Calculus in 2 or more Variables

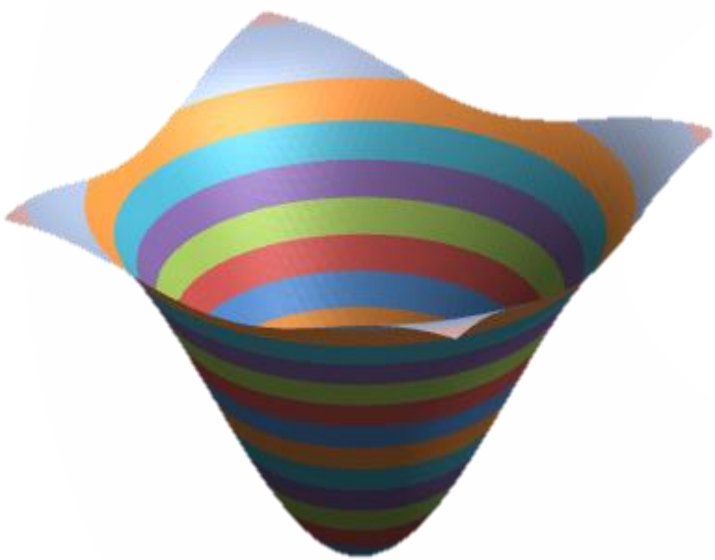
- graphical meaning
- calculating a gradient

First, a Note

While you're learning how to do these calculations for the first time, variables will be written as x, y, z instead of x_1, x_2, x_3 because it's a lot less confusing. However, be aware that in the “real world”, subscripts are more commonly used.

Graphical Meaning

Functions in 2 variables can be graphed in 3 dimensions:



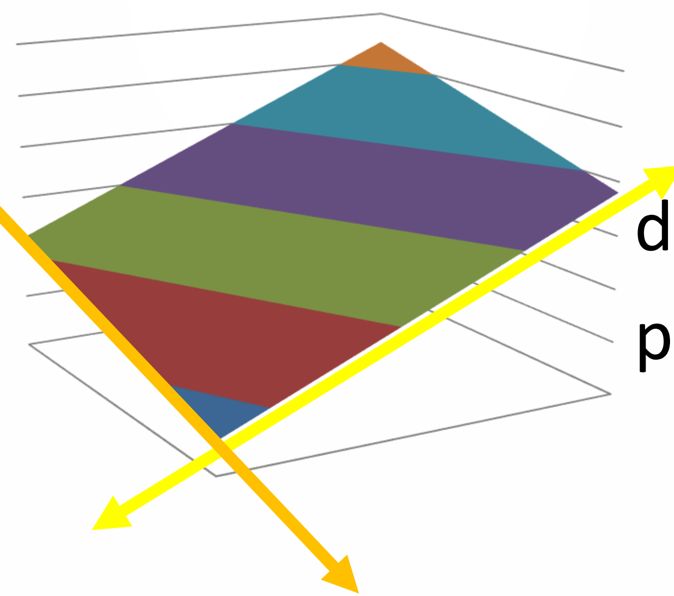
At each point on this graph, there is a slope; unlike calculus in 2 dimensions the slope is determined not by a line but by a plane.

Are there any locations on this graph where the slope appears to be zero? What does that mean?

Graphical Meaning

There are many ways to define the slope of a plane. One of them is to use a pair of slopes, z/x and z/y, where z is the vertical dimension and x and y are the two nonvertical dimensions.

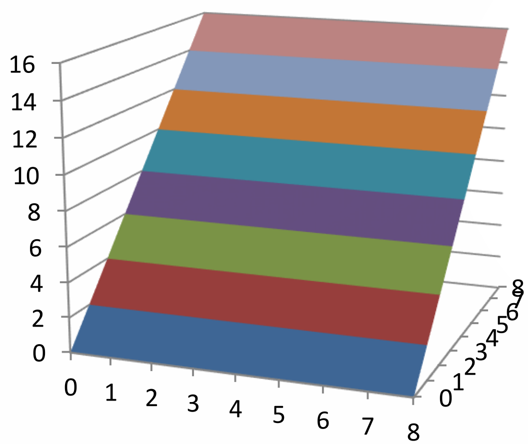
slope in this direction is negative, possibly $\frac{-1}{3}$

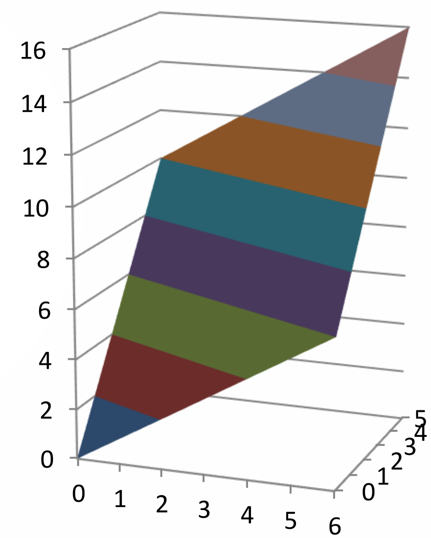


slope in this direction is positive, possibly $\frac{1}{2}$

Graphical Meaning

The graph below left shows a plane with a slope of 2 in one dimension and a slope of 0 in the other.

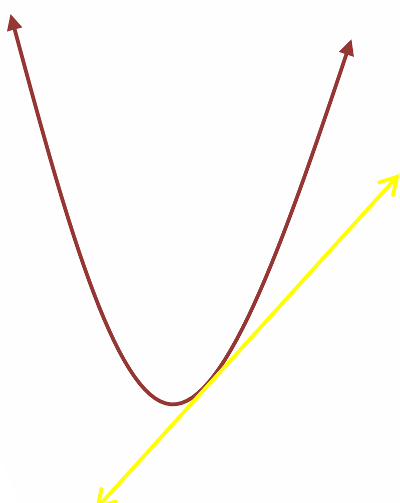




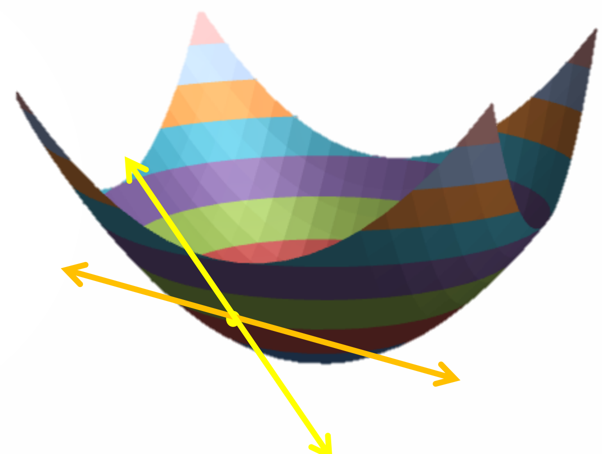
The graph above right shows a plane with a slope of 1 in one dimension and a slope of 2 in the other.

Graphical Meaning

So when we do calculus on 3D graphs, we're no longer looking for the slope of one line…



1-variable derivative



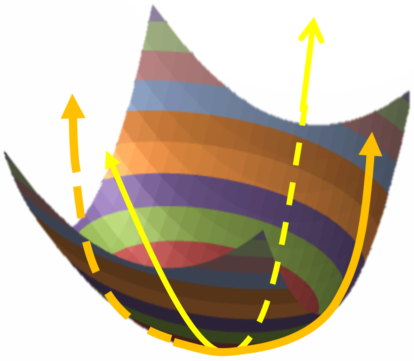
2-variable derivative

…we're looking for the slopes of two lines.

Finding the Gradient

The derivative in multiple dimensions is so different it has its own name: gradient.

When finding a gradient, we use a concept addressed in an earlier unit, of dealing with a 3D graph one cross-section at a time.



Finding the Gradient

The method we used in the second unit, called cyclic coordinate search, isolated each variable in turn by substituting constants for the other variables.

Then the function was minimized for the remaining variable and the process repeated.

Finding a gradient is pretty much the same.

Practice Problems 1 and 2

Consider the function

$$f(x, y, z) = 2x^2 + 3xy + 2z^2y - 4xyz$$

1. Rewrite this function substituting…

a) $x = 2, y = 4$

b) $x = -1, z = 3$

c) $y = -2, z = 100$

d) $x = 3, y = 2, z = -8$

2. Find the derivative of each function in 1a-d with respect to the remaining variable.

Practice Problem 3

Consider the function

$$f(x, y) = x^3 + 2x^2y - 3xy^2 + 8$$

Find the derivative with respect to x if…

a) $y = 2$ b) $y = 1$ c) $y = 100$

d) y is unknown, but a constant

Find the derivative with respect to y if…

e) $x = 4$ f) $x = -2$

g) x is unknown, but a constant

Finding the Gradient

When finding the gradient of a function in two variables, the procedure is:

1. Derive with respect to the first variable, treating the second as a constant
2. Derive with respect to the second variable, treating the first as a constant
3. Write the result as a vector $\begin{bmatrix} {}^{df}\!/_{dx} & {}^{df}\!/_{dy} \end{bmatrix}$

(These are called the partial derivatives of f.)

Finding the Gradient

So, if the function was $f(x, y) = x^2 + 3xy - y^3$,

- the derivative with respect to x (y is constant) is: $2x + 3y$
- the derivative with respect to y (x is constant) is: $3x - 3y^2$

Therefore the gradient is $\boxed{[2x + 3y \quad 3x - 3y^2]}$

Practice Problem 4

Find the gradient of:

a) $f(x, y) = 3x^2 + 4xy^2 - 2y + 7$

b) $f(x, y) = 7x^4 + 8y^3 - 3x^2y^2 + 12y$

c) $f(x, y) = x^2 + y^2$

d) $f(x, y) = 12xy$

Gradients Beyond 3D

The gradient in more variables is found in much the same way:

1. Derive with respect to the first variable, treating *all* others as constants
2. Repeat for all other variables
3. Write the resulting partial derivatives as a vector.

Practice Problem 5

a) Find the gradient of

$$f(x, y, z) = 2x^2 + 3xy + 2z^2y - 4xyz$$

b) Find the gradient of

$$f(x, y, z, w) = 2xy + 4yz - 3yz^2w$$