Math 1152 Lecture Notes

July 15, 2022

1 Differential Equations

Definition 1. A differential equation is an equation where the unknowns are *differentiable functions* and the equation relates one or more derivatives of the unknown function(s).

Example: Solve the equation $y' = 0$. Have you found all solutions?

Example: Solve the equation $y'' = 0$. Have you found all solutions?

Example: Solve the equation $y'' = y$. Have you found all solutions?

Definition 2. The order of a differential equation is the highest derivative of the unknown function occurring in the equation.

Example: The order of the equation $y'' = y$ is two. The order of the equation $y''' - y'' = \sin(x)$ is three.

Initial Value Problems

Differential Equations are, in some sense, a generalization of the indefinite integral. In keeping with this analogy, they always involve one or more unknown constants.

An nth order differential equation involves n-unknown constants.

Definition 3. An initial value problem is one in which an nth order differential equation is given, along with the values of the unknown function and its first $n-1$ derivatives at a given point (the "initial value").

Exercise 1. Solve the initial value problem

$$y'' = x,$$

$$y(0) = 2,$$

$$y'(0) = 1.$$

2 Separable Differential Equations

We can't solve every given differential equation (we can't even integrate every function), but large classes of differential equations are solvable. There are entire courses devoted to the subject.

Separable differential equations, on the other hand, we can solve.

Definition 4. A differential equation is **separable** if it is a first-order equation which takes the form $F(y, y') = G(x)$, where y is the unknown function we wish to solve for and x is the independent variable.

Separable differential equations may be solved via algebra and integration.

Exercise 2. Show that

$$y' = \frac{y \cdot (x-3)}{x-y}$$

is separable, then solve it.

3 Differential Equations and Power Series

Provided that a differential equation is "nice enough", we can solve it using Power Series. Here's how:

Exercise 3. Solve

$$y'' + y = 0$$

for x near 0 using the assumption that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

for some unknown sequence (a_n). Solving, here, means finding a closed formula for the a_n.

Exercise 4. Find a solution around $x = 2$ for the equation

$$y'' - xy = 0.$$

Not all differential equations will have Power Series solutions about a given point (not all functions do - $\frac{1}{x}$ doesn't have a Power Series centered at $x = 0$), but series still offer a powerful and highly general tool in solving ordinary differential equations.