Introduction to Linear Programming

- What is Linear Programming?
- Setting up an Initial Tableau

Definition and Use

Linear programming is used to solve optimization problems where all the constraints, as well as the objective function, are linear equalities or inequalities.

The methods were first developed in 1939 and used in military planning operations in World War 2. They were refined and published in 1947 by George Dantzig who called his method the “Simplex Method”.

Definition and Use

In lesson 1, you solved a linear programming problem (experienced and inexperienced workers) using a graph.

In the next few lessons you will learn to solve linear programming problems using matrices. Matrices have the following big advantages:

1. Can be used to solve problems in more than 2-3 variables (beyond our ability to graph)
2. Can be programmed into a computer

Setting Up Constraints

One of the problems that needs to be dealt with is that without a graph, systems of inequalities can be difficult to solve.

These were the constraints on the "workers" problem:

- $15x_1 + 10x_2 \leq 1200$ (hourly wages)
- $1x_1 + 2x_2 \leq 120$ (supervisor time)
- $x_1, x_2 \geq 0$ (positive variables)

Because of all the inequalities, it is very difficult to find the solution set without a graph.

Setting Up Constraints

The first step, therefore, is to convert inequality constraints to equality constraints.

In the example, the inequality

$$15x_1 + 10x_2 \leq 1200$$

implies that there is a difference (possibly 0) between the expression $15x_1 + 10x_2$ and the number 1200.

So, we introduce a variable (called a “slack variable”) to represent that difference:

$$15x_1 + 10x_2 + x_3 = 1200.$$

Setting Up Constraints

So,

$15x_1 + 10x_2 \leq 1200$ becomes

$$15x_1 + 10x_2 + x_3 = 1200$$

and $1x_1 + 2x_2 \leq 120$ becomes

$$1x_1 + 2x_2 + x_4 = 120$$

The two ≥ constraints are not converted, but it is assumed that all variables are positive.

Practice Problem 1

Convert the following inequalities into equalities of the form (expression) = (number).

a. $3x_1 + 5x_2 \leq 12$

b. $2x_1 - 4x_2 + x_3 \leq 200$

c. $6x_1 + x_2 \geq 150$

d. $-3x_1 + 2x_2 \leq 120{,}000$

e. $x_1 + 5x_2 + 2x_3 - 8x_4 + x_5 \geq 13.8$

f. $x_1 \leq 2x_2$

Setting Up the Objective Function

The next step is to convert the objective function so that its value can be placed into the initial matrix.

If you have "maximize $f = x_1 + 2x_2$", this would change to "$-x_1 - 2x_2 + f = 0$".

Practice Problem 2

Convert these objective functions to equations of the form (expression) = 0:

a. Maximize $f = 3x_1 + 12x_2$

b. Maximize $f = 4x_1 – x_2$

Convert this objective function to a “maximizing a negative” problem, then to (expression) = 0:

c. Minimize $f = 2x_1 + x_2$

Practice Problem 3

Convert the constraints and objective function of the following linear programming problem:

Maximize $f = 50x_1 + 20x_2 + 42x_3$

subject to $3x_1 + 2x_2 - x_3 \leq 100$

$2x_1 + x_2 + 4x_3 \leq 80$

Setting Up an Initial Tableau

Once you have your equations ready, you can put them into a matrix. Here's an example:

Maximize $-3x_1 - 4x_2 + f = 0$ (objective)

subject to $x_1 + x_2 + x_3 = 480$

$2x_1 + 3x_2 + x_4 = 1200$ (constraints)

becomes

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & f & \text{ans} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 480 \\ 2 & 3 & 0 & 1 & 0 & 1200 \\ -3 & -4 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Standard Maximum Form

The previous example is in "standard maximum form", meaning:

- the objective function is to be maximized, so the leading coefficients are negative in the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 480 \\ 2 & 3 & 0 & 1 & 0 & 1200 \\ -3 & -4 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- the constraints are all ≤, resulting in positive coefficients for the slack variables

Practice Problem 4

Using your answers to Practice Problem 3, set up an initial tableau.

Is it in standard maximum form?

Practice Problem 5

Convert this problem from words into inequalities and equations and then into a standard-maximum initial tableau.

A crafter builds chairs and desks. A desk takes 5 hours and 2 hardwood panels; chairs take 3 hours and 1 hardwood panel. He orders 10 panels and spends 22 hours a week on his craft. For storage reasons he must build at least 3 times as many chairs as desks. Desks can be sold for a profit of $400 and chairs for $100. Maximize his profit.