Homework (due Wednesday, November 5th)

Grading will take into account the *clarity* and *conciseness* of the solutions.

Problem 1 (NP-hardness)**.** Prove that the following problems are $\mathbf{NP}$-hard

- The halting problem $\text{HALT} = \{(\alpha, x) : M_\alpha \text{ halts on input } x\}$.
- The problem TQBF.
- The problem not-all-equal sat

$$\text{NAE-3-SAT} = \{(c_1, \ldots, c_n) \text{ where each } c_i \text{ is composed of 3 literals } c_i = (x_{i_1}, x_{i_2}, x_{i_3}) : \\ \text{there is an assignment s.t. the values assigned to the literals of every } c_i \text{ are not all equal}\}.$$

- The problem $\text{MAXCUT} = \{(G, k) : V_G = S \cup \bar{S} \text{ such that } |E_G(S, \bar{S})| \geq k\}$. V_G is the vertex set of G and $E_G(S, \bar{S})$ is the set of edges with one endpoint in S and the other not in S.

Are they $\mathbf{NP}$-complete?

Problem 2 (Reductions)**.** We saw that a language L is polynomial-time Karp reducible to L' if there is a polynomial-time computable functions f such that $x \in L$ iff $f(x) \in L'$. Give an example of a class that is closed under polynomial-time Karp reductions (i.e., if L is reducible to L' and $L' \in \mathcal{C}$, then $L \in \mathcal{C}$) and one that is not.

One can also define a polynomial-time Cook reduction. A language L is polynomial-time Cook reducible to L' if there is a polynomial-time Turing machine with an oracle L' that decides the language L. Show that with this definition of reduction (instead of the Karp reduction) the problem TAUTOLOGY is $\mathbf{NP}$-hard.

Problem 3 (Difference of $\mathbf{NP}$ problems)**.** Let $\mathbf{DP} = \{L_1 \setminus L_2 : L_1 \in \mathbf{NP} \text{ and } L_2 \in \mathbf{NP}\}$. Show that the problem $\text{EXACTINDSET} = \{(G, k) : \text{ the largest independent set of G has size exactly } k\}$ is $\mathbf{DP}$-complete for the usual polynomial-time Karp reductions.

Problem 4 (Classes with exponential resources)**.** Recall that $\mathbf{EXP} = \cup_{\ell \geq 1} \mathbf{DTIME}(2^{n^\ell})$. Find a complete problem for the class $\mathbf{EXP}$.

Show using a padding argument that $\mathbf{L} = \mathbf{P}$ implies that $\mathbf{PSPACE} = \mathbf{EXP}$.

Problem 5 (Downward self-reducibility)**.** A language A is called *downward self-reducible* if it can be solved using a polynomial time machine that is allowed to make queries (just as oracle Turing machines) that can decide whether a given word that is strictly smaller than the input is in A. More precisely, on input x, the machine has an oracle defined by the language $A^{<|x|} = A \cap \{0, 1\}^{|x|-1}$.

Give examples of languages that are downward self-reducible. Express the set of languages that are downward self-reducible using classes we defined in the lectures.

Problem 6 (Space hierarchy theorem)**.** State and prove a deterministic space hierarchy theorem. If you want to use a theorem about universal Turing machines, you should state it, but not necessarily prove it.

Problem 7 (Polynomial hierarchy)**.** Show that $\mathbf{P}^{\text{SAT}} = \mathbf{NP}^{\text{SAT}}$ implies that the polynomial hierarchy collapses.