Math 1152 Lecture Notes

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1 Trigonometric Integrals

Recall the following Trigonometric Identities:

Trigonometric Identities

Pythagorean Identties:

- $\cos^2(a) + \sin^2(a) = 1$
- $1 + \tan^2(a) = \sec^2(a)$
- $\cot^2(a) + 1 = \csc^2(a)$.

Double-Angle Formulas:

- $\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1$
- $\sin(2a) = 2\cos(a)\sin(a)$
- $\tan(2a) = \frac{2\tan[a]}{1-\tan^2(a)}$

Power-Reducing Formulas:

- $\sin^2(a) = \frac{1-2\cos(2a)}{2}$
- $\cos^2(a) = \frac{1+2\cos(2a)}{2}$
- $\tan^2(a) = \frac{1-2\cos(2a)}{1+2\cos(2a)}$

Now we will find the anti-derivatives of the trigonometric functions.

We already know the integrals of the sine and cosine functions. So lets start with tan and sec.

Exercise 1. Find $\int \tan(\theta)\, d\theta$.

Exercise 2. Find $\int \sec(\theta)\, d\theta$ using the substitution $u = \sec\theta + \tan\theta$.

Similarly, we find that

$$\int \csc(\theta)\,d\theta = -\ln\left(\csc(\theta) + \cot(\theta)\right) + C.$$

So we get

Integrals of Trigonometric Functions

- $\int \cos\theta\,d\theta = \sin\theta + C$
- $\int \sin\theta\,d\theta = -\cos\theta + C$
- $\int \tan\theta\,d\theta = -\ln\cos\theta + C$
- $\int \sec\theta\,d\theta = \ln\left(\sec\theta + \tan\theta\right) + C$
- $\int \csc\theta\,d\theta = -\ln\left(\csc\theta + \cot\theta\right) + C$
- $\int \cot\theta\,d\theta = \ln\sin\theta + C.$

Exercise 3. Find $\int \cos^5\theta\sin^3\theta\,d\theta$.

2 Trigonometric Substitution

We use the "algebra" of trigonometric functions to evaluate certain integrals.

- If we see $1 - x^2$

- If we see $1 + x^2$

- If we see $c^2 \pm x^2$

Trignometric Substitution using Sine

$$x = \sin(\theta)$$
$$1 - x^2 = y^2$$

where

$$y = \cos(\theta)$$
$$dx = \cos(\theta)\, d\theta$$

$$\int f(x)\, g(1 - x^2)\, dx = \int f(\sin\theta)\, g(\cos\theta)(\cos\theta)\, d\theta.$$

- Similar boxes can be made for other trigonometric substitutions.
- But it is probably easier to think about specific examples.

Exercise 4. Calculate
$$\int \sqrt{1-x^2}\,dx.$$

Exercise 5. Calculate
$$\int \sqrt{1+x^2}\,dx.$$

When another constant than 1 shows up, we could change it into a 1 using u-substitution, or we could multiply the Pythagorean identities through by that constant:

$$c^2 \cos^2 \theta + c^2 \sin^2 \theta = c^2$$

so

$$c^2 - c^2 \sin^2 \theta = c^2 \cos^2 \theta;$$

this tells us that if we have a pattern like $c^2 - x^2$, we should use the trigonometric substitution $x = c^2 \sin^2 \theta$.

Similar ideas work when we use a secant or tangent substitution.

Exercise 6. Calculate $\int \frac{1}{(x^2-25)^{3/2}}\, dx$.

When to use Which Trigonometric Substitutions

- $c^2 - x^2$:
- $c^2 + x^2$:
- $x^2 - c^2$:
- $x^2 + c^2$:

Exercise 7. Find $\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$ for $x \geq 5$.