Math 1152 Lecture Notes

May 23, 2022

1 Integration by Parts

Our main tool so far, and the first choice in our arsenal, is u-substitutions.

Question 1. Where do u-substitutions come from?

But u-substitutions will not always permit us to perform an integration, just as the Chain Rule isn't always the right rule for performing a differentiation.

Exercise 1.

a. Calculate $\int 2xe^{x^2}\,dx$.

b. Calculate $\int 2xe^{x}\,dx$.

The Product Rule for derivatives tells us that

$$(uv)' = u'v + uv'.$$

Question 2. What can the Product Rule tell us about integrals?

Integration by Parts

$$\int u\,dv = uv - \int v\,du.$$

This is a statement about functions and their derivatives; for definite integrals, what this means is that

Integration by Parts (definite integral version

$$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du.$$

Often, we can choose which term to set equal to u and which term to set equal to dv based upon which of them we can integrate, or which of them will be made simpler upon a differentiation:

Exercise 2. Calculate

$$\int 2xe^x\,dx.$$

Sometimes, Integration by Parts needs to be performed multiple times. Note that each time one performs Integration by Parts, a term gets differentiated.

Oftentimes, differentiation makes terms simpler, and if they're polynomial, they eventually disappear:

Exercise 3. Calculate

$$\int x^3 e^x \, dx.$$

Other times, differentiation is periodic, as is the case for differentiation of the sine and cosine functions and e^x.

Exercise 4. Consider $\int \cos(x)e^x \, dx$.. What are the first three derivatives of $\cos(x)$? What are the first three integrals of e^x?

In these cases, Integration by Parts may return us to a *multiple* the original integral together with some "boundary terms" (the uv terms in Integration by Parts) - this gives us an algebraic equation we can solve:

Exercise 5. Calculate $\int \cos(x)e^x \, dx$..

Occasionally, we can profitably use that for any function, f, $f = 1 \cdot f$ to apply Integration by Parts where it is not obvious to do so:

Exercise 6. Use Integration by Parts to find a formula for

$$\int \arctan(x)\, dx.$$

2 Trigonometric Integrals

Recall the following Trionometric Identities:

Trigonometric Identities

Pythagorean Identties:

- $\cos^2(a) + \sin^2(a) = 1$
- $1 + \tan^2(a) = \sec^2(a)$
- $\cot^2(a) + 1 = \csc^2(a)$.

Double-Angle Formulas:

- $\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1$
- $\sin(2a) = 2\cos(a)\sin(a)$
- $\tan(2a) = \frac{2\tan[a]}{1-\tan^2(a)}$

Power-Reducing Formulas:

- $\sin^2(a) = \frac{1-2\cos(2a)}{2}$
- $\cos^2(a) = \frac{1+2\cos(2a)}{2}$
- $\tan^2(a) = \frac{1-2\cos(2a)}{1+2\cos(2a)}$

Now we will find the anti-derivatives of the trigonometric functions.

We already know the integrals of the sine, cosine, and tangent functions. So lets start with sec.

Exercise 7. Find $\int \sec(\theta)\, d\theta$ using the subsitution $u = \sec\theta + \tan\theta$.

Similarly, we find that

$$\int \csc(\theta)\, d\theta = -\ln\left(\csc(\theta) + \cot(\theta)\right) + C.$$