Math 1152 Lecture Notes

July 25, 2022

1 Polar Coordinates

- What requirements should an alternate coordinate system meet?

- What reasons do we even have for introducing an alternate coordinate system?

When talking about parametric equations, we saw that we could parametrize the unit circle by considering

$$(\cos\theta, \sin\theta) .$$

And if we want to work with a bigger or smaller circle, we can change the radius by multiplying through by an r:

$$(r\cos\theta, r\sin\theta) .$$

We now have two parameters, r and θ, which give us a description of a 2-dimensional space.

What space is it?

Casting Polar Coordinates to Cartesian

$$\text{polar}(r, \theta) = \text{cartesian}(r\cos\theta, r\sin\theta).$$

Exercise 1. Graph the point polar$(2, \frac{5\pi}{4})$.

It is traditional to think of polar coordinates as describing the Euclidean plane, but with two defects:

1. The origin point cartesian$(0, 0)$ is either not represented, or is represented by infinitely many polar coordinates: all coordinates of the form

$$\text{polar}(0, \theta).$$

2. Polar descriptions of cartesian points are non-unique in two ways:

   (a) polar(r, θ) = polar$(r, \theta + 2k\pi)$

   (b) polar(r, θ) = polar$(-r, \theta + \pi)$.

   The usual solution to (2a) is to require that all angles be in $[0, 2\pi)$ - but then, there is a third, more subtle defect which pops up as we push the other one down:

   (c) If we apply the usual rule of giving all angles as $\theta \mod 2\pi$, then there is a discontinuity between the coordinates polar$(r, 2\pi - \epsilon)$ versus polar(r, ϵ) even though the "physical" points are close together.

   (In fact, if we allow $r = 0$, then there is a similar discontinuity as we pass through the origin).

There are two solutions to these defects: one is to say that we aren't actually giving coordinates for the cartesian plane, but some other space which then can be projected onto the cartesian plane; the second is to just sort of deal with it, trying to remember caveats as they come up.

Remark 1. Notice the symmetries in (2a) and (2b). We could consider them as stating that two different transformations on polar coordinates leave the underlying points fixed, and imagine taking a whole "plane" of (r, θ) and gluing them together according to these transformations.

Converting Between Polar Coordinates to Cartesian

$$x = r\cos\theta,$$

$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}.$$

Question 1. How do we find θ above?

Exercise 2. Conver the point polar$(2, \frac{5\pi}{4}$ to cartesian coordinates. Convert the point $(-2, 4)$ to polar coordinates.

2 Polar Curves

In cartesian coordinates, the coordinate lines are $x = constant$ and $y = constant$. These give horizontal and vertical lines.

In polar coordinates, the coordinate curves are $r = constant$ and $\theta = constant$. What shapes do these represent?

Generally, curves in polar coordinates will take the form

$$r = f(\theta).$$

Exercise 3. Graph $r = 2\cos\theta$. Convert $r = 2\cos\theta$ to cartesian coordinates. Convert $r = 2\cos\theta$ to a cartesian parametric equation.

Exercise 4. Graph $r = 1 + \sin(\theta)$ in polar coordinates.

Exercise 5. Graph $r = 3\sin(2\theta)$.

3 Common Polar Curves

1. **Line:** $\theta = k$
2. **Circle:** $r = a\cos\theta + b\sin\theta$
3. **Spiral:** $r = a + b\theta$
4. **Cardioid:** $r = a(1 \pm \sin\theta)$
5. **Cardioid:** $r = a(1 \pm \cos\theta)$
6. **Limacon:** $r = a\cos\theta + b$
7. **Limacon:** $r = a\sin\theta + b$
8. **Rose:** $r = a\cos(b\theta)$
9. **Rose:** $r = a\sin(b\theta)$