Math 1152 Lecture Notes

July 27, 2022

1 Calculus in Polar Coordinates

Derivatives

Consider a polar curve $r = f(\theta)$. What does $\frac{dr}{d\theta}$ represent?

What other derivative information is available to us?

Recall that

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Thus if $r = r(\theta)$,

$$\frac{dx}{d\theta} = r'(\theta)\cos\theta - r(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = r'(\theta)\sin\theta + r(\theta)\cos\theta.$$

By the Chain Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

(why?)

so we get the unpleasant-seeming formula

$$\frac{dy}{dx} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

Exercise 1. Consider the $r^2 = 4\cos(2\theta)$. Find all tangent lines passing through the origin (/ pole).

Integrals

Given a polar representation $r = f(\theta)$ for a curve, we can attempt a "Riemann Sum" of the form

$$\text{Area} \approx \sum_{k=1}^{n} \mathbf{A}\text{rea}(\theta^*)$$

where we break the region under the curve into *polar* rectangles.

Definition 1. A **polar rectangle** is a rectangle of the form $[a, b] \times [c, d]$ in polar coordinates, or equivalently,

$$a \leq r \leq b$$

$$c \leq \theta \leq d.$$

Exercise 2. What is the *shape* of a polar rectangle? What is its *area*?

So we've found that

Area of a polar rectangle

$$A = \frac{1}{2}r^2\Delta\theta.$$

With that in hand, we can form *polar integrals* from polar Riemann sums.

$$\begin{aligned} A &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2}\left(f(\theta_k^*)\right)^2\Delta\theta \\ &= \int_a^b \frac{1}{2}\left(f(\theta)\right)^2\, d\theta. \end{aligned}$$

In compact notation, $\boxed{A = \int \frac{1}{2}\, r^2\, d\theta.}$

Exercise 3. Verify this formula by using it to calculate the area of a circle with radius 2.

The area between curves works exactly the same in polar coordinates as it does in cartesian.

Exercise 4. Find the area between the cardioid $r = 2(1 + \cos\theta)$ and the circle $r = 4$.

Exercise 5. Find the area between the curves $r = 1 + \cos\theta$ and $r = 1 + \sin\theta$ as shown in the region below.