Non-Standard Constraints

- Setting up
- Phase 1
- Phase 2

Maximizing with Mixed Constraints

Some maximization problems contain mixed constraints, like this:

maximize	$3x_1 + 2x_2$	
subject to	$2x_1 + x_2 \leq 50$	(standard)
	$x_1 + 3x_2 \geq 15$	(greater than)
	$5x_1 + 6x_2 = 60$	(equality)

These can come in any combination, but in a typical problem more constraints are standard than nonstandard.

The Problem with Mixed Constraints

The first thing that needs to be done is to convert all constraints into equality constraints:

$2x_1 + x_2 \leq 50$	becomes	$2x_1 + x_2 + x_3 = 50$
$x_1 + 3x_2 \geq 15$	becomes	$x_1 + 3x_2 - x_4 = 15$
$5x_1 + 6x_2 = 60$	remains	$5x_1 + 6x_2 = 60$

But this causes some problems in the initial tableau, as shown on the next slide:

The Problem with Mixed Constraints

If written directly into a tableau, the modified constraints and objective function create:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & f & \text{ans} \end{array}$$

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 50 \\ 1 & 3 & 0 & -1 & 0 & 15 \\ 5 & 6 & 0 & 0 & 0 & 60 \\ -3 & -2 & 0 & 0 & 1 & 0 \end{bmatrix}$$

this solution is negative, therefore illegal

this row has no solution in it at all!

Fixing the Problem

To fix both problem rows, we need to add yet another variable to each one. It looks like this:

$x_1 + 3x_2 \geq 15$ was $x_1 + 3x_2 - x_4 = 15$;

now becomes $x_1 + 3x_2 - x_4 + x_5 = 15$

$5x_1 + 6x_2 = 60$ did not change;

now becomes $5x_1 + 6x_2 + x_6 = 0$

The variables x_5 and x_6 are called "artificial" variables because their value must equal zero. It is important to keep track of them in the tableau.

Fixing the Problem

When you put the new, improved constraints into the tableau it will look like this:

$$
\begin{array}{c}
\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5{}^* & x_6{}^* & f & \text{ans} \end{array} \\
\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 1 & 3 & 0 & -1 & 1 & 0 & 0 & 15 \\ 5 & 6 & 0 & 0 & 0 & 1 & 0 & 60 \end{bmatrix}
\end{array}
$$

If you switch these columns you can see the problem is (mostly) fixed!

Unfortunately, we created a new problem, because by definition x_5 and x_6 must be zero, and right now they're not. Plus, x_4 is still illegal.

Fixing the Problem

The first thing we have to do is minimize x_5 and x_6; their minimum value will be zero, which is what we want. We can do this by introducing a new, temporary objective function:

$$\text{minimize } g = x_5 + x_6$$

(note: same as maximize $g = -x_5 - x_6$)

This creates the tableau:

$x_5 + x_6 + g = 0$

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5^* & x_6^* & g & \text{ans} \end{array}$$
$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 1 & 3 & 0 & -1 & 1 & 0 & 0 & 15 \\ 5 & 6 & 0 & 0 & 0 & 1 & 0 & 60 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and new problems: no negative indicators.

Fixing the Problem

For the last fix, we need to replace the objective function row. To do that, we will need to find the sum of both problem rows and subtract that from the objective row:

from

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5{}^* & x_6{}^* & g & \text{ans} \end{array}$$

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 1 & 3 & 0 & -1 & 1 & 0 & 0 & 15 \\ 5 & 6 & 0 & 0 & 0 & 1 & 0 & 60 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

to

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 1 & 3 & 0 & -1 & 1 & 0 & 0 & 15 \\ 5 & 6 & 0 & 0 & 0 & 1 & 0 & 60 \\ -6 & -9 & 0 & 1 & 0 & 0 & 1 & -75 \end{bmatrix}$$

NO PROBLEMS!!

(maybe)

Fixing the Problem

In fact, the negative in the "answer" column is appropriate. Because we're minimizing by maximizing a negative, we should expect a negative value for our answer.

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 1 & 3 & 0 & -1 & 1^* & 0 & 0 & 15 \\ 5 & 6 & 0 & 0 & 0 & 1^* & 0 & 60 \\ -6 & -9 & 0 & 1 & 0 & 0 & 1 & -75 \end{bmatrix}$$

So, there are really no problems and we can finally begin. Phase 1 involves pivoting to solve the initial matrix, just like normal. Here our pivot will be at [2, 2].

*artificial variables

Summary of Setup Steps

The steps for setting up, in order:

1. Convert inequality constraints to equality constraints.
2. Add artificial variables to any constraints with a subtracted slack variable, or no slack variable.
3. Write an objective function that minimizes your artificial variables (from step 2) and put it all in a tableau.
4. Replace the last row of the tableau by adding all the problem rows and subtracting that from the last row.

Practice Problem 1

Convert this problem into a proper initial tableau.

Maximize $f = 10x_1 + 20x_2$

subject to $4x_1 + 5x_2 = 240$

$2x_1 + x_2 \leq 90$

$x_1 + 3x_2 \geq 120$

Note which variables are artificial, with a circle or star or by using a different letter to label the column.

Phase 1

With nonstandard constraints, setting up the initial tableau is the hardest part. Once you have that, you can pivot as usual.

After pivoting twice from the initial tableau, we are done with phase 1 and ready to move on to phase 2:

$$\begin{bmatrix} 0 & 0 & 27 & -21 & 21 & -15 & 0 & 765 \\ 0 & 27 & 0 & -15 & 15^* & -3 & 0 & 45 \\ 9 & 0 & 0 & 6 & -6 & 3^* & 0 & 90 \\ 0 & 0 & 0 & 0 & 27 & 27 & 27 & 0 \end{bmatrix}$$

no negative indicators

artificial variables no longer in basis, therefore = 0

Phase 1

So what was that all about, anyway?

- The simplex method always starts at the origin.
- In standard problems, the origin is always a corner of the feasible region.
- In problems with nonstandard constraints (like = or ≥), the origin is typically *not* in the feasible region.
- So, the first step is to get to the feasible region so we can maximize from there.

This is what the procedure in phase 1 accomplished.

Practice Problem 2

Use your pivoting program to complete phase 1 on the tableau from practice problem 1.

Phase 2

After phase 1, we are in the feasible region, but we still need to maximize the original objective function. In the example it was $3x_1 + 2x_2$. From the final phase 1 matrix, delete…

$$\begin{bmatrix} 0 & 0 & 27 & -21 & \cancel{21} & \cancel{-15} & 0 & 765 \\ 0 & 27 & 0 & -15 & \cancel{15^*} & \cancel{-3} & 0 & 45 \\ 9 & 0 & 0 & 6 & \cancel{-6} & \cancel{3^*} & 0 & 90 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{27} & \cancel{27} & \cancel{27} & \cancel{0} \end{bmatrix}$$

- the entire bottom row
- both artificial variable columns

Phase 2

Next, fill in the original objective function into the bottom row:

$$\begin{bmatrix} 0 & 0 & 27 & -21 & 0 & 765 \\ 0 & 27 & 0 & -15 & 0 & 45 \\ 9 & 0 & 0 & 6 & 0 & 90 \\ \boxed{-3 & -2 & 0 & 0 & 1 & 0} \end{bmatrix}$$

Finally, pivot as usual from this tableau until all negative indicators are gone.

The solution to this problem is a maximum value of 33.33 when $x_1 = 10$, $x_2 = 1.67$, and $x_3 = 28.33$.

Practice Problem 3

Complete phase 2 on your matrix from practice problem 2; the original objective function was to maximize $10x_1 + 20x_2$.

Nonstandard Constraints: All the Steps

1. Modify all constraints by adding or subtracting slack variables and adding artificial variables as needed.
2. Set up a matrix to minimize the sum of the artificial variables. Subtract (bottom row) – (sum of all problem rows) to get the phase 1 matrix.
3. Pivot as usual to maximize the phase 1 matrix.
4. Replace the objective row with the actual objective function and cross off the artificial variable columns.
5. Maximize as usual.

Practice Problem 4

Complete this problem from start to finish:

Maximize $f = 15x_1 + 10x_2 + 20x_3$

subject to $2x_1 + 4x_2 + x_3 \leq 20$

$3x_1 + x_2 + 5x_3 \geq 10$

A Note About Minimization

The actual procedures for minimization with nonstandard constraints (≤ or =) are complex.

More commonly, the initial tableau is set up and various pivot candidates are tried, with the goal of reaching the feasible region. After the feasible region is attained (in other words, all illegal solutions leave the basis and no new ones enter), the tableau can be optimized from there.