Course *Computer Assisted Proofs*

Homework to be returned the 24th Oct. 2013

Friedman's Translation

This homework is about the Π^0_2-conservativity of Peano arithmetic over Heyting arithmetic. The work is to give detailed proofs of all Theorems and Propositions of Section 1. It has to be returned in paper form to Colin Riba on **Friday 24th Oct. 2014**.

The goal of this homework is to show the following result:

- If a formula of the form $\forall x \exists y (a \doteq b)$ is provable in Peano arithmetic, then it is provable in Heyting arithmetic.

This will be done by using a technique due to Harvey Friedman.

We work in the systems HA and PA. Their definition is recalled in App. 3.

1 Friedman's Translation

Friedman's translation is a negative translation based on a notion of "parametrized negation".

Definition 1.1. *Let* $\mathtt{R}$ *be a formula. The* **parametrized negation** *is*

$$\neg_{\mathtt{R}} A \quad := \quad A \Rightarrow \mathtt{R}$$

We gather here some basic properties of the parametrized negation.

Proposition 1.2. *In intuitionistic logic,*

(i) $B \Rightarrow \neg_{\mathtt{R}} A \vdash A \Rightarrow \neg_{\mathtt{R}} B$

(ii) $A \vdash \neg_{\mathtt{R}} \neg_{\mathtt{R}} A$

(iii) $A \Rightarrow B \vdash \neg_{\mathtt{R}} B \Rightarrow \neg_{\mathtt{R}} A$

(iv) $A \Rightarrow B \vdash \neg_{\mathtt{R}} \neg_{\mathtt{R}} A \Rightarrow \neg_{\mathtt{R}} \neg_{\mathtt{R}} B$

(v) $\neg_{\mathtt{R}} \neg_{\mathtt{R}} \neg_{\mathtt{R}} A \vdash \neg_{\mathtt{R}} A$

We now define the parametrized translation.

Definition 1.3. *Let* $\mathtt{R}$ *be a formula. The* **paramterized negative translation** $A^{\neg_{\mathtt{R}}}$ *is defined by induction on* A *as follows:*

$$\bot^{\neg_{\mathtt{R}}} := \mathtt{R} \qquad \top^{\neg_{\mathtt{R}}} := \top \qquad (a \doteq b)^{\neg_{\mathtt{R}}} := \neg_{\mathtt{R}} \neg_{\mathtt{R}} (a \doteq b)$$

$$(A \wedge B)^{\neg_{\mathtt{R}}} := A^{\neg_{\mathtt{R}}} \wedge B^{\neg_{\mathtt{R}}} \qquad (A \Rightarrow B)^{\neg_{\mathtt{R}}} := A^{\neg_{\mathtt{R}}} \Rightarrow B^{\neg_{\mathtt{R}}}$$

$$(A \vee B)^{\neg_{\mathtt{R}}} := \neg_{\mathtt{R}} \neg_{\mathtt{R}} (A^{\neg_{\mathtt{R}}} \vee B^{\neg_{\mathtt{R}}})$$

$$(\forall x A)^{\neg_{\mathtt{R}}} := \forall x A^{\neg_{\mathtt{R}}} \qquad (\exists x A)^{\neg_{\mathtt{R}}} := \neg_{\mathtt{R}} \neg_{\mathtt{R}} (\exists x A^{\neg_{\mathtt{R}}})$$

Note that $(\neg A)^{\neg_{\mathtt{R}}} = \neg_{\mathtt{R}} A^{\neg_{\mathtt{R}}}$. We gather here the basic properties of the parametrized translation.

Proposition 1.4. *In intuitionistic logic,*

(i) $\vdash (A \vee \neg A)^{\neg_{\mathtt{R}}}$

(ii) $\mathtt{R} \vdash A^{\neg_{\mathtt{R}}}$

(iii) $\neg_{\mathtt{R}}\neg_{\mathtt{R}} A^{\neg_{\mathtt{R}}} \vdash A^{\neg_{\mathtt{R}}}$

It follows from these properties that $(\_)^{\neg_{\mathtt{R}}}$ translates classical logic into intuitionistic logic.

Theorem 1.5. *If* $\vdash A$ *is derivable in classical predicate logic and if no free variable of* $\mathtt{R}$ *occurs in the derivation, then* $\vdash A^{\neg_{\mathtt{R}}}$ *is derivable in intuitionistic predicate logic.*

In order to obtain Theorem 1.5 for arithmetic, it remains to show that HA proves the $\neg_{\mathtt{R}}$-translation of all its axioms.

Theorem 1.6. *If* $\mathsf{PA} \vdash A$ *and if no free variable of* $\mathtt{R}$ *occurs in the derivation, then* $\mathsf{HA} \vdash A^{\neg_{\mathtt{R}}}$.

It remains to use Friedman's trick in order to deduce the desired result.

Theorem 1.7 (Kreisel)**.** *If* $\mathsf{PA} \vdash \forall x \exists y (a \doteq b)$ *then* $\mathsf{HA} \vdash \forall x \exists y (a \doteq b)$.

2 Intuitionnistic Predicate Logic

$$\frac{}{\Delta, A \vdash A}\ (Ax) \qquad \frac{}{\Delta \vdash \top}\ (\top I) \qquad \frac{\Delta \vdash \bot}{\Delta \vdash A}\ (\bot E)$$

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \Rightarrow B}\ (\Rightarrow I) \qquad \frac{\Delta \vdash A \Rightarrow B \qquad \Delta \vdash A}{\Delta \vdash B}\ (\Rightarrow E)$$

$$\frac{\Delta \vdash A_1 \qquad \Delta \vdash A_2}{\Delta \vdash A_1 \wedge A_2}\ (\wedge I) \qquad \frac{\Delta \vdash A_1 \wedge A_2}{\Delta \vdash A_i}\ (\wedge_i E)$$

$$\frac{\Delta \vdash A_i}{\Delta \vdash A_1 \vee A_2}\ (\vee_i I) \qquad \frac{\Delta \vdash A_1 \vee A_2 \quad \Delta, A_1 \vdash B \quad \Delta, A_2 \vdash B}{\Delta \vdash B}\ (\vee E)$$

$$\frac{\Delta \vdash A}{\Delta \vdash \forall x A}\ (x \notin \Delta)\ (\forall I) \qquad \frac{\Delta \vdash \forall x A}{\Delta \vdash A[a/x]}\ (\forall E)$$

$$\frac{\Delta \vdash A[a/x]}{\Delta \vdash \exists x A}\ (\exists I) \qquad \frac{\Delta \vdash \exists x A \qquad \Delta, A \vdash B}{\Delta \vdash B}\ (x \notin \mathrm{FV}(\Delta, B))\ (\exists E)$$

where $a \in \mathcal{T}er(\mathcal{V}ar, \Sigma)$.

3 Arithmetic

3.1 The Language of Arithmetic

The signatue Σ_{Nat} consists of the zero-ary symbol $\mathsf{0}$, the unary symbol $\mathsf{S}(\_)$ and the binary symbols $\_ + \_$ and $\_ \times \_$.

Formulas of HA and PA are thus given by

$$\begin{array}{lcl} A, B & ::= & a \doteq b \quad | \quad \top \quad | \quad \bot \quad | \quad A \wedge B \quad | \quad A \vee B \quad | \quad A \Rightarrow B \\ & | & \forall x A \quad | \quad \exists x A \end{array}$$

where $a, b \in \mathcal{T}er(\mathcal{V}ar, \Sigma_{\mathsf{Nat}})$.

3.2 The Axioms of Arithmetic

Peano's Arithmetic (resp. **Heyting's Arithmetic**) is classical deduction (resp. intuitionistic deduction) with the following axioms augmented with the **Equality Axioms**:

Injectivity of $\mathsf{S}(\_)$.

$$\frac{}{\Delta \vdash \forall xy(\mathsf{S}(x) \doteq \mathsf{S}(y) \Rightarrow x \doteq y)}$$

Non Confusion.

$$\frac{}{\Delta \vdash \forall x \; \neg(\mathsf{S}(x) \doteq \mathsf{0})}$$

Induction Scheme. For all formula A:

$$\frac{}{\Delta \vdash A[\mathsf{0}/x] \Rightarrow \forall y(A[y/x] \Rightarrow A[\mathsf{S}(y)/x]) \Rightarrow \forall x A}$$

Equational Axioms.

$$\begin{array}{rclcrcl} x + \mathsf{0} & \doteq & x & \qquad & x + \mathsf{S}(y) & \doteq & \mathsf{S}(x) + y \\ x \times \mathsf{0} & \doteq & \mathsf{0} & \qquad & x \times \mathsf{S}(y) & \doteq & x + (x \times y) \end{array}$$