Math 1152 Lecture Notes

June 6, 2022

1 Series

A series is an "infinite sum":

Given a sequence (a_n), we can define a series $\sum_{n=1}^{\infty} a_n$ by

$$\sum_{n=1}^{\infty} a_n := \lim_{m\to\infty} \sum_{n=1}^{m} a_n.$$

(In fact, we can define a series starting anywhere, not just from $n = 1$).

The **partial sums** of the series are given by

$$s_m = \sum_{n=1}^{m} a_n;$$

so we have also that the value of a series is the limit of the values of its partial sums:

$$\sum_{n=1}^{\infty} a_n := \lim_{m\to\infty} s_m.$$

(If the limit exists, it's just a number - we don't want to say that a number and series are the same thing - really we mean either by "series" either the formal collection of symbols "$\sum_{n=1}^{\infty} a_n$" or the operaion of taking in a sequence (a_n) and outputting the limit of its partial sums).

A series **converges** if the associated limit does, otherwise it **diverges**.

It is not a terrible analogy to consider:

$$\text{Continuous function} :: \text{definite integral}$$
$$\text{Sequence} :: \text{Series}$$

- this isn't quite the right analogy, but it is close enough to a correct one to be valuable to us. You might consider the question of how to complete

$$f :: \frac{d}{dx} f$$
$$(a_n) :: ???$$

Operations on Series

Since a series is a combination of the algebraic operation of addition, and the analytic operation of taking a limit, things which behave nicely with respect to both behave nicely for series. In particular, from the limit laws and artihmetic we have

- Sums of Series: $\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

- Multiplying by a constant: $\sum_{n=1}^{\infty} c \cdot a_n = c \sum_{n=1}^{\infty} a_n$

We also have operations which only make sense for series:

- Splitting a series: $\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n = \sum_{n=1}^{k} a_n + \sum_{n=k+1}^{\infty} a_n$

  and

- Re-indexing a series: $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1} = \sum_{m=k+1}^{\infty} a_{m-k}$.

Exercise 1. Reindex the series $\sum_{m=100}^{\infty} a_m$ to begin at 0.

Geometric Series

Perhaps the most fundamental series is the Geometric Series:

$$\sum_{n=0}^{\infty} r^n.$$

When $r = 1/2$, it is easy to see that this series, started at $n = 1$ rather than $n = 0$, converges to 1.

What about for other r?

Convergence of Geometric Series

A geometric series $\sum_{n=0}^{\infty} r^n$

- Converges to $\frac{1}{1-r}$ if $|r| < 1$
- Diverges to ∞ if $|r| > 1$ or $r = 1$
- Diverges, oscilating between 1 and 0 if $r = -1$.

Telescoping Series

What we just saw is a special case of a Telescoping Series, which is a series which exhibits cancellation between any given term and later terms.

Examples:

Exercise 2. Show that $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ is telescoping and find its value.

Exercise 3. Show that $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ is telescoping and find its value.

In general, a series $\sum a_n$ is telescoping if there is some fixed integer k so that a_n can be written as $a_n = b_n - b_{n+k}$.

Divergence Test

If we add up infinitely many things, and all of the things are bigger than 1, then the sum has to be infinite.

If instead of 1, all of the terms is bigger than ϵ for some $\epsilon > 0$, the same is still true.

Even if only an infinite number of times there is a term bigger than ϵ, this still means that the sum diverges.

The Divergence Test

If $\sum a_n$ converges, then $a_n \to 0$.
This is equivalent to saying : If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

We can think of *summability* of a sequence (a_n) as a strictly stronger property than that $\lim_{n\to\infty} a_n = 0$, similarly to how differentiability of a function is stronger than continuity.

Harmonic Series

Does

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

converge?