Math 1152 Lecture Notes

July 11, 2022

1 Taylor Series

Recall the Taylor Polynomial Formula:

$$T_a^k(x) = f(a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

and the remainder formula

$$\left|R_a^k(x)\right| = \left|\frac{f^{(k+1)}(\boldsymbol{c})}{(k+1)!}(x-a)^{k+1}\right| \leq M\frac{|x-a|^{k+1}}{(k+1)!}$$

where $\left|f^{(k+1)}(c)\right| \leq M$ for all c in $[x, a]$.
Putting these together, we have

$$\boldsymbol{f(x) = T_a^k(x) + R_a^k(x)}$$

and

$$R_a^k(x) \to 0$$

if

$$M\frac{|x-a|^{k+1}}{(k+1)!} \to 0.$$

Taylor's Theorem

Suppose that $I = (a - r, a + r)$ and that on the interval I, for all k there is a constant M_k so that

$$\left| f^{(k)}(x) \right| \leq M_k$$

for all x. Then if

$$\lim_{k \to \infty} M_k \frac{|x - a|^{k+1}}{(k+1)!} = 0$$

the series

$$\sum_{n=0}^{\infty} a_n (x - a)^n$$

where

$$a_n = \frac{f^{(n)}(a)}{n!},$$

converges for $x \in I$, and further

$$f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n$$

for $x \in I$.

Remark 1. In particular, this holds if there is a universal constnat M so that for all k

$$\left| f^{(k)}(x) \right| \leq M.$$

Remark 2. Taylor's Theorem tells us what functions different Power Series converge to.

Examples

The following functions satisfy the requirement that all derivatives be bounded by a universal constant (M) on $(-\infty, \infty)$ and so that their Taylor Series converge on $(-\infty, \infty)$:

1. All polynomials.
2. e^x
3. $\cos(x)$
4. $\sin(x)$

The following functions satisfy the requirement of Taylor's Theorem on a restricted interval:

1. All polynomials.
2. $\log(x)$
3. $\frac{1}{1-x}$
4. $\tan(x)$.

Exercise 1. Show that the Taylor Series for $\cos(x)$ converges everywhere.

Exercise 2. Determine the Taylor Series centered at $a = 1$ for $\ln(x)$, then determine its interval of convergence.

Exercise 3. Explain why there exists a function $o(x)$ whose growthrate as $x \to 0$ is slower than that of x, such that for x near 0,

$$\sin(x) = x + o(x).$$

(This justifies the approximation $\sin(x) \approx x$ for $x \approx 0$.)

Compositions

Exercise 4. Determine the Taylor Series centered at 0 and radius of convergence for the following functions.

a. e^{x^2}

b. $\sin(2x)$

c. $\ln(1-x)$.