Math 1152 Lecture Notes

June 27, 2022

The Integral Test

What do we need to know in order to be able to say that the series $\sum a_n$ converges by the Integral Test?

What makes the integral test true?

Can we upgrade the qualitative information about convergence into quantitative information about the rate of convergence?

Using remainders, cont'd.

Exercise 1. Suppose that the series $\sum a_n$ has remainders which satisfy $|r_n| \leq \frac{1}{n^2}$.

How large must n be so that s_n is within 0.049 of its limit?

Exercise 2. Suppose that $a_n, b_n \geq 0$ and the tails for the series $\sum a_n$ are bounded by 2^{-n} and for the series $\sum b_n$ are bounded by $\log(n)$.

Does $\sum a_n$ converge? Does $\sum b_n$? Does $\sum (a_n b_n)^{\frac{1}{2}}$?

Exercise 3. Consider the series

(a) $\sum \frac{(-1)^n}{2^n}$

(b) $\sum \left(\frac{3}{4}\right)^n$

(c) $\sum \frac{2n^2-2n+1}{n^2(n-1)^2}$.

Which series converges fastest?

Series and Complex Numbers

Recall that a complex number is a number of the form $a + ib$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$ satisfies $i^2 = -1$ is the **imaginary unit**.

Complex numbers satisfy the exact same algebraic properties as the real numbers with respect to addition, subtraction, multiplication, and division.

Just like the reals, the complex numbers are closed under addition, subtraction, multiplication, division, exponentiation, and ... logarithms are complicated.

Exercise 4. Write $\frac{1+i}{1-i}$ in the form $a + ib$.

De Moivre's Formula lets us write a complex number in Polar Form:

De Movire's Formula

$$a + ib = r\left(\cos(\theta) + i\sin(theta)\right)$$

where

$$r = \sqrt{a^2 + b^2}$$

and

$$a = r\cos(\theta)$$

$$b = r\sin(\theta).$$

This is particularly useful for integer powers of complex numbers.

Exercise 5. Find $(1 + i)^{100}$.

Exercise 6. Find $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{in}$.

Exercise 7. For which values of z does

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$$

converge?

Exercise 8. For which values of z does

$$\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$$

converge?