Higher-Order Forward Guidance*

Marc Dordal i Carreras† Seung Joo Lee‡

August 19, 2024

Abstract

This paper presents a business cycle model that incorporates financial markets and endogenous financial volatility at the Zero Lower Bound (ZLB). We uncover three key insights: (i) Central banks can reduce excess financial volatility at the ZLB by credibly committing to future economic stabilization; (ii) Alternatively, a commitment to refrain from future stabilization can guide the economy towards more favorable equilibrium paths, highlighting a trade-off between future stabilization and reduced financial volatility at the ZLB; (iii) Maintaining some uncertainty about the timing of future stabilization is strictly superior to other forms of forward guidance commitments.

Keywords: Monetary Policy, Forward Guidance, Financial Volatility, Risk-Premium
JEL Codes: E32, E43, E44, E52, E62

*We appreciate Yuriy Gorodnichenko for his continuous mentorship. We are grateful to Jordi Galí, Nicolae Gârleanu, Pierre-Olivier Gourinchas, Chen Lian, Yang Lu, Albert Marcet, Maurice Obstfeld, Walker Ray, Alp Simsek, and participants at the Hong Kong junior macro group meeting and CREi-UPF macroeconomics seminar. This paper was previously circulated under the title 'Monetary Policy as a Financial Stabilizer'.
†Hong Kong University of Science and Technology (marcdordal@ust.hk).
‡Saïd Business School, Oxford University (seung.lee@sbs.ox.ac.uk).

1 Introduction

In the aftermath of the Great Recession and the recent Covid-19 pandemic, prolonged periods of constrained policy rates at the Zero Lower Bound (ZLB) have emphasized the need for alternative monetary interventions, particularly forward guidance. ZLB episodes are often marked by increased financial market volatility, exacerbated by the diminished effectiveness of conventional monetary policy tools. In this context, forward guidance extends beyond its traditional roles of conveying economic forecasts (Delphic guidance) and making policy commitments (Odyssean guidance) to serve as a tool for coordinating market participant actions and reducing overall economic uncertainty. In this paper, we introduce an analytically tractable framework to examine the effects of forward guidance policies at the ZLB, demonstrating that the central bank can strategically manage the intertemporal evolution of aggregate uncertainty to enhance welfare. For instance, by creating controlled uncertainty about future economic conditions, the monetary authority can effectively reduce current market volatility at the ZLB. We refer to such interventions as higher-order guidance, in contrast to traditional communication strategies that primarily focus on influencing the levels or expectations of economic variables.

Our paper builds on a model that integrates endogenous financial volatility within a Two-Agent New Keynesian (TANK) framework. The model features a representative stock market index that encapsulates the ownership rights to the profits of firms in the economy. A group of hand-to-mouth workers supplies labor to these firms, while a group of capitalists holds the economy's aggregate financial wealth, allocating it between consumption and portfolio choices. In equilibrium, the wealth of capitalists is directly affected by the stock market performance. In this environment, an increase in endogenous financial volatility raises market risk-premium, leading to depressed asset prices and wealth of capitalists and lowering in turn aggregate demand, whose fluctuation determines the endogenous financial volatility itself.[1] This dynamic creates a coordination challenge for economic agents that has the potential to lead to self-fulfilling shocks in volatility, resulting in an endogenous state of elevated financial volatility. While Lee and Dordal i Carreras (2024a) investigates the determinacy of the model's solutions under conventional monetary policy regimes in a nonlinear New Keynesian environment, this paper focuses on whether central bank forward guidance can steer agents towards equilibrium paths with lower financial volatility and

[1] A decline in aggregate demand leads to reduced firm profitability, negatively impacting both the stock market capitalization and the aggregate wealth of capitalists. Therefore, economic and financial market volatility are tightly connected in our framework.

quicker economic stabilization times at the ZLB.

Our analysis begins by exploring whether financial volatility intensifies when conventional monetary policy is constrained by the ZLB. We find that a credible commitment from the central bank to stabilize the economy *after* the ZLB period can also ensure that excess volatility does not appear *during* the ZLB. This conclusion is derived through backward induction: if the monetary authority credibly commits to stabilize the economy in a finite period of time, it rules out the possibility of catastrophic (or exuberant) scenarios that contribute to the economic volatility faced by the agents. As a result, this precludes the feasibility of the unfavorable coordination equilibrium paths that would initially lead to these scenarios.

We then analyze the benefits of various forward guidance strategies. In our framework, traditional forward guidance includes an Odyssean component, wherein the central bank credibly commits to maintaining the policy rate at zero for a period longer than minimally required by economic conditions. After this extended ZLB period, the central bank adopts a policy rule aimed at perfect stabilization outside the ZLB. The outcomes of this policy are consistent with those identified in previous research: by committing to a future period of accommodative policy rates, the central bank implicitly agrees to a temporary phase of positive excess demand and profits. This effect, driven by the forward-looking nature of stock markets, positively impacts current stock values, thereby boosting aggregate demand during the ZLB. This approach distributes the costs of the ZLB over time and is preferred from a welfare perspective. Moreover, the commitment to perfect future stabilization continues to mitigate excess financial volatility at the ZLB, as previously discussed.

The novel strategy we consider next explicitly leverages the agents' coordination problem to guide them toward an equilibrium with reduced financial and economic volatility at the ZLB. We term this approach *higher-order forward guidance*. For its execution, the central bank must relinquish the promise of perfect stabilization in the future: by committing not to enforce perfect stabilization at the conclusion of the Odyssean guidance period, the central bank enables the existence of multiple coordinated equilibria that were previously ruled out by backward induction. This strategy allows the central bank to guide agents towards equilibrium paths with low levels of volatility and risk premiums at the ZLB, thereby maximizing expected welfare beyond the capabilities of traditional forward guidance (which we identify as a limiting case of this strategy). However, this intervention has its trade-offs: by committing not to stabilize the business cycle after the ZLB period, the central bank risks significant future output gap deviations. Thus, our higher-order guidance weighs the lack of stabilization in the future economy against reduced financial volatility in the present while at the ZLB. Furthermore, we uncover that even the central bank's slight hint that perfect stabilization is not guaranteed at the conclusion of the Odyssean guidance period makes our higher-order forward guidance strategy viable.[2]

Finally, we discuss the options available to the central bank for enforcing its preferred equilibrium at the ZLB among the multiple possible solutions generated by the promise of a passive future monetary policy. We present an example where *off-equilibrium* threats of fiscal intervention —entailing zero transfers along the equilibrium path— are sufficient to establish the optimal higher-order guidance solution as the unique equilibrium of the model. Although we conjecture that other forms of equilibrium selection are possible, this result underscores the critical importance of coordination between monetary and fiscal authorities at the ZLB and the credibility placed by the public on the monetary authority's communications and commitments.

Featuring a demand-driven economy with perfectly rigid prices,[3] our framework emphasizes the significant impact of stock market performance on aggregate demand. Unlike prior studies, such as Akerlof and Yellen (1985), Blanchard and Kiyotaki (1987), Eggertsson and Krugman (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), and Schmitt-Grohé and Uribe (2016), who focus on demand-driven recessions due to deleveraging borrowers and aggregate demand externalities, our model triggers the ZLB episodes with a decrease in aggregate demand for risky assets, identified as a key driver of financial recessions by Caballero and Farhi (2017) and Caballero and Simsek (2020). In a similar way to Werning (2012), we assume the economy's exogenous and deterministic shift to the ZLB, here resulting from a shock that raises the risk premium in financial markets and reduces the demand for risky assets, resulting in a downward jump in the natural rate of interest to a negative territory. Our approach diverges from the literature by including an endogenous component to financial volatility, influenced by both the ZLB and forward guidance. Several previous works, such as Bloom (2009), Basu and Bundick

[2]To be specific, we prove that if the central bank promises there is a tiny probability that the business cycle might not be stabilized at the end of the Odyssean forward guidance period, then the higher-order forward guidance strategy becomes viable. Our model features a novel discontinuity in that regard: if the monetary authority achieves perfect stabilization with certainty after the ZLB period, we return to the traditional forward guidance case in which no excess volatility or risk premium is manipulated by the central bank. Even with a slight chance that perfect stabilization is relinquished, the central bank can engineer a better equilibrium with lower levels of financial volatility and risk premiums based on our higher-order forward guidance strategy.

[3]This assumption simplifies the analysis. An extended model with sticky prices à la Calvo (1983) produces qualitatively similar results.

(2017), and Bloom et al. (2018), suggest that uncertainty shocks can drive macroeconomic fluctuations. In particular, Basu and Bundick (2017) examine the stabilizing role of monetary policy in the presence of uncertainty shocks, highlighting the ZLB as a factor that exacerbates the decline in output and its components during periods of heightened uncertainty. While this literature similarly investigates the impact of uncertainty shocks on the business cycle, our work differs in several key aspects: whereas these studies primarily address exogenous uncertainty, we explore the role of endogenous volatility. Additionally, at the ZLB, we focus on the strategic creation of uncertainty in the future as a mechanism to reduce volatility at the present, demonstrating that central banks can engage in strategic intertemporal uncertainty management through equilibrium selection.

Papers including Eggertsson et al. (2003), Campbell et al. (2012, 2019), Del Negro et al. (2013), McKay et al. (2016), and Caballero and Farhi (2017) explore the implications of forward guidance at the ZLB from both theoretical and empirical perspectives. Our research distinguishes itself by focusing specifically on the impact of forward guidance on higher-order moments including the endogenous volatility of financial markets and the broader economy.[4]

Layout The structure of this paper is organized as follows: Section 2 presents the model. Section 3 discusses the incorporation of the ZLB into our framework. Section 4 examines the effectiveness of various forward guidance strategies. Section 5 provides concluding remarks.[5] The Online Appendix contains additional derivations and proofs. Specifically, Online Appendix H provides an analysis within the non-linear textbook New Keynesian model and demonstrates that the main equilibrium conditions and results are isomorphic to those of the model with financial markets presented here.

2 The Model

We begin by introducing a theoretical framework that facilitates the analysis of higher-order moments related to the aggregate financial and economic volatility of the economy.[6]

[4]Our approach, where central bank communications serve as an equilibrium coordination device, aligns well with the concept of 'open-mouth' operations at the ZLB described by Campbell and Weber (2019).

[5]Appendix I contains the parameter calibration, and derivations and proofs are detailed in Appendix II.

[6]Our results, except those in Section F, also hold in a non-linear version of the standard New Keynesian model (e.g., Woodford (2003) and Galí (2015)). We choose a Two-Agent New Keynesian (TANK) model representation because it clarifies the interaction among financial volatility, risk-premium, aggregate wealth, and aggregate demand, and allows us to study various macroprudential policies in a tractable way, as shown

2.1 Setting

We consider a continuous-time framework within a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$. The economy is composed of two equally sized agent groups: capitalists, characterized as neoclassical agents, and hand-to-mouth workers, conceptualized as Keynesian agents. This structure, closely aligned with the approach of Greenwald et al. (2014) and Caballero et al. (2024), assumes that all financial wealth is held by capitalists, while workers rely on labor income for consumption. The aggregate technology, denoted by A_t, introduces a single source of exogenous variation in the model and generates the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. The process evolves according to a geometric Brownian motion given by:

$$\frac{dA_t}{A_t} = \underbrace{g}_{\text{Growth}} dt + \underbrace{\sigma_t}_{\text{Fundamental risk}} dZ_t \ ,$$

where g represents the expected growth rate, and σ_t signifies the economy's *fundamental* risk, which we take as exogenous. For simplicity, σ_t is initially assumed constant and equal to σ in Section 2. Later in Section 3, we introduce a deterministic shift in σ_t to explore various scenarios involving the ZLB.

2.1.1 Firms

The economy features a unit measure of monopolistically competitive firms, each producing a unique intermediate good $y_t(i)$, for $i \in [0, 1]$. These intermediate firms contribute to the final good y_t through a Dixit-Stiglitz aggregation function with a substitution elasticity $\epsilon > 0$, as given by:

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} .$$

Each intermediate firm i employs a production function $y_t(i) = A_t(N_{W,t})^{\alpha} n_t(i)^{1-\alpha}$, where $N_{W,t}$ is the total labor in the economy, and $n_t(i)$ is the labor demand of firm i at time t. The inclusion of a production externality à la Baxter and King (1991) helps to align our model with observed asset price and wage co-movements, and does not alter other qualitative outcomes of our model.[7]

in Section F. Our analysis of a standard non-linear New Keynesian model is provided in Online Appendix H.

[7]In a model without Baxter and King (1991) externalities, increasing asset prices often correlate with lower wages, which is contrary to the empirical evidence (Chodorow-Reich et al., 2021) regarding the effects

Intermediate firms face a downward-sloping demand curve $y_i(p_t(i)|p_t, y_t)$, with $p_t(i)$ representing the price of their own good, and p_t and y_t the aggregate price index and output, respectively:

$$y_i(p_t(i)|p_t, y_t) = y_t \left(\frac{p_t(i)}{p_t} \right)^{-\epsilon} ,$$

where price index $p_t = \left(\int_0^1 p_t(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$ aggregates prices $\{p_t(i)\}$ from all intermediate goods. For tractability, we assume perfect price rigidity, $p_t(i) = p_t = \bar{p}$ for all t, i.[8] Thus, each firm produces an equal level of output $y_t(i) = y_t$ for all i, determined by demand.

2.1.2 Workers

A representative hand-to-mouth worker supplies labor to the intermediate firm producers, earning wage income $w_t N_{W,t}$ and spending it entirely on final good consumption. The representative worker maximizes:

$$\max_{C_{W,t}, N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t} \right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} , \quad \text{s.t.} \quad \bar{p} C_{W,t} = w_t N_{W,t} , \tag{1}$$

where $C_{W,t}$, $N_{W,t}$, and w_t stand for consumption, labor supply, and wage, respectively, with χ_0 being the inverse Frisch elasticity of labor supply. Following the approach of Mertens and Ravn (2011), we normalize consumption $C_{W,t}$ by technology A_t, which enables us to use a straightforward linearly additive utility function consistent with a balanced growth path.[9] Finally, under our rigid price assumption, equilibrium labor demand by each firm i, $\{n_t(i)\}$, aggregates linearly into total labor $N_{W,t}$, resulting in $n_t(i) = N_{W,t}$ for all i. Plugging this finding back into the production function, equilibrium output y_t simplifies to a linear function of total labor, $y_t = A_t N_{W,t}$.

of stock price hikes on aggregate demand, employment, and wages. The Baxter and King (1991) externality enables our calibration to reflect these empirical trends by linking higher asset prices and aggregate demand with increased labor demand and wages.

[8]The alternative assumption of sticky price-resetting à la Calvo (1983) does not significantly alter the model dynamics or the qualitative results presented in this paper.

[9]This standardization simplifies the analysis without altering the qualitative results of our model.

2.1.3 Financial Market and Capitalists

Unlike conventional New-Keynesian models where a representative household owns the economy's firms and receives lump-sum rebated profits, we assume firm profits are capitalized in the stock market through a representative index fund. Capitalists are faced with an optimal portfolio allocation problem, deciding between investing in a risk-free bond and the stock index at each moment t.

The aggregate nominal value of the stock index fund is represented by $\bar{p}A_tQ_t$, where Q_t is the normalized real index price. This price is endogenously determined and adapts to filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$, following the equation:

$$\frac{dQ_t}{Q_t} = \mu_t^q dt + \underbrace{\sigma_t^q}_{\substack{\text{Financial}\\ \text{volatility}}} dZ_t \ ,$$

with μ_t^q and σ_t^q representing the endogenous drift and volatility of the process, respectively. We interpret σ_t^q as a measure of financial uncertainty or disruption. Therefore, aggregate financial wealth $\bar{p}A_tQ_t$ evolves according to a geometric Brownian motion, characterized by a combined volatility of $\sigma + \sigma_t^q$. Notably, σ_t^q, determined in equilibrium, can be either positive or negative, indicating that aggregate real stock market value A_tQ_t might be more (or less) volatile than the technology process, $\{A_t\}$. When σ_t^q is negative, financial wealth volatility $\sigma + \sigma_t^q$ becomes smaller than the fundamental volatility σ.[10]

Alongside the stock market, we introduce a risk-free bond with a nominal interest rate i_t, set by the central bank. Bonds are assumed to be in zero net supply in equilibrium. A unit measure of identical capitalists decides how to allocate their wealth between risk-free bonds and the risky stock index. By holding the later, capitalists earn the profits from the intermediate goods sector, which are distributed as stock dividends, and benefit from stock price revaluations due to changes in A_t and Q_t. Given the competitive nature of financial markets, each capitalist takes the nominal risk-free rate i_t, the expected stochastic stock market return i_t^m, and the total risk level $\sigma + \sigma_t^q$ as given when making portfolio decisions.[11] If a capitalist invests a fraction θ_t of their nominal wealth a_t in the stock market, the total risk borne becomes $\theta_t a_t(\sigma + \sigma_t^q)$ over the interval $[t, t + dt]$. Thus, the portfolio's riskiness

[10]When $\sigma_t^q < 0$, we observe that $\text{Cov}_t\left(dA_t, dQ_t\right) = \sigma\sigma_t^q A_tQ_t dt < 0$, implying a negative covariance between TFP and asset prices.

[11]The competitive market assumption is crucial in our model for explaining inefficiencies stemming from the aggregate demand externality that each capitalist's financial investment decision imposes on the economy. For more details, see Farhi and Werning (2016).

is directly proportional to the investment share θ_t in the stock index. Capitalists, being risk-averse, demand a risk-premium compensation $i_t^m - i_t$ for investing in the risky index, which is determined in equilibrium. A representative capitalist solves the following problem:

$$\begin{aligned} &\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \ , \\ \text{s.t. } \ & da_t = \left(a_t \left(i_t + \theta_t (i_t^m - i_t)\right) - \bar{p} C_t\right) dt + \theta_t a_t \left(\sigma + \sigma_t^q\right) dZ_t \ , \end{aligned} \tag{2}$$

where ρ and C_t denote the subjective discount rate and final good consumption of capitalists, respectively. At each instant, the capitalist earns returns from both risk-free bond and risky stock investments, allocating their income towards consumption of the final good.

2.2 Equilibrium and Asset Pricing

The nominal state price density of capitalists, denoted as ξ_t^N, can be expressed as follows:

$$\xi_t^N = e^{-\rho t} \frac{1}{C_t} \frac{1}{\bar{p}} \ , \ \text{ where } \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right) = -i_t dt \ , \tag{3}$$

and the stochastic discount factor of capitalists between the present time t and a future time s is defined as $\frac{\xi_s^N}{\xi_t^N}$. The aggregate stock market wealth, $\bar{p} A_t Q_t$, is defined as the sum of discounted profit streams from the intermediate goods sector, priced using ξ_t^N, under the assumption that capitalists are the marginal investors in the stock market in equilibrium.

At time t, the total profit of the intermediate goods sector, denoted as D_t, is given by

$$D_t \equiv \bar{p} y_t - \underbrace{w_t N_{W,t}}_{=\bar{p} C_{W,t}} = \bar{p}(y_t - C_{W,t}) = \bar{p} C_t \ , \tag{4}$$

where $w_t N_{W,t}$, the wage income, is equivalent to the consumption expenditure of hand-to-mouth workers, given by $\bar{p} C_{W,t}$. Consequently, the total dividend is equal to the capitalists' aggregate consumption expenditure. Incorporating equation (4) into the asset pricing equation, we obtain

$$\bar{p} A_t Q_t = \mathbb{E}_t \frac{1}{\xi_t^N} \int_t^\infty \xi_s^N \underbrace{D_s}_{=\bar{p} C_s} ds = \frac{\bar{p} C_t}{\rho} \ , \tag{5}$$

which implies $C_t = \rho A_t Q_t$. It indicates that, in equilibrium, the rate of consumption by capitalists corresponds to a fixed proportion ρ of their aggregate financial wealth. From equations (4) and (5), the dividend yield of the stock market index fund is also constant and equal to ρ, which results in the equilibrium consumption of stock dividends by capitalists.

Agents of the same type (workers or capitalists) are identical and make symmetric decisions in equilibrium. Since bonds have a zero net supply, the capitalists' wealth share in the stock market, denoted as θ_t, must be equal to one for all t. This condition determines the equilibrium risk-premium demanded by capitalists. Drawing on equations (2), (3), and (5), the risk-premium is given by

$$\mathrm{rp}_t \equiv i_t^m - i_t = \left(\sigma + \sigma_t^q\right)^2 \ , \tag{6}$$

where rp_t increases with the total volatility $\sigma + \sigma_t^q$ of the aggregate financial wealth $\bar{p}A_tQ_t$. It is important to note that the wealth gain (or loss) of a capitalist equates to the nominal revaluation of the stock market index. Our equilibrium conditions in equations (5) and (6) are consistent with Merton (1971).

The equilibrium in the goods market and the expected stock return i_t^m are characterized as follows: Given that capitalists' consumption $C_t = \rho A_t Q_t$ holds in equilibrium, the final goods market equilibrium condition can be written as

$$\rho A_t Q_t + \frac{w_t}{\bar{p}} N_{W,t} = y_t = A_t N_{W,t} \ . \tag{7}$$

The nominal expected return on stocks, i_t^m, comprises the dividend yield from firm profits and the nominal stock price revaluation resulting from fluctuations in $\{A_t, Q_t\}$. In equilibrium, changes in i_t^m only affect nominal stock prices, as the dividend yield remains constant and equal to ρ. Defining $\{\mathbf{I}_t^m\}$ as the cumulative stock market return process, where $\mathbb{E}_t\left(dI_t^m\right) = i_t^m dt$, equation (8) decomposes i_t^m into its dividend yield and expected stock revaluation components as follows:

$$\begin{aligned}
d\mathbf{I}_t^m &= \frac{\overbrace{\cancel{\bar{p}}\left(\underbrace{y_t - \frac{w_t}{\bar{p}} N_{W,t}}_{=C_t}\right)}^{\text{Nominal dividend}}}{\underbrace{\cancel{\bar{p}} A_t Q_t}_{\text{Total stock market wealth}}} dt + \underbrace{\frac{d\left(\cancel{\bar{p}} A_t Q_t\right)}{\cancel{\bar{p}} A_t Q_t}}_{\text{Stock revaluation}} \\
&= \underbrace{\left(\rho + g + \mu_t^q + \sigma \sigma_t^q\right)}_{=i_t^m} dt + \underbrace{\left(\sigma + \sigma_t^q\right)}_{\text{Risk term}} dZ_t \ .
\end{aligned} \tag{8}$$

The real stock price Q_t is a pivotal factor in driving the business cycle in the model's equilibrium. An increase in Q_t leads to the higher consumption of capitalists, leading to higher wages, greater labor demand by firms, and consequently, increased consumption by all households.

Flexible Price Equilibrium In line with most of the literature, we adopt the equilibrium of the flexible price economy as the benchmark that guides the policy goals of the monetary authority. Details of this equilibrium are presented in Online Appendix A. Additionally, Online Appendix B outlines the necessary conditions for positive co-movement among the gaps in asset price, wage, labor supply, and consumption for both capitalists and workers. Here, 'gaps' refer to the log-deviations from the flexible price equilibrium. As illustrated in Online Appendix B, all the gaps are proportional to each other, and hereafter we write equilibrium conditions in asset price gap $\hat{Q}_t$.

In the flexible price equilibrium, denoted by the superscript n (indicating 'natural'), we obtain $\mu_t^{q,n} = \sigma_t^{q,n} = 0$, implying a constant natural stock price, Q_t^n. The natural interest rate, denoted by r_t^n, represents the real risk-free rate in the flexible price economy. In equilibrium, this rate remains constant, and is given by $r^n = \rho + g - \sigma^2$.

2.3 Gap Economy

In particular, we define the risk-premium gap as $\hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}_t^n$, where rp_t^n stands for the natural counterpart of the risk-premium. We also introduce the concept of the risk-adjusted natural rate, r_t^T, defined as:

$$r_t^T \equiv r_t^n - \frac{1}{2}\hat{rp}_t \ . \tag{9}$$

This rate adjusts the natural rate of return to account for the risk differential between rigid and flexible price economies, serving as an anchor for monetary policy in our model. For example, a positive risk-premium gap, $\hat{rp}_t > 0$, reduces the stock market portfolio demand of capitalists compared to the benchmark economy, potentially leading to a recession.

This effect is formally illustrated in equation (10) of Proposition 1, where a decline in r_t^T relative to the risk-free policy rate i_t fosters expectations of future asset price revaluations, which manifest through drops in current asset prices and the output gap. Note that in a linearized conventional New Keynesian model, the natural rate r_t^n appears in place of r_t^T in (10).

Proposition 1 (Dynamic IS Equation) *The dynamic IS equation of the model, expressed in terms of the asset price gap, is given by:*[12]

$$d\hat{Q}_t = (i_t - r_t^T)dt + \sigma_t^q dZ_t \ , \tag{10}$$

Proof. See Online Appendix C.

■

2.4 Monetary Policy and Equilibrium Uniqueness

We complete the model by incorporating a monetary policy rule. This rule, in conjunction with the dynamic IS equation defined in equation (10) and the implementation of forward guidance or other macroprudential measures, is necessary to determine the model's solution. The baseline policy rule is expressed as follows:

$$i_t = \max\left\{r_t^T + \phi_q \hat{Q}_t, \ 0\right\} \ , \tag{11}$$

where $\phi_q > 0$ satisfies the Taylor principle when not constrained by the ZLB.[13] Combining equations (10) and (11) when the ZLB is not binding, we obtain

$$\mathbb{E}_t \, d\hat{Q}_t = \phi_q \hat{Q}_t \ ,$$

which leads to perfect stabilization of the asset price gap, $\hat{Q}_t = 0$ for all t, as the unique rational expectations equilibrium of the economy outside the ZLB.[14] Section 3 discusses the stabilization and uniqueness properties of the model with a binding ZLB. Section 4 considers different forward guidance strategies that deviate from equation (11) by temporarily committing to a distinct set of passive policy rules (Odyssean guidance), whose stabilization and uniqueness properties are further discussed later.

[12] A conventional definition using the output gap leads to a comparable expression in our model, since both variables are proportional in equilibrium.

[13] In addition to the Taylor principle $\phi_q > 0$, Lee and Dordal i Carreras (2024a) establish that targeting the risk-adjusted natural rate or its risk-premium component is an additional necessary condition for ensuring equilibrium uniqueness in models incorporating higher-order terms in the dynamic IS equation.

[14] See Blanchard and Kahn (1980) and Buiter (1984) for a detailed presentation of the necessary conditions required for this uniqueness result.

3 The Zero Lower Bound

ZLB Recession Following Werning (2012), we consider a scenario where the interest rate reaches the ZLB due to a deterministic shift in the natural rate of interest, r_t^n. Specifically, we assume $\sigma_t = \bar{\sigma}$ for $0 \leq t \leq T$ and $\sigma_t = \underline{\sigma} < \bar{\sigma}$ for $t \geq T$. The TFP volatilities during these periods are such that the natural rate satisfies: $\underline{r} \equiv r^n(\bar{\sigma}) = \rho + g - \bar{\sigma}^2 < 0$ and $\bar{r} \equiv r^n(\underline{\sigma}) = \rho + g - \underline{\sigma}^2 > 0$, resulting in the ZLB binding in the first period. Without loss of generality, and as evident from the expression for r_t^n, we can alternatively consider shocks to the economy's growth rate g or the discount rate ρ as drivers of the ZLB spell. Our results also hold without loss of generality if T follows a stochastic distribution, which we illustrate in Online Appendix E. Therefore, we focus here on the simplest case where T is deterministic.

Recovery Without Guidance We begin our study of ZLB recessions by examining the benchmark scenario: economic recovery in the absence of forward guidance or other macroprudential policies. After period T, we assume that the monetary authority follows the Taylor rule presented in equation (11), achieving perfect economic stabilization defined by $\hat{Q}_t = 0$ for $t \geq T$. We infer by backward induction from equation (10) that perfect stabilization with certainty at T necessarily implies the absence of volatility in the asset price gap $\hat{Q}_t$ process in the preceding periods, $t < T$.[15] Therefore, it follows that $\sigma_t^q = 0$ and $r_t^T = \underline{r} < 0$ for $t < T$ whenever the monetary authority can credibly commit to follow the Taylor rule in equation (11) for $t \geq T$. In this scenario, the dynamics of $\hat{Q}_t$ according to (10) simplify to:

$$d\hat{Q}_t = -\underline{r}\, dt\ , \qquad \text{for } t < T\ , \tag{12}$$

with associated boundary condition $\hat{Q}_T = 0$ and initial asset price gap given by $Q_0 = \underline{r}\,T$. The trajectory of $\{\hat{Q}_t\}$ following equation (12) is illustrated in Figure 1.

The initial increase in σ_t from $\underline{\sigma}$ to $\bar{\sigma}$ raises the risk premium from $\text{rp}_2^n = (\underline{\sigma})^2$ to $\text{rp}_1^n = \bar{\sigma}^2$. This leads to a decline in asset prices $\hat{Q}_t$ because the ZLB prevents the risk-free rate from falling into negative territory, as would be necessary for complete stabilization.

[15] For instance, at $T - \Delta$, where Δ is an infinitesimally small time interval, $\sigma_{T-\Delta}^q = 0$ is the only rational solution to equation (10) consistent with $\hat{Q}_T = 0$ for any possible realization of the stochastic component of the TFP process, $dZ_{T-\Delta}$. This result deterministically pins down the asset price gap of the preceding period, $\hat{Q}_{T-\Delta}$, leading by backward induction to $\sigma_t^q = 0$ for $t \leq T$.

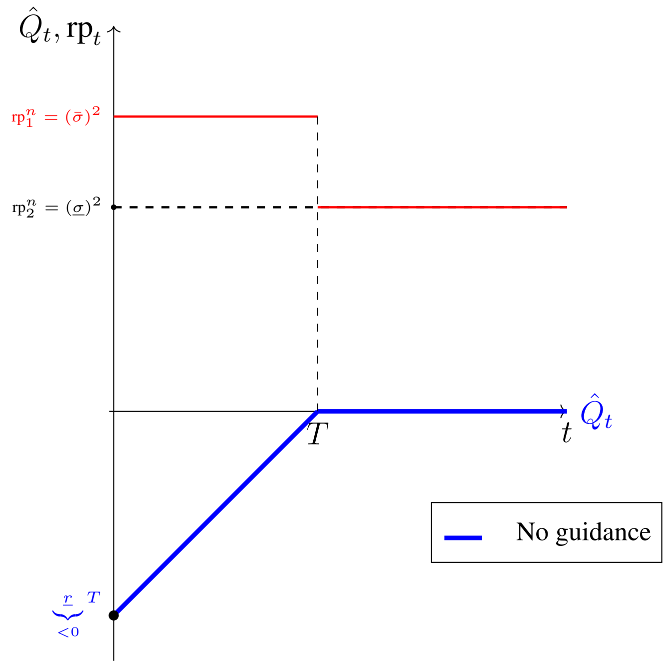

Figure 1: ZLB dynamics, economic recovery without guidance (Benchmark).

As a result, there is a diminished appetite among capitalists for stock market investments, leading to a reduction in both aggregate financial wealth and consumption demand.[16] This path is consistent with the dynamics described in Werning (2012) and Cochrane (2017), despite our model featuring a distinct IS equation (10) with endogenous volatility σ_t^q influencing the drift in the $\hat{Q}_t$ process, a departure from traditional New-Keynesian models. This result arises because ensuring future stabilization for $t \geq T$ effectively eliminates any excess endogenous volatility σ_t^q during a ZLB episode.

Remarks Central banks can prevent the emergence of endogenous volatility σ_t^q at the ZLB through a 'credible' commitment to stabilize the business cycle by a predetermined future date $T < +\infty$. Even if the monetary authority is constrained by the ZLB and thus unable to adhere to the policy rule outlined in (10), which directly targets the risk-premium, the additional financial stability costs resulting from policy inaction can be effectively managed, or even completely eliminated, by pledging to stabilize upon exiting the ZLB. One implication of this result is that the impact of the ZLB could vary significantly between

[16]While Caballero and Farhi (2017) demonstrate that an increased demand for safe assets can drive the economy into recession under ZLB constraints, our analysis suggests that it encourages investors to withdraw their wealth from the stock market, thus reducing stock market value and aggregate demand, akin to the findings of Caballero and Simsek (2020).

countries: those with monetary authorities committed to stabilization after the ZLB period may only face the demand-driven recession described in this Section. In contrast, countries lacking the capacity or willingness to stabilize in the future might incur additional costs due to potential increases in σ_t^q during a ZLB episode. Exploration of these scenarios is left for future research.

4 Forward Guidance

This section analyzes two different forward guidance strategies and explores the potential stabilization trade-offs involved in the use of these policy tools.

4.1 Traditional Forward Guidance

We define traditional forward guidance as the communication strategy where the central bank credibly commits to maintaining a zero policy rate for a duration of time $\hat{T}^{\text{TFG}} > T$ exceeding the initial period of high fundamental volatility. We further assume that the central bank reverts to the policy rule defined in equation (11) after the forward guidance period ends, resulting in a perfect stabilization of both the business cycle and financial markets for $t \geq \hat{T}^{\text{TFG}}$. Following from the backward induction rationale presented in Section 3, stabilization with certainty after $\hat{T}^{\text{TFG}}$ results in the absence of endogenous financial volatility, $\sigma_t^q = 0$, for $t < \hat{T}^{\text{TFG}}$. The dynamics of $\hat{Q}_t$ are described by

$$d\hat{Q}_t = \begin{cases} -\underline{r}\ dt\ , & \text{for } t < T\ , \\ -\bar{r}\ dt\ , & \text{for } T \leq t < \hat{T}^{\text{TFG}}\ , \end{cases} \tag{13}$$

with associated boundary condition $\hat{Q}_{\hat{T}^{\text{TFG}}} = 0$, resulting in an initial asset price gap of $\hat{Q}_0 = \underline{r}\,T + \overline{r}\,(\hat{T}^{\text{TFG}} - T)$.

The dynamics of $\{\hat{Q}_t\}$ governed by equation (13) are depicted in Figure 2. Traditional forward guidance induces an artificial economic boom between T and $\hat{T}^{\text{TFG}}$, thereby alleviating recessionary pressures within the interval $0 \leq t < T$. Specifically, traditional forward guidance increases asset prices between T and $\hat{T}^{\text{TFG}}$, which results in a narrower initial asset price gap $\hat{Q}_0$ due to the forward-looking nature of stock markets.

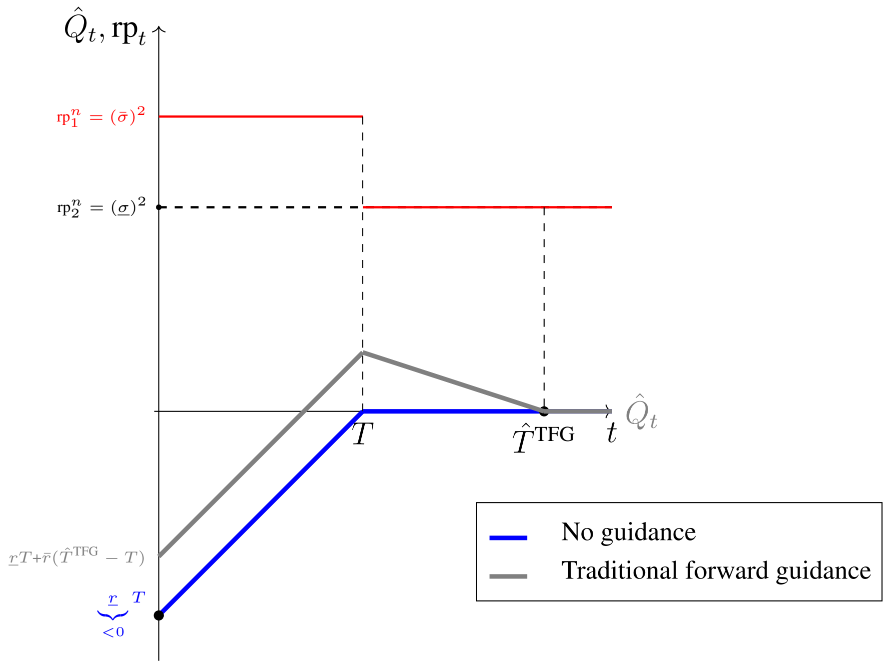

Figure 2: ZLB dynamics under traditional forward guidance.

Optimal Traditional Forward Guidance To determine the optimal forward guidance duration $\hat{T}^{\mathrm{TFG}}$, we minimize the quadratic welfare loss function represented by:[17]

$$\mathbb{L}^{Q}\left(\{\hat{Q}_t\}_{t\geq 0}\right) = \mathbb{E}_0 \int_0^{\infty} e^{-\rho t}\hat{Q}_t^2 \, dt \; , \tag{14}$$

subject to the dynamics outlined in equation (13). The first-order condition with respect to $\hat{T}^{\mathrm{TFG}}$ results in:

$$\int_0^{\infty} e^{-\rho t}\hat{Q}_t dt = 0 \; . \tag{15}$$

Section 4.5 presents a summary of the principal statistics and welfare gains resulting from the adoption of the optimal traditional forward guidance policy outlined in this discussion.

In the next section, we argue that central banks might voluntarily forgo perfect stabilization in the future to reduce financial volatility at the ZLB and potentially achieve higher welfare than with the traditional forward guidance policy described here. We term this approach a 'higher-order' forward guidance policy.

[17]The derivation of the quadratic welfare loss function in equation (14) is provided in Online Appendix I.

4.2 Higher-Order Forward Guidance

The principal cause of ZLB recessions in our model is an excessively high risk premium, driven by increased fundamental volatility σ_t. As a result, central banks might alternatively consider focusing on mitigating financial risk by guiding agents' actions toward a favorable trajectory for the asset price volatility $\{\sigma_t^q\}$ during the ZLB period, aiming to support asset prices and consumption demand.[18]

Context In the traditional forward guidance policy previously discussed, the central bank's commitment to perfect stabilization (with certainty) at $\hat{T}^{\text{TFG}}$ facilitates a smoother transition toward economic recovery. However, this approach prevents any deviation of σ_t^q from zero, its natural level, during the ZLB period, as depicted in Figure 3. This suggests that to sustain alternative equilibria where σ_t^q deviates from zero, the central bank must refrain from promising perfect stabilization upon exiting the ZLB at $\hat{T}^{\text{TFG}}$, as illustrated in Figure 4.

$$\boxed{\text{1. Central bank achieves perfect stabilization with certainty after } \hat{T}^{\text{TFG}} \text{ (i.e., } \hat{Q}_t = 0, \text{ for } t \geq \hat{T}^{\text{TFG}})}$$

$$\downarrow$$

$$\boxed{\text{2. } \hat{Q}_{\hat{T}^{\text{TFG}}} = 0 \text{ guarantees } \sigma_t^q = \sigma_t^{q,n} = 0, \text{rp}_t = \text{rp}_t^n \text{ for } t < \hat{T}^{\text{TFG}}}$$

Figure 3: Mechanism under traditional forward guidance.

$$\boxed{\neg 2.\ \sigma_t^q < \sigma_t^{q,n} = 0, \text{rp}_t < \text{rp}_t^n \text{ for } t < \hat{T}^{\text{TFG}}}$$

$$\downarrow$$

$$\boxed{\neg 1.\ \hat{Q}_{\hat{T}^{\text{TFG}}} \neq 0\text{: central bank commits not to perfectly stabilize the economy after } \hat{T}^{\text{TFG}}}$$

Figure 4: Mechanism under higher-order forward guidance.

Implementation We define $\hat{T}^{\text{HOFG}}$ as the duration of the zero policy rate under our 'higher-order' policy. We model the commitment constraint described in Figure 4 by assuming that after the forward guidance regime with i_t equal to zero ends at $\hat{T}^{\text{HOFG}}$, the

[18] The risk premium, rp_t, is given by $\text{rp}_t = (\bar{\sigma} + \sigma_t^q)^2$ for $t < T$ and $\text{rp}_t = (\underline{\sigma} + \sigma_t^q)^2$ for $T \leq t < \hat{T}^{\text{TFG}}$. Therefore, a negative σ_t^q can reduce the risk premium below its natural level, thereby improving asset prices and aggregate demand at the ZLB.

monetary authority implements a *passive policy rule* with i_t fixed at $\bar{r}$, allowing for the existence of multiple equilibria. The central bank then coordinates the economy's agents into an optimal path within the admissible solutions set, subject to the constraints: $\sigma_t^q = 0$ for $t \geq \hat{T}^{\text{HOFG}}$ and $\mathbb{E}_0 \hat{Q}_\infty = 0$. The latter is necessary to meet the economy's transversality condition, while the former simplifies the optimization problem by assuming the central bank ends its influence on financial market volatility at the conclusion of the forward guidance period. Together with the dynamic IS equation in (10), these constraints indicate that the asset price gap is initially expected to close, $\mathbb{E}_0 \hat{Q}_{\hat{T}^{\text{HOFG}}} = 0$, by the end of the forward guidance period at $\hat{T}^{\text{HOFG}}$. In Section 4.3, we relax the constraints on central bank behavior and assume that it permanently reverts to the active Taylor rule in equation (11) with a constant probability of less than one after $\hat{T}^{\text{HOFG}}$.

Formalism We denote the natural risk premiums as $\text{rp}_1^n \equiv \bar{\sigma}^2$ for $t < T$ (high fundamental volatility region), $\text{rp}_2^n \equiv \underline{\sigma}^2$ for $T \leq t < \hat{T}^{\text{HOFG}}$ (low fundamental volatility region), and $\text{rp}_3^n \equiv \underline{\sigma}^2$ for $t \geq \hat{T}^{\text{HOFG}}$ (low fundamental volatility region post-forward guidance period).[19]

$$\boxed{\neg 2.\ \sigma_t^q = \sigma_1^{q,L} < 0 \text{ for } t < T;\ \sigma_2^{q,L} < 0 \text{ for } T \leq t \leq \hat{T}^{\text{HOFG}};\ \sigma_t^{q,n} = 0 \text{ for } t > \hat{T}^{\text{HOFG}}}$$

$$\downarrow$$

$$\boxed{\neg 1.\ \hat{Q}_{\hat{T}^{\text{HOFG}}} \neq 0\text{: central bank pegs its policy rate } i_t = \bar{r} \text{ after } \hat{T}^{\text{HOFG}}}$$

Figure 5: Simplified higher-order forward guidance.

We can simplify the optimization problem by assuming that the central bank maintains consistent financial volatility and risk-premium levels within each regime. Specifically, financial volatility σ_t^q is set to be $\sigma_1^{q,L}$ for $t < T$, $\sigma_2^{q,L}$ for $T \leq t < \hat{T}^{\text{HOFG}}$, and zero for $t \geq \hat{T}^{\text{HOFG}}$. The risk-premia associated with each period are $\text{rp}_1 \equiv (\bar{\sigma} + \sigma_1^{q,L})^2 < \text{rp}_1^n$ for $t < T$, $\text{rp}_2 \equiv (\underline{\sigma} + \sigma_2^{q,L})^2 < \text{rp}_2^n$ for $T \leq t < \hat{T}^{\text{HOFG}}$, and $\text{rp}_3 \equiv (\underline{\sigma})^2$ for $t \geq \hat{T}^{\text{HOFG}}$.[20] This simplified problem is represented in Figure 5. Finally, the risk-adjusted natural rate in equation (9) is expressed as r_1^T for $t < T$ and r_2^T for $T \leq t < \hat{T}^{\text{HOFG}}$, each being a function

[19] Risk premium is defined as $\text{rp}_t = (\sigma_t + \sigma_t^q)^2$, and the expression for the natural level stems from the existence of zero endogenous financial volatility in a flexible price economy, where $\sigma_t^{q,n} = 0$ for all t.

[20] Proposition 2 later proves that $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ at the optimum. For illustration purposes, we assume these conditions are satisfied in the rest of the argument of this section.

of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, respectively:

$$\begin{aligned} r_1^T\left(\sigma_1^{q,L}\right) &\equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma}+\sigma_1^{q,L})^2}{2} > \underline{r} \equiv r_1^T(0) \text{ when } \sigma_1^{q,L} < 0 \ , \\ r_2^T\left(\sigma_2^{q,L}\right) &\equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{(\underline{\sigma}+\sigma_2^{q,L})^2}{2} > \bar{r} \equiv r_2^T(0) \text{ when } \sigma_2^{q,L} < 0 \ . \end{aligned} \tag{16}$$

From equation (16), we observe that lower risk premia during the forward guidance period up to $\hat{T}^{\text{HOFG}}$ lead to increased risk-adjusted rates and, consequently, higher values of the asset price gap $\{\hat{Q}_t\}$ along the expected equilibrium path (in comparison to a traditional forward guidance policy of the same duration). This results in reduction of the expected quadratic loss function in (14). However, as indicated by our IS equation (10), a σ_t^q different from zero introduces stochastic fluctuations in the trajectory of $\hat{Q}_t$, resulting in potential additional stabilization costs in the future. The green line in Figure 6 illustrates the expected trajectory (or deterministic component) of $\{\hat{Q}_t\}$ under a higher-order forward guidance policy as detailed in this section. The dashed lines alongside the expected path depict two possible sample paths that stem from stochastic variations in $\{\hat{Q}_t\}$.

In summary, central banks operating under our higher-order guidance with commitment face a trade-off between achieving lower risk premiums and higher asset price levels prior to $\hat{T}^{\text{HOFG}}$, and the subsequent costs of destabilization. This balancing act involves a careful choice of $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\text{HOFG}}$, as we discuss next. It will ultimately be the case that, due to the additional stabilization effects from negative $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, the optimal duration of the zero policy rate period $\hat{T}^{\text{HOFG}}$ decreases compared to $\hat{T}^{\text{TFG}}$.

Optimal Higher-Order Forward Guidance The initial asset price gap $\hat{Q}_0$ is determined by the condition $\mathbb{E}_0\hat{Q}_{\hat{T}^{\text{HOFG}}} = 0$ previously discussed and the dynamic IS equation in (10) as follows:

$$\hat{Q}_0 = r_1^T(\sigma_1^{q,L})\,T + r_2^T(\sigma_2^{q,L})\,(\hat{T}^{\text{HOFG}} - T) \ . \tag{17}$$

The central bank minimizes the loss function given by (14) by selecting the optimal values for $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\text{HOFG}}$. The formulation of the optimization problem is:

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Q}_t^2\, dt, \text{ s.t. } d\hat{Q}_t = \begin{cases} -r_1^T(\sigma_1^{q,L})dt + \sigma_1^{q,L}dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{q,L})dt + \sigma_2^{q,L}dZ_t, & \text{for } T \leq t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \geq \hat{T}^{\text{HOFG}}, \end{cases} \tag{18}$$

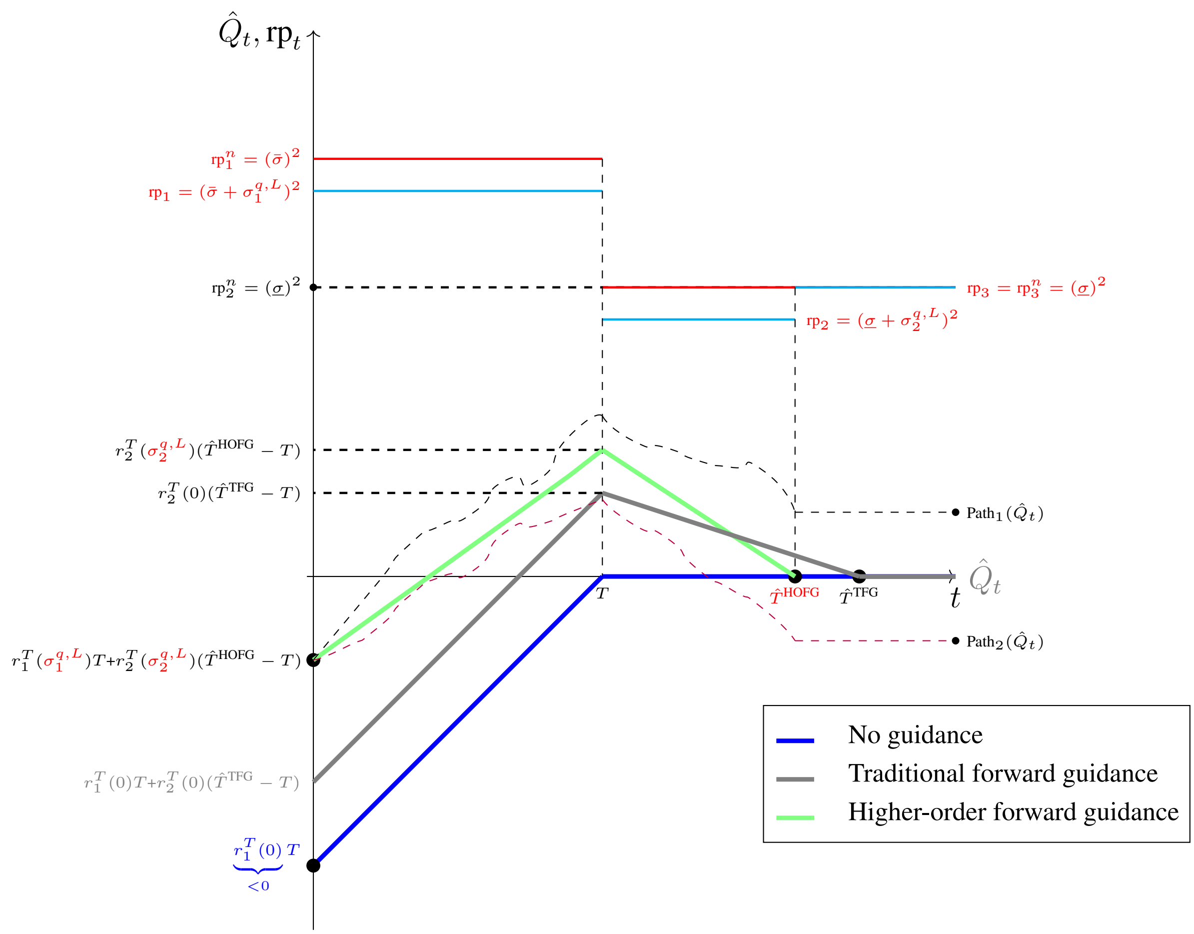

Figure 6: Intervention dynamics of $\{\hat{Q}_t\}$ with $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{\text{HOFG}} < \hat{T}^{\text{TFG}}$.

with $\hat{Q}_0$ determined by equation (17). The following Proposition 2 summarizes the resulting optimal commitment path for the central bank under higher-order forward guidance.

Proposition 2 (Optimal Commitment Path) *The solution to the central bank's higher-order forward guidance optimization problem in (18) results in an optimal commitment path characterized by $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. In addition, optimal higher-order forward guidance always results in an equal or lower expected quadratic loss than the traditional forward guidance discussed in Section 4.1.*

Proof. See Appendix II. The latter part follows from the fact that when $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}) = (0, 0, \hat{T}^{\text{TFG}})$, the trajectory of the asset price gap $\{\hat{Q}_t\}$ becomes identical to that of a traditional forward guidance policy with duration $\hat{T}^{\text{TFG}}$. Thus, an optimal choice of these parameters will always lead to an equal or lower value of the quadratic loss function presented in equation (14).

■

4.3 Higher-Order Forward Guidance *with* Stochastic Stabilization

In the previous section, we assumed that following the end of the forward guidance regime at $\hat{T}^{\text{HOFG}}$, the monetary authority would passively peg the policy rate i_t to the natural rate $\bar{r}$ and set σ_t^q to zero indefinitely. This setup allows for σ_t^q to deviate from zero during the ZLB period, as illustrated in Figure 6. Moving to this section, we relax these assumptions while maintaining the support for the existence of multiple equilibria provided by the earlier framework. Now, we assume that after forward guidance ends, the central bank not only follows the outlined passive rule but also commits to a stochastic return to the perfect stabilization rule in equation (11). This commitment is represented as a constant probability outcome determined by a Poisson process. Accordingly, $\hat{Q}_t$ after $\hat{T}^{\text{HOFG}}$ follows:

$$d\hat{Q}_t = -\hat{Q}_t d\Pi_t\,, \;\; \text{s.t.} \;\; d\Pi_t = \begin{cases} 1\,, & \text{with probability } \nu dt\,, \\ 0\,, & \text{with probability } 1-\nu dt\,, \end{cases}$$

where $d\Pi_t$ is a Poisson random variable, with rate parameter $\nu \geq 0$.[21] The central bank's optimization problem can be expressed as:

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Q}_t^2\, dt + \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t}\cdot e^{-\nu\left(t-\hat{T}^{\text{HOFG}}\right)}\cdot \hat{Q}_t^2\, dt\,,$$

$$\text{s.t.} \;\; d\hat{Q}_t = \begin{cases} -r_1^T(\sigma_1^{q,L})dt + \sigma_1^{q,L}dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{q,L})dt + \sigma_2^{q,L}dZ_t, & \text{for } T \leq t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \geq \hat{T}^{\text{HOFG}}, \end{cases} \tag{19}$$

with $\hat{Q}_0$ determined by equation (17). Proposition 3 outlines the optimal commitment path for the central bank under higher-order forward guidance with stochastic stabilization.

Proposition 3 (Optimal Commitment Path with Stochastic Stabilization) *The solution to the central bank's forward guidance optimization problem in (19) results in an optimal commitment path characterized by $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. In addition,*

[21] Here, ν is treated as an exogenous parameter determined by external factors. If the central bank could choose an optimal ν, it would select $\nu \to +\infty$, as demonstrated in Online Appendix D.

optimal higher-order forward guidance with a stochastic stabilization probability always results in an equal or lower expected quadratic loss than the traditional forward guidance discussed in Section 4.1.

Furthermore, an increased probability of stabilization, indicated by higher values of ν, leads to a reduction in the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, resulting in a decrease in risk premia at the ZLB.

Proof. See Online Appendix D. The first part of the proposition directly extends the results of Proposition 2 to a stochastic stabilization environment. The latter part of the proposition is based on the reduced costs of a more aggressive countercyclical policy at the ZLB when future stabilization is more likely.

■

Finally, Corollary 1 asserts that introducing a minimal degree of uncertainty about the timing of future stabilization in its communications is always optimal for the central bank, as it allows private agents to coordinate on the stochastic equilibrium with σ_t^q deviating from zero during the ZLB, as depicted in Figure 6. This approach facilitates the application of higher-order forward guidance, resulting in equilibrium paths that are strictly superior from a quadratic loss perspective compared to those under traditional forward guidance.

Corollary 1 (Discontinuity at the Limit) *The limit case where stabilization parameter ν equals $+\infty$ corresponds to the traditional forward guidance problem described in Section 4.1. As ν approaches $+\infty$ from the left, the central bank's expected quadratic loss function exhibits a discontinuity. Specifically, the expected quadratic loss is always lower when there is a non-zero probability of not achieving immediate stabilization by the end of the forward guidance period, $\hat{T}$. Formally:*

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right) < \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu = \infty\right) ,$$

where $\mathbb{L}^*\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right)$ represents the quadratic loss function defined in equation (14), evaluated at its optimum for an economy characterized by a Poisson rate ν.

Proof. See Online Appendix D. The intuition behind the statement's first part is that the probability of immediate stabilization upon exiting the forward guidance period at $\hat{T}^{\text{HOFG}}$ becomes one when $\nu = +\infty$, aligning with the scenario of the traditional forward guidance policy in Section 4.1. The second part is based on that higher-order guidance consistently results in an equal or lower expected quadratic loss compared to the traditional guidance, regardless of ν, as outlined in Proposition 3.

■

Equilibrium selection We next present an example of how the model's optimal higher-order solution can be implemented *in practice* from among the multiple existing ZLB equilibria through fiscal policy coordination.

4.4 Fiscal Policy Coordination

Fiscal intervention, alongside the forward guidance policies previously outlined, can enforce the optimal higher-order solution as the unique equilibrium of the model. While various approaches to such coordination are feasible, we focus specifically on a fiscal subsidy (or tax) that adjusts stock returns relative to the risk-free rate.[22,23]

We examine a subsidy scheme financed by withdrawals from the fiscal authority's monetary reserves, denoted F_t. The funds provided by the fiscal authority respond to unexpected shocks in stock returns, and for simplicity, we assume that the reserves are influenced solely by the transfers associated with the scheme. The process evolves as follows:

$$dF_t = -\theta_t a_t \tau_t dZ_t \ , \tag{20}$$

$$\text{with:} \quad F_0 = F_{0-} - \chi\theta_{0-}a_{0-} \ , \tag{21}$$

where τ_t and χ are variables determined by the fiscal authority. The subscript $_{0-}$ denotes the values of the respective variables prior to the ZLB shock, allowing for an initial jump in the subsidy process of size $\chi\theta_{0-}a_{0-}$. The flow budget constraint for capitalists becomes:

$$da_t = \left(a_t\left(i_t + \theta_t(i_t^m - i_t)\right) - \bar{p}C_t\right)dt + \theta_t a_t\left[(\sigma_t + \sigma_t^q) + \tau_t\right]dZ_t \ , \tag{22}$$

$$\text{with:} \quad \Delta a_0 = \chi\theta_{0-}a_{0-} + \bar{p}A_{0-}\Delta Q_0 \ , \tag{23}$$

[22] In our model, a subsidy (or tax) on stock investments operates similarly to a tax break or hike on capital income, a policy commonly employed by governments. We choose the model based on subsidy for simplicity in notation.

[23] Alternatively, these transfers can be interpreted as a stylized representation of the impact of large-scale asset-purchase (LSAP) programs on the relative returns of risky assets. The potential for balance sheet losses (and profits) from such policies, along with other direct or indirect effects of LSAPs, has led many authors to highlight the quasi-fiscal nature of these interventions. See, for example, Woodford (2016), Chionis et al. (2021), and Lee and Dordal i Carreras (2024b).

where $\Delta x_t \equiv x_t - x_{t-}$ represents the initial jump response of the corresponding variables following the ZLB shock. Equation (23) indicates that the change in the initial wealth of capitalists equals the change in the value of stocks, in which their wealth is fully allocated in equilibrium at time $0-$, plus the transfer from the fiscal authority.

Proposition 4 outlines the implementation of the subsidy scheme that enforces the optimal equilibrium paths discussed in Sections 4.2 and 4.3.

Proposition 4 (Fiscal Coordination and Unique Optimal Equilibrium) *In the ZLB environment described in Section 3, under the forward guidance policies of Sections 4.2 and 4.3, and with the variables $\{\tau_t, \chi\}$ governing the subsidy scheme as defined below, the optimal higher-order solution becomes the unique equilibrium of the model:*

$$\tau_t = (\sigma_t^{q,*} - \sigma_t^q) \ , \quad \textit{and} \quad \chi = \bar{p} A_{0-} \frac{Q_0^* - Q_0}{\theta_{0-} a_{0-}} \ , \tag{24}$$

where the star superscript denotes the variables under the optimal higher-order forward guidance path. Thus, $\hat{Q}_t = \hat{Q}_t^*$ and $\sigma_t^q = \sigma_t^{q,*}$ for all $t \geq 0$ in equilibrium.

Proof. See Appendix II. The intuition behind this result is that the transfer schedule in equation (24) targets two potential sources of deviation from the optimal path: (i) the initial stock price response, Q_0, to a shock that leads the economy to the ZLB, and (ii) the sensitivity σ_t^q of agents' responses to stochastic shocks along the transition path.

■

Under our proposed policy, fiscal intervention functions as an *off-equilibrium* threat by linking subsidies to deviations from the optimal path, resulting in zero transfers in equilibrium, as summarized in Corollary 2. We conjecture that alternative coordination mechanisms may exist, potentially involving components beyond fiscal intervention. However, as long as equilibrium selection takes the form of *off-equilibrium* threats, we can assess the properties and welfare benefits of higher-order guidance independently of the specific coordination mechanism employed by the monetary authority.

Corollary 2 (Zero Equilibrium Transfers) *Subsidies and tax transfers under the scheme outlined in Proposition 4 are zero in equilibrium,*

$$\begin{array}{ll} \tau_t = \chi = 0 & , \\ F_t = F_{0-} & , \end{array} \quad \forall t \geq 0 \ .$$

Proof. This follows directly from the equilibrium results of Proposition 4, with $Q_t = Q_t^*$ and $\sigma_t^q = \sigma_t^{q,*}$ for all $t \geq 0$, and the subsidy definitions in equation (24).
■

Finally, note that these results do not prevent the fiscal authority from also implementing more traditional fiscal transfer schemes to address output gap deviations caused by the ZLB. The effects of these policies, along with various implementation details, are discussed in Online Appendix F.

4.5 Welfare Comparison

For the quantitative evaluation of different forward guidance policies discussed in this paper, we simulate optimal commitment paths at the ZLB under three scenarios: (i) no forward guidance, (ii) traditional forward guidance, and (iii) higher-order forward guidance with varying probabilities of stabilization. The initial ZLB duration T is set at 20 quarters to reflect the lengthy ZLB periods that followed the global financial crisis. The Poisson rate parameter ν in the higher-order forward guidance policy is first calibrated to zero, denoting a zero probability of reverting to an active policy rule, and then to one, signifying the expectation of resuming an active policy rule one quarter after the forward guidance period concludes. The remaining model parameters are calibrated based on values commonly found in the literature, as detailed in Appendix Table I.1.

We define the loss function $\mathbb{L}$ as the quadratic output loss per quarter, and approximate it by:

$$\mathbb{L}^{Y}_{\text{Per-period}} \equiv \rho \int_0^{\infty} e^{-\rho t}\mathbb{E}_0\left(\hat{Y}_t^2\right) \approx \zeta^2 \cdot \rho \int_0^{\infty} e^{-\rho t}\frac{1}{s}\sum_{i=1}^{s}\left(\hat{Q}_t^{(i)}\right)^2 dt \ ,$$

where $\zeta > 0$ follows from the relationship $\hat{Y}_t = \zeta \hat{Q}_t$, as derived in equation (B.1) of Online Appendix B. Here, $\hat{Q}_t^{(i)}$ represents the i^{th} simulated stochastic sample path of the asset price gap.[24] We consider a scenario characterized by a one-time ZLB recession commencing in period zero, without any expectation of future recurrence. Therefore, $\mathbb{L}$ is to be interpreted as the expected conditional loss associated with a single ZLB episode.

Table 1 presents the results of our simulation, where $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ are expressed as percentages of the fundamental volatilities $\bar{\sigma}$ and $\underline{\sigma}$, respectively. The initial columns report the

[24] We use $s = 10^4$ randomly simulated sample paths to approximate the quadratic loss of the higher-order forward guidance policies.

Policy	**No guidance**	**Traditional**	**Higher-Order** (no stochastic stabilization)	**Higher-Order** (*with* stoch. stab., $\nu = 1$)
$\sigma_1^{q,L}$	0	0	-1.27%	-4.13%
$\sigma_2^{q,L}$	0	0	-0.24%	-3.79%
$\hat{T}^{\text{HOFG}}$	20	25.27	25.09	24.68
$\mathbb{L}^{Y}_{\text{Per-period}}$	7%	1.93%	1.81%	1.69%

Table 1: Policy comparisons.

effectiveness of traditional guidance, showing the central bank extending the ZLB for just over a year, reducing total loss by about five percentage points. These findings are aligned with existing literature (see Campbell et al. (2012, 2019), Del Negro et al. (2013), McKay et al. (2016)).[25] The last two columns provide summary statistics on optimal higher-order guidance implementation under the two stabilization regimes discussed above. The results are consistent with higher-order guidance characteristics described in Propositions 2 and 3. Higher-order guidance, compared to traditional policy, further reduces ZLB costs by a moderate 0.12%-0.24% per quarter through lower financial market volatility during the guidance period, and allows for an earlier exit from the ZLB. Finally, the last column reports that gains from higher-order guidance double when there is a positive probability of returning to full stabilization in the future.

Standard New Keynesian Model Our results in Section 3 and Section 4 hold in a non-linear version of the standard New Keynesian model (e.g., Woodford (2003) and Galí (2015)): even in a textbook New Keynesian model, higher aggregate endogenous volatility increases the degree of precautionary savings, depressing consumption demand and thereby inducing a recession. In this environment, a central bank has an incentive to choose an equilibrium with lower aggregate volatility during the ZLB periods, based on our higher-order forward guidance policy. We provide a detailed analysis of the textbook non-linear New Keynesian model in Online Appendix H.

[25]These studies also note the issue of traditional forward guidance being overly potent in plain vanilla New-Keynesian frameworks compared to empirical estimates. This paper does not include the quantitative adjustments proposed in the literature to address this discrepancy, focusing instead on the distinctions between traditional and higher-order forward guidance policies.

5 Conclusion

This paper explores the likelihood of increased financial volatility at the ZLB and finds that a credible commitment to future economic stabilization prevents excess volatility from developing. We then examine the effects of traditional forward guidance, defined as the monetary authority's promise to maintain a zero policy rate for an extended period. This commitment fosters expectations of higher future asset prices and aggregate demand, thereby increasing the market valuation of households' financial wealth and, consequently, their aggregate consumption at the ZLB.

Our findings suggest that a central bank may not always find it optimal to commit to perfectly stabilizing the business cycle in the future. By refusing to do so, the central bank permits alternative equilibrium paths with lower financial volatility at the ZLB and higher expected welfare. While this strategy is preferable from a welfare perspective, it involves trade-offs. Specifically, a lack of commitment or a positive degree of uncertainty about the timing of future stabilization enables the central bank to reduce financial volatility at the ZLB, but at the expense of potentially large and costly output gap deviations in the future.

This paper aims to provide valuable insights for academics and policymakers interested in the interplay between financial uncertainty and unconventional policies at the ZLB, notably forward guidance. We leave to future research the study of central banks' communication policies under alternative scenarios, such as private information about the state of the economy.

References

Akerlof, George and Janet L. Yellen, "A Near-Rational Model of the Business Cycle, With Wage and Price Inertia," *Quarterly Journal of Economics*, 1985, *100*, 823–838.

Alvarez-Cuadrado, Francisco, Ngo Van Long, and Markus Poschke, "Capital-labor substitution, structural change and the labor income share," *Journal of Economic Dynamics and Control*, 2018, *87*, 206–231.

Basu, Susanto and Brent Bundick, "Uncertainty Shocks in a Model of Effective Demand," *Econometrica*, 2017, *85* (3), 937–958.

Baxter, Marianne and Robert King, "Productive externalities and business cycles," *Working Paper*, 1991.

Blanchard, Olivier Jean and Charles M. Kahn, "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 1980, *48* (5), 1305–1311.

— **and Nobuhiro Kiyotaki**, "Monopolistic Competition and the Effects of Aggregate Demand," *American Economic Review*, 1987, *77* (4), 647–666.

Bloom, Nicholas, "The impact of uncertainty shocks," *Econometrica*, 2009, *77* (3), 623–685.

— **, Max Floetotto, Nir Jaimovich, Itay Saporta-Eksten, and Stephen J. Terry**, "Really Uncertain Business Cycles," *Econometrica*, 2018, *86* (3), 1031–1065.

Buiter, Willem H., "Saddlepoint Problems in Continuous Time Rational Expectations Models: A General Method and Some Macroeconomic Examples," *NBER Working Paper*, 1984.

Caballero, Ricardo J and Alp Simsek, "A Risk-centric Model of Demand Recessions and Speculation," *Quarterly Journal of Economics*, 2020, *135* (3), 1493–1566.

Caballero, Ricardo J. and Emmanuel Farhi, "The Safety Trap," *Review of Economic Studies*, 2017, *85* (1), 223–274.

Caballero, Ricardo, Tomas Caravello, and Alp Simsek, "Financial Conditions Targeting," *Working Paper*, 2024.

Calvo, Guillermo, "Staggered prices in a utility-maximizing framework," *Journal of Monetary Economics*, 1983, *12* (3), 383–398.

Campbell, Jeffrey R. and Jacob P. Weber, "Open Mouth Operations," *FRB of Chicago Working Paper No. WP-2018-3*, 2019.

— **, Charles L. Evans, Jonas DM Fisher, Alejandro Justiniano, Charles W. Calomiris, and Michael Woodford**, "Macroeconomic effects of federal reserve forward guidance [with comments and discussion]," *Brookings papers on economic activity*, 2012, pp. 1–80.

— **, Filippo Ferroni, Jonas D. M. Fisher, and Leonardo Melosi**, "The limits of forward guidance," *Journal of Monetary Economics*, 2019, *108*, 118—-134.

Chionis, Dionysios, Fotios Mitropoulos, and Antonios Sarantidis, "The Impact of Quantitative Easing Policy on the Government Debt and the NPLs of the Eurozone Periphery Countries," *Debt in Times of Crisis: Does Economic Crisis Really Impact Debt?*, 2021, pp. 55–76.

Chodorow-Reich, Gabriel, Plamen Nenov, and Alp Simsek, "Stock Market Wealth and the Real Economy : A Local Labor Market Approach," *American Economic Review*, 2021, *115* (5), 1613–57.

Cochrane, John, "The new-Keynesian liquidity trap," *Journal of Monetary Economics*, 2017, *92*, 47–63.

Comin, Diego A.and Javier Quintana, Tom G. Schmitz, and Antonella Trigari, "Revisiting Productivity Dynamics in Europe: A New Measure of Utilization-Adjusted TFP Growth," Technical Report, National Bureau of Economic Research 2023.

Dordal i Carreras, Marc, Olivier Coibion, Yuriy Gorodnichenko, and Johannes Wieland, "Infrequent but Long-Lived Zero-Bound Episodes and the Optimal Rate of Inflation," *Annual Review of Economics*, 2016, *8*, 497–520.

Eggertsson, Gauti and Paul Krugman, "Debt, Deleveraging, and the Liquidity Trap: A Fisher-Minsky-Koo approach," *Quarterly Journal of Economics*, 2012, *127* (3), 1469–1513.

Eggertsson, Gauti B et al., "Eggertsson, Gauti and Michael Woodford," *Brookings papers on economic activity*, 2003, *2003* (1), 139–233.

Farhi, Emmanuel and Iván Werning, "Dealing with the Trilemma: Optimal Capital Controls with Fixed Exchange Rates," *Working Paper*, 2012.

_ **and** _ , "A Theory of Macroprudential Policies in the Presence of Nominal Rigidities," *Econometrica*, 2016, *84* (5), 1645–1704.

_ **and** _ , "Fiscal Unions," *American Economic Review*, 2017, *107* (12), 3788–3834.

Galí, Jordi, *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications - Second Edition*, Princeton University Press, 2015.

Gandelman, Néstor and Rubén Hernández-Murillo, "Risk Aversion at the Country Level," *Federal Reserve Bank of St.Louis Working Paper*, 2014.

Greenwald, Daniel L., Martin Lettau, and Sydney C. Ludvigson, "Origins of Stock Market Fluctuations," *Working Paper*, 2014.

King, Robert G. and Sergio T. Rebelo, "Resuscitating Real Business Cycles," *Handbook of Macroeconomics, ed. John B. Taylor and Michael Woodford*, 1999, pp. 927–1007.

Korinek, Anton. and Alp Simsek, "Liquidity Trap and Excessive Leverage," *American Economic Review*, 2016, *106* (3), 699–738.

Lee, Seung Joo and Marc Dordal i Carreras, "Self-fulfilling Volatility and a New Monetary Policy," *Working Paper*, 2024.

_ and _ , "A Unified Theory of the Term-Structure and Monetary Stabilization," *Working Paper*, 2024.

McKay, Alisdair, Emi Nakamura, and Jón Steinsson, "The Power of Forward Guidance Revisited," *American Economic Review*, 2016, *106* (10), 3133—-3158.

Mertens, Karel and Morten O Ravn, "Understanding the aggregate effects of anticipated and unanticipated tax policy shocks," *Review of Economic dynamics*, 2011, *14* (1), 27–54.

Merton, Robert C, "Optimum consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, 1971, *3* (4), 373–413.

Negro, Marco Del, Marc Giannoni, and Christina Patterson, "The Forward Guidance Puzzle," *Federal Reserve Bank of New York Staff Report 574*, 2013.

Schmitt-Grohé, Stephanie and Martin Uribe, "Downward Nominal Wage Rigidity, Currency Pegs, and Involuntary Unemployment," *Journal of Political Economy*, 2016, *124* (5), 1466–1514.

Werning, Iván, "Managing a Liquidity Trap: Monetary and Fiscal Policy," *Working Paper*, 2012.

Woodford, Michael, *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, 2003.

_ , "Quantitative easing and financial stability," Technical Report, National Bureau of Economic Research 2016.

I Parameter Calibration

	Parameter Description	Value	Source
φ	Relative Risk Aversion	0.2	Within the admissible calibration ranges specified by Gandelman and Hernández-Murillo (2014).
χ_0	Inverse Frisch labor supply elasticity	0.25	See King and Rebelo (1999).
ρ	Subjective time discount factor	0.020	Target 2.8% natural rate.
g	TFP growth rate	0.0083	Annual growth rate of 3.3%, which corresponds to the US TFP growth rate from 2009 to 2020, as detailed in Table 8 of Comin et al. (2023).
$\underline{\sigma}$	TFP volatility, low volatility regime	0.009	See Dordal i Carreras et al. (2016).
$\bar{\sigma}$	TFP volatility, high volatility regime	0.209	Target -1.5% natural rate (ZLB recession).
T	ZLB duration (quarters)	20	A five-year ZLB duration, consistent with periods such as the Global Financial Crisis and the Great Recession. See Dordal i Carreras et al. (2016).
ν	Stabilization probability parameter	1	Target average duration $1/\nu$ of one quarter before returning to stabilization.
α	$1-$ Labor income share	0.4	See Alvarez-Cuadrado et al. (2018).
ϵ	Elasticity of substitution intermediate goods	7	Target steady-state mark-up of 16.7%. See Galí (2015).

Table I.1: Parameter calibration used in Section 4.

II Proofs and Derivations

Proof of Proposition 2. In the context outlined in Section 4.2, the central bank solves the following problem:[1]

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Q}_t^2 dt \ , \quad \text{s.t.} \ \ d\hat{Q}_t = \begin{cases} -\underbrace{r_1^T(\sigma_1^{q,L})}_{<0} dt + \sigma_1^{q,L} dZ_t \ , & \text{for } t < T \ , \\ -\underbrace{r_2^T(\sigma_2^{q,L})}_{>0} dt + \sigma_2^{q,L} dZ_t \ , & \text{for } T \le t < \hat{T}^{\text{HOFG}} \ , \\ 0 \ , & \text{for } t \ge \hat{T}^{\text{HOFG}} \ , \end{cases}$$

$$\text{with } \ \hat{Q}_0 = r_1^T(\sigma_1^{q,L})T + r_2^T(\sigma_2^{q,L})(\hat{T}^{\text{HOFG}} - T) \ , \tag{II.1}$$

where

$$r_1^T(\sigma_1^{q,L}) \equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma} + \sigma_1^{q,L})^2}{2} < 0 \ , \ r_2^T(\sigma_2^{q,L}) \equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{(\underline{\sigma} + \sigma_2^{q,L})^2}{2} > 0 \ .$$

After $\hat{T}^{\text{HOFG}}$, there are no additional fluctuation in $\hat{Q}_t$. Defining r_s^T as $r_1^T(\sigma_1^{q,L})$ for $s < T$ and as $r_2^T(\sigma_2^{q,L})$ for $T \le s \le \hat{T}^{\text{HOFG}}$, the process of $\hat{Q}_t$ can be articulated as follows:

$$\hat{Q}_t = \begin{cases} \underbrace{\int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds}_{\equiv \hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})} + \sigma_1^{q,L} \underbrace{Z_t}_{\sim N(0,t)} \ , & \text{for } \ t \le T \ , \\ \underbrace{\int_t^{\hat{T}^{\text{HOFG}}} r^T(s) ds}_{\equiv \hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})} + \sigma_1^{q,L} Z_T + \sigma_2^{q,L} \underbrace{W_{t-T}}_{\sim N(0,t-T)} \ , & \text{for } \ T < t \le \hat{T}^{\text{HOFG}} \ , \\ \sigma_1^{q,L} Z_T + \sigma_2^{q,L} \underbrace{W_{\hat{T}-T}}_{\sim N(0,\hat{T}-T)} = \hat{Q}_{\hat{T}^{\text{HOFG}}} , & \text{for } \ \hat{T}^{\text{HOFG}} < t \ . \end{cases} \tag{II.2}$$

where it is assumed that after $\hat{T}^{\text{HOFG}}$, central banks maintain $\sigma_t^q = \sigma_t^{q,n} = 0$. In this equation, Z_t, W_{t-T}, and $U_{\hat{T}-T}$ are independent Brownian motions. If we square each term in equation (II.2) and apply the expectation operator with respect to the information

[1]For this proof, it is implicitly assumed that $r_1^T(\sigma_1^{q,L}) < 0$ and $r_2^T(\sigma_2^{q,L}) > 0$ hold for the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, ensuring that the ZLB remains effective up to time T.

available at $t=0$, we obtain:

$$\mathbb{E}_0\,\hat{Q}_t^2 = \begin{cases} \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})^2 + \left(\sigma_1^{q,L}\right)^2 t\,, & \text{for } t \le T\,, \\ \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})^2 + \left(\sigma_1^{q,L}\right)^2 T + \left(\sigma_2^{q,L}\right)^2 (t-T)\,, & \text{for } T < t \le \hat{T}^{\mathrm{HOFG}}\,, \\ \left(\sigma_1^{q,L}\right)^2 T + \left(\sigma_2^{q,L}\right)^2 (\hat{T}^{\mathrm{HOFG}} - T)\,, & \text{for } \hat{T}^{\mathrm{HOFG}} < t\,. \end{cases} \tag{II.3}$$

If we substitute equation (II.3) into the central bank's loss function (14), the central bank's commitment problem can be expressed as follows:

$$\begin{aligned} &\min_{\hat{T}^{\mathrm{HOFG}},\sigma_1^{q,L},\sigma_2^{q,L}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Q}_t^2\,dt \\ &= \min_{\hat{T}^{\mathrm{HOFG}},\sigma_1^{q,L},\sigma_2^{q,L}} \int_0^{\hat{T}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})^2 dt + \left(\sigma_1^{q,L}\right)^2 \underbrace{\int_0^T t e^{-\rho t} dt}_{=\frac{1}{\rho^2}-\frac{1}{\rho^2}e^{-\rho T}-\cancel{\frac{T}{\rho}e^{-\rho T}}} + \left(\sigma_1^{q,L}\right)^2 T \underbrace{\int_T^\infty e^{-\rho t} dt}_{=\cancel{\frac{1}{\rho}e^{-\rho T}}} \\ &\quad + \left(\sigma_2^{q,L}\right)^2 \underbrace{\int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}(t-T) dt}_{=\cancel{-\frac{1}{\rho}(\hat{T}^{\mathrm{HOFG}}-T)e^{-\rho\hat{T}^{\mathrm{HOFG}}}}+\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^2}} + \left(\sigma_2^{q,L}\right)^2 (\hat{T}^{\mathrm{HOFG}}-T) \underbrace{\int_{\hat{T}^{\mathrm{HOFG}}}^\infty e^{-\rho t} dt}_{=\cancel{\frac{1}{\rho}e^{-\rho\hat{T}^{\mathrm{HOFG}}}}} \\ &= \min_{\hat{T},\sigma_1^{q,L},\sigma_2^{q,L}} \underbrace{\int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})^2 dt}_{\text{Deterministic fluctuations}} + \underbrace{\left(\sigma_1^{q,L}\right)^2 \frac{1}{\rho^2}(1-e^{-\rho T}) + \left(\sigma_2^{q,L}\right)^2 \left(\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^2}\right)}_{\text{Stochastic fluctuations}}\,. \end{aligned} \tag{II.4}$$

The central bank now has control over $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\mathrm{HOFG}}$, in addition to its conventional monetary policy tool $\{i_t\}$. Initially, we derive the first-order condition for $\hat{T}^{\mathrm{HOFG}}$, which is as follows:

$$2\cdot \underbrace{r_2^T(\sigma_2^{q,L})}_{>0} \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt + \left(\sigma_2^{q,L}\right)^2 \frac{1}{\rho} e^{-\rho\hat{T}^{\mathrm{HOFG}}} = 0\,, \tag{II.5}$$

from which we obtain

$$\int_0^\infty e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt = \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}}\|\sigma_1^{q,L}<0,\sigma_2^{q,L}<0)dt < 0\,. \tag{II.6}$$

The first-order condition for $\hat{T}^{\text{HOFG}}$ indicates that, at the optimum, the central bank reduces the value of $\hat{T}^{\text{HOFG}}$ compared to $\hat{T}^{\text{TFG}}$ (traditional forward guidance), as discussed in Section 4.1. This is because when the central bank utilizes traditional forward guidance and achieves perfect stabilization for $t \geq \hat{T}^{\text{TFG}}$, the expression above becomes

$$\int_0^{\hat{T}^{\text{TFG}}} e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}\|\sigma_1^{q,L}=\sigma_1^{q,n}=0,\sigma_2^{q,L}=\sigma_2^{q,n}=0)dt=0\;, \tag{II.7}$$

which is derived by plugging $\sigma_1^{q,L}=0$ and $\sigma_2^{q,L}=0$ into equation (II.5).

Given that at the optimum, $\sigma_1^{q,L}<0$ and $\sigma_2^{q,L}<0$ (which we will demonstrate),

$$\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}}\|\sigma_1^{q,L}=0,\sigma_2^{q,L}=0)<\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}}\|\sigma_1^{q,L}<0,\sigma_2^{q,L}<0)\;.$$

Therefore, we deduce from equation (II.1) that at the optimum, $\hat{T}^{\text{HOFG}}<\hat{T}^{\text{TFG}}$, as evidenced by comparing (II.7) with (II.6).

To characterize the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, a **variational argument** is required. This is because $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ influence the levels of $r_1^T(\sigma_1^{q,L})$, $r_2^T(\sigma_2^{q,L})$, and $\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})$. Specifically, we can derive:

$$\frac{\partial r_1^T(\sigma_1^{q,L})}{\partial\sigma_1^{q,L}}=-\left(\bar{\sigma}+\sigma_1^{q,L}\right)<0,\;\;\frac{\partial r_2^T(\sigma_2^{q,L})}{\partial\sigma_2^{q,L}}=-\left(\underline{\sigma}+\sigma_2^{q,L}\right)<0\;.$$

Determining $\sigma_1^{q,L}$ An increase in $\sigma_1^{q,L}$ leads to a decrease in $r_1^T(\sigma_1^{q,L})$, which alters the trajectory of $\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})$. This change is illustrated in Figure II.1, as depicted by the transition from the thick blue line to the dashed red line.

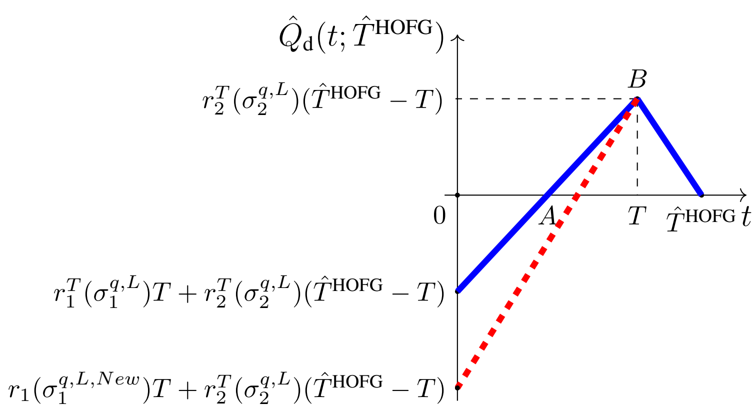

Figure II.1: Variation along $\sigma_1^{q,L}$. Increase to $\sigma_1^{q,L,New}>\sigma_1^{q,L}$.

Differentiating $\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}}) = \int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds$ with respect to $\sigma_1^{q,L}$, we obtain:

$$\frac{\partial \hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}{\partial \sigma_1^{q,L}} = \int_t^T -\left(\bar{\sigma}+\sigma_1^{q,L}\right) ds = -\left(\bar{\sigma}+\sigma_1^{q,L}\right)(T-t), \;\; \forall t \leq T \,.$$

To find optimal $\sigma_1^{q,L}$, we differentiate the objective function in (II.4) by $\sigma_1^{q,L}$ and obtain the following condition:

$$\left(\bar{\sigma}+\sigma_1^{q,L}\right)\int_0^T e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(T-t)dt = \left(\sigma_1^{q,L}\right)\frac{1-e^{-\rho T}}{\rho^2} \,,$$

from which we can prove that $\sigma_1^{q,L} < 0$ must be satisfied at the optimum, given that

$$\int_0^T e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(T-t)dt = \underbrace{\int_0^t e^{-\rho s}\hat{Q}_{\text{d}}(s;\hat{T}^{\text{HOFG}})ds(T-t)\Big|_0^T}_{=0} + \int_0^T \underbrace{\int_0^t e^{-\rho s}\hat{Q}_{\text{d}}(s;\hat{T}^{\text{HOFG}})ds}_{<0}\,dt < 0 \,,$$

where $\int_0^t e^{-\rho s}\hat{Q}_{\text{d}}(s;\hat{T}^{\text{HOFG}})ds < 0$ for $t \leq T$, as derived in equation (II.6).

Determining $\sigma_2^{q,L}$ An increase in $\sigma_2^{q,L}$ leads to a decrease in $r_2^T(\sigma_2^{q,L})$, which alters the shape of $\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})$. This effect is illustrated in Figure II.2 by the transition from the thick blue line to the dashed red line. To further analyze this, we differentiate $\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})$ with respect to $\sigma_2^{q,L}$ and obtain:

$$\frac{\partial \hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}{\partial \sigma_2^{q,L}} = \begin{cases} \int_T^{\hat{T}^{\text{HOFG}}} -\left(\underline{\sigma}+\sigma_2^{q,L}\right) ds = -\left(\underline{\sigma}+\sigma_2^{q,L}\right)(\hat{T}^{\text{HOFG}}-T)\,, & t < T\,, \\ \int_t^{\hat{T}^{\text{HOFG}}} -\left(\underline{\sigma}+\sigma_2^{q,L}\right) ds = -\left(\underline{\sigma}+\sigma_2^{q,L}\right)(\hat{T}^{\text{HOFG}}-t)\,, & T \leq t \leq \hat{T}^{\text{HOFG}}\,. \end{cases}$$

To find the optimal $\sigma_2^{q,L}$, we differentiate the objective function in (II.4) by $\sigma_2^{q,L}$ and obtain

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left(\int_0^T e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}}-T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}}-t)dt\right) = (\sigma_2^{q,L})\frac{e^{-\rho T}-e^{-\rho\hat{T}}}{\rho^2} \,,$$

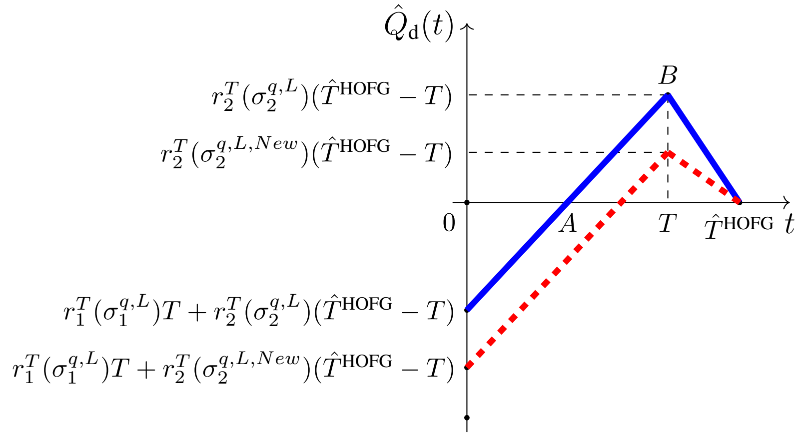

Figure II.2: Variation along $\sigma_2^{q,L}$. Increase to $\sigma_2^{q,L,New} > \sigma_2^{q,L}$.

from which we can demonstrate that at the optimum, $\sigma_2^{q,L} < 0$ must be satisfied, given that

$$
\begin{aligned}
&\int_0^T e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}} - t)dt \\
&< \int_0^T e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}} - T)dt \\
&= (\hat{T}^{\text{HOFG}} - T)\underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})dt}_{<0} < 0\,,
\end{aligned}
$$

where the final inequality is derived from equation (II.6). Hence, during periods of high TFP volatility (i.e., $t < T$) and low TFP volatility with forward guidance (i.e., $T \leq t \leq \hat{T}^{\text{HOFG}}$), a central bank aims to target financial volatility levels below those in a flexible price economy: $\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ and $\sigma_2^{q,L} < \sigma_2^{q,n} = 0$. Such intervention reduces the required risk premium and raises the asset price level $\hat{Q}_t$, thereby increasing output.

■

First-Order Conditions for $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\text{HOFG}}$ The deterministic component of the capitalists' asset gap process $\hat{Q}_t$, denoted as $\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})$, is defined as follows (with

$r_1^T(\sigma_1^{q,L})$ and $r_2^T(\sigma_2^{q,L})$ specified in equation (16)):

$$\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG}) = \int_t^{\hat{T}^{\rm HOFG}} r_s^T ds = \begin{cases} \underbrace{r_1^T(\sigma_1^{q,L})}_{<0}(T-t) + \underbrace{r_2^T(\sigma_2^{q,L})}_{>0}(\hat{T}^{\rm HOFG}-T), & \text{for } \forall t \leq T \ , \\ r_2^T(\sigma_2^{q,L})(\hat{T}^{\rm HOFG}-t), & \text{for } T \leq \forall t < \hat{T}^{\rm HOFG} \ , \end{cases}$$

from which we derive the following:

$$\begin{aligned} \int_0^{\hat{T}^{\rm HOFG}} e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})dt = & \int_0^T e^{-\rho t}\left[r_1^T(\sigma_1^{q,L})(T-t) + r_2^T(\sigma_2^{q,L})(\hat{T}^{\rm HOFG}-T)\right]dt \\ & + \int_T^{\hat{T}^{\rm HOFG}} e^{-\rho t} r_2^T(\sigma_2^{q,L})(\hat{T}^{\rm HOFG}-t)dt \ . \end{aligned} \tag{II.8}$$

The first condition for $\hat{T}^{\rm HOFG}$ can be written as

$$2 \cdot r_2^T(\sigma_2^{q,L}) \int_0^{\hat{T}^{\rm HOFG}} e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})dt + \left(\sigma_2^{q,L}\right)^2 \frac{e^{-\rho \hat{T}^{\rm HOFG}}}{\rho} = 0 \ , \tag{II.9}$$

where

$$\begin{aligned} \int_0^{\hat{T}^{\rm HOFG}} e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})dt = & r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}^{\rm HOFG}-T)\frac{1-e^{-\rho T}}{\rho} \\ & + r_2^T(\sigma_2^{q,L})\left[\frac{e^{-\rho \hat{T}^{\rm HOFG}}}{\rho^2} + \frac{\hat{T}^{\rm HOFG}-T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\right] \ , \end{aligned}$$

follows from equation (II.8). Combined with equation (II.9), the first-order condition for $\hat{T}^{\rm HOFG}$ is expressed as follows:

$$\begin{aligned} 2 \cdot r_2^T(\sigma_2^{q,L}) & \left[r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}^{\rm HOFG}-T)\frac{1-e^{-\rho T}}{\rho} \right. \\ & \left. + r_2^T(\sigma_2^{q,L})\left[\frac{e^{-\rho \hat{T}^{\rm HOFG}}}{\rho^2} + \frac{\hat{T}^{\rm HOFG}-T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\right]\right] + \left(\sigma_2^{q,L}\right)^2 \frac{e^{-\rho \hat{T}^{\rm HOFG}}}{\rho} = 0 \ . \end{aligned}$$

The first-order condition for $\sigma_1^{q,L}$ is expressed as

$$\left(\bar{\sigma} + \sigma_1^{q,L}\right) \int_0^T e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(T-t)dt = \left(\sigma_1^{q,L}\right) \frac{1-e^{-\rho T}}{\rho^2} \ , \tag{II.10}$$

where

$$\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(T-t)dt = r_1^T(\sigma_1^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho T}+\frac{T^2}{\rho}-\frac{2T}{\rho^2}+\frac{2}{\rho^3}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]. \tag{II.11}$$

Substituting equation (II.11) into equation (II.10), we arrive at:

$$(\bar{\sigma}+\sigma_1^{q,L})\left[r_1^T(\sigma_1^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho T}+\frac{T^2}{\rho}-\frac{2T}{\rho^2}+\frac{2}{\rho^3}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]\right] = (\sigma_1^{q,L})\frac{1-e^{-\rho T}}{\rho^2},$$

as the first-order condition for $\sigma_1^{q,L}$. Finally, the first-order condition for $\sigma_2^{q,L}$ is as follows:

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left((\hat{T}^{\mathrm{HOFG}}-T)\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt+\int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(\hat{T}^{\mathrm{HOFG}}-t)dt\right) = (\sigma_2^{q,L})\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^2},$$

Therefore, the first-order condition for $\sigma_2^{q,L}$ is expressed as:[2]

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left[\left[r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\frac{1-e^{-\rho T}}{\rho}\right](\hat{T}^{\mathrm{HOFG}}-T) + r_2^T(\sigma_2^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho\hat{T}^{\mathrm{HOFG}}}+\frac{(\hat{T}^{\mathrm{HOFG}}-T)^2}{\rho}e^{-\rho T}-\frac{2(\hat{T}^{\mathrm{HOFG}}-T)}{\rho^2}e^{-\rho T}+\frac{2}{\rho^3}e^{-\rho T}\right]\right] = \left(\sigma_2^{q,L}\right)\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^2}.$$

[2]We use the following properties of $\hat{Q}_d\left(t;\hat{T}^{\mathrm{HOFG}}\right)$:

$$\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt = r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\frac{1-e^{-\rho T}}{\rho},$$

and

$$\int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(\hat{T}^{\mathrm{HOFG}}-t)dt = r_2^T(\sigma_2^{q,L})\left[-\frac{2e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^3}+\frac{(\hat{T}^{\mathrm{HOFG}}-T)^2}{\rho}e^{-\rho T}-\frac{2(\hat{T}^{\mathrm{HOFG}}-T)}{\rho^2}e^{-\rho T}+\frac{2e^{-\rho T}}{\rho^3}\right].$$

Fiscal Policy Coordination - Derivations Integrating equation (20), we obtain an expression for the monetary reserves as:

$$F_t = \underbrace{F_{0-} - \chi\theta_{0-}a_{0-}}_{\equiv F_0} - \int_0^t \theta_s a_s \left(\sigma_s^{q,*} - \sigma_s^q\right) dZ_s \ . \tag{II.12}$$

From the first order condition of capitalists, and setting $\theta_t = 1$ in equilibrium, we can express the stock market's expected returns for $t \geq 0$ as:

$$i_t^m = i_t + \left[(\sigma_t + \sigma_t^q) + \tau_t\right]^2 \ . \tag{II.13}$$

Using equations (5), (24), (II.13) and the equilibrium condition $\theta_t = 1$ into equations (22) and (23), we obtain:

$$\frac{da_t}{a_t} = \left(i_t + \left(\sigma_t + \sigma_t^{q,*}\right)^2 - \rho\right) dt + \left(\sigma_t + \sigma_t^{q,*}\right) dZ_t \ , \tag{II.14}$$

$$a_0 = a_{0-} + \bar{p}A_{0-}\Delta Q_0^* = \underbrace{\bar{p}A_0 Q_0^*}_{=a_0^*} \ , \tag{II.15}$$

where the last equality in equation (II.15) follows from the fact that wealth is fully allocated into stocks in equilibrium, $a_{0-} = \bar{p}A_{0-}Q_{0-}$, and that TFP does not experience any jump at the limit, $A_0 = A_{0-}$. From equation (II.14) we obtain an expression for the ouput gap process as:

$$d\hat{Q}_t = \left(i_t - r_t^{T,*}\right) dt + \sigma_t^{q,*} dZ_t \ , \tag{II.16}$$

which aligns with the process of the output gap under the optimal forward guidance solution. We previously defined the volatility of the asset price gap as $\sigma_t^q = Var_t\left(\frac{dQ_t}{Q_t}\right)$, so equation (II.16) demonstrates that $\sigma_t^q = \sigma_t^{q,*}$ in equilibrium under the subsidy scheme. Similarly, equation (II.15) implies that $Q_0 = Q_0^*$, and threfore $\hat{Q}_0 = \hat{Q}_0^*$ in equilibrium. Finally, substituting these findings into equations (24) and (II.12), we establish the statements in Corollary 2.