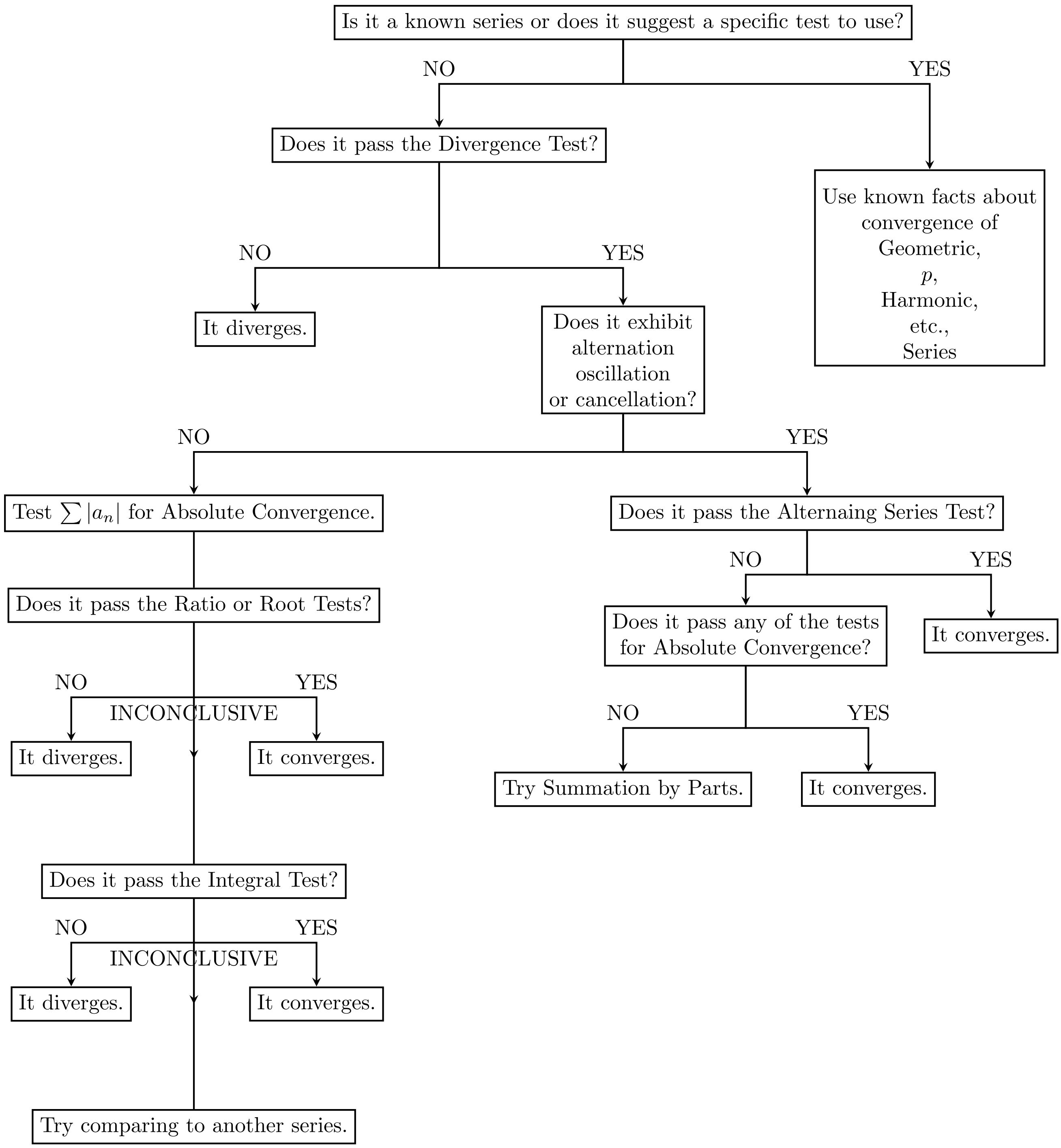
Math 1152 Lecture Notes

July 1, 2022

1 Flow Chart


Is it a known series or does it suggest a specific test to use?
NO
YES
Does it pass the Divergence Test?
Use known facts about convergence of Geometric, p, Harmonic, etc., Series
NO
YES
It diverges.
Does it exhibit alternation oscillation or cancellation?
NO
YES
Test $\sum |a_n|$ for Absolute Convergence.
Does it pass the Alternaing Series Test?
Does it pass the Ratio or Root Tests?
NO
YES
Does it pass any of the tests for Absolute Convergence?
It converges.
NO
YES
INCONCLUSIVE
It diverges.
It converges.
NO
YES
Try Summation by Parts.
It converges.
Does it pass the Integral Test?
NO
YES
INCONCLUSIVE
It diverges.
It converges.
Try comparing to another series.

2 More on Sequences, Cont'd

Last time, we said that a cluster point of a sequence (a_n) should be a point x with the property that every interval about x contains an element of (a_n). We then got an idea about what that should look like. There's just one problem. What if $x = a_k$ for some k?

Definition 1. A point x is a **cluster point** of a sequence (a_n) if every interval about x contains infinitely many points of (a_n).

We give a special name to the biggest and smallest cluster points.

Definition 2. The **lim-inf (or limit infimum) of** a_n, written $\boldsymbol{\liminf a_n}$ is L if L is a cluster point of (a_n) and $L \leq x$ for every other cluster point, x, of (a_n).
Similarly,

$$\boldsymbol{\liminf a_n = U}$$

if U is the largest cluster point of (a_n).

- How do we know that the liminf and limsup actually exist?

3 More on Series

The Ratio Comparison Test

If

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

for all large enough n, and $\sum b_n$ converges, then $\sum a_n$ converges.

Cauchy's Condensation Test

If $a_n \downarrow 0$, then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges.

Kummer's Tests

Let (D_n) denote any sequence of positive numbers, and

$$L = \liminf \left(D_n \frac{a_n}{a_{n+1}} - D_{n+1} \right).$$

If $L > 0$, then $\sum a_n$ converges.

If $L \leq 0$ **and** $\sum \frac{1}{D_n}$ diverges, then $\sum a_n$ diverges.

Exercise 1. What is Kummer's Test when $D_n \equiv 1$ for all n?

Raabe's Ratio Test

Let

$$L = \liminf n \left(\frac{a_n}{a_{n+1}} - 1 \right).$$

Then

1.) If $L > 1$, the series *converges*.

2.) If $L < 1$, the series *diverges*.

3.) If $L = 1$, the test is inconclusive.

Exercise 2. Determine whether

$$\sum \frac{n^n}{e^n n!}$$

converges.

4 Sequences, Series, and Complex Numbers

Consider the sequence (c_n) where $c_n = a_n + ib_n$. We say that $c_n \to z = a + ib$ if $a_n \to a$ and $b_n \to b$.

Exercise 3. If $c_n = a_n + ib_n$, $|a_n| \to s$ and $|b_n| \to t$, does $|c_n| \to \sqrt{s^2 + t^2}$?

Exercise 4. Find $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{in}$.

Exercise 5. For which values of z does

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

converge?

Exercise 6. For which values of z does

$$\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$$

converge?