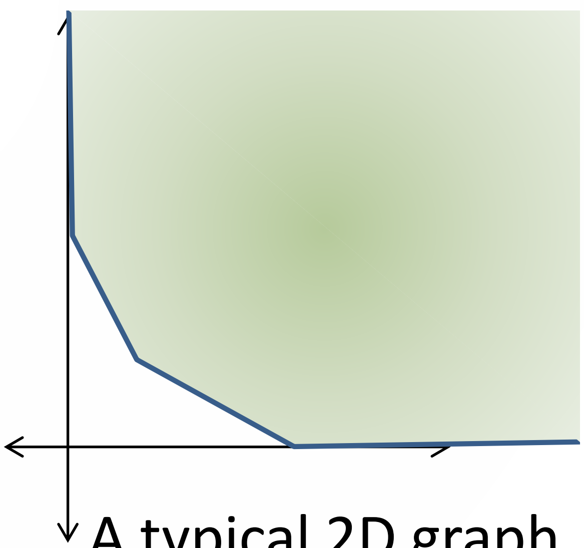
Minimization Using Duality

- Standard Minimization Problem
- Building the Dual Tableau
- Solving Standard Minimization

Standard Minimization Form

Standard minimization form involves a minimization problem where all constraints are ≥, for example:

$$\begin{aligned} \text{minimize} \quad & 3x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \geq 10 \\ & 2x_1 + 2x_2 \geq 8 \end{aligned}$$



A typical 2D graph

Standard Minimization Form

If we converted a standard minimization problem straight into a tableau it would look like this:

$$\underbrace{\begin{matrix} 1 & 3 \\ 2 & 2 \\ 3 & 4 \end{matrix}}\begin{matrix} -1 & 0 & 0 & 10 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & 1 & 0 \end{matrix}$$

"minimize $f = 3x_1 + 4x_2$" becomes "maximize $f = -3x_1 - 4x_2$", which converts to $3x_1 + 4x_2 + f = 0$

Constraints with ≥ mean we need to subtract slack variables, not add them

This tableau has all sorts of things wrong with it, including negative solutions which are illegal.

Duality

Fortunately, a standard minimization problem can be converted into a maximization problem with the same solution. The minimization problem and its corresponding maximization problem are called *duals* of each other.

The steps for using duality in the simplex method do not make much sense, but the method works.

Duality

The first step in solving a standard minimization problem using duality is to write the information into a matrix, ignoring everything you know about slack variables and objective functions.

minimize $f = 3x_1 + 4x_2$

subject to $x_1 + 3x_2 \geq 10$

$2x_1 + 2x_2 \geq 8$

$$\begin{array}{ccc} x_1 & x_2 & \text{ans} \end{array}$$

$$\begin{bmatrix} 1 & 3 & 10 \\ 2 & 2 & 8 \\ 3 & 4 & 0 \end{bmatrix}$$

Duality

The next step is to create the dual matrix, which starts with the *transpose* of the matrix we just created. "Transpose" means the first column becomes the first row, and so on:

$$\begin{bmatrix} 1 & 3 & 10 \\ 2 & 2 & 8 \\ 3 & 4 & 0 \end{bmatrix} \text{ becomes } \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 10 & 8 & 0 \end{bmatrix}$$

Duality

The next step is where it gets strange. Using the transposed matrix, fill it in as if it were a standard maximization matrix:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 0 & 1 & 0 & 4 \\ -10 & -8 & 0 & 0 & 1 & 0 \end{array}\right]$$

Then maximize as you normally would.

Duality

After pivoting twice, the finished matrix looks like this:

$$\begin{bmatrix} 0 & 4 & 3 & -1 & 0 & 5 \\ 12 & 0 & -6 & -6 & 0 & 6 \\ 0 & 0 & 12 & 36 & 12 & 180 \end{bmatrix}$$

The last step is to divide the last row so the “f” column becomes 1. Your new last row is:

$$[\underbrace{0 \qquad 0}_{\substack{\text{slack variables} \\ \text{1 and 2}}} \qquad \underset{x_1}{1} \qquad \underset{x_2}{3} \qquad \underset{1f\ =\ \text{solution}}{1} \qquad 15]$$

From this you can read the solution directly.

The minimum of 15 occurs at $x_1 = 1$, $x_2 = 3$.

Practice Problem 1

Convert these constraints into a dual matrix in standard maximum form:

Minimize $4x_1 + 2x_2 + 5x_3$

subject to $3x_1 + x_2 + 5x_3 \geq 15$

$2x_1 + 4x_2 + 2x_3 \geq 20$

Practice Problem 2

Using your pivoting program for standard maximum simplex, complete the steps to solve your tableau from problem 1.

2a. Write the final last row.

2b. Identify the values of: the slack variables; x_1, x_2, and x_3; and the minimized objective function.

Practice Problem 3

An office manager is equipping a new workspace with storage units. Traditional shelves can hold 9 cubic feet of material and provide 3 square feet of work area on top. Deep file cabinets can hold 12 cubic feet of material and provide 6 square feet of work area on top. The manager needs at least 50 cubic feet of storage and 36 square feet of work area. File cabinets cost $100 each and shelves $70. Minimize cost. Write a full answer in context.