Math 1152 Lecture Notes

July 18, 2022

1 Power Series and Diff. Eq., Cont'd.

Not all differential equations will have Power Series solutions about a given point (not all functions do - $\frac{1}{x}$ doesn't have a Power Series centered at $x = 0$), but series still offer a powerful and highly general tool in solving ordinary differential equations.

Theorem 1 (Existence Theorem for Analytic Coefficients)**.** Let x_0 be a real number and suppose that the coefficients a_k in the equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

have convergent power series expansions in an interval $[x_0 - r_0, x_0 + r]$ centered at x_0, with $r_0 > 0$. Then for any constants $c_0, \ldots, c_{n-1}$, there is a power-series solution, y, to the equation, satisfying $y^{(k)}(x_0) = c_k$ which converges on $[x_0 - r_0, x_0 + r_0]$.

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2 Separable Differential Equations, Cont'd.

One nice way to set up solutions to separable differential equations is the following:

$$
\begin{gathered}
y' = f(x)/g(y) \\
\Downarrow \\
\frac{dy}{dx} = f(x)/g(y) \\
\Downarrow \\
g(y)\,dy = f(x)\,dx \\
\Downarrow \\
\int g(y), dy = \int f(x)\,dx \\
\Downarrow \\
G(y) = F(x) + C \\
\Downarrow \\
y = H(x).
\end{gathered}
$$

Exercise 1. Find a solution to the equation

$$y' = (x^2 - 4)(3y + 2).$$

Newton's Law of Cooling states that the rate of change of an object's temperature is directly proportional to the difference between its temperature and that of the ambient environment.

In the following exercises, first, find a differential equation to describe the situation. Then, solve that differential equation.

Exercise 2. A pizza enters an $800°F$ oven. Suppose that the pizza is at a room-temperature of $70°F$. Browning occurs as part of the *Maillard reaction*, which takes place between $280°F$ and $330°F$. How long must the pizza remain in the oven for its crust to undergo the Maillard reaction? Is the Newton's Law of Cooling model accurate here?

Exercise 3. A pizza leaves an $800°F$ oven after its interior has reached an ideal $190°F$ - fully baked, but not dried out and tough (quickly reaching appropriate temperatures without drying out the pizza is the reason for cooking at high heats). In a $70°F$ room, how long until the interior has reached the pleasantly hot food temperature of $160°F$?

Definition 1. An **autonomous** differential equation is one in which the independent variable does not appear.

A **steady-state** or **equilibrium** solution to an autonomous differential equation is informally said to be a constant function which is the limit as $t \to \infty$ of a solution $y = f(t)$ to a separable differential equation, when such a constant exists.

Exercise 4. Solve the differential equation

$$y'y^2 - y = 0.$$

Does it possess a steady-state solution?

Exercise 5. Solve the differential equation

$$xy^2 + 2xy + x = yx^2y' - yy'.$$

Does it possess a steady-state solution?

Direction Fields

Given a differential equation of the form

$$y' = f(x, y),$$

we may (pointwise) plot solutions by

1. Determining a grid (or other set of coordinates to use in the plot)
2. Calculating the value of y' at each (x, y) above
3. Drawing a short, slanted line-segment whose slope corresponds to this value of y' at each such point.

(Under reasonable assumptions) we can visually read off the graphs of different solutions from the direction field.

Exercise 6. Sketch a slope-field for the equation

$$y' = y^2.$$

Can you tell whether the equation is autonomous from this direction field? Can you tell if there are any equilibrium solutions?