2. INDUCCIÓ

(10) $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \geq 1$$

Fem:

$n = 1$ es compleix la fórmula?

↳ sí! $$1^2 = \frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 6$$

$$\shortparallel$$

$$1$$

✓

Suposem que la fórmula val per $n-1$

$$\left(\text{i.e. } 1^2 + \dots + (n-1)^2 = \frac{(n-1)(n)(2(n-1)+1)}{6} \right)$$

Mirem si podem treure la fórmula per n

$$1^2 + 2^2 + \dots + n^2 = \underbrace{1^2 + 2^2 + \dots + (n-1)^2}_{\text{apliquem la fórmula}} + n^2 =$$

$$= \frac{(n-1)\,n\,(2(n-1)+1)}{6} + n^2 = \frac{(n-1)\cdot n\,(2(n-1)+1)+6n^2}{6} =$$

$$= \frac{n\,[(n-1)(2(n-1)+1)+6n]}{6} = \frac{n\,[2n^2 - n - 2n + 1 + 6n]}{6} =$$

$$= \frac{n\,(2n^2+3n+1)}{6} = \frac{n\,(n+1)\,(2n+1)}{6}$$

qed

11) $$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\dots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \quad \forall n\geq 1$$

Dem: Cas $n=1$ → trivial ✓

$$\underbrace{\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\dots+\frac{1}{(2(n-1)-1)(2(n-1)+1)}}_{\text{és "n-1"}} + \frac{1}{(2n-1)(2n+1)} =$$

$$= \frac{n-1}{2(n-1)+1} + \frac{1}{(2n-1)(2n+1)} =$$

$$= \frac{(n-1)(2n-1)(2n+1) + 2n-2+1}{(2n-1)(2n-1)(2n+1)} =$$

$$= \frac{\cancel{(2n-1)}\left[(n-1)(2n+1)+1\right]}{\cancel{(2n-1)}(2n-1)(2n+1)} = \frac{2n^2+n-2n-1+1}{(2n-1)(2n+1)} =$$

ja que era $n\geq 1$

$$= \frac{n\,\cancel{(2n-1)}}{\cancel{(2n-1)}(2n+1)} = \frac{n}{2n+1}$$

qed

(12) $\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n} \quad \forall\, n \geq 1$

Cas $n=1$ $\Rightarrow \frac{1}{2^1} \overset{?}{=} 2 - \frac{1+2}{2^1} = \frac{1}{2}$ ✓

Suposem: $\sum_{i=1}^{n-1} \frac{i}{2^i} = 2 - \frac{n+1}{2^{n-1}}$

$$\sum_{i=1}^{n} \frac{i}{2^i} = \sum_{i=1}^{n-1} \frac{i}{2^i} + \frac{n}{2^n} = 2 - \frac{n+1}{2^{n-1}} + \frac{n}{2^n} =$$

$$= 2 - \frac{2(n+1) - n}{2^n} = 2 - \frac{2n+2-n}{2^n} =$$

$$= 2 - \frac{n+2}{2^n}$$ qed

(13) $$\sum_{i=0}^{n} 2^i\, i!\,(2i+1) = 2^{n+1}(n+1)! - 1 \quad \forall n \geq 0$$

Cas $n=0$ → $\underbrace{2^0}_{1}\ \underbrace{0!}_{1}\ \underbrace{(2\cdot 0+1)}_{1} \stackrel{?}{=} \underbrace{2^1\ 1! - 1}_{1}$ ✓

Suposem per $n-1$ i vegem què passa amb n

$$\sum_{i=0}^{n} 2^i\, i!\,(2i+1) = \underbrace{\sum_{i=0}^{n-1} 2^i\, i!\,(2i+1)}_{} + 2^n\, n!\,(2n+1) =$$

$$= 2^n\, n! - 1 + 2^n\, n!\,(2n+1) =$$

$$= 2^n\, n!\,(+1+2n+1) - 1 = 2^n\, n!\,(2n+2) - 1 =$$

$$= 2^{n+1}(n+1)! - 1$$

qed

(14) $$\prod_{i=3}^{n}\left(1-\frac{2}{i}\right)=\frac{2}{n(n-1)} \quad \forall\, n\geq 3$$

Dem $n=3 \longrightarrow$ trivial

$$\prod_{i=3}^{n}\left(1-\frac{2}{i}\right)=\prod_{i=3}^{n-1}\left(1-\frac{2}{i}\right)\cdot\left(1-\frac{2}{n}\right)=$$

$$=\frac{2}{(n-1)(n-1-1)}\cdot\left(1-\frac{2}{n}\right)=\frac{2}{(n-1)\cancel{(n-2)}}\cdot\frac{\cancel{(n-2)}}{n}=$$

ja que $n\geq 3$

$$=\frac{2}{n(n-1)}$$ qed

(15) $a_0 = -2 \qquad a_n = 3a_{n-1} + 6 \quad \forall n \geq 1$

$\Downarrow$

$$a_n = 3^n - 3 \quad \forall n \geq 0$$

→ Cas $n = 0$ → $-2 = a_0 = 3^0 - 3 = -2$ ✓

→ Cas $n \geq 1$

$$a_n = 3a_{n-1} + 6 \overset{\text{hipòtesi d'inducció}}{=} 3 \cdot \left[3^{n-1} - 3\right] + 6 =$$

$$= 3^n - 9 + 6 = 3^n - 3$$

qed

(16) $4^n < n! \qquad \forall n \geq 9$

→ $n = 9$ → $4^9 < 9!$ → ho calculem i surt ✓

→ Suposem $4^{n-1} < (n-1)!$

$$4^n = 4 \cdot 4^{n-1} < 4 \cdot (n-1)! \underset{\text{ja que } n \geq 9 \geq 4}{\leq} n \cdot (n-1)! = n!$$

qed

(17) $$\sum_{i=0}^{n} \frac{1}{2i+1} \leq n/3 + 1 \qquad \forall n \geq 0$$

- Cas $n=0 \Rightarrow \frac{1}{2\cdot 0+1} \leq 0/3 + 1$ ✓

- Suposem $n-1$

$$\sum_{i=0}^{n} \frac{1}{2i+1} = \sum_{i=0}^{n-1} \frac{1}{2i+1} + \frac{1}{2n+1} \overset{\text{hipòtesi d'inducció}}{\leq}$$

$$\leq \frac{n-1}{3} + 1 + \frac{1}{2n+1} = \frac{(n-1)(2n+1)+3}{3(2n+1)} + 1 \leq$$

$$\leq \frac{(n-1)(2n+1) + 3\overbrace{\frac{2n+1}{3}}^{\geq 0}}{3(2n+1)} + 1 =$$

$$= \frac{(n-1)+1}{3} + 1 = n/3 + 1 \qquad \underline{\text{qed}}$$

(18) $(1+x)^n \geq 1+nx \quad \forall n \geq 0, \quad x \geq 0$

Cas $n=0 \rightarrow \underbrace{(1+x)^0}_{1} \overset{?}{\geq} 1+0\cdot x$ ✓

Suposem $(1+x)^{n-1} \geq 1+(n-1)x$

$$(1+x)^n = (1+x)(1+x)^{n-1} \geq (1+x)(1+(n-1)x) =$$

$$= (1+x)(1+nx-x) = 1+nx-x+x+nx^2-x^2 =$$

$$= 1+nx+\underbrace{x^2(n-1)}_{0 \leq} \geq 1+nx$$

$\forall x \geq 0$ i $n \geq 1$

qcd

(19) $\left(\frac{n}{e}\right)^n < n! \quad \forall n \geq 1$

Difícil? $\rightarrow$ ens donen una pista: $\left(1+\frac{1}{n}\right)^n < e \quad \forall n$

$n=1 \rightarrow \left(\frac{1}{e}\right)^1 < 1!$ ✓

Suposem $\left(\frac{n-1}{e}\right)^{n-1} < (n-1)!$

$$\left(\frac{n}{e}\right)^n = \frac{n}{e}\left(\frac{n}{e}\right)^{n-1} = n \cdot \frac{1}{e} \frac{n^{n-1}}{e^{n-1}} <$$

$$\left(1+\frac{1}{k}\right)^k < e \;\forall k \Rightarrow \left(\frac{k+1}{k}\right)^k < e \;\forall k \Rightarrow \frac{1}{e} < \frac{k^k}{(k+1)^k} \;\forall k$$

$$< n \cdot \frac{(n-1)^{n-1}}{\cancel{n^{n-1}}} \cdot \frac{\cancel{n^{n-1}}}{e^{n-1}} = n \frac{(n-1)^{n-1}}{e^{n-1}} < n \cdot (n-1)! =$$

hipòtesi inducció

$= n!$

<u>qed</u>

(20) $2^{3n+1} + 3 \cdot 5^{2n+1} = 17k \quad \forall n \geq 0$

$n=0 \Rightarrow 2 + 3 \cdot 5 = 17$ ✓

Suposem $n-1$: $2^{3n-2} + 3 \cdot 5^{2n-1} = 17k$

Vegem $2^{3n+1} + 3 \cdot 5^{2n+1}$

$$2^{3n+1} + 3 \cdot 5^{2n+1} = 2^{3n-2} \cdot \underbrace{2^{3}}_{=8} + 3 \cdot 5^{2n-1} \cdot \underbrace{5^{2}}_{=25} =$$

$$= 2^{3n-2} \overset{8)}{+ \dots +} 2^{3n-2} + 3 \cdot 5^{2n-1} \overset{25)}{+ \dots +} 3 \cdot 5^{2n-1} =$$

$$= 8 \cdot \underbrace{\left[2^{3n-2} + 3 \cdot 5^{2n-1} \right]}_{= \text{ hipòtesi inducció}} + \underbrace{17 \cdot 3 \cdot 5^{2n-1}}_{17 \cdot s} =$$

$8 \cdot 17 \cdot k$

$= 17r$ qed

(21) $\sum_{i=1}^{n} i! = 90k + 63 \quad \forall n \geq 5$

$\rightarrow n = 5 \Rightarrow 1! + 2! + 3! + 4! + 5! =$

$\rightarrow$ Suposem $\sum_{i=1}^{n-1} i! = 90k + 63$

Fem: $\sum_{i=1}^{n} i! = \sum_{i=1}^{n-1} i! + n! =$

$= 90k + 63 + n!$

Però $n > 5 \Rightarrow n! = \underbrace{720}_{5!} \cdot r$

$\Rightarrow \sum_{i=1}^{n} i! = 90k + 63 + 720 \cdot r =$

$= 90 [k + 8 \cdot r] + 63$

qed

(22) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Fem $A^2, A^3, \ldots$ i intuïm $A^n = \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$

Demostrem-lo per inducció

$n = 1 \longrightarrow$ ✓

Suposem $A^{n-1} = \begin{pmatrix} 1 & n-1 & \frac{(n-1)(n-2)}{2} \\ 0 & 1 & n-1 \\ 0 & 0 & 1 \end{pmatrix}$

$$A^n = A^{n-1} \cdot A = \begin{pmatrix} 1 & n-1 & \frac{(n-1)(n-2)}{2} \\ 0 & 1 & n-1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1+n-1 & \\ 0 & 1 & n-1+1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$n-1 + \frac{(n-1)(n-2)}{2} = \frac{\cancel{2n} - 2 + n^2 - \cancel{2n} - n + 2}{2} =$$

$$= \frac{n^2-n}{2} = \frac{n(n-1)}{2} \qquad \text{qed}$$

INDUCCIÓ COMPLETA

(39) $a_1 = 3$, $a_2 = 5$, $a_n = 3a_{n-1} - 2a_{n-2}$

↪ Demostrar $a_n = 2^n + 1 \quad \forall n \geq 1$

Dem

- $n = 1 \Rightarrow a_1 = 2^1 + 1 = 3$ ✓

- Suposem que per $1, 2, \ldots, n-1$ es complex $a_i = 2^i + 1$

Vegem $a_n = 2^n + 1$

$$a_n = 3a_{n-1} - 2a_{n-2} = 3\cdot\left(2^{n-1} + 1\right) - 2\left(2^{n-2} + 1\right) =$$

$$= 3\cdot 2^{n-1} + 3 - 2^{n-1} - 2 =$$

$$= 3\cdot 2^{n-1} - 2^{n-1} + 1 = 2^{n-1}(3-1) + 1 =$$

$$= 2^n + 1$$ qed

(40) $a_0 = 1$, $a_1 = 1$, $a_2 = 0$, $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3} + 2$

$\hookrightarrow \quad a_n = 2^n - n^2 \quad \forall\, n \geq 0$

$n=0 \rightarrow a_0 = 2^0 - 0^2 = 1$ ✓

Provem a_n

$$a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3} + 2 =$$

$$= 4\left[2^{n-1} - (n-1)^2\right] - 5\left[2^{n-2} - (n-2)^2\right] + 2\left[2^{n-3} - (n-3)^2\right] + 2 =$$

$$= \underbrace{\underbrace{2^{n+1} - 5\cdot 2^{n-2} + 2^{n-2}}_{2^{n-2}\left[\underbrace{2^3 - 5 + 1}_{4}\right]}}_{2^n} \underbrace{- 4(n-1)^2 + 5(n-2)^2 - 2(n-3)^2 + 2}_{= -n^2}$$

Fàcil veure-ho!

$n^2 \rightarrow -4 + 5 - 2 = -1 \Rightarrow -n^2$

$n \rightarrow +8 - 20 + 12 = 0 \Rightarrow 0 \cdot n$

$1 \rightarrow -4 + 20 - 18 + 2 = 0 \Rightarrow 0$

$$= 2^n - n^2$$

qed

(41) $a_0 = 1 \quad a_1 = 2 \quad a_2 = 6 \quad a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

$$\to \quad a_n = 1 - 2^n + 3^n \quad \forall n \geq 0$$

$n = 0 \to \checkmark$

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} =$$

$$= 6\left[1 - 2^{n-1} + 3^{n-1}\right] - 11\left[1 - 2^{n-2} + 3^{n-2}\right] + 6\left[1 - 2^{n-3} + 3^{n-3}\right] =$$

$$= 6 - 3 \cdot 2^n + 2 \cdot 3^n - 11 + 11 \cdot 2^{n-2} - 11 \cdot 3^{n-2} + 6 - 3 \cdot 2^{n-2} + 2 \cdot 3^{n-2} =$$

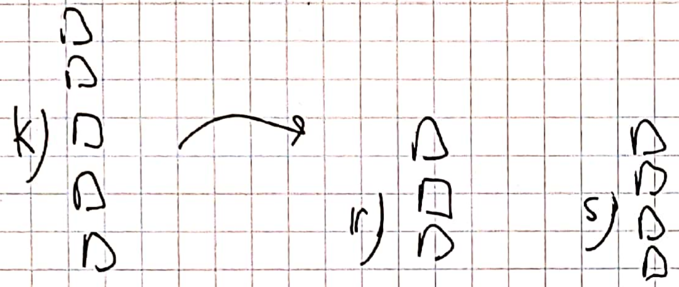
$$1$$

$$\underbrace{-3 \cdot 2^n}_{-12 \cdot 2^{n-2}} + 11 \cdot 2^{n-2} - 3 \cdot 2^{n-2} = -4 \cdot 2^{n-2} = -2^n$$

$$= 18 \cdot 3^{n-2} - 11 \cdot 3^{n-2} + 2 \cdot 3^{n-2} = 9 \cdot 3^{n-2} = 3^n$$

$$\Rightarrow \quad a_n = 1 - 2^n + 3^n \qquad \text{qed}$$

(42) Problema de la pila de caixes



$\rightarrow P(k) = r \cdot s + P(r) + P(s)$

Volem provar que $P(n) = \dfrac{n(n-1)}{2}$

$n=1 \rightarrow P(1) = 0$ ✓

$n=2 \rightarrow P(2) = 1$ ✓

$\vdots$

Fem $P(n)$

$$\underset{\substack{\uparrow \\ \alpha+\beta=n}}{P(n) =} \alpha\cdot\beta + P(\alpha) + P(\beta) = \alpha\cdot\beta + \frac{\alpha(\alpha-1)}{2} + \frac{\beta(\beta-1)}{2} =$$

$$= \frac{2\alpha\beta + \alpha^2 - \alpha + \beta^2 - \beta}{2} = \frac{\alpha^2 + 2\alpha\beta + \beta^2 - \alpha - \beta}{2} =$$

$$= \frac{(\alpha+\beta)^2 - (\alpha+\beta)}{2} = \frac{n^2 - n}{2}$$

qed

(43) $\forall n \geq 9 \quad n = 3t + 4s$ amb $t, s \geq 0$

important!!!

Són "nombre de monedes"

Dem.

$n = 9 \rightarrow 9 = 3 \cdot 3 + 4 \cdot 0$

$n = 10 \rightarrow 10 = 3 \cdot 2 + 4 \cdot 1$

$n = 11 \rightarrow 11 = 3 \cdot 1 + 4 \cdot 2$

Suposem vàlid per a 12, 13, ... n-1

Vegem:

$$n = \underbrace{n-3}_{\text{es pot aplicar la hipòtesi, ja que } n > 11} + 3 = 3t + 4s + 3 = 3(\underbrace{t+1}_{\geq 0}) + 4\underbrace{s}_{\geq 0}$$

ja que $t \geq 0$

(44) $f(1) = 2 \qquad f(n) = \left[f\left(\frac{n}{p}\right)\right]^2$ on $p=$ el primer més petit que $p \mid n$

$\hookrightarrow \quad n = \underbrace{p_1^{\alpha_1} \cdots p_k^{\alpha_k}}_{\text{descomposició en primers}} \Longrightarrow f(n) = 2^{2^{\alpha_1+\dots+\alpha_k}} \quad \forall n \geq 2$

<u>Dem</u>

$n=2 \longrightarrow f(2) = \left[f\left(\frac{2}{2}\right)\right]^2 = \left(f(1)\right)^2 = 2^2$ ✓

Per $n>2$

$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, on podem "reordenar" els primers $p_1 \cdots p_k$ per tal que $p_1 < p_2 < \dots < p_k$

$\Downarrow$

$f(n) = \left(f\left(\frac{n}{p_1}\right)\right)^2 \underset{\substack{\uparrow \\ \text{hipòtesi} \\ \text{inducció}}}{=} \left(2^{2^{\alpha_1-1+\alpha_2+\dots+\alpha_k}}\right)^2 =$

$= 2^{2^{\alpha_1+\alpha_2+\dots+\alpha_k}}$

<u>qed</u>

(45) $f(n) = n$ si $n \leq 2$, $f(n) = f(E(n/3)) + f(ES(\frac{2n}{3}))$ per $n > 2$

on $E(x) = \lfloor x \rfloor$, $ES(x) = \lceil x \rceil$

Provar que $f(n) \leq 2n-1 \quad \forall n \geq 1$

$n=1 \rightarrow f(1) = 1$

$n=2 \rightarrow f(2) = f(\underbrace{\lfloor 2/3 \rfloor}_{0}) + f(\underbrace{\lceil \frac{4}{3} \rceil}_{2}) = 0 + 2 \leq 2 \cdot 2 - 1$ ✓

Suposem $f(i) \leq 2i - 1 \quad \forall\, 2 < i < n$

Provem $f(n)$

$$f(n) = f\left(\lfloor n/3 \rfloor\right) + f\left(\left\lceil \frac{2n}{3} \right\rceil\right) \underset{\text{hipòtesi}}{\leq} 2\left\lfloor \frac{n}{3} \right\rfloor - 1 + 2\left\lceil \frac{2n}{3} \right\rceil - 1 =$$

$$= 2\left[\underbrace{\left\lfloor \frac{n}{3} \right\rfloor + \left\lceil \frac{2n}{3} \right\rceil}_{n} - 1\right] = 2n - 2 \leq 2n - 1$$

qed

(46) $f(1) = 1 \qquad f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 \qquad n>1$

Provar $\quad f(n) \leq \log_2 n + 1 \quad \forall\, n \geq 1$

$n = 1 \Rightarrow$

$$f(1) = 1 \leq \underbrace{\log_2 1}_{0} + 1 \quad \checkmark$$

Suposem $\quad f(K) \leq \log_2 K + 1 \quad \forall\, 1 < K < n$

Fem

$$f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 \underset{\text{hipòtesi}}{\leq} \log_2 \left\lfloor \frac{n}{2} \right\rfloor + 1$$

Si $n = \dot{2} \Rightarrow \left\lfloor \frac{n}{2} \right\rfloor = n/2$

$$\log_2 \left\lfloor \frac{n}{2} \right\rfloor + 1 = \log_2 n/2 + 1 = \log_2 n - \log_2 2 + 1 =$$

$$= \log_2 n - 1 + 1 \leq \log_2 n + 1 \quad \checkmark$$

Si $n = \dot{2} + 1 \quad \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$

$$\log_2 \left\lfloor \frac{n}{2} \right\rfloor + 1 = \log_2 \frac{n-1}{2} + 1 = \log_2 (n-1) - \log_2 2 + 1 =$$

$$= \log_2 (n-1) \underset{g(x) = \log_2 x \text{ és creixent.}}{\leq} \log_2 n \leq \log_2 n + 1 \quad \checkmark$$

qed

(47) $f(1) = 0 \qquad f(n) = 4 f\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + n$ per $n > 1$

Provar $f(n) \le n \log_4 n \quad \forall n \geq 1$

$n = 1 \to f(1) = 0 \le 1 \cdot \log_4 1 = 0$ ✓

Vegem $f(n)$

$$f(n) = 4 f\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + n \le 4\left[\left\lfloor \frac{n}{4} \right\rfloor \log_4 \left\lfloor \frac{n}{4} \right\rfloor\right] + n$$

Pot ser $n = 4k,\ 4k+1,\ 4k+2,\ 4k+3$

- $n = 4k \Rightarrow 4\left(\left\lfloor \frac{n}{4} \right\rfloor \log_4 \left\lfloor \frac{n}{4} \right\rfloor\right) + n = n \log_4 \frac{n}{4} + n =$

$$= n\left(\log_4 n - \underbrace{\log_4 4}_{1}\right) + n = \cancel{n \log_4 n}$$

$$= n \log_4 n \quad ✓$$

- $n = 4k+1 \Rightarrow \left\lfloor \frac{n}{4} \right\rfloor = \frac{n-1}{4}$

$$4\left[\frac{n-1}{4} \log_4 \frac{n-1}{4}\right] + n = (n-1)\left(\log_4(n-1) - \underbrace{\log_4 4}_{1}\right) + n =$$

$$= n \log_4(n-1) - \cancel{n} - \underbrace{\log_4(n-1)} + 1 + \cancel{n}$$

Com que $n > 1$ i $n = 4k+1$,
ha de ser $k \geq 1$

$\Rightarrow n-1 \geq 4 \Rightarrow \log_4(n-1) \geq 1$

Com que hem vist que $\log_4(n-1) \geq 1$

$$\Rightarrow \quad -\log_4(n-1) + 1 \leq 0$$

$$\Rightarrow \quad f(n) \leq n \log_4(n-1) - \log_4(n-1) + 1 \leq$$

$$\leq n \log_4(n-1) \underset{\substack{\uparrow \\ \text{log creixent}}}{\leq} n \log_4 n \quad \checkmark$$

Cal fer els altres casos!

$n = 4k+2$

$n = 4k+3$

48 $\forall n \geq 0$ es pot escriure en base 2

i.e. $n = \alpha_0 2^0 + \alpha_1 2^1 + \dots + \alpha_k 2^k$

$B_0 = \{0\}$ ← 0 escrit en base 2

$B_1 = \{1\}$ ← 1 escrit en base 2

$B_2 = \{0, 1\}$ ← 2 escrit en base 2 (es llegeix de dreta a esquerra)

⋮

$B_n = \{a_n, B_{\lfloor \frac{n}{2} \rfloor}\}$ on $\begin{cases} a_n = 0 & \text{si } n = \dot{2} \\ a_n = 1 & \text{si } n = \dot{2}+1 \end{cases}$

Hipòtesi d'inducció $\forall\, 1 < i < n$, $i = \alpha_0 2^0 + \dots + \alpha_k 2^k$

on $B_i = (a_0, \dots, a_k)$

Volem que $n = \alpha_0 2^0 + \dots + \alpha_k 2^k$

$B_n = (a_n, B_{\lfloor \frac{n}{2} \rfloor})$

~~$n = 2 a_n 2^0$~~

• $n = \dot{2} \Rightarrow a_n = 0 \Rightarrow n = 2 \cdot \left(\frac{n}{2}\right) = 0 \cdot 2^0 + 2\left(\frac{n}{2}\right) =$

$= 0 \cdot 2^0 + 2\left(\beta_0 2^0 + \dots + \beta_k 2^k\right) =$ ✓

• $n = \dot{2} + 1$

$$n = 1\cdot 2^0 + 2\left(\frac{n-1}{2}\right) = 1\cdot 2^0 + 2\left(\beta_0 2^0 + \beta_1 2^1 + \dots + \beta_k 2^k\right) =$$

$$= 1\cdot 2^0 + \beta_0 2^1 + \dots + \beta_k 2^{k+1}$$

qed

(19) $a_1 = 0$ $a_p = 1$ si p primer

$$a_{n\cdot m} \le 2a_n + a_m$$

Provar $a_{p_1^{\alpha_1}\cdots p_k^{\alpha_k}} \le 2\alpha_1 + \dots + 2\alpha_k - 1$

~~Els casos $n=1$, $n=0$~~

Cas $n = 2$ $\longrightarrow$ $1 = a_2 \le 2\cdot 1 - 1 = 1$ ✓

Suposem que es compleix per $2 < i < n$

$$a_n = a_{p_1^{\alpha_1}\cdots p_k^{\alpha_k}} \le 2a_{p_1} + a_{p_1^{\alpha_1-1}p_2^{\alpha_2}\cdots} =$$

$$= 2 + a_{p_1^{\alpha_1-1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}} \underset{\uparrow\ \text{hipòtesi}}{\le}$$

$$\le 2 + 2(\alpha_1 - 1) + 2\alpha_2 + \dots + 2\alpha_k - 1 =$$

$$= 2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_k - 1$$

qed

50 $a_0 = 1 \qquad a_n = a_0 - a_1 + \dots + (-1)^{n-1} a_{n-1}$ per $n \geq 0$

Provar $a_n = 0 \quad \forall n \geq 2$

$n=0 \rightarrow a_0 = 1$

$n=1 \rightarrow a_1 = a_0 = 1$

$n=2 \Rightarrow a_2 = a_0 - a_1 = 1-1 = 0$

Suposem $a_i = 0 \quad \forall \ 2 < i < n$

Vegem a_n

$$a_n = \underset{\substack{\uparrow \\ 1}}{a_0} - \underset{\substack{\uparrow \\ 1}}{a_1} + \underset{\substack{\uparrow \\ 0}}{a_2} + \underset{\substack{\uparrow \\ 0}}{\dots} + \underset{\substack{\uparrow \\ 0}}{\underbrace{(-1)^{n-1} a_{n-1}}} = 0$$

hipòtesi d'inducció

qed

(51)

a) $\forall n \geq 2 \qquad 1 + 1\cdot 1! + 2\cdot 2! + \dots + (n-1)\cdot(n-1)! = n!$

Provam

$n = 0 \rightarrow 1 + 1\cdot 1! = 2 = 2!$

Suposem que es compleix per $n-1$

$$\underbrace{1 + 1\cdot 1! + 2\cdot 2! + \dots + (n-2)(n-2)!}_{(n-1)!} + (n-1)(n-1)! =$$

hipòtesi

$$= (n-1)!\,(1 + n - 1) = n!$$

qed

b) $a_1 = 1, \quad a_n = a_1 + 2a_2 + \dots + (n-1)\,a_{n-1} \qquad \forall n \geq 2$

Proverem $a_n = \dfrac{n!}{2} \quad \forall n \geq 2$

- $n = 2 \rightarrow a_2 = a_1 = 1 = \dfrac{2!}{2}$ ✓

- Suposem $a_i = \dfrac{i!}{2} \quad \forall\; 2 < i < n$

$$a_n = \underbrace{a_1}_{1} + \underbrace{2a_2}_{2\cdot\frac{2!}{2}} + \dots + \underbrace{(n-1)\,a_{n-1}}_{(n-1)\frac{(n-1)!}{2}} = \frac{1 + 1 + 2\cdot 2! + \dots + (n-1)(n-1)!}{2} = \frac{n!}{2}$$

apartat a)

(52) $x_1 = 1 \quad x_{2r} = 2x_r \quad x_{2r+1} = 2x_{r+1}$ per $r \geq 0$

Provar que $x_n \leq n^2 \quad \forall n \geq 1$

- $n = 1 \rightarrow 1 \leq 1^2$ ✓
- Suposem $x_i \leq i^2 \quad \forall\ 1 < i < n$

x_n

A) $n = \dot{2} \Rightarrow x_n = 2x_{n/2} \leq 2\left(\frac{n}{2}\right)^2 = \frac{n^2}{2} \leq n^2$

B) $n = \dot{2}+1 \Rightarrow x_n = 2x_{\lceil n/2 \rceil} = 2x_{\frac{n+1}{2}} \leq$ $\quad \frac{n+1}{2} > 1$

$$\leq 2\left(\frac{n+1}{2}\right)^2 = \frac{(n+1)^2}{2} \leq n^2$$

¿o que $(n+1)^2 \leq 2n^2$

$\Updownarrow$

$n^2 + 2n + 1 \leq n^2 + n^2$

$\Updownarrow$

$\underbrace{2n+1 \leq n^2}$

Arrels: $1 \pm \sqrt{2} \Rightarrow$ És compleix quan $n \geq 1+\sqrt{2}$ ✓

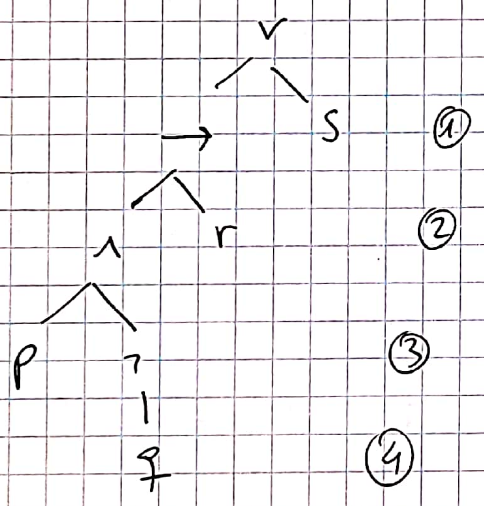
(13) $h(p) = 0$

$h(\neg\varphi) = 1 + h(\varphi)$

$h(\varphi * \psi) = \max(h(\varphi), h(\psi)) + 1$

a) El nombre de lletres proposicionals de φ és $\leq 2^{h(\varphi)}$

Ex. $(p \wedge \neg q \to r) \vee s = \varphi$



$\Rightarrow h(\varphi) = 4$

→ Cas $n = 0$ (i.e. $\varphi = p$) $\Rightarrow \#\varphi = 1 \leq 2^{h(\varphi)} = 1$ ✓

número de lletres

→ Suposem que es compleix $\#\varphi \leq 2^{h(\varphi)}$ $\forall\, h(\varphi) < n$

• Si $\varphi = \neg\alpha \Rightarrow h(\varphi) = 1 + h(\alpha)$

$\#\varphi = \#\alpha \leq 2^{h(\alpha)} = 2^{h(\varphi)-1} \leq 2^{h(\varphi)}$

ja que $h(\alpha) < n$

• Si $\varphi = \alpha * \beta$

$n = h(\varphi) = \text{màx}\left(h(\alpha), h(\beta)\right) + 1$

$\#\varphi = \#\alpha + \#\beta \leq 2^{h(\alpha)} + 2^{h(\beta)} \leq 2^{h(\varphi)}$

$h(\alpha), h(\beta) < n$

ja que és obvi que:

$2^{h(\alpha)} + 2^{h(\beta)} \leq 2^{\text{màx}(h(\alpha), h(\beta))+1}$

qed

b) $\forall n \geq 0 \quad \exists \varphi \mid h(\varphi) = n \quad i \quad \#\varphi = 2^{h(\varphi)}$

• Per $n = 0 \rightarrow \quad \exists \varphi = p \mid h(\varphi) = 0 \quad i \quad \#\varphi = 1 = 2^0$

• Suposem que es compleix per $0 < i < n$

↳ Això vol dir que $\exists \alpha \mid h(\alpha) = n-1 \quad i \quad \#\alpha = 2^{h(\alpha)}$

Llavors, nosaltres fem $\varphi = \alpha \wedge \alpha$

$\#\varphi = \#\alpha + \#\alpha = 2\#\alpha = 2 \cdot 2^{n-1} = 2^n$

i $h(\varphi) = \text{màx}\left(h(\alpha), h(\alpha)\right) + 1 = n-1+1 = n$

qed

c) $\forall n \geq 0 \quad \forall m \quad 1 \leq m \leq 2^n$

$\exists \varphi \quad h(\varphi) = n \quad \# \varphi = m$

Fem-lo per inducció.

- $n = 0 \rightarrow$ ¿ $\exists \varphi \mid h(\varphi) = 0 \quad \# \varphi = m = 1$?

  ja que $1 \leq m \leq 2^0 = 1$

  ✓

  Sí! és $\varphi = p$ ✓

- $n > 0$

  Suposem que $\exists \varphi \quad h(\varphi) = i \quad , \# \varphi = m$

  on $1 \leq m \leq 2^i$

Per no confondre'ns, estem dient,

- $n > 0$ (fixat)

$0 < k < n \rightarrow \forall m, \; 1 \leq m \leq 2^k \quad \exists \varphi_m \quad h(\varphi_m) = k \quad \# \varphi_m = m$

(això és la hipòtesi d'inducció)

Nosaltres volem demostrar que

$\underset{1 \leq m \leq 2^n}{\forall m} \quad \exists \varphi_m \quad h(\varphi_m) = n \quad , \# \varphi_m = m$

$\forall m \quad 1 \leq m \leq 2^n$ el podem escriure $m = 2^r + s$

on $r < n$

$s \leq 2^r$

$\swarrow$

Per hipòtesi d'inducció,

$$\exists \varphi_s \quad h(\varphi_s) = r \quad \# \varphi_s = s$$

Per altre banda, per b) sabem que

$$\exists \alpha \quad h(\alpha) = r \quad \# \alpha = 2^r$$

Llavors, podem fer $\alpha * \varphi_s$

$$h(\alpha * \varphi_s) = 1 + \text{màx}(\alpha, \varphi_s) = 1 + r$$

$$\#(\alpha * \varphi_s) = \#\alpha + \#\varphi_s = 2^r + s = m$$

Llavors, fem

$$\psi_m := \underbrace{\neg \neg \ldots \neg}_{n-r-1} (\alpha * \varphi_s)$$

que compleix $\# \psi_m = 2^r + s = m$

$h(\psi_m) = n - r - 1 + r + 1 = n$

qed