Math 1152 Lecture Notes

June 13, 2022

Announcement:

- We're adjusting the Calendar.
- There will be a new Calendar posted later today.
- There is no class next Monday, Juneteenth (classes and offices are closed)
- Written Homework 6 is moving to next Tuesday so that the week of the Exam is free of Written Homework.

1 Consolidating What We Know

Sequences

Monotone Convergence Theorem

Series

Telescoping Series

Geometric Series

Alternating Series

The Integral Test

The Cauchy-Schwarz Inequality

Summation by Parts

2 Absolute Convergence

Absolute Convergence implies usual convergence - why?

If $s_n = \sum_{k=0}^{n} |a_n|$, then (s_n) is increasing:

$$\Delta s_n = s_{n+1} - s_n = |a_n| > 0.$$

So by the Monotone Convergence Theorem, if we can *bound* (s_n) then we'll know it converges (this applies for a_n positive, or not - why?)

Another way of phrasing this is that when every $a_n \geq 0$, the only options are that either (s_n) converges, or that $s_n \uparrow \infty$.

3 Comparison Test

The Comparison Test

If $0 \leq a_n \leq b_n$, and $\sum b_n$ converges,

then $\sum a_n$ converges.

Exercise 1. Show that

$$\sum_{n=1} \infty \frac{1}{n^2 + n}$$

converges.

Can you think of a similar test which shows *divergence*?

Exercise 2. Determine whether

$$\sum \sqrt{\frac{n+5}{n^3+n^2+n+1}}$$

converges or diverges.

4 Limit Comparison Test

The Limit Comparison Test

$$\text{If} \quad \lim_{n\to\infty} \frac{|a_n|}{|b_n|} < \infty, \quad \text{and} \sum b_n \text{ converges absolutely,}$$

$$\text{then} \sum a_n \text{ converges.}$$

Exercise 3. Show that

$$\sum_{n=1}^{\infty} \frac{1 + \frac{1}{\sqrt{n}}}{n^2}$$

converges.

Exercise 4. Determine whether

$$\sum \sqrt{\frac{n+5}{n^2+n+1}}$$

converges or diverges.

5 Ratio Test

- Suppose that
$$0 < \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1.$$

- This is like a recurrence-relation for the sequence (a_n), except it's an inequality and occurs in the limit.

- Morally, we have that $a_{n+1} = ra_n$, which would mean that $a_n = r^n a_0$ if it were true for all n.

- And that would mean that we have a geometric series which converges.

- That's the intuition. How do we give it a frame that holds up?

- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$, then for any choice of s in $(r, 1)$ it has to be true that eventually, after some point N_0, `ALL` $\frac{a_{n+1}}{a_n} < s$ (why?)

- This new inequality is one which is true, and which we can use:

$$a_{n+1} < s a_n \text{ for all } n > N_0$$

so we can unwind from a_{N_0+m} all the way down to a_{N_0}:

$$a_{N_0+m} < s^m a_{N_0}.$$

- So

$$\begin{aligned} \sum_{n>0} a_n &= \sum_{n=1}^{N_0} a_n + \sum_{n=N_0+1}^{\infty} a_n \\ &= s_{N_0} + \sum_{m=1}^{\infty} a_{N_0+m} \end{aligned}$$

and $\sum_{m=1}^{\infty} a_{N_0+m}$ converges by the Comparison Test, since $a_{N_0+m} < s^m$ which are the terms of a geometric series if $s < 1$.

- Would this argument be faster/cleaner if we used the Limit Comparison Test instead?

The Ratio Test

Suppose that $a_n > 0$ and $a_{n+1}/a_n \to r$. Then if

- $r < 1$: $\sum a_n$ converges
- $r > 1$: $\sum a_n$ diverges
- $r = 1$: The Ratio Test is inconclusive.

Exercise 5. Does the series

$$\sum \frac{(n!)^2}{(2n)!}$$

converge or diverge?

When applying the Ratio Test to a term $a_n = \frac{p_n}{q_n}$, write

$$\frac{a_{n+1}}{a_n} = \frac{p_{n+1}}{p_n} \frac{q_n}{q_{n+1}}$$