Math 1152 Lecture Notes

May 27, 2022

1 Partial Fraction Decomposition

Part 1 : Quadratics

We now know how to find

- $\int \frac{1}{\sqrt{1-x^2}}\,dx$

- $\int (x^2 \pm a^2)^{k/2}\,dx$

- $\int \frac{1}{a^2+x^2}\,dx$

and

- $\int \frac{1}{x^2+2x+1}\,dx$.

But what about
$$\int \frac{1}{x^2+x+1}\,dx?$$

Idea : Complete the square.

To Integrate $\frac{1}{ax^2+bx+c}$ when the quadratic is irreducible:

1. Complete the square: Write $ax^2 + bx + c = a(x-h)^2 + k$.
2. Perform a u-substitution: $u = \sqrt{a}(x-h)$, $du = a\,dx$, so that
$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{\sqrt{a}} \int \frac{1}{u^2+k}\,du$$
3. Integrate to get $\int \frac{1}{u^2+k}\,du = \frac{1}{\sqrt{k}} \tan^{-1}\left(u/\sqrt{k}\right)$.
4. Plug in $u = \sqrt{a}(x-h)$ to get the answer.

- Why does the quadratic need to be irreducible?

- What can we do when the quadratic *does* factor?

Part 2 : Partial Fractions

The key insight:

$$\frac{1}{x-r} + \frac{1}{x-s} = \frac{r-s}{(x-r)(x-s)}.$$

Sums of linear fractions $\Rightarrow$ A single higher-degree fraction

Can this be reversed?

This is exactly what algebra is for.

Exercise 1. Write $\frac{1}{x(x-2)(x-3)}$ as a sum of linear recriprocal functions.

Idea: Write

$$\frac{p(x)}{(x-r_1)\cdots(x-r_n)} = \frac{A_1}{x-r_1} + \cdots + \frac{A_n}{x-r_n},$$

and solve for the unknown $A_1, \ldots, A_n$.

How to solve for $A_1, \ldots, A_n$:

Exercise 2. Find $\int \frac{x}{x^2-x-6}\,dx$.

There are some constraints, if this works:

- $p(x)$ must be a polynomial of degree less than n.
- Each of the r_i must be distinct. What if they weren't?
- Implicit constraint: The denominator is a product of linear factors. What if it were a combination of linear and quadratic factors?

For repeated roots: If the root r_i is repeated three times on the left, then include it three times on the right, each time with a higher power:

$$\text{example:} \quad \frac{1}{(x-r_i)^3} \to \frac{A}{x-r_i} + \frac{B}{(x-r_i)^2} + \frac{C}{(x-r_i)^3}.$$

Exercise 3. Find the partial fraction decomposition for $\frac{1}{x(x-2)^2}$.

For quadratic factors: If The factor is reducible, factor it as a product of linear factors. If it is irreducible, then the irreducible quadtratic as a denominator of one of the fractions on the right-hand side - the corresponding numerator should have the form $Ax+B$ in this case.

$$\frac{1}{x^2+1} \to \frac{Ax+B}{x^2+1}.$$

Exercise 4. Find the partial fraction decomposition of $\frac{1}{x^3+x}$,

Combine these rules for repeated, irreducible quadratics

Exercise 5. Find the partial fraction decomposition of $\frac{1}{x^5+x^4+x^3-x^2-x-1}$. (Hint: The denominator is $(x-1)(x^2+x+1)^2$).

To Find the Partial Fraction Decomposition

1. If the numerator has degree greater than or equal to the degree of the denominator (ie, it's 'improper'), use long division to rewrite it as the sum of a polynomial plus a proper rational function.
2. Factor the denominator into a product of distinct, repeated linear and irreducible quadratic factors as $(x-r_1)^{s_1}\cdots(x-r_n)^{s_n}\cdot q_1(x)^{t_1}\cdots q_m(x)^{t_m}$.
3. Then the P.F.D. is

$$\begin{array}{lll}
\dfrac{A_1}{x-r_1}+ & \cdots & +\dfrac{A_{s_1}}{(x-r_1)^{s_1}} \\
\vdots & & \\
+\dfrac{Z_n}{x-r_n}+ & \cdots & +\dfrac{Z_{s_n}}{(x-r_n)^{s_n}} \\
+\dfrac{A_1'x+B_1}{q_1(x)}+ & \cdots & +\dfrac{A_{t_1}'x+B_{t_1}}{q_1(x)^{t_1}} \\
\vdots & & \\
+\dfrac{Y_1'x+Z_1}{q_m(x)}+ & \cdots & +\dfrac{Y_{t_m}'x+Z_{t_m}}{q_m(x)^{t_1m}}
\end{array}$$

4. Solve for the unknowns in the numerators, either by multiplying through to clear denominators or setting up a system of equations, or by using the "Heaviside Cover-up Method", guessing good values of x to try plugging in, or a combination of approaches.

In other words:

- Work with a 'proper' rational function.
- Factor the denominator.
- Know what to do with distinct linear factors.
- Know what to do with irreducible quadratic factors.
- Know what to do with repeated factors.
- Know how to solve for the numerators.

Heaviside Coverup Method

Consider the original factored term

$$\frac{p(x)}{(x-r)(x-s)\cdots(x-t)}.$$

Find the value, r, of x which makes the linear term in one of the denominators zero.
Cover up that factor (ignore it/erase it) in the original fraction to get

$$\frac{p(x)}{(x-s)\cdots(x-t)}$$

then plug the value r in for x to get

$$\frac{p(r)}{(r-s)\cdots(r-t)}$$

This gives the value of A in the decomposition

$$\frac{p(x)}{(x-r)(x-s)\cdots(x-t)} = \frac{A}{x-r} + \frac{B}{x-s} + \cdots + \frac{C}{x-t}.$$

Exercise 6. Find the partial fraction decomposition of $\frac{1+x}{(x-2)(x-3)(x-4)}$ using the Heaviside Coverup Method.