Math 1152 Lecture Notes

June 15, 2022

1 Power Outage Interlude

What are integrals and sums good for?

- Measuring Size
- Transforming one object into another.

- Measuring the size of a transformation.

Examples:

Exercise 1. One student's scores up to time t while playing the popular videogame *Ghost Infestation!* were given by the formula

$$f(t) = \frac{1}{(1-t)^{\frac{1}{2}}}$$

while another's scores were given by

$$g(t) = 4t.$$

Which student is better overall at *Ghost Infestation!*?

ℓ^p norms for sums

$$\|f\|_{\ell^1} = \sum |f(n)|$$

$$\|f\|_{\ell^p} = \left(\sum |f(n)|^p\right)^{1/p}$$

$$\|f\|_{\ell^\infty} = \text{the largest value of } f(n).$$

L^p norms for integrals

$$\|f\|_{L^1} = \int_a^b |f(x)|\ dx$$

$$\|f\|_{L^p} = \left(\int_a^b |f(x)|^p\ dx\right)^{1/p}$$

$$\|f\|_{L^\infty} = \lim_{p\to\infty} \left(\int_a^b |f(x)|^p\ dx\right)^{1/p} = \text{ the biggest value } f \text{ achieves on } [a,b].$$

The *average* of f on $[0,1]$ is $\|f\|_{L^1}$. The maximum of f on $[0,1]$ is $\|f\|_{L^\infty}$. For p between 1 and ∞, $\|f\|_{L^p}$ represents an interpolation between these extremes.

Exercise 2. In a variant of *Ghost Infestation!*s rules, binding ghosts in different rooms earns different points: binding a_n ghosts in the nth room is worth

$$b_n = \frac{a_n}{n}$$

points. Is b_n summable if a_n is? Are there patterns (a_n) which lead to infinite scores under one ruleset but finite scores under the other?

Exercise 3. Consider the transformation, T, which takes a function f, and returns a new one, Tf, given by

$$Tf(x) = \int_1^\infty f(x)/x \, dx.$$

What conditions on f guarantee that Tf stays "small" if f is small?

Lengths of Curves, Detour

Recall our definition of the length of a curve $t \mapsto f(t)$ in the plane:

$$L =$$

- Does every finite curve have a continuous length?

- And if f' is an integrable function, does the formula

  $$L = \int_a^b \sqrt{1 + f'(t)^2}\, dt$$

  have to hold?

Lets see why the answer is “no”.

First, we have to talk about the Cantor Set.

This is a set defined by the infinite procedure of taking a collection of intervals and removing their middle thirds to obtain twice as many new intervals, then iterating.

Start with $[0, 1]$:

The endpoints of the intervals are all numbers which can be written as

$$\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots + \frac{a_n}{3^n}$$

where each a_i is either 0 or 2 -

the left-endpoints : $a_n = 0$,

right-endpoints : $a_n = 2$.

If we call the collection of intervals at the nth stage A_n, then the Cantor Set, $\mathcal{C}$, is the intersection of all of the A_n:

$$\mathcal{C} = \bigcap_{n=1}^{\infty} A_n.$$

What is its length?

What is the length of A_n?

A_n consists of ____________ intervals, each of length ____________, so the total size of A_n is

____

So $length(\mathcal{C}) \leq length(A_n) =$____________$\rightarrow$ ____________.

So the length of the Cantor set is 0 - but at the same time, it consists of all the points whose trinary expansions look like $0.a_1 a_2 \ldots a_n \ldots$ where each $a_i = 0$ or 2.

This is in one-to-one correspondence with the binary expansion of the points in $[0, 1]$ - there are as many points in $\mathcal{C}$ as there are in $[0, 1]$, even though the first has length 0 and the second has length 1.

So how does this give us a curve in $\mathbb{R}^2$ whose length is wrong?

Let $(c_n)_{n=0}^{\infty}$ be the left-endpoints of all the subintervals define A_n for any n, listed in order. These are the points which, writtein in trinary, are of the form $0.** \cdots * \mathbf{0}$, where everything except the last entry is either a 0 or a 2.

Let $f(0) = 0$ and $f(x) = 1 - \frac{1}{2^n}$ for all x between c_n and c_{n+1}, and $f(1) = 1$.

Claim: f is continuous.

Claim: $f'(x) = 0$ for all x not in $\mathcal{C}$.

Claim: $\int_0^1 \sqrt{1 + f'(x)^2}\, dx = 0$.

Claim: $L \neq 0$.