Math 1152 Lecture Notes

June 29, 2022

1 Sequences

Exercise 1. Suppose (a_n) satisfies $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{a_n + 2}$. Does (a_n) converge? If so, to what?

Exercise 2. Let $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for every n, and $a_n = 0.d_1 d_2 \cdots d_n$, that is,

$$a_n = \sum_{k=1}^{n} \frac{d_k}{10^k}.$$

What does a_n converge to? How do we know that it converges?

2 Subsequences

Recall that given a sequence $(a_n)_{n\in A}$ and an infinite subset $B \subset A$, we say that $(a_n)_{n\in B}$ is a subsequence of $(a_n)_{n\in A}$. Another way to write this is as follows: Let (s_k) be the increasing sequence of integers which belong to the set B. Then

$$(a_n)_{n\in B} = (a_{s_k})_{k\in\mathbb{N}}.$$

Subsequences will often be written in this form.

Recall the following important theorems on real-valued sequences.

Theorem 1. Every sequence has a monotonic subsequence.

Theorem 2 (Bolzano-Weierstrauss)**.** Every bounded sequence contains a convergent subsequence.

Definition 1. A point x is a **cluster point** of the sequence (a_n) if every interval centered at x contains a point a_n in the sequence.

Exercise 3. Show that if x is a cluster-point of (a_n), then for any $\epsilon > 0$, there are infinitely many n for which $a_n \in (x - \epsilon, x + \epsilon)$.

Exercise 4. Show that a point x is a cluster-point of the sequence (a_n) if and only if there is a subsequence (a_{n_k}) converging to x.

Theorem 3 (Cauchy Criterion for Convergence)**.** A sequence (a_n) converges if and only if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $m, n > N$,

$$|a_n - a_m| < \epsilon.$$

- This is useful for when we want to show a sequence converges but don't have a good guess for what the limit is, or don't care what the limit is.
- It is worth thinking about why this is equivalent to the usual definition of the limit of a sequence.
- A good warmup question is to show that " $\epsilon > 0$" can be replaced by "for any integer $M \in \mathbb{N}$" and "$|a_n - a_m| < \epsilon$" can be replaced with "$|a_n - a_m| < 1/M$."

3 Sequences, Series, and Complex Numbers

Recall that a complex number is a number of the form $a + ib$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$ satisfies $i^2 = -1$ is the **imaginary unit**.

Complex numbers satisfy the exact same algebraic properties as the real numbers with respect to addition, subtraction, multiplication, and division.

Just like the reals, the complex numbers are closed under addition, subtraction, multiplication, division, exponentiation, and ... logarithms are complicated.

Exercise 5. Write $\frac{1+i}{1-i}$ in the form $a + ib$.

De Moivre's Formula lets us write a complex number in Polar Form:

De Movire's Formula

$$a + ib = r\left(\cos(\theta) + i\sin(theta)\right)$$

where

$$r = \sqrt{a^2 + b^2}$$

and

$$a = r\cos(\theta)$$

$$b = r\sin(\theta).$$

This is particularly useful for integer powers of complex numbers.

Exercise 6. Find $(1 + i)^{100}$.

Consider the sequence (c_n) where $c_n = a_n + ib_n$. We say that $c_n \to z = a + ib$ if $a_n \to a$ and $b_n \to b$.

Exercise 7. If $c_n = a_n + ib_n$, $|a_n| \to s$ and $|b_n| \to t$, does $|c_n| \to \sqrt{s^2 + t^2}$?

Exercise 8. Find $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{in}$.

Exercise 9. For which values of z does

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$$

converge?

Exercise 10. For which values of z does

$$\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$$

converge?