# Math 1152 Lecture Notes

June 1, 2022

## 1 Improper Integrals

Which functions are integrable?

What even is our definition of an integrable function?

**Theorem 1.** If $f$ is continuous, then $f$ is integrable.

If $f$ is piecewise-continuous, then we break $f$ up into the intervals where it is continuous and sum the integral over each of those.

Does this cover all the functions we would want to integrate?

**What about $\frac{1}{x^{1/2}}$?**

- Our definition of Riemann sum rules out unbounded functions.
- So $\frac{1}{x^{1/2}}$ is not (Riemann) integrable over any interval containing 0.
- But despite that, we "know" what the integral should be: $\int_a^b f(x), dx = 2\sqrt{x}|_a^b$ even when $a < 0$ and $b > 0$.
- Another situation is if we want to integrate $\int_1^\infty f(x)\, dx$ - our Riemann sums are defined only over finite intervals, not $[a, \infty)$.
- Both of these problems are alleviated by using limits.

**Improper Integrals**

$$\int_a^\infty f(x)\, dx := \lim_{b\to\infty} \int_a^b f(x)\, dx,$$

and
If $f$ has a vertical assymptote at $c$ in $[a, b]$, then

$$\int_a^b f(x)\, dx = \left(\lim_{s\to c^-} \int_a^s f(x)\, dx\right) + \left(\lim_{s\to c^+} \int_s^b f(x)\, dx\right).$$

These two types of improper integrals can be combined with one another.
An improper integral **converges** if the limit exists and is finite, and **diverges** otherwise.

**Exercise 1.** Find $\int_0^1 \frac{1}{x^a}\, dx$ for different values of $a$. When does the integral exist? What about for $\int_1^\infty \frac{1}{x^a}$?

If an improper integral does not converge, weird behaviour is possible.

**Exercise 2.** Investigate

$$\int_{-1}^{1} \frac{1}{x}\,dx.$$

Does this integral converge or diverge?