Math 1152 Lecture Notes

May 16, 2022

1 Accumulated Cross Sections

- We found areas by decomposing into lines.
- How can we find *volumes*?

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Areas

- Known areas we can decompose into: rectangles
- Sketch of decomposition:
- Written as a Sum:
- Formula:

$$\text{Area} = \int_a^b \text{height}(t)\,dt$$

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Volumes

- Known volumes we can decompose into: boxes
- Sketch of decomposition:
- Written as a Sum:
- Formula:

$$\text{Volume} = \int_a^b \text{area}(t)\,dt$$

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Question 1. Does this idea work to give the correct volume for a rectangle?

Steps to find Volume via Cross-Sections

0) Sketch a picture.

1) Decide on the variable of integration, t (usually x or y).

2) Find the upper and lower bounds of integration, a and b.

3) Find the vertical or horizontal line-segments corresponding to the cross-section for a given value of the integration parameter t.

4) Sketch, if necessary, a diagram of the two-dimensional cross-section above a given line-segment, and find a formula, f, for its area.

5) Integrate $\int_a^b f(t)\,dt$.

Exercise 1. Draw a sketch of the solid whose base is bounded by the curves $y = x$ and $y = x^2$ and whose cross sections perpendicular to the x-axis are squares. Then find its volume.

Operational Skills:

- Vertical slices ⇔ Integrating ______________________________
- Horizontal slices ⇔ Integrating ______________________________
  Integrating ______________________________
- Modularize the problem: A very complicated problem doesn't have to be visualized and all parts known all at once. Each step can be treated as its own sub-problem.

Aside on Aphantasia

Can you visualize a circle inside of a square?

The circle should be just touching the sides of the square.

Maybe make the circle orange and the square green.

Not everyone has the same ability to see images which aren't there, but this wasn't really understood until the late 1800s.

The famous statistician Galton wrote:

``To my astonishment, I found that the great majority of the men of science to whom I first applied, protested that mental imagery was unknown to them, and they looked on me as fanciful and fantastic in supposing that the words 'mental imagery' really expressed what I believed everybody supposed them to mean. They had no more notion of its true nature than a colour-blind man who has not discerned his defect has of the nature of colour.''

This was not really rediscovered and understood until the 2000s.

Galton is perhaps best known for the *Galton-Watson* process in statistics and probability.

Exercise 2. Find the volume of a solid with a circular base of radius 9 and whose cross-sections perpendicular to the y-axis are equilateral triangles.

Exercise 3. Find the volume of a solid with a circular base of radius 9 and whose cross-sections perpendicular to the y-axis are right isosceles triangles with a leg on the base of the solid.

Common Area Formulas:

- ☐ Square: $A = base \times height$
- ☐ Semi-circle: $A = \frac{1}{2}\pi r^2$
- ☐ Equilateral Triangle: $A = \frac{\sqrt{3}}{4}s^2$
- ☐ Isosceles Right Triangle with sides of length s: $A = \frac{1}{2}s^2$.

2 Solids of Revolution - Washers

A special type of Accumulated Cross Section occurs when the cross-sections are all washers (also called annuli, the region between two concentric circles) with centers on the same line. In this case, the cross sections can be seen as the outcome of rotating a line-segment about that center line.

Exercise 4. Consider the region, R, between the lines $y = x^2$ and $y = x^3$ from $x = 0$ to $x = 1$. Suppose R is rotated in space around the line $x = 0$, creating a solid, S. What are the cross-sections of S perpendicular to the y-axis? What is the volume of S?

Area of a Washer with inner radius r and outer radius R

$$A = \pi \left(R^2 - r^2\right).$$

Volume of a Solid of Revolution - Washer Method (integrating dy)

Suppose that the region R lies between $y = a$ and $y = b$ and does not intersect the line $x = c$. Then if R is rotated about $x = c$, the volume of the resulting solid is given by

$$\int_a^b \pi \left(R^2(y) - r^2(y)\right)\, dy$$

where $R(t)$ and $r(t)$ are the inside and outside distances from the line $x = c$ to the region R with a y-coordinate of t.

Volume of a Solid of Revolution - Washer Method (integrating dx)

Suppose that the region R lies between $x = a$ and $x = b$ and does not intersect the line $y = c$. Then if R is rotated about $y = c$, the volume of the resulting solid is given by

$$\int_a^b \pi \left(R^2(x) - r^2(x)\right)\, dx$$

where $R(t)$ and $r(t)$ are the inside and outside distances from the line $y = c$ to the region R with a x-coordinate of t.

Exercise 5. Suppose that the circle $(x - 2)^2 + y^2 = 1$ is rotated about the x-axis to form a shape. If you can, think about what that shape is. Set up an integral that will represent the volume of this shape.

Observations:

- We do not need to be able to visualize the end-result in order to answer one of these questions.
- We first reduce to the two-dimensional region being rotated around a given line
- and we next reduce to the *line-segment* being rotated around that forms a given washer.
- **Washers occur when the line-segment is perpendicular to the axis of rotation.**