Math 1152 Lecture Notes

July 20, 2022

1 Parametric Equations

We might wish to use the tools of calculus to investigate a curve ...

... but that curve might not be a function.

This situation is easily fixed by changing perspective. Perhaps this curve actually *is* a function.

Ordinarily, we consider graphs where the domain lives on the horizontal axis and the range on the y-axis. But we could instead consider a curve where only the image is graphed in the xy-plane and the domain is implicit.

Parametric equations are equations (usually in this course given in x and y) involving variables which are defined by a separate parameter (often denoted t, or s, or θ).

If we take the curve as a fundamental object, then our usual way of thinking about it is to try to solve for y in terms of x, but we could instead think of both x and y as varying as we move along the curve. This is the difference between finding a cartesian description of the curve, and a parametric one.

Remark 1. In this view, the curve is the same curve regardless of what parametric description we obtain for it, even when those parametric descriptions are inequivalent as functions. We can create new parametric equations via substituing in transformations of the parameter, similar to thinking of two integrals as "equivalent" if they differ by a u-substitution.

Exercise 1. Find a parametric equation for the circle.

$$y(t) = \sin(t).$$

Exercise 2. Find another parametric equation for the circle.

Exercise 3. Modify your parametric equation for the circle to find a parametric equation for a spiral.

Convering Between Parametric and non-Parametric

Given a non-parametric equation

$$y = f(x)$$

one can always trivially make it parametric by setting

$$x(t) = t$$

$$y(t) = f(x(t)).$$

If f factors as $f = g \circ h$ where h has an inverse, then one can redistribute the complexity by writing

$$x(t) = h^{-1}(t),$$

$$y(t) = g(t)$$

- the implicit relationship between x and y is still that $y = f(x)$.

Exercise 4. Find a paremetric description of

$$y = \sqrt{x^3 + 1}.$$

To convert from parametric to non-parametric equations, we can to find an equation relating $x(t)$ and $y(t)$. That usually involves taking $y(t)$ and rewriting it until all of the t's which show up do so in blocks of $x(t)$, then replacing $x(t)$ by x.

Exercise 5. Find a non-parametric equation for

$$x(t) = x^2,$$

$$y(t) = x^4 - 2x^2 + 1.$$

Formally, we could always attempt to instead write

$$x = f(t)$$

$$y = g(t)$$

and then solve

$$t = f^{-1}(x),$$

$$y = g\left(f^{-1}(x)\right)$$

to obtain a non-parametric equation relating x and y - but this requires first that f be invertible (and that we can write down a formula for its inverse) and second that we perform the work of composing g with this inverse. Usually, this is less elegant and more work than alternatives.

Most often the best way to convert a parametric equation is halfway between the above two techniques - to solve for a common function of t in both equations

- for example, if we could find that $e^x = \sqrt{1+t}$ and $\frac{y-1}{6} = \sqrt{1+t}$, we could then conclude that $e^x = \frac{y-1}{6}$.

Exercise 6. Write

$$x(t) = 10\sin^2(t) + 2,$$

$$y(t) = 2\cos(t)$$

in non-parametric form by solving both for $\sin^2(t)$.