Math 1152 Lecture Notes

July 6, 2022

1 Linear Approximations

Recall the linear approximation to a function:

Linear Approximation

Given a differentiable function f, the linear approximation to f at a is the function
$$L_a(x) = f(a) + f'(a)(x - a).$$

- The linear approximation is a function which passes through the same point at a and has the same derivative as f at a.
- Does that mean that it's a *good* approximation to f near a? Whether any

  approximation in any context is good or not requires us to know something about the *error*.
  - If we know absolutely nothing, we do not know that approximating by the constant function 0, or 1, or 100 would not be better.
  - Our intuition convinces us that calculating some approximation based on data should be superior only because we so firmly expect there to be some reasonable, small bounds on the error even if we do not know them.
  - But that is still an insistence on there being control on the error, and just hoping that the control is good enough so that using a random approximation isn't superior. There is no universal theorem saying that any earnest attempt at approximation we make will work.
- What is the error in the linear approximation?

Theorem 1 (Mean Value Theorem)**.** If f is differentiable on (a, b) and continuous on $[a, b]$, then there is a point $c \in (a, b)$ so that

$$f(b) - f(a) = f'(c)(b - a).$$

Let $R_a(x) = R_a^{[1]}(x)$ be the error in the linear approximation, $R_a(x) = f(x) - L_a(x)$.

Linear Approximation - estimating the error

Let f be differentiable on the interval I containing a, and L_a be the linear approximation to f at a. Then if $|f'(x)| \leq M$ on I,

$$|R_a(x)| = |f(x) - L_a(x)| \leq 2M|x - a| \leq 2M|I|$$

where $|I|$ denotes the length of the interval I.

Exercise 1. Show that if there is no bound on the size of $|f'(x)|$ on I, then we cannot bound on the error in the linear approximation without replacing it with some other assumption.

Exercise 2. What's the error in approximating $e^{1/x}$ linearly on $[\epsilon, 1]$?

Quality of Approximation

The linear approximation is the best approximation by a function of the form $g(x) = m(x - a) + k$ for x in a small-enough interval about a (in fact, having a linear approximation of that quality is equivalent to f being differentiable). To make this precise, we need to decide on how to measure the distance between functions.

2 Higher-order approximations

Can we get a better approximation to a function by considering more derivatives?

Linear Approximation - estimating the error with a second derivative

Let f be twice differentiable on the interval I containing a, and L_a be the linear approximation to f at a. Then if $|f''(x)| \leq M$ on I,

$$|R_a(x)| = |f(x) - L_a(x)| \leq \frac{M}{2}|x-a|^2 \leq \frac{M}{2}|I|^2.$$

Idea: If more derivatives improve the bound on the error, use more derivatives to get a better approximation than $L_a(f)$.

Second Idea: Iterate the linear approximation.

Second Idea in more detail: If L_a captures the linear part of f, look at $f - L_a$ and try to capture more of what remains by using the second derivative.

Taylor Polynomials

The nth-order Taylor Polynomial to f at a is given by

$$T_a^n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^(n)(a)}{n!}(x-a)^n.$$

Exercise 3. What's the error to the nth-order Taylor Polynomial approximation to f?