Math 1152 - Power Series and ODES
Recitation Problem - Solution

July 15, 2022

Question. **(a)** Find a recurrence relation for a power series solution to the differential equation

$$y'' + xy' - y = 0;$$

(b) Solve that recurrence relation you found in part (a) to obtain the general form of the solution to the equation.

Solution
(a) Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n,$$

and

$$xy' = \sum_{n=1}^{\infty} n a_n x^n.$$

Since the above series starts at $n = 1$, we pull out the $n = 0$ terms from the other two series to get

$$y = a_0 + \sum_{n=1}^{\infty} a_n x^n$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n,$$

$$y'' = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Then

$$y'' + xy' - y = 0$$

becomes

$$
\begin{aligned}
2a_2 \quad &+ \quad \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n \\
&+ \quad \sum_{n=1}^{\infty} na_n x^n \\
-a_0 \quad &- \quad \sum_{n=1}^{\infty} a_n x^n \\
&= 0
\end{aligned}
$$

or

$$2a_2 - a_0 + \sum_{n=1}^{\infty}((n+2)(n+1)a_{n+2} + na_n - a_n) = 0.$$

Equating coefficients to 0, we get

$$2a_2 = a_0,$$

and

$$(n+2)(n+1)a_{n+2} + na_n - a_n = 0, \text{ for } n \geq 1$$

or

$$\boxed{\begin{gathered} 2a_2 = a_0, \\ \text{and} \quad a_{n+2} = \frac{n-1}{(n+2)(n+1)}a_n \text{ for } n \geq 1. \end{gathered}}$$

(b) Solving from writing out first a_0, then a_2, a_4, and finding the pattern, this gives us

$$a_{2n} = \frac{(2n-3)(2n-1)\cdots(3)(1)}{(2n)!}a_2$$

and as for odd terms, when $n = 1$ this gives

$$a_3 = \frac{1-1}{\cdots}a_1 = 0$$

which inducts up to

$$a_{2n+1} = 0 \text{ when } \ n \geq 3.$$

Using that $a_2 = a_0/2$, this gives

$$y = a_1x + a_0 + \sum_{n=1}^{\infty}\frac{(2n-3)(2n-1)\cdots(3)(1)}{(2n)!}\frac{a_0}{2}x^{2n}$$

or $$\boxed{y = a_1x + a_0\left(1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n-3)(2n-1)\cdots(3)(1)}{(2n)!}x^{2n}\right).}$$