Math 1152 Lecture Notes

June 8, 2022

1 Series, Cont'd.

Now that we have the tool of Telescoping Series in hand, we can revisit our argument about Geometric Series and see how they are telescoping:

In general, a series $\sum a_n$ is telescoping if there is some fixed integer k so that a_n can be written as $a_n = b_n - b_{n+k}$.

Divergence Test

If we add up infinitely many things, and all of the things are bigger than 1, then the sum has to be infinite.

If instead of 1, all of the terms are bigger than ϵ for some $\epsilon > 0$, the same is still true.

Even if only an infinite number of times there is a term bigger than ϵ, this still means that the sum diverges.

The Divergence Test

If $\sum a_n$ converges, then $a_n \to 0$.
This is equivalent to saying : If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

We can think of *summability* of a sequence (a_n) as a strictly stronger property than that $\lim_{n\to\infty} a_n = 0$, similarly to how differentiability of a function is stronger than continuity.

The first test we should perform when determining whether a series converges or diverges is the Divergence Test.

Remember, the Divergence Test *will not* tell us that a series converges.

Exercise 1. Does the series $\sum \frac{n^2+1}{\log(n)+\log^2(n)}$ converge?

Harmonic Series

Does

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

converge?

2 Series and Integrals

We can take what we just saw for the series $\sum \frac{1}{n}$ and generalize it.

The crucial thing was that the function $\frac{1}{x}$ is continuous and decreasing.

The Integral Test

If f is a continuous and decreasing positive function then

$$\int_{k+1}^{\infty} f(x)\,dx \leq \sum_{n=k}^{\infty} f(n) \leq \int_{k}^{\infty} f(x)\,dx.$$

In particular, $\sum f(n)$ converges or diverges depending only on whether $\int_a^\infty f(x)\,dx$ does.

Exercise 2. For which values of p does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge or diverge?

Exercise 3. For which values of q does $\sum_{n=1}^{\infty} \frac{1}{n \log^q n}$ converge or diverge?

There is no single function or growth rate f which tells us the point at which $\frac{1}{nf(n)}$ converges or diverges - we can always keep adding more logs, and then convergence or divergence depends on the power of the logs. And if any series converges or diverges, one can always find a new convergent or a new divergent series with worse growth rates which still converge or diverge.

$$\sum \frac{1}{n \log n}, \sum \frac{1}{n \log n (\log \log n)}, \sum \frac{1}{n \log n (\log \log n)(\log \log \log n) \ldots}$$

all diverge, while

$$\sum \frac{1}{n \log n}, \sum \frac{1}{n \log n (\log \log n)^{1+\epsilon}}, \sum \frac{1}{n \log n (\log \log n)(\log \log \log n)^{1+\epsilon}} \ldots$$

all converge.

“Stieltjes” Sums

We said that there is an analogy between $\sum_{n=0}^{\infty} a_n$ and $\int_0^{\infty} f(x)\,dx$.

We can make this analogy more precise if we think about what is “missing” in the series : an analog of dx.

Given a sequence (b_n), define $\Delta b_n = b_{n+1} - b_n$.

Now consider series of the form

$$\sum_{n=1}^{\infty} a_n \, \Delta b_n.$$

Exercise 4. If for some (b_n), all series $\sum_{n=1}^{\infty} a_n$ are the same as the series $\sum_{n=1}^{\infty} a_n \, \Delta b_n$, what must (b_n) be?

Now we can interpret all series as *discrete* integrals.

We have already seen that reindexing is an analog of u-substitutions. Is there an analog of Integration by Parts?

Abel's Partial Summation Formula

$$\sum_{n=0}^{k} a_n \, \Delta b_n = (a_k b_{k+1} - a_0 b_0) - \sum_{n=0}^{k-1} b_{n+1} \Delta a_n$$

We can think of the Δb_n as putting a "weight" on the interval $[n, n+1]$, equal to how much the sequence (b_n) grows from n to $n+1$.

Aside: Is there an analog of this for integrals?

Think about Riemann sums,

$$\sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

but now allow the "mass" of the interval to be different than its length, say as determined by another function, g:

$$\sum_{k=1}^{n} f(x_k^*) \left(g(x_{k+1}) - g(x_k) \right) =: \sum_{k=1}^{n} f(x_k^*) \, \Delta g(x_k).$$

These sums, if they converge, are said to converge to the Riemann-Stieltjes integral

$$\int_a^b f(x) \, dg(x).$$

We can re-interpret Integration by Parts in terms of Stieltjes integrals:

$$\int_a^b f \, dg = fg|_a^b - \int_a^b g \, df$$

How can we check that this is true, as Riemann-Stieltjes integrals?

Does this give you any ideas for new kinds of derivatives?