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set #6

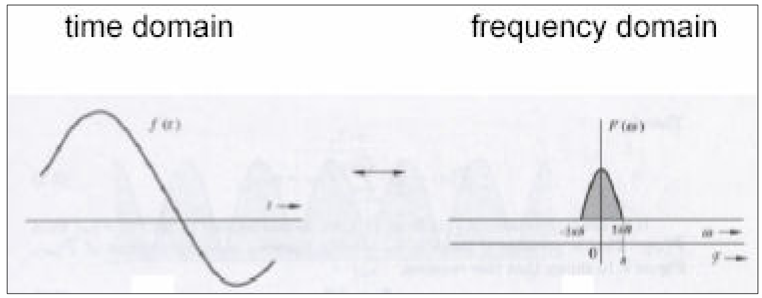
lecture #:	date:	professor:
17	October 16th, 2006	Dr. Swain
18	October 18th, 2006	Dr. Swain

announcements:

- *Please note that this NTC Set is unedited!*
- *GOOD LUCK ON THE MIDTERM EVERYONE!!!*

Fourier Transforms

- For **periodic signals**, they can be represented as a sum of sinusoidal waveforms in a **Fourier series**, where the frequencies of the sinusoidal waveforms are harmonics of a fundamental frequency component. (ie. the ratio of their frequencies are whole numbers to the fundamental frequency)
- For **non-periodic signals**, they can be represented as a sum of sinusoidal waveforms in a **Fourier transform**, which involving complex numbers, and component sinusoidal waveforms can be of any frequency (ie. the frequencies of the component waveforms can be infinitestimally close to each other)
- **Fourier transforms** represent these complex signals as a function of their frequencies
- The **bandwidth** of a Fourier transform is the range of frequencies covered in the positive part of the spectrum (ie. the difference between the highest positive frequency component and the lowest positive frequency component)
- The amplitude of the frequency component in the Fourier transform reflects the importance of that frequency's contribution to the real time signal



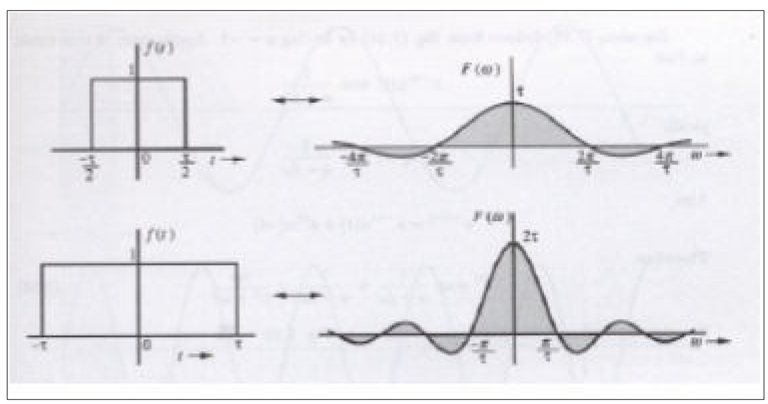
There are **three properties of Fourier transforms** that we should know:

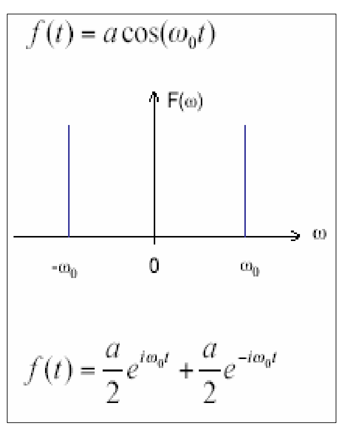
Property 1: *The Fourier transform is always symmetric around ω=0 for a real signal.*
- I.e. The positive frequencies is symmetrical to the negative frequencies

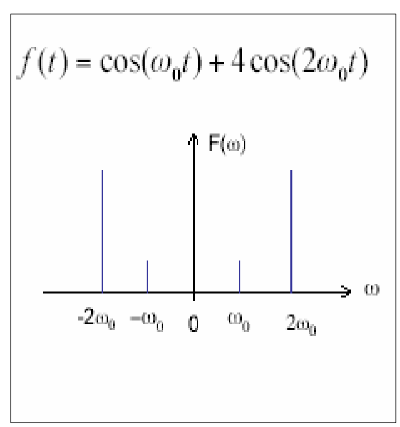
Property 2: *The shorter lived the signal, the larger the effective bandwidth of the signal, and vice-versa.*
- The more quickly the signal changes, the more frequency components you need to describe that quick change.
- Pulse width α 1/(effective bandwidth)

A) A time-limited signal has contribution at all frequencies.
- If in real time, the signal decays to zero, it would require many frequencies to describe the changes that take place in real time.

B) A frequency limited signal is an everlasting signal.
- Conversely, if in real time the signal does NOT decay to zero and is periodic, then its Fourier transform would have a finite number of frequencies.
- Example 1: The cosine function is finite in frequency space, and is an everlasting signal – the cosine







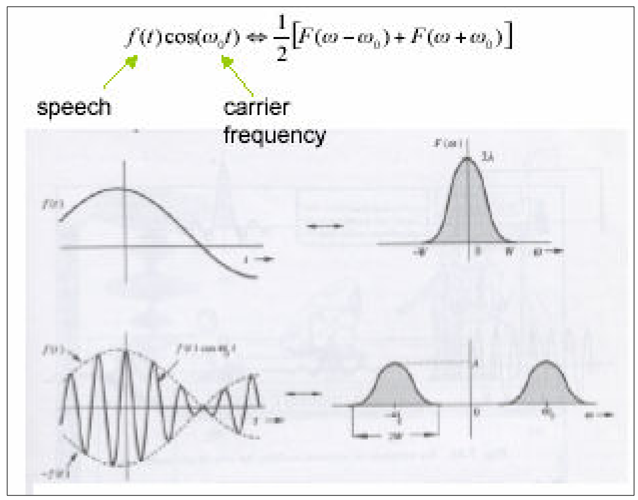
Fourier transform has exactly one frequency (ω_0), and in real time, the cosine waveform does not decay to zero (is everlasting). The height of the peak at ω_0 is the amplitude of the cosine function.

- Example 2: A function made of two sinusoids, $f(t) = \cos(\omega_0 t) + 4\cos(2\omega_0 t)$ – The Fourier transform allows us to see the two frequencies that make up this signal (ω_0 and $2\omega_0$). The heights of the peaks represent the amplitudes of the corresponding cosine waves (amplitude of 1 and 4).

Property 3: *Both amplitude, $|F(\omega)|$, and phase arg $F(\omega)$, contribute to the representation of the signal in the frequency domain.*

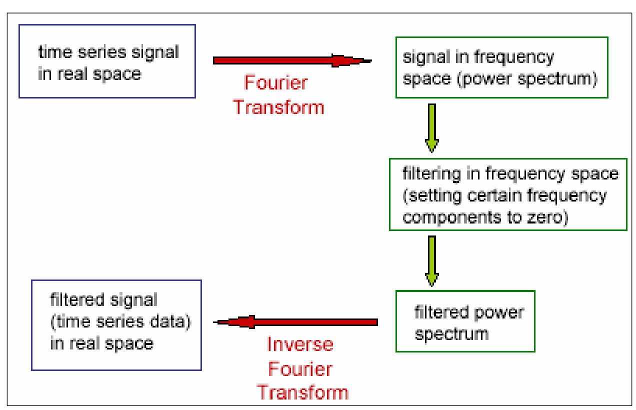
- $|F(\omega)|$ = **amplitude** of the Fourier transform
- arg $F(\omega)$ = **phase** of the Fourier transform (the **argument**)
- Changing the phase of the Fourier transform means adding time delays to the signal in the time domain.

Amplitude Modulation



- This figure shows a complicated signal *f(t)* represented in Fourier space $F(\omega)$ with a bandwidth of *W* and amplitude of *2A*.
- Suppose this signal *f(t)* was a recording of human speech. This signal is then multiplied by a cosine function $\cos(\omega_0 t)$, where ω_0 is the **carrier frequency**.
- The new signal, $f(t)\cos(\omega_0 t)$ is represented in Fourier space.
- Notice how the frequency spectrum has been split into two parts, with peaks at $-\omega_0$ and $+\omega_0$ (the carrier frequency), a bandwidth of *2W*, and amplitude of *A*.
- Hence, $f(t)\cos(\omega_0 t) \Leftrightarrow \frac{1}{2}[F(\omega-\omega_0) + F(\omega+\omega_0)]$
- **Application:** This is interesting because by multiplying the original power spectrum (aka. Fourier transform) by $\cos(\omega_0 t)$, the average frequency is shifted. This is useful because this is how radio signals can be cleanly transmitted. When we listen to the radio, the frequency we tune our radios to is the **carrier frequency**.

Digital Filtering

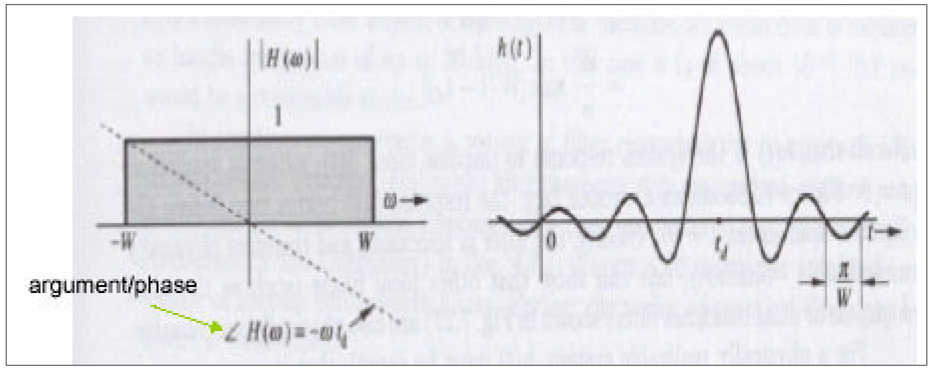


- In Fourier space, we can do things that are almost impossible to do in real space; one of these things is **digital filtering**.
- **Digital filtering** sets certain frequency components to zero, taking them out of the real time signal, while leaving others intact.
- __**Four types of filtering:**__
  1) **Low pass filter:** Only *low* frequencies 'pass' and all high frequencies are eliminated.
     - Most commonly used in biology, used to filter fluctations in the signal due to noise, etc.
  2) **High pass filter:** Only *high* frequencies 'pass' and all low frequencies are eliminated.
     - Preserves noise effects and sudden changes in the original signal.
  3) **Band pass filter:** Only frequencies within a *selected range* "pass" and the rest are eliminated.

4) **Band reject (notch) filter:** Frequencies of a *selected range* are eliminated, while the rest pass.

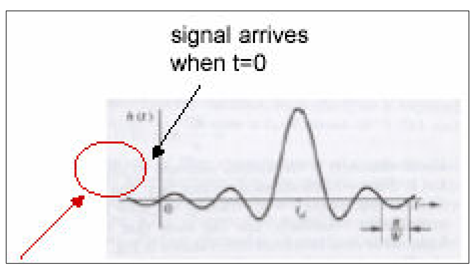
Practical Filters

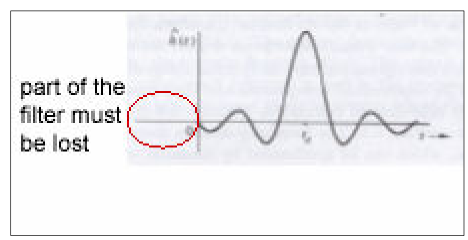
- In everyday application, engineers strive to design a desirable filter for various equipments to modify signals in real time. For example, microphones have filters that eliminate frequencies that could correspond to static noise in real time, producing better sound quality. This is challenging because ideal filters are **non-causal.**
- This example shows the application of a **low-pass filter** where only frequencies below *W* are allowed to pass.
- [Figure] The graph on the left side shows the Fourier transform function of the low-pass filter. All frequencies below *W* are set to an amplitude of 1 (ie. they are allowed to pass), while all other frequencies are set to an amplitude of 0 (ie. they are eliminated, filtered out).



- The dotted line shows the argument/phase of the Fourier transform. Just know that the definition of the phase is given by the formula: $< \mathbf{H(\omega) = -\omega_0 t_d}$ (c.f. the phase θ of a sinusoid) in $f(t) = \sin(\omega t + \theta)$
- The figure on the right shows the *inverse* Fourier transform of the same low-pass filter, ie. the representation of the filter in real time.
- This filter would be applied to the real signal (e.g. human speech). When the signal function is multiplied by the filter function, all signal frequencies below W would be kept (mult. by 1), while all frequencies above W would be eliminated (mult. by 0).

- In these graphs, representing the inverse of the filter Fourier transform, ie. the filter applied in real time, t=0 is when the signal (e.g. speech) starts. t_d is the delay time when the filter is applied to the signal.

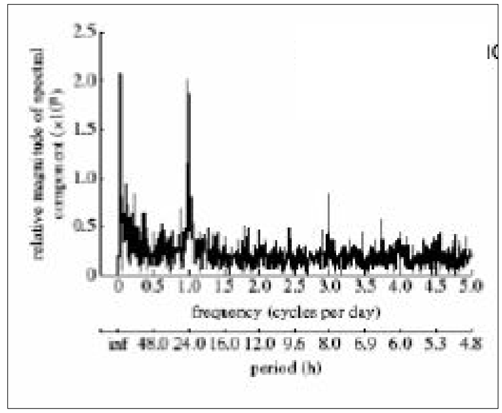




- The filter needs to be applied before the signal has arrived, there has to be a time delay, t_d. **Non-causal.**
- You can think of it like this: because the Fourier transform is frequency-limited, in real time it is everlasting. As a result, if you were to apply the signal at t_d, part of the filter function would be lost (whatever chunk of the filter is present before t=0, the time when the signal starts). The lost of a part of the filter means that the signal is not as well processed as it could be.
- If we lengthen t_d, then we can keep more of the filter function and apply it to the signal. To an extreme, we could increase t_d so that the entire signal is finished before applying the filter in its entirety. (e.g. finish the speech and apply the filter to the recording of the speech) However, this delay means that we will have to a long time, and it is impractical in real-time applications, for example in a speech to a live audience.
- The compromise: ↑t_d ↑waiting time ↑quality of filtering; or else ↓t_d ↓waiting time ↓quality of filtering
- Part of what engineers and designers have to do is determine what the best t_d value would be for the filter. Microphones usually have a t_d on the order of 1 ms.

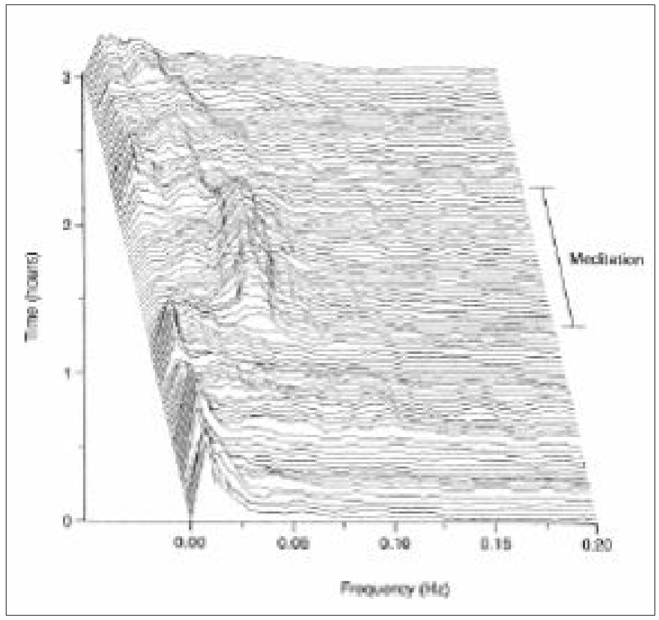
Applications of Fourier Transforms

Example 1: Diving Behaviour of Whale Sharks



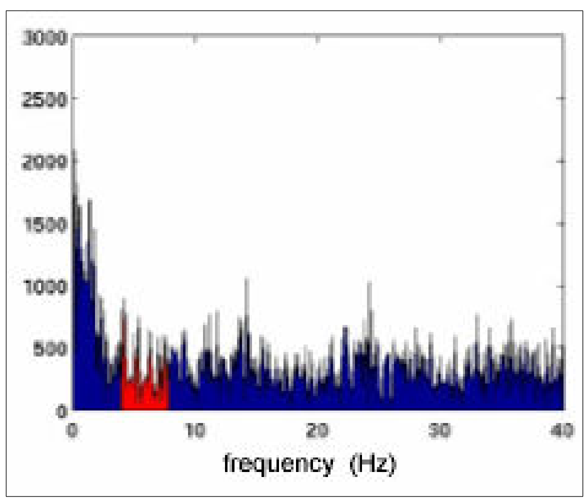
- Whale sharks with attached pressure devices were monitored for their diving patterns over a period of 206 days. The resulting data was analyzed using the Fourier transform method.
- The power spectrum (aka. Fourier transform) shows the frequencies of the dives made by the sharks, in units of *cycles/day*.
- The prominient peaks of the power spectrum at *0.030, 1.0 and 3.0 cycles/day* show that the sharks made rhythmic deep dives *every month, every day and every 8 hours*, respectively. (Also described as circa-lunar, diel, and ultradian dives, respectively).

Example 2: Analysis of Heart Rates

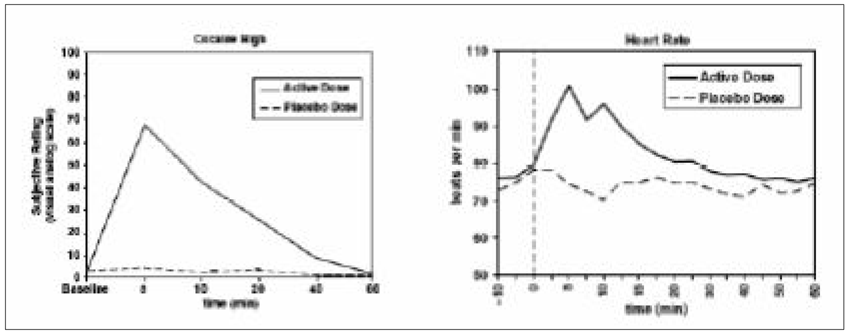


- The heart rate of a meditating person(s) was measured over a few hours to demonstrate the significant changes in heart rate dynamics during meditation. The collected data was analyzed using the Fourier transform method (aka. spectral analysis).
- [Fig.] The Fourier transform of collected data. This graph represents many power spectra juxtaposed next to each other. Each individual trace of the power spectra was produced using 15 minutes of the collected data, representing the data frequencies (horizontal axis) present in that 15-minute window. The amplitude of each frequency is represented by the curvature of each power spectrum (vertical axis) Each subsequent trace of the power spectra starts slightly after the previous power spectra, corresponding to a slightly later "15-minute window" in the collected signal data (deep axis, time of meditation).
- The power spectra shows that for the period of meditation (~1.3 to 2.1 hours), there is indeed a significant change in the frequencies of the heart rate dynamics. Further analysis of the subject's electrocardiogram (not shown) shows a correlating change in respiration.
- The results suggest that meditation may not be only an autonomically quiescent state.

Example 3: Responses to Smoking Cocaine



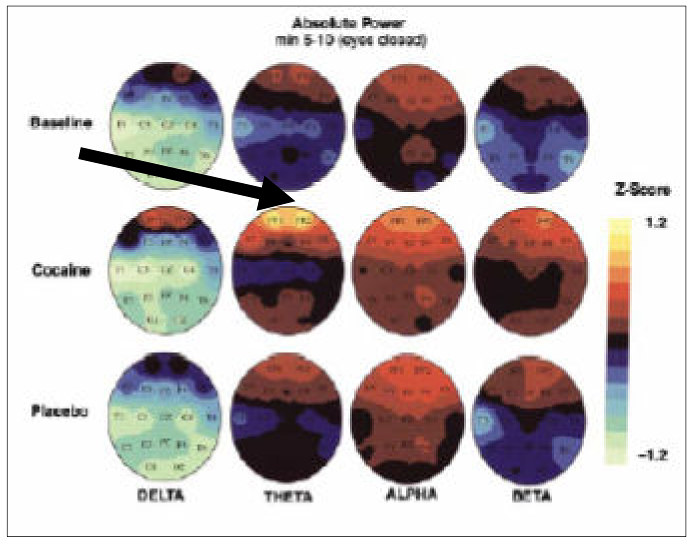
- Cocaine was administered to cocaine-dependent subjects and their EEG was measured.
- On the left, the EEG is represented as a Fourier transform, showing the frequency components of the recorded EEG.
- **Absolute theta power** is the area under the Fourier transform curve between 4-7 Hz, corresponding to the theta waves. **Relative theta power** would be the ratio of the area of the curve



of 4-7 Hz to the area of the rest of curve.

- The graph below shows that the subject experienced a cocaine high 5 minutes into the experiment; the recorded heart rate also peaks at 5 minutes, correlating with the subject's experience of a high. Subjects administered with placebo did not experience any physiological or psychological effects.
- The data was further analyzed by calculating the Z-score. The Z-score is a normalized way of measuring how far a specific measured data point deviates from the distribution's mean.
- **Z-score = (data point – mean of data points) / (standard deviation of data points)**

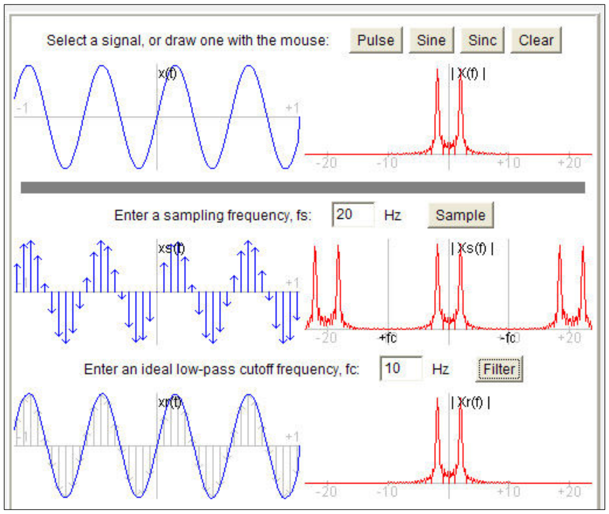
- This figure show the results from the analyzed EEGs collected from various parts of the cerebral cortex. Notably, the prefrontal cortex showed an increase in absolute theta power (higher Z-score), indicating that prefrontal cortex's EEG is involved in response to acute cocaine.



Practical Fourier Transforms: The Sampling Theorem

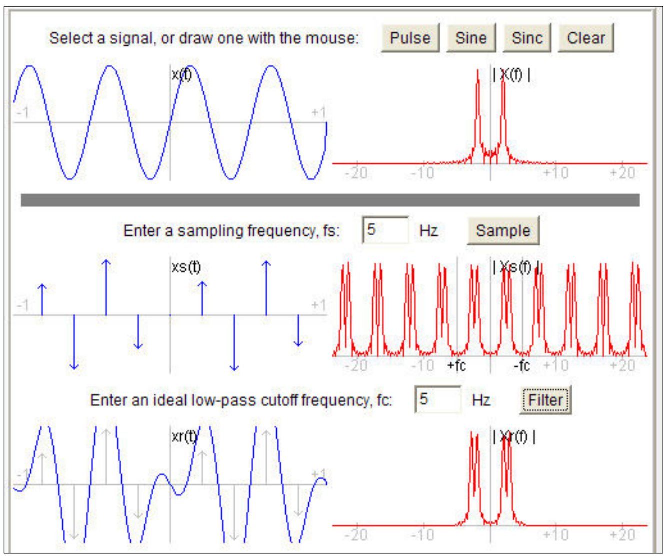
1. *A band limited signal of bandwidth* B *Hz can be reconstructed exactly from its samples if they are taken uniformly at a sampling rate greater than or equal to 2B samples per second.*
2. Similarly – *By sampling at a rate of* F_s, *the maximum signal bandwidth (the maximum frequency in the signal) that can at best be accurately measured is* $F_s/2$.

- The **Nyquist rate** is the minimum sampling rate that will reconstruct an accurate signal.
- **Nyquist rate = 2*B** (where B is the bandwidth of the signal)
- **Nyquist sampling interval = 1/(2*B)** , or the reciprocal of the Nyquist rate
- That is to say, each wavelength needs at least two sampling data points to be accurately represented.
- Sampling takes place every finite time interval, **T**
- **Sampling rate Fs = 1/T; Sampling frequency $\omega_s = 2\pi/T$** (where T is the finite sampling interval)
- [Fig.] Because of the finite limitations, the sampling process leads to changes of the Fourier space – the Fourier transform of the signal becomes periodic, being repeated at the sampling frequency, F_s
- This is easily fixable by application of a <u>low-pass filter</u> to eliminate frequencies beyond the bandwidth.



Visit www.jhu.edu/~signals/sampling/Hz/index.html for a Java demonstration.

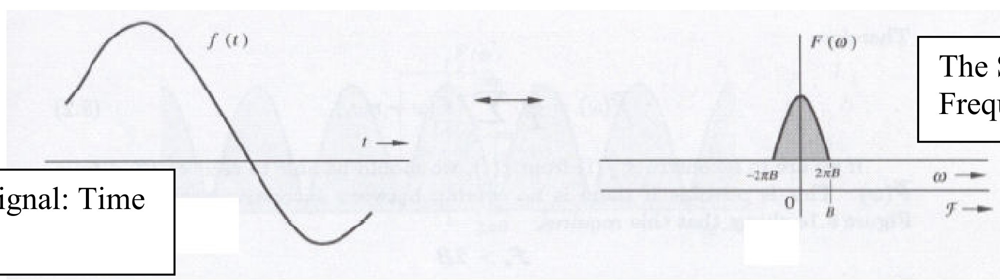
- This figure shows a sampling frequency above the Nyquist rate. The original Fourier transform and sine wave are conserved. Notice that the Fourier transform repeats itself at a frequency of 20 Hz, which is the sampling frequency.



- When sampling occurs below the Nyquist rate, the Fourier transform and sine wave are distorted.

- When sampling at the Nyquist rate, the Fourier transform waveform is repeated exactly side by side.
- However, if the sampling rate is less than the Nyquist rate ie. $F_s < 2B$, then the signal becomes distorted. The two adjacent Fourier transform waveforms intrude each other, the higher frequencies of the Fourier transform are lost, and **aliasing** occurs.
- **Aliasing** is a type of distortion that occurs when sampling high frequencies at a low sampling rate below the Nyquist frequency

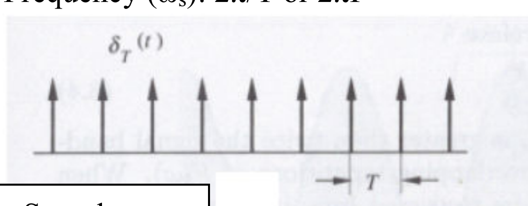
- **Review from LAST CLASS:**
    - **Pratical Fourier Transformations: The Sampling Theorem**
- **Background Info: What is the pupose of sampling?**
- In the sampling process, a continuous time signal is converted to a discrete time signal. In the reconstruction process, the continuous signal is recovered from the discrete signal. This has applications in the interconversion of digital and analog signals. Example: How a sound wave is converted into a digital signal and then to a sound wave when you talk on the telephone.
- **What are the conditions for sampling?**
  - A band limited signal of bandwidth “B Hz” can be reconstructed exactly from its samples if they are taken uniformly at a sample rate greater than or equal to 2B samples per second.
    - E.g. If we have a signal that spans 10 Hz (bandwidth = 10Hz) then we can reconstruct the original signal from it’s fourier transformation if we “sample” the signal at a frequency at least twice that of the original bandwidth. In this case, it would have to be sampled at 20Hz.
      - The minimum sampling rate is called the **Nyquist rate** and is equal to **2B (Hz)**
      - The minimum sampling interval (in seconds) **Nyquist sampling interval** is equal to **1/2B (seconds) → aka the receiprical of the rate.**
  - Similarily, by sampling at a frequency of F_s, the maxiumum signal bandwidth (the maxiumum frequency in the signal) that can at best be accurately measured is $F_s/2$. This is the exact same thing as said above, you just have to know that $F_s = 2B$.
    - E.g. If we take the same example as above we would choose to sample at 20Hz and the maximum bandwidth we could measure would be 10 Hz.



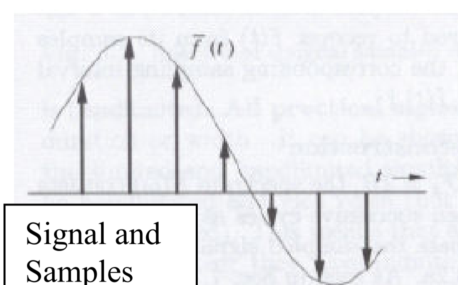
The Signal: Time Space

The Signal: Frequency Space

- Sampling Frequency (ω_s): $2\pi/T$ or $2\pi F$

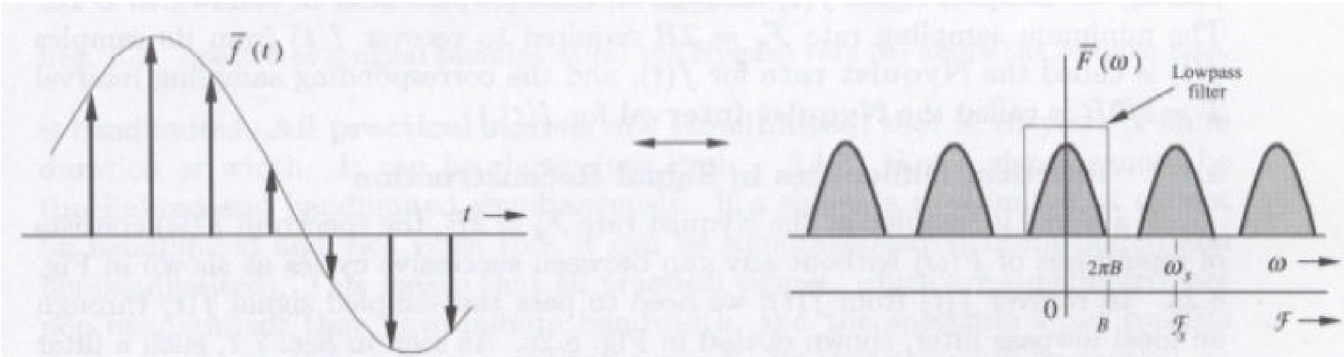


The Samples



Signal and Samples

- The sampled signal and it's fourier transform:
  - Sampling the signal leads to a periodic (everlasting) Fourier transform.



Why does sampling generate multiple copies of the power spectrum?

$f(t)$

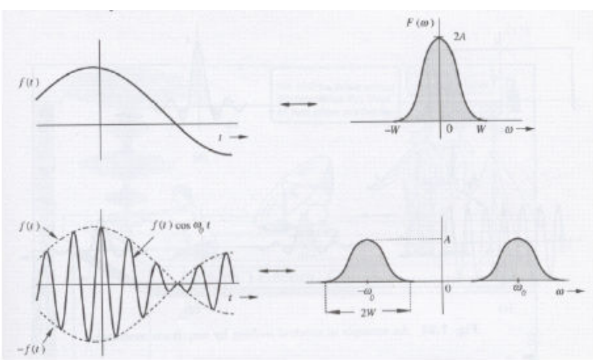
$$= \sum_{n=-\infty}^{\infty} D_n e^{2\pi i n F_s t}$$ represent as a Fourier series

with $F_s = \frac{1}{T}$

$$= f(t) \times \sum_{n=-\infty}^{\infty} D_n e^{2\pi i n F_s t}$$

$$= \sum_{n=-\infty}^{\infty} D_n f(t) e^{2\pi i n F_s t}$$

- These forumals suggest that the spectrum of the baseband signal being sampled is shifted and repeated forever at integral multiples of the sampling frequency, f_S.
- This means that the fundaments signal is being multiplied by an interger and this is what causes the repeats.
- The important thing to remember is that it is replicated with the same frequency as the sampling frequency.



Remember: Multiplying by a sinusoid

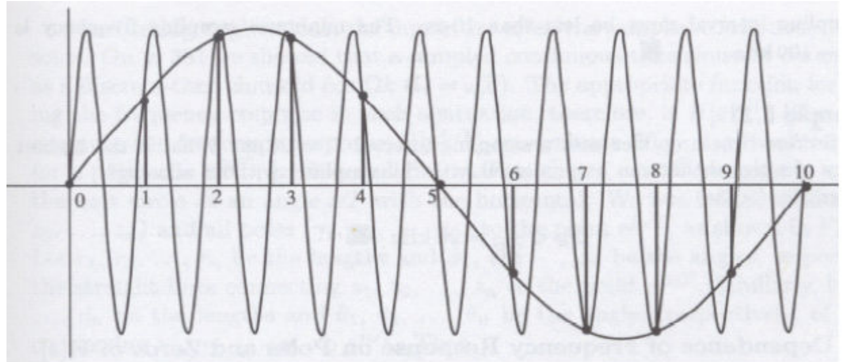
- As we have seen in previous lectures when you mutiply by a sinusoid the freuquency is shifter so that you get two signals with a bandwidth of 2W and an Amplitude of A.

- We can now pass the signal through a low pass filter which will isolate only one copy of the signal. If this is done properly, when you reverse transform this signal you will restore your original signal.

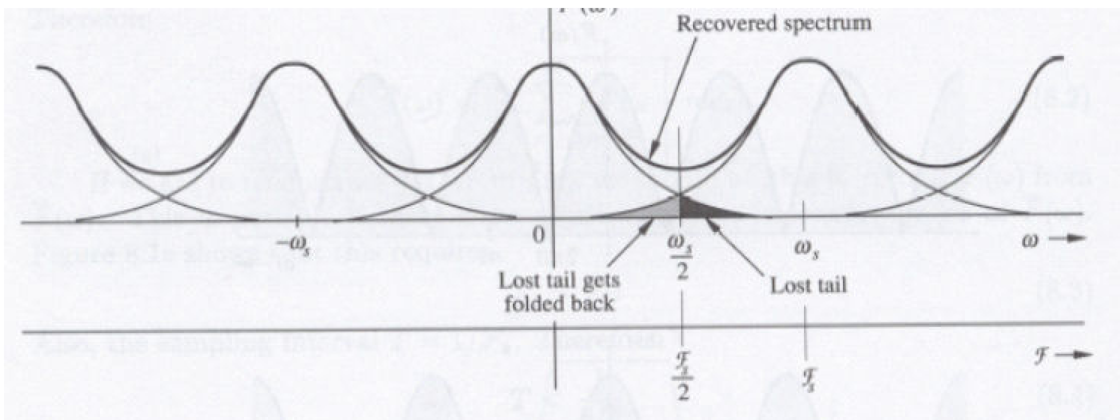
- **What happens when you sample below the Nyquist rate?**

○ What if ω_S is less than $2\pi B$ (i.e. less than the Nyquist rate)
    - $F_s = 1/T < 2B$
  - When this happens the signals start to **overlap** and become distorted. When this happens **aliasing** occurs.

- **Aliasing in the time domain:**
  - Aliasing gives rise to the appearance of spurious (false) higher frequencies in the time domain (from spectral folding in the frequency domain). See high frequency signal detected on the right - this fits the curve completely.

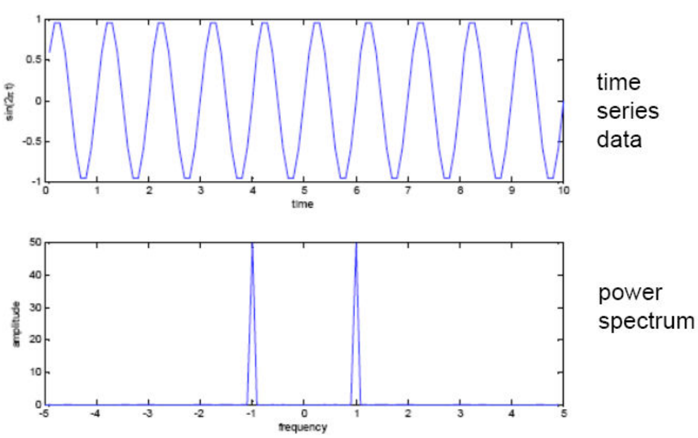


- **Aliasing in the frequency domain:**



  - In the frequency domain – overlapping of the signals occur. In the above case, the tail of F(ω) is lost when $F>F_s/2$ or when $F<F_s/2$
  - The lost tail folds back around the folding frequency $F_s/2$. This spectral folding is known as **aliasing.**

➢ **Note:**
  - All real world signals are time limited and therefore have contributions at all frequencies.
  - Aliasing will always occur, but can be controlled by passing the signal first through a low pass anti-aliasing filter before sampling.
  - When this is done the band will be limited, but some of the signal may be lost.



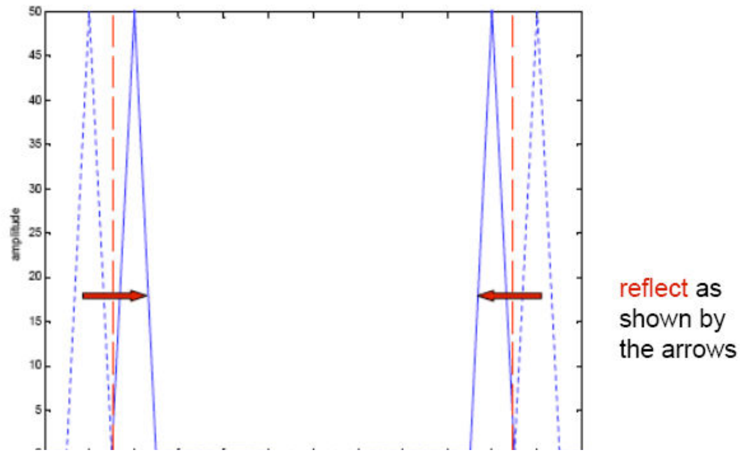
$\tau = 0.1$

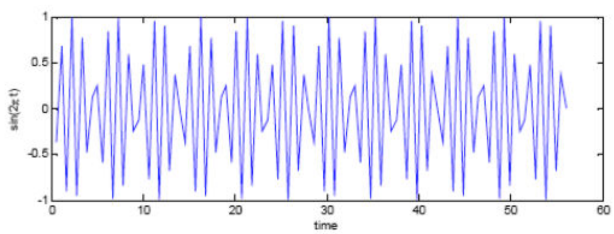
- **Example of Aliasing:**
  - Sin $(2\pi t)$ → Has a frequency of 1 Hz.
- We choose a sampling interval of 0.1 s because this would lead to a F_s of 10Hz (by $Fs=1/\tau$) which is more than enough to meet the minimum sampling rate. (In this case the minimum sampling rate would be 2Hz (2B)).

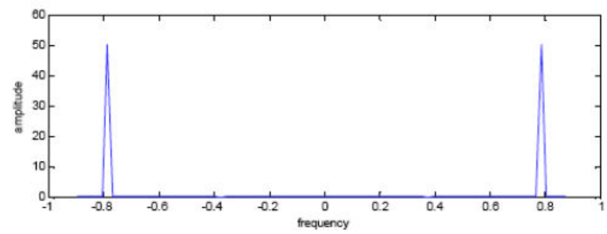
- If we change τ to 0.56s what is the new power spectrum?

$$\frac{F_S}{2} = \frac{1}{2\tau} = \frac{1}{2 \times 0.56} = 0.89$$

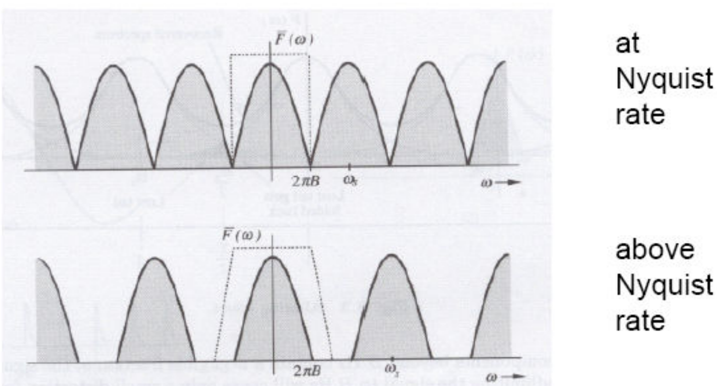
  - By using the equating to the left we can conclude that with $\tau = 0.56$s the sampling rate will be 0.89Hz.
  - We can draw in this new maxium frequency with the red dotted lines.
  - We can then reflect the frequencies which are higher than $1/2\tau$ and lower than $-1/2\tau$ around the dotted line.
  - This is due to **spectral folding**.
  - When the power spectrum is translated back into time series data you can see that the signal has changed drastically. →





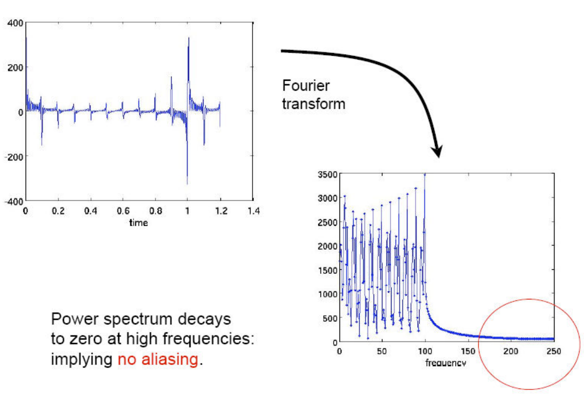


τ = 0.56



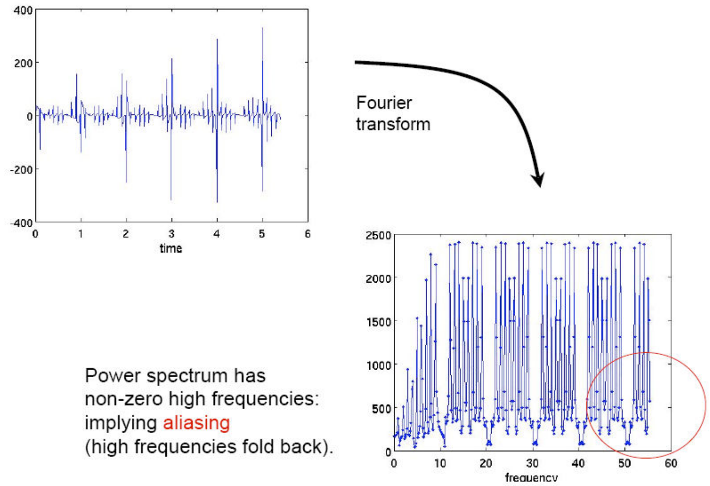
- (See diagram above) It is better to sample above the Nyquist rate then at the nyquist rate.
- A real world low-pass filter may not have a sharp cutoff between the high and low frequencies so the more spaced out the samples the more likely that the low pass filter will be able to isolate only one sample.

- How do you detect an aliased signal?



- If the power spectrum decays at high frequencies there is a good chance that there is no aliasing

- If the power spectrum has high frequencies which are non-zero then this implies that there is aliasing occurring.
- These high frequencies are the result of folding back of the signal.



Summary of Sampling:

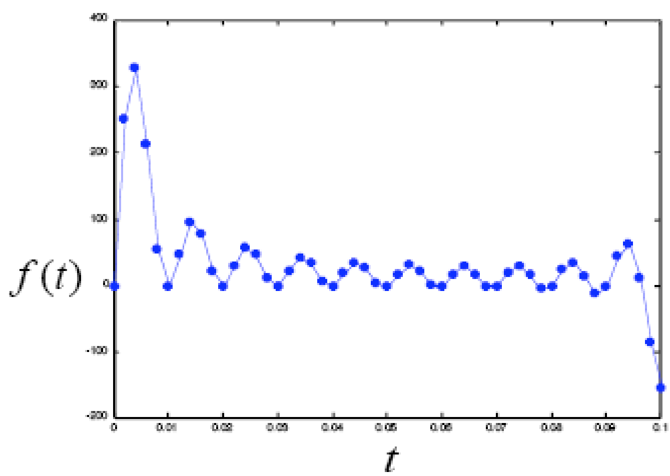
- Sampling leads to the replication of the signal in fourier space.
- The power spectrum is repeated at multiples of the sampling frequency.
- A sampled signal is put through a low pass filter to remove the effects of the replications.
- For an aliased signal, high frequency components fold back (around half the sampling frequency), altering the power spectrum and so distorting the signal in the time domain.
- Experimentally, anti-aliasing filters are applied to the signal before sampling. High frequency components of the signal are lost, but the signal is band limited and therefore can be sampled at an appropriate rate.

- **Discrete Fourier Transforms for sampled data** (aka how the computer does it):
  - The signal is expressed as a *finite* sum of complex sinusoids:

$$f(t) \approx \sum_k F_k e^{2\pi i f_k t}$$

formula not on exam

- Using ω=2πf
- We have to determine the values of the amplitude, F_k, and the frequencies, f_s, that describe the signal.
- **Determining the Frequencies:**



- The frequencies of the sinusoids that we can measure in the sampled data are determined by: **how often we sample the original signal** and **the total number of data points we generate.**
- N = total number of data points, e.g. 50 in the figure
- τ= sampling interval, e.g. 0.002 seconds

$$f_k = -\frac{1}{2\tau}, -\frac{\frac{N}{2}-1}{N\tau}, \cdots, 0, \cdots, \frac{\frac{N}{2}-1}{N\tau}, \frac{1}{2\tau}$$

$$= \frac{k}{N\tau} \quad \text{where } k \text{ goes from } k = -\frac{N}{2} \text{ to } k = \frac{N}{2} \text{ in steps of } 1$$

- **Example:**
  - Suppose N = 4
  - $\tau = 0.1$s

then

  - $N\tau = 4 \times 0.1 = 0.4$ or 2/5

And

  - $f_k = k/(N\tau) = 5k/2$ (with k being one of {-2,-1,0,1,2} ← because k=-N/2 to k=N/2 in steps of 1)
- Consequently, the frequencies that we can measure are {-5,-2.5,0,2.5,5} in Hz
- Why is the maxiumum frequency 5Hz? The sampling frequency determines the min and max frequencies. The sampling frequency is 10Hz (1/0.1s) which is the span from -5Hz to 5Hz.

- **Determining the Amplitudes:**
  - The size (absolute value of) the amplitude, F_k, indicates whether a complex sinusoid with frequency f_s is present in the signal.
  - You can use the **Discrete Fourier Transform** to find the amplitudes:
  - Where j runs through all the data points in the signal.
  - The discrete Fourier transform is implemented via the Fast Fourier Transform (FFT) Algorithm.

$$F_k = \sum_j f(t_j)\, e^{2\pi ijk/N}$$

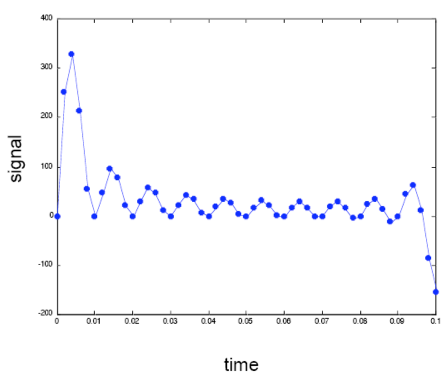
formula not on exam

Note that when $k=0$,

$$F_0 = \sum_j f(t_j)$$

is the sum of all the data points.

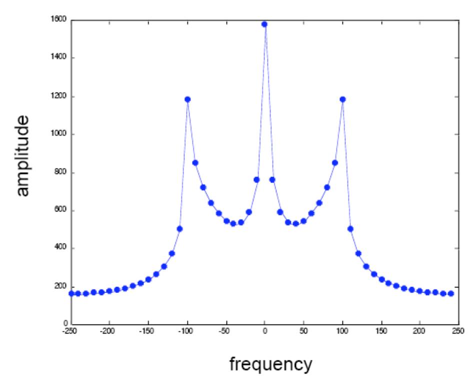
- Example:
  - Sampled Data:



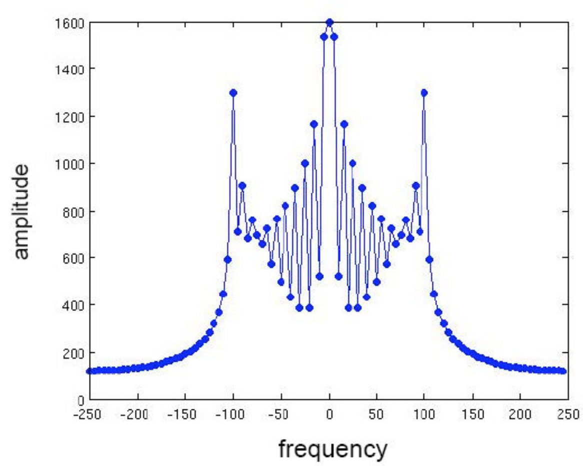
$\tau = 0.002$s

Power Spectrum:

The power spectrum is a plot of the absolute value of the fourier transform against the frequency. i.e $|F(\omega)|$



- Doubling the number of data points, doubles the frequency resolution of the power spectrum but **does not change** the maximum frequency.



Properties of the Discrete Fourier Transform:

- To increase the accuracy of the Fourier transform, we need to decrease the sampling interval between sample points (sample more frequently).
- To increase the resolution of the Fourier transform, we need more data points.

- Extra data points can be added by adding zeros to the original data, called zero padding. This procedure allows us to "see" more of the fourier transform (**but the accuracy of the fourier transform is always set by the sampling interval)** as the points in the frequency domain become closer together.
- Know this disctinction → Accuracy – Determined by the sampling frequency. Resolution – Determined by the number of data points.

- **<u>Summary:</u>**
    - The discrete fourier transform (DFT) is used to numerically calculate the Fourier transform of a data set.
    - The sampling interval determines the accuracy of the transform.
    - The number of data points taken determines the resolution of the transform (how much of the details we can see, but the accuracy of these details is set by the sampling interval.)

- **<u>Final Summary:</u>**
- A signal can be represented in the time or frequency domain. Both are equivalent, and one may be more convenient than the other for a particular task.
- A periodic signal can be expressed as a fourier series.
- Any **periodic** signal can be expressed as a Fourier series.
- Any signal can be expressed as a Fourier transform.
- To express a data set, rather than a mathematical expression, in the frequency domain, the discrete Fourier transform via the Fast Fourier Transform (FFT) algorithm is always used.