Math 1152 Lecture Notes

July 22, 2022

1 Last time

Parametric equations parametrize an equation in variables $x, y, \cdots$ by making explicit the parameter(s) $t, s, \cdots$ on which the variables depend. This makes it easy to see, for instance, that a curve like

$$\gamma(t) = \begin{cases} x(t) = t^2 + \sin(t) \\ y(t) = t^3 + \cos(t) \end{cases}$$

is one-dimensional.

The idea of making explicit some hidden parameters which underly relationships between multiple varialbes is also a thing which shows up, in a different form, in statistics.

2 Calculus and parametric equations

Suppose that $\gamma(t) = (x(t), y(t))$. To talk about limits, we want to be able to talk about how close $\gamma(t)$ and $\gamma(s)$ are. Given two points in the plane, their distance is given by the distance formula, so we define

$$|\gamma(t) - \gamma(s)| = \sqrt{|x(t) - x(s)|^2 + |y(t) - y(s)|^2}.$$

Then we can say

$$\lim_{t \to s} \gamma(t) = \left(\lim_{t \to s} x(t), \lim_{t \to s} y(t) \right)$$

not as a *definition*, but as a *theorem*, with our definition of limit being that

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$\lim_{t \to s} \gamma(t) = L$ if for every $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ so that

$$|\gamma(t) - L| < \epsilon$$

whenever

$$|t - s| < \delta.$$

Theorem 1. If $\lim_{t \to s} x(t) = a$ and $\lim_{t \to s} y(t) = b$, then

$$\lim_{t \to s} \gamma(t) = (a, b).$$

Continuity

We define continuity as usual: γ is continuous at s if $\lim_{t\to s}\gamma(t) = \gamma(s)$.

Exercise 1. Explain why $\gamma = (x, y)$ is continuous if and only if x and y are continuous.

Question 1. Are there so-called "space-filling" curves?

Derivatives

Here's an attempt at a definition of derivative for a parametrized curve γ :

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

In order for this to work, we have just one problem : we need to know about operations of addition (or subtraction) between curves and multiplying (or rather, dividing) them by real numbers.

Vectors

A Vector Space is a set, X, endowed with a binary operation, $+$, and an identity element, 0, such that for any $u, v, w \in V$

1. $0 + v = v$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$

together with an operation $\cdot : \mathbb{R} \cdot V \to V$ satisfying that for any real numbers (called **scalars**), a, b,

4. $1 \cdot v = v$
5. $a \cdot (b \cdot v) = (ab) \cdot v$
6. $v + (-1) \cdot v = 0$
7. $a \cdot v + b \cdot v = (a + b) \cdot v$.

Elements of a vector space are called **vectors**.

Exercise 2. $\mathbb{R}^2$ is a vector space. So is $\mathbb{R}$. So are the polynomials of degree less than any fixed d. So is the space of all continuous functions.

So lets reinterpret our parametrized curve γ as mapping $\mathbb{R}$ into the vector space $\mathbb{R}^2$. Then

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

makes sense, and we can see that it is just

$$\begin{aligned}\gamma'(t) &= \lim_{h \to 0} \frac{(x(t+h) - x(t), y(t+h) - y(t))}{h} \\ &= \lim_{h \to 0} \left(\frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h} \right) \\ &= (x'(t), y'(t)) \,.\end{aligned}$$

Geometrically, this also matches what we want from the derivative:

Exercise 3. Find the equation for the tangent lines to the curve $\gamma(t) = (\cos(t), \sin(t))$.

3 Lengths of curves, revisited