Math 1152 Lecture Notes

June 10, 2022

1 Recap

Series cannot converge if $a_n \nrightarrow 0$, since in that case there are not-vanishing a_n for arbitrarily large n, which will throw the value of sums far away from where they're otherwise converging. (This is the Divergence Test).

But even if $a_n \to 0$, convergence might still fail. We saw this happens in the case of the Harmonic Series, by looking at the Integral Test.

There are two ways that convergence of the partial sums s_n can happen - either the *magnitude* of the a_n shrinks to 0 fast enough (consider the case where all of the $a_n = 0$ after some n - certainly this converges, since then the partial sums s_n are eventually constant!)

or there is *cancellation* - some of the terms are positive, some are negative, and this causes the terms to effectively grow much less quickly than otherwise.

Even though

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{diverges}$$

maybe

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \qquad \text{converges?}$$

So we can ask when does $\sum_{n=1}^{\infty} a_n$ converge even though $\sum_{n=1}^{\infty} |a_n|$ diverges,

and separately, when does $\sum_{n=1}^{\infty} |a_n|$ converge?

Definition 1. We say that the series $\sum a_n$ converges **absolutely** if

$$\sum |a_n| < \infty$$

and that the series converges **conditionally** if only

$$\sum a_n < \infty.$$

Our first tool for understanding absolute convergence is, like with applying Integration by Parts to integrals, best used when there are *two* terms in the sum:

The Cauchy-Schwarz Inequality

$$\left(\sum |a_n b_n|\right)^2 \leq \left(\sum a_n^2\right)\left(\sum b_n^2\right).$$

Exercise 1. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

The Alternating Series Test

Suppose that $a_n \geq 0$, $a_n \to 0$, and a_n is decreasing (for example, if $a_n = \frac{1}{n}$). Consider

$$\sum (-1)^n a_n.$$

Does this sum converge?

Alternating Series Test

If a_n is positive, decreasing, and converges to 0, then

$$\sum (-1)^n a_n$$

converges.

Exercise 2. Does $\sum_{n=1}^{\infty} \sin(n)/n$ converge by the Alternating Series test? For which a does $\sum_{n=1}^{\infty} \sin(an)/n$ converge by this test?

Exercise 3. Does

$$\sum_{n\geq 4} \frac{(-1)^{n+2}(1-n)}{3n-n^2}$$

converge?

What do we do when we think there's enough cancellation for a series to converge, but it isn't alternating in sign between every other term?

If we can group terms together into new ones which alternate sign and the conditions of the Alternating Series Test still hold, then great!

Otherwise, we use Summation by Parts:

Exercise 4. Let Does

$$\sum(-1)^{n^2}/n$$

converge?