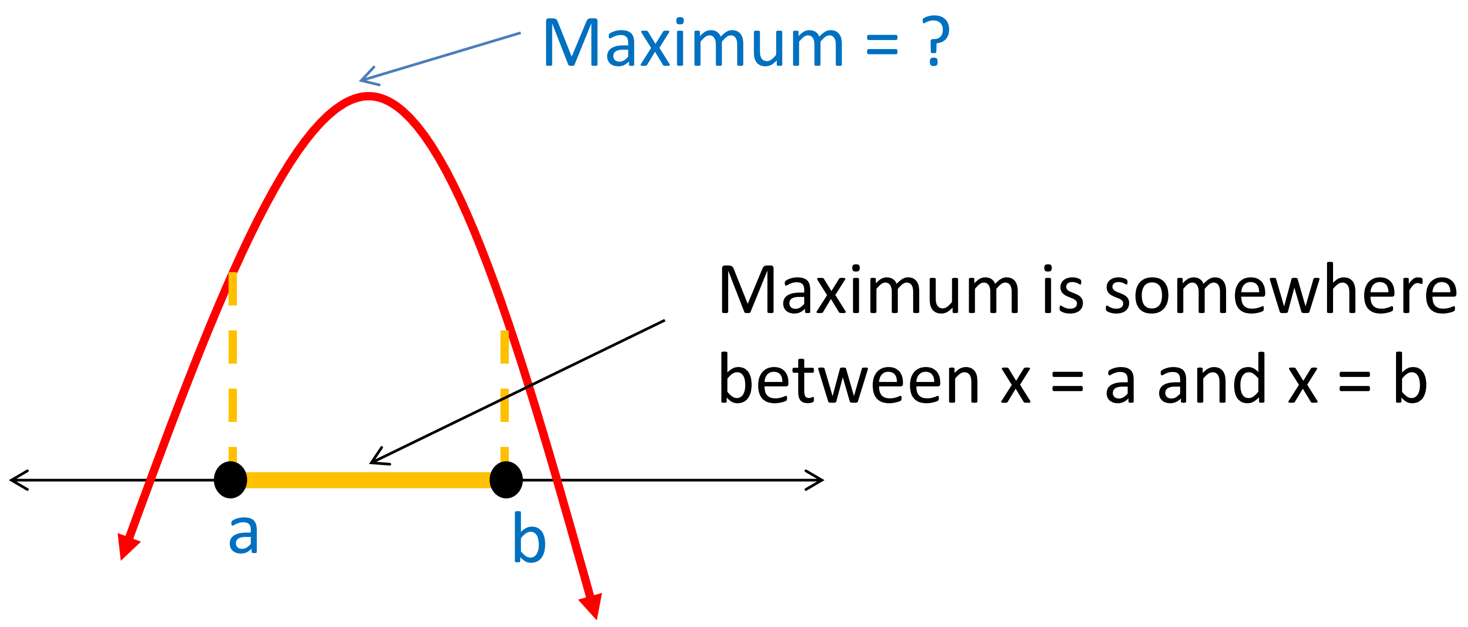
Iteration and Recursion 3

Finding a 3-Point Interval

The Premise

If your goal is to optimize (maximize or minimize) the value of an equation, one of the very first steps is to find an interval where that value might occur.



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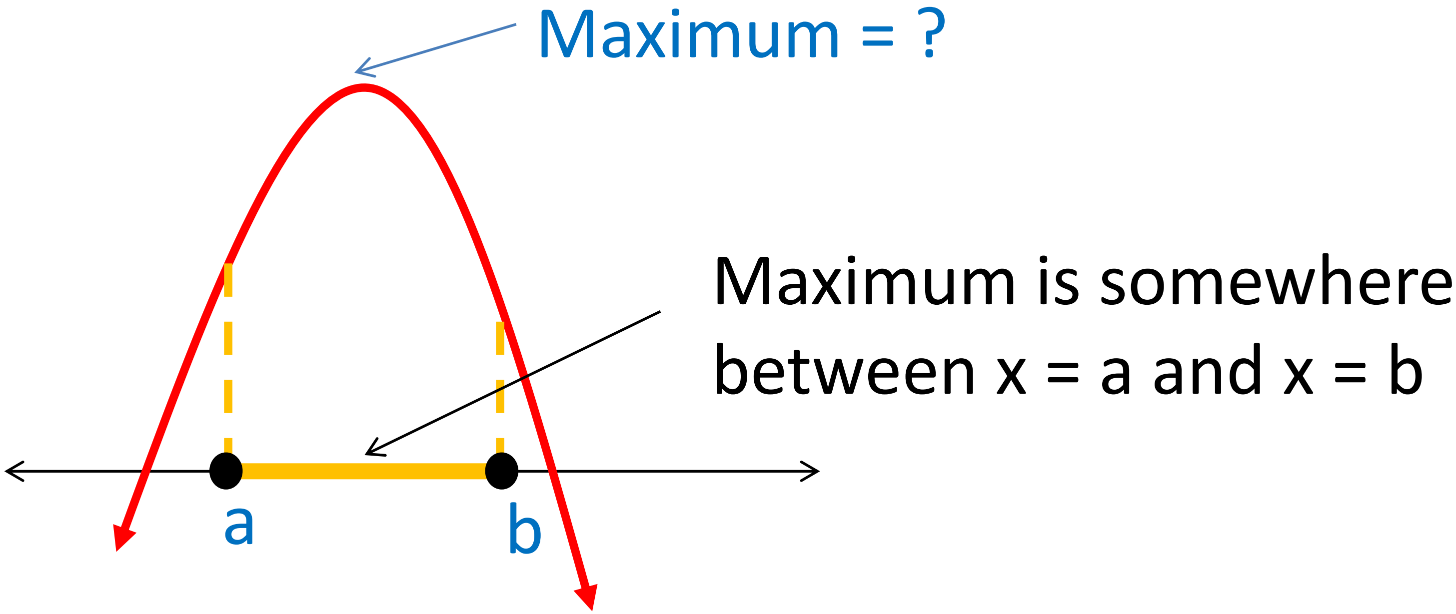
If your goal is to optimize (maximize or minimize) the value of an equation, one of the very first steps is to find an interval where that value might occur. This is obvious if you know the graph…



The Premise

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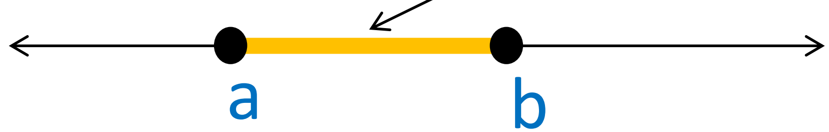
Maximum = ?

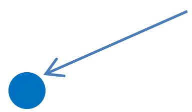
Maximum is somewhere between x = a and x = b



… but what if you don’t?

The first step

The first thing you do is find a point. Any point. Since you have no clues about the graph or its shape, you're flying blind. Pick your favorite number (π?) and plug it in.

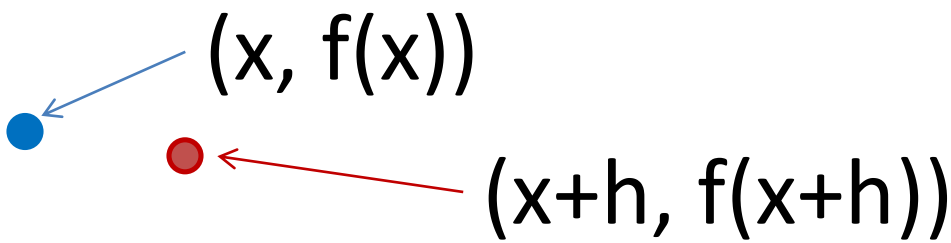


$(x, f(x))$

Notice that we still have no information about the shape of the graph.

The next step

Next, you find a second point pretty close to the first. You can do this by guessing a second number, but you can save steps later by just picking an interval value, which I'll call *h*. your second point will have an x-coordinate h units larger than the first.

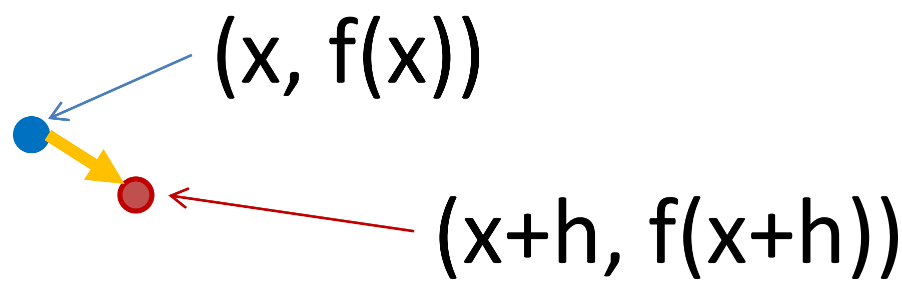


So if you chose x = 2 and h = 0.5, you'll have points at (2, f(2)) and (2.5, f(2.5)).

The Initial Test

Once you have the second point, you compare it to the first.

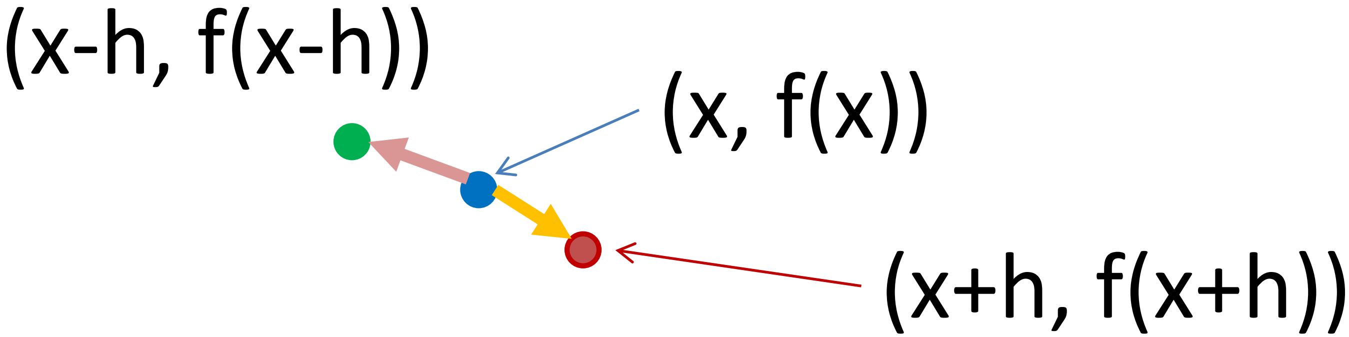
- If you're seeking a minimum, you would like your second point to be lower (smaller y) than the first.



- If you're seeking a maximum, you want your second point to be higher (larger y).

The Initial Test

If the second point fails the test, change your h to –h and try again with (x, f(x)) and (x-h, f(x-h)).



If the graph is decreasing to the right, it's usually increasing to the left, and vice-versa.

Practice Problem 1

1. For the given function $f(x)$, starting point x, and interval h, find the three points $(x, f(x))$, $(x+h, f(x+h))$, and $(x-h, f(x-h))$. What is the direction of increase (left or right)? What is the direction of decrease?

a. $f(x) = x^2 + 3$, $x = 4$, $h = 0.1$

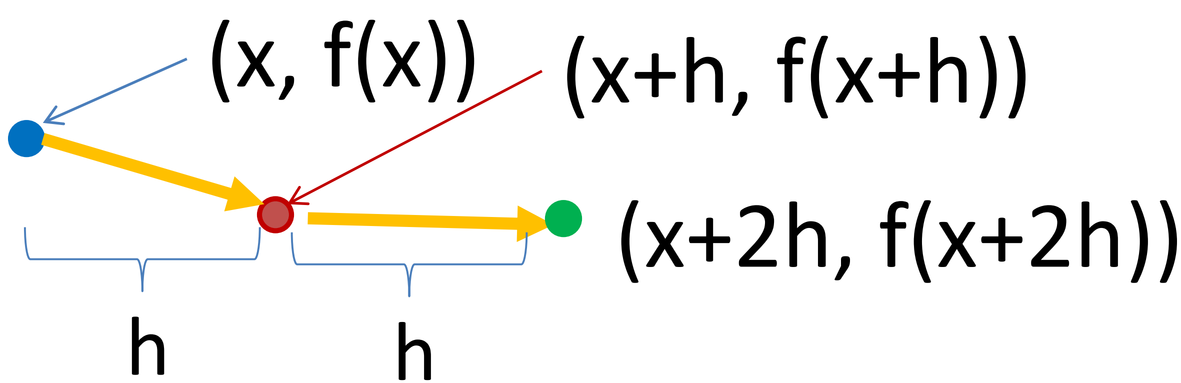
b. $f(x) = 2x^3 – 4x^2 + 17$, $x = 2$, $h = 1$

c. $f(x) = e^x – 4x + 2$, $x = 0$, $h = 0.5$

After the initial test

Once you know your direction, take another step in that direction using the same value of h.

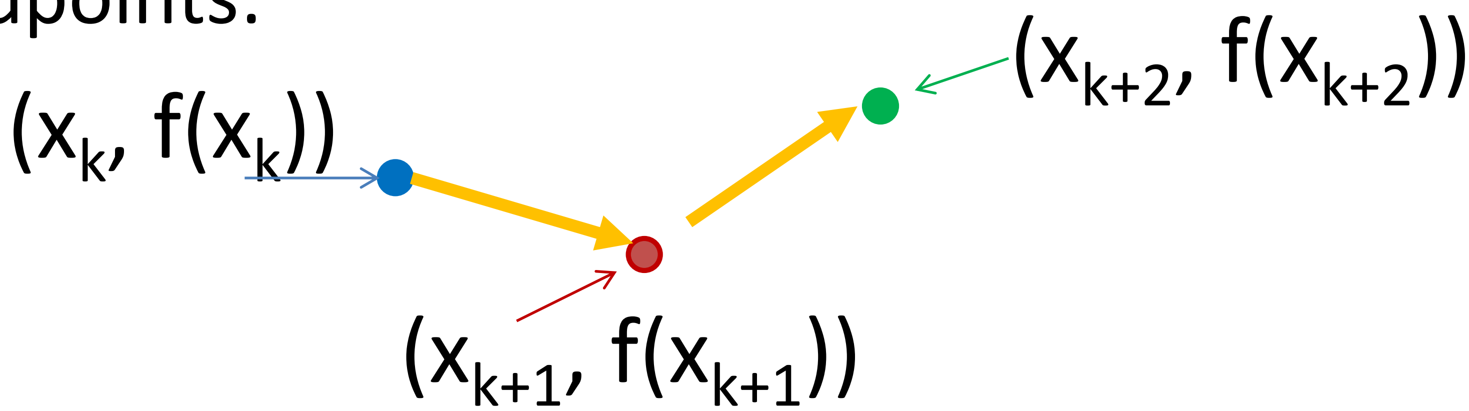
If you were minimizing a decreasing function it might look like this:



Now, you have 3 points.

The Goal

If you're looking for a minimum, your ultimate goal is to find an interval of three points such that the point in the middle is lower than both endpoints:



When you get this type of picture, you know there is a minimum somewhere in the interval between x_k and x_{k+2}.

The Loop

1. Start with (x, f(x)).
2. Find (x+h, f(x+h)) and check if it's going the right direction. If not, find (x-h, f(x-h)).
3. Take one more step h or –h.
4. Check if you have reached the goal (middle point higher/lower than both endpoints).
   – If yes, stop.
   – If no, return to step 3.

Practice Problem 2

2. For the given function f(x), starting point x, and interval h, use the steps to find a 3-point interval *without referencing a graph*:

a) $f(x) = x^2 - 4x$, $x = 0$, $h = 0.6$, interval contains the minimum

b) $f(x) = x^3 - 5x^2 + 3x + 2$, $x = 2$, $h = 0.5$, interval contains the maximum

The problem with h

In the practice problems, h was chosen for you on purpose, because determining h brings up a pervasive issue:

- If h is too big, your margin of error is large and you risk skipping over the maximum or minimum completely.
- If h is too small, the calculations become too numerous.

The Loop, Modified

1. Start with (x, f(x)).
2. Find (x+h, f(x+h)) and check if it's going the right direction. If not, find (x-h, f(x-h)).
3. Take one more step h or –h.
4. Check if you have reached the goal (middle point higher/lower than both endpoints).
   - If yes, stop.
   - If no, increase h and return to step 3.

The problem with h

One solution is to start with small values of h (like 0.1) and then increase h as the loop continues to iterate.

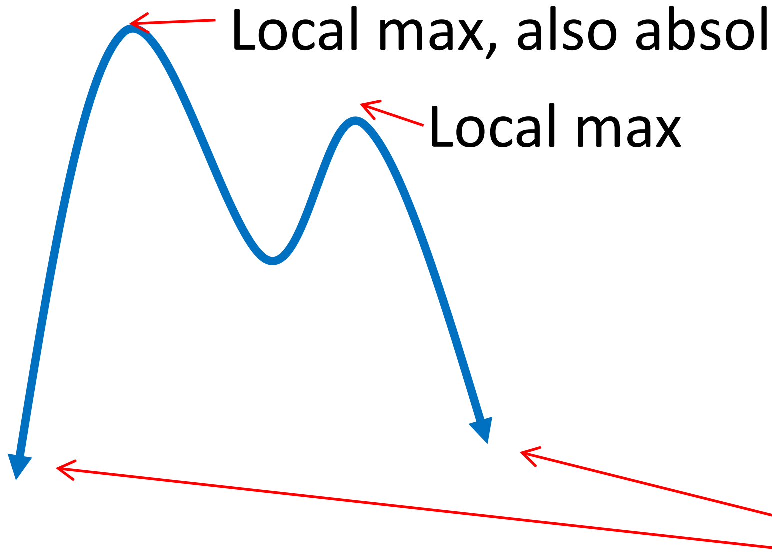
Doubling h is troublesome because it gets too big too fast: h, 2h, 4h, 8h, 16h, 32h, 64h, 128h…

Multiplying h by 1.5 makes ugly decimals: h, 1.5h, 2.25h, 3.375h, 5.0625h, 7.59375h…

So a good compromise is to use Fibonacci numbers: h, h, 2h, 3h, 5h, 8h, 13h, 21h…

Problems with the procedure

This procedure will only find a local minimum or maximum, not necessarily the absolute minimum or maximum.



Also, if the function is unbounded, you might not find an interval at all.

Problems with the procedure

There is no easy way to prevent the first error (local vs. absolute), but some advanced methods have been developed to do so.

For the second error (unboundedness), you can either introduce a boundary (like x must be between ___ and ___), or you can stop the process if you appear to be heading off towards infinity, and start again with a different x.

Practice Problems 3 and 4

Choose an x and h and use the iterative procedure to find a 3-point interval. Show your steps!

3. $f(x) = x^3 - 10x^2 - 400x + 4000$, interval contains a maximum.

4. $f(x) = e^x - 2x\sin x - 20x$, interval contains a minimum.

You may graph to *check* your work, but not to start it.