Math 1152 Lecture Notes

July 8, 2022

1 Power Series

A Taylor Polynomial has the form

$$T_a^k(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^(k)(a)}{k!}(x-a)^k$$

$$= \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In the limit as $k \to \infty$, we get an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n.$$

There are a number of natural questions to ask, such as

- Does this infinite series converge to the value $f(x)$? If not always, when and where does it do so? For which f and for which x?
- Can all functions be written as infinite series?
- Can we use the structure of the infinite series to derive any special properties of the functions they converge to?
- If the infinite series is a limit of polynomials, does that mean that the function it converges to has any special properties in common with polynomials?

In order to answer some of these questions, it helps to temporarily forget about Taylor Polynomials, and just focus on series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n.$$

Such series are called **Power Series**.

Definition 1. A **Power Series** is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n;$$

the **center** of the series is at the point a; when $a = 0$, the series

$$\sum_{n=0}^{\infty} a_n x^n$$

is also called a MacLauren Series.

Exercise 1. Let

$$f(x) = \sum_{n=0}^{\infty} x^n.$$

For which x does the series $f(x)$ converge?

Definition 2. The **radius of convergence** of a power series $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is the value $r \geq 0$ such that for all x in $(a-r, a+r)$, $f(x)$ converges.

The interval $(a-r, a+r)$ is known as the **(open) interval of convergence** of the series. If conditional convergence happens at the endpoints of the open interval of convergence, then that largest closed or half-open interval is also referred to as the interval of convergence.

1. The interval of convergence is given by the reciprical of the ratio in the Ratio Test:

2. If x is within the the open interval of convergence, then the series $f(x)$ converges absolutely (and in fact, **uniformly**).

3. If x lies outside the interval of convergence, then the series $f(x)$ diverges.

Exercise 2. Find the interval of convergence of the following series.

a. $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n}$

b. $g(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{n!}$

c. $h(z) = \sum_{n=0}^{\infty} \frac{(i)^n x^{2n}}{(2n)!}$

Operations on Power Series

- Sums

- Products

- Compositions

- Derivatives

- Integrals