Name: ______________________________

Math 1152 Summer 2022 Midterm 2

Due: 11:59pm, July 5, 2022

Before Starting

- Complete Problems 1 and 2.
- Choose another **2** of the following problems to complete in addition to problems 1 and 2. You will not receive additional credit for completing additional problems beyond these two.
- This exam should reflect your work only and is not a group assignment.
- This exam is open-notes.
- This exam is due at **11:59pm, Tuesday July 5**.
- Full work must be shown for all problems you submit.
- **Fill in the circles below corresponding to the 2 additional problems you choose to have graded. Problems 1 and 2 have already been filled in for you.**

I am submitting for grading:

● **Question 1**
● **Question 2**
○ **Question 3**
○ **Question 4**
○ **Question 5**
○ **Question 6**
○ **Question 7**
○ **Question 8**
○ **Question 9**
○ **Question 10**
○ **Question 11**

Question 1.

Suppose that (a_n) is a sequence satisfying the condition that

$$a_n = \frac{3}{n+2} - \frac{3}{n+4}.$$

a. Determine if the series $\sum_{k=1}^{\infty} a_k$ converges or diverges, if it converges, determine its value, and bubble in the appropriate data below.

The series:			
	Ⓐ Diverges	Ⓑ Converges to 0	Ⓒ Converges to 1
	Ⓓ Converges to $\frac{1}{6}$	Ⓔ Converges to $\frac{25}{24}$	Ⓕ Converges to $\frac{7}{4}$
	Ⓖ More Information is Needed		

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Harmonic Series Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Divergence Test

b. Let

$$s_k = \sum_{n=1}^{k} a_n.$$

Determine if $\sum_{k=1}^{\infty} s_k$ converges or diverges, and bubble in the appropriate data below.

The series:

Ⓐ Diverges	Ⓑ Converges to 0	Ⓒ Converges to 1
Ⓓ Converges to $\frac{1}{6}$	Ⓔ Converges to $\frac{25}{24}$	Ⓕ Converges to $\frac{7}{4}$
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Ⓓ Harmonic Series Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Divergence Test

Question 2.

Consider the series $S = \sum a_n$.

a. Let

$$r_k = \sum_{n=k}^{\infty} a_n.$$

If S **converges**, then

$\lim_{k\to\infty} r_k =$	Ⓐ 0	Ⓑ < 1	Ⓒ 1	Ⓓ Not enough information to answer.

If S **diverges**, then

$\lim_{k\to\infty} r_k =$	Ⓐ 0	Ⓑ < 1	Ⓒ 1	Ⓓ Not enough information to answer.

Explain why below.

b. Let

$$t_k = \sum_{n=k}^{\infty} |a_n|.$$

If $\sum |a_n|$ **converges**, then

$\lim_{k\to\infty} t_k =$	Ⓐ 0	Ⓑ < 1	Ⓒ 1	Ⓓ Not enough information to answer.

If $\sum |a_n|$ **diverges**, then

$\lim_{k\to\infty} t_k =$	Ⓐ > 0	Ⓑ < 1	Ⓒ 1	Ⓓ Not enough information to answer.

c. Invent a test for convergence for a series $\sum a_n$ based on $\boldsymbol{t_k}$ using your answer above. **Explain why it works below.**

d. Invent a test for convergence for a series $\sum a_n$ based on the series $\sum |a_n|$ using your answer above. **Explain why it works below.**

Question 3.

Determine which of the follow series converge or diverge. **If more information is needed, state what information would make the series converge or diverge**.

Justify your answer by stating which test for convergence or divergence you have used, and showing that its requirements are met.

If you do not know how to show convergence of one of these series, that is okay - it may be possible to earn full credit on this problem without providing a perfect answer.

3(a).

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$$

The series:	Ⓐ Diverges	Ⓑ Converges	Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(b).

$$\sum_{n=1}^{\infty} \frac{3n(n+1)(n+2)}{n^3\sqrt{n}}$$

The series:

Ⓐ Diverges	Ⓑ Converges	Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(c).

$$\sum_{n=2}^{\infty} \frac{n!}{6^n}$$

The series:	Ⓐ Diverges	Ⓑ Converges	Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(d).

$$\sum_{n=0}^{\infty} \frac{n^n}{e^n n!}$$

The series:	Ⓐ Diverges	Ⓑ Converges	Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(e).

$$\sum_{n=2}^{\infty} \frac{1}{n^s \log(n)}$$

The series: Ⓐ Diverges Ⓑ Converges Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(f).

$$\sum_{n=1}^{\infty} a^{1/n} - 1$$

The series:	Ⓐ Diverges	Ⓑ Converges	Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(g).

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$$

The series:			
	Ⓐ Diverges	Ⓑ Converges	Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(h).

$$\sum_{n=2}^{\infty} \frac{(-2)^n}{\log(n)}$$

The series:	Ⓐ Diverges	Ⓑ Converges	Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(i).

$$\sum_{n=2}^{\infty} \frac{1}{n \log^t(n)}$$

The series:	Ⓐ Diverges	Ⓑ Converges	Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(j).

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - n}$$

The series:

Ⓐ Diverges	Ⓑ Converges	Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(k).

$$\sum_{n=2}^{\infty} \frac{1}{n^s \log^t(n)}$$

The series:

Ⓐ Diverges	Ⓑ Converges	Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

3(l).

$$\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$$

The series: Ⓐ Diverges Ⓑ Converges Ⓒ More Information is Needed

Work:

The **convergence test** you used to get an answer was the :

Ⓐ Geometric Series Test	Ⓑ Telescoping Series Test	Ⓒ Alternating Series Test
Ⓓ Divergence Test	Ⓔ Integral Test	Ⓕ Root Test
Ⓖ Comparison Test	Ⓗ Ratio Test	Ⓘ Other (see above)

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Question 4.

State the Summation by Parts formula (as was covered, for instance, at the end of the Integral Test lecture). Then, show that if $0 \leq b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1$ and $m \leq \sum_{n=1}^{k} a_n \leq M$ for $k = 1, \ldots, L$ then

$$b_1 m \leq \sum_{n=1}^{L} a_n b_n \leq b_1 M.$$

Question 5.

a. Find a formula for $\sum_{n=-k}^{k} r^n$.

b. We will see later that for any x,

$$e^{ix} = \cos(x) + i\sin(x).$$

What does this tell us $e^{ix} - e^{-ix}$ is? What about $e^{ix} + e^{-ix}$?

c. Use parts (a) and (b) to show that

$$\sum_{n=-k}^{k} e^{inx} = \frac{\sin\left(\left(k+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}.$$

d. Why does this mean that

$$1 + 2\sum_{n=1}^{k} \cos(kx) = \frac{\sin\left(\left(k+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}?$$

e. For a fixed value of x, what is the biggest that $\sum_{n=0}^{k} \cos(kx)$ can be? What is the smallest? **(Your answer should have an x in it).**

f. Show that

$$\sum_{n=0}^{k} \frac{\cos(n)}{n}$$

converges.

Question 6.

Let f be the function given by the formula

$$f(n) = n^p.$$

a. Is there a simple non-constant function G such that $G\big(f(n)\big) =$ a constant for all n?

If so, what is that constant?

b. For which values of p does

$$\sum f(n)$$

converge? For which values of p does it diverge?

c. Using your answer to Part a., create a new test for convergence of a series

$$\sum a_n;$$

for inspiration, you may want to think about what we did to show the Ratio and Root tests.

Extra space for Question 6.

Question 7.

Consider the series

$$\sum_{n=1}^{\infty} a_n$$

where

$$a_n = \frac{(n!)^2}{(2n)!}.$$

a. Show that $\sum_{n=1}^{\infty} a_n$ converges.

b. The well-understood series to which we compare the series above to power the Convergence Test that you used is a

Ⓐ Geometric Series	Ⓑ Telescoping Series	Ⓒ Alternating Series	Ⓓ Harmonic Series
Ⓔ p-Series	Ⓕ Power Series	Ⓖ None of the above	

c. What is the constant which shows up in the series above (ie, if the general form of that series is $\sum 1/n^p$ or $\sum r^n$ or $\sum x/log(n)$, what is the particular value of the constant p, r, x, etc. in this case?)

d. By considering just the tail

$$t_N = \sum_{n=N}^{\infty} a_n$$

for very large N, and comparing this to the corresponding tail of the kind of series in part b. above, we can get a bound on the size of the t_N and so control the rate of convergence.

For large N, what is the bound we obtain on $|t_N|$?

Question 8.

a. What is the size of the tail of the series

$$\sum \left(\frac{4}{9}\right)^n ?$$

b. What is the size of the tail of the series

$$\sum \frac{1}{n^{4/3}}?$$

c. Using your answers to parts (a) and (b), what is the best size you can get for the tail of the series

$$\sum \left(\frac{2}{3}\right)^n \frac{1}{n^{2/3}}?$$

Does the Integral Test help here?

Question 9.

a. We've seen that the Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Adapt the argument we gave in class, in order to get an approximate value for

$$\sum_{k=1}^{n} \frac{1}{k}.$$

Answer: $\displaystyle\sum_{k=1}^{n} \frac{1}{k} \approx f(n)$

where $f(n) =$ $\boxed{}$

What is the difference between the upper and lower bound of your approximation for $\sum_{k=1}^{n} \frac{1}{k}$ when $n = 1000$?

(A) 1	(B) 0.5	(C) 0.1	(D) 0.05	(E) 0.01
(F) 0.005	(G) 0.001	(H) None of the above		

b. Does

$$\sum_{n=1}^{\infty} \log\left(\frac{1}{n!}\right)$$

converge or diverge?

Question 10.

(a). Show that the sequence given by

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots$$

converges and find its limit.

(b). Find

$$\lim_{n\to\infty} \frac{n^2 + 2^n}{\sqrt{n^2 \sin(n) + 4^n}}.$$

Question 11.

Show that $\sum a_n^2 \leq \left(\sum |a_n|\right)^2$.

Remember to go back to the front page and fill in the circle to indicate which problems you are submitting for grading.