

FRIEDMAN's translation

Marc CHEVALIER
Thomas PELLISSIER TANON

October 22, 2014

1 FRIEDMAN's Translation

Definition 1. Let R be a formula. The *parametrized negation* is

$$\neg_R := A \Rightarrow R$$

We gather here some basic properties of the parametrized negation

Proposition 2. In intuitionistic logic,

- (i) $B \Rightarrow \neg_R A \vdash A \Rightarrow \neg_R B$
- (ii) $A \vdash \neg_R \neg_R A$
- (iii) $A \Rightarrow B \vdash \neg_R B \Rightarrow \neg_R A$
- (iv) $A \Rightarrow B \vdash \neg_R \neg_R A \Rightarrow \neg_R \neg_R B$
- (v) $\neg_R \neg_R \neg_R A \vdash \neg_R A$

Proof. (i)

$$\frac{\frac{B \Rightarrow \neg_R A, A, B \vdash A}{B \Rightarrow \neg_R A, A, B \vdash A} \quad \frac{\frac{B \Rightarrow \neg_R A, A, B \vdash B}{B \Rightarrow \neg_R A, A, B \vdash B \Rightarrow A \Rightarrow R} \quad \frac{B \Rightarrow \neg_R A, A, B \vdash A \Rightarrow R}{B \Rightarrow \neg_R A, A, B \vdash A \Rightarrow R}}{\frac{B \Rightarrow \neg_R A, A, B \vdash R}{B \Rightarrow \neg_R A, A \vdash \neg_R B} \quad \frac{B \Rightarrow \neg_R A, A \vdash \neg_R B}{B \Rightarrow \neg_R A \vdash A \Rightarrow \neg_R B}}$$

(ii)

$$\frac{\frac{A, \neg_R A \vdash A \Rightarrow R}{A, \neg_R A \vdash R} \quad \frac{A, \neg_R A \vdash A}{A \vdash \neg_R \neg_R A}}$$

(iii)

$$\frac{\frac{A \Rightarrow B, \neg_R B, A \vdash B \Rightarrow R}{A \Rightarrow B, \neg_R B, A \vdash B \Rightarrow R} \quad \frac{\frac{A \Rightarrow B, \neg_R B, A \vdash A \Rightarrow B}{A \Rightarrow B, \neg_R B, A \vdash A} \quad \frac{A \Rightarrow B, \neg_R B, A \vdash A}{A \Rightarrow B, \neg_R B, A \vdash A}}{\frac{A \Rightarrow B, \neg_R B, A \vdash R}{A \Rightarrow B, \neg_R B \vdash \neg_R A} \quad \frac{A \Rightarrow B, \neg_R B \vdash \neg_R A}{A \Rightarrow B \vdash \neg_R B \Rightarrow \neg_R A}}$$

(iv)

$$\frac{\frac{A \Rightarrow B, \neg_R \neg_R A, \neg_R B, A \vdash A \Rightarrow B}{(\Pi_0) : A \Rightarrow B, \neg_R \neg_R A, \neg_R B, A \vdash B}}{\frac{A \Rightarrow B, \neg_R \neg_R A, \neg_R B, A \vdash A \Rightarrow B}{(\Pi_0) : A \Rightarrow B, \neg_R \neg_R A, \neg_R B, A \vdash B}}$$

$$\frac{\frac{A \Rightarrow B, \neg_R \neg_R A, \neg_R B, A \vdash B \Rightarrow R}{(\Pi_1) : A \Rightarrow B, \neg_R \neg_R A, \neg_R B \vdash A \Rightarrow R}}{\frac{A \Rightarrow B, \neg_R \neg_R A, \neg_R B, A \vdash R}{(\Pi_1) : A \Rightarrow B, \neg_R \neg_R A, \neg_R B \vdash A \Rightarrow R}}$$

$$\frac{\frac{\frac{A \Rightarrow B, \neg_R \neg_R A, \neg_R B \vdash A \Rightarrow R \Rightarrow R}{(\Pi_2) : \neg_R \neg_R \neg_R A, A, A \Rightarrow R \vdash R}}{\frac{A \Rightarrow B, \neg_R \neg_R A, \neg_R B \vdash R}{(\Pi_2) : \neg_R \neg_R \neg_R A, A, A \Rightarrow R \vdash R}}}{\frac{A \Rightarrow B, \neg_R \neg_R A, \neg_R B \vdash R}{(\Pi_2) : \neg_R \neg_R \neg_R A, A, A \Rightarrow R \vdash R}}$$

(v)

$$\frac{\frac{\neg_R \neg_R \neg_R A, A, A \Rightarrow R \vdash A \Rightarrow R}{(\Pi_2) : \neg_R \neg_R \neg_R A, A, A \Rightarrow R \vdash R}}{\frac{\neg_R \neg_R \neg_R A, A, A \Rightarrow R \vdash R}{(\Pi_2) : \neg_R \neg_R \neg_R A, A, A \Rightarrow R \vdash R}}$$

$$\frac{\frac{\frac{\neg_R \neg_R \neg_R A, A \vdash A \Rightarrow R \Rightarrow R \Rightarrow R}{(\Pi_2) : \neg_R \neg_R \neg_R A, A \vdash A \Rightarrow R \Rightarrow R}}{\frac{\neg_R \neg_R \neg_R A, A \vdash A \Rightarrow R \Rightarrow R}{(\Pi_2) : \neg_R \neg_R \neg_R A, A \vdash A \Rightarrow R \Rightarrow R}}}{\frac{\neg_R \neg_R \neg_R A, A \vdash R}{(\Pi_2) : \neg_R \neg_R \neg_R A, A \vdash \neg_R A}}$$

□

We now define the parametrized translation.

Definition 3. Let R be a formula. The *parametrized negative translation* A^{\neg_R} is defined by induction on A as follows:

$$\begin{aligned} \perp^{\neg_R} &:= R & \top^{\neg_R} &:= \top & (a \doteq b)^{\neg_R} &:= \neg_R \neg_R (a \doteq b) \\ (A \wedge B)^{\neg_R} &:= A^{\neg_R} \wedge B^{\neg_R} & (A \Rightarrow B)^{\neg_R} &:= A^{\neg_R} \Rightarrow B^{\neg_R} \\ (A \vee B)^{\neg_R} &:= \neg_R \neg_R (A^{\neg_R} \vee B^{\neg_R}) \\ (\forall x A)^{\neg_R} &:= \forall x A^{\neg_R} & (\exists x A)^{\neg_R} &:= \neg_R \neg_R (\exists x A^{\neg_R}) \end{aligned}$$

Note that $(\neg A)^{\neg_R} = \neg_R A^{\neg_R}$. We gather here the basic properties of the parametrized translation.

Proposition 4. In intuitionistic logic,

- (i) $\vdash (A \vee \neg A)^{\neg_R}$
- (ii) $R \vdash A^{\neg_R}$
- (iii) $\neg_R \neg_R A^{\neg_R} \vdash A^{\neg_R}$

Proof. (i) We proof this property by induction. We will use intensively these lemmas:

$$\frac{\frac{\frac{\overline{\neg A, A \vdash \neg A} \quad \overline{\neg A, A \vdash A}}{\neg A, A \vdash \perp}}{\neg A, A \vdash \mathbf{R}}}{(\Psi_1) : \neg A \vdash \neg_{\mathbf{R}} A}$$

$(\Psi_2) : A \vdash \neg_{\mathbf{R}} \neg_{\mathbf{R}} A$ (proposition 2 (ii))

$$\frac{\frac{\frac{\overline{\neg_{\mathbf{R}} A \vdash A \Rightarrow \mathbf{R}} \quad \overline{\neg_{\mathbf{R}} A \vdash A}}{\neg_{\mathbf{R}} A \vdash \mathbf{R}}}{(\Psi_3) : \vdash \neg_{\mathbf{R}} \neg_{\mathbf{R}} A}$$

We will often use this last one with the weakening.

- $A = \top$.

$$\frac{\frac{\frac{\overline{\perp \vdash \perp}}{\perp \vdash \mathbf{R}}}{\vdash (\neg_{\mathbf{R}} \perp)}}{\vdash (\neg_{\mathbf{R}} \top) \vee (\neg_{\mathbf{R}} \perp)} \quad \frac{}{\vdash (\top \vee \neg \top)^{\neg_{\mathbf{R}}}}$$

- $A = \perp$.

$$\frac{\frac{\frac{\overline{\perp \vdash \perp}}{\perp \vdash \mathbf{R}}}{\vdash (\neg_{\mathbf{R}} \perp)}}{\vdash (\perp \Rightarrow \mathbf{R}) \vee (\neg_{\mathbf{R}} \top)} \quad \frac{}{\vdash (\perp \vee \neg \perp)^{\neg_{\mathbf{R}}}}$$

- $A = (a \dot{=} b)$.

$$\frac{\frac{\text{decidability of QF formulas}}{\vdash (a \dot{=} b) \vee (\neg(a \dot{=} b))} \quad \frac{\Psi_1}{(a \dot{=} b) \vdash (a \dot{=} b) \quad \neg(a \dot{=} b) \vdash \neg_{\mathbf{R}}(a \dot{=} b)}}{(\Pi_3) : \vdash (a \dot{=} b) \vee (\neg_{\mathbf{R}}(a \dot{=} b))}$$

$$\frac{\frac{\frac{\Psi_2}{(a \dot{=} b) \vdash \neg_{\mathbf{R}} \neg_{\mathbf{R}}(a \dot{=} b)}}{(a \dot{=} b) \vdash \neg_{\mathbf{R}} \neg_{\mathbf{R}}(a \dot{=} b) \vee (\neg_{\mathbf{R}} \neg_{\mathbf{R}} \neg_{\mathbf{R}}(a \dot{=} b))} \quad \frac{\frac{\Psi_2}{\neg_{\mathbf{R}}(a \dot{=} b) \vdash \neg_{\mathbf{R}} \neg_{\mathbf{R}} \neg_{\mathbf{R}}(a \dot{=} b)}}{\neg_{\mathbf{R}}(a \dot{=} b) \vdash \neg_{\mathbf{R}} \neg_{\mathbf{R}}(a \dot{=} b) \vee (\neg_{\mathbf{R}} \neg_{\mathbf{R}} \neg_{\mathbf{R}}(a \dot{=} b))} \quad \Pi_3}{\vdash \neg_{\mathbf{R}} \neg_{\mathbf{R}}(a \dot{=} b) \vee (\neg_{\mathbf{R}} \neg_{\mathbf{R}} \neg_{\mathbf{R}}(a \dot{=} b))} \quad \frac{}{\vdash ((a \dot{=} b) \vee \neg(a \dot{=} b))^{\neg_{\mathbf{R}}}}$$

- $A = B \vee C$

$$IH_B : \vdash (B \vee \neg B)^{\neg_{\mathbf{R}}}$$

$$IH_C : \vdash (C \vee \neg C)^{\neg_{\mathbf{R}}}$$

$$\frac{\frac{\overline{B^{\neg_{\mathbf{R}}} \vee C^{\neg_{\mathbf{R}}} \vdash B^{\neg_{\mathbf{R}}} \vee C^{\neg_{\mathbf{R}}}}} \quad \frac{\overline{\neg_{\mathbf{R}} B^{\neg_{\mathbf{R}}}, \neg_{\mathbf{R}} C^{\neg_{\mathbf{R}}}, B^{\neg_{\mathbf{R}}} \vdash B^{\neg_{\mathbf{R}}}}} \quad \frac{\overline{\neg_{\mathbf{R}} B^{\neg_{\mathbf{R}}}, \neg_{\mathbf{R}} C^{\neg_{\mathbf{R}}}, C^{\neg_{\mathbf{R}}} \vdash C^{\neg_{\mathbf{R}}}}}{\neg_{\mathbf{R}} B^{\neg_{\mathbf{R}}}, \neg_{\mathbf{R}} C^{\neg_{\mathbf{R}}}, B^{\neg_{\mathbf{R}}} \vdash \mathbf{R}} \quad \neg_{\mathbf{R}} B^{\neg_{\mathbf{R}}}, \neg_{\mathbf{R}} C^{\neg_{\mathbf{R}}}, C^{\neg_{\mathbf{R}}} \vdash \mathbf{R}}}{(\Pi_6) : \neg_{\mathbf{R}} B^{\neg_{\mathbf{R}}}, \neg_{\mathbf{R}} C^{\neg_{\mathbf{R}}}, (B^{\neg_{\mathbf{R}}} \vee C^{\neg_{\mathbf{R}}}) \vdash \mathbf{R}}$$

$$\begin{array}{c}
\frac{IH_B}{\vdash (B \vee \neg B)^{\neg R}} \\
\frac{\vdash B^{\neg R} \vee \neg_R B^{\neg R} \quad \frac{\Pi_5}{B^{\neg R} \vdash (B^{\neg R} \wedge C^{\neg R}) \vee \neg_R (B^{\neg R} \wedge C^{\neg R})} \quad \Pi'_5}{\vdash (B^{\neg R} \wedge C^{\neg R}) \vee \neg_R (B^{\neg R} \wedge C^{\neg R})} \\
\frac{\Psi_3}{\vdash ((B \wedge C) \vee \neg(B \wedge C))^{\neg R}}
\end{array}$$

- $A = B \Rightarrow C$

$$(IH_B) : \vdash (B \vee \neg B)^{\neg R}$$

$$(IH_C) : \vdash (C \vee \neg C)^{\neg R}$$

We use here the property (ii) $R \vdash A^{\neg R}$. It is possible to do so because the proof of (ii) does not relies on the proof of (i).

$$\begin{array}{c}
\frac{\neg_R B^{\neg R}, \neg_R C^{\neg R}, B^{\neg R} \vdash B^{\neg R} \quad \neg_R B^{\neg R}, \neg_R C^{\neg R}, B^{\neg R} \vdash \neg_R B^{\neg R}}{(ii) : \neg_R B^{\neg R}, \neg_R C^{\neg R}, B^{\neg R} \vdash R} \\
\frac{\neg_R B^{\neg R}, \neg_R C^{\neg R}, B^{\neg R} \vdash C^{\neg R}}{\neg_R B^{\neg R}, \neg_R C^{\neg R} \vdash B^{\neg R} \Rightarrow C^{\neg R}} \\
(\Pi_8) : \neg_R B^{\neg R}, \neg_R C^{\neg R} \vdash (B \Rightarrow C)^{\neg R}
\end{array}$$

$$\begin{array}{c}
\frac{B^{\neg R}, \neg_R C^{\neg R}, B^{\neg R} \Rightarrow C^{\neg R} \vdash B^{\neg R} \quad B^{\neg R} \Rightarrow C^{\neg R} \vdash B^{\neg R} \Rightarrow C^{\neg R}}{B^{\neg R}, \neg_R C^{\neg R}, B^{\neg R} \Rightarrow C^{\neg R} \vdash C^{\neg R}} \quad \frac{\neg_R C^{\neg R} \vdash \neg_R C^{\neg R}}{\neg_R C^{\neg R} \vdash \neg_R C^{\neg R}} \\
\frac{B^{\neg R}, \neg_R C^{\neg R}, B^{\neg R} \Rightarrow C^{\neg R} \vdash R}{B^{\neg R}, \neg_R C^{\neg R} \vdash \neg_R (B^{\neg R} \Rightarrow C^{\neg R})} \\
(\Pi_7) : B^{\neg R}, \neg_R C^{\neg R} \vdash \neg_R (B \Rightarrow C)^{\neg R}
\end{array}$$

$$\frac{\Pi_8}{(\Pi'_8) : \neg_R B^{\neg R}, \neg_R C^{\neg R} \vdash (B \Rightarrow C)^{\neg R} \vee \neg_R (B \Rightarrow C)^{\neg R}}$$

$$\begin{array}{c}
\frac{IH_B}{\vdash (B \vee \neg B)^{\neg R}} \\
\frac{\vdash B^{\neg R} \vee \neg_R B^{\neg R} \quad \frac{\Pi_7}{B^{\neg R}, \neg_R C^{\neg R} \vdash (B \Rightarrow C)^{\neg R} \vee \neg_R (B \Rightarrow C)^{\neg R}} \quad \Pi'_8}{(\Pi_6) : \neg_R C^{\neg R} \vdash (B \Rightarrow C)^{\neg R} \vee \neg_R (B \Rightarrow C)^{\neg R}}
\end{array}$$

$$\begin{array}{c}
\frac{IH_C}{\vdash (C \vee \neg C)^{\neg R}} \quad \frac{\frac{C^{\neg R}, B^{\neg R} \vdash C^{\neg R}}{C^{\neg R} \vdash B^{\neg R} \Rightarrow C^{\neg R}}}{C^{\neg R} \vdash (B \Rightarrow C)^{\neg R}} \quad \frac{\Pi_6}{\neg_R C^{\neg R} \vdash (B \Rightarrow C)^{\neg R} \vee \neg_R (B \Rightarrow C)^{\neg R}} \\
\frac{\vdash (B \Rightarrow C)^{\neg R} \vee \neg_R (B \Rightarrow C)^{\neg R}}{\vdash \neg_R \neg_R ((B \Rightarrow C)^{\neg R} \vee \neg_R (B \Rightarrow C)^{\neg R})} \\
\vdash ((B \Rightarrow C) \vee \neg(B \Rightarrow C))^{\neg R}
\end{array}$$

- $A = \forall x B$

$$IH_B : \vdash (B \vee \neg B)^{\neg R}$$

$$\begin{array}{c}
\frac{IH_B}{\vdash (B \vee \neg_R B)^{\neg_R}} \quad \frac{\overline{B^{\neg_R} \vdash B^{\neg_R}}}{\overline{B^{\neg_R} \vdash \forall x B^{\neg_R}}} \quad \frac{\overline{\forall x B^{\neg_R} \vdash B^{\neg_R}} \quad \overline{\neg_R B^{\neg_R} \vdash \neg_R B^{\neg_R}}}{\overline{\neg_R B^{\neg_R}, \forall x B^{\neg_R} \vdash R}} \\
\frac{\vdash B^{\neg_R} \vee \neg_R B^{\neg_R}}{B^{\neg_R} \vdash (\forall x B)^{\neg_R} \vee \neg_R (\forall x B)^{\neg_R}} \quad \frac{\overline{\neg_R B^{\neg_R} \vdash \neg_R (\forall x B^{\neg_R})}}{\neg_R B^{\neg_R} \vdash (\forall x B)^{\neg_R} \vee \neg_R (\forall x B)^{\neg_R}} \\
\hline
\vdash (\forall x B)^{\neg_R} \vee \neg_R (\forall x B)^{\neg_R} \\
\vdash \neg_R \neg_R ((\forall x B)^{\neg_R} \vee \neg_R (\forall x B)^{\neg_R}) \\
\vdash ((\forall x B) \vee \neg (\forall x B))^{\neg_R}
\end{array}$$

- $A = \exists x B$
 $IH_B : \vdash (B \vee \neg B)^{\neg_R}$

$$\begin{array}{c}
\frac{\overline{\exists x B^{\neg_R} \vdash \exists x B^{\neg_R}}}{\exists x B^{\neg_R} \vdash \exists x B^{\neg_R}} \quad \frac{\overline{B^{\neg_R} \vdash B^{\neg_R}} \quad \overline{\neg_R B^{\neg_R}, \vdash \neg_R B^{\neg_R}}}{\overline{\neg_R B^{\neg_R}, \exists x B^{\neg_R} \vdash R}} \\
\frac{\overline{\neg_R B^{\neg_R}, \exists x B^{\neg_R} \vdash R}}{\neg_R B^{\neg_R} \vdash \neg_R \exists x B^{\neg_R}} \\
\frac{\neg_R B^{\neg_R} \vdash \neg_R \neg_R \neg_R (\exists x B^{\neg_R})}{(\Pi_{10}) : \neg_R B^{\neg_R} \vdash (\exists x B)^{\neg_R} \vee \neg_R (\exists x B)^{\neg_R}}
\end{array}$$

$$\begin{array}{c}
\frac{IH_B}{\vdash (B \vee \neg_R B)^{\neg_R}} \quad \frac{\overline{B^{\neg_R} \vdash B^{\neg_R}}}{\overline{B^{\neg_R} \vdash \exists x B^{\neg_R}}} \\
\frac{\vdash B^{\neg_R} \vee \neg_R B^{\neg_R}}{B^{\neg_R} \vdash (\exists x B)^{\neg_R} \vee \neg_R (\exists x B)^{\neg_R}} \quad \Pi_{10} \\
\hline
\vdash (\exists x B)^{\neg_R} \vee \neg_R (\exists x B)^{\neg_R} \\
\vdash \neg_R \neg_R ((\exists x B)^{\neg_R} \vee \neg_R (\exists x B)^{\neg_R}) \\
\vdash ((\exists x B) \vee \neg (\exists x B))^{\neg_R}
\end{array}$$

(ii) We proof this property by induction on A and we denote by IH the induction hypothesis.

- $A = \perp$

$$\overline{R \vdash R}$$

- $A = \top$

$$\overline{R \vdash \top}$$

- $A = (a \doteq b)$

$$\frac{\overline{R, \neg_R (a \doteq b) \vdash R}}{R \vdash (a \doteq b)^{\neg_R}}$$

- $A = B \wedge C$

$$\frac{\frac{IH}{R \vdash B^{\neg_R}} \quad \frac{IH}{R \vdash C^{\neg_R}}}{R \vdash B^{\neg_R} \wedge C^{\neg_R}}$$

- $A = B \Rightarrow C$

$$\frac{\frac{IH}{R, B^{\neg_R} \vdash C^{\neg_R}}}{R \vdash B^{\neg_R} \Rightarrow C^{\neg_R}}$$

- $A = B \vee C$

$$\frac{\overline{R, \neg_R (B^{\neg_R} \vee C^{\neg_R}) \vdash R}}{R \vdash \neg_R \neg_R (B^{\neg_R} \vee C^{\neg_R})}$$

- $A = \forall xB$

$$\frac{\frac{IH}{R \vdash B^{\neg R}}}{R \vdash \forall xB^{\neg R}}$$

- $A = \exists xB$

$$\frac{\frac{R, \neg R(\exists xB^{\neg R}) \vdash R}{R \vdash \neg R \neg R(\exists xB^{\neg R})}}$$

So, $R \vdash A^{\neg R}$

(iii) We proof this property by induction on A and we denote by IH the induction hypothesis.

- $A = \perp$

$$\frac{\frac{\neg R \neg R R \vdash R \Rightarrow R \Rightarrow R}{\neg R \neg R \perp^{\neg R} \vdash \perp^{\neg R}} \quad \frac{\frac{\neg R \neg R R, R \vdash R}{\neg R \neg R R \vdash R \Rightarrow R}}{\neg R \neg R \perp^{\neg R} \vdash \perp^{\neg R}}$$

- $A = \top$

$$\frac{\frac{\neg R \neg R \top^{\neg R} \vdash \top}{\neg R \neg R \top^{\neg R} \vdash \top^{\neg R}}}$$

- $A = (a \doteq b)$

$$\frac{\text{Proposition 2 (v)}}{\frac{\neg R \neg R \neg R(\neg R(a \doteq b)) \vdash \neg R(\neg R(a \doteq b))}{\neg R \neg R(a \doteq b)^{\neg R} \vdash (a \doteq b)^{\neg R}}}$$

- $A = B \wedge C$

$$\frac{\frac{\neg R \neg R(B \wedge C)^{\neg R}, \neg R B^{\neg R}, (B \wedge C)^{\neg R} \vdash \neg R B^{\neg R}}{\neg R \neg R(B \wedge C)^{\neg R}, \neg R B^{\neg R}, (B \wedge C)^{\neg R} \vdash B^{\neg R} \wedge C^{\neg R}} \quad \frac{\neg R \neg R(B \wedge C)^{\neg R}, \neg R B^{\neg R}, (B \wedge C)^{\neg R} \vdash B^{\neg R}}{(\Theta_2) : \neg R \neg R(B \wedge C)^{\neg R}, \neg R B^{\neg R} \vdash \neg R(B \wedge C)^{\neg R}}$$

$$\frac{\frac{\neg R \neg R(B \wedge C)^{\neg R}, \neg R B^{\neg R} \vdash \neg R \neg R(B \wedge C)^{\neg R}}{\neg R \neg R(B \wedge C)^{\neg R}, \neg R B^{\neg R} \vdash R} \quad \Theta_2}{(\Theta_3) : \neg R \neg R(B \wedge C)^{\neg R} \vdash \neg R \neg R B^{\neg R}}$$

$$\frac{\frac{IH_B}{\neg R \neg R(B \wedge C)^{\neg R}, \neg R \neg R B^{\neg R} \vdash B^{\neg R}}}{\Theta_3 \quad \frac{\neg R \neg R(B \wedge C)^{\neg R} \vdash \neg R \neg R B^{\neg R} \Rightarrow B^{\neg R}}{(\Theta_4) : \neg R \neg R(B \wedge C)^{\neg R} \vdash B^{\neg R}}}$$

$$\frac{\frac{\neg R \neg R(B \wedge C)^{\neg R}, \neg R C^{\neg R}, (B \wedge C)^{\neg R} \vdash \neg R C^{\neg R}}{\neg R \neg R(B \wedge C)^{\neg R}, \neg R C^{\neg R}, (B \wedge C)^{\neg R} \vdash B^{\neg R} \wedge C^{\neg R}} \quad \frac{\neg R \neg R(B \wedge C)^{\neg R}, \neg R C^{\neg R}, (B \wedge C)^{\neg R} \vdash B^{\neg R} \wedge C^{\neg R}}{\neg R \neg R(B \wedge C)^{\neg R}, \neg R C^{\neg R}, (B \wedge C)^{\neg R} \vdash C^{\neg R}}}{(\Theta_5) : \neg R \neg R(B \wedge C)^{\neg R}, \neg R C^{\neg R} \vdash \neg R(B \wedge C)^{\neg R}}$$

$$\frac{\frac{\neg_R \neg_R (B \wedge C)^{\neg_R}, \neg_R C^{\neg_R} \vdash \neg_R \neg_R (B \wedge C)^{\neg_R} \quad \Theta_5}{\neg_R \neg_R (B \wedge C)^{\neg_R}, \neg_R C^{\neg_R} \vdash R}}{(\Theta_6) : \neg_R \neg_R (B \wedge C)^{\neg_R} \vdash \neg_R \neg_R C^{\neg_R}}$$

$$\frac{\frac{IH_C}{\neg_R \neg_R (B \wedge C)^{\neg_R}, \neg_R \neg_R C^{\neg_R} \vdash C^{\neg_R}}}{\Theta_6 \quad \neg_R \neg_R (B \wedge C)^{\neg_R} \vdash \neg_R \neg_R C^{\neg_R} \Rightarrow C^{\neg_R}} \quad (\Theta_7) : \neg_R \neg_R (B \wedge C)^{\neg_R} \vdash C^{\neg_R}$$

$$\frac{\Theta_4 \quad \Theta_7}{\frac{\neg_R \neg_R (B \wedge C)^{\neg_R} \vdash B^{\neg_R} \wedge C^{\neg_R}}{\neg_R \neg_R (B \wedge C)^{\neg_R} \vdash (B \wedge C)^{\neg_R}}}$$

- $A = B \Rightarrow C$

$$\frac{\frac{todo}{\neg_R \neg_R (B \Rightarrow C)^{\neg_R}, B^{\neg_R} \vdash C^{\neg_R}}}{\neg_R \neg_R (B \Rightarrow C)^{\neg_R} \vdash B^{\neg_R} \Rightarrow C^{\neg_R}} \quad \neg_R \neg_R (B \Rightarrow C)^{\neg_R} \vdash (B \Rightarrow C)^{\neg_R}$$

- $A = B \vee C$

$$\frac{\text{Proposition 2 (v)}}{\frac{\neg_R \neg_R \neg_R (\neg_R (B^{\neg_R} \vee C^{\neg_R})) \vdash \neg_R (\neg_R (B^{\neg_R} \vee C^{\neg_R}))}{\neg_R \neg_R (B \vee C)^{\neg_R} \vdash (B \vee C)^{\neg_R}}}$$

- $A = \forall x B$

$$\frac{\frac{IH}{\neg_R \neg_R \forall x B^{\neg_R} \vdash B^{\neg_R}}}{\neg_R \neg_R \forall x B^{\neg_R} \vdash \forall x B^{\neg_R}}$$

- $A = \exists x B$

$$\frac{\text{Proposition 2 (v)}}{\frac{\neg_R \neg_R \neg_R (\neg_R (\exists x B^{\neg_R})) \vdash \neg_R (\neg_R (\exists x B^{\neg_R}))}{\neg_R \neg_R (\exists x B)^{\neg_R} \vdash (\exists x B)^{\neg_R}}}$$

So, $\neg_R \neg_R A^{\neg_R} \vdash A^{\neg_R}$

□

Theorem 5. If $\vdash A$ is derivable in classical predicate logic and if no free variable of R occurs in the derivation, then $\vdash A^{\neg_R}$ is derivable in intuitionistic predicate logic.

Proof. Every rules except excluded middle are the same so we just keep them (and we replace all expression X to X^{\neg_R}) in order to get a derivation of $\vdash A^{\neg_R}$ from a derivation of $\vdash A$. It works because there is no free occurrences of R .

For the excluded middle rule we rewrite:

$$\frac{\frac{\vdots}{\Gamma, \neg A \vdash \perp}}{\Gamma \vdash A}$$

to:

$$\frac{\frac{\text{proposition 4 (i)}}{\vdash (A \wedge \neg A)^{\neg_R}} \quad \frac{\frac{\vdots}{\Gamma, \neg_R A^{\neg_R} \vdash \perp}}{\Gamma, \neg_R A^{\neg_R} \vdash A^{\neg_R}}}{\Gamma \vdash (A \wedge \neg A)^{\neg_R} \quad \Gamma, A^{\neg_R} \vdash A^{\neg_R} \quad \Gamma, \neg_R A^{\neg_R} \vdash A^{\neg_R}} \quad \Gamma \vdash A^{\neg_R}$$

□

