

# On the complexity of recognizing S-composite and S-prime graphs

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## ARTICLE INFO

### Article history:

Received 7 May 2012

Received in revised form 2 November 2012

Accepted 5 November 2012

Available online 22 November 2012

### Keywords:

Graph

S-prime

S-composite

Path- $k$ -coloring

Cartesian product

NP-complete

CoNP-complete

## ABSTRACT

S-prime graphs are graphs that cannot be represented as nontrivial subgraphs of nontrivial Cartesian products of graphs, i.e., whenever it is a subgraph of a nontrivial Cartesian product graph it is a subgraph of one of the factors. A graph is S-composite if it is not S-prime. Although linear time recognition algorithms for determining whether a graph is prime or not with respect to the Cartesian product are known, it remained unknown if a similar result holds also for the recognition of S-prime and S-composite graphs.

In this contribution the computational complexity of recognizing S-composite and S-prime graphs is considered. Klavžar et al. [S. Klavžar, A. Lipovec, M. Petkovšek, On subgraphs of Cartesian product graphs. *Discrete Math.*, 244 (2002) 223–230] proved that a graph is S-composite if and only if it admits a nontrivial path- $k$ -coloring. The problem of determining whether there exists a path- $k$ -coloring for a given graph is shown to be NP-complete even for  $k = 2$ . This in turn is utilized to show that determining whether a graph is S-composite is NP-complete and thus, determining whether a graph is S-prime is CoNP-complete. A plenty of other problems are shown to be NP-hard, using the latter results.

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## 1. Introduction and preliminaries

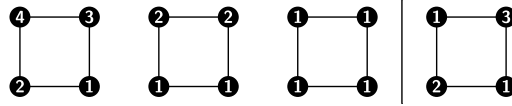
A graph  $S$  is said to be *S-prime* ( $S$  stands for “subgraph”) w.r.t. to an arbitrary graph product  $\star$  if for all graphs  $G$  and  $H$  with  $S \subseteq G \star H$  holds:  $S \subseteq H$  or  $S \subseteq G$ . A graph is *S-composite* if it is not S-prime. The only S-prime graphs w.r.t. the direct product are complete graphs or complete graphs minus an edge [13]. The only S-prime graphs w.r.t. the strong product and the lexicographic product are the single vertex graph  $K_1$ , the disjoint union  $K_1 \cup K_1$  and the complete graph on two vertices  $K_2$  [11,12].

Not much is known, however, about the structure of S-prime graphs w.r.t. the Cartesian product. Examples include the complete graphs  $K_n$  with  $n \geq 1$  vertices and the complete bipartite graphs  $K_{m,n}$  with  $m \geq 2$ ,  $n \geq 3$ . Another class of Cartesian S-prime graphs are so-called diagonalized Cartesian products of S-prime graphs [5], which in turn play an important role in finding approximate strong product graphs; see [4]. Several interesting characterizations of (basic) S-prime graphs due to Lamprey and Barnes [11,12], Klavžar et al. [9,10] and Brešar [1] are known. However, although those characterizations are established and graphs can be recognized as prime (or factorizable) w.r.t. the Cartesian product in linear time [8], it remained unknown if a similar result holds also for the recognition of S-prime and S-composite graphs. We will show in this contribution that the problem of determining whether a graph is S-composite, resp., S-prime w.r.t. the Cartesian product is NP-complete, resp., CoNP-complete. Moreover, using this result we are able to show the NP-hardness of several other problems.

Before we proceed, we introduce some notation. We consider finite, simple, connected and undirected graphs  $G = (V, E)$  with vertex set  $V$  and edge set  $E$ . A graph  $H$  is a *subgraph* of a graph  $G$ , in symbols  $H \subseteq G$ , if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ .

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**Fig. 1.** Possible path- $k$ -coloring of a square  $Q_2$  for  $k = 1, 2, 4$ . A possible well-coloring that is not a path-3-coloring is shown on the right-hand side graph [5].

If  $H \subseteq G$  and all pairs of adjacent vertices in  $G$  are also adjacent in  $H$  then  $H$  is called an *induced* subgraph. The subgraph of a graph  $G$  that is induced by a vertex set  $W \subseteq V(G)$  is denoted by  $\langle W \rangle$ . We define the *neighborhood* of vertex  $v$  as the set  $N(v) = \{x \in V(G) \mid (v, x) \in E\}$ . A *complete graph*  $K_n$  is a graph in which every vertex is adjacent to every other vertex.

In the following we will consider the Cartesian product only. Therefore, the terms S-prime and S-composite refer to this product from here on.

The Cartesian product  $G \square H$  has vertex set  $V(G \square H) = V(G) \times V(H)$ ; two vertices  $(g_1, h_1), (g_2, h_2)$  are adjacent in  $G \square H$  if  $(g_1, g_2) \in E(G)$  and  $h_1 = h_2$ , or  $(h_1, h_2) \in E(H)$  and  $g_1 = g_2$ . The one-vertex complete graph  $K_1$  serves as a unit, as  $K_1 \square H = H$  for all graphs  $H$ . A Cartesian product  $G \square H$  is called *trivial* if  $G \simeq K_1$  or  $H \simeq K_1$ . A graph  $G$  is *prime* with respect to the Cartesian product if it has only a trivial Cartesian product representation. The Cartesian product is associative. Therefore, a vertex  $x$  of a Cartesian product  $\square_{i=1}^n G_i$  is properly “coordinatized” by the vector  $c(x) := (c_1(x), \dots, c_n(x))$  whose entries are the vertices  $c_i(x)$  of its factor graphs  $G_i$ . Two adjacent vertices in a Cartesian product graph therefore differ in exactly one coordinate. The Cartesian product  $Q_n = \square_{i=1}^n K_2$  is called *hypercube*. W.l.o.g. we assume that  $c(x) \in \{0, 1\}^n$  for all  $x \in V(Q_n)$ . For detailed information about product graphs we refer the interested reader to [3,6] or [7].

For our purposes, the characterization of S-composite graphs in terms of particular colorings [9] is of direct interest. A  $k$ -coloring of  $G$  is a surjective mapping  $\mathcal{C} : V(G) \rightarrow \{1, \dots, k\}$ . This coloring need not be proper, i.e., adjacent vertices may obtain the same color. A path  $P$  in  $G$  is *well-colored* by  $\mathcal{C}$  if for any two consecutive vertices  $u$  and  $v$  of  $P$  we have  $\mathcal{C}(u) \neq \mathcal{C}(v)$ . Following [9], we say that  $\mathcal{C}$  is a *path- $k$ -coloring* of  $G$  if  $\mathcal{C}(u) \neq \mathcal{C}(v)$  holds for the endpoints of every well-colored  $u, v$ -path  $P$  in  $G$ ; see Fig. 1. For  $k = 1$  and  $k = |V|$  there are trivial path- $k$ -colorings: For  $k = 1$  the coloring is constant and hence there are no well-colored paths. On the other hand if a different color is used for every vertex, then every path, of course, has distinctly colored endpoints. A path- $k$ -coloring is nontrivial if  $2 \leq k \leq |V(G)| - 1$ .

**Theorem 1.1** ([9]). *A connected graph  $G$  is S-composite if and only if there exists a nontrivial path- $k$ -coloring of  $G$ .*

## 2. Results

We start our exposition with some known results on path- $k$ -colorings. We continue to examine basic properties of path-2-colorings of hypercubes and so-called joint graphs as well as properties of path- $k$ -colorings of so-called  $k$ -extended joint graphs. These results will be used to show that the problem of determining whether there exists a path- $k$ -coloring for a given graph is NP-complete. Hence, there is no polynomial time algorithm solving this problem unless  $P = NP$ . Using the latter result we can easily infer the NP-completeness of the problem to decide whether a graph is S-composite and therefore, we can conclude that determining whether a graph is S-prime is CoNP-complete. Moreover, other decision problems concerning particular graph properties that are equivalent to decide whether a graph is S-composite are therefore NP-hard. Those problems are stated at the end of this section. For detailed information about (Co)NP-completeness we refer the interested reader to [2].

### 2.1. Path- $k$ -colorings, Hypercubes, the Joint Graph $G(m)$ and the $k$ -Extended Joint Graph $G(m, k)$

**Lemma 2.1** ([5]). *Let  $H \subseteq G$  and suppose  $\mathcal{C}$  is a path- $k$ -coloring of  $G$ . Then the restriction  $\mathcal{C}|_{V(H)}$  of  $\mathcal{C}$  on  $V(H)$  is a path- $l$ -coloring of  $H$ ,  $l \leq k$ . Moreover, if  $V(H) = V(G)$  and  $\mathcal{C}$  is a nontrivial path- $k$ -coloring of  $G$ , then it is also a nontrivial path- $k$ -coloring of  $H$ .*

**Lemma 2.2** ([5]). *Let  $\mathcal{C}$  be a path- $k$ -coloring of the Cartesian product  $G = \square_{i=1}^n S_i$  of S-prime graphs  $S_i$  and suppose there are two vertices with maximal distance in  $G$  that have the same color. Then  $\mathcal{C}$  is constant on  $G$ , i.e.,  $k = 1$ .*

**Lemma 2.3** ([5]). *The hypercube  $Q_2 = K_2 \square K_2$  has no path-3-coloring. In particular, every path-2-coloring of  $Q_2$  has adjacent vertices with the same color.*

From Theorem 9 (Path- $k$ -coloring of Cartesian products of S-prime Graphs) in [5] we can easily derive the next lemma.

**Lemma 2.4.** *Any nontrivial path- $k$ -coloring of a hypercube  $Q_3$  is either a path-2-coloring or a path-4-coloring.*

In the following lemma a simple but useful result concerning nontrivial path- $k$ -colorings of hypercubes  $Q_3$  is proved; see also Fig. 2.

**Lemma 2.5.** *Let  $\mathcal{C}$  be a nontrivial path- $k$ -coloring of the hypercube  $Q_3 = (V, E)$ . Then  $\mathcal{C}(x) \neq \mathcal{C}(y)$  for all vertices  $x, y \in V$  with distance  $d(x, y) = 3$ .*

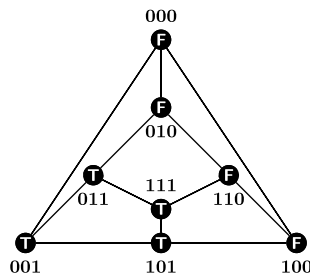


Fig. 2. Shown is a hypercube  $Q_3$ , together with its coordinatized vertices and a possible path-2-coloring.

If  $k = 2$  then for all vertices  $x \in V$  there are two different vertices  $u, v \in N(x)$  with color  $\mathcal{C}(u) = \mathcal{C}(v) = \mathcal{C}(x)$  and one vertex  $w \in N(x)$  with  $\mathcal{C}(w) \neq \mathcal{C}(x)$ .

If  $k = 4$  then for all vertices  $x \in V$  there is one vertex  $u \in N(x)$  with color  $\mathcal{C}(u) = \mathcal{C}(x)$  and there is a well-colored path from  $u$  to  $y$ , where  $y \in V$  with  $d(x, y) = 3$ .

**Proof.** Let  $x$  and  $y$  be two vertices with maximal distance  $d(x, y) = 3$ . Since  $Q_3$  is the Cartesian product of  $S$ -prime graphs and  $\mathcal{C}$  is a nontrivial path- $k$ -coloring, Lemma 2.2 implies that  $\mathcal{C}(x) \neq \mathcal{C}(y)$ .

Let  $\mathcal{C}$  be a path-2-coloring of the hypercube  $Q_3$ . Let  $N(x) = \{u, v, w\}$  be the set of adjacent vertices to  $x$  and  $y$  be the vertex with maximal distance  $d(x, y) = 3$  to  $x$ . Due to the three paths  $\langle\{v, x, w\}\rangle$ ,  $\langle\{v, x, u\}\rangle$  and  $\langle\{u, x, w\}\rangle$ , we can conclude that there is at most one vertex in  $N(x)$  with color  $\mathcal{C}(y)$ , otherwise  $\mathcal{C}$  would not be a path-2-coloring. Assume that all vertices  $u, v, w$  have color  $\mathcal{C}(x)$ . Consider the squares  $\langle\{x, u, a, v\}\rangle$  and  $\langle\{x, u, b, w\}\rangle$  in  $Q_3$ . Lemmas 2.1 and 2.3 imply that  $\mathcal{C}(a) = \mathcal{C}(b) = \mathcal{C}(x)$ . Since  $\mathcal{C}(x) \neq \mathcal{C}(y)$  we can conclude that the path  $a - y - b$  is well-colored, contradicting that  $\mathcal{C}$  is a path-2-coloring. Thus, exactly one vertex contained in  $N(x)$  has color  $\mathcal{C}(y)$  and the other two have color  $\mathcal{C}(x)$ .

Let  $\mathcal{C}$  be a path-4-coloring of the hypercube  $Q_3$ . Assume there is a vertex  $x \in V$  such that for each vertex  $z \in N(x) = \{u, v, w\}$  holds  $\mathcal{C}(z) \neq \mathcal{C}(x)$ . Note, in this case it holds  $\mathcal{C}(z) \neq \mathcal{C}(z')$  for each  $z, z' \in N(x)$  with  $z \neq z'$ , otherwise there is a well-colored path  $z - x - z'$ , a contradiction. Consider the square  $\langle\{x, u, a, v\}\rangle$  in  $Q_3$ . Thus, by Lemmas 2.1 and 2.3 the vertex  $a$  must obtain the fourth color, namely  $\mathcal{C}(w)$ . Hence, there is a well-colored path  $a - u - x - w$  with  $\mathcal{C}(w) = \mathcal{C}(a)$ , a contradiction. Therefore, there is a vertex  $u \in N(x)$  with color  $\mathcal{C}(u) = \mathcal{C}(x)$ . It remains to show that there is a well-colored path from  $u$  to  $y$ , where  $y \in V$  with  $d(x, y) = 3$ . Note, we have  $\mathcal{C}(u) \neq \mathcal{C}(y)$ . Assume there is no well-colored path from  $u$  to  $y$ . Thus, the two shortest paths  $u - z - y$  and  $u - z' - y$  are not well-colored and therefore,  $\mathcal{C}(z), \mathcal{C}(z') \in \{\mathcal{C}(u), \mathcal{C}(y)\}$ . If the vertices  $z$  and  $z'$  have both color  $\mathcal{C}(u)$ , resp.,  $\mathcal{C}(y)$  there is a well-colored path  $z - y - z'$ , resp.,  $z - u - z'$ , a contradiction. Thus, one vertex must have color  $\mathcal{C}(u)$ , say the vertex  $z$ , while the other vertex  $z'$  has color  $\mathcal{C}(y)$ . Consider the square  $\langle\{x, u, z, v\}\rangle$  and the square  $\langle\{x, u, z', v'\}\rangle$  in  $Q_3$ . By construction and Lemma 2.3, vertex  $v$  must get color  $\mathcal{C}(x)$  and vertex  $v'$  the color  $\mathcal{C}(y)$ . Again, Lemma 2.3 implies that the vertex  $a$  contained in the square  $\langle\{v, z, y, a\}\rangle$  must get color  $\mathcal{C}(y)$  and we obtain a valid path-2-coloring, a contradiction since we assumed to have a path-4-coloring. Therefore, in any path-4-coloring one of the vertices  $z$  or  $z'$  gets a color different from  $\mathcal{C}(u)$  and  $\mathcal{C}(y)$  and hence, one of the paths  $u - z - y$  or  $u - z' - y$  is well-colored.  $\square$

Now, we establish the so-called joint graph  $G(m)$  and the  $k$ -extended joint graph  $G(m, k)$ , that will become a powerful tool for proving the NP-hardness, as we shall see later; see also Figs. 3 and 4.

**Definition 2.6** (Joint Graph  $G(m)$ ). Let  $\mathcal{Q} = \{Q^1, \dots, Q^m\}$  be a set of hypercubes  $Q_3$  and let the respective coordinates of vertices  $v \in V(Q^j)$  be denoted by  $c^j(v) \in \{0, 1\}^3$ . Based on the set  $\mathcal{Q}$  we define the joint graph  $G(m) = (V, E)$  as follows:

$$V = \bigcup_{i=1}^m V(Q^i)$$

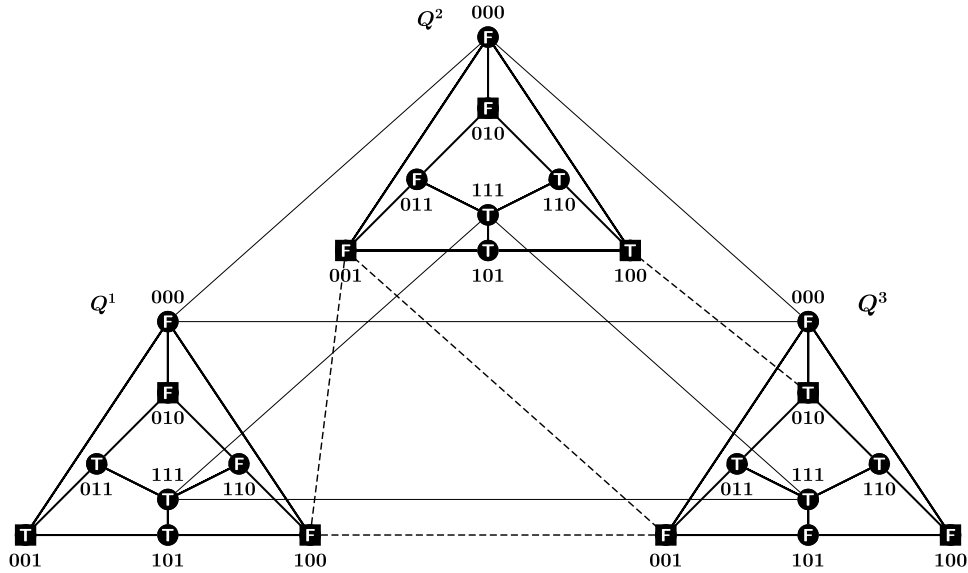
and

$$E = \bigcup_{i=1}^m E(Q_i) \cup E_0 \cup E_1 \cup E',$$

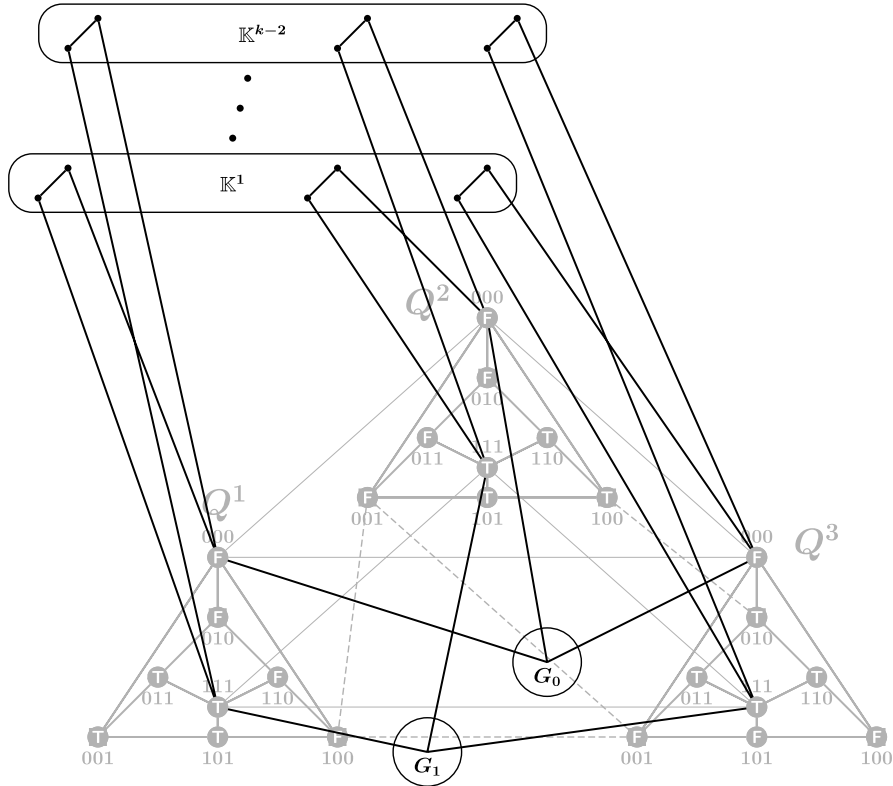
where for  $k \in \{0, 1\}$  the edge set  $E_k$  is defined as the set  $\{(u, v) \mid c^i(u) = c^i(v) = (kkk), i \neq j\}$  and  $E'$  denotes a set of arbitrarily added edges between vertices  $u \in V(Q^i)$  and  $v \in V(Q^j)$  with  $i \neq j$  and  $c^i(u), c^i(v) \in \{(001), (010), (100)\}$  with the restriction that each such vertex  $u \in V(Q^i)$  is connected to at most one other vertex  $v \in V(Q^j)$  for each  $j$ .

In other words, the joint graph  $G(m)$  consists of  $m$  disjoint copies of hypercubes  $Q_3$ , where all pairwise different vertices with coordinates (000) are forced to be adjacent resulting in a complete subgraph  $K_m$ . These edges are contained in  $E_0$ . In the same way vertices with coordinates (111) are connected by edges contained in  $E_1$  and thus, the induced subgraph of these vertices results in a complete subgraph  $K_m$  as well. Moreover, additional edges contained in  $E'$  between vertices with coordinates (001), (010) and (100) between different copies of these hypercubes can be placed in a restricted way; see also Fig. 3.

**Definition 2.7** ( $k$ -Extended Joint Graph  $G(m, k)$ ). Let  $G(m)$  be a given joint graph,  $k$  be an arbitrary integer and  $V_0, V_1 \subset V(G(m))$  be the set of vertices with coordinates (000) and (111) in  $G(m)$ , respectively. Moreover, let  $\mathcal{K} = \{K^1, \dots, K^{k-2}\}$



**Fig. 3.** Shown is a joint graph  $G(m)$  based on the set  $\mathcal{Q} = \{Q^1, Q^2, Q^3\}$  together with a path-2-coloring  $c: V(G(m)) \rightarrow \{F, T\}$ . In particular,  $G(m)$  reflects the monotone 1-in-3 SAT formula  $\psi = \{L_1, L_2, L_3\}$  with clauses  $L_1 = (b_1, b_2, b_3)$ ,  $L_2 = (b_1, b_4, b_5)$  and  $L_3 = (b_1, b_4, b_6)$ . Each clause  $L_i$  is identified with the hypercube  $Q^i$ . Each of the variables  $b_{i1}, b_{i2}, b_{i3} \in L_i$  is uniquely identified with one of the vertices in  $V(Q^i)$  that have coordinates (001), (010), resp., (100) (highlighted by square vertices). Edges  $(u, v)$  between different hypercubes are added, whenever  $c^i(u) = c^i(v) \in \{(000), (111)\}$  (thin-lined edges) or the Boolean variables the vertices are associated to are identical (dashed-lined edges).



**Fig. 4.** Shown is the  $k$ -extended joint graph  $G(m, k)$  for the joint graph  $G(m)$  shown in Fig. 3. The graphs  $G_0$  and  $G_1$  are complete graphs of size  $k - m + 1$ . Each vertex of  $G_i$  is connected to each vertex with respective coordinates (iii) for  $i = 0, 1$ . Thus, the vertices with respective coordinates (iii) together with the vertices of  $G_i$  induce a complete subgraph of size  $k + 1$ . Finally,  $k - 2$  disjoint complete graphs  $K^i \in \mathcal{K}$  of size  $2m + k + 1$  are added to  $G(m)$ . Vertices  $v$  and  $w$  with coordinates (000), resp., (111) contained in each  $Q^i \in \mathcal{Q}$  are connected to arbitrary but different vertices  $x, y \in V(K^i)$  for each  $K^i \in \mathcal{K}$ . Those edges  $(v, x)$  and  $(w, y)$  are summarized in the set  $F^i$ .

be the set of  $k - 2$  complete subgraphs of size  $2m + k + 1$ . With  $G_0 = (W_0, E(G_0))$  and  $G_1 = (W_1, E(G_1))$  we denote two further disjoint complete graphs of size  $k - m + 1$ , while  $G_0 = G_1 = \emptyset$  if  $k - m + 1 \leq 0$ . The  $k$ -extended joint graph  $G(m, k) = (V, E)$  consists of

$$V = \bigcup_{i=1}^{k-2} V(K^i) \cup V(G(m)) \cup W_0 \cup W_1$$

and

$$E = \bigcup_{i=1}^{k-2} E(K^i) \cup E(G(m)) \cup E(G_0) \cup E(G_1) \cup F_0 \cup F_1 \cup F'$$

where for  $i \in \{0, 1\}$  the edge set  $F_i$  is defined as the set  $\{(u, v) \mid u \in V_i, v \in W_i\}$ . The edges in  $F'$  connect all vertices  $v \in V_0 \cup V_1$  to vertices  $w \in V(K^i)$ , while different vertices of  $G(m)$  are connected to different vertices  $w \in V(K^i)$ , for each  $i = 1, \dots, k - 2$ .

In other words, the  $k$ -extended joint graph  $G(m, k)$  consists of the graph  $G(m)$ ,  $k - 2$  disjoint copies of complete graphs  $K^j \in \mathcal{K}$  of size  $2m + k + 1$  and two disjoint copies of complete graphs  $G_0 =$  and  $G_1$  of size  $k - m + 1$ . All pairwise different vertices with coordinates (000) and (111) are connected to pairwise different vertices in  $V(K^j)$  for each  $K^j \in \mathcal{K}$ . These edges are contained in  $F'$ . Hence, any pair of vertices  $(u, v) \in V(Q^i)$  with coordinates (000) and (111) is mapped to an edge in  $K^j$ , while such pairs of vertices of different hypercubes are mapped to different non-incident edges in  $K^j$  for each  $K^j \in \mathcal{K}$ . The edges in  $F_i$  connect all vertices with coordinates (iii) to all vertices contained in  $G_i$ ,  $i = 0, 1$ . Hence,  $\langle V_0 \cup W_0 \rangle \simeq \langle V_1 \cup W_1 \rangle \simeq K_{k+1}$ . Note,  $G(m) = G(m, 2)$  for  $m > 2$ .

**Lemma 2.8.** Let  $G(m)$  be a joint graph for the set of hypercubes  $\mathcal{Q}$  and let  $m > 2$ . If  $\mathcal{C}$  is a path-2-coloring of  $G(m)$  then all vertices  $v$  and  $w$  connecting different hypercubes  $Q^i, Q^j \in \mathcal{Q}$ , i.e.,  $(v, w) \in E_0 \cup E_1 \cup E'$ , must have the same color  $\mathcal{C}(v) = \mathcal{C}(w)$ . Furthermore,  $\mathcal{C}(v) \neq \mathcal{C}(w)$  for all vertices  $v, w$  with  $c^i(v) = (000)$  and  $c^i(w) = (111)$ ,  $i, j \in \{1, \dots, m\}$ .

**Proof.** First, we show that all vertices with  $c^i(v) = (000)$ , resp.,  $c^i(w) = (111)$  share the same color, while  $\mathcal{C}(v) \neq \mathcal{C}(w)$  for all  $v, w$  with  $c^i(v) = (000)$  and  $c^i(w) = (111)$ . Since all vertices  $v, v'$  with coordinates (000) are connected via edges  $(v, v') \in E_0$  they induce a complete subgraph  $K_m$ . Since  $m > 2$ , we can conclude that  $\mathcal{C}(v) = \mathcal{C}(v')$ , otherwise  $\mathcal{C}$  would not be a path-2-coloring of  $G(m)$ . Analogously, we have  $\mathcal{C}(w') = \mathcal{C}(w)$  for all vertices  $w, w'$  with coordinates (111).

Now, assume for contradiction that  $\mathcal{C}(v) = \mathcal{C}(w)$  for some vertex  $v$  with coordinates  $c^i(v) = (000)$  and  $w$  with  $c^i(w) = (111)$ . Using the previous arguments we can conclude that for all  $Q^j \in \mathcal{Q}$  it is the case that  $\mathcal{C}(v') = \mathcal{C}(w')$  for  $v', w' \in V(Q^j)$  with  $c^i(v') = (000)$  and  $c^i(w') = (111)$ . Lemmas 2.1 and 2.5 imply that all vertices within the induced subgraph  $\langle V(Q^j) \rangle$  must have same color, this holds for all  $Q^j \in \mathcal{Q}$  and therefore  $\mathcal{C}(v) = \mathcal{C}(w)$  for all  $v, w \in V(G(m))$ , contradicting that  $\mathcal{C}$  is a path-2-coloring of  $G(m)$ .

It remains to show that for vertices  $a, b$  with  $(a, b) \in E'$  it is the case that  $\mathcal{C}(a) = \mathcal{C}(b)$ . Let  $a \in V(Q^j)$  and  $b \in V(Q^i)$ . Since  $\mathcal{C}(v) \neq \mathcal{C}(w)$  for  $v, w \in V(Q^j)$  with  $c^j(v) = (000)$  and  $c^j(w) = (111)$ , it follows from Lemma 2.1 that each induced subgraph  $\langle V(Q^j) \rangle$  must be “proper” path-2-colored. Thus, Lemma 2.5 implies that there is a vertex  $a' \in N(a) \cap V(Q^j)$  with  $\mathcal{C}(a') \neq \mathcal{C}(a)$ . Hence, if  $\mathcal{C}(b) \neq \mathcal{C}(a)$  the path  $a' - a - b$  would be well-colored but  $\mathcal{C}(a') = \mathcal{C}(b)$ , a contradiction.  $\square$

**Lemma 2.9.** Let  $\mathcal{C}$  be a nontrivial path- $k$ -coloring of the  $k$ -extended joint graph  $G(m, k)$ ,  $k > 2$ . Then  $\mathcal{C}|_{V(G(m))}$  is a path-2-coloring of the underlying joint graph  $G(m)$ .

**Proof.** Let  $\mathcal{C}$  be a nontrivial path- $k$ -coloring of  $G(m, k)$ ,  $k > 2$ . First we show that all vertices  $v \in V(Q^i)$  and  $w \in V(Q^j)$  with coordinates  $c^i(v) = (000)$  and  $c^j(w) = (111)$  must have different colors,  $\mathcal{C}(v) \neq \mathcal{C}(w)$ . Assume  $\mathcal{C}(v) = \mathcal{C}(w)$  for some vertices  $v \in V(Q^i)$  with coordinates (000) and  $w \in V(Q^j)$  with coordinates (111). Since the vertices in  $G_0$  are connected to all vertices with coordinates (000) and since those vertices together with  $G_0$  induce a complete graph  $K_{k+1}$  all vertices within  $G_0$  and all vertices with coordinates (000) must have the same color. Analogously, all vertices within  $G_1$  and all vertices with coordinates (111) must have the same color. Therefore,  $\mathcal{C}(v) = \mathcal{C}(w)$  for all vertices with coordinates (000) and (111). Lemmas 2.1 and 2.5 imply that each  $Q^i \in \mathcal{Q}$  can then only be trivial path- $k$ -colored, i.e., either path-1-colored or path-8-colored. Since we assumed  $\mathcal{C}(v) = \mathcal{C}(w)$  each  $Q^i \in \mathcal{Q}$  must be path-1-colored and hence, all vertices within each  $Q^i \in \mathcal{Q}$  have the same color. Therefore, the subgraph  $\langle \bigcup_{i=1}^m V(Q^m) \cup V(G_0) \cup V(G_1) \rangle$  is colored with just one color. Thus, the remaining  $k - 1$  colors can only be assigned to vertices in complete subgraphs of  $\mathcal{K}$ . Since each such subgraph has  $2m + k + 1$  vertices and by Lemma 2.1 it follows that all vertices within such a complete graph can only be colored with one color. Since there are only  $k - 2$  such complete graphs at most  $k - 2$  of the remaining  $k - 1$  colors can be used. Thus one might get a path- $(k - 1)$ -coloring, but no path- $k$ -coloring. Hence, we can conclude  $\mathcal{C}(v) \neq \mathcal{C}(w)$  for all vertices with coordinates  $c^i(v) = (000)$  and  $c^j(w) = (111)$ .

The above statements and Lemma 2.1 directly imply that the subgraph  $G(m)$  and in particular all  $Q^i \in \mathcal{Q}$  must be path- $l$ -colored using at least two colors, i.e.,  $2 \leq l \leq k$ .

We continue to show that for any vertex  $w \in V(K^i)$  of each complete graph  $K^i \in \mathcal{K}$  and for all vertices  $v \in V(Q^j)$  of each  $Q^j \in \mathcal{Q}$  holds  $\mathcal{C}(w) \neq \mathcal{C}(v)$ . Assume first, there is a vertex  $v \in V(Q^j)$  with coordinates (000) or (111) such that  $\mathcal{C}(w) = \mathcal{C}(v)$  for some vertex  $w \in V(K^i)$ . Without loss of generality let  $v$  have coordinates (111). Let  $u \in V(Q^j)$  be the vertex with coordinates (000). Clearly, since  $K^i$  is a complete subgraph of size  $2m + k + 1$  all vertices within  $K^i$  must obtain the color  $\mathcal{C}(v)$ , in particular the vertex  $x \in V(K^i)$  that is connected to vertex  $u$  via the edge  $(u, x) \in F'$ . By Lemmas 2.1



and 2.4 and since  $\mathcal{C}(u) \neq \mathcal{C}(v)$  (as shown before), the subgraph  $Q^j$  can only be path-2-colored, path-4-colored or path-8-colored. If  $Q^j$  is path-2-colored then Lemma 2.5 implies that there is a vertex  $z \in N(u)$  with  $\mathcal{C}(z) = \mathcal{C}(v) = \mathcal{C}(x)$  and hence the well-colored path  $z - u - x$ , a contradiction. If  $Q^j$  is path-4-colored then Lemma 2.5 implies that there is a vertex  $z \in N(v)$  with  $\mathcal{C}(z) = \mathcal{C}(v) = \mathcal{C}(x)$  and that there is a well-colored path from  $z$  to  $u$ . Thus, there is a well-colored path from  $z$  to  $x$ , using the edge  $(u, x)$ , a contradiction. If  $Q^j$  is path-8-colored then any path within  $Q^j$  from  $v$  over  $u$  to  $x$  is well-colored, while  $\mathcal{C}(v) = \mathcal{C}(x)$ , a contradiction. Hence, for all vertices  $w \in V(K^i)$  of each  $K^i \in \mathcal{K}$  and for all vertices  $v \in V(Q^j)$  with coordinates (000) or (111) of each  $Q^j \in \mathcal{Q}$  holds  $\mathcal{C}(w) \neq \mathcal{C}(v)$ . Now let  $v \in V(Q^j)$  be an arbitrary vertex with coordinates different from (000) or (111) and assume that  $\mathcal{C}(w) = \mathcal{C}(v)$  for some vertex  $w \in V(K^i)$ . Let  $x$  and  $y$  denote the vertices of  $Q^j$  with coordinates (000) and (111), respectively. As argued before, each vertex within  $K^i$  must obtain the color  $\mathcal{C}(v)$ , in particular the vertices  $z, z' \in V(K^i)$  that are connected to  $x$ , resp.,  $y$  via the edges  $(z, x), (z', y) \in F'$ . Since  $\mathcal{C}(x) \neq \mathcal{C}(y)$ ,  $\mathcal{C}(x) \neq \mathcal{C}(w) = \mathcal{C}(v)$  and  $\mathcal{C}(y) \neq \mathcal{C}(w) = \mathcal{C}(v)$  this vertex  $v$  must be colored with a third color  $\mathcal{C}(w)$  different from  $\mathcal{C}(x)$  and  $\mathcal{C}(y)$ . However, vertex  $v$  has either distance one to vertex  $x$  or to vertex  $y$ . Thus, there is always a well-colored path  $v - x - z$  or  $v - y - z'$ , a contradiction.

Finally, we show that for complete graphs  $K^i, K^j \in \mathcal{K}, i \neq j$  holds  $\mathcal{C}(v) \neq \mathcal{C}(w)$  for any vertex  $v \in V(K^i)$  and  $w \in V(K^j)$ . Note that any path- $k$ -coloring within each  $K^i$  must be constant since each  $K^i \in \mathcal{K}$  is of size  $2m + k + 1$ . If  $\mathcal{K} = \{K^1\}$ , i.e. if  $k = 3$ , then there is nothing to show. In particular, as argued before all vertices within the subgraph  $K^1$  must obtain a third color different from colors used in  $G(m)$  and therefore, by Lemma 2.1, the subgraph  $G(m) \subseteq G(m, 3)$  must be path-2-colored whenever  $G(m, 3)$  is path-3-colored. Assume  $|\mathcal{K}| \geq 2$ , i.e.,  $k > 3$ , and  $\mathcal{C}(v) = \mathcal{C}(w)$  for  $v \in V(K^i), w \in V(K^j), i \neq j$ . Let  $u \in V(Q^1)$  be the vertex with coordinates (000). By construction, this vertex is connected to respective vertices  $z \in V(K^i)$  and  $z' \in V(K^j)$  via edges  $(u, z), (u, z') \in F'$ . As shown before,  $\mathcal{C}(u) \neq \mathcal{C}(z)$  and  $\mathcal{C}(u) \neq \mathcal{C}(z')$  and hence, there is a well-colored path  $z - u - z'$  with  $\mathcal{C}(z) = \mathcal{C}(z')$ , a contradiction.

To summarize, we have shown that for any path- $k$ -coloring of the  $k$ -extended joint graph  $G(m, k)$  each of the  $k - 2$  complete graphs in  $\mathcal{K}$  must obtain different colors, while the coloring within each such complete graph is constant. Thus,  $k - 2$  of the  $k$  colors are used. Moreover, none of the used  $k - 2$  colors can be reused in any  $Q^j \in \mathcal{Q}$  and thus neither in  $G(m) \subseteq G(m, k)$ . Moreover, we showed that  $\mathcal{C}(v) \neq \mathcal{C}(w)$  for all vertices with coordinates  $\mathcal{C}(v) = (000)$  and  $\mathcal{C}(w) = (111)$  and therefore the two unused colors must occur in all  $Q^j \in \mathcal{Q}$  and hence in  $G(m) \subseteq G(m, k)$ . The above statements and Lemma 2.1 directly imply that if  $G(m, k)$  is path- $k$ -colored then the subgraph  $G(m)$  must be path-2-colored.  $\square$

## 2.2. Computational complexity

Now, we are able to prove the NP-completeness for deciding whether a graph has a path- $k$ -coloring. For this the next well-known Problem MONOTONE 1-IN-3 SAT and theorem will be crucial.

### Problem. MONOTONE 1-IN-3 SAT

*Input:* Given a set  $U$  of Boolean variables and a set of clauses  $\psi = \{L_1, \dots, L_m\}$  over  $U$  such that for all  $i = 1, \dots, m$  holds:  $|L_i| = 3$  and  $L_i$  contains no negated variables.

*Question:* Is there a truth assignment to  $\psi$  such that each  $L_i$  contains exactly one true variable?

**Theorem 2.10** ([14]). MONOTONE 1-IN-3 SAT is NP-complete.

### Problem. PATH- $k$ -COLORING (P- $k$ -COL)

*Input:* Given an arbitrary connected graph  $G = (V, E)$  and an integer  $K$  with  $2 \leq K \leq |V| - 1$ .

*Question:* Is there a nontrivial path- $k$ -coloring for  $G, k \leq K$ .

We shortly summarize the main steps for proving the NP-completeness of P- $k$ -COL. After verifying that P- $k$ -COL  $\in$  NP, we will show by reduction from MONOTONE 1-IN-3 SAT that P- $k$ -COL is NP-hard: For a given instance  $\psi = (L_1, \dots, L_m)$  of MONOTONE 1-IN-3 SAT we identify each clause  $L_i$  with a hypercube  $Q^i \in \mathcal{Q}$ . Moreover, each variable  $b \in L_i$  is identified with a unique vertex in  $Q^i$ . Finally, for an arbitrary integer  $k \geq 2$  we construct the  $k$ -extended joint graph  $G(m, k)$  as in Definition 2.7 and show that  $\psi$  has a truth assignment if and only if  $G(m, k)$  has a nontrivial path- $k$ -coloring. Note, since  $G(m, 2) = G(m)$  one can directly conclude that the problem of determining whether a graph has a nontrivial path-2-coloring is NP-complete.

**Theorem 2.11.** P- $k$ -COL is NP-complete.

**Proof.** First we show that P- $k$ -COL  $\in$  NP. Let  $\mathcal{C}$  be a given  $k$ -coloring on  $G = (V, E)$ . We must show that one can verify in polynomial time if  $\mathcal{C}$  is a path- $k$ -coloring. For this, we first remove all edges  $(u, v) \in E$  with  $\mathcal{C}(u) = \mathcal{C}(v)$  and obtain a new graph  $G' = (V, E')$ ,  $E' \subseteq E$ . Thus, for all edges  $(u, v) \in E'$  holds  $\mathcal{C}(u) \neq \mathcal{C}(v)$ . Therefore, in each connected component of  $G'$  all vertices are connected via a well-colored path. Hence, one has to verify that in none of the connected components of  $G'$  some color occurs twice which can be done in polynomial time in the number of edges and vertices.

We will show by reduction from MONOTONE 1-IN-3 SAT that P- $k$ -COL is NP-hard. Let  $\psi = (L_1, \dots, L_m)$  be an arbitrary instance of MONOTONE 1-IN-3 SAT. Each clause  $L_i$  will be identified with a hypercube  $Q_3$  denoted by  $Q^i$ . Let  $\mathcal{Q} = \{Q^1, \dots, Q^m\}$  be the set of these hypercubes. Each of the variables  $b_1, b_2, b_3 \in L_i$  is identified with one vertex  $v_1, v_2, v_3 \in V(Q^i)$  that have

coordinates (001), (010) and (100), respectively. Different variables  $b_r, b_s \in L_i$  are identified with different vertices. We can now create the joint graph  $G(m)$  based on the set  $\mathcal{Q}$  as in Definition 2.6, whereby we add an edge  $(v, w) \in E'$  between two different hypercubes  $Q^i$  and  $Q^j$  if there are two variables  $b \in L_i$  and  $b' \in L_j$  with  $b = b'$ . For an arbitrary integer  $k \geq 2$  we extend the graph  $G(m)$  to the graph  $G(m, k)$  as in Definition 2.7. Clearly, this reduction can be done in polynomial time in the number  $m$  of clauses and the integer  $k$ .

We will show in the following that  $G(m, k)$  has a path- $k$ -coloring if and only if  $\psi$  has a truth assignment. Let  $\psi = (L_1, \dots, L_m)$  have a truth assignment. Thus, in each clause  $L_i$  exactly one variable is true (**T**) and two of the variables are false (**F**). We will show that the corresponding joint graph  $G(m)$  has a path-2-coloring  $\mathcal{C} : V(G(m)) \rightarrow \{\mathbf{F}, \mathbf{T}\}$ , which then leads directly to a path- $k$ -coloring of  $G(m, k)$ . Thus, we first start to color the graph  $G(m) \subseteq G(m, k)$ . Let  $W^j \subset V(Q^j)$  be the set of vertices contained in  $G(m)$  with coordinates  $c^j(v) \in \{(001), (010), (100)\}$ . We assign to each vertex  $v \in W^j$  the respective color **T** or **F** according to the truth value of the unique Boolean variable contained in  $L_j$  that is associated to vertex  $v$ . Hence, exactly two of the vertices contained in  $W^j$  get color **F** and one gets color **T**. Since all such vertices in  $W^j$  are adjacent to the respective vertex  $x \in V(Q^j)$  with coordinates (000) we must set  $\mathcal{C}(x) = \mathbf{F}$ . To obtain a proper path-2-coloring Lemma 2.8 implies that  $\mathcal{C}(y) \neq \mathcal{C}(x)$  for the vertex  $y$  with coordinates (111), thus we set  $\mathcal{C}(y) = \mathbf{T}$ . The remaining vertices in  $V(Q^j)$  can now easily be colored in a unique way regarding Lemma 2.3 in order to obtain a path-2-coloring of  $\langle V(Q^j) \rangle$ . All vertices in  $G(m)$  are colored in this way w.r.t. to their corresponding induced subgraphs  $\langle V(Q^j) \rangle$  they are contained in. By construction, edges contained in  $E_0$ , resp.,  $E_1$  connect only vertices with the same color **F**, resp., **T**. The same holds for edges contained in  $E'$ , since  $\psi$  has a truth assignment and only those vertices are connected by edges  $e \in E'$  if they represent the same variable in different clauses. Therefore, all well-colored paths of  $G(m)$  are entirely included in the respective subgraphs  $\langle V(Q^j) \rangle$ ,  $j = 1, \dots, m$  which are path-2-colored. Hence, the joint graph  $G(m)$  has a path-2-coloring. In order to obtain a path- $k$ -coloring of the the graph  $G(m, k)$  we first observe that the vertices in  $G_0$ , must get color **F**, since all vertices with coordinates (000) induce together with the vertices in  $G_0$  a complete graph  $K_{k+1}$  and thus, only the trivial path-1-coloring can be used within this subgraph. Analogously, all vertices in  $G_1$  get color **T**. By the same arguments, each  $\mathbf{K}^i \in \mathcal{K}$  can only be path-1-colored since each  $\mathbf{K}^i$  is of size  $2m + k + 1$ . Hence, we color each  $\mathbf{K}^i \in \mathcal{K}$  with one of the remaining  $k - 2$  colors, different graphs contained in  $\mathcal{K}$  obtain different colors. It is easy to see that in this way new obtained well-colored paths can only be found in the induced subgraphs  $(\bigcup_{i=1}^{k-2} V(\mathbf{K}^i) \cup V(Q^l))$  for fixed  $l = 1, \dots, m$ . In particular, these new well-colored paths are either edges  $(x, y) \in E'$  or paths of the form  $x - v - y$  with  $x \in V(\mathbf{K}^i)$  for some  $\mathbf{K}^i \in \mathcal{K}$ ,  $v \in V(Q^l)$  and either  $y \in V(Q^l)$  or  $y \in V(\mathbf{K}^j)$ ,  $j \neq i$ ; see also Fig. 4. By construction, for all these cases holds  $\mathcal{C}(x) \neq \mathcal{C}(y)$  and hence, we obtain a path- $k$ -coloring of  $G(m, k)$ .

Conversely, suppose that  $G(m, k)$  has a path- $k$ -coloring  $\mathcal{C}'$ . We will show that the corresponding set of clauses  $\psi = (L_1, \dots, L_m)$  has a truth assignment. By Lemma 2.9, the restriction  $\mathcal{C} = \mathcal{C}'|_{V(G(m))}$  is a path-2-coloring of  $G(m)$ . Let  $\mathcal{C} : V(G(m)) \rightarrow \{\mathbf{F}, \mathbf{T}\}$  be such a path-2-coloring of the joint graph  $G(m) \subseteq G(m, k)$ . Lemma 2.8 implies that  $\mathcal{C}(v) \neq \mathcal{C}(w)$  for all vertices  $v, w \in V(G(m))$  with coordinates  $c^j(v) = (000)$  and  $c^l(w) = (111)$ , while  $\mathcal{C}(v) = \mathcal{C}(w)$  if  $(v, w) \in E_0 \cup E_1$ . Thus, w.l.o.g. we can assume that  $\mathcal{C}(v) = \mathbf{F}$  and  $\mathcal{C}(w) = \mathbf{T}$  for all vertices  $v$  and  $w$  with coordinates  $c^j(v) = (000)$  and  $c^l(w) = (111)$ , respectively. Let  $v \in V(Q^j)$  be the vertex with coordinates (000). Lemmas 2.1 and 2.5 imply that there are two vertices in  $N(v) \cap V(Q^j)$  with color **F** and one with color **T** for all  $Q^j \in \mathcal{Q}$  in  $G(m)$ . Note, the vertices in  $N(v) \cap V(Q^j)$  have respective coordinates (001), (010), and (100) and are by construction uniquely identified with the respective Boolean variables in  $L_j$ . Hence, we assign to each Boolean variable in each clause  $L_j$  a value “true” (**T**), resp., “false” (**F**), depending on the color of the corresponding vertex in  $V(Q^j)$ . Therefore, two of the Boolean variables get the value **F** and one gets value **T** in each clause  $L_j$ . Finally, applying Lemma 2.8 again we can conclude that  $\mathcal{C}(v) = \mathcal{C}(w)$  for all vertices  $v, w$  with  $(v, w) \in E'$ . Those edges connect vertices whenever the corresponding Boolean variables in the different clauses are identical. Since those vertices share the same color, we can conclude that the assignment of the values **T**, resp. **F**, to the Boolean variables does not lead to any conflicts and therefore to a valid truth assignment for  $\psi$ .  $\square$

We finish this contribution by stating several results concerning the complexity of other problems that result from the previous theorem.

#### Problem. CHECK S-COMPOSITE

*Input:* Given an arbitrary connected graph  $G = (V, E)$ .

*Question:* Is  $G$  S-composite?

**Theorem 2.12.** CHECK S-COMPOSITE is NP-complete.

**Proof.** To prove that CHECK S-COMPOSITE  $\in$  NP we must verify that a proposed embedding of the given graph  $G = (V, E)$  into a nontrivial Cartesian product  $G_1 \square G_2$  maps some edges of  $G$  to copies of edges of  $G_1$  and some edges to copies of edges of  $G_2$ . This task can obviously be done in polynomial time in the number of edges of  $G$ .

The NP-hardness follows directly by using the same arguments as in the proof of Theorem 1 in [9]. The authors showed (indirectly) the NP-hardness of CHECK S-COMPOSITE by constructing a polynomial time reduction from P- $k$ -COL and by proving that for any path- $k$ -coloring ( $2 \leq k \leq |V| - 1$ ) a nontrivial embedding of  $G$  into the Cartesian product of complete graphs  $K_k \square K_t$  ( $t \geq 2$ ) with  $k \leq K$  can be found and vice versa.  $\square$

There are other characterizations of S-composite graphs as they can also be defined in terms of edge labelings and nontrivial embeddings into Hamming graphs, that is, Cartesian products of complete graphs.

**Theorem 2.13** ([10]). *Let  $G = (V, E)$  be a connected graph. The following statements are equivalent:*

1.  $G$  is S-composite
2.  $G$  is 2-labelable, i.e.,  $E$  can be labeled with two labels such that on any induced cycle of  $G$  on which both labels appear, the labels change at least three times while passing the cycle.
3.  $G$  is a nontrivial subgraph of a Hamming graph with two factors.

Using the latter results and the constructions given in [10] we can formulate the following problems and theorem.

**Problem. 2-LABELABLE**

*Input:* Given an arbitrary connected graph  $G = (V, E)$ .

*Question:* Is  $G$  2-labelable?

**Problem. 2-FACTOR-HAMMING EMBEDDING**

*Input:* Given an arbitrary connected graph  $G = (V, E)$ .

*Question:* Is  $G$  a nontrivial subgraph of a Hamming graph with two factors?

**Theorem 2.14.** *The Problems 2-LABELABLE and 2-FACTOR-HAMMING EMBEDDING are NP-hard.*

Finally, since CHECK S-COMPOSITE is NP-complete its complementary decision problem CHECK S-PRIME is CoNP-complete.

**Problem. CHECK S-PRIME**

*Input:* Given an arbitrary connected graph  $G = (V, E)$ .

*Question:* Is  $G$  S-prime?

**Theorem 2.15.** *CHECK S-PRIME is CoNP-complete.*

## Acknowledgments

I thank Peter F. Stadler, Lydia Ostermeier, Daniel Stöckel and Jiong Guo for several useful remarks that helped to improve the presentation. This work was supported in part by the *Deutsche Forschungsgemeinschaft* (DFG) Project STA850/11-1 within the EUROCORES Programme EuroGIGA (project GReGAS) of the European Science Foundation.

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