FRIEDMAN's translation

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1 FRIEDMAN's Translation

Definition 1. *Let R be a formula. The **parametrized negation** is*

$$\neg_R := A \Rightarrow R$$

We gather here some basic properties of the parametrized negation

Proposition 2. *In intuitionisctic logic,*

(i) $B \Rightarrow \neg_R A \vdash A \Rightarrow \neg_R B$

(ii) $A \vdash \neg_R \neg_R A$

(iii) $A \Rightarrow B \vdash \neg_R B \Rightarrow \neg_R A$

(iv) $A \Rightarrow B \vdash \neg_R \neg_R A \Rightarrow \neg_R \neg_R B$

(v) $\neg_R \neg_R \neg_R A \vdash \neg_R A$

Proof. (i)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow \neg_R A, A, B \vdash A} \quad \dfrac{\dfrac{}{B \Rightarrow \neg_R A, A, B \vdash B} \quad \dfrac{}{B \Rightarrow \neg_R A, A, B \vdash B \Rightarrow A \Rightarrow R}}{B \Rightarrow \neg_R A, A, B \vdash A \Rightarrow R}}{B \Rightarrow \neg_R A, A, B \vdash R}}{B \Rightarrow \neg_R A, A \vdash \neg_R B}}{B \Rightarrow \neg_R A \vdash A \Rightarrow \neg_R B}$$

(ii)

$$\dfrac{\dfrac{\dfrac{}{A, \neg_R A \vdash A \Rightarrow R} \quad \dfrac{}{A, \neg_R A \vdash A}}{A, \neg_R A \vdash R}}{A \vdash \neg_R \neg_R A}$$

(iii)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow B, \neg_R B, A \vdash B \Rightarrow R} \quad \dfrac{\dfrac{}{A \Rightarrow B, \neg_R B, A \vdash A \Rightarrow B} \quad \dfrac{}{A \Rightarrow B, \neg_R B, A \vdash A}}{A \Rightarrow B, \neg_R B, A \vdash B}}{A \Rightarrow B, \neg_R B, A \vdash R}}{A \Rightarrow B, \neg_R B \vdash \neg_R A}}{A \Rightarrow B \vdash \neg_R B \Rightarrow \neg_R A}$$

(iv)

$$\dfrac{\dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash A \Rightarrow B} \quad \dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash A}}{(\Pi_0) : A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash B}$$

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash B \Rightarrow \mathtt{R}} \quad \dfrac{\Pi_0}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash B}}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B, A \vdash \mathtt{R}}}{(\Pi_1) : A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B \vdash A \Rightarrow \mathtt{R}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B \vdash A \Rightarrow \mathtt{R} \Rightarrow \mathtt{R}} \quad \dfrac{\Pi_1}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B \vdash A \Rightarrow \mathtt{R}}}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A, \neg_{\mathtt{R}}B \vdash \mathtt{R}}}{A \Rightarrow B, \neg_{\mathtt{R}}\neg_{\mathtt{R}}A \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}B}}{A \Rightarrow B \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}A \Rightarrow \neg_{\mathtt{R}}\neg_{\mathtt{R}}B}$$

(v)

$$\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A, A \Rightarrow \mathtt{R} \vdash A \Rightarrow \mathtt{R}} \quad \dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A, A \Rightarrow \mathtt{R} \vdash A}}{(\Pi_2) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A, A \Rightarrow \mathtt{R} \vdash \mathtt{R}}$$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A \vdash A \Rightarrow \mathtt{R} \Rightarrow \mathtt{R} \Rightarrow \mathtt{R}} \quad \dfrac{\dfrac{\Pi_2}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A, A \Rightarrow \mathtt{R} \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A \vdash A \Rightarrow \mathtt{R} \Rightarrow \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A, A \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}A \vdash \neg_{\mathtt{R}}A}$$

□

We now define the parametrized translation.

Definition 3. *Let $\mathcal{R}$ be a formula. The* ***parametrized negative translation*** $A^{\neg_{\mathcal{R}}}$ *is defined by induction on A as follows:*

$$\begin{array}{c}
\bot^{\neg_{\mathcal{R}}} := \mathcal{R} \qquad \top^{\neg_{\mathcal{R}}} := \top \qquad (a \doteq b)^{\neg_{\mathcal{R}}} := \neg_{\mathcal{R}}\neg_{\mathcal{R}}(a \doteq b) \\
(A \wedge B)^{\neg_{\mathcal{R}}} := A^{\neg_{\mathcal{R}}} \wedge B^{\neg_{\mathcal{R}}} \qquad (A \Rightarrow B)^{\neg_{\mathcal{R}}} := A^{\neg_{\mathcal{R}}} \Rightarrow B^{\neg_{\mathcal{R}}} \\
(A \vee B)^{\neg_{\mathcal{R}}} := \neg_{\mathcal{R}}\neg_{\mathcal{R}}(A^{\neg_{\mathcal{R}}} \vee B^{\neg_{\mathcal{R}}}) \\
\forall x A)^{\neg_{\mathcal{R}}} := \forall x A^{\neg_{\mathcal{R}}} \qquad \exists x A^{\neg_{\mathcal{R}}} := \neg_{\mathcal{R}}\neg_{\mathcal{R}}(\exists x A^{\neg_{\mathcal{R}}})
\end{array}$$

Note that $(\neg A)^{\neg_{\mathtt{R}}} = \neg_{\mathtt{R}} A^{\neg_{\mathtt{R}}}$. We gather here the basic properties of the parametrized translation.

Proposition 4. *In intuitionistic logic,*

(i) $\vdash (A \vee \neg A)^{\neg_{\mathcal{R}}}$

(ii) $\mathcal{R} \vdash A^{\neg_{\mathcal{R}}}$

(iii) $\neg_{\mathcal{R}}\neg_{\mathcal{R}} A^{\neg_{\mathcal{R}}} \vdash A^{\neg_{\mathcal{R}}}$

Proof. (i) We proof this property by induction. We will use intensively these lemmas:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A, A \vdash \neg A} \qquad \dfrac{}{\neg A, A \vdash A}}{\neg A, A \vdash \bot}}{\neg A, A \vdash \mathtt{R}}}{(\Psi_1) : \neg A \vdash \neg_{\mathtt{R}} A}$$

$(\Psi_2) : A \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}} A$ (proposition 2 *(ii)*)

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}} A \vdash A \Rightarrow \mathtt{R}} \qquad \dfrac{\vdots}{\neg_{\mathtt{R}} A \vdash A}}{\neg_{\mathtt{R}} A \vdash \mathtt{R}}}{(\Psi_3) : \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}} A}$$

We will often use this last one with the weakening.

- $A = \top$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\bot \vdash \bot}}{\bot \vdash \mathtt{R}}}{\vdash (\neg_{\mathtt{R}} \bot)}}{\dfrac{\vdash (\neg_{\mathtt{R}} \top) \vee (\neg_{\mathtt{R}} \bot)}{\vdash (\top \vee \neg \top)^{\neg_{\mathtt{R}}}}}$$

- $A = \bot$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\bot \vdash \bot}}{\bot \vdash \mathtt{R}}}{\vdash (\neg_{\mathtt{R}} \bot)}}{\dfrac{\vdash (\bot \Rightarrow \mathtt{R}) \vee (\neg_{\mathtt{R}} \top)}{\vdash (\bot \vee \neg \bot)^{\neg_{\mathtt{R}}}}}$$

- $A = (a \doteq b)$.

$$\dfrac{\dfrac{\text{decidability of QF formulas}}{\vdash (a \doteq b) \vee (\neg (a \doteq b))} \qquad \dfrac{}{(a \doteq b) \vdash (a \doteq b)} \qquad \dfrac{\Psi_1}{\neg (a \doteq b) \vdash \neg_{\mathtt{R}} (a \doteq b)}}{(\Pi_3) : \vdash (a \doteq b) \vee (\neg_{\mathtt{R}} (a \doteq b))}$$

$$\dfrac{\dfrac{\dfrac{\Psi_2}{(a \doteq b) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}} (a \doteq b)}}{(a \doteq b) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}} (a \doteq b) \vee (\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}} (a \doteq b))} \quad \dfrac{\dfrac{\Psi_2}{\neg_{\mathtt{R}} (a \doteq b) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}} (a \doteq b)}}{\neg_{\mathtt{R}} (a \doteq b) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}} (a \doteq b) \vee (\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}} (a \doteq b))} \quad \Pi_3}{\dfrac{\vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}} (a \doteq b) \vee (\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}} (a \doteq b))}{\vdash ((a \doteq b) \vee \neg (a \doteq b))^{\neg_{\mathtt{R}}}}}$$

- $A = B \vee C$
  $IH_B : \vdash (B \vee \neg B)^{\neg_{\mathtt{R}}}$
  $IH_C : \vdash (C \vee \neg C)^{\neg_{\mathtt{R}}}$

$$\dfrac{\dfrac{}{B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}} \vdash B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}} \quad \dfrac{\dfrac{}{\neg_{\mathtt{R}} B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}} C^{\neg_{\mathtt{R}}}, B^{\neg_{\mathtt{R}}} \vdash B^{\neg_{\mathtt{R}}}}}{\neg_{\mathtt{R}} B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}} C^{\neg_{\mathtt{R}}}, B^{\neg_{\mathtt{R}}} \vdash \mathtt{R}} \quad \dfrac{\dfrac{}{\neg_{\mathtt{R}} B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}} C^{\neg_{\mathtt{R}}}, C^{\neg_{\mathtt{R}}} \vdash C^{\neg_{\mathtt{R}}}}}{\neg_{\mathtt{R}} B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}} C^{\neg_{\mathtt{R}}}, C^{\neg_{\mathtt{R}}} \vdash \mathtt{R}}}{(\Pi_6) : \neg_{\mathtt{R}} B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}} C^{\neg_{\mathtt{R}}}, (B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vdash \mathtt{R}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_6}{\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}}, (B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vdash \mathtt{R}}}{\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}})}}{\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vdash \mathtt{R}}}{\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}})}}{(\Pi_7) : \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}})}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash (C \vee \neg C)^{\neg_{\mathtt{R}}}}}{\vdash (C \vee \neg_{\mathtt{R}} C)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, C^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vdash C^{\neg_{\mathtt{R}}})}}{\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, C^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vdash (B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}})}}{\Psi_3}}{\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, C^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}})}}{\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, C^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}} \quad \Pi_7}{(\Pi_8) : \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}})}$$

$$\dfrac{\Pi_8}{(\Pi_9) : \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}})}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash (B \vee \neg_{\mathtt{R}} B)^{\neg_{\mathtt{R}}}}}{\vdash B^{\neg_{\mathtt{R}}} \vee \neg B^{\neg_{\mathtt{R}}}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vdash B^{\neg_{\mathtt{R}}}}}{B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vdash B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}}}{\Psi_3}}{B^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}})}}{B^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}})} \quad \Pi_9}{\vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}}) \vee \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \vee C^{\neg_{\mathtt{R}}})}}{\vdash (B \vee C)^{\neg_{\mathtt{R}}} \vee (\neg_{\mathtt{R}}(B \vee C)^{\neg_{\mathtt{R}}})}}{\vdash ((B \vee C) \vee (\neg_{\mathtt{R}}(B \vee C))^{\neg_{\mathtt{R}}}}$$

- $A = B \wedge C$
  $(IH_B) : \vdash (B \vee \neg B)^{\neg_{\mathtt{R}}}$
  $(IH_C) : \vdash (C \vee \neg C)^{\neg_{\mathtt{R}}}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B^{\neg_{\mathtt{R}}}, B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}} \vdash B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}}}}{B^{\neg_{\mathtt{R}}}, B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}} \vdash C^{\neg_{\mathtt{R}}}}}{B^{\neg_{\mathtt{R}}}, \neg C^{\neg_{\mathtt{R}}}, B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}} \vdash \mathtt{R}}}{B^{\neg_{\mathtt{R}}}, \neg C^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}})}}{(\Pi_4) : B^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}} \vdash (B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}}) \vee \neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}})}$$

$$\dfrac{\dfrac{\dfrac{IH_C}{\vdash (C \vee \neg C)^{\neg_{\mathtt{R}}}}}{\vdash (C^{\neg_{\mathtt{R}}} \vee \neg_{\mathtt{R}} C^{\neg_{\mathtt{R}}})} \quad \dfrac{\dfrac{\dfrac{}{B^{\neg_{\mathtt{R}}}, C^{\neg_{\mathtt{R}}} \vdash B^{\neg_{\mathtt{R}}}} \quad \dfrac{}{B^{\neg_{\mathtt{R}}}, C^{\neg_{\mathtt{R}}} \vdash C^{\neg_{\mathtt{R}}}}}{B^{\neg_{\mathtt{R}}}, C^{\neg_{\mathtt{R}}} \vdash B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}}}}{B^{\neg_{\mathtt{R}}}, C^{\neg_{\mathtt{R}}} \vdash (B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}}) \vee \neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}})} \quad \Pi_4}{\Pi_5 : B^{\neg_{\mathtt{R}}} \vdash (B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}}) \vee \neg_{\mathtt{R}}(B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}})}$$

$$
\begin{prooftree}
\AxiomC{IH_B}
\UnaryInfC{$\vdash (B \vee \neg B)^{\neg \mathrm{R}}$}
\UnaryInfC{$\vdash B^{\neg \mathrm{R}} \vee \neg_{\mathrm{R}} B^{\neg \mathrm{R}}$}
\AxiomC{Π_5}
\noLine
\UnaryInfC{$B^{\neg \mathrm{R}} \vdash (B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}}) \vee \neg_{\mathrm{R}}(B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}})$}
\AxiomC{}
\UnaryInfC{$B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}} \vdash B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}}$}
\UnaryInfC{$B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}} \vdash B^{\neg \mathrm{R}}$}
\UnaryInfC{$\neg_{\mathrm{R}} B^{\neg \mathrm{R}}, B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}} \vdash \mathrm{R}$}
\UnaryInfC{$\neg_{\mathrm{R}} B^{\neg \mathrm{R}} \vdash \neg_{\mathrm{R}}(B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}})$}
\UnaryInfC{$\neg_{\mathrm{R}} B^{\neg \mathrm{R}} \vdash (B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}}) \vee \neg_{\mathrm{R}}(B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}})$}
\TrinaryInfC{$\vdash (B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}}) \vee \neg_{\mathrm{R}}(B^{\neg \mathrm{R}} \wedge C^{\neg \mathrm{R}})$}
\UnaryInfC{Ψ_3}
\UnaryInfC{$\vdash ((B \wedge C) \vee \neg(B \wedge C))^{\neg \mathrm{R}}$}
\end{prooftree}
$$

- $A = B \Rightarrow C$
  $(IH_B) : \vdash (B \vee \neg B)^{\neg \mathrm{R}}$
  $(IH_C) : \vdash (C \vee \neg C)^{\neg \mathrm{R}}$
  We use here the property *(ii)* $\mathrm{R} \vdash A^{\neg \mathrm{R}}$. It is possible to do so because the proof of *(ii)* does not relies on the proof of *(i)*.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\neg_{\mathrm{R}} B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}}, B^{\neg \mathrm{R}} \vdash B^{\neg \mathrm{R}}$}
\AxiomC{}
\UnaryInfC{$\neg_{\mathrm{R}} B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}}, B^{\neg \mathrm{R}} \vdash \neg_{\mathrm{R}} B^{\neg \mathrm{R}}$}
\BinaryInfC{$(ii) : \neg_{\mathrm{R}} B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}}, B^{\neg \mathrm{R}} \vdash R$}
\UnaryInfC{$\neg_{\mathrm{R}} B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}}, B^{\neg \mathrm{R}} \vdash C^{\neg \mathrm{R}}$}
\UnaryInfC{$\neg_{\mathrm{R}} B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}} \vdash B^{\neg \mathrm{R}} \Rightarrow C^{\neg \mathrm{R}}$}
\UnaryInfC{$(Pi_8) : \neg_{\mathrm{R}} B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}} \vdash (B \Rightarrow C)^{\neg \mathrm{R}}$}
\end{prooftree}
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$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}}, B^{\neg \mathrm{R}} \Rightarrow C^{\neg \mathrm{R}} \vdash B^{\neg \mathrm{R}}$}
\AxiomC{}
\UnaryInfC{$B^{\neg \mathrm{R}} \Rightarrow C^{\neg \mathrm{R}} \vdash B^{\neg \mathrm{R}} \Rightarrow C^{\neg \mathrm{R}}$}
\BinaryInfC{$B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}}, B^{\neg \mathrm{R}} \Rightarrow C^{\neg \mathrm{R}} \vdash C^{\neg \mathrm{R}}$}
\AxiomC{}
\UnaryInfC{$\neg_{\mathrm{R}} C^{\neg \mathrm{R}} \vdash \neg_{\mathrm{R}} C^{\neg \mathrm{R}}$}
\BinaryInfC{$B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}}, B^{\neg \mathrm{R}} \Rightarrow C^{\neg \mathrm{R}} \vdash R$}
\UnaryInfC{$B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}} \vdash \neg_{\mathrm{R}}(B^{\neg \mathrm{R}} \Rightarrow C^{\neg \mathrm{R}})$}
\UnaryInfC{$(\Pi_7) : B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}} \vdash \neg_{\mathrm{R}}(B \Rightarrow C)^{\neg \mathrm{R}}$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{IH_B}
\UnaryInfC{$\vdash (B \vee \neg B)^{\neg \mathrm{R}}$}
\UnaryInfC{$\vdash B^{\neg \mathrm{R}} \vee \neg_{\mathrm{R}} B^{\neg \mathrm{R}}$}
\AxiomC{Π_7}
\noLine
\UnaryInfC{$B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}} \vdash (B \Rightarrow C)^{\neg \mathrm{R}} \vee \neg_{\mathrm{R}}(B \Rightarrow C)^{\neg \mathrm{R}}$}
\AxiomC{Π_8}
\noLine
\UnaryInfC{$\neg_{\mathrm{R}} B^{\neg \mathrm{R}}, \neg_{\mathrm{R}} C^{\neg \mathrm{R}} \vdash (B \Rightarrow C)^{\neg \mathrm{R}} \vee \neg_{\mathrm{R}}(B \Rightarrow C)^{\neg \mathrm{R}}$}
\TrinaryInfC{$(\Pi_6) : \neg_{\mathrm{R}} C^{\neg \mathrm{R}} \vdash (B \Rightarrow C)^{\neg \mathrm{R}} \vee \neg_{\mathrm{R}}(B \Rightarrow C)^{\neg \mathrm{R}}$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{IH_C}
\UnaryInfC{$\vdash (C \vee \neg C)^{\neg \mathrm{R}}$}
\UnaryInfC{$\vdash C^{\neg \mathrm{R}} \vee \neg_{\mathrm{R}} C^{\neg \mathrm{R}}$}
\AxiomC{}
\UnaryInfC{$C^{\neg \mathrm{R}}, B^{\neg \mathrm{R}} \vdash C^{\neg \mathrm{R}}$}
\UnaryInfC{$C^{\neg \mathrm{R}} \vdash B^{\neg \mathrm{R}} \Rightarrow C^{\neg \mathrm{R}}$}
\UnaryInfC{$C^{\neg \mathrm{R}} \vdash (B \Rightarrow C)^{\neg \mathrm{R}}$}
\UnaryInfC{$C^{\neg \mathrm{R}} \vdash (B \Rightarrow C)^{\neg \mathrm{R}} \vee \neg_{\mathrm{R}}(B \Rightarrow C)^{\neg \mathrm{R}}$}
\AxiomC{Π_6}
\noLine
\UnaryInfC{$\neg_{\mathrm{R}} C^{\neg \mathrm{R}} \vdash (B \Rightarrow C)^{\neg \mathrm{R}} \vee \neg_{\mathrm{R}}(B \Rightarrow C)^{\neg \mathrm{R}}$}
\TrinaryInfC{$\vdash (B \Rightarrow C)^{\neg \mathrm{R}} \vee \neg_{\mathrm{R}}(B \Rightarrow C)^{\neg \mathrm{R}}$}
\UnaryInfC{$\vdash \neg_{\mathrm{R}} \neg_{\mathrm{R}}((B \Rightarrow C)^{\neg \mathrm{R}} \vee \neg_{\mathrm{R}}(B \Rightarrow C)^{\neg \mathrm{R}})$}
\UnaryInfC{$\vdash ((B \Rightarrow C) \vee \neg(B \Rightarrow C))^{\neg \mathrm{R}}$}
\end{prooftree}
$$

- $A = \forall x B$
  $IH_B : \vdash (B \vee \neg B)^{\neg \mathrm{R}}$

$$\dfrac{\dfrac{\dfrac{IH_B}{\vdash (B \vee \neg_{\mathtt{R}} B)^{\neg \mathtt{R}}}}{\vdash B^{\neg \mathtt{R}} \vee \neg_{\mathtt{R}} B^{\neg \mathtt{R}}} \quad \dfrac{\dfrac{\dfrac{}{B^{\neg \mathtt{R}} \vdash B^{\neg \mathtt{R}}}}{B^{\neg \mathtt{R}} \vdash \forall x B^{\neg \mathtt{R}}}}{B^{\neg \mathtt{R}} \vdash (\forall x B)^{\neg \mathtt{R}} \vee \neg_{\mathtt{R}} (\forall x B)^{\neg \mathtt{R}}} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\forall x B^{\neg \mathtt{R}} \vdash B^{\neg \mathtt{R}}} \quad \dfrac{}{\neg_{\mathtt{R}} B^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}} B^{\neg \mathtt{R}}}}{\neg_{\mathtt{R}} B^{\neg \mathtt{R}}, \forall x B^{\neg \mathtt{R}} \vdash R}}{\neg_{\mathtt{R}} B^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}} (\forall x B^{\neg \mathtt{R}})}}{\neg_{\mathtt{R}} B^{\neg \mathtt{R}} \vdash (\forall x B)^{\neg \mathtt{R}} \vee \neg_{\mathtt{R}} (\forall x B)^{\neg \mathtt{R}}}}{\dfrac{\vdash (\forall x B)^{\neg \mathtt{R}} \vee \neg_{\mathtt{R}} (\forall x B)^{\neg \mathtt{R}}}{\dfrac{\vdash \neg_{\mathtt{R}} \neg_{\mathtt{R}} ((\forall x B)^{\neg \mathtt{R}} \vee \neg_{\mathtt{R}} (\forall x B)^{\neg \mathtt{R}})}{\vdash ((\forall x B) \vee \neg (\forall x B))^{\neg \mathtt{R}}}}}$$

- $A = \exists x B$
  $IH_B : \vdash (B \vee \neg B)^{\neg \mathtt{R}}$

$$\dfrac{\dfrac{\dfrac{IH_B}{\vdash (B \vee \neg_{\mathtt{R}} B)^{\neg \mathtt{R}}}}{\vdash B^{\neg \mathtt{R}} \vee \neg_{\mathtt{R}} B^{\neg \mathtt{R}}} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{B^{\neg \mathtt{R}} \vdash B^{\neg \mathtt{R}}}}{B^{\neg \mathtt{R}} \vdash \exists x B^{\neg \mathtt{R}}}}{B^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}} \neg_{\mathtt{R}} (\exists x B^{\neg \mathtt{R}})}}{B^{\neg \mathtt{R}} \vdash (\exists x B)^{\neg \mathtt{R}} \vee \neg_{\mathtt{R}} (\exists x B)^{\neg \mathtt{R}})} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x B^{\neg \mathtt{R}} \vdash \exists x B^{\neg \mathtt{R}}} \quad \dfrac{\dfrac{}{B^{\neg \mathtt{R}} \vdash B^{\neg \mathtt{R}}} \quad \dfrac{}{\neg_{\mathtt{R}} B^{\neg \mathtt{R}}, \vdash \neg_{\mathtt{R}} B^{\neg \mathtt{R}}}}{\neg_{\mathtt{R}} B^{\neg \mathtt{R}}, B^{\neg \mathtt{R}} \vdash R}}{\neg_{\mathtt{R}} B^{\neg \mathtt{R}}, \exists x B^{\neg \mathtt{R}} \vdash R}}{\neg_{\mathtt{R}} B^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}} \exists x B^{\neg \mathtt{R}}}}{\neg_{\mathtt{R}} B^{\neg \mathtt{R}} \vdash \neg_{\mathtt{R}} \neg_{\mathtt{R}} \neg_{\mathtt{R}} (\exists x B^{\neg \mathtt{R}})}}{\neg_{\mathtt{R}} B^{\neg \mathtt{R}} \vdash (\exists x B)^{\neg \mathtt{R}} \vee \neg_{\mathtt{R}} (\exists x B)^{\neg \mathtt{R}})}}{\dfrac{\vdash (\exists x B)^{\neg \mathtt{R}} \vee \neg_{\mathtt{R}} (\exists x B)^{\neg \mathtt{R}})}{\dfrac{\vdash \neg_{\mathtt{R}} \neg_{\mathtt{R}} ((\exists x B)^{\neg \mathtt{R}} \vee \neg_{\mathtt{R}} (\exists x B)^{\neg \mathtt{R}}))}{\vdash ((\exists x B) \vee \neg (\exists x B)^{\neg \mathtt{R}}}}}$$

(ii) We proof this property by induction on A and we denote by IH the induction hypothesis.

- $A = \bot$

$$\dfrac{}{\mathtt{R} \vdash \mathtt{R}}$$

- $A = \top$

$$\dfrac{}{\mathtt{R} \vdash \top}$$

- $A = (a \doteq b)$

$$\dfrac{\dfrac{}{\mathtt{R}, \neg_{\mathtt{R}} (a \doteq b) \vdash \mathtt{R}}}{\mathtt{R} \vdash (a \doteq b)^{\neg \mathtt{R}}}$$

- $A = B \wedge C$

$$\dfrac{\dfrac{IH}{\mathtt{R} \vdash B^{\neg \mathtt{R}}} \quad \dfrac{IH}{\mathtt{R} \vdash C^{\neg \mathtt{R}}}}{\mathtt{R} \vdash B^{\neg \mathtt{R}} \wedge C^{\neg \mathtt{R}}}$$

- $A = B \Rightarrow C$

$$\dfrac{\dfrac{IH}{\mathtt{R}, B^{\neg \mathtt{R}} \vdash C^{\neg \mathtt{R}}}}{\mathtt{R} \vdash B^{\neg \mathtt{R}} \Rightarrow C^{\neg \mathtt{R}}}$$

- $A = B \vee C$

$$\dfrac{\dfrac{}{\mathtt{R}, \neg_{\mathtt{R}} (B^{\neg \mathtt{R}} \vee C^{\neg \mathtt{R}}) \vdash \mathtt{R}}}{\mathtt{R} \vdash \neg_{\mathtt{R}} \neg_{\mathtt{R}} (B^{\neg \mathtt{R}} \vee C^{\neg \mathtt{R}})}$$

- $A = \forall x B$

$$\dfrac{\dfrac{IH}{\mathtt{R} \vdash B^{\neg \mathtt{R}}}}{\mathtt{R} \vdash \forall x B^{\neg \mathtt{R}}}$$

- $A = \exists x B$

$$\dfrac{\dfrac{}{\mathtt{R}, \neg_{\mathtt{R}} (\exists x B^{\neg \mathtt{R}}) \vdash \mathtt{R}}}{\mathtt{R} \vdash \neg_{\mathtt{R}} \neg_{\mathtt{R}} (\exists x B^{\neg \mathtt{R}})}$$

So, $\mathtt{R} \vdash A^{\neg_{\mathtt{R}}}$

(iii) We proof this property by induction on A and we denote by IH the induction hypothesis.

- $A = \bot$

$$\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\mathtt{R} \vdash \mathtt{R} \Rightarrow \mathtt{R} \Rightarrow \mathtt{R}} \quad \dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\mathtt{R}, \mathtt{R} \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\mathtt{R} \vdash \mathtt{R} \Rightarrow \mathtt{R}}}{\dfrac{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\mathtt{R} \vdash \mathtt{R}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\bot^{\neg_{\mathtt{R}}} \vdash \bot^{\neg_{\mathtt{R}}}}}$$

- $A = \top$

$$\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\top^{\neg_{\mathtt{R}}} \vdash \top}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\top^{\neg_{\mathtt{R}}} \vdash \top^{\neg_{\mathtt{R}}}}$$

- $A = (a \doteq b)$

$$\dfrac{\dfrac{\text{Proposition 2 } \textit{(v)}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}(\neg_{\mathtt{R}}(a \doteq b)) \vdash \neg_{\mathtt{R}}(\neg_{\mathtt{R}}(a \doteq b))}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(a \doteq b)^{\neg_{\mathtt{R}}} \vdash (a \doteq b)^{\neg_{\mathtt{R}}}}$$

- $A = B \wedge C$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, (B \wedge C)^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}} \quad \dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, (B \wedge C)^{\neg_{\mathtt{R}}} \vdash B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, (B \wedge C)^{\neg_{\mathtt{R}}} \vdash B^{\neg_{\mathtt{R}}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}, (B \wedge C)^{\neg_{\mathtt{R}}} \vdash \mathtt{R}}}{(\Theta_2) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}}$$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}} \quad \Theta_2}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}} \vdash \mathtt{R}}}{(\Theta_3) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}}}$$

$$\dfrac{\Theta_3 \quad \dfrac{\dfrac{IH_B}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}} \vdash B^{\neg_{\mathtt{R}}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}B^{\neg_{\mathtt{R}}} \Rightarrow B^{\neg_{\mathtt{R}}}}}{(\Theta_4) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}} \vdash B^{\neg_{\mathtt{R}}}}$$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}}, (B \wedge C)^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}}} \quad \dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}}, (B \wedge C)^{\neg_{\mathtt{R}}} \vdash B^{\neg_{\mathtt{R}}} \wedge C^{\neg_{\mathtt{R}}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}}, (B \wedge C)^{\neg_{\mathtt{R}}} \vdash C^{\neg_{\mathtt{R}}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}}, (B \wedge C)^{\neg_{\mathtt{R}}} \vdash \mathtt{R}}}{(\Theta_5) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}}$$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}} \quad \Theta_5}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}} \vdash \mathtt{R}}}{(\Theta_6) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}}}$$

$$\dfrac{\Theta_6 \quad \dfrac{\dfrac{IH_C}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}}, \neg_{\mathtt{R}}\neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}} \vdash C^{\neg_{\mathtt{R}}}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}} \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}C^{\neg_{\mathtt{R}}} \Rightarrow C^{\neg_{\mathtt{R}}}}}{(\Theta_7) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}(B \wedge C)^{\neg_{\mathtt{R}}} \vdash C^{\neg_{\mathtt{R}}}}$$

$$\dfrac{\dfrac{\Theta_4 \quad \Theta_7}{\neg_R\neg_R(B\wedge C)^{\neg_R} \vdash B^{\neg_R} \wedge C^{\neg_R}}}{\neg_R\neg_R(B\wedge C)^{\neg_R} \vdash (B\wedge C)^{\neg_R}}$$

- $A = B \Rightarrow C$

$$\dfrac{\dfrac{\dfrac{\textit{todo}}{\neg_R\neg_R(B\Rightarrow C)^{\neg_R}, B^{\neg_R} \vdash C^{\neg_R}}}{\neg_R\neg_R(B\Rightarrow C)^{\neg_R} \vdash B^{\neg_R} \Rightarrow C^{\neg_R}}}{\neg_R\neg_R(B\Rightarrow C)^{\neg_R} \vdash (B\Rightarrow C)^{\neg_R}}$$

- $A = B \vee C$

$$\dfrac{\dfrac{\text{Proposition 2 } (v)}{\neg_R\neg_R\neg_R(\neg_R(B^{\neg_R} \vee C^{\neg_R})) \vdash \neg_R(\neg_R(B^{\neg_R} \vee C^{\neg_R}))}}{\neg_R\neg_R(B\vee C)^{\neg_R} \vdash (B\vee C)^{\neg_R}}$$

- $A = \forall x B$

$$\dfrac{\dfrac{IH}{\neg_R\neg_R\forall x B^{\neg_R} \vdash B^{\neg_R}}}{\neg_R\neg_R\forall x B^{\neg_R} \vdash \forall x B^{\neg_R}}$$

- $A = \exists x B$

$$\dfrac{\dfrac{\text{Proposition 2 } (v)}{\neg_R\neg_R\neg_R(\neg_R(\exists x B^{\neg_R})) \vdash \neg_R(\neg_R(\exists x B^{\neg_R}))}}{\neg_R\neg_R(\exists x B)^{\neg_R} \vdash (\exists x B)^{\neg_R}}$$

So, $\neg_R\neg_R A^{\neg_R} \vdash A^{\neg_R}$

□

Theorem 5. *If $\vdash A$ is derivable in classical predicate logic and if no free variable of R occurs in the derivation, then $\vdash A^{\neg_R}$ is derivable in intuitionisctic predicate logic.*

Proof. Every rules except excluded middle are the same so we just keep them (and we replace all expression X to $X^{\neg_R}$) in order to get a derivation of $\vdash A^{\neg_R}$ from a derivation of $\vdash A$. It works because there is no free occurrences of R.

For the excluded middle rule we rewrite:

$$\dfrac{\dfrac{\vdots}{\Gamma, \neg A \vdash \bot}}{\Gamma \vdash A}$$

to:

$$\dfrac{\dfrac{\dfrac{\text{proposition 4 } (i)}{\vdash (A\wedge\neg A)^{\neg_R}}}{\Gamma \vdash (A\wedge\neg A)^{\neg_R}} \quad \dfrac{}{\Gamma, A^{\neg_R} \vdash A^{\neg_R}} \quad \dfrac{\dfrac{\vdots}{\Gamma, \neg_R A^{\neg_R} \vdash \bot}}{\Gamma, \neg_R A^{\neg_R} \vdash A^{\neg_R}}}{\Gamma \vdash A^{\neg_R}}$$

□

Theorem 6. *If $PA \vdash A$ and if no free variable of R occurs in the derivation, then $HA \vdash A^{\neg_R}$.*

Proof. 1. Injectivity of S

$$\dfrac{\dfrac{}{\neg_R\neg_R S(x)\doteq S(y), \neg_R x\doteq y, S(x)\doteq S(y) \vdash S(x)\doteq S(y)} \quad \dfrac{}{\vdash S(x)\doteq S(y) \Rightarrow x\doteq y}}{(\Xi_1) : \neg_R\neg_R S(x)\doteq S(y), \neg_R x\doteq y, S(x)\doteq S(y) \vdash x\doteq y}$$

$$\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y), \neg_{\mathtt{R}}x\doteq y, S(x)\doteq S(y) \vdash \neg_{\mathtt{R}}x\doteq y} \quad \Xi_1}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y), \neg_{\mathtt{R}}x\doteq y, S(x)\doteq S(y) \vdash \mathtt{R}}}{(\Xi_2) : \neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y), \neg_{\mathtt{R}}x\doteq y \vdash \neg_{\mathtt{R}}S(x)\doteq S(y)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y)} \quad \Xi_2}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y), \neg_{\mathtt{R}}x\doteq y \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y) \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}x\doteq y}}{\vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y) \Rightarrow \neg_{\mathtt{R}}\neg_{\mathtt{R}}x\doteq y}}{\vdash \forall xy(\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq S(y) \Rightarrow \neg_{\mathtt{R}}\neg_{\mathtt{R}}x\doteq y)}}{\vdash (\forall xy(S(x)\doteq S(y) \Rightarrow x\doteq y))^{\neg_{\mathtt{R}}}}$$

2. Non confusion

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0 \vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0, S(x)\doteq 0 \vdash S(x)\doteq 0} \quad \dfrac{\dfrac{}{\vdash \neg\forall x S(x)\doteq 0}}{\vdash \neg S(x)\doteq 0}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0, S(x)\doteq 0 \vdash \bot}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0, S(x)\doteq 0 \vdash \mathtt{R}}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0 \vdash \neg_{\mathtt{R}}S(x)\doteq 0}}{\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0 \vdash \mathtt{R}}}{\vdash \neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0}}{\vdash \forall x\neg_{\mathtt{R}}\neg_{\mathtt{R}}\neg_{\mathtt{R}}S(x)\doteq 0}}{\vdash (\forall x\neg S(x)\doteq 0)^{\neg_{\mathtt{R}}}}$$

3. Induction Scheme

$$\dfrac{\dfrac{}{\vdash A[0/x]^{\neg_{\mathtt{R}}} \Rightarrow \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \Rightarrow A^{\neg_{\mathtt{R}}}} \quad \dfrac{}{A[0/x]^{\neg_{\mathtt{R}}} \vdash A[0/x]^{\neg_{\mathtt{R}}}}}{(\Xi_3) : A[0/x]^{\neg_{\mathtt{R}}}, \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \vdash \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \Rightarrow A^{\neg_{\mathtt{R}}}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Xi_3 \quad \dfrac{}{\forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \vdash \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}})}}{A[0/x]^{\neg_{\mathtt{R}}}, \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \vdash A^{\neg_{\mathtt{R}}}}}{A[0/x]^{\neg_{\mathtt{R}}}, \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \vdash \forall x A^{\neg_{\mathtt{R}}}}}{A[0/x]^{\neg_{\mathtt{R}}} \vdash \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \Rightarrow \forall x A^{\neg_{\mathtt{R}}}}}{\vdash A[0/x]^{\neg_{\mathtt{R}}} \Rightarrow \forall y(A[y/x]^{\neg_{\mathtt{R}}} \Rightarrow A[S(y)/x]^{\neg_{\mathtt{R}}}) \Rightarrow \forall x A^{\neg_{\mathtt{R}}}}}{\vdash (A[0/x] \Rightarrow \forall y(A[y/x] \Rightarrow A[S(y)/x]) \Rightarrow \forall x A)^{\neg_{\mathtt{R}}}}$$

$\square$

Theorem 7. *If $PA \vdash \forall x, \exists y : (a\doteq b)$ then $HA \vdash \forall x, \exists y : (a\doteq b)$.*

Proof. Let us write $F\forall x, G$ where G is a Σ_1^0 formula. By using $(\forall E)$, if F if provable with PA, so G too. As G is Σ_1^0, G is provable with HA. By using $(\forall I)$, we deduce that F is provable with HA too. $\square$