\partial t} + 3\bar{\rho}\frac{\dot{a}}{a} = 0 \tag{3}$$

Now this is a separable differential equation (DE), solved simply by

$$\frac{1}{\bar{\rho}} d\bar{\rho} = -3\frac{1}{a} da \tag{4}$$

Integrating from $t_0 \to t$, and naming $f(t) = f, f(t_0) = f_0$, gives us

$$
\begin{aligned}
ln\left(\bar{\rho}\right) - ln\left(\bar{\rho_0}\right) &= -3\left[ln\left(a\right) - ln\left(a_0\right)\right] \\
ln\left(\bar{\rho}\right) &= ln\left(\bar{\rho_0}\right) + ln\left[\frac{a}{a_0}\right]^{-3} \\
\bar{\rho} &= \bar{\rho}_0 \frac{a^{-3}}{a_0^{-3}} \\
&\boxed{a_0 = 1} \\
\bar{\rho} &= \bar{\rho}_0 a^{-3}
\end{aligned}
$$

(1.2)

We introduce perturbations in all relevant quantities. Velocity will be without vector notation for simplicity.

$$
\rho = \rho_0 + \delta\rho, \quad \phi = \phi_0 + \delta\phi, \quad v = v_0 + \delta v, \quad p = p_0 + \delta p, \quad \delta = \frac{\delta\rho}{\bar{\rho}} \tag{5}
$$

Euler equation

The Euler equation reads

$$
\frac{dv}{dt} = -\frac{1}{\rho}\nabla p - \nabla\phi. \tag{6}
$$

We introduce the perturbations into the equation. First, we look at the right hand side (RHS):

$$
\begin{aligned}
-\frac{1}{\rho}\nabla p - \nabla\phi &= -\frac{1}{\rho_0 + \delta\rho}\nabla(p_0 + \delta p) - \nabla(\phi_0 + \delta\phi) \\
&= -\frac{1}{\rho_0}\left(\frac{1}{1 + \frac{\delta\rho}{\rho_0}}\right)(\nabla p_0 + \nabla\delta p) - \nabla\phi_0 - \nabla\delta\phi
\end{aligned}
$$

Now for small perturbations, we have that $\frac{\delta\rho}{\rho_0} \ll 1$. We then get

$$
-\frac{1}{\rho}\nabla p - \nabla\phi = -\frac{1}{\rho_0}\nabla p_0 - \frac{1}{\rho_0}\nabla\delta p - \nabla\phi_0 - \nabla\delta\phi. \tag{7}
$$

Now the left hand side (LHS):

$$
\begin{aligned}
\frac{dv}{dt} &= \frac{\partial v}{\partial t} + v \cdot \nabla v \\
&= \frac{\partial}{\partial t}(v_0 + \delta v) + (v_0 + \delta v) \cdot \nabla (v_0 + \delta v) \\
&= \frac{\partial v_0}{\partial t} + \frac{\partial \delta v}{\partial t} + v_0 \cdot \nabla v_0 + \delta v \cdot \nabla v_0 + v_0 \cdot \nabla \delta v + \delta v \cdot \nabla \delta v
\end{aligned}
$$

Now applying the definition of total derivative, we get that

$$
\frac{dv_0}{dt} = \frac{\partial v_0}{\partial t} + v_0 \cdot \nabla v_0 \ , \ \frac{d\delta v}{dt} = \frac{\partial v}{\partial t} + (v_0 + \delta v) \cdot \nabla \delta v \tag{8}
$$

And we're left with

$$
\frac{dv}{dt} = \frac{dv_0}{dt} + \frac{d\delta v}{dt} + \delta v \cdot \nabla v_0 \tag{9}
$$

We then get the full equation

$$
\begin{aligned}
\frac{dv_0}{dt} + \frac{d\delta v}{dt} + \delta v \cdot \nabla v_0 &= = -\frac{1}{\rho_0}\nabla p_0 - \frac{1}{\rho_0}\nabla \delta p - \nabla \phi_0 - \nabla \delta \phi \\
\left(\frac{dv_0}{dt}\right) + \frac{d\delta v}{dt} + \delta v \cdot \nabla v_0 &= -\frac{1}{\rho_0}\nabla \delta p - \nabla \delta \phi \left(-\frac{1}{\rho_0}\nabla p_0 - \nabla \phi_0\right)
\end{aligned}
$$

We see that the bracketed quantities are the unperturbed ones, and must obey the unperturbed Euler equation, thus canceling each other out. We get as a final equation

$$
\frac{d\delta v}{dt} + \delta v \cdot \nabla v_0 = -\frac{1}{\rho_0}\nabla \delta p - \nabla \delta \phi. \tag{10}
$$

Poisson equation

The original Poisson equation reads

$$
\nabla^2 \phi = 4\pi G \rho. \tag{11}
$$

Inserting the perturbed quantities, we get

$$
\begin{aligned}
\nabla^2(\phi_0 + \delta\phi) &= 4\pi G(\rho_0 + \delta\rho) \\
\nabla^2 \phi_0 + \nabla^2 \delta\phi &= 4\pi G \rho_0 + 4\pi G \delta\rho.
\end{aligned}
$$

Again, the unperturbed quantities ϕ_0 and ρ_0 must follow the original Poisson equation, and we're left with

$$\nabla^2 \delta\phi = 4\pi G \delta\rho. \tag{12}$$

Exercise 2

(2.1)

We start from the relation

$$\frac{H^2}{H_0^2} = \sum_i \Omega_{i,0} \left(\frac{a}{a_0}\right)^{-3(1+\omega_i)} \tag{13}$$

We have that $\omega_m = 0$ and $\omega_\Lambda = -1$. With $a_0 = 1$, we get that

$$\begin{aligned}
\frac{H^2}{H_0^2} &= \left[\Omega_m a^{-3} + \Omega_\Lambda\right] \\
H^2 &= H_0^2 \left[\Omega_m a^{-3} + \Omega_\Lambda\right] \\
\frac{\dot{a}^2}{a^2} &= H_0^2 \left[\Omega_m a^{-3} + \Omega_\Lambda\right] \\
\frac{\dot{a}}{a} &= H_0 \sqrt{\left[\Omega_m a^{-3} + \Omega_\Lambda\right]}
\end{aligned}$$

Now inserting the values we get

$$\begin{aligned}
(\Omega_m, \Omega_\Lambda) = (1.0, 0.0) &: \frac{\dot{a}}{a} = H_0 \sqrt{a^{-3}} = H_0 a^{-\frac{3}{2}} \\
(\Omega_m, \Omega_\Lambda) = (0.3, 0.7) &: \frac{\dot{a}}{a} = H_0 \sqrt{[0.3a^{-3} + 0.7]} \\
(\Omega_m, \Omega_\Lambda) = (0.8, 0.2) &: \frac{\dot{a}}{a} = H_0 \sqrt{[0.8a^{-3} + 0.2]}
\end{aligned}$$

(2.2)

We start from the given differential equation, ignoring the pressure term because the perturbation is much larger than the Jeans length

$$\frac{d^2\delta}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\delta}{dt} = \delta 4\pi G \rho \tag{14}$$

From the Friedmann equations in a flat universe, we know that

$$\rho = \frac{3H^2}{8\pi G}, \tag{15}$$

such that we get

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} = \frac{3}{2}H^2\delta. \tag{16}$$

Now we know from the previous exercise that

$$H = \sqrt{H_0^2\left[\Omega_m a^{-3} + \Omega_\Lambda\right]}, \tag{17}$$

and

$$\dot{a} = \sqrt{a^2 H_0^2\left[\Omega_m a^{-3} + \Omega_\Lambda\right]} = \sqrt{H_0^2\left[\Omega_m a^{-1} + \Omega_\Lambda a^2\right]} \tag{18}$$

We split the second order DE into two first order ones, by introducing $u = \frac{d\delta}{dt}$, which gives

$$\frac{du}{dt} + 2Hu = \frac{3}{2}H^2\delta \tag{19}$$

or

$$\frac{du}{dt} = \frac{3}{2}H^2\delta - 2Hu. \tag{20}$$

Using Euler-Cromer to numerically integrate, we get the integration scheme

$$u_{i+1} = u_i + \left(\frac{3}{2}H^2\delta - 2Hu_i\right)\cdot dt \tag{21}$$

$$\delta_{i+1} = \delta_i + u_{i+1}\cdot dt. \tag{22}$$

We change the integration variable from $t \to a$ by noting

$$\frac{da}{dt} = \dot{a} \quad \Rightarrow dt = \frac{da}{\dot{a}} \tag{23}$$

We use the expressions in equations 17 and 18 to find H and $\dot{a}$. Since we assume that $\delta \propto a$ in the early universe, we also assume that $\dot{\delta} = \dot{a}$ as the boundary condition. The results are shown in figures 1 and 2.

We see that early on that we have the case where $\delta \propto a$. However, in the cases of $\Omega_\Lambda \neq 0$, the cosmological constant starts to dominate more and more, driving the expansion of the universe. Therefore δ will decrease, as matter is driven apart and "smeared" out over the growing universe.

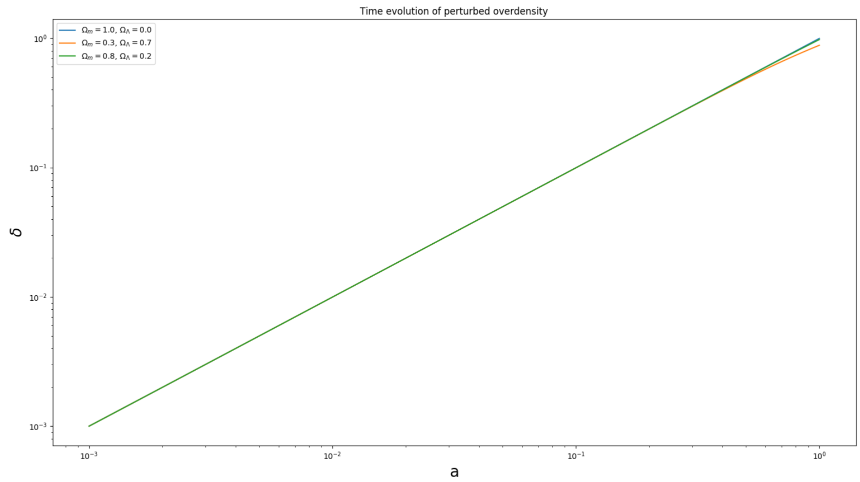

Figure 1: Time evolution of the perturbed overdensity, plotted as a function of the scale factor.

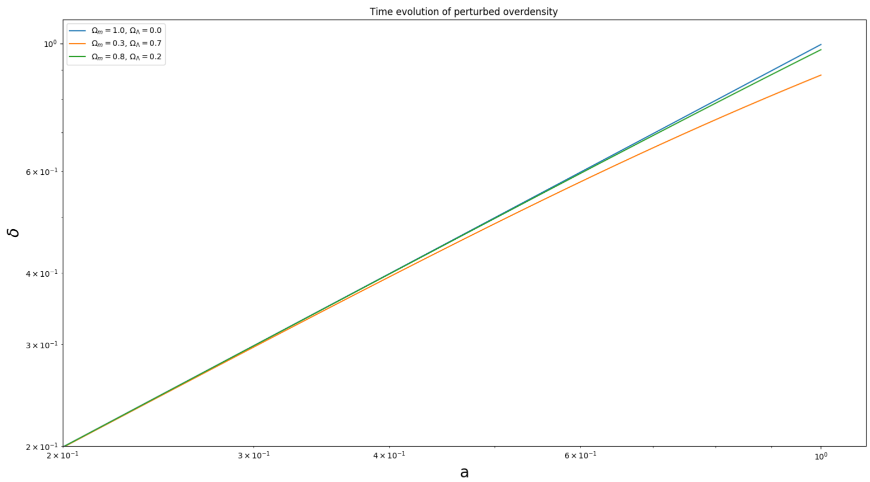

Figure 2: Close up of the last part of the time evolution of the perturbed overdensity.

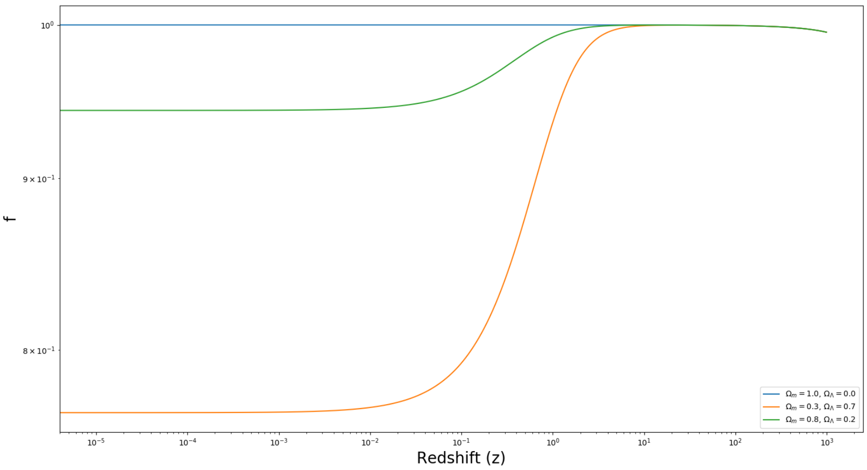

Figure 3: Growth factor f plotted as a function of redshift.

(2.3)

We solve this numerically. A numerical approach gives that

$$f_i = \frac{\Delta ln(\delta)}{\Delta ln(a)} = \frac{ln(\delta_{i+1}) - ln(\delta_i)}{ln(a_{i+1}) - ln(a_{i+1})} = \frac{ln\left(\frac{\delta_{i+1}}{\delta_i}\right)}{ln\left(\frac{a_{i+1}}{a_i}\right)} \tag{24}$$

Also, from the relation

$$a = \frac{1}{1+z} \tag{25}$$

we get that

$$z = \frac{1}{a} - 1. \tag{26}$$

The results are shown in figure 3.

Exercise 3

(3.1)

For the gas, we use the expression for the adiabatic processes, namely

$$TV^{\gamma-1} = constant, \tag{27}$$

where

$$\gamma = \frac{f+2}{f} \tag{28}$$

and f is the degrees of freedom. For a mono-atomic gas, $f = 3$, and we get $\gamma = \frac{5}{3}$. The volume will expand along the universe, so we know that $V \propto a^3$ (each dimension scales with a). We then get

$$\begin{aligned}
T_{gas} V^{\frac{5}{3}-1} &= C_1 \\
T_{gas} V^{\frac{2}{3}} &= C_1 \\
T_{gas} a^2 &= C_2 \\
T_{gas} &= \frac{C_2}{a^2}
\end{aligned}$$

Now to find C_2, we estimate that the temperature at recombination ($z = 1100$) was $T = 3000K$, we get

$$C_2 = T_{gas,recbomb} a_{recomb}^2 = 3000K \cdot \left(\frac{1}{1+1100}\right)^2 = 2.47837 \cdot 10^{-3} K \approx 2.5 \cdot 10^{-3} K. \tag{29}$$

This means we get the final expression

$$T_{gas} = \frac{2.5 \cdot 10^{-3} K}{a^2} \tag{30}$$

Now for the radiation, we use start of with the photon energy

$$\begin{aligned}
E_\gamma &= \frac{hc}{\lambda} \\
k_b T_\gamma &= \frac{hc}{\lambda} \\
T_\gamma &= \frac{hc}{k_b \lambda} \equiv \frac{C_1}{\lambda}
\end{aligned}$$

Now for photons, we know their wavelength gets shifted along with the expansion of the universe, namely

$$\lambda \propto a. \tag{31}$$

We then get

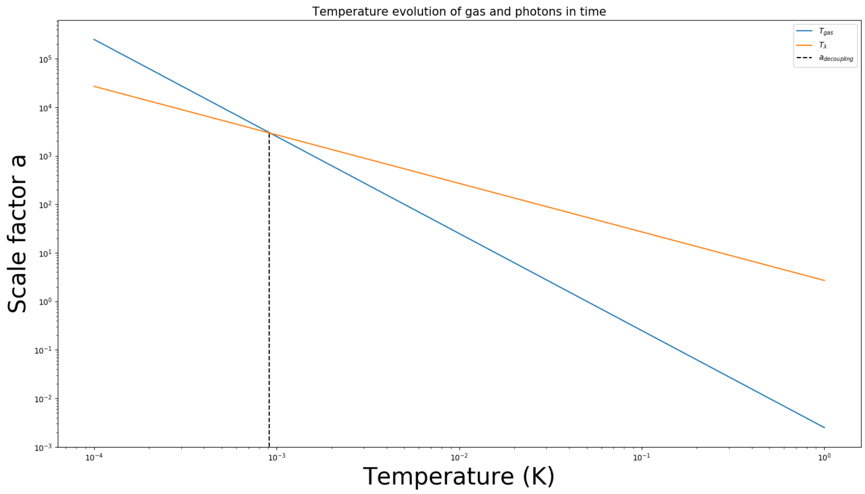

Figure 4: Time evolution of the temperature, plotted as a function of scale factor a.

$$T_\gamma = \frac{C_2}{a}. \tag{32}$$

Again, to find C_2, we use that $T_\gamma = 3000K$ when $z = 1100$, and we get

$$C_2 = T_\gamma a = 3000K \cdot \left(\frac{1}{1+1100}\right) = 2.72480K \approx 2.7 \tag{33}$$

So we get

$$T_\gamma = \frac{2.7K}{a}. \tag{34}$$

Plotting this gives the result shown in figure 4. The model shows that the temperatures are decoupled before $z = 1100$, however this is not the case. We should have that $T_\gamma = T_{gas}$ before decoupling.

(3.1)

The Jeans length and -mass is given, as a function of the sound speed, as

$$\lambda_J = c_s\sqrt{\frac{\pi}{G\rho}}, \quad M_J = \frac{\pi^{5/2}}{6G^{3/2}\rho^{1/2}} \cdot c_s^3. \tag{35}$$

Before decoupling, the sound speed for both the gas and the photons is

$$c_s = \frac{c}{\sqrt{3}}. \tag{36}$$

We know that $\rho = \rho_0 a^{-3} = \rho_0 (1+z)^3$, so we get before decoupling that

$$\lambda_J = \frac{c}{\sqrt{3}} \sqrt{\frac{\pi}{G\rho_0 (1+z)^3}}$$

$$\lambda_J = (1+z)^{-3/2} \sqrt{\frac{c^2 \pi}{3G\rho_0}}. \tag{37}$$

For the Jeans mass, we get

$$M_J = (1+z)^{-3/2} \frac{\pi^{5/2}}{6G^{3/2}\rho_0^{1/2}} \cdot \left(\frac{c}{\sqrt{3}}\right)^3$$

After decoupling, we have that the sound speed is generally given as

$$c_s = \sqrt{\frac{k_b T}{\mu m_p}}. \tag{38}$$

We use our previous relation for the temperature in the sound speed:

$$c_{s,gas} = \sqrt{\frac{k_b T}{\mu m_p}}$$

$$c_{s,gas} = \sqrt{\frac{k_b \left(\frac{2.5 \cdot 10^{-3} K}{a^2}\right)}{\mu m_p}}$$

$$c_{s,gas} = \frac{0.05}{a} \sqrt{\frac{k_b \cdot K}{\mu m_p}}$$

$$c_{s,gas} = 0.05 (1+z) \sqrt{\frac{k_b \cdot K}{\mu m_p}}$$

Now inserting in the Jeans length, and noting that $\rho = \rho_0 a^{-3} = \rho_0 (1+z)^3$, we get

$$\lambda_J = 0.05 (1+z) \sqrt{\frac{k_b \cdot K}{\mu m_p}} \sqrt{\frac{\pi}{G\rho_0 (1+z)^3}}$$

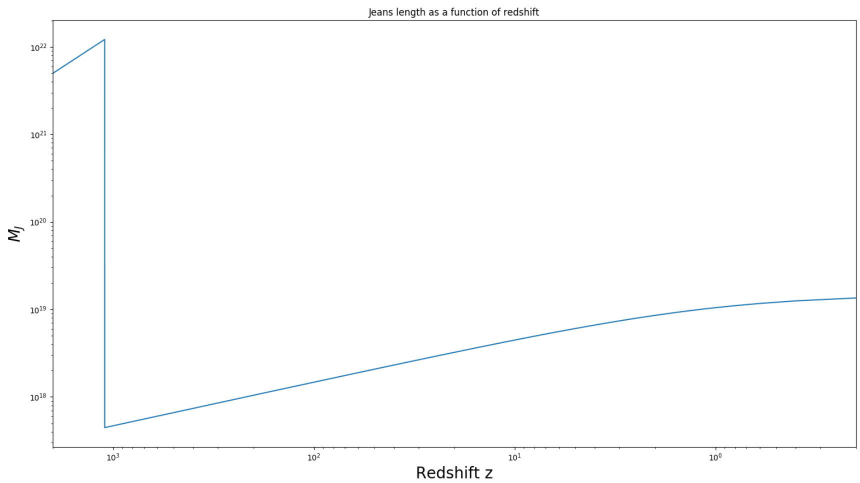

Figure 5: Jeans length plotted as a function of redshift.

$$\lambda_J = 0.05\,(1+z)^{-1/2}\sqrt{\frac{k_b \cdot K}{\mu m_p}\frac{\pi}{G\rho_0}} \tag{39}$$

We do the same for the Jeans mass, and we get

$$M_J = \frac{\pi^{5/2}}{6G^{3/2}\left(\rho_0\,(1+z)^3\right)^{1/2}} \cdot \left(0.05\,(1+z)\sqrt{\frac{k_b \cdot K}{\mu m_p}}\right)^3$$
$$M_J = (1+z)^{3/2}\,\frac{1.25 \cdot 10^{-4}\pi^{5/2}}{6G^{3/2}\rho_0^{1/2}}\left(\frac{k_b \cdot K}{\mu m_p}\right)^{3/2}$$

The results from plotting the Jeans length and -mass is shown in figures 5 and 6.

We see a considerable drop in both the length and mass at decoupling, which is due to the drastic drop in the sound speed for the gas (5 orders of magnitude).

Exercise 4

We have the DE

$$\ddot{R} = -\frac{GM}{R^2} \tag{40}$$

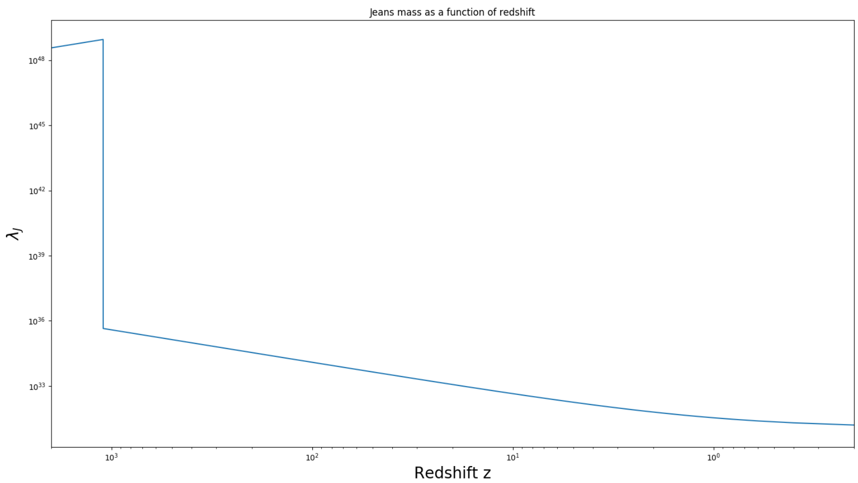

Figure 6: Jeans mass plotted as a function of redshift.

And the assumed parametric solutions

$$R = A\left(1 - cos\theta\right), \quad t = B\left(\theta - sin\theta\right), \quad A^3 = GMB^2. \tag{41}$$

We first note that

$$\frac{dt}{d\theta} = B\left(1 - cos\theta\right) \quad \Rightarrow \frac{d\theta}{dt} = \frac{1}{B\left(1 - cos\theta\right)}, \tag{42}$$

and that

$$\frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \frac{Asin\theta}{B\left(1 - cos\theta\right)}. \tag{43}$$

Now realizing that we can apply the chain rule on equation 40 if we multiply by $\frac{dR}{dt}$:

$$\frac{dR}{dt}\frac{d^2R}{dt^2} = -\frac{GM}{R^2}\frac{dR}{dt}$$
$$\frac{1}{2}\frac{d}{dt}\left(\frac{dR}{dt}\right)^2 = \frac{d}{dt}\left(\frac{GM}{R}\right)$$

Integrating both sides now gives us

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 + C_1 = \frac{GM}{R} + C_2$$
$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{GM}{R} = C_2 - C_1.$$

We see that the expression is a kind energy expression, energy per mass. We rename $C_2 - C_1 \equiv K$, and we get

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{GM}{R} = K. \tag{44}$$

Now inserting the parametric solutions for R and $\frac{dR}{dt}$, we get

$$K = \frac{1}{2}\left[\frac{Asin\theta}{B\left(1 - cos\theta\right)}\right]^2 - \frac{GM}{A\left(1 - cos\theta\right)}$$
$$K = \frac{1}{2}\frac{A^2}{B^2}\frac{sin^2\theta}{\left(1 - cos\theta\right)^2} - \frac{GM}{A\left(1 - cos\theta\right)}$$

$$\boxed{sin^2\theta = 1 - cos^2\theta = \left(1 + cos\theta\right)\left(1 - cos\theta\right)}$$

$$K = \frac{1}{2}\frac{A^2}{B^2}\frac{\left(1 + cos\theta\right)}{\left(1 - cos\theta\right)} - \frac{GM}{A\left(1 - cos\theta\right)}$$
$$K = \frac{A^3\left(1 + cos\theta\right) - 2GMB^2}{2AB^2\left(1 - cos\theta\right)} \tag{45}$$

Now since K is an integration constant, we know that it must be constant for all values of θ. Therefore, we can insert a value to get an expression for K. For simplicity, we use $\theta = \pi$:

$$K = -\frac{GM}{2A} \tag{46}$$

Now inserting this back into equation 45

$$
\begin{aligned}
-\frac{GM}{2A} &= \frac{A^3\left(1+cos\theta\right)-2GMB^2}{2AB^2\left(1-cos\theta\right)} \\
\frac{2GMB^2}{2AB^2\left(1-cos\theta\right)}-\frac{GM}{2A} &= \frac{A^3\left(1+cos\theta\right)}{2AB^2\left(1-cos\theta\right)} \\
GM\left(\frac{1}{A\left(1-cos\theta\right)}-\frac{1}{2A}\right) &= \frac{A^2\left(1+cos\theta\right)}{2B^2\left(1-cos\theta\right)} \\
GM\left(\frac{1+cos\theta}{2A\left(1-cos\theta\right)}\right) &= \frac{A^2\left(1+cos\theta\right)}{2B^2\left(1-cos\theta\right)} \\
\frac{A^2}{B^2} &= \frac{GM}{A}.
\end{aligned}
$$

This reproduces the result

$$A^3 = GMB^2 \tag{47}$$

which shows the parametric solutions are correct.

Exercise 5

From the last result, we already know that

$$v = \frac{dR}{dt} = \frac{Asin\theta}{B\left(1-cos\theta\right)}. \tag{48}$$

From the parametric solution for R, we also see that $R_{max} = 2A$ when $\theta = \pi$. We know that we have virialization at half of R_{max}, so we get

$$R_{vir} = \frac{1}{2}R_{max} = A \tag{49}$$

Now equating this with our parametric solution for R gives us

$$
\begin{aligned}
R_{vir} &= R \\
A &= A\left(1-cos\theta\right) \\
cos\theta &= 0
\end{aligned}
$$

This gives the solutions $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. The first solution corresponds to halfway to R_{max}. But we know virialization happens on the way back, so the solution of interest here is $\theta = \frac{3\pi}{2}$. Inserting this into equation 48 yields

$$
\begin{aligned}
v &= \frac{Asin\left(\frac{3\pi}{2}\right)}{B\left(1 - cos\left(\frac{3\pi}{2}\right)\right)} \\
&= -\frac{A}{B} \\
\Rightarrow v^2 &= \frac{A^2}{B^2}
\end{aligned}
$$

Now using the relation found in equation 49 and that $\frac{A^2}{B^2} = \frac{GM}{A}$ from equation 41, we get

$$v^2 = \frac{GM}{R_{vir}} \tag{50}$$

$$v = \pm\sqrt{\frac{GM}{R_{vir}}} \tag{51}$$

Exercise 6

For finding the gravitational binding energy, we imagine s shell trying to escape the gravitational potential of the mass within the rest of the sphere. The energy required for a shell is given by

$$dU = -G\frac{m_{shell}m_{sphere}}{r}. \tag{52}$$

We have the mass for a shell with thickness dr given as

$$m_{shell} = 4\pi r^2 \rho dr, \tag{53}$$

and the mass of a general sphere with radius r as

$$m_{sphere} = \frac{4}{3}\pi r^3 \rho. \tag{54}$$

We insert into equation 52:

$$
\begin{aligned}
dU &= -G\frac{4\pi r^2 \rho \cdot \frac{4}{3}\pi r^3 \rho}{r} dr \\
&= -G\left(\frac{4}{3}\pi\rho\right)(4\pi\rho)\, r^4 dr.
\end{aligned}
$$

Now integrating from the center to the edge of the sphere, $0 \rightarrow R$, we get

$$
\begin{aligned}
U(R) = \int_0^R dU &= -G\left(\frac{4}{3}\pi\rho\right)(4\pi\rho)\int_0^R r^4 dr \\
&= -G\left(\frac{4}{3}\pi\rho\right)(4\pi\rho)\frac{1}{5}R^5 \\
&= -\frac{3G}{5R}\left(\frac{4}{3}\pi\rho R^3\right)\left(\frac{4}{3}\pi\rho R^3\right)
\end{aligned}
$$

$$
U = -\frac{3GM^2}{5R} \tag{55}
$$