Higher-Order Forward Guidance*

Marc Dordal i Carreras† Seung Joo Lee‡

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Abstract

This paper provides a model of the business cycle that incorporates financial markets and endogenously generates periods of increased financial uncertainty at the Zero Lower Bound (ZLB). Within this framework, forward guidance is identified as a crucial mechanism for coordinating the actions of market participants, guiding the economy towards optimal equilibrium paths with lower financial volatility and enhanced welfare. Our research reveals two significant insights: (i) Central banks, by credibly pledging future economic stabilization, can mitigate excess financial market volatility at the ZLB; (ii) Alternatively, a central bank's commitment not to stabilize the economy in the future can direct the economy towards more favorable equilibrium paths with reduced endogenous volatility at the ZLB, thus presenting an interesting trade-off between future stability and current financial uncertainty. Finally, an examination of alternative fiscal policies reveals that measures aimed at encouraging increased investment in risky assets can stimulate economic activity at the ZLB by positively impacting aggregate household financial wealth.

Keywords: Monetary Policy, Forward Guidance, Macroprudential Policy, Financial Volatility, Risk-Premium

JEL Codes: E32, E43, E44, E52, E62

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†Hong Kong University of Science and Technology (marcdordal@ust.hk).

‡Saïd Business School, Oxford University (seung.lee@sbs.ox.ac.uk).

1 Introduction

In the aftermath of the Great Recession and the recent Covid-19 pandemic, prolonged periods of constrained policy rates at the Zero Lower Bound (ZLB) have underscored the need for alternative monetary interventions, notably forward guidance. ZLB episodes are often characterized by heightened economic and financial uncertainty, exacerbated by the reduced efficacy of conventional monetary policy tools. In this context, forward guidance goes beyond its traditional roles of conveying economic forecasts (delphic guidance) and making policy commitments (odyssean guidance), and evolves into a tool for coordinating market participant actions and reducing overall economic uncertainty. This paper provides an analytically tractable framework for examining the effects of unconventional policies at the ZLB, and investigates the impact of central bank forward guidance on economic uncertainty and welfare.

Our paper builds on the model proposed by Lee and Dordal i Carreras (2023), which integrates endogenous financial risk within a New Keynesian framework. This model features a representative stock market index that encapsulates the ownership rights to the profits of firms in the economy. A group of hand-to-mouth workers supplies labor to these firms, while a group of capitalists holds the aggregate financial wealth of the economy, allocating it between consumption and financial investment. In equilibrium, the wealth of these capitalists is directly related with the stock market's performance. As a result, a decline in aggregate demand, leading to reduced firm profitability, can negatively impact both the stock market capitalization and the aggregate wealth of capitalists. This dynamic creates a coordination challenge for economic agents and can lead to self-fulfilling demand shocks, resulting in an endogenous state of elevated financial risk. While Lee and Dordal i Carreras (2023) investigates the determinacy of the model's solutions under conventional monetary policy regimes, the present paper focuses on whether central bank forward guidance can steer agents towards equilibrium paths with lower financial uncertainty and quicker economic stabilization times.

Our analysis begins by exploring whether financial instability intensifies when conventional monetary policy is constrained by the ZLB. We discover that a credible commitment from the central bank to stabilize the economy *after* the ZLB period can also ensure financial stability *during* the ZLB. This conclusion is derived through backward induction: if the central bank credibly commits to stabilize the economy in a finite period of time, it rules out the possibility of catastrophic (or exuberant) scenarios that contribute to the economic uncertainty faced by the agents of the model. As a result, this precludes the feasibility of the unfavorable coordination equilibrium paths that would initially lead to these scenarios.

The paper then proceeds to analyze the benefits of various forward guidance strategies. Within our framework, traditional forward guidance is defined as possessing an Odyssean component, where the central bank can credibly commit to maintaining the policy rate at zero for an extended period of time beyond the minimum dictated by economic conditions. Following this is a policy rule consistent with perfect stabilization outside the ZLB, as assumed in the previous paragraph. The outcomes of this strategy align with those identified in the prior research: by committing to a future period of accommodative policy rates, the central bank implicitly agrees to a temporary phase of positive excess demand and profits. This effect, owing to the forward-looking nature of stock markets, positively influences stock values at present, thereby raising aggregate demand at the ZLB. Such an approach spreads the costs of the ZLB over time, and is preferred when considering the quadratic welfare costs of fluctuations in the output gap. In addition, the commitment to perfect stabilization in the future precludes the appearance of excess financial volatility at the ZLB, as previously discussed.

The next strategy we consider explicitly leverages the agents' coordination problem to direct them towards an equilibrium with reduced financial and economic volatility at the ZLB. This approach is termed the higher-order forward guidance strategy. For its execution, the central bank must relinquish the promise of perfect stabilization in the future: by not committing to enforce perfect stabilization at the conclusion of the Odyssean guidance period, the central bank makes possible the existence of coordinated equilibriums that were previously ruled out by backward induction. This strategy allows the central bank to guide agents towards equilibrium paths with low uncertainty, thereby maximizing expected welfare beyond the capabilities of traditional forward guidance (which we identify as a limiting case of this strategy). However, this intervention has its trade-offs: by not committing to stabilize the economy after the ZLB period, the central bank risks significant future output gap deviations. Thus, higher-order guidance weighs the uncertainty of future equilibrium outcomes against reduced financial volatility in the present.

Finally, we analyze two macroprudential policies at the ZLB designed to incentivize investors to assume more financial risk, thereby raising asset prices and aggregate demand: (i) a subsidy on risky asset investments (or equivalently, a reduction in capital gains taxes), and (ii) fiscal redistribution among agents. The first policy illustrates that a temporary subsidy on holding risky assets at the ZLB enhances their Sharpe ratio, leading to higher asset prices and increased aggregate financial wealth of the economy. This surge in financial wealth boosts the aggregate demand of capitalists and alleviates the severity of recessions, as well as the welfare costs associated with the ZLB. However, our study emphasizes the need to consider the varying marginal propensities to consume across households when selecting the optimal funding sources for the subsidy. In a hypothetical scenario where the subsidy is financed by non-distortionary taxation on hand-to-mouth workers, the policy's effectiveness is completely nullified: the increase in financial wealth and aggregate demand induced by the subsidy is exactly offset by a reduction in workers' consumption, which negatively impacts firm profitability and stock market capitalization. In this context, the second macroprudential policy examined focuses on the effects of fiscal transfers at the ZLB from capitalists with a low marginal propensity to consume to workers with a high marginal propensity to consume. As expected, this transfer leads to an increase in the economy's aggregate demand. Our study contributes to the existing literature by showing that such redistribution fosters increased demand through another channel: the initial rise in demand from workers' consumption boosts firm profitability, which in turn increases the financial wealth of capitalists and their willingness to invest in risky assets.

Featuring a demand-driven economy with perfectly rigid prices,[1] our framework emphasizes the significant impact of stock market performance on aggregate demand. Unlike prior studies, such as Akerlof and Yellen (1985), Blanchard and Kiyotaki (1987), Eggertsson and Krugman (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), and Schmitt-Grohé and Uribe (2016), who focus on demand-driven recessions due to deleveraging borrowers and aggregate demand externalities, our model triggers the ZLB episodes with a decrease in aggregate demand for risky assets, identified as a key driver of financial recessions by Caballero and Farhi (2017) and Caballero and Simsek (2020). Following Werning (2012), we assume that the economy's shift to the ZLB results from an exogenous shock that raises the risk premium in financial markets and reduces the demand for risky assets, resulting in a downward jump in the natural rate of interest to a negative territory. Our approach diverges from the literature by including an endogenous component to financial volatility, influenced by both the ZLB and forward guidance. Papers including Eggertsson et al. (2003), Campbell et al. (2012, 2019), Del Negro et al. (2013), McKay et al. (2016), and Caballero and Farhi (2017) explore the implications of forward guidance at the ZLB from both theoretical and empirical perspectives. Our research distinguishes itself by

[1] This assumption simplifies the analysis. An extended model with sticky prices à la Calvo (1983) produces qualitatively similar results.

focusing specifically on the impact of forward guidance on higher-order moments including the endogenous volatility of financial markets and the broader economy.[2] In addition, our study of macroprudential policies at the ZLB, while building on existing literature, e.g., Lorenzoni (2008), Farhi and Werning (2012, 2016, 2017), and Korinek and Simsek (2016), places a stronger emphasis on the interplay between asset prices and aggregate demand.

The structure of this paper is organized as follows: Section 2 presents the model. Section 3 examines the effectiveness of various forward guidance strategies and other macroprudential policies at the ZLB. Section 4 provides concluding remarks.

2 The Model

We start by presenting a theoretical framework adapted from Lee and Dordal i Carreras (2023) that allows for the analysis of higher-order moments associated with aggregate financial volatility and the practical implications for monetary and fiscal policy.

2.1 Setting

We consider a continuous-time framework within a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$. The economy is composed of two equally sized agent groups: capitalists, characterized as neoclassical agents, and hand-to-mouth workers, conceptualized as Keynesian agents. This structure, closely aligned with the approach of Greenwald et al. (2014), assumes that all financial wealth is held by capitalists, while workers rely on labor income for consumption. The aggregate technology, denoted by A_t, introduces a single source of exogenous variation in the model and generates the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. The process evolves according to a geometric Brownian motion given by:

$$\frac{dA_t}{A_t} = \underbrace{g}_{\text{Growth}} dt + \underbrace{\sigma_t}_{\text{Fundamental risk}} dZ_t \ ,$$

where g represents the expected growth rate, and σ_t signifies the economy's *fundamental* risk, which we take as exogenous. For simplicity, σ_t is initially assumed constant ($\sigma_t = \sigma$) in Section 2. Later, in Section 3, we introduce a deterministic shift in σ_t to explore various scenarios involving the ZLB on nominal interest rates.

[2] Our approach, where central bank communications serve as an equilibrium coordination device, aligns well with the concept of 'open-mouth' operations at the ZLB described by Campbell and Weber (2019).

2.1.1 Firms and Workers

The economy features a unit measure of monopolistically competitive firms, each producing a unique intermediate good $y_t(i)$, for $i \in [0, 1]$. These intermediate firms contribute to the final good y_t through a Dixit-Stiglitz aggregation function with a substitution elasticity $\epsilon > 0$, as shown by:

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} .$$

Each intermediate firm i employs a production function $y_t(i) = A_t(N_{W,t})^{\alpha} n_t(i)^{1-\alpha}$, where $N_{W,t}$ is the total labor in the economy, and $n_t(i)$ is the labor demand of firm i at time t. The inclusion of a production externality à la Baxter and King (1991) helps to align our model with observed asset price and wage co-movements, and does not alter other qualitative outcomes of our model.[3]

Intermediate firms face a downward-sloping demand curve $y_i(p_t(i)|p_t, y_t)$, with $p_t(i)$ representing the price of their own good, and p_t, y_t the aggregate price index and output, respectively:

$$y_i(p_t(i)|p_t, y_t) = y_t \left(\frac{p_t(i)}{p_t} \right)^{-\epsilon} .$$

The price index p_t aggregates prices $\{p_t(i)\}$ from all intermediate goods as $p_t = \left(\int_0^1 p_t(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$. For tractability, we assume perfect price rigidity; thus, $p_t(i) = p_t = \bar{p}$ for all t, i.[4] Thus, each firm produces an equal level of output $y_t(i) = y_t$ for all i, determined by demand.

A representative hand-to-mouth worker supplies labor to these firms, earning wage income $w_t N_{W,t}$ and spending it entirely on final good consumption. The worker maximizes:

$$\max_{C_{W,t}, N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t} \right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} , \quad \text{s.t.} \quad \bar{p} C_{W,t} = w_t N_{W,t} , \tag{1}$$

[3] In a model without Baxter and King (1991) externalities, increasing asset prices often correlate with lower wages, which is contrary to the empirical evidence (Chodorow-Reich et al., 2021) regarding the effects of stock price hikes on aggregate demand, employment, and wages. The Baxter and King (1991) externality enables our calibration to reflect these empirical trends by linking higher asset prices and aggregate demand with increased labor demand and wages.

[4] Lee and Dordal i Carreras (2023) introduces sticky price-resetting following Calvo (1983), which does not significantly alter model dynamics.

where $C_{W,t}$, $N_{W,t}$, and w_t are his consumption, labor supply, and wage, respectively, with χ_0 being the inverse Frisch elasticity of labor supply.[5] Under our rigid price assumption, equilibrium labor demand by each firm i, $\{n_t(i)\}$, aggregates linearly into total labor $N_{W,t}$, resulting in $n_t(i) = N_{W,t}$ for all i. Plugging this back into the production function, equilibrium output y_t simplifies to a linear function of total labor, $y_t = A_t N_{W,t}$.

2.1.2 Financial Market and Capitalists

Unlike conventional New-Keynesian models where a representative household owns the economy's firms and receives lump-sum rebated profits, we assume firm profits are capitalized in the stock market through a representative index fund. Capitalists are faced with an optimal portfolio allocation problem, deciding between investing in a risk-free bond and the stock index at each moment t.

The aggregate nominal financial wealth of the economy is represented by $\bar{p}A_tQ_t$, where Q_t is the normalized real index price. This price is endogenously determined and adapts to filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$, following the equation:

$$\frac{dQ_t}{Q_t} = \mu_t^q dt + \underbrace{\sigma_t^q}_{\substack{\text{Financial}\\ \text{volatility}}} dZ_t \ ,$$

with μ_t^q and σ_t^q representing the endogenous drift and volatility of the process, respectively. We interpret σ_t^q as a measure of financial uncertainty or disruption. Therefore, aggregate financial wealth $\bar{p}A_tQ_t$ evolves according to a geometric Brownian motion, characterized by a combined volatility of $\sigma + \sigma_t^q$. Notably, σ_t^q, determined in equilibrium, can be either positive or negative, indicating that aggregate real wealth A_tQ_t might be less volatile than the TFP process, $\{A_t\}$.

Alongside the stock market, we introduce a risk-free bond with a nominal interest rate i_t, set by the central bank. Bonds are assumed to be in zero net supply in equilibrium. A unit measure of identical capitalists decides how to allocate their wealth between this risk-free bond and the risky stock index. By holding the later, capitalists earn the profits from the intermediate goods sector, which are distributed as stock dividends, and benefit from stock price revaluations due to changes in A_t and Q_t. Given the competitive nature of financial markets, each capitalist takes the nominal risk-free rate i_t, the expected stochastic stock

[5] Consumption $C_{W,t}$ is normalized by the aggregate TFP A_t due to trending economic growth, a standardization that does not affect our model's qualitative results.

market return i_t^m, and the total risk level $\sigma + \sigma_t^q$ as given when making portfolio decisions.[6] If a capitalist invests a fraction θ_t of their wealth a_t in the stock market, the total risk borne is $\theta_t a_t(\sigma + \sigma_t^q)$ over the interval $[t, t + dt]$. Thus, the portfolio's riskiness is directly proportional to the investment share θ_t in the index. Capitalists, being risk-averse, demand a risk-premium compensation $i_t^m - i_t$ for investing in the risky index, which is determined in equilibrium. Each capitalist, with nominal wealth a_t and exhibiting log-utility, solves the following problem:

$$\begin{aligned} &\max_{C_t, \theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \ , \\ \text{s.t. } \ & da_t = \left(a_t \left(i_t + \theta_t (i_t^m - i_t)\right) - \bar{p} C_t\right) dt + \theta_t a_t \left(\sigma + \sigma_t^q\right) dZ_t \ , \end{aligned} \tag{2}$$

where ρ and C_t denote the subjective discount rate and final good consumption of capitalists, respectively. At each instant, the capitalist earns returns from both risk-free bond and risky stock investments, allocating their income towards consumption of the final good.

2.2 Equilibrium and Asset Pricing

Due to the logarithmic utility function of capitalists, their nominal state price density, denoted as ξ_t^N,[7] can be expressed as follows:

$$\xi_t^N = e^{-\rho t} \frac{1}{C_t} \frac{1}{\bar{p}} \ , \quad \text{where } \ \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right) = -i_t dt \ , \tag{3}$$

where the stochastic discount factor between the current time t and a future time s is defined as $\frac{\xi_s^N}{\xi_t^N}$. The aggregate stock market wealth, $\bar{p} A_t Q_t$, is defined as the sum of discounted profit streams from the intermediate goods sector, priced using ξ_t^N, under the assumption that capitalists are the marginal investors in the stock market in equilibrium.

At time t, the total profit of the intermediate goods sector, denoted as D_t, is given by

$$D_t \equiv \bar{p} y_t - \underbrace{w_t N_{W,t}}_{= \bar{p} C_{W,t}} = \bar{p}(y_t - C_{W,t}) = \bar{p} C_t \ , \tag{4}$$

where $w_t N_{W,t}$, the wage income, is equivalent to the consumption expenditure of hand-to-

[6] The competitive market assumption is crucial in our model for explaining inefficiencies stemming from the aggregate demand externality that each capitalist's financial investment decision imposes on the economy. For more details, see Farhi and Werning (2016).

[7] A superscript N indicates a nominal state-price density, whereas a superscript r signifies a real one.

mouth workers, denoted as $\bar{p}C_{W,t}$. Consequently, the total dividend is equal to the capitalists' aggregate consumption expenditure.

Incorporating equation (4) into the asset pricing equation, we obtain

$$\bar{p}A_tQ_t = \mathbb{E}_t \frac{1}{\xi_t^N} \int_t^\infty \xi_s^N \underbrace{D_s}_{=\bar{p}C_s} ds = \frac{\bar{p}C_t}{\rho} \ , \tag{5}$$

which implies $C_t = \rho A_t Q_t$. This equation indicates that, in equilibrium, the rate of consumption by capitalists corresponds to a fixed proportion ρ of their aggregate financial market wealth. From equations (4) and (5), the dividend yield of the stock market index is also constant and equal to ρ, which results in the equilibrium consumption of stock dividends by capitalists.

Agents of the same type (worker or capitalist) are identical and make symmetric decisions in equilibrium. Since bonds have zero net supply, a capitalist's wealth share in the stock market, denoted as θ_t, must be $\theta_t = 1$ for all t. This condition determines the equilibrium risk-premium demanded by capitalists. Drawing on equations (2), (3), and (5), the risk-premium is given by

$$\mathrm{rp}_t \equiv i_t^m - i_t = \left(\sigma + \sigma_t^q\right)^2 \ , \tag{6}$$

where rp_t increases with the endogenous volatility σ_t^q. It is important to note that the wealth gain or loss of a capitalist equates to the nominal revaluation of the stock market index. Our equilibrium conditions in equations (5) and (6) are consistent with Merton (1971).

The equilibrium in the goods market and the expected stock return i_t^m are characterized as follows: Given that capitalists' consumption $C_t = \rho A_t Q_t$ holds in equilibrium, the final goods market equilibrium condition can be written as

$$\rho A_t Q_t + \frac{w_t}{\bar{p}} N_{W,t} = y_t = A_t N_{W,t} \ . \tag{7}$$

The nominal expected return on stocks, i_t^m, comprises the dividend yield from firm profits and the nominal stock price revaluation, or capital gain, resulting from fluctuations in $\{A_t, Q_t\}$. In our model, the dividend yield is always equal to ρ, the discount rate for capitalists. Therefore, changes in i_t^m only affect nominal stock prices, as the dividend yield remains constant. Defining $\{\mathbf{I}_t^m\}$ as the cumulative stock market return process, where $\mathbb{E}_t\left(dI_t^m\right) = i_t^m dt$, equation (8) decomposes i_t^m into dividend yield and stock revaluation in equilibrium:

$$d\mathbf{I}_t^m = \frac{\overbrace{\bar{p}\left(\underbrace{y_t - \frac{w_t}{\bar{p}} N_{W,t}}_{=C_t}\right)}^{\text{Nominal dividend}}}{\underbrace{\bar{p} A_t Q_t}_{\text{Total capital market wealth}}} dt + \underbrace{\frac{d\left(\bar{p} A_t Q_t\right)}{\bar{p} A_t Q_t}}_{\text{Capital gain}} \tag{8}$$

$$= \underbrace{\left(\rho + g + \mu_t^q + \sigma\sigma_t^q\right)}_{=i_t^m} dt + \underbrace{\left(\sigma + \sigma_t^q\right)}_{\text{Risk term}} dZ_t \ .$$

Equilibrium The model's equilibrium encompasses several key elements: workers' optimization as derived from equation (1), capitalists' optimization per equations (5) and (6), the equilibrium in the goods market as defined in equation (7), and the determination of the return on risky stocks as specified in equation (8). Additionally, a monetary policy rule is required to complete the model framework.

The real stock price Q_t is a pivotal factor in driving the business cycle in the equilibrium. An increase in the stock price Q_t enhances the consumption of capitalists, leading to higher wages, greater labor demand by firms, and consequently, increased consumption by households.

Flexible Price Equilibrium In line with most of the literature, we adopt the equilibrium of the flexible price economy as the benchmark that guides the policy goals of the monetary authority. Details of this equilibrium are presented in Appendix B. Additionally, Appendix B.1 outlines the necessary conditions for positive co-movement among the gaps in asset price, wage, labor supply, and consumption for both capitalists and workers. Here, 'gaps' refer to the log-deviations from the flexible price equilibrium.

In the flexible price equilibrium, denoted by the superscript n (indicating 'natural'), we obtain $\mu_t^{q,n} = \sigma_t^{q,n} = 0$, implying a constant natural stock price, Q_t^n. The natural interest rate, denoted as r_t^n, refers to the real risk-free rate in the flexible price economy. It is given by $r_t^n = r^n = \rho + g - \sigma^2$ as in the literature.[8]

[8]Note that r_t^n is constant if σ_t is constant, e.g., $\sigma_t = \sigma$ as we assume in Section 2.

2.3 Gap Economy

The economy's dynamic IS equation, written in terms of the asset price gap $\hat{Q}_t$,[9] incorporates significant modifications from the standard New-Keynesian framework. We initially define the risk-premium $\mathrm{rp}_t \equiv (\sigma + \sigma_t^q)^2$ in the context of our rigid price equilibrium, along with its counterpart in the flexible price economy, denoted as $\mathrm{rp}_t^n \equiv \sigma^2$ where $\sigma_t^{q,n} = 0$. These definitions are detailed in Appendix B. The risk-premium gap is then formulated as $\hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}_t^n$. We also introduce the concept of the risk-adjusted natural rate, r_t^T, defined as:

$$r_t^T \equiv r_t^n - \frac{1}{2}\hat{rp}_t \ . \tag{9}$$

This rate acts as an anchor for monetary policy, as illustrated in Proposition 1. For instance, a positive risk-premium gap $\hat{rp}_t > 0$ diminishes the stock market portfolio demand of capitalists relative to the benchmark economy, potentially leading to a recession. Consequently, this affects the anchor rate r_t^T, which monetary policy aims to stabilize according to equation (10).

Proposition 1 (Asset Price Gap Process: IS Equation)

$$d\hat{Q}_t = d\hat{C}_t = (i_t - r_t^T)dt + \sigma_t^q dZ_t \ , \tag{10}$$

where r_t^T takes the role of the natural rate r_t^n of conventional macroeconomic models.

Proof. See Lee and Dordal i Carreras (2023).

■

Capitalists are exposed to a total risk of $(\sigma + \sigma_t^q)$ from the stock market, and in equilibrium demand a risk-premium of $(\sigma + \sigma_t^q)^2$.[10] In the flexible price equilibrium, the natural rate is established as $r_t^n = \rho + g - \sigma^2$, where σ_t^q is set to $\sigma_t^{q,n} = 0$. Consequently, the expected real return on the stock market is computed as $r_t^n + \sigma^2 - \frac{1}{2}\sigma^2$, where the term $\frac{1}{2}\sigma^2$ originates from the quadratic variation inherent in the second-order Taylor expansion of the cumulative stock returns process.[11] In our rigid price economy with endogenous volatility σ_t^q, the risk premium changes from σ^2 to $(\sigma + \sigma_t^q)^2$. Therefore, with a policy rate

[9] A conventional definition using the output gap leads to a comparable expression in our model, since both variables are proportional in equilibrium.

[10] This result is a consequence of the logarithmic utility preferences of capitalists.

[11] Refer to Ito's lemma for an explanation of the adjustments required to account for quadratic variations in stochastic calculus.

of i_t, the expected real return on the stock market is formulated as $i_t + \frac{1}{2}(\sigma + \sigma_t^q)^2$. If this return deviates from its natural level, $r_t^n + \frac{1}{2}\sigma^2$, the asset price gap $\hat{Q}_t$ endogenously adjusts. This adjustment induces real distortions via its influence on aggregate demand. Notably, the endogenous financial volatility, σ_t^q, impacts the risk-premium gap $\hat{rp}_t$, the risk-adjusted natural rate r_t^T, and subsequently the drift of the asset price gap $\{\hat{Q}_t\}$ process, which governs the fluctuation of all other gap variables over time.

Inserting $r_t^T = r^n - \frac{1}{2}(\sigma + \sigma_t^q)^2 + \frac{1}{2}\sigma^2$ into equation (10) yields:

$$d\hat{Q}_t = d\hat{C}_t = \left(i_t - r^n + \frac{1}{2}(\sigma + \sigma_t^q)^2 - \frac{1}{2}\sigma^2\right) dt + \sigma_t^q dZ_t \ , \tag{11}$$

highlighting the role of endogenous volatility σ_t^q in the drift term and adding complexity to the analysis.

Monetary Policy and Equilibrium Uniqueness To close the model, we need to introduce monetary policy rules, which with the IS equation (10) will constitute our equilibrium written in gap variables. In a linearized version of the model, we usually have the IS equation given by

$$d\hat{Q}_t = d\hat{C}_t = (i_t - r^n)dt + \sigma_t^q dZ_t \ , \tag{12}$$

where, in contrast to our non-linear IS equation (11), the endogenous volatility σ_t^q does not appear in the drift given by $i_t - r^n$. In that case, we know that Taylor rules $i_t = r^n + \phi_q \hat{Q}_t$, with $\phi_q > 0$ would achieve perfect stabilization as unique equilibrium: plugging Taylor rules into (12), we obtain

$$\mathbb{E}_t d\hat{Q}_t = \phi_q \hat{Q}_t \ ,$$

which, due to Blanchard and Kahn (1980), leads to $\hat{Q}_t = 0$ as unique rational expectations equilibrium.

In Lee and Dordal i Carreras (2023), we uncovered that conventional Taylor rules $i_t = r^n + \phi_q \hat{Q}_t$, with $\phi_q > 0$, which would achieve perfect stabilization in linearized canonical New-Keynesian models, do not guarantee a unique equilibrium under our IS equation (11), and there is always a chance that a self-fulfilling volatility σ_0^q arises.[12] Instead we suggested

[12]The crux of the problem is that σ_t^q, which we use as a proxy for financial instability, is itself an endogenous variable to be determined in equilibrium, with no guarantee that it equates its natural level $\sigma_t^{q,n} = 0$.

that monetary policy should target risk-premium directly, as in

$$i_t = r_t^T + \phi_q \hat{Q}_t = r^n + \phi_q \hat{Q}_t - \frac{1}{2}\hat{rp}_t, \;\; \phi_q > 0 \;, \tag{13}$$

in order to achieve perfectly stabilized path, i.e., $\hat{Q}_t = 0$ for $\forall t \geq 0$, as unique equilibrium: plugging (13) into (10), we obtain

$$\mathbb{E}_t d\hat{Q}_t = \phi_q \hat{Q}_t \;,$$

which, due to Blanchard and Kahn (1980), leads to $\hat{Q}_t = 0$ as unique rational expectations equilibrium. In the presence of first-order effects of endogenous volatility σ_t^q on business cycle levels through the financial wealth channel, the monetary authority should independently respond to fluctuations in σ_t^q and risk-premium $\hat{rp}_t$ for stabilization purposes.

3 Zero Lower Bound and Forward Guidance

During the zero lower bound (ZLB) episodes, economies typically experience recessions, characterized by declines in both asset prices $\hat{Q}_t$ and key business cycle indicators. To mitigate the adverse effects of ZLB-induced recessions, central banks globally have adopted strategies such as 'forward guidance'. This section examines how these strategies, including a novel approach focusing on endogenous volatility σ_t^q and the risk premium gap $\hat{rp}_t$, can stabilize both the economy and financial markets. We also discuss the potential stabilization trade-offs associated with the deployment of these monetary tools.

ZLB Scenario Following the approach of Werning (2012), we consider a scenario where the interest rate is brought to the ZLB by a deterministic shift in the fundamental volatility of the economy between $t = 0$ and T. We consider the case where $\sigma_t = \bar{\sigma}$ for $0 \leq t \leq T$ and $\sigma_t = \underline{\sigma} < \bar{\sigma}$ for $t \geq T$. More specifically, we assume: TFP volatilites during these periods are such that the real natural rate of the economy satisfies: $\underline{r} \equiv r^n(\bar{\sigma}) = \rho + g - \bar{\sigma}^2 < 0$ and $\bar{r} \equiv r^n(\underline{\sigma}) = \rho + g - \underline{\sigma}^2 > 0$, resulting in the ZLB binding from $t = 0$ until $t = T$.

3.1 Zero Lower Bound Recession

After $t = T$, we assume that the monetary authority can always follow the modified Taylor rule presented in equation (13), achieving perfect economic stabilization defined by $\hat{Q}_t = 0$

for $t \geq T$.[13]

It follows by backward induction from equation (10) that perfect stabilization with certainty at T, $\hat{Q}_T = 0$, necessarily implies the absence of volatility in the asset price gap $\hat{Q}_t$ process in the preceding periods, $t < T$.[14] Therefore, if follows that $\sigma_t^q = \sigma_t^{q,n} = 0$ and $r_t^T = \underline{r} < 0$ for $t \leq T$ whenever the monetary authority can credibly commit to follow the modified Taylor rule in (13) for $t \geq T$. In this scenario, the dynamics of $\hat{Q}_t$ according to (10) simplifies to:

$$d\hat{Q}_t = \underbrace{-\underline{r}}_{>0} dt \ , \qquad \text{for } t < T \ , \tag{14}$$

with associated boundary condition $\hat{Q}_T = 0$ and initial asset price gap given by $Q_0 = \underline{r}\, T$. The trajectory of $\{\hat{Q}_t\}$ following (14) is illustrated in Figure 1. This path is consistent with the dynamics described in Werning (2012) and Cochrane (2017), despite our model featuring a distinct IS equation (10) with endogenous volatility σ_t^q influencing the drift in the $\hat{Q}_t$ process, a departure from traditional New-Keynesian models. This result arises because ensuring future stabilization for $t \geq T$ effectively eliminates any excess endogenous volatility σ_t^q during a ZLB episode, resulting in $\sigma_t^q = \sigma_t^{q,n} = 0$ for $t \leq T$.

Notice that while the dynamics of $\{\hat{Q}_t\}$ resemble those in Werning (2012) and Cochrane (2017), the underlying driving forces differ. In our case, the ZLB constraint becomes active for $t \leq T$, following a surge in σ_t from $\underline{\sigma}$ to $\bar{\sigma}$. This change increases the risk premium from $\text{rp}_2^n = (\underline{\sigma})^2$ to $\text{rp}_1^n = \bar{\sigma}^2$, as illustrated in Figure 1. Consequently, asset prices $\hat{Q}_t$ decline due to the ZLB preventing the risk-free rate from falling into the negative values required for full stabilization. This leads to a diminished interest from capitalists in stock market investments, culminating in a decrease in both aggregate financial wealth and consumption demand.[15]

Conversely, in a standard log-linearized New-Keynesian model, the ZLB causes agents to deleverage and curtail consumption, collectively diminishing aggregate output due to an

[13]For the period up to $t \leq T$, due to $\sigma_t = \bar{\sigma}$ leading to $r^n(\bar{\sigma}) = \rho + g - \bar{\sigma}^2$ being less than zero, the monetary authority is restricted by the ZLB and cannot implement the rule in equation (13).

[14]For instance, at $T - \Delta$, where Δ is an infinitesimally small time interval, $\sigma_{T-\Delta}^q = 0$ is the only rational solution to equation (10) consistent with $\hat{Q}_T = 0$ for any possible realization of the stochastic component of the TFP process, $dZ_{T-\Delta}$. This result deterministically pins down the asset price gap of the preceding period, $\hat{Q}_{T-\Delta}$, leading by backward induction to $\sigma_t^q = 0$ for $t \leq T$.

[15]While Caballero and Farhi (2017) demonstrate that a heightened demand for safe assets can propel the economy into recession under ZLB constraints, our analysis suggests that it prompts investors to withdraw their wealth from the stock market, reducing stock market value and aggregate demand similarly to Caballero and Simsek (2020).

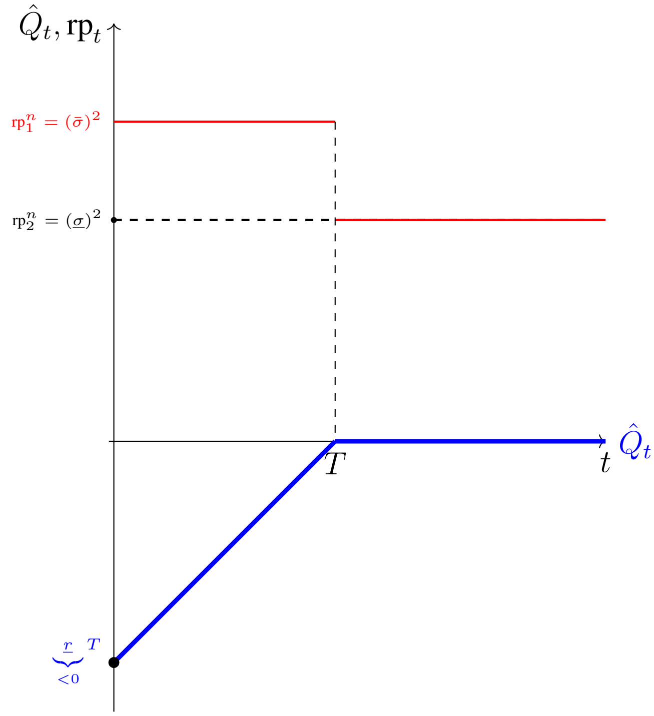

Figure 1: ZLB dynamics (Benchmark)

aggregate demand externality.[16]

Remark Central banks can prevent the emergence of endogenous volatility σ_t^q at the ZLB through a 'credible' commitment to stabilize the business cycle by a predetermined future date $T < +\infty$. Even if the monetary authority is constrained by the ZLB and thus unable to adhere to the policy rule outlined in (10), which directly targets the risk-premium, the additional financial stability costs resulting from policy inaction can be effectively managed, or even completely eliminated, by pledging to stabilize upon exiting the ZLB. One implication is that the impact of the ZLB could vary significantly between countries: those with monetary authorities committed to stabilization after the ZLB period may only face the demand-driven recession previously described. In contrast, countries lacking the capacity or willingness to stabilize in the future might incur additional costs due to potential increases in σ_t^q during a ZLB episode.

[16]See Farhi and Werning (2012) for further details on this channel.

3.2 Traditional Forward Guidance

This section examines traditional forward guidance policy where the central bank commits to maintaining a zero policy rate for a duration of time $\hat{T} > T$ exceeding the initial period of high fundamental volatility. Within our model, forward guidance is a powerful tool, echoing the findings of Werning (2012) and Cochrane (2017). For this baseline case, we assume that the central bank reverts to the policy rule defined in equation (13) after the forward guidance period ends, resulting in a perfect stabilization of both the business cycle and financial markets for $t \geq \hat{T}$. Following from the backward induction rationale presented in Section 3, stabilization with certainty after $\hat{T}$ results in the absence of endogenous financial volatility $\sigma_t^q = \sigma_t^{q,n} = 0$ for $t \leq \hat{T}$. The dynamics of $\hat{Q}_t$ are described by

$$d\hat{Q}_t = \begin{cases} \underbrace{-\underline{r}}_{>0} \, dt \ , & \text{for } t \leq T \ , \\ \underbrace{-\bar{r}}_{<0} \, dt \ , & \text{for } T \leq t \leq \hat{T} \ , \end{cases} \tag{15}$$

with associated boundary condition $\hat{Q}_{\hat{T}} = 0$ and initial gap given by $Q_0 = \underline{r}\, T + \bar{r}\, (\hat{T} - T)$.

The dynamics of $\{\hat{Q}_t\}$ governed by equations (15) are depicted in Figure 2. Traditional forward guidance induces an artificial economic boom between T and $\hat{T}$, thereby alleviating recessionary pressures within the interval $0 \leq t \leq T$. Specifically, traditional forward guidance increases asset prices, $\hat{Q}_t$, from T to $\hat{T}$, which results in a narrower initial asset price gap due to the forward-looking nature of stock markets. In the context of a standard New-Keynesian framework, forward guidance proves beneficial as it enhances household consumption and income from T to $\hat{T}$, consequently boosting consumption prior to T. Given that $\sigma_t^q = 0$ for $t \leq \hat{T}$, the traditional forward guidance policy, when implemented with commitment, mirrors the effects seen in a canonical New-Keynesian model by generating future increases in aggregate demand.

Optimal Traditional Forward Guidance To determine the optimal duration of forward guidance, denoted as $\hat{T}$, we minimize the quadratic loss function represented by:

$$L(\{\hat{Q}_t\}_{t\geq 0}) = \mathbb{E}_0 \int_0^\infty e^{-\rho t} \hat{Q}_t^2 \, dt \ , \tag{16}$$

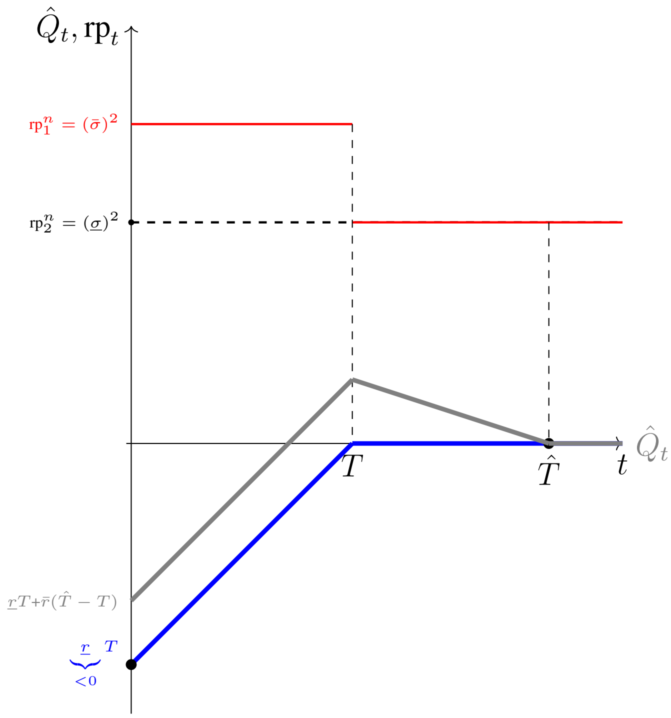

Figure 2: ZLB dynamics under traditional forward guidance.

constrained by the dynamics outlined in equation (15). The first-order condition with respect to $\hat{T}$ results in:

$$\int_0^\infty e^{-\rho t}\hat{Q}_t dt = 0 \ . \tag{17}$$

The gray path in Figure 2 depicts a trajectory for the asset price gap $\{Q_t\}$ that aligns with condition (17).

Note that heterogeneous costs associated with the ZLB reemerge under forward guidance. Central banks capable of credibly committing to stabilize the business cycle and financial markets after $\hat{T}$, that is, $\hat{Q}_t = 0$ for $t \geq \hat{T}$, ensure $\sigma_t^q = 0$ for $t \leq \hat{T}$. In other words, they avoid adverse 'financial volatility costs' from implementing such policies, leading to unequivocally positive outcomes for welfare and business cycle stabilization. This conclusion, however, may shift significantly if a central bank cannot commit to perfect stabilization after $\hat{T}$. Here, the economy faces potential endogenous volatility σ_t^q, elevating risk-premium and intensifying recession.[17] In such a scenario, extending the duration of the ZLB period *voluntarily* can be perilous. By maintaining a passive monetary stance until $\hat{T}$, the central bank may amplify costs arising from the economy's inherent financial volatil-

[17]For further details, see Lee and Dordal i Carreras (2023).

ity—particularly when these costs peak due to the absence of monetary policy response.[18] We posit this as a novel insight into the trade-offs of forward guidance policy, potentially explaining the cautious approach central bankers globally take towards implementing such policies *in practice*.

In the next Section 3.3, in contrast to the above Section 3.1 and Section 3.2, we will argue that central banks might voluntarily forgo perfect stabilization in the future but achieve even better welfare than under the traditional forward guidance policy described in Section 3.2. It requires that central banks have a commitment power enough to make private agents coordinate on proper endogenous volatility σ_t^q levels they deem optimal even if the policy rate is kept at zero. We call this type of policy, in which central banks sacrifice future stabilization in order to manipulate current endogenous volatility σ_t^q and risk premium rp_t during a ZLB episode, a 'higher-order forward guidance' policy.

3.3 Higher-Order Forward Guidance under Commitment

During the Global Financial Crisis (GFC) and afterwards, central banks around the world purchased large amounts of assets in financial markets, which mitigated collapses in asset prices and brought down levels of risk-premia for a variety of assets.[19] As our framework's Ricardian structure does not allow the central bank's balance sheet quantities to affect the equilibrium, we turn our eyes to a different type of policy that can prop up asset markets and the business cycle: a central bank commitment to passive financial stabilization *in the future* in exchange for lower financial volatility and risk premium *at* the ZLB.

General idea In this section we explore an alternative commitment path for a central bank at the ZLB, which potentially yields greater expected welfare than the conventional forward guidance previously examined. For that purpose, we assume that the central bank is capable of coordinating the economic agents on one of the several admissible paths for the endogenous volatility $\{\sigma_t^q\}$ which are consistent with the constraints imposed by the dynamic IS equation (10) and the associated boundary conditions. In the traditional forward guidance approach outlined in Section 3.2, the central bank's ability to achieve perfect

[18] It's important to note that although the effectiveness of forward guidance policies is constrained when a central bank cannot commit to future stabilization, expected benefits from the policy remain positive.

[19] See Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), and Gorodnichenko and Ray (2018) for empirical evidence on how the Fed's QE1 and QE2 programs affected asset prices and risk-premia in financial markets, e.g., Gagnon et al. (2011) present the evidence of asset purchases leading to reductions in interest rates on a range of securities, which reflects reductions in the levels of risk premia.

stabilization with certainty after $\hat{T}$, as depicted in Figure 2, influences financial volatility levels during the ZLB period, including the forward guidance interval between T and $\hat{T}$. Specifically, we determined that $\sigma_t^q = \sigma_t^{q,n} = 0$ for the durations $T \leq t \leq \hat{T}$ (the forward guidance period) and $t < T$ (the ZLB period). Consequently, the risk-premium level prior to $\hat{T}$ is equated to its natural counterpart, rp_t^n, resulting in $\hat{rp}_t = 0$ for all $t \leq \hat{T}$. This concept is further elucidated in the following diagram.

$$\boxed{\text{1. Central bank achieves perfect stabilization with certainty after } \hat{T} \text{ (i.e., } \hat{Q}_t = 0, \forall t \geq \hat{T}\text{)}}$$

$$\downarrow$$

$$\boxed{\text{2. } \hat{Q}_{\hat{T}} = 0 \text{ guarantees } \sigma_t^q = \sigma_t^{q,n} = 0,\ \mathrm{rp}_t = \mathrm{rp}_t^n \text{ for } t \leq \hat{T}}$$

Figure 3: Mechanism under traditional forward guidance.

We pose the following question: Can a central bank commit to less-than-certain stabilization after the conclusion of the forward guidance period (i.e., for $t > \hat{T}$ in Figure 2), while orchestrating an equilibrium path with reduced risk-premium rp_t levels during both the ZLB period (up to T) and the forward guidance phase (from T to $\hat{T}$)? From the central banks' perspective, the risk premium, influenced by the fundamental volatility σ_t (equal to $\bar{\sigma}$ for $t \leq T$ and $\underline{\sigma}$ for $T \leq t \leq \hat{T}$) of the $\{A_t\}$ process, is excessively high during the ZLB episode. Such high volatility leads to significant asset price drops, thrusting the economy into a severe recession. Therefore, central banks may find that reducing σ_t^q levels (or, equivalently, risk-premium levels) from $\sigma^{q,n} = 0$ would alleviate the crisis during the ZLB by bolstering asset prices, thereby stimulating aggregate demand.[20] This strategy aligns with central banks' endeavors to infuse liquidity into financial markets through Large-Scale Asset Purchase (LSAP) policies, recapitalization, government guarantees for financial institutions, and various bailout measures primarily aimed at diminishing financial uncertainty. In our model, this reduction in uncertainty is achieved through the central bank's commitment to deviate from a fully stabilized trajectory in the future (post-$\hat{T}$).

The possibility of σ_t^q deviating from $\sigma_t^{q,n} = 0$ leads to stochastic fluctuations in the asset price gap $\hat{Q}_t$ up to $\hat{T}$.[21] Consequently, achieving $\hat{Q}_{\hat{T}} = 0$ with certainty is not feasible for

[20] Note that the risk premium rp_t is defined as $\mathrm{rp}_t = (\bar{\sigma} + \sigma_t^q)^2$ for $t \leq T$ and $\mathrm{rp}_t = (\underline{\sigma} + \sigma_t^q)^2$ for $T \leq t \leq \hat{T}$. Consequently, a negative σ_t^q can lower risk-premium levels and enhance asset prices and aggregate demand.

[21] From equation (10), a deviation of σ_t^q from $\sigma^{q,n} = 0$ induces a stochastic movement $\sigma_t^q dZ_t$ in the asset price gap process $\{\hat{Q}_t\}$ until $\hat{T}$.

the central bank. Therefore, the hypothesized strategy of reducing $\{\sigma_t^q\}$ below $\{\sigma_t^{q,n} = 0\}$ until $\hat{T}$ for stabilization purposes will only be effective if the central bank pre-commits to not seeking perfect stabilization even after the end of the forward guidance period and the return to normal economic conditions.[22,23] This rationale can be viewed as the inverse of the logic in Figure 3 and is depicted in Figure 4.

In essence, when crafting an improved equilibrium path, the central bank faces a trade-off between stimulating the economy during the ZLB by reducing the endogenous volatility $\{\sigma_t^q\}$, and foregoing perfect stabilization in the future once economic fundamentals normalize. This approach effectively entails trading future stability for an immediate reduction in uncertainty at the ZLB.

$\neg$2. $\sigma_t^q < \sigma_t^{q,n} = 0$, $\text{rp}_t < \text{rp}_t^n$ for $t \leq \hat{T}$

$\downarrow$

$\neg$1. $\hat{Q}_{\hat{T}} \neq 0$: central bank commits not to perfectly stabilize the economy after $\hat{T}$

Figure 4: Financial Market Intervention and Stabilization.

It is important to recognize that the central bank aims to lower both the financial volatility and the risk-premium even after the TFP volatility σ_t reverts to $\underline{\sigma}$ for $t \geq T$, while adhering to the forward guidance rate prescription ($i_t = 0$) until $t = \hat{T}$. This strategy is intended to elevate $\hat{Q}_t$ between T and $\hat{T}$, thereby increasing the asset price $\hat{Q}_t$ during periods of high TFP volatility ($\sigma_t = \bar{\sigma}$ for $t \leq T$), given the forward-looking nature of $\hat{Q}_t$. With the introduction of a new method (altering the endogenous volatility σ_t^q) for stimulating the economy both during and after the ZLB, the traditional forward guidance approach outlined in Section 3.2 becomes less essential for supporting $\hat{Q}_t$. Consequently, the duration of this forward guidance, $\hat{T}$, should be reduced when implementing this intervention.

Formalism Let $\hat{T}'$ be the forward guidance duration under this newly engineered path. We define levels of the natural risk-premium as $\text{rp}_1^n \equiv \bar{\sigma}^2$, $\text{rp}_2^n \equiv \underline{\sigma}^2$, and $\text{rp}_3^n \equiv \underline{\sigma}^2$ for the

[22] In attempts to influence business cycle trajectories by lowering the 'excessively high' risk-premium during the ZLB, central banks must consider the impact of such trajectories at the conclusion of the forward guidance period, as the risk-premium is tied to the endogenous volatility σ_t^q.

[23] For instance, we assume that following the end of the forward guidance period, the central bank adopts a passive policy rule with just $i_t = r^n(\underline{\sigma})$, which allows for multiple equilibria. In Section 3.3, we identify one such equilibrium where we have $\sigma_t^q = \sigma^{q,n} = 0$ after the forward guidance period concludes (after $\hat{T}$ in Figure 2), with $\mathbb{E}_0\hat{Q}_\infty = 0$. Hence, $\hat{Q}_t$ remains at $\hat{Q}_{\hat{T}}$ beyond $t \geq \hat{T}$ with $i_t = r^n(\underline{\sigma})$.

respective time intervals: $t \leq T$, $T < t \leq \hat{T}'$, and $t \geq \hat{T}'$.[24]

For simplicity, we assume that the central bank maintains consistent financial volatility and risk-premium levels within each regime: specifically, the financial volatility σ_t^q is set at $\sigma_1^{q,L}$ from 0 to T (the high TFP volatility region), $\sigma_2^{q,L}$ from T to $\hat{T}'$ (the low TFP volatility region under forward guidance), and is zero after $\hat{T}'$ (the low TFP volatility region at the conclusion of the forward guidance period). The premise that $\sigma_t^q = 0$ after $\hat{T}'$ implies that the central bank ceases to influence financial markets once forward guidance concludes. We will demonstrate that at the optimum, $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}' < \hat{T}$, as discussed.

Therefore, under this newly engineered path, the risk-premium levels $\text{rp}_t = (\sigma_t + \sigma_t^q)^2$ in each time interval become $\text{rp}_1 \equiv (\bar{\sigma} + \sigma_1^{q,L})^2 < \text{rp}_1^n$ for $t \leq T$, $\text{rp}_2 \equiv (\underline{\sigma} + \sigma_2^{q,L})^2 < \text{rp}_2^n$ for $T \leq t \leq \hat{T}'$, and $\text{rp}_3 \equiv (\underline{\sigma})^2$ after $\hat{T}'$.[25] As the policy intervention lowers the economy's total risk, risk-premium levels fall, and asset price and business cycle levels rise in response.

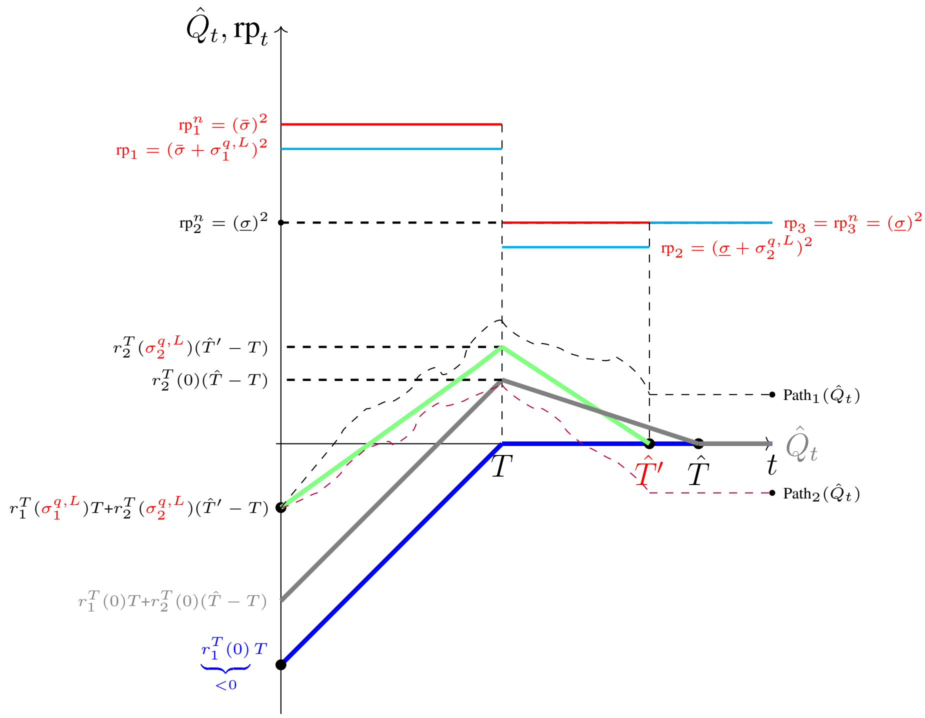

Figure 5: Possible Intervention Dynamics of $\{\hat{Q}_t\}$ with $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}' < \hat{T}$.

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[24]This delineation stems from the definition $\text{rp}_t = (\sigma_t + \sigma_t^q)^2$ and the zero endogenous financial volatility in a flexible price economy, where $\sigma_t^{q,n} = 0$ for all t.

[25]We will eventually prove $\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ and $\sigma_2^{q,L} < \sigma^{q,n} = 0$ at optimum. For illustration purposes, we assume these conditions are satisfied in the rest of the argument in Section 3.3.

From equation (9), we can express the risk-adjusted natural rates r_1^T (for $t \le T$) and r_2^T (for $T \le t \le \hat{T}'$), which enter the dynamic IS equation (10), as functions of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, respectively, as

$$\begin{aligned} r_1^T(\sigma_1^{q,L}) &\equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma} + \sigma_1^{q,L})^2}{2} > \underline{r} \equiv r^n(\bar{\sigma}) = r_1^T(0) \;\text{ when }\; \sigma_1^{q,L} < 0 \;, \\ r_2^T(\sigma_2^{q,L}) &\equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{(\underline{\sigma} + \sigma_2^{q,L})^2}{2} > \bar{r} \equiv r^n(\underline{\sigma}) = r_2^T(0) \;\text{ when }\; \sigma_2^{q,L} < 0 \;. \end{aligned} \tag{18}$$

From (18), we see $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ imply $r_1^T > \underline{r}$ and $r_2^T > \bar{r}$, which yield higher levels of $\{\hat{Q}_t\}$ during and after the ZLB *on average*. That would be the reason why central banks want to push down $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ from zero, but from (10) we see $\sigma_t^q \neq 0$ creates a stochastic fluctuation of $\hat{Q}_t$, which brings additional costs in terms of stabilization. Therefore, central banks face a trade-off between time-0 volatilities of the business cycle in the future (i.e., after the forward guidance policy is over) and at the ZLB, i.e., lowering σ_t^q at the ZLB requires uncertain $\hat{Q}_t$ after forward guidance is over, when it engineers the new commitment path.

Monetary Policy and Equilibrium Selection The central bank is assumed to follow the passive policy rate rule as follows: $i_t = 0$ for $t \le \hat{T}'$, and $i_t = \bar{r}$. To pin down equilibrium paths,[26] we assume that, at $t = 0$, the monetary authority anchors an expected value of $\hat{Q}_t$ in the far future, e.g., $\mathbb{E}_0 \hat{Q}_\infty$, at zero. Since after $\hat{T}'$, when the traditional forward guidance policy is over, the central bank just pegs the policy rate at the natural rate of interest: $i_t = \bar{r}$, $\mathbb{E}_0 \hat{Q}_\infty = 0$ is equivalent to $\mathbb{E}_0 \hat{Q}_{\hat{T}'} = 0$.

The gray line in Figure 5 represents the trajectory of $\{\hat{Q}_t\}$ under the traditional forward guidance strategy discussed in Section 3.2. In contrast, the green line depicts the expected path (or deterministic component) of $\{\hat{Q}_t\}$ when the central bank implements a new strategy with $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}' < \hat{T}$. With $\sigma_1^{q,L} \neq 0$ and $\sigma_2^{q,L} \neq 0$, stochastic variations in $\{\hat{Q}_t\}$ around this deterministic path emerge, as shown by the two possible sample paths (dashed lines) in Figure 5. These stochastic fluctuations result in welfare losses with respect to the optimal expected path. Additionally, we note that with $\sigma_2^{q,L} < 0$ and $\mathbb{E}_0 \hat{Q}_{\hat{T}'} = 0$, the average $\hat{Q}_t$ level is higher from T to $\hat{T}'$ compared to the gray forward guidance path. The higher expected value of $\hat{Q}_t$ between T and $\hat{T}'$ increases $\hat{Q}_0$ by the forward looking nature

[26] There are multiple equilibria if a central bank does not use the policy rule in (13), and thus we need to select one equilibrium.

of asset prices. Moreover, $\sigma_1^{q,L} < 0$ for $t \leq T$ further boosts the average $\hat{Q}_t$ level during the high fundamental volatility period before T, resulting in a smaller decline of $\hat{Q}_0$ compared to the (gray) forward guidance path depicted in Figure 5.

In summary, central banks operating under commitment need to navigate a trade-off between achieving higher asset price and output levels prior to $\hat{T}'$ and ensuring future stabilization after the ZLB. This balancing act involves careful manipulation of $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}')$. The next step in our analysis is to verify whether our conjecture in Section 3.3 aligns with the optimal commitment solution.

Central Bank's Optimal Commitment Path To minimize the loss function in (16), the central bank strategically selects optimal values for $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}'$. These selections are guided by the functions $r_1^T(\cdot)$ and $r_2^T(\cdot)$ defined in equation (18). Following $\hat{T}'$, the central bank adopts a *passive monetary policy*, anchoring the interest rate at $i_t = r_2^T(0) = r^n(\underline{\sigma})$, and does not actively pursue economic stabilization. The initial asset price level $\hat{Q}_0$ is determined by the condition $\mathbb{E}_0 \hat{Q}_{\hat{T}'} = 0$, as follows:

$$\hat{Q}_0 = \underbrace{r_1^T(\sigma_1^{q,L})}_{<0} T + \underbrace{r_2^T(\sigma_2^{q,L})}_{>0} (\hat{T}' - T) \ .$$

In summary, the central bank confronts the following optimization problem:

$$\min_{\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}'} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \hat{Q}_t^2 \, dt \, , \quad \text{s.t.} \;\; d\hat{Q}_t = \begin{cases} -\underbrace{r_1^T(\sigma_1^{q,L})}_{<0} dt + \sigma_1^{q,L} dZ_t, & \text{for } t < T \ , \\ -\underbrace{r_2^T(\sigma_2^{q,L})}_{>0} dt + \sigma_2^{q,L} dZ_t, & \text{for } T \leq t < \hat{T}' \ , \\ 0, \;\; \text{for } t \geq \hat{T}' \ , \end{cases}$$
$$\text{with } \; \hat{Q}_0 = \underbrace{r_1^T(\sigma_1^{q,L})}_{<0} T + \underbrace{r_2^T(\sigma_2^{q,L})}_{>0} (\hat{T}' - T) \, . \tag{19}$$

The traditional forward guidance path discussed in Section 3.2 aligns with the scenario where $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}') = (0, 0, \hat{T})$. Therefore, an optimal combination of $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}')$ results in a lower level of the (quadratic) loss function in (16).

It turns out that our hypothesis in Section 3.3 that the optimal solution features $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}' < \hat{T}$ holds, as formally presented in the following Proposition 2.

Proposition 2 (Optimal Commitment Path) *The solution to the central bank's forward guidance optimization problem in (19) results in an optimal commitment path characterized by $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}' < \hat{T}$.*

We simulate the solutions to the stochastic optimal commitment paths in (19) with a T value of 4.5 quarters and using model parameters calibrated from Lee and Dordal i Carreras (2023). The loss function is approximated by:

$$\int_0^\infty e^{-\rho t}\mathbb{E}_0\hat{Q}_t^2\,dt \simeq \int_0^\infty e^{-\rho t}\frac{1}{s}\sum_{i=1}^{s}\left(\hat{Q}_t^{(i)}\right)^2 dt\ , \tag{20}$$

where $\hat{Q}_t^{(i)}$ represents the i^{th} realized sample path.[27] Our results indicate that when $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}$ are optimally selected, the loss function decreases by 0.4239%, representing a moderate improvement with respect to traditional forward guidance. In the simulation, the conditions $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}' < \hat{T}$ are all met at the optimal solution,[28] consistent with Proposition 2. The optimal commitment path involves only slight adjustments in the endogenous volatility (and risk-premium) because upon exit of the high volatility period in T, there is no possibility of hitting the ZLB, increasing the cost of destabilization when the central bank's policy becomes passive after $\hat{T}'$. In more realistic scenarios in which the economy might stochastically fall back into the ZLB, the central bank might need to manipulate financial market volatilities and risk-premia to a greater extent. This analysis highlights the critical considerations for a central bank when attempting to alter the business cycle trajectory by manipulating financial market variables during the ZLB period.

In the next Section 3.4, we shift focus to explore potential macroprudential interventions from the fiscal perspective, aimed at raising asset prices $\hat{Q}_t$ and stabilizing the business cycle during a ZLB crisis.

3.4 Macroprudential Policies

In this section, we examine two types of macroprudential policies designed to stimulate the economy at the ZLB. Firstly, we consider a fiscal subsidy aimed at encouraging capitalists to undertake higher levels of risk, thereby boosting asset prices and other real economic activities. Secondly, we explore the impact of direct fiscal transfers from capitalists to

[27] We use $s = 1000$ sample paths in our simulation.

[28] The simulation results are $\hat{T}' = 5.612 < \hat{T} = 5.614$, $\sigma_1^{q,L} = -1.4325 \times 10^{-4} < 0$, and $\sigma_2^{q,L} = -1.0467 \times 10^{-6} < 0$.

hand-to-mouth workers, who typically exhibit a higher marginal propensity to consume (MPC). This policy is shown to increase overall stock market dividends, and consequently, asset prices $\hat{Q}_t$ and consumption levels. To assess the impact of macroprudential policies on the business cycle, forward guidance is excluded from our analysis in this section. We maintain the same scenario as outlined in Section 3.1, and assume that monetary policy reverts to the perfect stabilization rule specified in equation (13) for $t \geq T$.

3.4.1 Fiscal Subsidy on Stock Market Investment

In the period up to T, where $r_t^n = \underline{r} < 0$ and monetary policy is restricted by the ZLB, the risk-premium level $\mathrm{rp}_1^n = \bar{\sigma}^2$ required by capitalists leads to a reduction in asset prices, $\hat{Q}_t$. To counteract this, we propose a subsidy policy aimed at incentivizing capitalists' holdings of the risky stock market index. This intervention is expected to increase $\hat{Q}_t$, thereby addressing the aggregate demand externalities responsible for dragging the economy into a ZLB recession.[29]

We begin by examining a government subsidy for risky stock market index purchases.[30] Specifically, instead of the usual expected return i_t^m, a capitalist earns an expected return of $(1+\tau)i_t^m$ for every dollar invested in the stock market, where $\tau \geq 0$ is the stock subsidy. To fund this intervention, the government imposes a 'lump-sum' tax T_t on capitalists. Consequently, a capitalist solves the optimization problem with a modified flow budget constraint given by:

$$\begin{aligned}
&\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t \, dt \\
&\text{s.t. } \; da_t = \left(a_t \left(i_t + \theta_t((1+\tau)i_t^m - i_t)\right) - \bar{p}C_t - T_t\right) dt + \theta_t a_t \left(\bar{\sigma} + \sigma_t^q\right) dZ_t \, .
\end{aligned} \tag{21}$$

Capitalists finance the stock market subsidy by paying a tax T_t equal to $\tau a_t \theta_t i_t^m$. Setting

[29] Numerous studies have examined the link between externalities (e.g., pecuniary or aggregate-demand) and macroprudential policies. Notable references include Caballero and Krishnamurthy (2001), Lorenzoni (2008), Farhi et al. (2009), Bianchi and Mendoza (2010), Jeanne and Korinek (2010), Stein (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), Dàvila and Korinek (2018), among others.

[30] In our model, a subsidy for stock investments functions similarly to a tax break on capital income, a policy commonly implemented *in practice* by governments. We opt for the subsidy model for simplicity in notation.

$\theta_t = 1$, we can express the stock market's expected return in equilibrium as follows:

$$i_t^m = \frac{i_t + (\bar{\sigma} + \sigma_t^q)^2}{1+\tau} = \underbrace{\rho}_{\substack{\text{Dividend} \\ \text{yield}}} + \underbrace{g + \mu_t^q + \sigma_t \sigma_t^q}_{\text{Capital gain}} . \tag{22}$$

As detailed in Section 3.1, given that $\sigma_t^q = 0$ and $i_t = 0$ for $t \leq T$, equation (22) simplifies to

$$i_t^m = \frac{\bar{\sigma}^2}{1+\tau} ,$$

which is lower than $\bar{\sigma}^2$ and inversely proportional to τ. Thus, a positive subsidy rate $\tau > 0$ increases $\hat{Q}_t$ along the path up to time T, when the economy achieves stabilization $\hat{Q}_T = 0$. Proposition 3 summarizes this result.

Proposition 3 (Fiscal Subsidy on Stock Market Expected Return) *Under the ZLB environment of Section 3.1, where a fiscal subsidy $\tau \geq 0$ is applied to the expected return of stock markets, the IS equation for $\hat{Q}_t$ during the period $t \leq T$ can be expressed as:*

$$d\hat{Q}_t = -\left(\underbrace{\underline{r}}_{\equiv r^n(\bar{\sigma})<0} + \underbrace{\frac{\tau}{1+\tau}\bar{\sigma}^2}_{>0} \right) dt , \tag{23}$$

for $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2 < 0$. After time T, the central bank perfectly stabilizes the economy and eliminates the volatility in asset prices, $\sigma_t^q = \sigma_t^{q,n} = 0$, for all $t \geq T$. When $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2 > 0$, the subsidy lifts the economy out of the ZLB and immediate stabilization becomes possible by adhering to the policy rule outlined in equation (13)*.*

In equation (23), a positive subsidy $\tau > 0$ increases the effective natural rate from $\underline{r}$ to $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2$. This rise narrows the gap between the ZLB and the 'effective' natural rate, consequently raising $\hat{Q}_t$ relative to the scenario described in Section 3.1. It is important to note that as τ approaches infinity, the expression $\underline{r}+\frac{\tau}{1+\tau}\bar{\sigma}^2$ converges to $\underline{r}+\bar{\sigma}^2 = \rho+g > 0$. In this situation, the economy moves away from the ZLB and the monetary authority can achieve perfect stabilization by adhering to the policy rule outlined in equation (13):

$$i_t = \underbrace{\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2}_{>0} + \phi_q \hat{Q}_t - \frac{1}{2}\hat{rp}_t, \quad \phi_q > 0 . \tag{24}$$

A subsidy on risky asset holdings (or equivalently, a tax cut on capital gains) effectively raises the demand for stock market investments among capitalists. This policy stimulates both financial markets and real economic activity, particularly when operating under the constraints of the ZLB.

Taxing the Hand-to-Mouth Workers Consider an alternative scenario where the government funds the stock market subsidy τ by imposing a lump-sum tax T_t on hand-to-mouth workers. Under this policy, the budget constraint of the workers (1) becomes:

$$\frac{w_t}{\bar{p}} N_{W,t} = C_{W,t} + \frac{T_t}{\bar{p}} \,. \tag{25}$$

Hand-to-mouth workers, characterized by a marginal propensity to consume of one, experience a proportional reduction in their consumption due to taxation. This decrease adversely impacts stock dividends and prices, $\hat{Q}_t$. In this context, the formula for the stock market's expected return i_t^m is as follows:

$$i_t^m = \underbrace{\frac{y_t - \frac{w_t}{\bar{p}} N_{W,t}}{A_t Q_t}}_{\text{Dividend yield}} + \mathbb{E}_t \left[\frac{d(\cancel{\bar{p}} A_t Q_t)}{\cancel{\bar{p}} A_t Q_t} \frac{1}{dt} \right] = \underbrace{\rho - \tau i_t^m}_{\text{Dividend yield}} + \mathbb{E}_t \left[\frac{d(\cancel{\bar{p}} A_t Q_t)}{\cancel{\bar{p}} A_t Q_t} \frac{1}{dt} \right] , \tag{26}$$

where we assume an equilibrium tax T_t equal to $\tau i_t^m \bar{p} A_t Q_t$. Our analysis reveals that the effect on asset prices $\hat{Q}_t$ of the decrease in workers' consumption due to the tax T_t is exactly counterbalanced by the positive impact of stock market subsidies. As a result, there is no net effect on the dynamics of $\{\hat{Q}_t\}$ during the ZLB period, other than a redistribution of wealth from workers to capitalists. This key insight is summarized in Proposition 4.

Proposition 4 (Fiscal Subsidy and Tax on Workers) *The policy outlined in Section 3.4.1, which subsidizes the expected return on risky stock markets and is financed by a lump-sum tax on workers, does not affect the $\{\hat{Q}_t\}$ dynamics during a ZLB episode. This policy results in dynamics identical to those depicted in Figure 1.*

3.4.2 Fiscal Redistribution

Consider a redistribution policy in the form of a fiscal transfer $T_t > 0$ from capitalists to hand-to-mouth workers during a ZLB episode.[31] This policy increases aggregate demand due to the high marginal propensity to consume of workers and, in turn, the total dividends paid by the stock market index. The expected return on the stock market i_t^m then becomes:

$$i_t^m = \frac{y_t - \frac{w_t}{\bar{p}} N_{W,t}}{A_t Q_t} + \mathbb{E}_t \left[\frac{d(\cancel{p} A_t Q_t)}{\cancel{p} A_t Q_t} \frac{1}{dt} \right] = \rho + \underbrace{\frac{T_t}{\bar{p} A_t Q_t}}_{>0} + \mathbb{E}_t \left[\frac{d(\cancel{p} A_t Q_t)}{\cancel{p} A_t Q_t} \frac{1}{dt} \right] .$$

Assuming capitalists finance this transfer T_t by paying a portion φ of their wealth a_t, the dividend yield increases to $\rho + \varphi$ from a baseline yield (before transfers) of ρ. This adjustment raises the effective natural rate of interest from $\underline{r}$ to $\underline{r} + \varphi$, resulting in an increase in asset prices $\hat{Q}_t$ and a narrower output gap during a ZLB episode. Proposition 5 summarizes this result.

Proposition 5 (Fiscal Redistribution) *In the ZLB environment presented in Section 3.1, and under a redistribution scheme where a $\varphi \geq 0$ portion of capitalists' wealth is transferred to hand-to-mouth workers, the dynamic IS equation for $\hat{Q}_t$ becomes:*

$$d\hat{Q}_t = -(\underbrace{\underline{r}}_{<0} + \varphi)\, dt \ , \tag{27}$$

for $\underline{r} + \varphi < 0$. After time T, the central bank perfectly stabilizes the economy and eliminates the volatility in asset prices, $\sigma_t^q = \sigma_t^{q,n} = 0$, for all $t \geq T$. When $\underline{r} + \varphi > 0$, fiscal transfers lift the economy out of the ZLB and immediate stabilization is possible by adhering to the policy rule outlined in equation (13), with $\underline{r} + \varphi$ as the effective natural rate.

From capitalists' perspective, this policy effectively reduces their expected wealth growth by φ, taking the expected stock market return i_t^m as given. At the ZLB, i_t^m does not react to fiscal transfers due to the the binding constraint on the policy rate i_t.[32] As a result, the equilibrium growth rates of capitalists' wealth and the stock price index fall by φ, leading to a

[31] A policy subsidizing firms' payroll, financed through a lump-sum tax T_t on capitalists, produces identical results. When firms incur net payroll costs of $w_t N_{W,t} - T_t$, the consequent rise in employment effectively creates a transfer of income equivalent to T_t to the workers. We opt for the direct transfer formulation for simplicity in notation.

[32] Note from the capitalists' optimization that risk-premium rp_t is given by $\bar{\sigma}^2$ during the ZLB, and $i_t^m = i_t + \mathrm{rp}_t$.

less pronounced initial drop in asset prices $\hat{Q}_0$ at the start of the ZLB episode. Therefore, fiscal transfers to workers with a high marginal propensity to consume not only enhance aggregate demand but also create additional wealth effects which manifest through increases in dividend yields and asset prices, $\hat{Q}_t$.

4 Conclusion

This paper begins by exploring the likelihood of increased financial volatility at the ZLB, and finds that a credible commitment to economic stabilization in the future prevents excess financial volatility from developing. Next, we examine the effects of traditional forward guidance, defined by the central bank's promise to maintain a zero policy rate for an extended period of time. We show that this commitment fosters expectations of higher future asset prices and aggregate demand, thereby increasing the market valuation of households' financial wealth and, consequently, their aggregate consumption at the ZLB.

Our findings also suggest that a central bank might not always find it optimal to commit to perfectly stabilizing the business cycle in the future. By refusing to do so, the central bank opens up the possibility for alternative equilibrium paths with lower financial volatility at the ZLB and higher expected welfare. While this strategy is preferable from a welfare perspective, it involves trade-offs. Specifically, a lack of a commitment to future stabilization allows the central bank to reduce financial volatility at the ZLB, but at the expense of potentially large and costly output gap deviations in the future.

Finally, our analysis investigates the efficacy of alternative fiscal policies at the ZLB, such as subsidies for risky asset investments and fiscal redistribution among households. We show that both policies have the potential to augment the households' risk bearing capacity, resulting in a higher valuation of their financial wealth and consequently, an increase in aggregate consumption demand.

This paper aims to provide valuable insights for academics and policymakers interested in the interplay between financial uncertainty and unconventional policies at the ZLB, notably forward guidance. We leave to future research the study of central banks' communications policy under alternative scenarios, such as imperfect commitment or private information about the state of the economy.

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A Derivations and Proofs

Proof of Proposition 2. In the context outlined in Section 3.3, the central bank solves the following problem:[1]

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}'} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Q}_t^2 dt \ , \quad \text{s.t. } d\hat{Q}_t = \begin{cases} -\underbrace{r_1^T(\sigma_1^{q,L})}_{<0} dt + \sigma_1^{q,L} dZ_t \ , & \text{for } t < T \ , \\ -\underbrace{r_2^T(\sigma_2^{q,L})}_{>0} dt + \sigma_2^{q,L} dZ_t \ , & \text{for } T \le t < \hat{T}' \ , \\ 0 \ , & \text{for } t \ge \hat{T}' \ , \end{cases}$$

$$\text{with } \hat{Q}_0 = r_1^T(\sigma_1^{q,L})T + r_2^T(\sigma_2^{q,L})(\hat{T}' - T) \ , \tag{A.1}$$

where

$$r_1^T(\sigma_1^{q,L}) \equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma} + \sigma_1^{q,L})^2}{2} < 0 \ , \ r_2^T(\sigma_2^{q,L}) \equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{(\underline{\sigma} + \sigma_2^{q,L})^2}{2} > 0 \ .$$

After $\hat{T}'$, there are no additional fluctuation in $\hat{Q}_t$. Defining r_s^T as $r_1^T(\sigma_1^{q,L})$ for $s < T$ and as $r_2^T(\sigma_2^{q,L})$ for $T \le s \le \hat{T}'$, the process of $\hat{Q}_t$ can be articulated as follows:

$$\hat{Q}_t = \begin{cases} \underbrace{\int_t^{\hat{T}'} r_s^T ds}_{\equiv \hat{Q}_d(t;\hat{T}')} + \sigma_1^{q,L} \underbrace{Z_t}_{\sim N(0,t)} \ , & \text{for } \ t \le T \ , \\ \underbrace{\int_t^{\hat{T}'} r^T(s) ds}_{\equiv \hat{Q}_d(t;\hat{T}')} + \sigma_1^{q,L} Z_T + \sigma_2^{q,L} \underbrace{W_{t-T}}_{\sim N(0,t-T)} \ , & \text{for } \ T < t \le \hat{T}' \ , \\ \sigma_1^{q,L} Z_T + \sigma_2^{q,L} \underbrace{W_{\hat{T}-T}}_{\sim N(0,\hat{T}-T)} = \hat{Q}_{\hat{T}'}, & \text{for } \ \hat{T}' < t \ . \end{cases} \tag{A.2}$$

where it is assumed that after $\hat{T}'$, central banks maintain $\sigma_t^q = \sigma_t^{q,n} = 0$. In this equation, Z_t, W_{t-T}, and $U_{\hat{T}-T}$ are independent Brownian motions. If we square each term in equation (A.2) and apply the expectation operator with respect to the information available at

[1] For this proof, it is implicitly assumed that $r_1^T(\sigma_1^{q,L}) < 0$ and $r_2^T(\sigma_2^{q,L}) > 0$ hold for the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, ensuring that the ZLB remains effective up to time T.

$t = 0$, we obtain:

$$\mathbb{E}_0\, \hat{Q}_t^2 = \begin{cases} \hat{Q}_{\mathrm{d}}(t;\hat{T}')^2 + \left(\sigma_1^{q,L}\right)^2 t\,, & \text{for } t \leq T\,, \\ \hat{Q}_{\mathrm{d}}(t;\hat{T}')^2 + \left(\sigma_1^{q,L}\right)^2 T + \left(\sigma_2^{q,L}\right)^2 (t-T)\,, & \text{for } T < t \leq \hat{T}'\,, \\ \left(\sigma_1^{q,L}\right)^2 T + \left(\sigma_2^{q,L}\right)^2 (\hat{T}'-T)\,, & \text{for } \hat{T}' < t\,. \end{cases} \tag{A.3}$$

If we substitute equation (A.3) into the central bank's loss function (16), the central bank's commitment problem can be expressed as follows:

$$\begin{aligned} &\min_{\hat{T}',\sigma_1^{q,L},\sigma_2^{q,L}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Q}_t^2\, dt \\ &= \min_{\hat{T}',\sigma_1^{q,L},\sigma_2^{q,L}} \int_0^{\hat{T}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')^2 dt + \left(\sigma_1^{q,L}\right)^2 \underbrace{\int_0^T te^{-\rho t}dt}_{=\frac{1}{\rho^2}-\frac{1}{\rho^2}e^{-\rho T}-\cancel{\frac{T}{\rho}e^{-\rho T}}} + \left(\sigma_1^{q,L}\right)^2 T \underbrace{\int_T^\infty e^{-\rho t}dt}_{=\cancel{\frac{1}{\rho}e^{-\rho T}}} \\ &\quad + \left(\sigma_2^{q,L}\right)^2 \underbrace{\int_T^{\hat{T}'} e^{-\rho t}(t-T)dt}_{=-\cancel{\frac{1}{\rho}(\hat{T}-T)e^{-\rho\hat{T}'}}+\frac{e^{-\rho T}-e^{-\rho\hat{T}'}}{\rho^2}} + \left(\sigma_2^{q,L}\right)^2 (\hat{T}'-T) \underbrace{\int_{\hat{T}'}^\infty e^{-\rho t}dt}_{=\cancel{\frac{1}{\rho}e^{-\rho\hat{T}'}}} \\ &= \min_{\hat{T},\sigma_1^{q,L},\sigma_2^{q,L}} \underbrace{\int_0^{\hat{T}'} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')^2 dt}_{\text{Deterministic fluctuations}} + \underbrace{\left(\sigma_1^{q,L}\right)^2 \frac{1}{\rho^2}(1-e^{-\rho T}) + \left(\sigma_2^{q,L}\right)^2 \left(\frac{e^{-\rho T}-e^{-\rho\hat{T}'}}{\rho^2}\right)}_{\text{Stochastic fluctuations}}\,. \end{aligned} \tag{A.4}$$

The central bank now has control over $\sigma_1^{q,L}, \sigma_2^{q,L}$, and $\hat{T}'$, in addition to its conventional monetary policy tool $\{i_t\}$. Initially, we derive the first-order condition for $\hat{T}'$, which is as follows:

$$2 \cdot \underbrace{r_2^T(\sigma_2^{q,L})}_{>0} \int_0^{\hat{T}'} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')dt + \left(\sigma_2^{q,L}\right)^2 \frac{1}{\rho}e^{-\rho\hat{T}'} = 0\,, \tag{A.5}$$

from which we obtain

$$\int_0^\infty e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')dt = \int_0^{\hat{T}'} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}' \| \sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0)dt < 0\,. \tag{A.6}$$

The first-order condition for $\hat{T}'$ indicates that, at the optimum, the central bank reduces the value of $\hat{T}'$ compared to $\hat{T}$ (traditional forward guidance), as discussed in Section 3.2. This is because when the central bank utilizes traditional forward guidance and achieves perfect stabilization for $t \geq \hat{T}$, the expression above becomes

$$\int_0^{\hat{T}} e^{-\rho t} \hat{Q}_{\mathrm{d}}(t; \hat{T} \| \sigma_1^{q,L} = \sigma_1^{q,n} = 0, \sigma_2^{q,L} = \sigma_2^{q,n} = 0) dt = 0 \;, \tag{A.7}$$

which is derived by plugging $\sigma_1^{q,L} = 0$ and $\sigma_2^{q,L} = 0$ into equation (A.5).

Given that at the optimum, $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ (which we will demonstrate),

$$\hat{Q}_{\mathrm{d}}(t; \hat{T}' \| \sigma_1^{q,L} = 0, \sigma_2^{q,L} = 0) < \hat{Q}_{\mathrm{d}}(t; \hat{T}' \| \sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0) \;.$$

Therefore, we deduce from equation (A.1) that at the optimum, $\hat{T}' < \hat{T}$, as evidenced by comparing (A.7) with (A.6).

To characterize the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, a **variational argument** is required. This is because $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ influence the levels of $r_1^T(\sigma_1^{q,L})$, $r_2^T(\sigma_2^{q,L})$, and $\hat{Q}_{\mathrm{d}}(t; \hat{T}')$. Specifically, we can derive:

$$\frac{\partial r_1^T(\sigma_1^{q,L})}{\partial \sigma_1^{q,L}} = -\left(\bar{\sigma} + \sigma_1^{q,L}\right) < 0, \;\; \frac{\partial r_2^T(\sigma_2^{q,L})}{\partial \sigma_2^{q,L}} = -\left(\underline{\sigma} + \sigma_2^{q,L}\right) < 0 \;.$$

Determining $\sigma_1^{q,L}$ An increase in $\sigma_1^{q,L}$ leads to a decrease in $r_1^T(\sigma_1^{q,L})$, which alters the trajectory of $\hat{Q}_{\mathrm{d}}(t; \hat{T}')$. This change is illustrated in Figure A.1, as depicted by the transition from the thick blue line to the dashed red line.

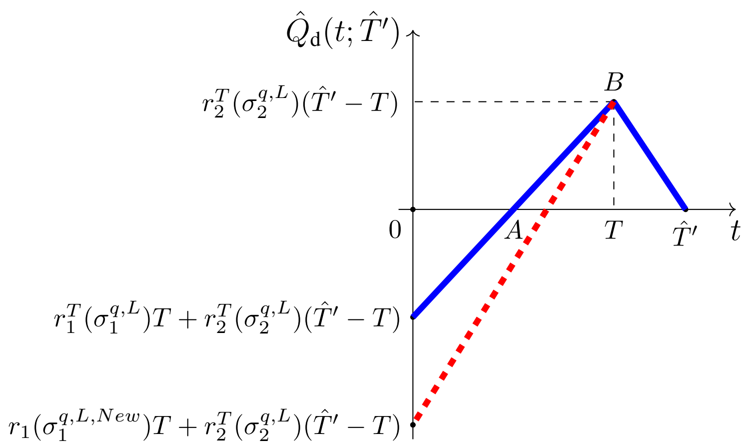

Figure A.1: Variation along $\sigma_1^{q,L}$. Increase to $\sigma_1^{q,L,New} > \sigma_1^{q,L}$.

Differentiating $\hat{Q}_{\mathrm{d}}(t;\hat{T}') = \int_t^{\hat{T}'} r_s^T ds$ with respect to $\sigma_1^{q,L}$, we obtain:

$$\frac{\partial \hat{Q}_{\mathrm{d}}(t;\hat{T}')}{\partial \sigma_1^{q,L}} = \int_t^T -\left(\bar{\sigma} + \sigma_1^{q,L}\right) ds = -\left(\bar{\sigma} + \sigma_1^{q,L}\right)(T-t), \;\; \forall t \le T\ .$$

To find optimal $\sigma_1^{q,L}$, we differentiate the objective function in (A.4) by $\sigma_1^{q,L}$ and obtain the following condition:

$$\left(\bar{\sigma} + \sigma_1^{q,L}\right) \int_0^T e^{-\rho t} \hat{Q}_{\mathrm{d}}(t;\hat{T}')(T-t)dt = \left(\sigma_1^{q,L}\right) \frac{1-e^{-\rho T}}{\rho^2}\ ,$$

from which we can prove that $\sigma_1^{q,L} < 0$ must be satisfied at the optimum, given that

$$\int_0^T e^{-\rho t} \hat{Q}_{\mathrm{d}}(t;\hat{T}')(T-t)dt = \underbrace{\int_0^t e^{-\rho s} \hat{Q}_{\mathrm{d}}(s;\hat{T}')ds \cdot (T-t)\Big|_0^T}_{=0} + \int_0^T \underbrace{\int_0^t e^{-\rho s} \hat{Q}_{\mathrm{d}}(s;\hat{T}')ds}_{<0}\, dt < 0\ ,$$

where $\int_0^t e^{-\rho s} \hat{Q}_{\mathrm{d}}(s;\hat{T}')ds < 0$ for $t \le T$, as derived in equation (A.6).

Determining $\sigma_2^{q,L}$ An increase in $\sigma_2^{q,L}$ leads to a decrease in $r_2^T(\sigma_2^{q,L})$, which alters the shape of $\hat{Q}_{\mathrm{d}}(t;\hat{T}')$. This effect is illustrated in Figure A.2 by the transition from the thick blue line to the dashed red line. To further analyze this, we differentiate $\hat{Q}_{\mathrm{d}}(t;\hat{T}')$ with respect to $\sigma_2^{q,L}$ and obtain:

$$\frac{\partial \hat{Q}_{\mathrm{d}}(t;\hat{T}')}{\partial \sigma_2^{q,L}} = \begin{cases} \int_T^{\hat{T}'} -\left(\underline{\sigma} + \sigma_2^{q,L}\right) ds = -\left(\underline{\sigma} + \sigma_2^{q,L}\right)(\hat{T}'-T)\,, & t < T\ , \\ \int_t^{\hat{T}'} -\left(\underline{\sigma} + \sigma_2^{q,L}\right) ds = -\left(\underline{\sigma} + \sigma_2^{q,L}\right)(\hat{T}'-t)\,, & T \le t \le \hat{T}'\ . \end{cases}$$

To find the optimal $\sigma_2^{q,L}$, we differentiate the objective function in (A.4) by $\sigma_2^{q,L}$ and obtain

$$\left(\underline{\sigma} + \sigma_2^{q,L}\right) \left(\int_0^T e^{-\rho t} \hat{Q}_{\mathrm{d}}(t;\hat{T}')(\hat{T}'-T)dt + \int_T^{\hat{T}'} e^{-\rho t} \underbrace{\hat{Q}_{\mathrm{d}}(t;\hat{T}')}_{>0}(\hat{T}'-t)dt \right) = (\sigma_2^{q,L}) \frac{e^{-\rho T} - e^{-\rho \hat{T}}}{\rho^2}\ ,$$

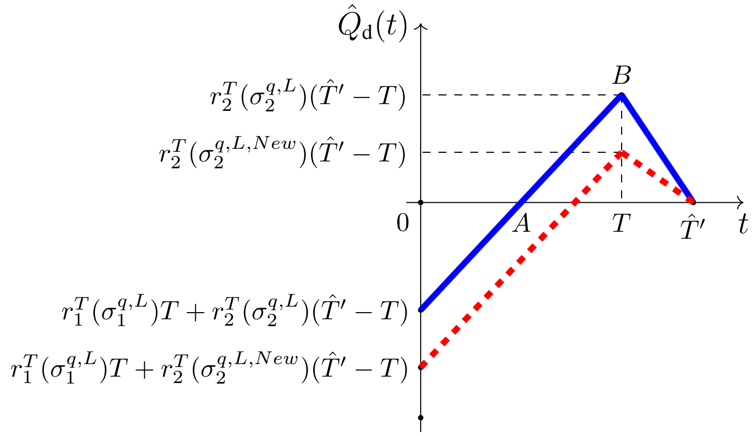

Figure A.2: Variation along $\sigma_2^{q,L}$. Increase to $\sigma_2^{q,L,New} > \sigma_2^{q,L}$.

from which we can demonstrate that at the optimum, $\sigma_2^{q,L} < 0$ must be satisfied, given that

$$
\begin{aligned}
&\int_0^T e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}')(\hat{T}' - T)dt + \int_T^{\hat{T}'} e^{-\rho t}\underbrace{\hat{Q}_{\text{d}}(t;\hat{T}')}_{>0}(\hat{T}' - t)dt \\
&< \int_0^T e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}')(\hat{T}' - T)dt + \int_T^{\hat{T}'} e^{-\rho t}\underbrace{\hat{Q}_{\text{d}}(t;\hat{T}')}_{>0}(\hat{T}' - T)dt \\
&= (\hat{T}' - T)\underbrace{\int_0^{\hat{T}'} e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}')dt}_{<0} < 0 ,
\end{aligned}
$$

where the final inequality is derived from equation (A.6). Hence, we have proven that during periods of high TFP volatility (i.e., $t < T$) and low TFP volatility with forward guidance (i.e., $T \leq t \leq \hat{T}'$), a central bank aims to target financial volatility levels below those in a flexible price economy: $\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ and $\sigma_2^{q,L} < \sigma_2^{q,n} = 0$. Such intervention reduces the required risk premium and raises the asset price level $\hat{Q}_t$, thereby increasing output.

■

First-Order Conditions for $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}'$ The deterministic component of the capitalists' asset gap process $\hat{Q}_t$, denoted as $\hat{Q}_{\text{d}}(t;\hat{T}')$, is defined as follows (with $r_1^T(\sigma_1^{q,L})$ and

$r_2^T(\sigma_2^{q,L})$ specified in equation (18)):

$$\hat{Q}_{\mathrm{d}}(t;\hat{T}') = \int_t^{\hat{T}'} r_s^T ds = \begin{cases} \underbrace{r_1^T(\sigma_1^{q,L})}_{<0}(T-t) + \underbrace{r_2^T(\sigma_2^{q,L})}_{>0}(\hat{T}'-T), & \text{for } \forall t \leq T \ , \\ r_2^T(\sigma_2^{q,L})(\hat{T}'-t), & \text{for } T \leq \forall t < \hat{T}' \ , \end{cases}$$

from which we derive the following:

$$\int_0^{\hat{T}'} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')dt = \int_0^T e^{-\rho t}\left[r_1^T(\sigma_1^{q,L})(T-t) + r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\right]dt + \int_T^{\hat{T}'} e^{-\rho t} r_2^T(\sigma_2^{q,L})(\hat{T}'-t)dt \ . \tag{A.8}$$

The first condition for $\hat{T}'$ can be written as

$$2 \cdot r_2^T(\sigma_2^{q,L}) \int_0^{\hat{T}'} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')dt + \left(\sigma_2^{q,L}\right)^2 \frac{e^{-\rho\hat{T}'}}{\rho} = 0 \ , \tag{A.9}$$

where

$$\begin{aligned}\int_0^{\hat{T}'} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')dt =& r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\frac{1-e^{-\rho T}}{\rho} \\ &+ r_2^T(\sigma_2^{q,L})\left[\frac{e^{-\rho\hat{T}'}}{\rho^2} + \frac{\hat{T}'-T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\right] \ ,\end{aligned}$$

follows from equation (A.8). Combined with equation (A.9), the first-order condition for $\hat{T}'$ is expressed as follows:

$$\begin{aligned}2 \cdot r_2^T(\sigma_2^{q,L})\Bigg[& r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\frac{1-e^{-\rho T}}{\rho} \\ &+ r_2^T(\sigma_2^{q,L})\left[\frac{e^{-\rho\hat{T}'}}{\rho^2} + \frac{\hat{T}'-T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\right]\Bigg] + \left(\sigma_2^{q,L}\right)^2 \frac{e^{-\rho\hat{T}'}}{\rho} = 0 \ .\end{aligned}$$

The first-order condition for $\sigma_1^{q,L}$ is expressed as

$$\left(\bar{\sigma} + \sigma_1^{q,L}\right)\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')(T-t)dt = \left(\sigma_1^{q,L}\right)\frac{1-e^{-\rho T}}{\rho^2} \ , \tag{A.10}$$

where

$$\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')(T-t)dt = r_1^T(\sigma_1^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho T}+\frac{T^2}{\rho}-\frac{2T}{\rho^2}+\frac{2}{\rho^3}\right] \\ + r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right] . \tag{A.11}$$

Substituting equation (A.11) into equation (A.10), we arrive at the following result:

$$(\bar{\sigma}+\sigma_1^{q,L})\left[r_1^T(\sigma_1^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho T}+\frac{T^2}{\rho}-\frac{2T}{\rho^2}+\frac{2}{\rho^3}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]\right] \\ = (\sigma_1^{q,L})\frac{1-e^{-\rho T}}{\rho^2} ,$$

as the first-order condition for $\sigma_1^{q,L}$. Finally, the first-order condition for $\sigma_2^{q,L}$ is as follows:

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left((\hat{T}'-T)\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')dt+\int_T^{\hat{T}'} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')(\hat{T}'-t)dt\right) = (\sigma_2^{q,L})\frac{e^{-\rho T}-e^{-\rho\hat{T}'}}{\rho^2} ,$$

where

$$\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')dt = r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\frac{1-e^{-\rho T}}{\rho} ,$$

and

$$\int_T^{\hat{T}'} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}')(\hat{T}'-t)dt = r_2^T(\sigma_2^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho\hat{T}'}+\frac{(\hat{T}'-T)^2}{\rho}e^{-\rho T}-\frac{2(\hat{T}'-T)}{\rho^2}e^{-\rho T}+\frac{2}{\rho^3}e^{-\rho T}\right] .$$

Therefore, the first-order condition for $\sigma_2^{q,L}$ is expressed as follows:

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left[\left[r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\frac{1-e^{-\rho T}}{\rho}\right](\hat{T}'-T) \\ + r_2^T(\sigma_2^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho\hat{T}'}+\frac{(\hat{T}'-T)^2}{\rho}e^{-\rho T}-\frac{2(\hat{T}'-T)}{\rho^2}e^{-\rho T}+\frac{2}{\rho^3}e^{-\rho T}\right]\right] \\ = \left(\sigma_2^{q,L}\right)\frac{e^{-\rho T}-e^{-\rho\hat{T}'}}{\rho^2} .$$

A.1 Section 3.4

Proof of Proposition 3. We begin by solving the capitalist's problem presented in equation (21), considering a subsidy rate τ on stock market investments for $t \leq T$:

$$
\begin{aligned}
&\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \\
&\text{s.t. } da_t = (a_t(i_t + \theta_t((1+\tau)\, i_t^m - i_t)) - \bar{p}C_t - T_t)dt + \theta_t a_t \left(\bar{\sigma} + \sigma_t^q\right) dZ_t \ .
\end{aligned} \tag{A.12}
$$

Since the subsidy τ is financed through a lump-sum tax on capitalists, the dividend process in equation (4) and the stock market valuation equation (5) remain unchanged. As a result, $\bar{p}C_t = \rho a_t$ and $C_t = \rho A_t Q_t$. Equilibrium taxes T_t equal to $\tau i_t^m a_t$, and the budget constraint in equation (A.12) becomes

$$
\begin{aligned}
\frac{dC_t}{C_t} = \frac{da_t}{a_t} &= ((1+\cancel{\tau})i_t^m - \rho - \cancel{\tau i_t^m})dt + (\bar{\sigma} + \sigma_t^q)dZ_t \\
&= (i_t^m - \rho)dt + (\bar{\sigma} + \sigma_t^q)dZ_t \ ,
\end{aligned} \tag{A.13}
$$

where we used equilibrium condition $\theta_t = 1$. Since $\xi_t^N = e^{-\rho t}\frac{1}{\bar{p}C_t}$, we obtain:

$$
\begin{aligned}
\frac{d\xi_t^N}{\xi_t^N} &= -\rho dt - \frac{dC_t}{C_t} + \left(\frac{dC_t}{C_t}\right)^2 \\
&= -\cancel{\rho} dt - [(i_t^m - \cancel{\rho})dt + (\bar{\sigma} + \sigma_t^q)dZ_t] + (\bar{\sigma} + \sigma_t^q)^2 dt \\
&= -\left[i_t^m - (\bar{\sigma} + \sigma_t^q)^2\right] dt - (\bar{\sigma} + \sigma_t^q)dZ_t \ .
\end{aligned} \tag{A.14}
$$

The subsidy τ on the expected return i_t^m alters the original Euler equation $\mathbb{E}_t \frac{d\xi_t^N}{\xi_t^N} = -i_t dt$. Consequently, as derived from equation (A.14), the revised expression is:

$$
\mathbb{E}_t \frac{d\xi_t^N}{\xi_t^N} = -\left[(1+\tau) i_t^m - (\bar{\sigma} + \sigma_t^q)^2\right] = -i_t dt \ ,
$$

from which we obtain equation (22):

$$
i_t^m = \frac{i_t + (\bar{\sigma} + \sigma_t^q)^2}{1+\tau} = \frac{\bar{\sigma}^2}{1+\tau} \ ,
$$

where the final equality results from substituting $i_t = 0$ and $\sigma_t^q = 0$ into the equation. From equation (A.13),it follows that:

$$\frac{dC_t}{C_t} = (i_t^m - \rho)dt + \bar{\sigma} dZ_t = \left(\frac{\bar{\sigma}^2}{1+\tau} - \rho\right) dt + \bar{\sigma} dZ_t \ , \tag{A.15}$$

with which we obtain

$$d \ln C_t = \left(\frac{\bar{\sigma}^2}{1+\tau} - \rho - \frac{\bar{\sigma}^2}{2}\right) dt + \bar{\sigma} dZ_t \ .$$

Finally, by using equation (B.6) from Appendix B, we derive the natural counterpart to the above expression:

$$d \ln C_t^n = \left(\underbrace{\bar{r}}_{<0} - \rho + \frac{\bar{\sigma}^2}{2}\right) + \bar{\sigma} dZ_t \ . \tag{A.16}$$

Combining both expressions, we obtain the dynamic IS equation in (23).

■

Proof of Proposition 4. By equation (26), the condition that characterizes the equilibrium stock market return i_t^m is given by:

$$i_t^m = \frac{y_t - \overbrace{\frac{w_t}{\bar{p}} N_{W,t}}^{=C_{W,t} + \frac{T_t}{\bar{p}}}}{A_t Q_t} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} = \underbrace{\rho - \tau i_t^m}_{\text{Dividend yield}} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \ ,$$

from which we obtain $(1+\tau) i_t^m = \rho + g + \mu_t^q$ using $\sigma_t^q = 0$. Since $(1+\tau) i_t^m = \bar{\sigma}^2$ by equation (22), we infer that μ_t^q remains constant in comparison to the scenario without subsidy, conditional on $i_t = 0$ and $\sigma_t^q = 0$. Therefore, the subsidy policy does not alter the $\{\hat{Q}_t\}$ process. To align this intuition with the mathematical representation, we begin by examining the process for C_t, which is different from that in equation (A.15), as capitalists are now exempt from paying taxes T_t:

$$\begin{aligned} \frac{dC_t}{C_t} &= ((1+\tau) i_t^m - \rho) dt + \bar{\sigma} dZ_t \\ &= (\bar{\sigma}^2 - \rho) dt + \bar{\sigma} dZ_t \ . \end{aligned}$$

Given that the previous expression remains unchanged in the presence of subsidy τ, it can be inferred that a policy subsidizing the expected return of the stock market and financed by a lump-sum tax on workers does not impact the $\{\hat{Q}_t\}$ process. Consequently, the dynamics of $\{\hat{Q}_t\}$ are identical to those in an economy without this policy.

■

Proof of Proposition 5. A fiscal transfer $T_t > 0$ from capitalists to hand-to-mouth workers increases the aggregate dividends in the financial market. This results in a reduced need for expected future capital gains, which translates into higher asset prices $\hat{Q}_t$ at the ZLB. The expected stock market return i_t^m under these circumstances is given by:

$$
\begin{aligned}
i_t^m &= \frac{A_t N_{W,t} - \overbrace{\frac{w_t}{\bar{p}} N_{W,t}}^{=C_{W,t}-\frac{T_t}{\bar{p}}}}{A_t Q_t} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t}\frac{1}{dt} = \rho + \underbrace{\frac{T_t}{\bar{p} A_t Q_t}}_{>0} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t}\frac{1}{dt} \\
&= \rho + \varphi + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t}\frac{1}{dt} ,
\end{aligned}
$$

where the last equality follows from T_t being equal to $\varphi \bar{p} A_t Q_t$ in equilibrium.

To derive equation (27), we start from the capitalists' optimization problem:

$$
\begin{aligned}
&\max_{C_t, \theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \\
&\text{s.t. } da_t = (a_t(i_t + \theta_t(i_t^m - i_t)) - \bar{p} C_t - T_t)dt + \theta_t a_t(\bar{\sigma} + \sigma_t^q)dZ_t ,
\end{aligned}
$$

which features equilibrium conditions for C_t and θ_t identical to those described in equations (5) and (6), together with $\sigma_t^q = 0$. As a result, $C_t = \rho \bar{p} A_t Q_t$ and $i_t^m = i_t + (\bar{\sigma} + \sigma_t^q)^2$ follows. In an equilibrium where $\sigma_t^q = 0$ and i_t is constrained by the ZLB, the wealth process for capitalists is given by:

$$
\frac{dC_t}{C_t} = \frac{da_t}{a_t} = (i_t^m - \rho - \varphi)\, dt + \bar{\sigma}_t dZ_t = (\bar{\sigma}^2 - \varphi - \rho)dt + \bar{\sigma}_t dZ_t ,
$$

from which we derive

$$
d \ln C_t = \left(\frac{\bar{\sigma}^2}{2} - \varphi - \rho \right) dt + \bar{\sigma}_t dZ_t .
$$

Subtracting the process for C_t^n in equation (A.16) yields the dynamic IS equation in (27).

■

B Flexible Price Equilibrium

This section derives the flexible price equilibrium of the model, establishing it as the benchmark for economic and welfare analysis. We begin by revisiting the Fisherian identity, incorporating an inflation premium linked to wealth volatility into the relation. Lemma 1 summarizes the modified identity.

Lemma 1 (Inflation Premium) *The real interest rate of the economy is given by:*

$$r_t = i_t - \pi_t + \overbrace{\sigma_t^p \underbrace{\left(\sigma + \sigma_t^p + \sigma_t^q\right)}_{\textit{Wealth volatility}}}^{\textit{Inflation Premium}} \, . \tag{B.1}$$

Proof of Lemma 1. The financial wealth of capitalists is equal to the value of the stock market index, $a_t = p_t A_t Q_t$, which follows from bonds being in zero net supply and capitalists being symmetric and identical in equilibrium. We start by stating capitalist's nominal state-price density ξ_t^N, which satisfies the following condition:

$$\frac{d\xi_t^N}{\xi_t^N} = -i_t dt - (\sigma + \sigma_t^q) dZ_t \; ,$$

and the real state price density ξ_t^r, which is given by

$$\xi_t^r = e^{-\rho t} \frac{1}{C_t} = p_t \xi_t^N \; . \tag{B.2}$$

Utilizing equations (2) and (3), and considering that $\theta_t = 1$ in equilibrium, the application of Ito's Lemma to equation (B.2) yields the following expression:

$$\frac{d\xi_t^r}{\xi_t^r} = \left(\underbrace{\pi_t - i_t - \sigma_t^p \left(\sigma + \sigma_t^q + \sigma_t^p\right)}_{= -r_t} \right) dt - (\sigma + \sigma_t^q) dZ_t \; ,$$

resulting in the modified Fisherian identity detailed in equation (B.1).

■

Definition 1 *Let* $\chi^{-1} \equiv \dfrac{1-\varphi}{\chi_0 + \varphi}$ *represent the effective labor supply elasticity of workers, conditional on their optimal consumption decision.*

Proposition 6 summarizes the dynamics of the real wage, asset price, natural interest rate r_t^n, and the consumption process of capitalists within the flexible price equilibrium.

Proposition 6 (Flexible Price Equilibrium) *In the flexible price equilibrium,*[2] *the following results are obtained:*

1. *The real wage is proportional to aggregate technology A_t, and given by*

$$\frac{w_t^n}{p_t} = \frac{(\epsilon-1)(1-\alpha)}{\epsilon} A_t \ .$$

2. *The equilibrium asset price Q_t^n is constant and given by*

$$Q_t^n = \frac{1}{\rho}\left(\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)^{\frac{1}{\chi}}\left(1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \ , \quad \textit{and} \quad \mu_t^{q,n} = \sigma_t^{q,n} = 0 \ .$$

3. *The natural interest rate r_t^n is constant and defined as $r_t^n \equiv r^n = \rho + g - \sigma^2$. The consumption of capitalists evolves according to the following equation:*

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma dZ_t = \underbrace{\left(r^n - \rho + \sigma^2\right)}_{\equiv \mu_t^{c,n}} dt + \underbrace{\sigma}_{\equiv \sigma_t^{c.n}} dZ_t \ .$$

Proof of Proposition 6. Starting with the optimization problem of intermediate firms, the presence of an externality à la Baxter and King (1991) imposes extra steps on the aggregation process of individual decisions across firms. Utilizing the production function, the employed labor of firm i can be expressed as

$$n_t(i) = \left(\frac{y_t(i)}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \ ,$$

where we defined $E_t \equiv (N_{W,t})^{\alpha}$. At any given time t, each intermediate firm i determines the optimal price $p_t(i)$ to maximize its profits,

$$\max_{p_t(i)} p_t(i)\left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} y_t - w_t \left(\frac{y_t}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} \ , \tag{B.3}$$

taking the aggregate demand of the economy y_t as given. In the flexible price equilibrium, all firms charge the same price, $p_t(i) = p_t$ for all i, and hire the same amount of labor,

[2] Variables in the flexible price (i.e., natural) equilibrium are denoted with the superscript n.

$n_t(i) = N_{w,t}$ for all i. From the first-order condition (B.3), we obtain the real wage as

$$\frac{w_t^n}{p_t} = \frac{\epsilon-1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} N_{W,t}{}^{\frac{\alpha}{1-\alpha}} = \frac{\epsilon-1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} \left(\frac{w_t^n}{p_t^n}\right)^{\frac{\alpha}{\chi(1-\alpha)}} A_t^{\frac{-\alpha}{\chi(1-\alpha)}} ,$$

which can be further simplified to the following expression:

$$\frac{w_t^n}{p_t} = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{\chi(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\chi\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{\chi-\alpha}{\chi(1-\alpha)-\alpha}} .$$

Aggregate production in the flexible price equilibrium is linear, $y_t = A_t N_{W,t}$. Therefore, we obtain:

$$y_t = A_t \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{1-\frac{\alpha}{\chi}}{\chi(1-\alpha)-\alpha}} A_t^{-\frac{1}{\chi}} .$$

The previous expression allows us to write the natural level of output y_t^n and the natural real wage $\frac{w_t^n}{p_t}$ as

$$y_t^n = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t \text{ and } \frac{w_t^n}{p_t} = \frac{\epsilon-1}{\epsilon}(1-\alpha) A_t ,$$

from which we obtain

$$N_{W,t}^n = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} \text{ and } C_{W,t}^n = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t . \tag{B.4}$$

In equilibrium, the combined consumption of capitalists and workers equates to the total final output, as detailed in equation (7). Following from equation (B.4), we obtain:

$$\rho A_t Q_t^n + \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t .$$

where we defined Q_t^n to be the natural stock price. Therefore, we obtain an expression for Q_t^n as

$$Q_t^n = \frac{1}{\rho}\left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) ,$$

and $C_t^n = \rho A_t Q_t^n$. Since Q_t^n is constant in equilibrium, its process in a flexible price economy exhibits neither drift nor volatility, which implies $\mu_t^{q,n} = \sigma_t^{q,n} = 0$. To determine the natural interest rate r_t^n, we start from the capital gain component outlined in equation (8). The application of Ito's lemma yields:

$$\mathbb{E}_t \frac{d\left(p_t A_t Q_t\right)}{p_t A_t Q_t} \frac{1}{dt} = \pi_t + \underbrace{\mu_t^q}_{=0} + g + \underbrace{\sigma_t^q}_{=0} \sigma_t^p + \sigma \left(\sigma_t^p + \underbrace{\sigma_t^q}_{=0} \right) .$$

Given a constant dividend yield equal to ρ, applying expectations to both sides of equation (8) and combining this expression with the equilibrium condition presented in equation (6) results in:

$$i_t^m = \rho + \pi_t + g + \sigma \sigma_t^p = i_t + \left(\sigma + \sigma_t^p\right)^2 .$$

Inserting the previous expression into the Fisherian identity in equation (B.1), we express the natural rate of interest r_t^n as

$$r_t^n = i_t - \pi_t + \sigma_t^p \left(\sigma + \underbrace{\sigma_t^{q,n}}_{=0} + \sigma_t^p \right) = \rho + g - \sigma^2 , \tag{B.5}$$

which is a function of structural parameters, including σ, thereby proving the final point of Proposition 6. As the consumption of capitalists C_t^n is directly proportional to the level of technology A_t, it follows that:

$$\frac{dC_t^n}{C_t^n} = g dt + \sigma dZ_t = \left(r_t^n - \rho + \sigma^2\right) dt + \sigma dZ_t , \tag{B.6}$$

where the last equality is derived using equation equation (B.5).

■

B.1 Co-movements between gap variables

The following Lemma 2 demonstrates that Assumption 1 serves as a sufficient condition for the model to exhibit the empirical regularities of positive co-movements between asset prices and various business cycle variables, such as real wage and consumption (of capitalists and workers), as observed in data.[3]

[3] For plausible calibration, see Table 1 of Appendix A in Lee and Dordal i Carreras (2023).

Assumption 1 (Labor Supply Elasticity) *The effective labor supply elasticity of workers satisfies:* $\chi^{-1} > \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}$.

Lemma 2 (Positive comovement) *Under Assumption 1, the consumption gaps of capitalists C_t and workers $C_{W,t}$, employment $N_{W,t}$, and the real wage $\frac{w_t}{p_t}$ exhibit joint positive comovement. This relationship is approximated up to a first-order as follows:*

$$\hat{Q}_t = \hat{C}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \widehat{\frac{w_t}{p_t}} = \frac{1}{1+\chi^{-1}} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right) \widehat{C_{W,t}} \ ,$$

and is related to the output gap of the economy by:

$$\hat{y}_t = \zeta \hat{Q}_t \ , \ \text{ where } \ \zeta \equiv \chi^{-1} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1} > 0 \ . \tag{B.7}$$

Proof of Lemma 2. From $C_t = \rho A_t Q_t$, we obtain $\hat{C}_t = \hat{Q}_t$. We start from the flexible price economy's good market equilibrium condition, which can be written as

$$A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} = \rho A_t Q_t^n + \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} \ , \tag{B.8}$$

where $\frac{w_t^n}{p_t^n}$ is the real wage in the flexible price economy. We subtract equation (B.8) from the analogous good market condition in the sticky price economy, which yields the following result:

$$A_t \left(\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}\right) \frac{1}{A_t^{\frac{1}{\chi}}} = (C_t - C_t^n) + \left(\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}\right) \frac{1}{A_t^{\frac{1}{\chi}}} \ .$$

Dividing both sides of the previous expression by $y_t^n \equiv A_t^{1-\frac{1}{\chi}} (\frac{w_t^n}{p_t^n})^{\frac{1}{\chi}}$, we obtain:

$$\underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}{\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{1}{\chi}\widehat{\frac{w_t}{p_t}}} = \underbrace{\frac{C_t^n}{A_t^{1-\frac{1}{\chi}} \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}} \hat{C}_t + \underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}}{A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\left(1+\frac{1}{\chi}\right)\widehat{\frac{w_t}{p_t}}} \ ,$$

which can be written as

$$\frac{1}{\chi}\frac{\widehat{w_t}}{p_t} = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)\hat{C}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\underbrace{\left(1+\frac{1}{\chi}\right)\frac{\widehat{w_t}}{p_t}}_{=\hat{C^w}(t)} .$$

The previous expression together with $\hat{C}_t = \hat{Q}_t$ leads to

$$\hat{Q}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0}\frac{\widehat{w_t}}{p_t} = \underbrace{\frac{1}{1+\chi^{-1}}\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0}\widehat{C_{W,t}} .$$

Finally, equation (B.7) follows by combining the previous expression together with the market clearing condition $y_t = C_t + C_{W,t}$.

■