Math 1152 Lecture Notes

June 3, 2022

1 Sequences

Sequences are

- Functions from $\mathbb{R}$ to $\mathbb{R}$ (or to other spaces ...)
- Conceptually simpler than functions on $\mathbb{R}$
- The best way to think about limits in general.

Definition 1. Given an infinite subset $S \subset \mathbb{N}$, a **sequence** $(a_n)_{n \in S}$ or $\{a_n\}_{n \in S}$, sometimes written $\{a_n\}_{n=1}^{\infty}$ or just (a_n) if $S = \mathbb{N}$ or the set S is either understood from context or irrelevant, is a function from S to $\mathbb{R}$, whose value at n is a_n.

Exercise 1. Let $f(x) = x^2$. Define a sequence (a_n) by the formula

$$a_n = f(n).$$

What are the first few terms of (a_n)?

Sequences may be given by explicit formulas or *implicitly* via recursive formulas. In the example above, the formula was explicit.

Explicit **Implicit**

Exercise 2. Suppose that $q_1 = 1$ and that $q_{n+1} = 1 + \frac{1}{q_n}$. Write out the first few terms of this sequence. Is there an obvious way to convert it from recursive to explicit?

Exercise 3. Let (a_n) be a sequence and suppose that $a_0 = 0$ and $a_{n+1} = a_n + 1$. Find an explicit formula for a_n.

Exercise 4. Let (b_m) be a sequence and suppose that $b_0 = 1$ and $b_{m+1} = 2 * b_m$. Find an explicit formula for b_m.

Exercise 5. Let (c_p) be a sequence and suppose that $c_0 = 1$ and $c_{p+1} = c_{p-1}c_p$. Find an explicit formula for c_p.

2 Sequences and Sums

We can add together sequences:

$$\{a_n\}_{n\in S} + \{b_n\}_{n\in S} = \{(a+b)_n\}_{n\in S}.$$

And we can perform other binary operations on sequences, just like we do with functions.

More interestingly, we can form new sequences via partial sums:

Exercise 6. Let $a_k = \frac{1}{k^2}$ for $k \in \mathbb{N}$. Define

$$s_n = \sum_{k=1}^{n} a_k.$$

What are the first few terms of (s_n)? If we graph them, do we notice anything?

Sums like the above, called **series**, are a significant topic which we will talk more about next time.

3 Sequences and limits

We saw in the above example that it appeared that the sequence s_n was converging towards some value.

Definition 2. The **limit** of a sequence (a_n) is a number L, if it exists, such that for any number $\epsilon > 0$, there is an N in $\mathbb{N}$ such that

$$|a_n - L| < \epsilon$$

for ann $n > N$.

Informally,

Definition 3. The **limit** of a sequence (a_n) is a number L, if it exists, such a_n will be as close as we want to L provided that n is large enough.

Exercise 7. Let $a_n = \frac{1}{n^2}$. What is $\lim_{n\to\infty} a_n$?

Remark 1. This is the same definition as that given for the limit at infinity of a function $f : \mathbb{R} \to \mathbb{R}$. If we think of sequences as special cases of real functions, then everything we know about limits of functions applies to limits of sequences.

What's the *difference* between a limit of a sequence and the limit of a function?

Exercise 8. Let $f(x) = \cos(\pi x)$ and $a_n = f(n)$. Compare

$$\lim_{x\to\infty} f(x)$$

and

$$\lim_{n\to\infty} a_n.$$

Subsequences

Definition 4. If S is an infinite subset of $T \subset \mathbb{N}$, then the sequence $(a_n)_{n\in S}$ is a **subsequence** of the sequence $(a_n)_{n\in T}$.

Exercise 9. Let $f(x) = \cos(\pi x)$ and $a_n = f(n)$.

We saw that $a_n = (-1)^n$. We can think of (a_n) as built out of two interleaving subsequences:

$$(a_n)_{n\in Evens}$$

and

$$(a_n)_{n\in Odds}.$$

What's the limit of the first sequence? What's the limit of the second?

Theorem 1. A bounded, monotonic sequence always has a limit.

Theorem 2. Every bounded sequence has a bounded, monotonic subsequence.

Remark 2. If a sequence is not bounded, it either converges to $+\infty$, to $-\infty$, or decomposes into two or three subsequences, one of which converges to $\pm\infty$, one of which is bounded, and possibly one of which converges to $\mp\infty$.

Exercise 10. Write down some monotonic sequences. Which are bounded and which aren't? can you think of an unbounded sequence which decomposes into a part which converges to ∞ and a part which is bounded?

4 Tools for limits

We can completely understand limits of *functions* via sequences

Suppose that $\lim_{x\to a} f(x) = L$. Let (a_n) be a sequence with $a_n \to a$. Then $f(a_n) \to L$.

Conversely, suppose that $f(a_n) \to L$ for every sequence (a_n) with $a_n \to a$. Then $f(x) \to L$. Why?

Growth Rates

Definition 5. Given two sequences (a_n) and (b_n), we say that a_n **grows more slowly** than b_n, written $a_n \ll b_n$, if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Equivalently, we say that b_n grows more quickly than a_n and that $b_n \gg a_n$. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

for a non-zero constant c, then we say that a_n and b_n have **comparable growth rates**.

Theorem 3. For any $p, q > 0$ and $b > 1$,

$$\ln^p(n) \ll n^p \ll b^n \ll n! \ll n^n.$$

Exercise 11. Find $\lim_{n\to\infty} \frac{2^n}{n!}$.

Dominant Term Analysis

Definition 6. Given a limit of the form

$$\lim_{n\to\infty} \frac{p_1(n) + \cdots + p_m(n)}{q_1(n) + \cdots q_k(n)}$$

the **dominant terms** are the terms in the numerator and denominator with the greatest growth rates.

If the dominant terms are $p(n)$ and $q(n)$, then the limit is equal to

$$\lim_{n\to\infty} \frac{p(n)}{q(n)}.$$

Exercise 12. Find

$$\lim_{n\to\infty} \frac{n^2 + 3n + 1}{\sqrt{n} + n!}.$$

Squeeze Theorem

Theorem 4. If $a_n < b_n < c_n$ for all n, and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then

$$\lim_{n\to\infty} b_n = L.$$

Exercise 13. Find

$$\lim_{n\to\infty} \frac{n^2 + \cos(n)}{\sqrt{n^2 + \sin(n)}}.$$