Math 1152 Lecture Notes

July 13, 2022

1 Operations on Power Series

Sums

Within the smaller of the two radii of convergence, the sum of two equicentral series $\sum a_n x^n$ and $\sum b_n x^n$ is the "series of the sums": $\sum (a_n + b_n)x^n$; this may require reindexing:

Exercise 1. Find the power series for $\cos(x) + i\sin(x)$.

Products

Within the smaller of the two radii of two equicentral series, the product is given by

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

where

$$c_n = \sum_{k=0}^{n} a_{n-k} b_k.$$

Exercise 2. Write out the first few terms of the product above by hand to see why.

Note that multiplying by x^k has the impact of shifting a series' coefficients by k:

Exercise 3. Find the series for $\frac{x}{1-x}$.

Compositions

Composition of two series is valid, though this may recenter a series and shrink its radius; generally, there isn't a very good way to compute the formula for the coefficients of the composition except "by hand". We will usually look at compositions of a known series with a very simple, polynomial, "inside" series.

Exercise 4. Determine the Taylor Series centered at 0 and radius of convergence for the following functions.

a. e^{x^2}

b. $\sin(2x)$

c. $1/(1+x)$.

Derivatives

Within the interval of convergence of a series, the derivative of the series is the "series of derivatives":

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n.$$

Upon reindexing, this simplifies to

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx}(n+1)a_{n+1} x^n.$$

Going by definitions, this requires the interchange of two limit processes (that for the derivative, and that for the infinite sum) - this works because the error term in the infinite sum is bounded *uniformly* within any closed subset of the interval of convergence -

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{N} a_n x^n + R(x)$$

where $|R(x)| < \epsilon$ when N is big enough - regardless of which x within the closed subset of the interval of convergence is chosen -

Exercise 5. Find the series for $\frac{1}{(1-x)^2}$.

Integrals

Just like for derivatives, integrals work term-by-term within the radius of convergence, for the same reasons.

$$\int \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \int a_n x^n$$

or

$$\int \sum_{n=0}^{\infty} a_n x^n = a_{-1} + \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n;$$

here, a_{-1} is the unspecificied constant of integration.

Exercise 6. Find the series for $\ln(x)$.

Putting it all together

Exercise 7. Find the series for $\arctan(x)$.

Exercise 8. Find the series for $(x-1)/(1-x^2)$.