Math 1152 Lecture Notes

June 17, 2022

1 Last time

The Comparison Test

Suppose that $0 \leq a_n \leq b_n$.

- If $\sum b_n$ converges, then
$$\sum a_n \text{ converges.}$$
- If $\sum a_n$ diverges, then
$$\sum b_n \text{ diverges.}$$

The Limit Comparison Test

If $a_n, b_n \geq 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} < \infty$,

then

$$\sum a_n \text{ and } \sum b_n \text{ either both converge or both diverge.}$$

Exercise 1. Determine whether

$$\sum \frac{n^{.01}}{\sqrt[100]{n^{1}01}}$$

converges or diverges.

2 Ratio Test

- Suppose that
$$0 < \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1.$$

- This is like a recurrence-relation for the sequence (a_n), except it's an inequality and occurs in the limit.

- Morally, we have that $a_{n+1} = ra_n$, which would mean that $a_n = r^n a_0$ if it were true for all n.

- And that would mean that we have a geometric series which converges.

- That's the intuition. How do we give it a frame that holds up?

- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$, then for any choice of s in $(r, 1)$ it has to be true that eventually, after some point N_0, `ALL` $\frac{a_{n+1}}{a_n} < s$ (why?)

- This new inequality is one which is true, and which we can use:

$$a_{n+1} < sa_n \text{ for all } n > N_0$$

so we can unwind from a_{N_0+m} all the way down to a_{N_0}:

$$a_{N_0+m} < s^m a_{N_0}.$$

- So

$$\begin{aligned}\sum_{n>0} a_n &= \sum_{n=1}^{N_0} a_n + \sum_{n=N_0+1}^{\infty} a_n \\ &= s_{N_0} + \sum_{m=1}^{\infty} a_{N_0+m}\end{aligned}$$

and $\sum_{m=1}^{\infty} a_{N_0+m}$ converges by the Comparison Test, since $a_{N_0+m} < s^m$ which are the terms of a geometric series if $s < 1$.

- Would this argument be faster/cleaner if we used the Limit Comparison Test instead?

The Ratio Test

Suppose that $a_n > 0$ and $a_{n+1}/a_n \to r$. Then if

- $r < 1$: $\sum a_n$ converges
- $r > 1$: $\sum a_n$ diverges
- $r = 1$: The Ratio Test is inconclusive.

Exercise 2. Does the series

$$\sum \frac{(n!)^2}{(2n)!}$$

converge or diverge?

When applying the Ratio Test to a term $a_n = \frac{p_n}{q_n}$, write

$$\frac{a_{n+1}}{a_n} = \frac{p_{n+1}}{p_n} \frac{q_n}{q_{n+1}}$$

3 The Root Test

Exercise 3. Show that $\lim_{n\to\infty} \sqrt[n]{C} = 1$.

The Root Test

If $a_n \geq 0$ and

$$\lim_{n\to\infty} \sqrt[n]{a_n} < 1$$

then $\sum a_n$ converges.

Converseley, the series diverges if the above limit is > 1.

Exercise 4. Show that the Root Test holds.

Exercise 5. Use the Root Test to determine for which values of $x > 0$ the series

$$\sum_{n=0}^{\infty} nx^n$$

converges.

4 More exercises

In each example, determine whether to use the Root or Ratio Tests, and then apply them.

- $\sum \frac{1}{n!}$

- $\sum \left(\frac{n-1}{2n+3}\right)^n$

- $\sum \frac{10^n}{n!}$

- $\sum \frac{(\ln n)^{2n}}{n^n}$

- $\sum \frac{n^2}{2^n}$