Practice 2

Estimation of a potential using Random Walks

In this practice we are trying to find the electrostatic potential distribution inside of a square box, or a more complicated geometry using random walks.

More specifically, we are trying to find the solution to a Laplace equation in two dimensions, since the electrostatic potential inside an area bounded by a square box satisfies the Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{2.1}$$

In absence of the boundary conditions the solution to the Laplace equation is given by a Gaussian distribution function. This probability distribution also arises in a Random walk, thus we can reformulate the problem of finding the electrostatic potential in a box as a random walk problem.

Methodology

I defined a class called `Grid` since we need a way to keep track of multiple magnitudes for each point in the grid that makes the square box. These are

- The value of the potential $V(x, y)$.
- The value of the gradient (for the third task) $E(x, y)$.
- Which points in the grid are boundaries, e.g., the edges in the first part; or the plate condensers in the second part.
- The number of times we visit a specific point in the grid during a random walk.

This last point is very useful if we want our code to be more efficient and require less random walks. From the central limit theorem we know that our approximation of the solution for this problem will be better if we do more random walks for each point in the grid (the reduced variance decreases with $\sqrt{N}$).

We can make use of the fact that random walks are uncorrelated by definition and, instead of only keeping track of the starting position of the random walk when estimating the value of the potential, we save the entire path that the random walk followed until reaching a boundary. Then, we normalize the estimation of the potential by the number of times we visit each point in the grid. This way, we have more statistics with less random walks.

2.1 First Task

In this first part, we consider a rectangular (square) box with boundary conditions at the edges:

- Potential at the left edge: $V(0, y) = +10$.
- Potential at any other edge: $V(x, 0) = V(x, 1) = V(1, y) = +1$.

Notice that we normalize the box's dimensions to the region $(x, y) \in [0, 1]^2$.

2.1.1 Results

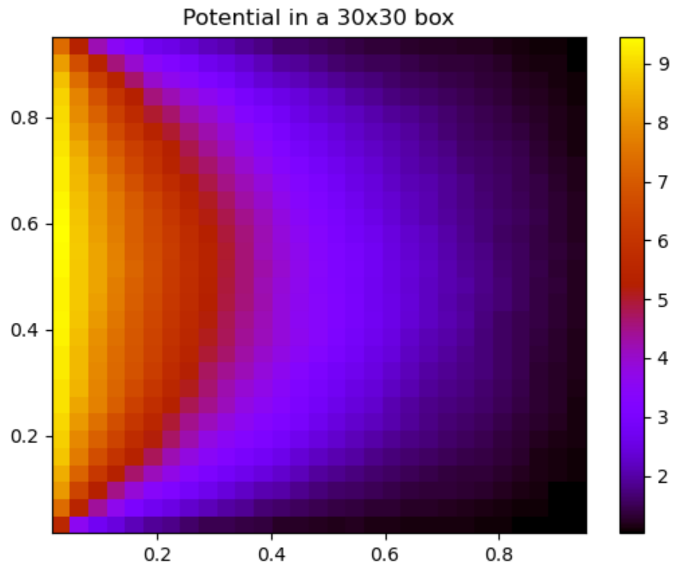

Figure 2.1: Estimation of the potential in a 30×30 grid.

We estimate and plot the potential for a 30×30 grid. As we can see in Figure 2.1, the resulting potential resembles the exact solution of Equation 2.1 with only 100 random walks for each point in the grid.

2.2 Second Task

For the second part, we will consider again the same rectangular box, but the boundary conditions will now change to:

- Potential at the edges: $V(0, y) = V(x, 0) = V(x, 1) = V(1, y) = 0$.
- Potential at the left vertical bar: $+1$.
- Potential at the right vertical bar: -1.

The bars are placed approximately $x_{bar} \in \{0.25, 0.75\}$ inside the box and centered vertically. Their lengths are half the box's, and their width is assumed to be infinitely small (at least relative to the box width).

2.2.1 Results

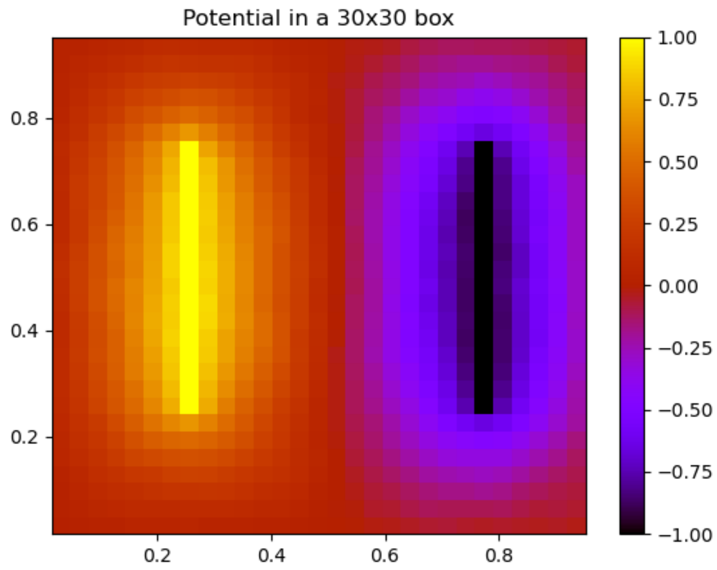

Figure 2.2: Estimation of the potential in a 30×30 grid.

In this case, we observe that the potential is only different than 0 when close to one of the two plate condensers, and the center of the box has 0 potential because the charges in the vertical bars are equal in magnitude but opposite in sign, so their contributions cancel out.

Once again, using only 100 random walks we are able to obtain a solution without too much noise. However, for the last part we will need much more precision in our estimation, since we will be approximating the gradient of the potential in the box.

2.3 Third Task

As we just mentioned, using the same box conditions as in the previous part, we will estimate the gradient at each point. For that we will use the following expression:

$$E = \sqrt{\left[\frac{V(x+\Delta x, y) - V(x,y)}{\Delta x}\right]^2 + \left[\frac{V(x, y+\Delta y) - V(x,y)}{\Delta y}\right]^2} = \sqrt{E_x^2 + E_y^2}$$

The width of the displacement should be very small compared to the size of the condenser. For that reason, we will increase the number of grid points in our box to 100×100.

We will also change the value of the potential at the vertical bars to $+10, -10$, respectively as well as modifying a bit their dimensions (length of the condenser ~ 50 units, 3 units wide). Finally, we can bring the bars closer to each other so that the gradient is larger between them $x_{bar} \in \{0.33, 0.66\}$.

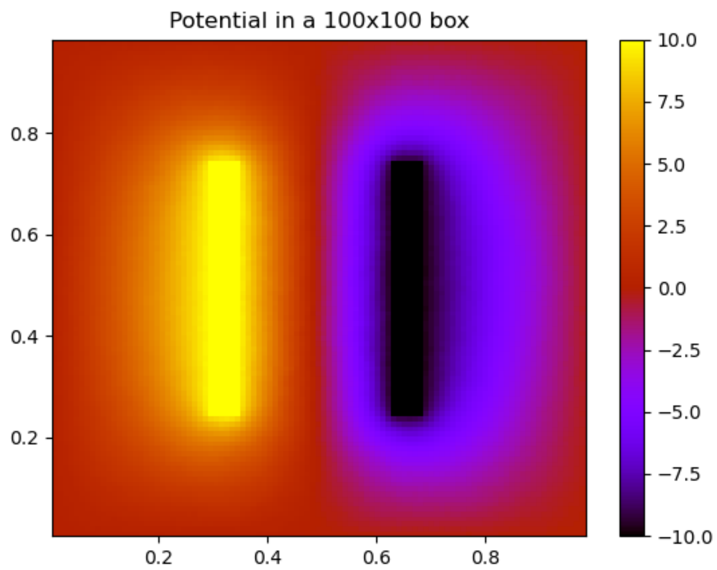

Figure 2.3: Estimation of the potential in a 100×100 grid.

2.3.1 Results

Comparing Figures 2.3-2.4 to the previous part, we can see that the obtained potential is even smoother, this is because we have a finer grid of points. On the other hand, by bringing both plates closer, we can now notice more easily that the middle part of the box is free from the potential due to the bars having opposite charges.

The gradient shown in Figure 2.4 was obtained from the previous potential within the square box by calculating its numerical gradient. This plot showcases the spatial variation and even the directionality of the electric field inside the box as the strength of the field changes across different regions.

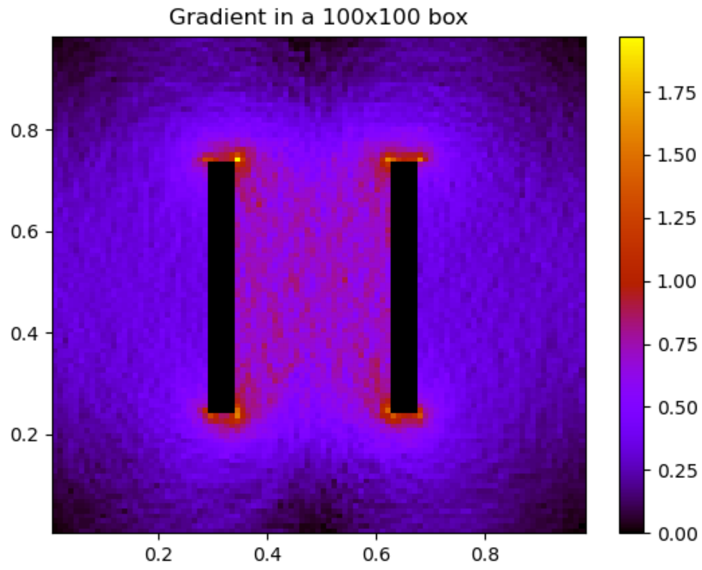

Figure 2.4: Estimation of the gradient in a 100 × 100 grid.