3.2. Relacions d'equivalència

(10) $A \subseteq \mathbb{N}$ $\quad s: \mathbb{N} \longrightarrow \mathbb{N}$

$x \longmapsto$ "suma dels dígits en base 10"

$$xRy \Leftrightarrow s(x) = s(y)$$

a) R és d'equivalència

b) Fer A/R per $A = \{4, 44, 42, \ldots\}$

Dem

a) Cal veure:

1) Reflexiva
2) Simètrica
3) Transitiva

1) $s(x) = s(x)$ → obvi!

2) $xRy \Rightarrow yRx$ → obvi! $(s(x) = s(y) \Rightarrow s(y) = s(x))$

3) $xRy \wedge yRz \Rightarrow xRz$ → obvi!

$$\left.\begin{array}{l} s(x) = s(y) \\ s(y) = s(z) \end{array}\right\} \Rightarrow s(x) = s(z) \checkmark$$

b) $A = \{4, 44, 42, 22, 36, 8, 11, 35, 13, 15, 17, 18, 51, 33, 6\}$

[4] [8] [6] [4] [9] [8] [2] [8] [4] [6] [8] [9] [6] [6] [6]

$A/R = \{[2], [4], [6], [8], [9]\}$

(91) $B, C, \quad B \subseteq C \quad . \quad A = P(C)$

$x, y \in A \qquad xRy \Leftrightarrow x \cup B = y \cup B$

a) R és d'equivalència

b) Cs $C = \{1, 2, 3, 4\}$, $B = \{1, 2\}$

Dem

a) Cal veure

1) Reflexiva

2) Simètrica

3) Transitiva

1) xRx ja que $x \cup B = x \cup B$ ✓

2) $xRy \overset{?}{\Longrightarrow} yRx$

Cert, ja que $x \cup B = y \cup B \Rightarrow y \cup B = x \cup B$ ✓

3) $\left.\begin{array}{l} xRy \\ yRz \end{array}\right\} \overset{?}{\Longrightarrow} xRz$

Tenim $\left.\begin{array}{l} x \cup B = y \cup B \\ y \cup B = z \cup B \end{array}\right\} \Rightarrow x \cup B = z \cup B$

(ja que "=" és transitiva)

(Nota: De fet, "=" és relació d'equivalència ✓)

b) $C = \{1, 2, 3, 4\}$ $B = \{1, 2\}$

$$P(C) = \{ \underset{[\emptyset]}{\emptyset}, \underset{[\{1\}]}{\{1\}}, \underset{[\{1\}]}{\{2\}}, \underset{[\{3\}]}{\{3\}}, \underset{[\{4\}]}{\{4\}}, \underset{[\{1\}]}{\{1,2\}}, \underset{[\{3\}]}{\{1,3\}}, \underset{[\{4\}]}{\{1,4\}}, \underset{[\{4\}]}{\{2,4\}},$$

$$\underset{[\{3,4\}]}{\{3,4\}}, \underset{[\{3\}]}{\{2,3\}}, \underset{[\{4\}]}{\{1,2,4\}}, \underset{[\{3\}]}{\{1,2,3\}}, \underset{[\{3,4\}]}{\{2,3,4\}}, \underset{[\{3,4\}]}{\{1,2,3,4\}} \}$$

$$\Rightarrow \quad P(C)/R = \{ [\emptyset], [\{1\}], [\{3\}], [\{4\}], [\{3,4\}] \}$$

92 $x, y \in \mathbb{Z}$ $\quad xRy \overset{(def)}{\Longleftrightarrow} x^2 + 3y = y^2 + 3x$

a) R ... d'equivalenza

b) Calcolare $[0], [1], [2], [3], [4]$ ... $[n]$ ... $\mathbb{Z}/R$

a) ① Riflessiva $\rightarrow$ ✓ $\quad x^2 + 3x = x^2 + 3x$

② Simmetrica $\rightarrow$ ✓ $\quad x^2 + 3y = y^2 + 3x \Rightarrow y^2 + 3x = x^2 + 3y$

③ Transitiva $\rightarrow$

$$\left.\begin{aligned} x^2 + 3y &= y^2 + 3x \\ y^2 + 3z &= z^2 + 3y \end{aligned}\right\} \underset{\text{sommando}}{\Rightarrow} x^2 + \cancel{3y} + \cancel{y^2} + 3z = \cancel{y^2} + 3x + z^2 + \cancel{3y}$$

$$\Downarrow$$

$$x^2 + 3z = z^2 + 3x$$

b) • $[0]$ = " Tots els $x \in \mathbb{Z}$ $x^2 + 3\cdot 0 = 0^2 + 3x$ "

O sigui, $x^2 = 3x \Rightarrow x = 0$ ó $x = 3$

Per tant, $[0] = \{0, 3\}$

• $[1]$: $x^2 + 3 = 1^2 + 3x \Rightarrow x^2 - 3x + 2 = 0$

$$\Rightarrow x = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 4/2 = 2 \\ 2/2 = 1 \end{cases}$$

$[1] = \{1, 2\}$

• $[2]$: $x^2 + 3\cdot 2 = 2^2 + 3x \Rightarrow x^2 - 3x + 2 = 0$

$$\Rightarrow [2] = [1] = \{1, 2\}$$

• $[3]$: $x^2 + 3\cdot 3 = 3^2 + 3x \Rightarrow x^2 - 3x = 0 \Rightarrow x = \begin{cases} 0 \\ 3 \end{cases}$

$$\Rightarrow [3] = [0] = \{0, 3\}$$

• $[4]$: $x^2 + 3\cdot 4 = 4^2 + 3\cdot x \Rightarrow x^2 - 3x - 4 = 0$

$$\Rightarrow x = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} = \begin{cases} 4 \\ -1 \end{cases}$$

$$\Rightarrow [4] = \{4, -1\}$$

$[n]$

Ha de ser $x \in \mathbb{Z}$ | $n^2+3x = x^2+3n$

① Sigui, $x^2 - 3x + 3n - n^2 = 0$

$$x = \frac{3 \pm \sqrt{9 - 4(3n-n^2)}}{2}$$

on $9 - 4(3n-n^2) = 9 - 12n + 4n^2 = (2n-3)^2$

Per tant,

$$x = \frac{3 \pm (2n-3)}{2} = \begin{cases} \frac{3+2n-3}{2} = n \\ \frac{3-2n+3}{2} = \frac{6-2n}{2} = 3-n \end{cases}$$

Així que $[n] = \{n, 3-n\}$

93 $xRy \overset{\text{def}}{\iff}$ mateixa part entera x, y $(x, y \in \mathbb{R})$

a) R és equivalència

b) $[1'2]$, $[\pi]$, $[-1'2]$

c) $\mathbb{R}/R$

a) - Reflexiva ? → sí $\lfloor x \rfloor = \lfloor x \rfloor$ ← part entera

- Simètrica ? → sí ja que "=" és simètric

- Transitiva ? → sí, ja que "=" ho és.

b) $[1'2] = \{ x \in \mathbb{R} \mid 1 \leq x < 2 \}$

$[\pi] = \{ x \in \mathbb{R} \mid 3 \leq x < 4 \}$

$[-1'2] = \{ x \in \mathbb{R} \mid -2 \leq x < -1 \}$ ← Recordem que $\lfloor x \rfloor$ = part entera "per sota"

c) $\mathbb{R}/_R = \{ [x] \mid x \in \mathbb{Z} \}$

(94) $A = \mathbb{Z} \times \mathbb{Z}$ $\quad (x,y) R (x',y') \overset{def}{\Longleftrightarrow} x \cdot y = x' \cdot y'$

a) Equivalència

b) $(1,0), (1,1), (2,1), (2,3), (-2,4)$

c) $A/_R$

a) ① Reflexiva → $x \cdot y = x \cdot y \quad \forall x,y$

② Simètrica → $x \cdot y = x' y' \Rightarrow x' y' = x \cdot y$ ✓

③ Transitiva → $\left.\begin{matrix} x \cdot y = z \cdot t \\ z \cdot t = u \cdot w \end{matrix}\right\} \Rightarrow x \cdot y = u \cdot w$ ✓

(b)

$$[(1,0)] = \{(x,y) \mid x\cdot y = 0\} = \{(x,0) \mid x\in\mathbb{Z}\} \cup \{(0,x) \mid x\in\mathbb{Z}\}$$

$$[(1,1)] = \{(x,y) \mid x\cdot y = 1\} = \{(1,1), (-1,-1)\}$$

$$[(2,1)] = \{(x,y) \mid x\cdot y = 2\} = \{(1,2),(2,1),(-1,-2)\}$$

$$[(2,3)] = \{(x,y) \mid x\cdot y = 6\} = \{(1,6),(2,3),(3,2),(6,1), (-1,-6),(-2,-3),(-3,-2),(-6,-1)\}$$

$$[(-2,4)] = \{(x,y) \mid x\cdot y = -8\} = \{(-1,8),(-2,4),(-4,2),(-8,1), (1,-8),(2,-4),(4,-2),(8,-1)\}$$

(c) Per tant,

$$[(a,b)] = \left\{(1,ab), \left(p_1, \frac{ab}{p_1}\right), \left(p_1^2, \frac{ab}{p_1^2}\right), \dots\right\}$$

Si $a\cdot b = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ ← descomposició en primers

¿quants elements té $[(a,b)]$?

↳ tants com divisors tingui $a\cdot b$

$a\cdot b = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \Rightarrow$ divisors $a\cdot b = (\alpha_1+1)\cdot \dots \cdot(\alpha_k+1)$

(d) Seguint, $A/R = \{(1,\alpha) \mid \alpha \in \mathbb{Z}\} \sim \mathbb{Z}$

(95) R rel. equivalència a A

$x, y \in A$ Veure que són equivalents:

a) $\overline{x} \cap \overline{y} \neq \emptyset$

b) $\overline{x} \subseteq \overline{y}$

c) $\overline{x} \cap \overline{y} = \overline{y}$

d) $\exists z \in A \mid \overline{x} \cup \overline{y} \subseteq \overline{z}$

e) $\exists z \in A \mid \overline{x} \subseteq \overline{z} \subseteq \overline{y}$

f) $\forall z \in A, \ \overline{z} \subseteq \overline{x} \Rightarrow \overline{y} \subseteq \overline{z}$

Gairebé tautològic !!

Tot implica $\overline{x} = \overline{y}$

Dem

a ⇒ b) Suposo $x \in \overline{x}$

Com que $\overline{x} \cap \overline{y} \neq \emptyset$

$\exists z \in \overline{x}$ i $z \in \overline{y}$

Però llavors, $\overline{z} \subseteq \overline{y}$ i com que $z \in \overline{x}$, $\overline{z} = \overline{x}$

(ja que és equivalència)

$$t \in \overline{x} \Rightarrow tRx \underset{xRz}{\Rightarrow} tRz \underset{zRy}{\Rightarrow} tRy \Rightarrow t \in \overline{y}$$

b ⇒ c) $\overline{x} \cap \overline{y} \subseteq \overline{y}$ sempre

Veiem $\overline{y} \subseteq \overline{x} \cap \overline{y}$

~~$y \in \overline{y}$ si fos $y \notin \overline{y} \cap \overline{x}$, llavors $y \notin \overline{x}$~~

~~Tenim $\overline{x} \subseteq \overline{y}$ Però $x \in \overline{x}$ ja que és d'equivalència Per tant~~

~~xcty~~ $t \in \overline{y} \Rightarrow tRy \Rightarrow tRx \Rightarrow t \in \overline{x}$

$x \in \overline{x} \subseteq \overline{y}$

Për rrjedhojë yRx

$c \Rightarrow d)$ $\overline{x} \cap \overline{y} = \overline{y} \Rightarrow \overline{y} \subseteq \overline{x} \Rightarrow \overline{x} \cup \overline{y} = \overline{x}$

Si afër fund $z \in \overline{x}$, kemi $z \in A$ i $\overline{x} \cup \overline{y} \subseteq \overline{z}$

$d \Rightarrow e)$ Kemi $z \mid \overline{x} \cup \overline{y} \subseteq \overline{z}$

Atëherë ndërtoj z që ems compleks për $\overline{x} \subseteq \overline{z} \subseteq \overline{y}$

ja për:

→ Dihet: $\overline{x} \subseteq \overline{z}$ ✓

→ Vegem $\overline{z} \subseteq \overline{y}$

$t \in \overline{z} \Rightarrow tRz \Rightarrow tRy \Rightarrow t \in \overline{y}$

$y \in \overline{y} \subseteq \overline{x} \cup \overline{y} \subseteq \overline{z}$

Për rrjedhojë yRz

$e \Rightarrow f)$ ~~Re ekzistës z ndërmjet dy rrjeve~~

~~$\exists z \in A \mid \overline{z} \subseteq \overline{x} \wedge \overline{y} \not\subseteq \overline{z}$~~

Vdem $\forall z$ $\overline{z} \subseteq \overline{x} \Rightarrow \overline{y} \subseteq \overline{z}$

Sipas $t \in \overline{y} \Rightarrow \exists u \in A \mid \overline{x} \subseteq \overline{u} \subseteq \overline{y} \Rightarrow$ ~~$xRy \Rightarrow tRy$~~ tRy i $xRy \Rightarrow tRx \xRightarrow{\overline{z} \subseteq \overline{x}} tRz$ ✓

krijo

$f \Rightarrow e)$

Com que $\forall z \in A \quad \overline{z} \subseteq \overline{x} \Rightarrow \overline{y} \subseteq \overline{z}$

prenent $z := x$, tenim que $\overline{x} \subseteq \overline{x}$

i llavors $\overline{y} \subseteq \overline{x}$

Per tant, $\overline{x} \cap \overline{y} = \overline{x} \neq \emptyset$

(96) $x, y, z \in A$. R equivalència

$$\overline{z} \subseteq \overline{x} \cap \overline{y} \overset{?}{\Rightarrow} \overline{x} \cup \overline{y} \subseteq \overline{z}$$

<u>Dem,</u>

$t \in \overline{x} \cup \overline{y} \Rightarrow t R x$ ó $t R y$

Però $t R x \Rightarrow t R z \Rightarrow t \in \overline{z}$

sabem que $\forall w \in \overline{z}$, $w \in \overline{x} \cap \overline{y} \Rightarrow w \in \overline{x} \Rightarrow w R x$

$w R t \ \forall t \in \overline{x}$

i simètricament,

$t R y \Rightarrow \dots \Rightarrow t \in \overline{z}$ ✓

(97) $\cancel{[x]} = \cancel{[y]} \cancel{\Leftrightarrow x R y}$

"Suposo" que l'exercici demana:

$$\overline{x} \subseteq \overline{y} \Rightarrow \overline{x} = \overline{y}$$

Dem només cal veure $\overline{y} \subseteq \overline{x}$

$$t \in \overline{y} \Rightarrow t R y \Rightarrow t R x \Rightarrow t \in x \checkmark$$

ja que $x R y$ d ser $\overline{x} \subseteq \overline{y}$

(98) $\left.\begin{array}{l} \overline{x} \cap \overline{y} \neq \emptyset \\ \overline{x} \cap \overline{z} \neq \emptyset \end{array}\right| \Rightarrow \overline{x} \cap \overline{z} \neq \emptyset$

$\left.\begin{array}{l} \exists\, t \in \overline{x} \text{ i } t \in \overline{y} \\ \exists\, u \in \overline{x} \text{ i } u \in \overline{z} \end{array}\right\} \Rightarrow t R x$ i $t R y \Rightarrow x R y$

$\Downarrow$

$\overline{x} = \overline{y}$

$u R x$ i $u R z$

$\Downarrow$

$x R z \Rightarrow \overline{x} = \overline{z}$

Clàrament,

$$\overline{x} = \overline{z} = \overline{y}$$

(99) S reflexiva i transitiva, A conjunt

Definim R, $xRy \overset{(def)}{\Longleftrightarrow} xSy \wedge ySx$

a) Veure que R és d'equivalència

b) $A/_R$ definim T com: $\bar{x} T \bar{y} \overset{(def)}{\Longleftrightarrow} xSy$

Veure que:

b1) Està ben definida:

$$xSy, xRx', yRy' \Rightarrow x'Sy'$$

b2) S és reflexiva, antisimètrica i transitiva

Dem

a) ① reflexiva: sí, ja que S ho és

② simètrica:

$$xRy \overset{?}{\Rightarrow} yRx$$

sí, ja que $xRy \Rightarrow xSy \wedge ySx \Rightarrow ySx \wedge xSy \Rightarrow$ ✓

la "$\wedge$" és simètrica

③ transitiva

$$\left.\begin{matrix} xRy \\ yRz \end{matrix}\right\} \Rightarrow \left.\begin{matrix} xSy \wedge ySx \\ ySz \wedge zSy \end{matrix}\right\} \Rightarrow \left.\begin{matrix} xSy \wedge ySz \\ zSy \wedge ySx \end{matrix}\right\} \overset{S \text{ transitiva}}{\Rightarrow} \left.\begin{matrix} xSz \\ zSx \end{matrix}\right\} \Rightarrow xRz$$ ✓

b1) $\left.\begin{array}{l} xSy \\ xRx' \\ yRy' \end{array}\right\} \Longrightarrow \overset{①}{\left(\begin{array}{l} xSy \\ xSx' \\ x'Sx \\ ySy' \\ y'Sy \end{array}\right)} ② \overset{S \text{ transitive}}{\Longrightarrow} x'Sy'$

b2) • Reflexiva?

$\overline{x}\,T\,\overline{x} \iff xSx$ cert ✓ (ja que S reflexiva)

• Antisimètrica?

$\left.\begin{array}{l} \overline{x}\,T\,\overline{y} \\ \overline{y}\,T\,\overline{x} \end{array}\right\} \overset{?}{\Longrightarrow} \overline{x} = \overline{y}$ (ja que $\overline{x} \neq \overline{y} \Rightarrow \overline{x}\not T\,\overline{y}$ ó $\overline{y}\not T\,\overline{x}$)

$\left.\begin{array}{l} \overline{x}\,T\,\overline{y} \\ \overline{y}\,T\,\overline{x} \end{array}\right\} \Longrightarrow \begin{array}{l} xSy \\ ySx \end{array} \Longrightarrow xRy \Longrightarrow \overline{x} = \overline{y}$

• Transitiva?

$\left.\begin{array}{l} \overline{x}\,T\,\overline{y} \\ \overline{y}\,T\,\overline{z} \end{array}\right\} \overset{?}{\Longrightarrow} \overline{x}\,T\,\overline{z}$

$\left.\begin{array}{l} \overline{x}\,T\,\overline{y} \\ \overline{y}\,T\,\overline{z} \end{array}\right\} \Longrightarrow \left.\begin{array}{l} xSy \\ ySz \end{array}\right\} \overset{S \text{ transitiva}}{\Longrightarrow} xSz \Longrightarrow \overline{x}\,T\,\overline{z}$

(107) $\mathbb{R}\times\mathbb{R}$

$$(x,y)\, R\, (z,t) \overset{def}{\Longleftrightarrow} |x|+|y| = |z|+|t|$$

a) R d'equivalència

- Reflexiva

$|x|+|y| = |x|+|y|$ (evidentment) $\longrightarrow (x,y)\, R\, (x,y)$

- Simètrica

$|x|+|y| = |y|+|x|$ ✓

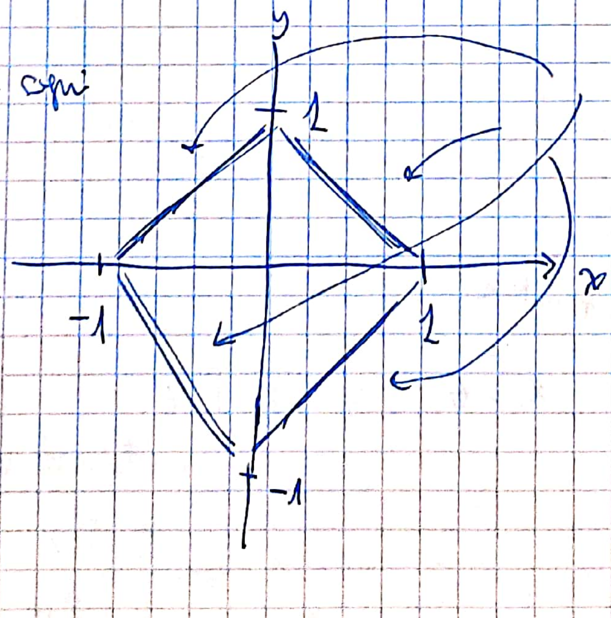
- Transitiva

$$\left.\begin{array}{l} |x|+|y| = |z|+|t| \\ |z|+|t| = |u|+|w| \end{array}\right\} \Rightarrow |x|+|y| = |u|+|w|$$

Per tant, $\left.\begin{array}{l}(x,y)\, R\, (z,t) \\ (z,t)\, R\, (u,w)\end{array}\right\} \Rightarrow (x,y)\, R\, (u,w)$

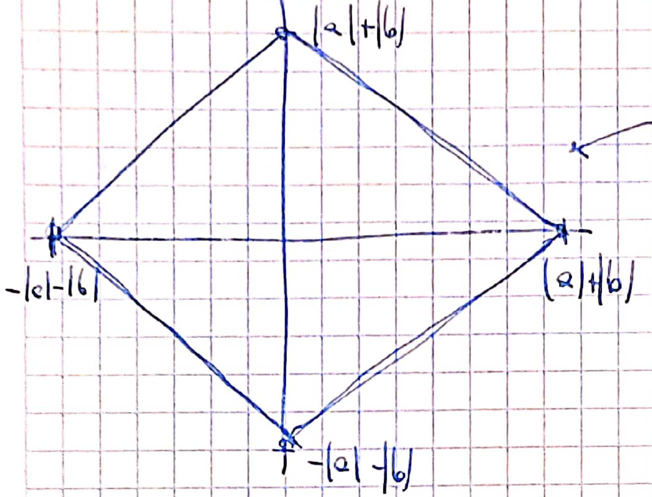
b) $[(1,0)] \leftarrow$ són els $(x,y) \mid |x|+|y| = 1$

Aquí



aquests 4 segments són els únics punts (x,y) que compleixen $|x|+|y| = 1$

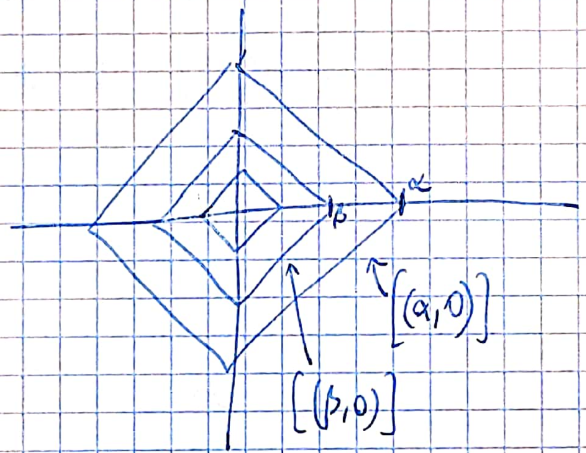
c) $[(a,b)] = \{ (x,y) \mid |x| + |y| = |a| + |b| \}$



és aquest quadrat

d) El conjunt quocient $\mathbb{R}\times\mathbb{R}/_R$ és

e)



$$\mathbb{R}\times\mathbb{R}/_R = \left\{ \overline{(\alpha,0)} \mid \alpha \in \mathbb{R} \right\} \sim \mathbb{R}$$

el podem "assimilar" a $\mathbb{R}$

encara que és diferent!!

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$f: \mathbb{R} \times \mathbb{R} \longrightarrow \{-1, 0, 1\}$

$$f(x,y) = \begin{cases} 1 & \text{si } x \cdot y > 0 \\ 0 & \text{si } x \cdot y = 0 \\ -1 & \text{si } x \cdot y < 0 \end{cases}$$

$(x,y) \, R \, (u,v) \overset{\text{def}}{\iff} f(x,y) = f(u,v)$

a) Equivalència

b c) Classes : $\mathbb{R}\times\mathbb{R} / R$

Dem

a) Reflexiva $(x,y) \, R \, (x,y)$?

sí ! ja que $f(x,y) = f(x,y)$

• Simètrica $(x,y) \, R \, (u,v) \overset{?}{\Rightarrow} (u,v) \, R \, (x,y)$

$f(x,y) = f(u,v) \Rightarrow f(u,v) = f(x,y)$ ✓

• Transitiva

$$\left.\begin{array}{l} (x,y) \, R \, (u,v) \\ (u,v) \, R \, (t,z) \end{array}\right\} \overset{?}{\Rightarrow} (x,y) \, R \, (t,z)$$

$$\left.\begin{array}{l} f(x,y) = f(u,v) \\ f(u,v) = f(t,z) \end{array}\right\} \Rightarrow f(x,y) = f(t,z)$$ ✓

b) c)

Les classes són:

$$"0" = [(0,1)] = \{(x,y) \mid x\cdot y = 0\} = \{(x,y) \mid x=0 \text{ ó } y=0\}$$

$$[(0,1)] = [(1,0)]$$

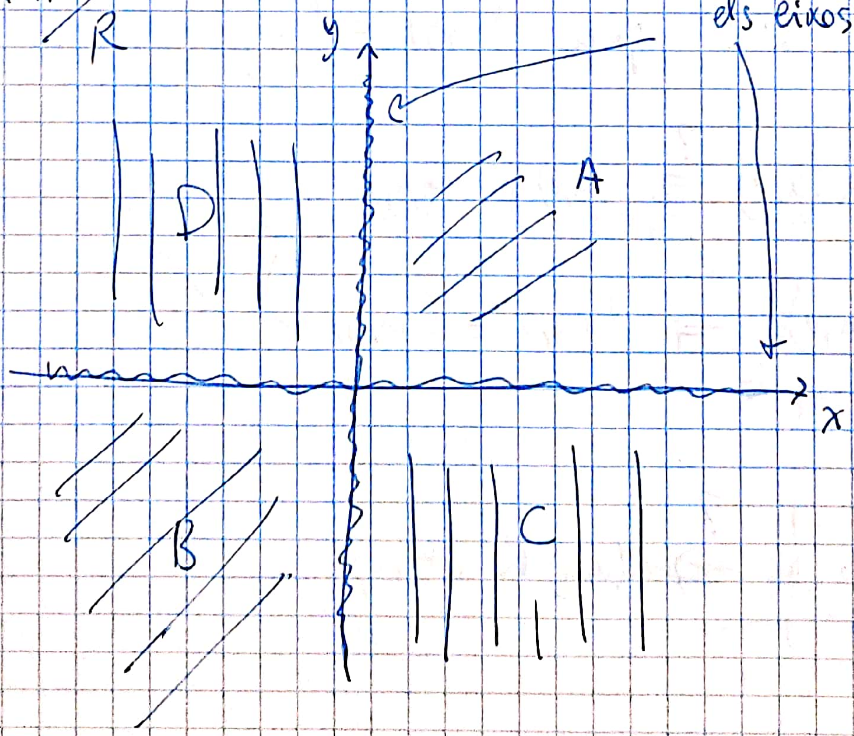
$$"1" = [(1,1)] = \{(x,y) \mid x\cdot y > 0\} = \left\{(x,y) \;\middle|\; \begin{array}{l} x>0 \text{ i } y>0 \\ \text{ó} \\ x<0 \text{ i } y<0 \end{array}\right\} =$$

$$= \{(x,y) \mid x,y>0\} \cup \{(x,y) \mid x,y<0\}$$

$$"-1" = [(-1,1)] = \{(x,y) \mid x\cdot y < 0\} =$$

$$= \{(x,y) \mid x<0 \text{ i } y>0\} \cup \{(x,y) \mid x>0 \text{ i } y<0\}$$

$\mathbb{R}\times\mathbb{R}/R$



$A \cup B = "1"$

$C \cup D = "-1"$