Math 1152 Lecture Notes

May 13, 2022

1 Area Between Curves, Cont'd.

Steps to finding the area between curves:

1. Solve algebraically for where the curves intersect one another.
2. Sketch a rough graph or solve the inequality in each region between intersection points to determine which curve is on top in each region.
3. Set up a (sum of) integral(s) over the region(s) which represent the area.
4. Evaluate the integral(s).

Exercise 1 (Multiple regions.)**.** Let $f(x) = \sin(\pi x)$ and $g(x) = x$. Find the area bounded by the curves $y = f(x)$, and $y = g(x)$ for x between 0 and 2.

Exercise 2 (More than two curves.)**.** Find the area bounded by the curves $y = 2x$, $y = 3 - x^2$, and $y = 1 - x$.

Observation:

- Area shouldn't change under rotations, so the area between curves shouldn't care whether y is a function of x or x is a function of y : if $x = f^{-1}(y)$ and $x = g^{-1}(y)$, then ...

- Sometimes it is easier, or might only seem possible, to compute an area integral by flipping perspective and making it about "integrating dy" instead of "dx".

Exercise 3. Find the area between the curves $y = \ln(x)$ and the line passing through the points $(1, 0)$ and $(e, 1)$.

Question 1. How does the area between curves relate to the area under the curve?

Exercise 4.

a. Find the area under the curve $y = \sin(x)$ from $x = 0$ to $x = 2\pi$.

b. Find the area between the curves $y = \sin(x)$ and $y = 0$ from $x = 0$ to $x = 2\pi$.

Exercise 5. Find the area between the curves $x = 2y$ and $x = y^2 - x$.

2 Accumulated Cross Sections

- We found areas by decomposing into lines.
- How can we find *volumes*?

Areas

- Known areas we can decompose into: rectangles
- Sketch of decomposition:
- Written as a Sum:
- Formula:

$$\text{Area} = \int_a^b \text{height}(t)\,dt$$

Volumes

- Known volumes we can decompose into: boxes
- Sketch of decomposition:
- Written as a Sum:
- Formula:

$$\text{Volume} = \int_a^b \text{area}(t)\,dt$$

Question 2. Does this idea work to give the correct volume for a rectangle?

Steps to find Volume via Cross-Sections

0) Sketch a picture.

1) Decide on the variable of integration, t (usually x or y).

2) Find the upper and lower bounds of integration, a and b.

3) Find the vertical or horizontal line-segments corresponding to the cross-section for a given value of the integration parameter t.

4) Sketch, if necessary, a diagram of the two-dimensional cross-section above a given line-segment, and find a formula, f, for its area.

5) Integrate $\int_a^b f(t)\,dt$.

Exercise 6. Draw a sketch of the solid whose base is bounded by the curves $y = x$ and $y = x^2$ and whose cross sections perpendicular to the x-axis are squares. Then find its volume.

Operational Skills:

- Vertical slices $\Leftrightarrow$ Integrating ________________
- Horizontal slices $\Leftrightarrow$ Integrating ________________ Integrating ________________
- Modularize the problem: A very complicated problem doesn't have to be visualized and all parts known all at once. Each step can be treated as its own sub-problem.

Aside on Aphantasia

Can you visualize a circle inside of a square?

The circle should be just touching the sides of the square.

Maybe make the circle orange and the square green.

Not everyone has the same ability to see images which aren't there, but this wasn't really understood until the late 1800s.

The famous statistician Galton wrote:

``To my astonishment, I found that the great majority of the men of science to whom I first applied, protested that mental imagery was unknown to them, and they looked on me as fanciful and fantastic in supposing that the words 'mental imagery' really expressed what I believed everybody supposed them to mean. They had no more notion of its true nature than a colour-blind man who has not discerned his defect has of the nature of colour.''

This was not really rediscovered and understood until the 2000s.

Galton is perhaps best known for the *Galton-Watson* process in statistics and probability.

Exercise 7. Find the volume of a solid with a circular base of radius 9 and whose cross-sections perpendicular to the y-axis are equilateral triangles.

Exercise 8. Find the volume of a solid with a circular base of radius 9 and whose cross-sections perpendicular to the y-axis are right isosceles triangles with a leg on the base of the solid.

Common Area Formulas:

- ☐ Square: $A = base \times height$
- ☐ Semi-circle: $A = \frac{1}{2}\pi r^2$
- ☐ Equilateral Triangle: $A = \frac{\sqrt{3}}{4}s^2$
- ☐ Isosceles Right Triangle with sides of length s: $A = \frac{1}{2}s^2$.