FRIEDMAN's translation

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1 FRIEDMAN's Translation

Definition 1. *Let* R *be a formula. The* ***parametrized negation*** *is*

$$\neg_R := A \Rightarrow R$$

We gather here some basic properties of the parametrized negation

Proposition 2. *In intuitionisctic logic,*

(i) $B \Rightarrow \neg_R A \vdash A \Rightarrow \neg_R B$

(ii) $A \vdash \neg_R \neg_R A$

(iii) $A \Rightarrow B \vdash \neg_R B \Rightarrow \neg_R A$

(iv) $A \Rightarrow B \vdash \neg_R \neg_R A \Rightarrow \neg_R \neg_R B$

(v) $\neg_R \neg_R \neg_R A \vdash \neg_R A$

Proof. (i)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow \neg_{\mathtt{R}} A, A, B \vdash A} \quad \dfrac{\dfrac{}{B \Rightarrow \neg_{\mathtt{R}} A, A, B \vdash B} \quad \dfrac{}{B \Rightarrow \neg_{\mathtt{R}} A, A, B \vdash B \Rightarrow A \Rightarrow R}}{B \Rightarrow \neg_{\mathtt{R}} A, A, B \vdash A \Rightarrow \mathtt{R}}}{B \Rightarrow \neg_{\mathtt{R}} A, A, B \vdash \mathtt{R}}}{B \Rightarrow \neg_{\mathtt{R}} A, A \vdash \neg_{\mathtt{R}} B}}{B \Rightarrow \neg_{\mathtt{R}} A \vdash A \Rightarrow \neg_{\mathtt{R}} B}$$

(ii)

$$\dfrac{\dfrac{\dfrac{}{A, \neg_{\mathtt{R}} A \vdash A \Rightarrow \mathtt{R}} \quad \dfrac{}{A, \neg_{\mathtt{R}} A \vdash A}}{A, \neg_{\mathtt{R}} A \vdash \mathtt{R}}}{A \vdash \neg_{\mathtt{R}} \neg_{\mathtt{R}} A}$$

(iii)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}} B, A \vdash B \Rightarrow \mathtt{R}} \quad \dfrac{\dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}} B, A \vdash A \Rightarrow B} \quad \dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}} B, A \vdash A}}{A \Rightarrow B, \neg_{\mathtt{R}} B, A \vdash B}}{A \Rightarrow B, \neg_{\mathtt{R}} B, A \vdash \mathtt{R}}}{A \Rightarrow B, \neg_{\mathtt{R}} B \vdash \neg_{\mathtt{R}} A}}{A \Rightarrow B \vdash \neg_{\mathtt{R}} B \Rightarrow \neg_{\mathtt{R}} A}$$

(iv)

$$\dfrac{\dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash A \Rightarrow B} \quad \dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash A}}{(\Pi_0) : A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash B}$$

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash B \Rightarrow \mathtt{R}} \quad \dfrac{\Pi_0}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash B}}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash \mathtt{R}}}{(\Pi_1) : A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B \vdash A \Rightarrow \mathtt{R}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B \vdash A \Rightarrow \mathtt{R} \Rightarrow \mathtt{R}} \quad \dfrac{\Pi_1}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B \vdash A \Rightarrow \mathtt{R}}}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B \vdash \mathtt{R}}}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}B}}{A \Rightarrow B \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}A \Rightarrow \neg_{\mathtt{R}}\neg_{\mathtt{R}}B}$$

(v)

$$\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A, A \Rightarrow \mathtt{R} \vdash A \Rightarrow \mathtt{R}} \quad \dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A, A \Rightarrow \mathtt{R} \vdash A}}{(\Pi_2) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A, A \Rightarrow \mathtt{R} \vdash \mathtt{R}}$$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A \vdash A \Rightarrow \mathtt{R} \Rightarrow \mathtt{R} \Rightarrow \mathtt{R}} \quad \dfrac{\dfrac{\Pi_2}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A, A \Rightarrow \mathtt{R} \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A \vdash A \Rightarrow \mathtt{R} \Rightarrow \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A \vdash \neg_{\mathtt{R}}A}$$

$\square$

We now define the parametrized translation.

Definition 3. *Let R be a formula. The* ***parametrized negative translation*** *$A^{\neg_R}$ is defined by induction on A as follows:*

$$\begin{array}{c}
\bot^{\neg_R} := R \qquad \top^{\neg_R} := \top \qquad (a \dot{=} b)^{\neg_R} := \neg_R\neg_R(a \dot{=} b) \\
(A \wedge B)^{\neg_R} := A^{\neg_R} \wedge B^{\neg_R} \qquad (A \Rightarrow B)^{\neg_R} := A^{\neg_R} \Rightarrow B^{\neg_R} \\
(A \vee B)^{\neg_R} := \neg_R\neg_R(A^{\neg_R} \vee B^{\neg_R}) \\
\forall x A)^{\neg_R} := \forall x A^{\neg_R} \qquad \exists x A^{\neg_R} := \neg_R\neg_R(\exists x A^{\neg_R})
\end{array}$$

Note that $(\neg A)^{\neg_{\mathtt{R}}} = \neg_{\mathtt{R}} A^{\neg_{\mathtt{R}}}$. We gather here the basic properties of the parametrized translation.

Proposition 4. *In intuitionistic logic,*

(i) $\vdash (A \vee \neg A)^{\neg_R}$

(ii) $R \vdash A^{\neg_R}$

(iii) $\neg_R\neg_R A^{\neg_R} \vdash A^{\neg_R}$

Proof. (i) We proof this property by induction. We will use intensively these lemmas:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A, A \vdash \neg A} \quad \dfrac{}{\neg A, A \vdash A}}{\neg A, A \vdash \bot}}{\neg A, A \vdash \mathrm{R}}}{(\Psi_1) : \neg A \vdash \neg_{\mathrm{R}} A}$$

$$\dfrac{\dfrac{\dfrac{}{A, \neg_{\mathrm{R}} A \vdash A \Rightarrow \mathrm{R}} \quad \dfrac{}{A, \neg_{\mathrm{R}} A \vdash A}}{A, \neg_{\mathrm{R}} A \vdash \mathrm{R}}}{(\Psi_2) : A \vdash \neg_{\mathrm{R}} \neg_{\mathrm{R}} A}$$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathrm{R}} A \vdash A \Rightarrow \mathrm{R}} \quad \dfrac{\vdots}{\neg_{\mathrm{R}} A \vdash A}}{\neg_{\mathrm{R}} A \vdash \mathrm{R}}}{(\Psi_3) : \vdash \neg_{\mathrm{R}} \neg_{\mathrm{R}} A}$$

We will often use this last one with the weakening

- $A = \top$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\bot \vdash \bot}}{\bot \vdash \mathrm{R}}}{\vdash (\neg_{\mathrm{R}} \bot)}}{\vdash (\neg_{\mathrm{R}} \top) \vee (\neg_{\mathrm{R}} \bot)}}{\vdash (\top \vee \neg \top)^{\neg_{\mathrm{R}}}}$$

- $A = \bot$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\bot \vdash \bot}}{\bot \vdash \mathrm{R}}}{\vdash (\neg_{\mathrm{R}} \bot)}}{\vdash (\bot \Rightarrow \mathrm{R}) \vee (\neg_{\mathrm{R}} \top)}}{\vdash (\bot \vee \neg \bot)^{\neg_{\mathrm{R}}}}$$

- $A = (a \doteq b)$.

$$\dfrac{\dfrac{\text{decidability of QF formulas}}{\vdash (a \doteq b) \vee (\neg (a \doteq b))} \quad \dfrac{}{(a \doteq b) \vdash (a \doteq b)} \quad \dfrac{\Psi_1}{\neg (a \doteq b) \vdash \neg_{\mathrm{R}} (a \doteq b)}}{(\Pi_3) : \vdash (a \doteq b) \vee (\neg_{\mathrm{R}} (a \doteq b))}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Psi_2}{(a \doteq b) \vdash \neg_{\mathrm{R}} \neg_{\mathrm{R}} (a \doteq b)}}{(a \doteq b) \vdash \neg_{\mathrm{R}} \neg_{\mathrm{R}} (a \doteq b) \vee (\neg_{\mathrm{R}} \neg_{\mathrm{R}} \neg_{\mathrm{R}} (a \doteq b))} \quad \dfrac{\dfrac{\Psi_2}{\neg_{\mathrm{R}} (a \doteq b) \vdash \neg_{\mathrm{R}} \neg_{\mathrm{R}} \neg_{\mathrm{R}} (a \doteq b)}}{\neg_{\mathrm{R}} (a \doteq b) \vdash \neg_{\mathrm{R}} \neg_{\mathrm{R}} (a \doteq b) \vee (\neg_{\mathrm{R}} \neg_{\mathrm{R}} \neg_{\mathrm{R}} (a \doteq b))} \quad \Pi_3}{\vdash \neg_{\mathrm{R}} \neg_{\mathrm{R}} (a \doteq b) \vee (\neg_{\mathrm{R}} \neg_{\mathrm{R}} \neg_{\mathrm{R}} (a \doteq b))}}{\vdash ((a \doteq b) \vee \neg (a \doteq b))^{\neg_{\mathrm{R}}}}$$

- $A = B \vee C$
  $IH_B : \vdash (B \vee \neg B)^{\neg_{\mathrm{R}}}$
  $IH_C : \vdash (C \vee \neg C)^{\neg_{\mathrm{R}}}$

$$\dfrac{\dfrac{}{B^{\neg_{\mathrm{R}}} \vee C^{\neg_{\mathrm{R}}} \vdash B^{\neg_{\mathrm{R}}} \vee C^{\neg_{\mathrm{R}}}} \quad \dfrac{\dfrac{}{\neg_{\mathrm{R}} B^{\neg_{\mathrm{R}}}, \neg_{\mathrm{R}} C^{\neg_{\mathrm{R}}}, B^{\neg_{\mathrm{R}}} \vdash B^{\neg_{\mathrm{R}}}}}{\neg_{\mathrm{R}} B^{\neg_{\mathrm{R}}}, \neg_{\mathrm{R}} C^{\neg_{\mathrm{R}}}, B^{\neg_{\mathrm{R}}} \vdash \mathrm{R}} \quad \dfrac{\dfrac{}{\neg_{\mathrm{R}} B^{\neg_{\mathrm{R}}}, \neg_{\mathrm{R}} C^{\neg_{\mathrm{R}}}, C^{\neg_{\mathrm{R}}} \vdash C^{\neg_{\mathrm{R}}}}}{\neg_{\mathrm{R}} B^{\neg_{\mathrm{R}}}, \neg_{\mathrm{R}} C^{\neg_{\mathrm{R}}}, C^{\neg_{\mathrm{R}}} \vdash \mathrm{R}}}{(\Pi_6) : \neg_{\mathrm{R}} B^{\neg_{\mathrm{R}}}, \neg_{\mathrm{R}} C^{\neg_{\mathrm{R}}}, (B^{\neg_{\mathrm{R}}} \vee C^{\neg_{\mathrm{R}}}) \vdash \mathrm{R}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_6}{\neg_{\mathtt{R}}B^{\neg\mathtt{R}}, \neg_{\mathtt{R}}C^{\neg\mathtt{R}}, (B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash \mathtt{R}}}{\neg_{\mathtt{R}}B^{\neg\mathtt{R}}, \neg_{\mathtt{R}}C^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}}{\neg_{\mathtt{R}}B^{\neg\mathtt{R}}, \neg_{\mathtt{R}}C^{\neg\mathtt{R}}, \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash \mathtt{R}}}{\neg_{\mathtt{R}}B^{\neg\mathtt{R}}, \neg_{\mathtt{R}}C^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}}{(\Pi_7) : \neg_{\mathtt{R}}B^{\neg\mathtt{R}}, \neg_{\mathtt{R}}C^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash (C \vee \neg C)^{\neg\mathtt{R}}}}{\vdash (C \vee \neg_{\mathtt{R}} C)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}B^{\neg\mathtt{R}}, C^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash C^{\neg\mathtt{R}})}}{\neg_{\mathtt{R}}B^{\neg\mathtt{R}}, C^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash (B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}}{\Psi_3}}{\neg_{\mathtt{R}}B^{\neg\mathtt{R}}, C^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}}{\neg_{\mathtt{R}}B^{\neg\mathtt{R}}, C^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}} \quad \Pi_7}{(\Pi_8) : \neg_{\mathtt{R}}B^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}$$

$$\dfrac{\Pi_8}{(\Pi_9) : \neg_{\mathtt{R}}B^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash (B \vee \neg_{\mathtt{R}} B)^{\neg\mathtt{R}}}}{\vdash B^{\neg\mathtt{R}} \vee \neg B^{\neg\mathtt{R}}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash B^{\neg\mathtt{R}}}}{B^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}}}{\Psi_3}}{B^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}}{B^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})} \quad \Pi_9}{\vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}}{\vdash (B \vee C)^{\neg\mathtt{R}} \vee (\neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}})}$$

- $A = B \wedge C$
  $(IH_B) : \vdash (B \vee \neg B)^{\neg\mathtt{R}}$
  $(IH_C) : \vdash (C \vee \neg C)^{\neg\mathtt{R}}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B^{\neg\mathtt{R}}, B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}} \vdash B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}}}}{B^{\neg\mathtt{R}}, B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}} \vdash C^{\neg\mathtt{R}}}}{B^{\neg\mathtt{R}}, \neg C^{\neg\mathtt{R}}, B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}} \vdash \mathtt{R}}}{B^{\neg\mathtt{R}}, \neg C^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}})}}{(\Pi_4) : B^{\neg\mathtt{R}}, \neg_{\mathtt{R}}C^{\neg\mathtt{R}} \vdash (B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}})}$$

$$\dfrac{\dfrac{\dfrac{IH_C}{\vdash (C \vee \neg C)^{\neg\mathtt{R}}}}{\vdash (C^{\neg\mathtt{R}} \vee \neg_{\mathtt{R}} C^{\neg\mathtt{R}})} \quad \dfrac{\dfrac{\dfrac{}{B^{\neg\mathtt{R}}, C^{\neg\mathtt{R}} \vdash B^{\neg\mathtt{R}}} \quad \dfrac{}{B^{\neg\mathtt{R}}, C^{\neg\mathtt{R}} \vdash C^{\neg\mathtt{R}}}}{B^{\neg\mathtt{R}}, C^{\neg\mathtt{R}} \vdash B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}}}}{B^{\neg\mathtt{R}}, C^{\neg\mathtt{R}} \vdash (B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}})} \quad \Pi_4}{\Pi_5 : B^{\neg\mathtt{R}} \vdash (B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}})}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{IH_B}{\vdash (B \vee \neg B)^{\neg\mathtt{R}}}}{\vdash B^{\neg\mathtt{R}} \vee \neg_{\mathtt{R}} B^{\neg\mathtt{R}}} \quad \dfrac{\Pi_5}{B^{\neg\mathtt{R}} \vdash (B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}})} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}} \vdash B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}}}}{B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}} \vdash B^{\neg\mathtt{R}}}}{\neg B^{\neg\mathtt{R}}, B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}} \vdash \mathtt{R}}}{\neg B^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}})}}{\neg B^{\neg\mathtt{R}} \vdash (B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}})}}{\vdash (B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}}) \vee \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}})}}{\Psi_3}}{\vdash ((B \wedge C) \vee \neg(B \wedge C))^{\neg\mathtt{R}}}$$

- $A = B \Rightarrow C$
  $(IH_B) : \vdash (B \vee \neg B)^{\neg\mathtt{R}}$
  $(IH_C) : \vdash (C \vee \neg C)^{\neg\mathtt{R}}$

(ii) We proof this property by induction on A and we denote by IH the induction hypothesis.

- $A = \bot$

$$\dfrac{}{\mathtt{R} \vdash \mathtt{R}}$$

- $A = \top$

$$\dfrac{}{\mathtt{R} \vdash \top}$$

- $A = (a \doteq b)$

$$\dfrac{\dfrac{}{\mathtt{R}, \neg_{\mathtt{R}}(a \doteq b) \vdash \mathtt{R}}}{\mathtt{R} \vdash (a \doteq b)^{\neg\mathtt{R}}}$$

- $A = B \wedge C$

$$\dfrac{\dfrac{IH}{\mathtt{R} \vdash B^{\neg\mathtt{R}}} \quad \dfrac{IH}{\mathtt{R} \vdash C^{\neg\mathtt{R}}}}{\mathtt{R} \vdash B^{\neg\mathtt{R}} \wedge C^{\neg\mathtt{R}}}$$

- $A = B \Rightarrow C$

$$\dfrac{\dfrac{IH}{\mathtt{R}, B^{\neg\mathtt{R}} \vdash C^{\neg\mathtt{R}}}}{\mathtt{R} \vdash B^{\neg\mathtt{R}} \Rightarrow C^{\neg\mathtt{R}}}$$

- $A = B \vee C$

$$\dfrac{\dfrac{}{\mathtt{R}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash \mathtt{R}}}{\mathtt{R} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}$$

- $A = \forall x B$

$$\dfrac{\dfrac{IH}{\mathtt{R} \vdash B^{\neg\mathtt{R}}}}{\mathtt{R} \vdash \forall x B^{\neg\mathtt{R}}}$$

- $A = \exists x B$

$$\dfrac{\dfrac{}{\mathtt{R}, \neg_{\mathtt{R}}(\exists x B^{\neg\mathtt{R}}) \vdash \mathtt{R}}}{\mathtt{R} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(\exists x B^{\neg\mathtt{R}})}$$

So, $\mathtt{R} \vdash A^{\neg\mathtt{R}}$

(iii) We proof this property by induction on A and we denote by IH the induction hypothesis.

- $A = \bot$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\mathtt{R} \vdash \mathtt{R} \Rightarrow \mathtt{R} \Rightarrow \mathtt{R}} \quad \dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\mathtt{R}, \mathtt{R} \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\mathtt{R} \vdash \mathtt{R} \Rightarrow \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\mathtt{R} \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\bot^{\neg\mathtt{R}} \vdash \bot^{\neg\mathtt{R}}}$$

- $A = \top$

$$\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\top^{\neg\mathtt{R}} \vdash \top}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\top^{\neg\mathtt{R}} \vdash \top^{\neg\mathtt{R}}}$$

- $A = (a \doteq b)$

$$\frac{\frac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}(a \doteq b), \neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b)} \quad \frac{}{\neg_{\mathtt{R}}(a \doteq b), \neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b) \vdash \neg_{\mathtt{R}}(a \doteq b)}}{(\Theta_1) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}(a \doteq b), \neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b) \vdash \mathtt{R}}$$

$$\frac{\frac{\frac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}(a \doteq b) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b)} \quad \frac{\Theta_1}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}(a \doteq b) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b)}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}(a \doteq b) \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b)^{\neg \mathtt{R}} \vdash (a \doteq b)^{\neg \mathtt{R}}}$$

- $A = B \wedge C$

$$\frac{\frac{\frac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}B^{\neg \mathtt{R}}, (B \wedge C)^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}}B^{\neg \mathtt{R}}} \quad \frac{\frac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}B^{\neg \mathtt{R}}, (B \wedge C)^{\neg \mathtt{R}} \vdash B^{\neg \mathtt{R}} \wedge C^{\neg \mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}B^{\neg \mathtt{R}}, (B \wedge C)^{\neg \mathtt{R}} \vdash B^{\neg \mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}B^{\neg \mathtt{R}}, (B \wedge C)^{\neg \mathtt{R}} \vdash \mathtt{R}}}{(\Theta_2) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}B^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}}$$

$$\frac{\frac{\frac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}B^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}} \quad \Theta_2}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}B^{\neg \mathtt{R}} \vdash \mathtt{R}}}{(\Theta_3) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}B^{\neg \mathtt{R}}}$$

$$\frac{\Theta_3 \quad \frac{\frac{IH_B}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}\neg_{\mathtt{R}}B^{\neg \mathtt{R}} \vdash B^{\neg \mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}B^{\neg \mathtt{R}} \Rightarrow B^{\neg \mathtt{R}}}}{(\Theta_4) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}} \vdash B^{\neg \mathtt{R}}}$$

$$\frac{\frac{\frac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}C^{\neg \mathtt{R}}, (B \wedge C)^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}}C^{\neg \mathtt{R}}} \quad \frac{\frac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}C^{\neg \mathtt{R}}, (B \wedge C)^{\neg \mathtt{R}} \vdash B^{\neg \mathtt{R}} \wedge C^{\neg \mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}C^{\neg \mathtt{R}}, (B \wedge C)^{\neg \mathtt{R}} \vdash C^{\neg \mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}C^{\neg \mathtt{R}}, (B \wedge C)^{\neg \mathtt{R}} \vdash \mathtt{R}}}{(\Theta_5) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}C^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}}$$

$$\frac{\frac{\frac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}C^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}} \quad \Theta_5}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}C^{\neg \mathtt{R}} \vdash \mathtt{R}}}{(\Theta_6) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}C^{\neg \mathtt{R}}}$$

$$\frac{\Theta_6 \quad \frac{\frac{IH_C}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}}, \neg_{\mathtt{R}}\neg_{\mathtt{R}}C^{\neg \mathtt{R}} \vdash C^{\neg \mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}C^{\neg \mathtt{R}} \Rightarrow C^{\neg \mathtt{R}}}}{(\Theta_7) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}} \vdash C^{\neg \mathtt{R}}}$$

$$\frac{\frac{\Theta_4 \quad \Theta_7}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}} \vdash B^{\neg \mathtt{R}} \wedge C^{\neg \mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg \mathtt{R}} \vdash (B \wedge C)^{\neg \mathtt{R}}}$$

- $A = B \Rightarrow C$

$$\dfrac{\dfrac{\dfrac{todo}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \Rightarrow C)^{\neg\mathtt{R}}, B^{\neg\mathtt{R}} \vdash C^{\neg\mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \Rightarrow C)^{\neg\mathtt{R}} \vdash B^{\neg\mathtt{R}} \Rightarrow C^{\neg\mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \Rightarrow C)^{\neg\mathtt{R}} \vdash (B \Rightarrow C)^{\neg\mathtt{R}}}$$

- $A = B \vee C$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})} \quad \dfrac{}{B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}} \vdash B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}), (B \vee C)^{\neg\mathtt{R}}, B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}} \vdash \mathtt{R}}}{(\Theta_8) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}), (B \vee C)^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}$$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})} \quad \Theta_8}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}), (B \vee C)^{\neg\mathtt{R}} \vdash \mathtt{R}}}{(\Theta_9) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash \neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}}} \quad \Theta_9}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}}) \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg\mathtt{R}} \vee C^{\neg\mathtt{R}})}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \vee C)^{\neg\mathtt{R}} \vdash (B \vee C)^{\neg\mathtt{R}}}$$

- $A = \forall x B$

$$\dfrac{\dfrac{IH}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\forall x B^{\neg\mathtt{R}} \vdash B^{\neg\mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\forall x B^{\neg\mathtt{R}} \vdash \forall x B^{\neg\mathtt{R}}}$$

- $A = \exists x B$

$$\dfrac{\dfrac{\dfrac{}{(\exists x B)^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}\exists x B^{\neg\mathtt{R}}} \quad \dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}(\exists x B^{\neg\mathtt{R}}) \vdash \neg_{\mathtt{R}}(\exists x B^{\neg\mathtt{R}})} \quad \dfrac{}{\exists x B^{\neg\mathtt{R}} \vdash \exists x B^{\neg\mathtt{R}}}}{\neg_{\mathtt{R}}(\exists x B^{\neg\mathtt{R}}), (\exists x B)^{\neg\mathtt{R}}, \exists x B^{\neg\mathtt{R}} \vdash \mathtt{R}}}{\neg_{\mathtt{R}}(\exists x B^{\neg\mathtt{R}}), (\exists x B)^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\exists x B^{\neg\mathtt{R}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(\exists x B)^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(\exists x B^{\neg\mathtt{R}}), (\exists x B)^{\neg\mathtt{R}} \vdash \mathtt{R}}}{(\Theta_{10}) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(\exists x B)^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(\exists x B^{\neg\mathtt{R}}) \vdash \neg_{\mathtt{R}}(\exists x B)^{\neg\mathtt{R}}}$$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(\exists x B)^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(\exists x B)^{\neg\mathtt{R}}} \quad \Theta_{10}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(\exists x B)^{\neg\mathtt{R}}, \neg_{\mathtt{R}}(\exists x B^{\neg\mathtt{R}}) \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(\exists x B)^{\neg\mathtt{R}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(\exists x B^{\neg\mathtt{R}})}$$

So, $\neg_{\mathtt{R}}\neg_{\mathtt{R}}A^{\neg\mathtt{R}} \vdash A^{\neg\mathtt{R}}$

$\square$

Theorem 5. *If $\vdash A$ is derivable in classical predicate logic and if no free variable of $\mathtt{R}$ occurs in the derivation, then $\vdash A^{\neg\mathtt{R}}$ is derivable in intuitionisctic predicate logic.*

Proof. $\square$

Theorem 6. *If $PA \vdash A$ and if no free variable of $\mathtt{R}$ occurs in the derivation, then $HA \vdash A^{\neg\mathtt{R}}$.*

Proof. 1. Injectivity of S

$$\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y), \neg_{\mathtt{R}}x\doteq y, S(x)\doteq S(y) \vdash S(x)\doteq S(y)} \quad \dfrac{}{\vdash S(x)\doteq S(y) \Rightarrow x\doteq y}}{(\Xi_1) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y), \neg_{\mathtt{R}}x\doteq y, S(x)\doteq S(y) \vdash x\doteq y}$$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y), \neg_{\mathtt{R}}x\doteq y, S(x)\doteq S(y) \vdash \neg_{\mathtt{R}}x\doteq y} \quad \Xi_1}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y), \neg_{\mathtt{R}}x\doteq y, S(x)\doteq S(y) \vdash \mathtt{R}}}{(\Xi_2) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y), \neg_{\mathtt{R}}x\doteq y \vdash \neg_{\mathtt{R}}S(x)\doteq S(y)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y)} \quad \Xi_2}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y), \neg_{\mathtt{R}}x\doteq y \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}x\doteq y}}{\vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y) \Rightarrow \neg_{\mathtt{R}}\neg_{\mathtt{R}}x\doteq y}}{\vdash \forall xy(\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y) \Rightarrow \neg_{\mathtt{R}}\neg_{\mathtt{R}}x\doteq y)}}{\vdash (\forall xy(S(x)\doteq S(y) \Rightarrow x\doteq y))^{\neg_{\mathtt{R}}}}$$

2. Non confusion

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0 \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0, S(x)\doteq 0 \vdash S(x)\doteq 0} \quad \dfrac{\dfrac{}{\vdash \neg\forall x S(x)\doteq 0}}{\vdash \neg S(x)\doteq 0}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0, S(x)\doteq 0 \vdash \bot}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0, S(x)\doteq 0 \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0 \vdash \neg_{\mathtt{R}}S(x)\doteq 0}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0 \vdash \mathtt{R}}}{\vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0}}{\dfrac{\vdash \forall x\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0}{\vdash (\forall x\neg S(x)\doteq 0)^{\neg_{\mathtt{R}}}}}$$

3. Induction Scheme

$$\dfrac{\dfrac{}{\vdash A[0/x]^{\neg_{\mathtt{R}}} \Rightarrow \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \Rightarrow A^{\neg_{\mathtt{R}}}} \quad \dfrac{}{A[0/x]^{\neg_{\mathtt{R}}} \vdash A[0/x]^{\neg_{\mathtt{R}}}}}{(\Xi_3) : A[0/x]^{\neg_{\mathtt{R}}}, \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \vdash \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \Rightarrow A^{\neg_{\mathtt{R}}}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Xi_3 \quad \dfrac{}{\forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \vdash \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}})}}{A[0/x]^{\neg_{\mathtt{R}}}, \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \vdash A^{\neg_{\mathtt{R}}}}}{A[0/x]^{\neg_{\mathtt{R}}}, \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \vdash \forall x A^{\neg_{\mathtt{R}}}}}{A[0/x]^{\neg_{\mathtt{R}}} \vdash \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \Rightarrow \forall x A^{\neg_{\mathtt{R}}}}}{\dfrac{\vdash A[0/x]^{\neg_{\mathtt{R}}} \Rightarrow \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \Rightarrow \forall x A^{\neg_{\mathtt{R}}}}{\vdash (A[0/x] \Rightarrow \forall y(A[y/x] \Rightarrow A[S(y)/x]) \Rightarrow \forall x A)^{\neg_{\mathtt{R}}}}}$$

□

Theorem 7. *If PA $\vdash \forall x, \exists y : (a\doteq b)$ then HA $\vdash \forall x, \exists y : (a\doteq b)$.*

Proof. Let us write $F\forall x, G$ where G is a Σ_1^0 formula. By using $(\forall E)$, if F if provable with PA, so G too. As G is σ_1^0, G is provable with HA. By using $(\forall I)$, we deduce that F is provable with HA too. □