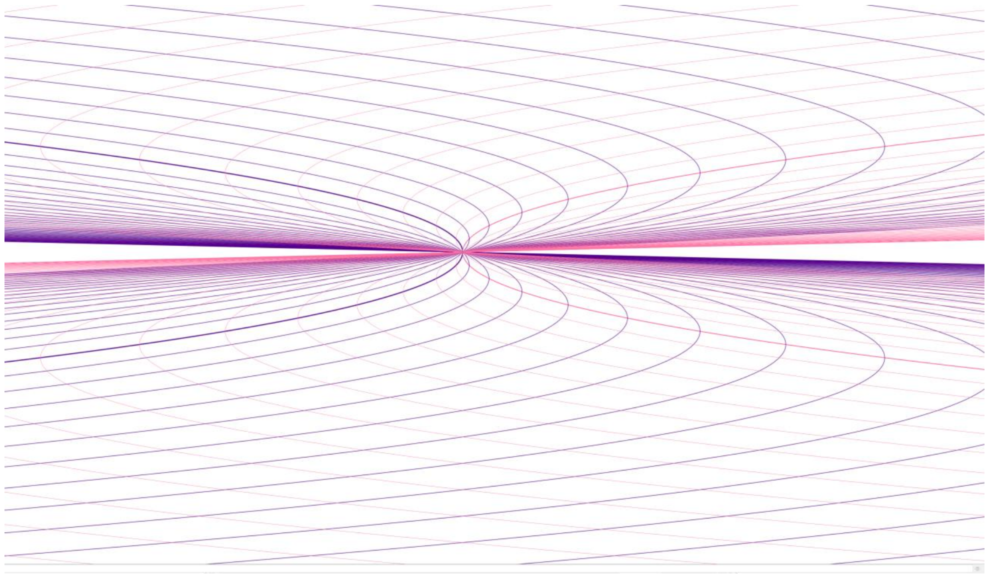
OFFSET IN POLYNOMIALS



Sumário

1. Introduction to OFFSET (shift or phase or synchronism)

The concept of offset in this work is used to represent and quantify the shift, the phase or synchronicity between infinite integer sequences in polynomials. Another term, which could be used, for integer sequences, would be "shift" to maintain the same analogy that is used in programming language commands like "shift string". Even though, whatever the finite sequence of elements that will represent a real given sequence (finite or infinite), its elements may or may not be shifted, and yet retain the same properties.

The term offset is being used along this study because OEIS adopted this term. It is important to notice that any infinite integer sequence has no start and no end. The only start and end are our finite representations of the sequence. We may have infinite many sequence representations of the same infinite sequence. Also, these representations may be shifted one from another, and then the offset concept will be applied.

The concept of offset is present in all equations of type $x = X(y)$ that produce x results as a function of variable y (index). Apparently, in an equation where there is no offset variation is the type $x = X(y) = constant$. But, any expression where x varies as a function of y, offsetting is possible. This is because in any expression we can substitute the index y by a new index $h = y + k$, or, more comprehensively, substitute y by $h = H(k)$ which can be not only linear, but any desired infinite function.

A very simple example of offset is how we usually define even numbers. We say that x is even if $x = 2y$. Very simple like that. And also, we usually want to make the expressions as simple as possible. We could also say that x is even if $x = 2y + 2$ or $x = 2y + 6$ or $x = 2y + 1234567890$ or $x = 2y + 2k + 2t$, etc. Mathematically all are perfectly correct. As we can choose for simplicity, we always choose to say only $x = 2y$.

What about odd numbers? Would it be x = 2y + 1 or x = 2y-1? Or would it still be x = 2y + 3 or x = 2y-3? What to choose? Here too, simplicity is always taken into consideration. We always choose $x = 2y - 1$ because of the simple form of counting we adopt called Tally counting. Thus, we will have odd numbers for all indexes $y = 1, 2, 3, 4, 5, 6, \ldots$ only in the formula $x = 2y - 1$. In the formula $x = 2y + 1$, when we count the indexes according to Tally, the first odd will be 3 and odd 1 will not have been considered. Thus, $x = 2y + 1$ is an offset of $x = 2y - 1$ because only $2y - 1$ covers all odd generated by the Tally count from ZERO.

When we refer to the offset of a sequence generated by a quadratic of the type $x = ay^2 + by + c$, we also mean that there is a displacement of the terms (elements of the sequence) generated on the X-axis in relation to the terms (or elements) of index y. In XY-plane offset between two identical sequences is equivalent to moving the quadratic curve only along the Y-axis in an integer shift without moving along the X-axis. The same with any other polynomial.

When we approach synchronicity between polynomials, we will also refer to offset as being a phase in the same way as we refer to angles or phases in complex numbers. This means that the same sequence of integers in the form of a polynomial or exponential can be represented by infinite many equations. It is also like fractions. We can write it in the simplest form or any other infinite many not simplified form.

2. Infinite many formulas, infinite many quadratics curves, only one sequence

See the behavior represented in the 3 sequences below shown in the 3 tables with 8 equations each where $x = ay^2 + by + c$. The first table represents the sequence A165900 Values of Fibonacci polynomial, the second table represents the sequence A002378 Oblong (or promic, pronic, or heteromecic) numbers, and the last table represents A002061 Central polygonal numbers.

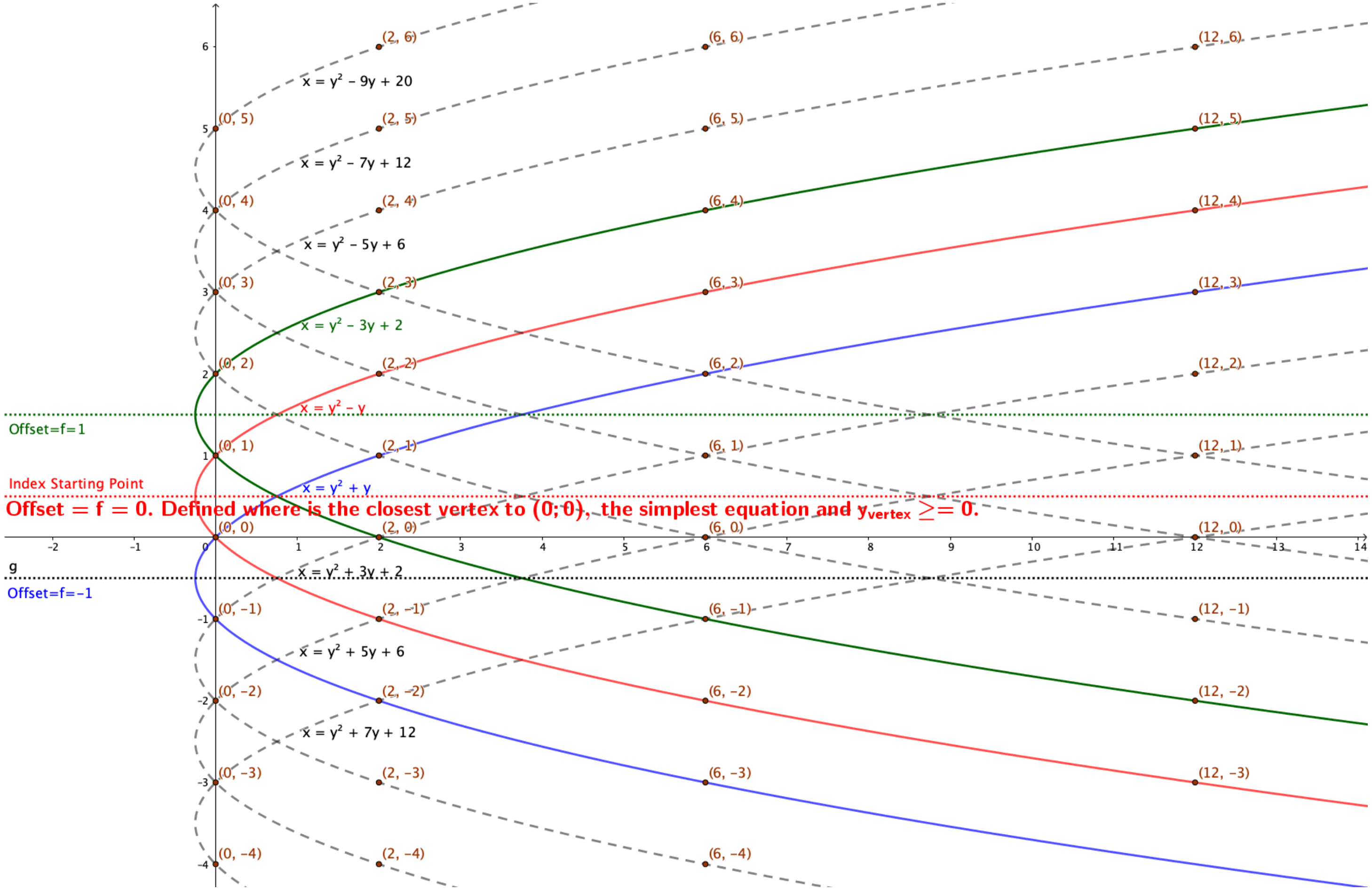
			A165900 Values of Fibonacci polynomial							
Δ			5	5	5	5	5	5	5	5
C. G.			0,2361	0,2361	0,2361	0,2361	0,2361	0,2361	0,2361	0,2361
x_vertex			-1,25	-1,25	-1,25	-1,25	-1,25	-1,25	-1,25	-1,25
x_focus			-1	-1	-1	-1	-1	-1	-1	-1
Latus Rectum			1	1	1	1	1	1	1	1
y_vertex			-3,5	-2,5	-1,5	-0,5	0,5	1,5	2,5	3,5
f			-4	-3	-2	-1	0	1	2	3
a			1	1	1	1	1	1	1	1
b			7	5	3	1	-1	-3	-5	-7
c			11	5	1	-1	-1	1	5	11
		10	181	155	131	109	89	71	55	41
		9	155	131	109	89	71	55	41	29
		8	131	109	89	71	55	41	29	19
		7	109	89	71	55	41	29	19	11
		6	89	71	55	41	29	19	11	5
		5	71	55	41	29	19	11	5	1
		4	55	41	29	19	11	5	1	-1
k		3	41	29	19	11	5	1	-1	-1
j		2	29	19	11	5	1	-1	-1	1
i	x_3	1	19	11	5	1	-1	-1	1	5
h	x_2	0	11	5	1	-1	-1	1	5	11
g	x_1	-1	5	1	-1	-1	1	5	11	19
f		-2	1	-1	-1	1	5	11	19	29
e		-3	-1	-1	1	5	11	19	29	41
		-4	-1	1	5	11	19	29	41	55
		-5	1	5	11	19	29	41	55	71
		-6	5	11	19	29	41	55	71	89
		-7	11	19	29	41	55	71	89	109
		-8	19	29	41	55	71	89	109	131
		-9	29	41	55	71	89	109	131	155
		-10	41	55	71	89	109	131	155	181

			A002378 Oblong (or promic, pronic, or heteromecic) numbers							
Δ			1	1	1	1	1	1	1	1
C. G.			0	0	0	0	0	0	0	0
x_vertex			-0,25	-0,25	-0,25	-0,25	-0,25	-0,25	-0,25	-0,25
x_focus			0	0	0	0	0	0	0	0
Latus Rectum			1	1	1	1	1	1	1	1
y_vertex			-3,5	-2,5	-1,5	-0,5	0,5	1,5	2,5	3,5
f			-4	-3	-2	-1	0	1	2	3
a			1	1	1	1	1	1	1	1
b			7	5	3	1	-1	-3	-5	-7
c			12	6	2	0	0	2	6	12
		10	182	156	132	110	90	72	56	42
		9	156	132	110	90	72	56	42	30
		8	132	110	90	72	56	42	30	20
		7	110	90	72	56	42	30	20	12
		6	90	72	56	42	30	20	12	6
		5	72	56	42	30	20	12	6	2
		4	56	42	30	20	12	6	2	0
k		3	42	30	20	12	6	2	0	0
j		2	30	20	12	6	2	0	0	2
i	x_3	1	20	12	6	2	0	0	2	6
h	x_2	0	12	6	2	0	0	2	6	12
g	x_1	-1	6	2	0	0	2	6	12	20
f		-2	2	0	0	2	6	12	20	30
e		-3	0	0	2	6	12	20	30	42
		-4	0	2	6	12	20	30	42	56
		-5	2	6	12	20	30	42	56	72
		-6	6	12	20	30	42	56	72	90
		-7	12	20	30	42	56	72	90	110
		-8	20	30	42	56	72	90	110	132
		-9	30	42	56	72	90	110	132	156
		-10	42	56	72	90	110	132	156	182

			A002061 Central polygonal numbers							
Δ			-3	-3	-3	-3	-3	-3	-3	-3
C. G.			#NÚM!	#NÚM!	#NÚM!	#NÚM!	#NÚM!	#NÚM!	#NÚM!	#NÚM!
x_vertex			0,75	0,75	0,75	0,75	0,75	0,75	0,75	0,75
x_focus			1	1	1	1	1	1	1	1
Latus Rectum			1	1	1	1	1	1	1	1
y_vertex			-3,5	-2,5	-1,5	-0,5	0,5	1,5	2,5	3,5
f			-4	-3	-2	-1	0	1	2	3
a			1	1	1	1	1	1	1	1
b			7	5	3	1	-1	-3	-5	-7
c			13	7	3	1	1	3	7	13
		10	183	157	133	111	91	73	57	43
		9	157	133	111	91	73	57	43	31
		8	133	111	91	73	57	43	31	21
		7	111	91	73	57	43	31	21	13
		6	91	73	57	43	31	21	13	7
		5	73	57	43	31	21	13	7	3
		4	57	43	31	21	13	7	3	1
k		3	43	31	21	13	7	3	1	1
j		2	31	21	13	7	3	1	1	3
i	x_3	1	21	13	7	3	1	1	3	7
h	x_2	0	13	7	3	1	1	3	7	13
g	x_1	-1	7	3	1	1	3	7	13	21
f		-2	3	1	1	3	7	13	21	31
e		-3	1	1	3	7	13	21	31	43
		-4	1	3	7	13	21	31	43	57
		-5	3	7	13	21	31	43	57	73
		-6	7	13	21	31	43	57	73	91
		-7	13	21	31	43	57	73	91	111
		-8	21	31	43	57	73	91	111	133
		-9	31	43	57	73	91	111	133	157
		-10	43	57	73	91	111	133	157	183

Perceive in each table occur a shift (phase/offset) of the sequence as long as we change the starting element x_1 at index $y = -1$ or starting element x_2 at index y=0 or starting element x_3 at index $y = 1$. This means that, for each table, we are changing by a unit the y_{vertex} from each quadratic sequence and maintain the same value for x_{vertex}. In any case, for each table, the sequence of the elements will always be kept. There is no risk to occur inversions, missing or scramble between elements. Always the result of the sequence of the elements are exactly the same because the coefficients (a, b, c) for each equation will be adjusted accordingly as we will study next.

Just as illustration, see in the figure below how the sequence A002378 Oblong (or promic, pronic, or heteromecic) numbers behave in XY plane:



The other two curves sequence A165900 Values of Fibonacci polynomial and A002061 Central polygonal numbers are exactly the same as A002378 Oblong (or promic, pronic, or heteromecic) numbers, but one shifted one unit to the negative side of X-axis and other one unit shifted to the positive side of the X-axis, respectively.

It is clear that in all cases there is no right or wrong choice in which curve or formula we have to choose to get the integers sequence desired. For each table set, all curves and all formulas are correct and generate the same sequence of integers. This happens, because as far as we adjust the index y all equations generate the same sequence.

The big difference between them is in simplicity. Simplicity in writing and working with. After all, it becomes much easier to work with simplified fractions, small expressions, small numbers, small angles, than large. Now we will include to work with small offset: offset zero.

3. Basic concept

When we have 3 elements (x_1, x_2, x_3) of a quadratic, we mean that each of them is the x result of the equation $x = ay^2 + by + c$. In this case, x_n is the resulting value for each value of y_n. In this case, y_n may be called the index. For each index we can calculate the value x of the element of a quadratic. In fact, we could express the equation showing x as function of y:

$$x = X(y) = ay^2 + by + c$$

where, (a, b, c) are the fixed coefficients and y is the variable index.

Therefore, when we speak about a sequence of 3 elements, we have defined perfectly the curve of a quadratic. We mean that the 3 elements are generated for a sequence of 3 integers values of the index y. In other words, we have a sequence of 3 elements of the quadratic (x_1, x_2, x_3) where each of them was obtained by a different index (y_1, y_2, y_3):

$$x_1 = ay_1^2 + by_1 + c$$

$$x_2 = ay_2^2 + by_2 + c$$

$$x_3 = ay_3^2 + by_3 + c$$

Where y_1, y_2, y_3 are sequential indexes.

Now, to be <u>simpler</u>, we usually take sequential indexes as <u>consecutive integer indexes</u>. So, for consecutive integer indexes y_1, y_2, y_3, we have:

$$y_3 = y_2 + 1 = y_1 + 2$$

Now, considering generically, we have an index y_n to obtain $x_n = ay_n^2 + by_n + c$, then,

$$\begin{aligned}
&\ldots\\
&for\ y = -3, then\ x = e = 9a - 3b + c\\
&for\ y = -2, then\ x = f = 4a - 2b + c\\
&for\ y = -1, then\ x = g = a - b + c\\
&for\ y = 0, then\ x = h = c\\
&for\ y = 1, then\ x = i = a + b + c\\
&for\ y = 2, then\ x = j = 4a + 2b + c\\
&for\ y = 3, then\ x = k = 9a + 3b + c\\
&\ldots
\end{aligned}$$

Here, perceive that we do not need necessarily 3 consecutive index generating 3 consecutive elements to find the 3 coefficients (a, b, c). Any combination of 3 equations will be enough to find the 3 coefficients (a, b, c). But, in order to be <u>simpler</u>, we will work with 3 consecutive indexes. Likewise, in a straight line: any 2 points can define it exactly. It will be <u>simpler</u> to use 2 consecutive elements.

Now, the question is: which 3 consecutive indexes should we take from infinite many possibilities in order to be <u>simpler</u>? Should be <u>simpler</u> to use indexes (1,2,3) or indexes (0,1,2) or other set? Is there a set that is simplest than all the others?

4. Analyzing the alternatives

If we consider only the 7 equations above listed as examples and keep the <u>simpler</u> approach using sequential consecutive indexes, we can choose any set of 3 equations from the 5 possibilities below:

- Set 1 of 5:
  - $for\ y_1 = -3, then\ x_1 = e = 9a - 3b + c$
  - $for\ y_2 = -2, then\ x_2 = f = 4a - 2b + c$
  - $for\ y_3 = -1, then\ x_3 = g = a - b + c$
- Set 2 of 5:
  - $for\ y_1 = -2, then\ x_1 = f = 4a - 2b + c$
  - $for\ y_2 = -1, then\ x_2 = g = a - b + c$
  - $for\ y_3 = 0, then\ x_3 = h = c$
- Set 3 of 5:
  - $for\ y_1 = -1, then\ x_1 = g = a - b + c$
  - $for\ y_2 = 0, then\ x_2 = h = c$
  - $for\ y_3 = 1, then\ x_3 = i = a + b + c$
- Set 4 of 5:
  - $for\ y_1 = 0, then\ x_1 = h = c$
  - $for\ y_2 = 1, then\ x_2 = i = a + b + c$
  - $for\ y_3 = 2, then\ x_3 = j = 4a + 2b + c$
- Set 5 of 5:
  - $for\ y_1 = 1, then\ x_1 = i = a + b + c$
  - $for\ y_2 = 2, then\ x_2 = j = 4a + 2b + c$
  - $for\ y_3 = 3, then\ x_3 = k = 9a + 3b + c$

In all of them we set the first index as being y_1, the second index as being y_2 and third index a being y_3.

Always the first index will be y_1 because of Tally counting.

Note that, each set will produce different coefficients (a, b, c) which will result in different quadratic equations.

Each set of consecutive index (y_1, y_2, y_3) will generate 3 different consecutive elements (x_1, x_2, x_3) of exactly the same quadratic integer sequence. But, because all different consecutive elements generates different quadratic equations, then this will be reflected as a different quadratic curve in plane XY. The result will produce exactly the same sequence of elements x_n shifted one step (one element) from the next set. This is the offset.

4.1. Set 5 of 5: NumberSpiral.com

If we look at Robert Sacks' NumberSpiral study, we will conclude it was used the option Set 5 of 5:

$$for\ y_1 = 1, then\ x_1 = i = a + b + c$$
$$for\ y_2 = 2, then\ x_2 = j = 4a + 2b + c$$
$$for\ y_3 = 3, then\ x_3 = k = 9a + 3b + c$$

So, when NumberSpiral created 3 simultaneous equations for elements named (i, j, k), they set $y_1 = 1$, once ***"the first number, $x_1 = i$, is generated when we plug $y = 1$ into $x = ay^2 + by + c$ expression. The next number, $x_2 = j$, since it's the very next number in the sequence after $x_1 = i$, is generated by $y = 2$. And similarly, $x_3 = k$ is generated by $y = 3$."***

Then, the result found is:

$$a = \frac{i - 2j + k}{2} = \frac{x_1 - 2x_2 + x_3}{2}$$
$$b = j - i - 3a = x_2 - x_1 - 3a$$
$$c = i - a - b = x_1 - a - b$$

or,

$$a = \frac{i - 2j + k}{2} = \frac{x_1 - 2x_2 + x_3}{2}$$
$$b = \frac{-5i + 8j - 3k}{2} = \frac{-5x_1 + 8x_2 - 3x_3}{2}$$
$$c = 3i - 3j + k = 3x_1 - 3x_2 + x_3$$

Where the final expression of quadratic is given by:

$$x = \left(\frac{i - 2j + k}{2}\right)y^2 + \left(\frac{-5i + 8j - 3k}{2}\right)y + (3i - 3j + k)$$

Then,

$$X_{Set\ 5\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{-5x_1 + 8x_2 - 3x_3}{2}\right)y + (3x_1 - 3x_2 + x_3)$$

Perceive that:

- NumberSpiral used $(y_1 = 1; y_2 = 2; y_3 = 3)$
- If we set $(y_1 = 2; y_2 = 3; y_3 = 4)$ we get the same coefficient $a = \frac{i - 2j + k}{2}$, but less simple factors for coefficients b and c than the factors in NumberSpiral set $(y_1 = 1; y_2 = 2; y_3 = 3)$;
- If we set $(y_1 = 0; y_2 = 1; y_3 = 2)$ we get also the same coefficient $a = \frac{i - 2j + k}{2}$, but simpler factors for coefficients b and c than the factors in NumberSpiral set $(y_1 = 1; y_2 = 2; y_3 = 3)$;

So, the remaining question: given a sequence of 3 elements (x_1, x_2, x_3) of a quadratic, is this the simplest form to express the equation of a quadratic? Let's continue…

4.2. Set 4 of 5:

- Set 4 of 5:
  - $for\ y_1 = 0, then\ x_1 = h = c$
  - $for\ y_2 = 1, then\ x_2 = i = a + b + c$
  - $for\ y_3 = 2, then\ x_3 = j = 4a + 2b + c$

Then,

$$c = x_1$$
$$x_2 = a + b + x_1$$
$$x_3 = 4a + 2b + x_1$$

Then

$$2x_2 = 2a + 2b + 2x_1$$
$$x_3 - 2x_2 = 4a + 2b + x_1 - 2a - 2b - 2x_1$$
$$x_3 - 2x_2 = 2a - x_1$$
$$a = \frac{x_1 - 2x_2 + x_3}{2}$$

Then,

$$x_2 = a + b + x_1$$
$$x_2 = \frac{x_1 - 2x_2 + x_3}{2} + b + x_1$$
$$2x_2 = x_1 - 2x_2 + x_3 + 2b + 2x_1$$
$$4x_2 = 3x_1 + x_3 + 2b$$
$$b = \frac{-3x_1 + 4x_2 - x_3}{2}$$

Then,

$$X_{Set\ 4\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{-3x_1 + 4x_2 - x_3}{2}\right)y + x_1$$

4.3. Set 3 of 5:

- Set 3 of 5:
  - $for\ y_1 = -1, then\ x_1 = g = a - b + c$
  - $for\ y_2 = 0, then\ x_2 = h = c$
  - $for\ y_3 = 1, then\ x_3 = i = a + b + c$

Then,

$$c = x_2$$
$$x_1 = a - b + x_2$$
$$x_3 = a + b + x_2$$

Then,

$$x_1 + x_3 = 2a + 2x_2$$
$$a = \frac{x_1 - 2x_2 + x_3}{2}$$

Then,

$$x_3 - x_1 = 2b$$
$$b = \frac{-x_1 + x_3}{2}$$

Then,

$$X_{Set\ 3\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{-x_1 + x_3}{2}\right)y + x_2$$

4.4. Set 2 of 5:

- Set 2 of 5:

- $for\ y_1 = -2, then\ x_1 = f = 4a - 2b + c$
- $for\ y_2 = -1, then\ x_2 = g = a - b + c$
- $for\ y_3 = 0, then\ x_3 = h = c$

Then,

$$c = x_3$$
$$x_1 = 4a - 2b + x_3$$
$$x_2 = a - b + x_3$$

Then,

$$x_1 - 2x_2 = 4a - 2b + x_3 - 2a + 2b - 2x_3$$
$$x_1 - 2x_2 = 2a - x_3$$
$$a = \frac{x_1 - 2x_2 + x_3}{2}$$

Then,

$$x_1 - 4x_2 = 4a - 2b + x_3 - 4a + 4b - 4x_3$$
$$x_1 - 4x_2 = 2b - 3x_3$$
$$b = \frac{x_1 - 4x_2 + 3x_3}{2}$$

Then,

$$X_{Set\ 2\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{x_1 - 4x_2 + 3x_3}{2}\right)y + x_3$$

4.5. Set 1 of 5:

- Set 1 of 5:
  - $for\ y_1 = -3, then\ x_1 = e = 9a - 3b + c$
  - $for\ y_2 = -2, then\ x_2 = f = 4a - 2b + c$
  - $for\ y_3 = -1, then\ x_3 = g = a - b + c$

Then,

$$x_2 - x_3 = 4a - 2b + c - a + b - c = 3a - b$$
$$b = 3a - x_2 + x_3$$

Then,

$$x_1 = 9a - 3b + c = 9a - 3(3a - x_2 + x_3) + c = 3x_2 - 3x_3 + c$$
$$x_2 = 4a - 2b + c = 4a - 2(3a - x_2 + x_3) + c = 4a - 6a + 2x_2 - 2x_3 + c = -2a + 2x_2 - 2x_3 + c$$
$$x_1 - x_2 = 3x_2 - 3x_3 + c - (-2a + 2x_2 - 2x_3 + c) = 3x_2 - 3x_3 + c + 2a - 2x_2 + 2x_3 - c = x_2 - x_3 + 2a$$
$$a = \frac{x_1 - 2x_2 + x_3}{2}$$

Then,

$$b = 3a - x_2 + x_3 = 3\left(\frac{x_1 - 2x_2 + x_3}{2}\right) - x_2 + x_3 = \frac{3x_1 - 6x_2 + 3x_3 - 2x_2 + 2x_3}{2}$$
$$b = \frac{3x_1 - 8x_2 + 5x_3}{2}$$

Then,

$$c = x_3 - a + b = x_3 - \frac{x_1 - 2x_2 + x_3}{2} + \frac{3x_1 - 8x_2 + 5x_3}{2} = \frac{2x_3 - x_1 + 2x_2 - x_3 + 3x_1 - 8x_2 + 5x_3}{2} = \frac{2x_1 - 6x_2 + 6x_3}{2}$$

$$X_{Set\ 1\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{3x_1 - 8x_2 + 5x_3}{2}\right)y + (x_1 - 3x_2 + 3x_3)$$

4.6. Summary of equation's behavior:

Then, the 5 set together can be summarized as:

$$X_{Set\ 1\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{3x_1 - 8x_2 + 5x_3}{2}\right)y + (x_1 - 3x_2 + 3x_3)$$
$$X_{Set\ 2\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{x_1 - 4x_2 + 3x_3}{2}\right)y + x_3$$
$$X_{Set\ 3\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{-x_1 + x_3}{2}\right)y + x_2$$
$$X_{Set\ 4\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{-3x_1 + 4x_2 - x_3}{2}\right)y + x_1$$
$$X_{Set\ 5\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{-5x_1 + 8x_2 - 3x_3}{2}\right)y + (3x_1 - 3x_2 + x_3)$$

4.7. Conclusion

We can have the simplest equation for a quadratic function. It will be defined the simplest quadratic equation as being:

$$X_{Set\ 3\ of\ 5}(y) = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{x_3 - x_1}{2}\right)y + x_2$$

5. Coefficients (a, b, c), Determinant, Y_vertex and X_vertex

Once we have defined our general most simple 2nd degree polynomial equation as being $x = ay^2 + by + c = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{-x_1 + x_3}{2}\right)y + x_2$, then,

5.1. Coefficient a

$$a = \frac{x_1 - 2x_2 + x_3}{2}$$

5.2. Coefficient b

$$b = \frac{-x_1 + x_3}{2}$$

5.3. Coefficient c

$$c = x_2$$

5.4. Y_vertex equation

$$y_{vertex} = -\frac{b}{2a} = -\frac{\frac{-x_1 + x_3}{2}}{2\frac{x_1 - 2x_2 + x_3}{2}}$$

$$y_{vertex} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3}$$

5.5. X_vertex equation

$$x_{vertex} = -\frac{b^2 - 4ac}{4a} = -\frac{\left(\frac{-x_1 + x_3}{2}\right)^2 - 4\left(\frac{x_1 - 2x_2 + x_3}{2}\right)x_2}{4\left(\frac{x_1 - 2x_2 + x_3}{2}\right)} = \frac{4\left(\frac{x_1 - 2x_2 + x_3}{2}\right)x_2 - \left(\frac{-x_1 + x_3}{2}\right)^2}{2(x_1 - 2x_2 + x_3)} = \frac{2(x_1 - 2x_2 + x_3)x_2 - \frac{(-x_1 + x_3)^2}{4}}{2(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8x_1x_2 - 16x_2^2 + 8x_2x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(x_1 - 2x_2 + x_3)}$$

$$= \frac{8x_1x_2 - 16x_2^2 + 8x_2x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(x_1 - 2x_2 + x_3)} = \frac{-x_1^2 - 16x_2^2 - x_3^2 + 8x_1x_2 + 8x_2x_3 + 2x_1x_3}{8(x_1 - 2x_2 + x_3)} = \frac{-x_1^2 - (4x_2)^2 - x_3^2 + 2x_1(4x_2) + 2(4x_2)x_3 + 2x_1x_3}{8(x_1 - 2x_2 + x_3)}$$

Or more simplified way:

$$x_{vertex} = -\frac{b^2 - 4ac}{4a} = c - \frac{b^2}{4a} = x_2 - \frac{\left(\frac{-x_1 + x_3}{2}\right)^2}{4\left(\frac{x_1 - 2x_2 + x_3}{2}\right)} = x_2 - \frac{\frac{(-x_1 + x_3)^2}{4}}{2(x_1 - 2x_2 + x_3)}$$

$$x_{vertex} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)}$$

5.6. Determinant equation

Note that

$$x_{vertex} = -\frac{b^2 - 4ac}{4a} = -\frac{\Delta}{4a}$$

And,

$$x_{vertex} = \frac{-x_1^2 - 16x_2^2 - x_3^2 + 8x_1x_2 + 8x_2x_3 + 2x_1x_3}{8(x_1 - 2x_2 + x_3)} = -\frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{\frac{16(x_1 - 2x_2 + x_3)}{2}} = -\frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{16a}$$

$$x_{vertex} = -\frac{\left(\frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4}\right)}{4a}$$

So,

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4}$$

$$\Delta = \frac{x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2(4x_2)x_3 - 2x_1x_3}{4}$$

We know that,

$$(X - Y - Z)^2 = X^2 + Y^2 + Z^2 - 2XY - 2XZ + 2YZ$$

So,

$$X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = (X - Y - Z)^2 - 4YZ$$

Then,

$$X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = (X - Y - Z + 2\sqrt{YZ})(X - Y - Z - 2\sqrt{YZ})$$

Now, being $X = x_1, Y = 4x_2, Z = x_3$, then:

$$x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2x_1x_3 - 2(4x_2)x_3 = \left(x_1 - 4x_2 - x_3 + 2\sqrt{4x_2x_3}\right)\left(x_1 - 4x_2 - x_3 - 2\sqrt{4x_2x_3}\right)$$

$$x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2x_1x_3 - 2(4x_2)x_3 = (x_1 - 4x_2 - x_3 + 4\sqrt{x_2x_3})(x_1 - 4x_2 - x_3 - 4\sqrt{x_2x_3})$$

$$x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2x_1x_3 - 2(4x_2)x_3 = (x_1 - x_3 - 4(x_2 - \sqrt{x_2x_3}))(x_1 - x_3 - 4(x_2 + \sqrt{x_2x_3}))$$

$$x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2x_1x_3 - 2(4x_2)x_3 = (x_1 - x_3 - 4\sqrt{x_2}(\sqrt{x_2} - \sqrt{x_3}))(x_1 - x_3 - 4\sqrt{x_2}(\sqrt{x_2} + \sqrt{x_3}))$$

$$numerator(\Delta) = (x_1 - x_3 - 4\sqrt{x_2}(\sqrt{x_2} - \sqrt{x_3}))(x_1 - x_3 - 4\sqrt{x_2}(\sqrt{x_2} + \sqrt{x_3}))$$

6. Composite generator and Prime sequences seeker

Let's define **quadratic composite generator as being the equation $x = ay^2 + by + c$ which generates zero or a finite number of primes elements.**
From our last determinant equation given by $\Delta = \frac{x_1^2+(4x_2)^2+x_3^2-2x_1(4x_2)-2(4x_2)x_3-2x_1x_3}{4}$ we can conclude that:

- Determinant will be a square only when numerator is a square.
- There will be a quadratic composite generator when $numerator(\Delta) = (x_1 - x_3 - 4\sqrt{x_2}(\sqrt{x_2} - \sqrt{x_3}))(x_1 - x_3 - 4\sqrt{x_2}(\sqrt{x_2} + \sqrt{x_3})) = N^2 = square$.
- If just one of the 3 elements (x_1, x_2, x_3) is zero, then $numerator(\Delta) = \left(x_1 - x_3 - 4\sqrt{x_2}(\sqrt{x_2} - \sqrt{x_3})\right)\left(x_1 - x_3 - 4\sqrt{x_2}(\sqrt{x_2} + \sqrt{x_3})\right)$will be a perfect square.
- This is why any element ZERO in Paraboctys will generate infinite many composite quadratic elements with only a finite number of primes.
- Notice that these composite generators will "cut" sequences of primes in Paraboctys and contribute to limit the size of prime sequences.

Conclusion: the best way to compare prime sequences sizes is comparing the sequences generated by a combination between a selection of 2 primes in 1st degree, 3 primes in 2nd degree, 4 primes in 3rd degree, and so on. See the chapter xxx how we created a prime sequences seeker.

6.1. Elements ZERO and UNIT as generators

From these findings, we can conclude:

- ZERO generates COMPOSITE GENERATORS polynomials.
- UNIT generates PRIME GENERATORS polynomials.

7. Behavior of Y_vertex, coefficient a, Determinant and X_vertex

Let's see the behavior of these parameters in six particular cases:

7.1. When $x_1 = x_2$

$$y_{vertex} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 - x_3}{2x_1 - 4x_1 + 2x_3} = \frac{x_1 - x_3}{-2x_1 + 2x_3} = -\frac{x_1 - x_3}{2x_1 - 2x_3} = -\frac{1}{2}$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - 2x_1 + x_3}{2} = \frac{-x_1 + x_3}{2}$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_1^2 + x_3^2 - 8x_1x_1 - 8x_1x_3 - 2x_1x_3}{4} = \frac{9x_1^2 + x_3^2 - 10x_1x_3}{4} = \frac{(-x_1 + x_3)^2 + 8x_1^2 - 8x_1x_3}{4} = \frac{(-x_1 + x_3)^2 - (-8x_1^2 + 8x_1x_3)}{4} = \frac{(-x_1 + x_3)^2 - 8x_1(-x_1 + x_3)}{4}$$

$$= (-x_1 + x_3)\frac{(-x_1 + x_3) - 8x_1}{4} = (-x_1 + x_3)\frac{x_3 - 9x_1}{4}$$

$$x_{vertex} = -\frac{\Delta}{4a} = -\frac{(-x_1 + x_3)\frac{x_3 - 9x_1}{4}}{4\frac{-x_1 + x_3}{2}} = -\frac{\frac{x_3 - 9x_1}{4}}{2} = -\frac{x_3 - 9x_1}{8} = \frac{9x_1 - x_3}{8}$$

$$x_{vertex} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(x_1 - 2x_1 + x_3)} = \frac{8(-x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(-x_1 + x_3)} = \frac{-8x_1^2 + 8x_1x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(-x_1 + x_3)} = \frac{-8x_1^2 + 8x_1x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(-x_1 + x_3)} = \frac{-9x_1^2 + 10x_1x_3 - x_3^2}{8(-x_1 + x_3)}$$

$$= \frac{9x_1^2 - 10x_1x_3 + x_3^2}{8(x_1 - x_3)} = \frac{(x_1 - x_3)^2 + 8x_1^2 - 8x_1x_3}{8(x_1 - x_3)} = \frac{(x_1 - x_3)^2 + 8x_1(x_1 - x_3)}{8(x_1 - x_3)} = \frac{(x_1 - x_3) + 8x_1}{8} = \frac{9x_1 - x_3}{8}$$

$$x_{vertex} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_1 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_1 + x_3)} = x_1 - \frac{(x_3 - x_1)^2}{8(-x_1 + x_3)} = \frac{8(-x_1 + x_3)x_1 - (x_3 - x_1)^2}{8(-x_1 + x_3)} = \frac{-8x_1^2 + 8x_1x_3 - (x_3^2 + x_1^2 - 2x_1x_3)}{8(-x_1 + x_3)} == \frac{-8x_1^2 + 8x_1x_3 - x_3^2 - x_1^2 + 2x_1x_3}{8(-x_1 + x_3)} = \frac{-9x_1^2 + 10x_1x_3 - x_3^2}{8(-x_1 + x_3)} = \frac{9x_1^2 - 10x_1x_3 + x_3^2}{8(x_1 - x_3)}$$

$$= \frac{(x_1 - x_3)^2 + 8x_1^2 - 8x_1x_3}{8(x_1 - x_3)} = \frac{(x_1 - x_3)^2 + 8x_1(x_1 - x_3)}{8(x_1 - x_3)} = \frac{(x_1 - x_3) + 8x_1}{8} = \frac{9x_1 - x_3}{8}$$

7.2. When $x_1 = -x_2$

$$y_{vertex} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 - x_3}{2x_1 + 4x_1 + 2x_3} = \frac{x_1 - x_3}{6x_1 + 2x_3} = \frac{1}{2}\left(\frac{x_1 - x_3}{3x_1 + x_3}\right)$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 + 2x_1 + x_3}{2} = \frac{3x_1 + x_3}{2}$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_1^2 + x_3^2 + 8x_1x_1 + 8x_1x_3 - 2x_1x_3}{4} = \frac{25x_1^2 + x_3^2 + 6x_1x_3}{4} = \frac{(3x_1 + x_3)^2 + 16x_1^2}{4}$$

$$x_{vertex} = -\frac{\Delta}{4a} = -\frac{\frac{(3x_1 + x_3)^2 + 16x_1^2}{4}}{4\frac{3x_1 + x_3}{2}} = -\frac{\frac{(3x_1 + x_3)^2 + 16x_1^2}{4}}{2(3x_1 + x_3)} == -\frac{(3x_1 + x_3)^2 + 16x_1^2}{8(3x_1 + x_3)} = -\left(\frac{(3x_1 + x_3)}{8} + \frac{2x_1^2}{(3x_1 + x_3)}\right)$$

$$x_{vertex} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{-8(x_1 + 2x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(x_1 + 2x_1 + x_3)} = \frac{-8(3x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(3x_1 + x_3)} = \frac{-24x_1^2 - 8x_1x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(3x_1 + x_3)} = \frac{-24x_1^2 - 8x_1x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(3x_1 + x_3)} = \frac{-25x_1^2 - 6x_1x_3 - x_3^2}{8(3x_1 + x_3)}$$

$$= -\frac{25x_1^2 + 6x_1x_3 + x_3^2}{8(3x_1 + x_3)} = -\frac{(3x_1 + x_3)^2 + 16x_1^2}{8(3x_1 + x_3)} = -\left(\frac{(3x_1 + x_3)}{8} + \frac{2x_1^2}{(3x_1 + x_3)}\right)$$

$$x_{vertex} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = -x_1 - \frac{(x_3 - x_1)^2}{8(x_1 + 2x_1 + x_3)} = -x_1 - \frac{(x_3 - x_1)^2}{8(3x_1 + x_3)} = \frac{-8(3x_1 + x_3)x_1 - (x_3 - x_1)^2}{8(3x_1 + x_3)} = \frac{-24x_1^2 - 8x_1x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(3x_1 + x_3)} = \frac{-24x_1^2 - 8x_1x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(3x_1 + x_3)} = \frac{-25x_1^2 - 6x_1x_3 - x_3^2}{8(3x_1 + x_3)}$$

$$= -\frac{(3x_1 + x_3)^2 + 16x_1^2}{8(3x_1 + x_3)} = -\left(\frac{(3x_1 + x_3)}{8} + \frac{2x_1^2}{(3x_1 + x_3)}\right)$$

7.3. When $x_1 = x_3$

$$y_{vertex} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = 0$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - 2x_2 + x_1}{2} = \frac{2x_1 - 2x_2}{2} = x_1 - x_2$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_2^2 + x_1^2 - 8x_1x_2 - 8x_2x_1 - 2x_1x_1}{4} = \frac{16x_2^2 - 16x_1x_2}{4} = 4x_2^2 - 4x_1x_2 = 4x_2(x_2 - x_1)$$

$$x_{vertex} = -\frac{\Delta}{4a} = -\frac{4x_2(x_2 - x_1)}{4(x_1 - x_2)} = x_2$$

$$x_{vertex} = \frac{8(x_1 - 2x_2 + x_3)x_2 + (x_1 - x_3)^2}{2(x_1 - 2x_2 + x_3)} = \frac{8(2x_1 - 2x_2)x_2}{2(2x_1 - 2x_2)} = \frac{8(2x_1 - 2x_2)x_2}{8(2x_1 - 2x_2)} = x_2$$

$$x_{vertex} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(x_1 - x_1)^2}{8(x_1 - 2x_2 + x_1)} = x_2$$

7.4. When $x_1 = -x_3$

$$y_{vertex} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 + x_1}{2x_1 - 4x_2 - 2x_1} = \frac{2x_1}{-4x_2} = -\frac{x_1}{2x_2}$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - 2x_2 - x_1}{2} = -x_2$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_2^2 + x_1^2 - 8x_1x_2 + 8x_2x_1 + 2x_1x_1}{4} = \frac{4x_1^2 + 16x_2^2}{4} = x_1^2 + 4x_2^2$$

$$x_{vertex} = -\frac{\Delta}{4a} = -\frac{x_1^2 + 4x_2^2}{4(-x_2)} = \frac{x_1^2 + 4x_2^2}{4x_2}$$

$$x_{vertex} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_2 - x_1)x_2 - (-x_1 - x_1)^2}{8(x_1 - 2x_2 - x_1)} = \frac{8(-2x_2)x_2 - (-2x_1)^2}{8(-2x_2)} = \frac{-16x_2^2 - 4x_1^2}{-16x_2} = \frac{4x_2^2 + x_1^2}{4x_2}$$

$$x_{vertex} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(-x_1 - x_1)^2}{8(x_1 - 2x_2 - x_1)} = x_2 - \frac{4x_1^2}{-16x_2} = x_2 + \frac{x_1^2}{4x_2} = \frac{4x_2^2 + x_1^2}{4x_2}$$

7.5. When $x_2 = x_3$

$$y_{vertex} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 - x_2}{2x_1 - 2x_2} = \frac{1}{2}$$
$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - x_2}{2}$$
$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_2^2 + x_2^2 - 8x_1x_2 - 8x_2x_2 - 2x_1x_2}{4} = \frac{x_1^2 + 9x_2^2 - 10x_1x_2}{4} = \frac{(x_1 - x_2)^2 + 8x_2^2 - 8x_1x_2}{4} = \frac{(x_1 - x_2)^2 + 8x_2(x_2 - x_1)}{4} = \frac{(x_1 - x_2)^2 - 8x_2(x_1 - x_2)}{4}$$
$$= (x_1 - x_2)\frac{(x_1 - x_2) - 8x_2}{4} = (x_1 - x_2)\frac{x_1 - 9x_2}{4}$$
$$x_{vertex} = -\frac{\Delta}{4a} = -\frac{(x_1 - x_2)\frac{x_1 - 9x_2}{4}}{4\frac{x_1 - x_2}{2}} = -\frac{x_1 - 9x_2}{8} = \frac{-x_1 + 9x_2}{8}$$
$$x_{vertex} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_2 + x_2)x_2 - (-x_1 + x_2)^2}{8(x_1 - 2x_2 + x_2)} = \frac{8(x_1 - x_2)x_2 - (-x_1 + x_2)^2}{8(x_1 - x_2)} = \frac{8(x_1 - x_2)x_2 - (-x_1 + x_2)^2}{8(x_1 - x_2)} = \frac{8(x_1 - x_2)x_2 - (-x_1 + x_2)(-x_1 + x_2)}{8(x_1 - x_2)} = \frac{8(x_1 - x_2)x_2 + (x_1 - x_2)(-x_1 + x_2)}{8(x_1 - x_2)}$$
$$= \frac{8x_2 + (-x_1 + x_2)}{8} = \frac{-x_1 + 9x_2}{8}$$
$$x_{vertex} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(x_2 - x_1)^2}{8(x_1 - 2x_2 + x_2)} = x_2 - \frac{(x_2 - x_1)^2}{8(x_1 - x_2)} = x_2 + \frac{(x_2 - x_1)^2}{8(x_2 - x_1)} = x_2 + \frac{x_2 - x_1}{8} = \frac{-x_1 + 9x_2}{8}$$

7.6. When $x_2 = -x_3$

$$y_{vertex} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 + x_2}{2x_1 - 4x_2 - 2x_2} = \frac{1}{2}\left(\frac{x_1 + x_2}{x_1 - 3x_2}\right)$$
$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - 2x_2 - x_2}{2} = \frac{x_1 - 3x_2}{2}$$
$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_2^2 + x_2^2 - 8x_1x_2 + 8x_2x_2 + 2x_1x_2}{4} = \frac{x_1^2 + 25x_2^2 - 6x_1x_2}{4} = \frac{(x_1 - 3x_2)^2 + 16x_2^2}{4}$$
$$x_{vertex} = -\frac{\Delta}{4a} = -\frac{\frac{(x_1 - 3x_2)^2 + 16x_2^2}{4}}{4\frac{x_1 - 3x_2}{2}} = -\frac{\frac{(x_1 - 3x_2)^2 + 16x_2^2}{4}}{2(x_1 - 3x_2)} = -\frac{(x_1 - 3x_2)^2 + 16x_2^2}{8(x_1 - 3x_2)} = -\left(\frac{(x_1 - 3x_2)}{8} + \frac{2x_2^2}{(x_1 - 3x_2)}\right)$$
$$x_{vertex} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_2 - x_2)x_2 - (-x_1 - x_2)^2}{8(x_1 - 2x_2 - x_2)} = \frac{8(x_1 - 3x_2)x_2 - (-x_1 - x_2)^2}{8(x_1 - 3x_2)} = \frac{8x_1x_2 - 24x_2^2 - (x_1^2 + x_2^2 + 2x_1x_2)}{8(x_1 - 3x_2)} = \frac{8x_1x_2 - 24x_2^2 - x_1^2 - x_2^2 - 2x_1x_2}{8x_1 - 24x_2} = \frac{-x_1^2 - 25x_2^2 + 6x_1x_2}{8x_1 - 24x_2}$$
$$= \frac{-x_1^2 - 25x_2^2 + 6x_1x_2}{8x_1 - 24x_2}$$
$$x_{vertex} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(-x_2 - x_1)^2}{8(x_1 - 2x_2 - x_2)} = x_2 - \frac{(-x_2 - x_1)^2}{8(x_1 - 3x_2)} = \frac{8x_2(x_1 - 3x_2) - (x_1 + x_2)^2}{8(x_1 - 3x_2)} =$$

8. Offset mechanism in quadratics

When we study the general quadratic equation $x = ay^2 + by + c$ we learn that:

- As the second derivative of the equation is only proportional to the coefficient a, then coefficient a is the only one responsible for opening or closing the "mouth" of the quadratic. No other coefficient will change the “mouth” shape of a quadratic. In fact, this is given by “latus rectum”;

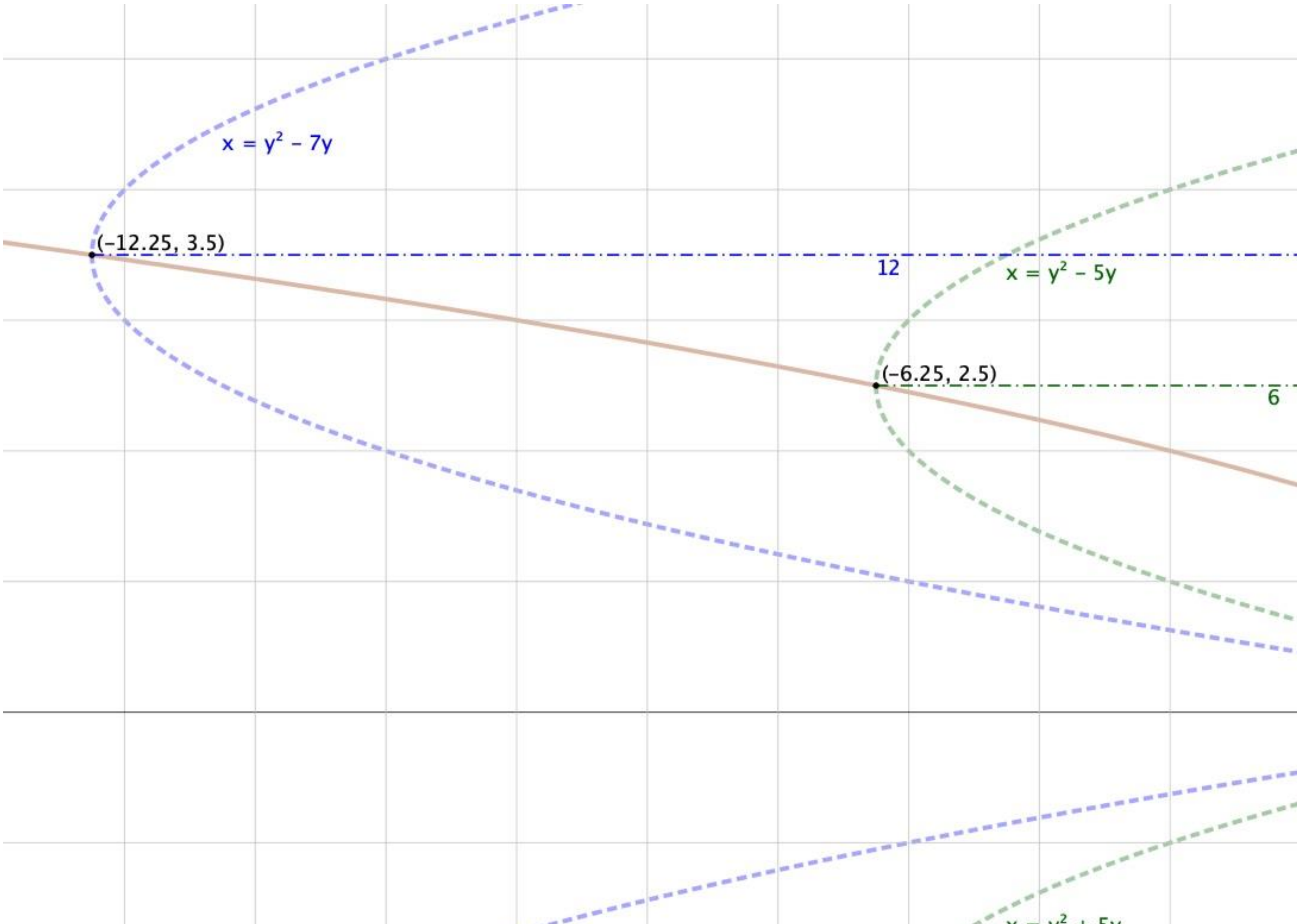
$$Latus\ Rectum = LR = absolute\ value\ of\ \left(\frac{1}{a}\right)$$

  - Note that here is where a quadratic is born. Whatever the equation of a quadratic, it will always be of the form $x = ay^2$.
  - The form of a quadratic always starts at the vertex in the dot $(X, Y) = (0,0)$ with the orientation of the opening of its “mouth” determined by coefficient a. This is because the form $x = ay^2$ of a quadratic contains the coefficients b and c zeroed and therefore $x_{vertex} = 0$ and $y_{vertex} = 0$. No matter the value of the coefficient a, the vertex from the form $x = ay^2$ of a quadratic never leaves the position (0,0).
- If we want to change the position of the vertex from (0,0) we have to use the others two coefficients (b, c) properly.

- Once we have coefficient a determined and constant, knowing that $y_{vertex} = -\frac{b}{2a}$, then, the coefficient b is the only responsible for the shift (offset/phase) of the vertex of the quadratic along the Y$-$axis. Coefficients a, c do not change the vertex along Y-axis.
- Because $x_{vertex} = -\frac{b^2-4ac}{4a}$, then changes in coefficient b also causes a shift of the vertex along the X-axis.
- In other words, to shift the vertex along the Y-axis we have to vary only the coefficient b. But, when we vary the coefficient b for the vertex to reach the desired value of y_{vertex}, we automatically change the value of x_{vertex} proportional to the square of the coefficient b.

- Once the general quadratic equation $x = (something\ 2nd\ degree) + c$, then, the coefficient c is responsible only for the shift of the vertex along the X-axis.
  - This also can be seen in the equation $x_{vertex} = -\frac{b^2}{4a} + c$.
  - So, if we want to cancel the vertex shift caused by the coefficient b on the X-axis, we can compensate using properly the coefficient c that only changes shift on the X-axis.

See in the figure the offset mechanism showed for A002378 Oblong (or promic, pronic, or heteromecic) numbers:



x = y² – 7y
(–12.25, 3.5)
12
x = y² – 5y
(–6.25, 2.5)
6

Considering general quadratic equation

$$X(y) = x = ay^2 + by + c$$

when we shift the curve along the Y-axis in steps of a unit, we are not changing the values that x assumes in the X-axis. For example, when we shift the quadratic $x = y^2 + 5y + 6$ in 5 units step up along the Y-axis, then if initially, X(-1) had the value 2 now X(-1+5)=X(4) is also 2. So, we can express the general quadratic equation above as:

$$X(y \pm h) = a(y \pm h)^2 + b(y \pm h) + c$$

where

$$x_{vertex}(y \pm h) = x_{vertex}(y)$$

So, we can express:

$$X(y \pm h) = a(y \pm h)^2 + b(y \pm h) + c$$
$$X(y \pm h) = a(y^2 \pm 2hy + h^2) + by \pm bh + c$$
$$X(y \pm h) = ay^2 \pm 2ahy + ah^2 + by \pm bh + c$$
$$X(y \pm h) = ay^2 + (b \pm 2ah)y + (ah^2 \pm bh + c)$$

Note that the quadratic curve shift along the Y-axis is represented here by variable h in $X(y \pm h)$. In all cases, we are considering absolute value of h to express the existing shift. So, we are predicting:

$X(y + h) = ay^2 + (b + 2ah)y + (ah^2 + bh + c)$ to be a shift in the curve in the NEGATIVE DIRECTION along the Y-axis, and

$X(y - h) = ay^2 + (b - 2ah)y + (ah^2 - bh + c)$ to be a shift in the curve in the POSITIVE DIRECTION along the Y-axis.

9.1. X_vertex mechanism with offset

Note that the quadratic shift along the Y-axis is represented here by variable h in $X(y \pm h)$. Always the shift value is a positive value. So, we are predicting the quadratic X(y + h) to be a shift in the NEGATIVE direction in the Y-axis and X(y-h) to be a shift in the POSITIVE direction of the Y-axis. In both cases, x_{vertex} will be exactly the same. See the confirmation:

For

$$X(y) = x = ay^2 + by + c$$

Then,

$$x_{vertex}(y) = -\frac{b^2}{4a} + c$$

And, for:

$$X(y \pm h) = a(y \pm h)^2 + b(y \pm h) + c$$
$$X(y \pm h) = ay^2 + (b \pm 2ah)y + (ah^2 \pm bh + c)$$

Then,

$$x_{vertex}(y \pm h) = -\frac{(b \pm 2ah)^2}{4a} + (ah^2 \pm bh + c)$$
$$x_{vertex}(y + h) = -\frac{(b + 2ah)^2}{4a} + (ah^2 + bh + c)$$
$$x_{vertex}(y + h) = \frac{4a(ah^2 + bh + c)}{4a} - \frac{(b + 2ah)^2}{4a}$$
$$x_{vertex}(y + h) = \frac{4a^2h^2 + 4abh + 4ac - b^2 - 4a^2h^2 - 4abh}{4a}$$
$$x_{vertex}(y + h) = c - \frac{b^2}{4a} = x_{vertex}(y)$$
$$x_{vertex}(y - h) = -\frac{(b - 2ah)^2}{4a} + (ah^2 - bh + c)$$

$$x_{vertex}(y-h)=\frac{4a(ah^2-bh+c)}{4a}-\frac{(b-2ah)^2}{4a}$$
$$x_{vertex}(y-h)=\frac{4a^2h^2-4abh+4ac-b^2+4a^2h^2-4abh}{4a}$$
$$x_{vertex}(y-h)=c-\frac{b^2}{4a}=x_{vertex}(y)$$

9.2. Y_vertex mechanism with offset

$$X(y)=x=ay^2+by+c$$
$$y_{vertex}=-\frac{b}{2a}$$

Or, we can express:

$$y_{vertex}(h=0)=-\frac{b}{2a}$$
$$X(y\pm h)=a(y\pm h)^2+b(y\pm h)+c$$
$$X(y\pm h)=ay^2+(b\pm 2ah)y+(ah^2\pm bh+c)$$
$$y_{vertex}(h)=-\frac{b\pm 2ah}{2a}$$
$$y_{vertex}(h)=-\frac{b}{2a}\mp h$$
$$y_{vertex}(h)=y_{vertex}(h=0)\mp h$$

So, there is an inversion of sign between y_{vertex} and $offset=h$.

This confirms for $h>0$:

- $X(y+h)=ay^2+(b+2ah)y+(ah^2+bh+c)$ to be a shift in the curve in the NEGATIVE DIRECTION along the Y-axis, and
- $X(y-h)=ay^2+(b-2ah)y+(ah^2-bh+c)$ to be a shift in the curve in the POSITIVE DIRECTION along the Y-axis.

When we increase offset in positive values we decrease the y_{vertex} value.

So, for any $X(y\pm h)=ay^2+(b\pm 2ah)y+(ah^2\pm bh+c)$, where $abs(b\pm 2ah)>abs(b)$ we can always find a new equation where h=0 and will represent the same sequence.

9.3. Quadratics roots mechanism with Offset

Another way to see the effect of the offset is by analyzing what happens to the roots of quadratics as we vary the offset.

See the formula of the roots of the quadratic equation: $x=ay^2+by+c$.

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x_{1,2}=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$
$$x_{1,2}=\frac{-b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$
$$x_{1,2}=\frac{-b}{2a}\pm\sqrt{\frac{\Delta}{4a^2}}$$
$$x_{1,2}=\frac{-b}{2a}\pm\sqrt{\frac{\Delta}{4a}*\frac{1}{a}}$$
$$x_{1,2}=y_{vertex}\pm\sqrt{-x_{vertex}*latus\ rectum}$$

When we vary offset of a quadratic, coefficient a and x_{vertex} are constant. The ony variable is y_{vertex}. So,

$$x_{1,2} = y_{vertex} \pm \text{constant}$$

The roots vary proportionally with y_{vertex} and opposite with h.

9.4. Coefficient c mechanism with Offset

The roots are given by

$$x_{1,2} = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$x_{1,2} = \frac{-b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}}$$
$$x_{1,2} = \frac{-b}{2a} \pm \sqrt{\left(-\frac{b}{2a}\right)^2 - \frac{c}{a}}$$
$$x_{1,2} = y_{vertex} \pm \sqrt{y_{vertex}^2 - \frac{c}{a}}$$

But, we know that

$$\sqrt{y_{vertex}^2 - \frac{c}{a}} = constant = \sqrt{\frac{\Delta}{4a^2}}$$

Then,

$$y_{vertex}^2 - \frac{c}{a} = constant = k = \frac{\Delta}{4a^2}$$
$$y_{vertex}^2 - \frac{c}{a} = k$$
$$ay_{vertex}^2 - c = ak$$
$$c = ay_{vertex}^2 - ak$$
$$c = a(y_{vertex}^2 - k)$$

Because a and k are constant, then, coefficient c is only proportional to y_{vertex}^2.

Remember that

- When in offset zero at $h = 0$, then $x(y = 0) = a * 0 + b * 0 + c = x_2$. When we move the curve along Y-axis from this initial position, coefficient c will change proportionally with y_{vertex}^2.
- Another way to see, coefficient c is only dependent with the square of y_{vertex} position.

10. Definition of offset f=0 (offset zero) and offset f≠0

Now we will define offset f=0 (offset zero):

Offset ZERO is the closest y_{vertex} to $y = 0$ as possible in the simplest equation possible.

From this definition:

- For $y_{vertex} = -1$, offset is $f = -1$.
- For $y_{vertex} = -\frac{3}{4}$, offset is $f = -1$.

- For $y_{vertex} = -\frac{1}{2}$, offset is $f = -1$.
- For $y_{vertex} = -\frac{1}{4}$, offset is $f = 0$.
- For $y_{vertex} = 0$, offset is $f = 0$.
- For $y_{vertex} = \frac{1}{4}$, offset is $f = 0$.
- For $y_{vertex} = \frac{1}{2}$, offset is $f = 0$.
- For $y_{vertex} = \frac{3}{4}$, offset is $f = 1$.
- For $y_{vertex} = 1$, offset is $f = 1$.

In this definition, we are determining offset f to follow y_{vertex}. When y_{vertex} increases, the offset f will increase and vice-versa. So, offset is opposite to the shift h imposed to the index in the polynomial equation:

$$f = -h$$

10.1. Condition to offset zero

Another consequence of this definition is the value of offset f to follow y_{vertex}. In any integer sequence, y_{vertex} varies in unit steps.
So, the closest y_{vertex} to zero will be in the range:

$$-0.5 < y_{vertex}(@offset\ zero) \leq 0.5$$
$$-0.5 < \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} \leq 0.5$$
$$-1 < \frac{x_1 - x_3}{x_1 - 2x_2 + x_3} \leq 1$$
$$-x_1 + 2x_2 - x_3 < x_1 - x_3 \leq x_1 - 2x_2 + x_3$$

From

$$x_1 - x_3 \leq x_1 - 2x_2 + x_3$$
$$-2x_3 \leq -2x_2$$

Then,

$$x_3 \geq x_2$$

From

$$-x_1 + 2x_2 - x_3 < x_1 - x_3$$
$$2x_2 < 2x_1$$

Then,

$$x_2 < x_1$$

So, the quadratic condition to result in offset zero is:

$$x_3 \geq x_2 < x_1$$

So, x_2 is the "vertex integer" because it is the closest integer to x_{vertex}. This means that once determined x_2, any $x_1 > x_2$ and $x_3 \geq x_2$ will define the quadratic curve with offset zero.
So, when offset increase, the "h" displacement decreases and vice-versa.

11. OFFSET equation

Note that the shift or offset of a quadratic integer sequence $x = ay^2 + by + c$ causes change only on y_{vertex}. The value of x_{vertex} remains constant.

At the same time, the coefficient a do not change with offset and remains always constant.

Then, we can get a general equation of offset following these steps:

1. Let's define the equation where -0.5 $< y_{vertex}(@offset\ zero) \leq$ 0.5 is given by:

$$X_{offset=0}(y) = X_{-0.5<y_{vertex}\leq 0.5}(y) = Xº(y) = ay^2 + bºy + cº$$

2. Then,

$$y_{vertex@offset=0} = -\frac{bº}{2a}$$
$$x_{vertex@offset=0} = \frac{-\Delta}{4a} = -\frac{bº^2 - 4acº}{4a}$$

3. So,

$$x_{offset\neq zero} = a(y+h)^2 + bº(y+h) + cº$$
$$x_{offset\neq zero} = a(y^2 + h^2 + 2hy) + bºy + bºh + cº$$
$$x_{offset\neq zero} = ay^2 + ah^2 + 2ahy + bºy + bºh + cº$$
$$x_{offset\neq zero} = ay^2 + (2ahy + bºy) + (ah^2 + bºh + cº)$$

Which result in the general offset equation:

$$x_{offset\neq zero} = ay^2 + (bº + 2ah)y + (ah^2 + bºh + cº)$$

4. Notice, when we apply offset, there is no alteration in coefficient a. Now, the new b and c coefficients are:

$$b = bº + 2ah$$
$$c = ah^2 + bºh + cº$$
$$y_{vertex@offset\neq 0} = -\frac{b}{2a} = -\frac{bº + 2ah}{2a} = -\frac{bº}{2a} - h$$
$$y_{vertex@offset\neq 0} = y_{vertex@offset=0} - h$$
$$x_{vertex@offset\neq 0} = x_{vertex@offset=0}$$
$$-\frac{b^2 - 4ac}{4a} = -\frac{bº^2 - 4acº}{4a}$$
$$b^2 - 4ac = bº^2 - 4acº$$
$$(2ah + bº)^2 - 4a(ah^2 + bºh + cº) = bº^2 - 4acº$$
$$4a^2h^2 + bº^2 + 4ahbº - 4a^2h^2 - 4abºh - 4acº = bº^2 - 4acº$$
$$4a^2h^2 + bº^2 + 4ahbº - 4a^2h^2 - 4abºh - 4acº = bº^2 - 4acº$$

5. From $y_{vertex@offset\neq 0} = -\frac{bº}{2a} - h = y_{vertex@offset\neq 0} - h$ equation, we can see that as far as we increase $h > 0$, $y_{vertex@offset\neq 0}$ decrease from $y_{vertex@offset=0}$.

6. So, if we want offset parameter to follow y_{vertex} direction, we have to change the signal between offset and parameter h:

$$offset = f = -h$$

So, from our general offset equation

$$x = ay^2 + (bº - 2af)y + (af^2 - bºf + cº)$$

we will have

$$y_{vertex} = -\frac{bº - 2af}{2a}$$
$$y_{vertex} = f - \frac{bº}{2a}$$

and

$$x_{vertex} = -\frac{\Delta}{4a} = -\frac{(bº - 2af)^2 - 4a(af^2 - bºf + cº)}{4a} = \frac{4a(af^2 - bºf + cº) - (bº - 2af)^2}{4a} = \frac{4a^2f^2 - 4abºf + 4acº - (bº^2 - 4abºf + 4a^2f^2)}{4a} = \frac{4a^2f^2 - 4abºf + 4acº - bº^2 + 4abºf - 4a^2f^2}{4a}$$
$$= \frac{4a^2f^2 - 4abºf + 4acº - bº^2 + 4abºf - 4a^2f^2}{4a} = \frac{4acº - bº^2}{4a}$$

$$x_{vertex} = -\frac{bº^2 - 4acº}{4a} = cº - \frac{bº^2}{4a}$$

From x_{vertex} and y_{vertex} formulas above, we can deduct:

Once a, b^{o} and c^{o} are fixed values x_{vertex} is a fixed value for any offset.

Only y_{vertex} varies in function of offset f.

Starting from the vertex, when moving on the quadratic along one of the two possible directions along Y-axis, we will increase or decrease the value of the index y. In any of the quadratics of our example we will always arrive exactly at the same values of x. This means that any one of them generates the same sequence of integers numbers.

Summary:

$$x = ay^2 + (b^\circ - 2af)y + (af^2 - b^\circ f + c^\circ)$$

$$y_{vertex}(offset \neq 0) = -\frac{b}{2a} = -\frac{b^\circ - 2af}{2a} = -\frac{b^\circ}{2a} + f$$

$$y_{vertex}(offset \neq 0) = y_{vertex}(offset = 0) + f$$

$$a = a^\circ$$

$$b = b^\circ - 2af$$

$$c = af^2 - b^\circ f + c^\circ$$

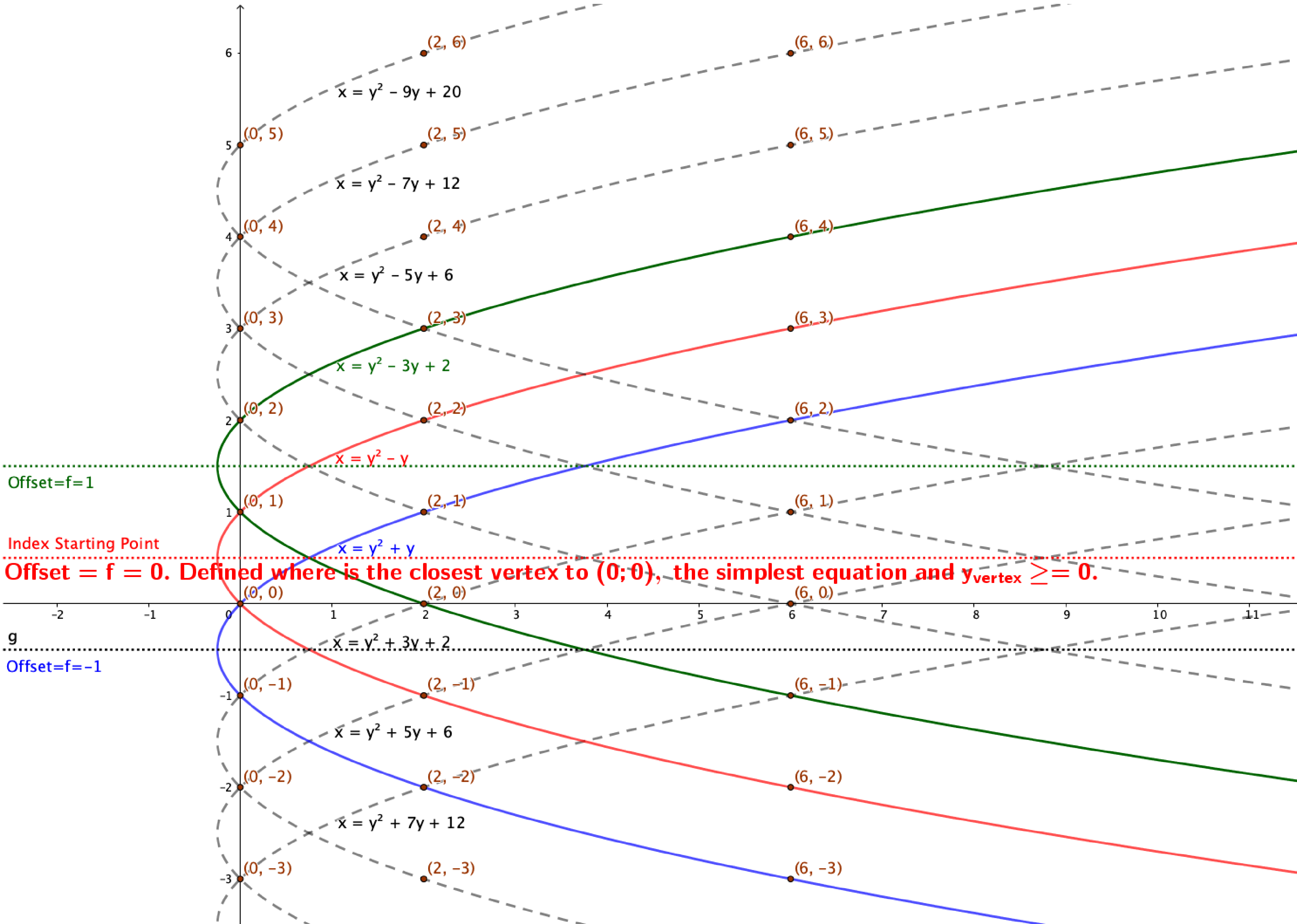
$$a^\circ = a$$

$$b^\circ = b + 2af$$

$$c^\circ = c - af^2 + b^\circ f = af^2 + bf + c$$

11.1. Analysis

From the figure:


x = y² − 9y + 20
x = y² − 7y + 12
x = y² − 5y + 6
x = y² − 3y + 2
x = y² − y
x = y² + y
x = y² + 3y + 2
x = y² + 5y + 6
x = y² + 7y + 12
Offset=f=1
Index Starting Point
Offset = f = 0. Defined where is the closest vertex to (0;0), the simplest equation and $y_{vertex} >= 0$.
g
Offset=f=−1
(2, 6)
(6, 6)
(0, 5)
(2, 5)
(6, 5)
(0, 4)
(2, 4)
(6, 4)
(0, 3)
(2, 3)
(6, 3)
(0, 2)
(2, 2)
(6, 2)
(0, 1)
(2, 1)
(6, 1)
(0, 0)
(2, 0)
(6, 0)
(0, −1)
(2, −1)
(6, −1)
(0, −2)
(2, −2)
(6, −2)
(0, −3)
(2, −3)
(6, −3)

We have:

- In the curve $x = y^2 - 7y + 12$ from vertex given by ($x_{vertex} = -0.25$; $y_{vertex} = 3.5$) we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 - 5y + 6$ from vertex given by ($x_{vertex} = -0.25$; $y_{vertex} = 2.5$) we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 - 3y + 2$ from vertex given by ($x_{vertex} = -0.25$; $y_{vertex} = 1.5$) we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 - y$ from vertex given by ($x_{vertex} = -0.25$; $y_{vertex} = 0.5$) we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 + y$ from vertex given by ($x_{vertex} = -0.25$; $y_{vertex} = -0.5$) we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 + 3y + 2$ from vertex given by ($x_{vertex} = -0.25$; $y_{vertex} = -1.5$) we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 + 5y + 6$ from vertex given by ($x_{vertex} = -0.25$; $y_{vertex} = -2.5$) we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 + 7y + 12$ from vertex given by ($x_{vertex} = -0.25$; $y_{vertex} = -3.5$) we get the sequence $(0, 2, 6, 12, 20, \dots)$

All of them generate the same integer sequence of numbers as far as we increase or decrease y value from y_{vertex} as reference.

Also, note that in these cases y_{vertex} varies by ONE-unit increments, and x_{vertex} remains constant in all cases.

Notice that when we fix a value of index $y = y_n$ as the same for all curves, each equation will generate a different element value in X-axis. That's why, when we want to synchronize the sequence generated by them, we have to consider offset in the equation.

11.2. Offset possibilities

Notice here that we have 2 simplest equations:

$x = y^2 - y$ from $Vertex = (x_{vertex} = -0.25;\ y_{vertex} = 0.5)$ we get the sequence $(0, 2, 6, 12, 20, \dots)$, and

$x = y^2 + y$ from $Vertex = (x_{vertex} = -0.25;\ y_{vertex} = -0.5)$ we get the sequence $(0, 2, 6, 12, 20, \dots)$.

Which one to choose as being offset zero?

We have to define just one equation as being $offset = f = 0$. How to choose?

11.2.1. Option 1

If we define $x = y^2 + y$ with $y_{vertex} = -0.5$ as being $offset = f = 0$, then $x = y^2 - y$ with $y_{vertex} = 0.5$ would have $offset = f = 1$.

Consequently, the whole set would be like this:

- In the curve $x = y^2 - 7y + 12$ we have $y_{vertex} = 3.5$ and $f = 4$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $3 \le y < \infty$
- In the curve $x = y^2 - 5y + 6$ we have $y_{vertex} = 2.5$ and $f = 3$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $2 \le y < \infty$
- In the curve $x = y^2 - 3y + 2$ we have $y_{vertex} = 1.5$ and $f = 2$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $1 \le y < \infty$
- In the curve $x = y^2 - y$ we have $y_{vertex} = 0.5$ and $f = 1$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $0 \le y < \infty$
- **In the curve $x = y^2 + y$ we have $y_{vertex} = -0.5$ and $f = 0$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-1 \le y < \infty$**
- In the curve $x = y^2 + 3y + 12$ we have $y_{vertex} = -1.5$ and $f = -1$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-2 \le y < \infty$
- In the curve $x = y^2 + 5y + 12$ we have $y_{vertex} = -2.5$ and $f = -2$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-3 \le y < \infty$
- In the curve $x = y^2 + 7y + 12$ we have $y_{vertex} = -3.5$ and $f = -3$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-4 \le y < \infty$

In this choice the equation seems to be expressed as:

$$f = ceiling(y_{vertex}) = ceiling\left(-\frac{b}{2a}\right) = \left\lceil -\frac{b}{2a} \right\rceil$$

11.2.2. Option 2

If we define $x = y^2 - y$ with $y_{vertex} = 0.5$ as being $offset = f = 0$, then $x = y^2 + y$ with $y_{vertex} = -0.5$ would have offset $f = -1$.

Consequently, the whole set would be like this:

- In the curve $x = y^2 - 7y + 12$ we have $y_{vertex} = 3.5$ and $f = 3$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $3 \le y < \infty$
- In the curve $x = y^2 - 5y + 6$ we have $y_{vertex} = 2.5$ and $f = 2$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $2 \le y < \infty$
- In the curve $x = y^2 - 3y + 2$ we have $y_{vertex} = 1.5$ and $f = 1$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $1 \le y < \infty$
- **In the curve $\boldsymbol{x = y^2 - y}$ we have $\boldsymbol{y_{vertex} = 0.5}$ and $\boldsymbol{f = 0}$. Sequence $\boldsymbol{(0, 2, 6, 12, 20, \dots)}$ will appear in index sequence $\boldsymbol{0 \le y < \infty}$**
- In the curve $x = y^2 + y$ we have $y_{vertex} = -0.5$ and $f = -1$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-1 \le y < \infty$
- In the curve $x = y^2 + 3y + 12$ we have $y_{vertex} = -1.5$ and $f = -2$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-2 \le y < \infty$
- In the curve $x = y^2 + 5y + 12$ we have $y_{vertex} = -2.5$ and $f = -3$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-3 \le y < \infty$
- In the curve $x = y^2 + 7y + 12$ we have $y_{vertex} = -3.5$ and $f = -4$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-4 \le y < \infty$

In this choice the equation seems to be expressed as:

$$f = floor(y_{vertex}) = floor\left(-\frac{b}{2a}\right) = \left\lfloor -\frac{b}{2a} \right\rfloor$$

Notice, in this case, the reasoning of OEIS always starting the indexes where y=0 for any sequence, here coincides with offset zero.

11.2.3. Comparisons between the options

From our definition: “offset zero is the equation which results in y_{vertex} most close to $y = 0$” then, we expect to see this behavior in yellow:

y_vertex	Option 1 floor(y_vertex)	Option 2 ceiling(y_vertex)	Option 3 round(y_vertex)	Behavior expected in offset
3,5	3	4	4	3
3,25	3	4	3	3
3	3	3	3	3
2,75	2	3	3	3
2,5	2	3	3	2
2,25	2	3	2	2
2	2	2	2	2
1,75	1	2	2	2
1,5	1	2	2	1
1,25	1	2	1	1
1	1	1	1	1
0,75	0	1	1	1
0,5	0	1	1	0
0,25	0	1	0	0
0	0	0	0	0
-0,25	-1	0	0	0
-0,5	-1	0	0	-1
-0,75	-1	0	-1	-1
-1	-1	-1	-1	-1
-1,25	-2	-1	-1	-1
-1,5	-2	-1	-1	-2
-1,75	-2	-1	-2	-2
-2	-2	-2	-2	-2
-2,25	-3	-2	-2	-2
-2,5	-3	-2	-2	-3
-2,75	-3	-2	-3	-3
-3	-3	-3	-3	-3
-3,25	-4	-3	-3	-3
-3,5	-4	-3	-3	-4

(Note: I noticed that in MS-Excel round(-0.5)=-1. I think it is wrong and should be round(-0.5)=0 to sustain http://mathworld.wolfram.com/StaircaseFunction.html . Am I correct?)

MS-Excel

y_vertex	Option 1 floor(y_vertex)	Option 2 ceiling(y_vertex)	Option 3 round(y_vertex)	Behavior expected in offset
3,5	3	4	4	3
3,25	3	4	3	3
3	3	3	3	3
2,75	2	3	3	3
2,5	2	3	3	2
2,25	2	3	2	2
2	2	2	2	2
1,75	1	2	2	2
1,5	1	2	2	1
1,25	1	2	1	1
1	1	1	1	1
0,75	0	1	1	1
0,5	0	1	1	0
0,25	0	1	0	0
0	0	0	0	0
-0,25	-1	0	0	0
-0,5	-1	0	-1	-1
-0,75	-1	0	-1	-1
-1	-1	-1	-1	-1
-1,25	-2	-1	-1	-1
-1,5	-2	-1	-2	-2
-1,75	-2	-1	-2	-2
-2	-2	-2	-2	-2
-2,25	-3	-2	-2	-2
-2,5	-3	-2	-3	-3
-2,75	-3	-2	-3	-3
-3	-3	-3	-3	-3
-3,25	-4	-3	-3	-3
-3,5	-4	-3	-4	-4

In conclusion, there is no current formula to deal with offset.

12. Implementing a new function roundz(y) to get the formal offset formula

The closest formula is the round formula that should be changed to meet $Offset = f = 0$ only when $-0.5 < y_{vertex}(@offset = 0) \leq +0.5$.

It is like a magnet of y_{vertex} to be rounded to integer numbers. In other words,

$$Offset = f = 0 \text{ when } -0.5 < y_{vertex}(@offset = 0) = -\frac{b}{2a} \leq +0.5$$

So,

$$f = 0 \text{ when } \left[(-0.5 + infinitesimal) \leq -\frac{b}{2a} \leq +0.5\right]$$

Expanding:

$$f = -3 \text{ when } \left[(-3.5 + infinitesimal) \leq -\frac{b}{2a} \leq -2.5\right]$$

$$f = -2 \text{ when } \left[(-2.5 + infinitesimal) \leq -\frac{b}{2a} \leq -1.5\right]$$

$$f = -1 \text{ when } \left[(-1.5 + infinitesimal) \leq -\frac{b}{2a} \leq -0.5\right]$$

$$f = 0 \text{ when } \left[(-0.5 + infinitesimal) \leq -\frac{b}{2a} \leq 0.5\right]$$

$$f = 1 \text{ when } \left[(0.5 + infinitesimal) \leq -\frac{b}{2a} \leq 1.5\right]$$

$$f = 2 \text{ when } \left[(1.5 + infinitesimal) \leq -\frac{b}{2a} \leq 2.5\right]$$

$$f = 3 \text{ when } \left[(2.5 + infinitesimal) \leq -\frac{b}{2a} \leq 3.5\right]$$

In the computer, these calculations are almost equivalent to use the function ROUND, but just one very important difference. Although the definition

$$round\left(\frac{odd}{2}\right) = \left(integer\left(\frac{odd}{2}\right) + 1\right)$$

in our case we want all the properties of ROUND function, but modified to

$$roundz\left(\frac{odd}{2}\right) = integer\left(\frac{odd}{2}\right)$$

where ROUNDZ is the name for this new ROUND function modified.

So, in our tables in MS-Excel we defined formula of offset as being:

$$f = round\left(-\frac{b}{2a} - infinitesimal\right)$$

where in infinitesimal we put 1E-15.

So,

$$offset = f = round\left(-\frac{b}{2a} - infinitesimal\right) = round\left(f - \frac{b^{o}}{2a} - infinitesimal\right)$$

Or using a new function RoundZ (Round Minus Infinitesimal or Round to Zero)

$$offset = f = roundz\left(-\frac{b}{2a}\right) = roundz\left(f - \frac{b^{o}}{2a}\right)$$

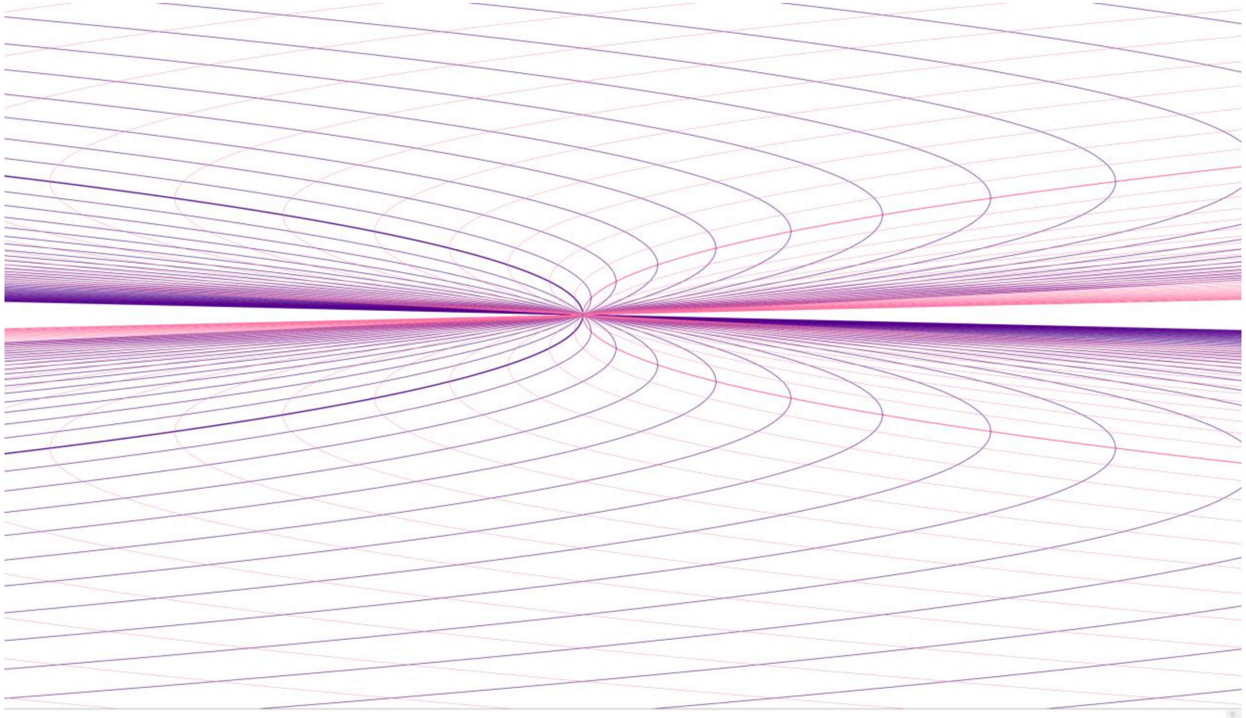
If we have the 3 consecutive elements x_1, x_2, x_3 we can write the general offset equation as being:

$$offset = f = roundz\left(f - \frac{b^{o}}{2a}\right) = roundz\left(f - \frac{x_3 - x_1}{2(x_1 - 2x_2 + x_3)}\right)$$

Given that

$$\begin{gathered} \ldots \\ round(1.5) = 2 \\ roundz(1.5) = 1 \\ round(1) = 1 \\ roundz(1) = 1 \\ round(0.5) = 1 \\ roundz(0.5) = 0 \\ round(0) = 0 \\ roundz(0) = 0 \\ round(-0.5) = 0 \\ roundz(-0.5) = -1 \\ round(-1) = -1 \\ roundz(-1) = -1 \\ round(-1.5) = -1 \\ roundz(-1.5) = -2 \\ \ldots \end{gathered}$$

And since we are dealing with infinity, we cannot round infinite sequences of digits in math as the way we round in engineering.
The whole engineering we know is based on finitude.
The only common possible rounding between infinite and finite, (coincidence which occur between math and engineering), is 0.999999 … = 1.000000 …
It is also where round and roundz returns the same result.



General 2nd degree polynomial equation

$$x = ay^2 + by + c$$

Then, given 3 consecutive dots (x_1, x_2, x_3), the simplest equation is

$$x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right) y^2 + \left(\frac{-x_1 + x_3}{2}\right) y + x_2$$

Where the simplest coefficients are:

$$a = \frac{x_1 - 2x_2 + x_3}{2}$$

$$b = \frac{-x_1 + x_3}{2}$$

$$c = x_2$$

And

$$y_{vertex} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3}$$

$$determinant = \Delta = b^2 - 4ac = \frac{x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2(4x_2)x_3 - 2x_1x_3}{4}$$

$$x_{vertex} = -\frac{\Delta}{4a} = c - \frac{b^2}{4a} = x_2 - \frac{(-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)}$$

$$Latus\ Rectum = \left|\frac{1}{a}\right| = \left|\frac{2}{x_1 - 2x_2 + x_3}\right|$$

Offset of this equation is given by:

$$offset = f = roundz(y_{vertex}) = roundz\left(-\frac{b}{2a}\right)$$

$$x = ay^2 + by + c = x = ay^2 + (bº - 2af)y + (af^2 - bºf + cº)$$

Then,

$a = aº$	$aº = a$
$b = bº - 2af$	$bº = b + 2af$
$c = aºf^2 - bºf + cº$	$cº = af^2 + bf + c$

Condition	Y_vertex	Type	Coef. a	Coef. b	
	$-\frac{b}{2a}$		a	b	
	$\frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3}$		$\frac{x_1 - 2x_2 + x_3}{2}$	$\frac{-x_1 + x_3}{2}$	
$\boldsymbol{x_1 = x_2}$	$-\frac{\mathbf{1}}{\mathbf{2}}$	**DES** |a|=|b|≠0	$\frac{\boldsymbol{-x_1 + x_3}}{\mathbf{2}} = \boldsymbol{b}$	$\frac{\boldsymbol{-x_1 + x_3}}{\mathbf{2}} = \boldsymbol{a}$	

$x_2 = x_3$	$\frac{1}{2}$	DES |a|=|b|≠0	$\frac{x_1 - x_2}{2} = -b$	$\frac{-x_1 + x_2}{2} = -a$	
$x_1 = x_3$	0	SUB |a|>|b|=0	$x_1 - x_2$	0	
$x_1 = -x_2$	$\frac{1}{2}\left(\frac{x_1 - x_3}{3x_1 + x_3}\right)$	ACC |a|≠|b|≠0	$\frac{3x_1 + x_3}{2}$	$\frac{-x_1 + x_3}{2}$	
$x_2 = -x_3$	$\frac{1}{2}\left(\frac{x_1 + x_2}{x_1 - 3x_2}\right)$	ACC |a|≠|b|≠0	$\frac{x_1 - 3x_2}{2}$	$\frac{-x_1 - x_2}{2}$	
$x_1 = -x_3$	$-\frac{x_1}{2x_2}$	ACC |a|≠|b|≠0	$-x_2$	$-x_1$	