Math 1152 Lecture Notes

May 20, 2022

1 Lengths of Curves

- Consider a parabola, say $y = x^2$. What is its length from $(0, 0)$ to $(1, 1)$?

- How is this similar to when we went from talking about areas between curves to volumes via accumulated cross sections, and how is it different?

- To find a means of calculating length, lets use the same ideas which lead to our formula's for area and volume:

  - Approximate the shape by one whose measure you know: line-segments

  - Use the information given to determine the shape of the approximating segments - a line-segment is determined by

    a. its starting and ending points and

    b. its slope

  - We get the starting and ending points of each line-segment from the subdivision of the interval - so we already know (a).

  - How do we decide what the slopes should be?

- Now we need a formula for the length of a line-segment whose x-coordinates go from x_k to x_{k+1} and whose slope is m:

$$\text{length} = \sqrt{\Delta x^2 + m^2 \Delta x^2}.$$

- So we break the curve from 0 to 1 into pieces approximated by line-segments, each of which has length l_k, and add up the lengths to approximate the length of the curve. This gives us

$$\text{Length} \approx \sum_{k=1}^{n} l_k = \sum_{k=1}^{n} \sqrt{\Delta x^2 + f'(x_k^*)^2 \Delta x^2}.$$

- Is this a Riemann sum?

- It is, if we factor the Δx out of the square-root sign:

$$\text{Length} \approx \sum_{k=1}^{n} \sqrt{1 + f'(x_k^*)^2} \Delta x.$$

- What integral does this Riemann sum converge to?

- $$\text{Length} = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

- So can we find the length of a segment of the parabola?

  Exercise 1. Find the length of the segment of the parabola $y = x^2$ from $x = 0$ to $x = 1$.

  Exercise 2. Let $g(y) = \left(\frac{2}{3} + y^2\right)^{\frac{3}{2}} - 2$. Find the length of the curve $x = g(y)$ from $y = 0$ to $y = 2$.

- Are there any important points we've missed? Do we now have a complete understanding of lengths of curves?