Operations on Matrices

- Definition and Use
- Basic operations
- Multiplication
- Inverses

Definition of Matrix

A matrix is an array of objects, usually numbers. You used some in your work in Unit 2.

$$\begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

All of these are examples of matrices.

The first two are "square", the first 2x2 and the second 3x3. The third one is 3x2, the fourth one a "row" matrix at 1x3, and the fifth one a "column" matrix at 3x1.

Use of Matrices

Sometimes, matrices are just lists of numbers and a matrix is the best way to organize them.

More frequently, the numbers stand for something. You used matrices where the numbers stood for coordinate points; they can also stand for coefficients of equations, among other things.

Use of Matrices

For example, the system of equations

$$4x_1 - 5x_2 = 13$$

$$2x_1 + x_2 = 7$$

can be inserted into a matrix that looks like this:

$$\left[\begin{array}{cc|c} 4 & -5 & 13 \\ 2 & 1 & 7 \end{array}\right]$$

In matrix (array) form, the system of equations is much easier for a computer to work with.

Basic Operations on Matrices

- Matrices that are the same size can be added or subtracted:

$$\begin{bmatrix} 3 & 5 \\ 2 & 0 \\ 6 & -8 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 3 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -1 & -4 \\ 8 & -13 \end{bmatrix}$$

- A matrix can also be multiplied easily by a constant:

$$2 \cdot \begin{bmatrix} 3 & 1 & -2 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -4 \\ 4 & 10 & 0 \end{bmatrix}$$

Matrices in Julia

To enter the previous example into Julia, you could type in:

```
A = [3 1 -2; 2 5 0]
2*A
```

or just

```
2*[3 1 -2; 2 5 0]
```

Spaces separate the numbers across a row, and a semicolon marks the break between rows.

Practice Problem 1

1a. Calculate by hand:

$$3 \cdot \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 8 & 5 \end{bmatrix} - 4 \cdot \begin{bmatrix} -1 & 3 \\ 5 & 0 \end{bmatrix}$$

Then check your result using Julia.

1b. Insert this system of equations into a matrix:

$$3x_1 + 2x_2 - x_3 = 14$$

$$2x_1 - 2x_2 + 5x_3 = 22$$

$$-x_1 + x_2 - 2x_3 = -5$$

Multiplying Matrices

In contrast, multiplying two matrices together is not so easy. And, although size determines whether two matrices can be multiplied, they do not have to be the same size. Let’s look at:

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -3 & 5 \\ 1 & 7 \end{bmatrix}$$

which can, in fact, be multiplied.

Multiplying Matrices

Multiplying can be made easier by offsetting the first matrix below the second. You may not switch the order; multiplication does not commute (ie, it does not have the property that A · B = B · A).

$$\begin{array}{cc} & \begin{bmatrix} -3 & 5 \\ 1 & 7 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 0 & 5 \end{bmatrix} & \begin{bmatrix} & \\ & \\ & \end{bmatrix} \end{array}$$

The answer will go into the space between them and will be the same size as that space.

Multiplying Matrices

$$\begin{bmatrix} -3 & 5 \\ 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \text{-5} & \\ & \\ & \end{bmatrix}$$

Each space in the answer matrix is found using the corresponding row from the first matrix and column from the second.

To find the first space, you would multiply -3·2 and add 1·1, which is -6 + 1 = -5.

Multiplying Matrices

Using the same procedure you can find the rest of the values.

$$\begin{bmatrix} -3 & 5 \\ 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -5 & 17 \\ & \\ & \end{bmatrix}$$

Multiplying Matrices

Using the same procedure you can find the rest of the values.

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -3 & 5 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 17 \\ 7 & 23 \\ 5 & 35 \end{bmatrix}$$

the answer

Multiplying Matrices

Now take the same two matrices, but try multiplying them in reverse:

$$\begin{bmatrix} -3 & 5 \\ 1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 0 & 5 \end{bmatrix}$$

Multiplication is impossible because the dimensions don’t match properly.

And even if the dimensions do match, you will usually get different answers for AB and BA.

Practice Problem 2

Multiply the following by hand, then check using Julia:

2a. $\begin{bmatrix} 5 & 1 & 4 \\ -3 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ -3 & 8 \\ 0 & -4 \end{bmatrix}$

2b. $\begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -3 & 8 \end{bmatrix}$

2c. $\begin{bmatrix} 0 & 1 \\ -3 & 8 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix}$

2d. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix}$

2e. $\begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Practice Problem 3

3a. Multiply:

$$\begin{bmatrix} 4 & -2 & 1 \\ 3 & 0 & -1 \\ -2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3b. Then set it equal to $\begin{bmatrix} 21 \\ 12 \\ 14 \end{bmatrix}$

(Often, systems of equations are abbreviated AX = B where A, X, and B are all matrices as shown here.)

Inverses/Dividing

Because multiplication is so complex, dividing is not possible in the traditional way.

However, a square matrix can be multiplied by its *inverse* which mimics division, just like multiplying by ½ mimics dividing by 2.

Inverses/Dividing

The inverse of A is defined as a matrix such that $A \cdot A^{-1} = A^{-1} \cdot A = \mathtt{I}$, where $\mathtt{I}$ is a square matrix with 1's across the diagonal and 0's everywhere else.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix $\mathtt{I}$ is known as the *identity matrix* because $\mathtt{I}A = A\mathtt{I} = A$ for all square matrices A. It works much like the number "1" in multiplication.

Inverses/Dividing

The two important properties so far:

$$\boxed{A^{-1}\cdot A = A\cdot A^{-1} = I \qquad\qquad I\cdot A = A\cdot I = A}$$

We can use these to “divide”, as follows:

example: AX = B, solve for X.

$\underbrace{A^{-1}\cdot A}X = A^{-1}\cdot B$ (note that the order matters here: can’t use $B\cdot A^{-1}$)

$\underbrace{I\cdot X} = A^{-1}\cdot B$

$X = A^{-1}\cdot B$

Solved!

Inverses/Dividing

Inverses can be found by hand, but that goes beyond the scope of this class. For now, all you need to know is:

```
inv(A)
```

which is the Julia command for the inverse of A.

Practice Problem 4

Find the inverse of…

4a. $\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 4 \\ -1 & 1 & 5 \end{bmatrix}$

4b. $\begin{bmatrix} 4 & 2 \\ 10 & 5 \end{bmatrix}$

4c. Find the solution to this system of equations using inverses:

$$2x_1 + x_2 - 3x_3 + x_4 = 12$$
$$x_1 - 2x_2 - 6x_4 = -28$$
$$-3x_1 + 2x_2 - x_3 + 3x_4 = 10$$
$$-x_1 + x_3 - 2x_4 = -13$$