4. FUNCIONS

(2) $a: \mathbb{Z} \to \mathbb{Z}$

$n \longmapsto n-1$

a injectiva $\Leftrightarrow$ $\left(a(n) = a(m) \Rightarrow n = m \right)$

$a(n) = a(m) \Rightarrow n-1 = m-1 \Rightarrow n = m$ ∴ ✓ ∴ injectiva

a exhaustiva $\Leftrightarrow$ $\forall m \in \mathbb{Z}\ \exists n \in \mathbb{Z} \mid a(n) = m$

$\forall m \in \mathbb{Z}$ trobem $n := m+1$ que compleix $a(n) = a(m+1) =$
$= m+1-1 = m$

∴ ✓

Per tant, a és bijectiva

$b: n \longmapsto n^2+1$

injectiva? $n^2+1 = m^2+1 \Rightarrow n^2 = m^2 \Rightarrow n = \pm m$

$\Rightarrow$ No és injectiva!

ja que $n \neq 0 \Rightarrow n \neq -n$ però $b(n) = b(-n)$!

exhaustiva? caldria que $\forall m \in \mathbb{Z}\ \exists n \mid m = n^2+1$

o sigui, $n = \sqrt{m-1}$ però no existeix sempre!

(Ex: $m = 0$)

$c: n \longmapsto n^3$

injectiva $\Leftrightarrow$ $(n^3 = m^3 \Rightarrow n = m)$

cert ✓

exhaustiva $\Leftrightarrow$ $\forall m \in \mathbb{Z} \ \exists n \mid n^3 = m$

Sí, $n := \sqrt[3]{m}$ que sempre existeix.

Per tant, és bijectiva

$d: n \longmapsto E(n/2)$

injectiva?

↳ Clarament no ho és

Ex. $d(2) = d(3)$ i $2 \neq 3$!

exhaustiva?

$\forall m \in \mathbb{Z} \ \exists n \mid m = E(n/2)$??

Sí! $n := \cancel{m/2} \ 2 \cdot m$ compleix que

$E\left(\frac{2 \cdot m}{2}\right) = E(m) = m \quad \forall m \in \mathbb{Z}$

ja que m és enter!

③ $f: \mathbb{N} \to \mathbb{N}$ injectiva

$g: \mathbb{N} \to \mathbb{N}$

$x \longmapsto 2f(x)+3$

g injectiva $\Leftarrow\Rightarrow \left[\forall m, n \in \mathbb{N} \left(g(m) = g(n) \Rightarrow m = n \right) \right]$

$g(m) = g(n) \Rightarrow 2f(m)+3 = 2f(n)+3 \Rightarrow$

$\Rightarrow f(m) = f(n) \underset{f \text{ injectiva}}{\Rightarrow} m = n$ ✓

④ $A, B, \; B \neq \emptyset$

$f: A \times B \to A$ és exhaustiva

$(x, y) \longmapsto x$

$\forall x \in A \; \exists \; z := (x, y)$ on $\underset{\text{possible, ja que } B \neq \emptyset}{y \in B}$ $\mid$ $x = f(z)$ ✓

⑤ $A, B, \quad b \in B$

$f: A \to A \times B$ és injectiva

$x \longmapsto (x, b)$

$f(x) = f(y) \Rightarrow (x, b) = (y, b) \Rightarrow x = y \wedge b = b$

és el que volíem   sí

⑥ $f: \mathbb{Z} \to \mathbb{Z}$ exhaustiva

$g: \mathbb{Z} \to \mathbb{Z}$

$x \longmapsto f(x+1) - 3$

g exhaustiva $\iff \forall y \in \mathbb{Z} \; \exists x \in \mathbb{Z} \mid g(x) = y$

Com que $g(x) = f(x+1) - 3$

⚠ No està dit que f sigui bijectiva!!

Si agafem $x := f^{-1}(y+3) - 1$

$\exists z \mid f(z) = y+3$

entesa com una antiimatge def

Comprovem

$g(x) = g\left(f^{-1}(y+3) - 1\right) = f\left(f^{-1}(y+3) - 1 + 1\right) - 3 =$

$= f f^{-1}(y+3) - 3 = y + 3 - 3 = y$ ✓

(14) $f: (-\infty, 3) \longrightarrow \mathbb{R}$

$x \longmapsto \ln(6-2x)$

a) → ben definida
b) → bijectiva
c) → $f^{-1} = ?$

a) $\forall x \in (-\infty, 3) \ \exists \ \ln(6-2x)$ → sí ✓ (ben definida)

~~b) $\ln(\underbrace{6-2x}_{\alpha}) = \ln(\underbrace{6-2y}_{\beta}) \Rightarrow e^{\alpha} = e^{\beta} \Rightarrow \alpha = \beta$~~

~~(ja que e^x és injectiva)~~

~~$\Rightarrow 6-2x = 6-2y$~~

b) $\ln(6-2x) = \ln(6-2y) \Rightarrow e^{\ln(6-2x)} = e^{\ln(6-2y)} \Rightarrow$

e^x ben definida

$\equiv (x=y \Rightarrow e^x = e^y)$

$\Rightarrow 6-2x = 6-2y \Rightarrow x = y$ ✓ (injectiva)

definició de logaritme

Exhaustiva: $\forall y \in \mathbb{R} \ \exists x \mid y = \ln(6-2x)$

Sí, fent $x = \dfrac{-e^y + 6}{2}$ compleix ✓

Per tant, $\exists f^{-1}$, i aquesta és:

$$f^{-1}(y) = \frac{-e^{y} - 6}{2}$$

(15) $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$

$x \longmapsto \sin x$

a) ben definida: $\sin x$ està ben definit $\exists$ a $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

i $-1 \leq \sin x \leq 1 \quad \forall x \in \mathbb{R}$

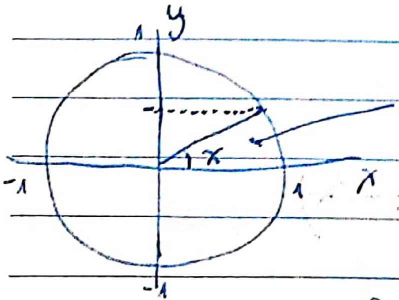
b) injectiva:

$\sin x = \sin y \Longrightarrow x = y + \pi \cdot k$ per $k \in \mathbb{Z}$

Però $x, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Longrightarrow k = 0$ i $x = y$

exhaustiva

$\forall y \in [-1, 1] \quad \exists x \mid \sin x = y$?



Sí! $\forall y \in [-1, 1]$ trobem un angle x que compleix $\sin x = y$

$$f^{-1}(y) = \arcsin(y)$$

16) $f : [10, 11] \to [10, 11]$

$$x \longmapsto \frac{\cos(\pi x - 10\pi)}{2} + \frac{21}{2}$$

e) ben definida?

↳ si $\forall x \in \mathbb{R} \quad \exists f(x)$ ✓

↳ $x \in [10, 11] \Rightarrow f(x) \in [10, 11]$

ja que $x = 10 + t$, amb $t \in [0, 1]$

$$f(x) = \frac{\cos(\cancel{10\pi} + \pi t - \cancel{10\pi})}{2} + \frac{21}{2}$$

$\cos(\pi t) \in [-1, 1]$ si $t \in [0, 1]$

Per tant $\frac{\cos \pi t}{2} \in \left[\frac{-1}{2}, \frac{1}{2}\right]$

i llavors, $\frac{\cos \pi t}{2} + \frac{21}{2} \in \left[\frac{-1+21}{2}, \frac{1+21}{2}\right]$ ✓

$$\frac{-1+21}{2} = 10, \quad \frac{1+21}{2} = 11$$

b) injectiva?

$f(x) = f(y) \overset{?}{\Longrightarrow} x = y$

$\cos(\pi x - 10\pi)$ és bijectiva a $x \in [10, 11]$

(ja que $\cos(\pi z)$ és bijectiva a $[-1, 1]$)

Per tant, clarament f és bijectiva ✓

c) $f^{-1}(y) = \arccos\left[\left(y - \frac{21}{2}\right) \cdot 2\right] = \arccos(2y - 21)$ ✓

17) $f: X \to X$ $\quad X = \{1, 2, \dots 99, 100\}$

$x \mapsto 2x$ si $1 \leq x \leq 50$

$x \mapsto 2(x-51)+1$ si $51 \leq x \leq 100$

a) ben definida

$1 \leq x \leq 50 \Rightarrow 2x \in X$ ✓

$51 \leq x \leq 100 \Rightarrow 2\underbrace{(x-51)}_{\geq 0} + 1 \leq 100$ ✓

b) injectiva? $f(x) = f(y) \Rightarrow x = y$?

- Si $1 \le x \le 50$ i $1 \le y \le 50$

$f(x) = f(y) \Rightarrow 2x = 2y \Rightarrow x = y$ ✓

- Si $51 \le x \le 100$ i $51 \le y \le 100$

$f(x) = f(y) \Rightarrow 2(x-51)+1 = 2(y-51)+1 \Rightarrow x = y$ ✓

- Si $1 \le x \le 50$ i $51 \le y \le 100$

$f(x) = f(y) \Rightarrow 2x = 2(y-51)+1 \Rightarrow 2x = 2y - 101 \Rightarrow$

$\Rightarrow 2(x-y) = -101 \Rightarrow$ impossible !!

ja que $2 \in \mathbb{Z}$, $x-y \in \mathbb{Z}$

però $2 \nmid 101$

- Si $51 \le x \le 100$ i $1 \le y \le 50$ — simètric

Exhaustiva?

$\forall y \in X$, si $y = \dot{2}$ $\Rightarrow$ $x := y/2$

si $y = \dot{2}+1 \Rightarrow x := \frac{y-1}{2} + 51$

Aparte és f^{-1}

(27) $f: \mathbb{Z} \to \mathbb{Z}$

$x \longmapsto x$ si $x = \dot{2}$

$x \longmapsto x+1$ si $x = \dot{2}+1$

a)

$A = \{-2, -1, 0, 1, 2, 3, 4\}$

$f(A) = \{-2, 0, 2, 4\}$

$f(\mathbb{N}) = f(\mathbb{N}_{parells}) \cup f(\mathbb{N}_{senars}) =$

$= f(\mathbb{N}_{parells}) \cup f(\mathbb{N}_{parells}) = \{2k \mid k \in \mathbb{N}\}$

$f(\mathbb{Z}) = f(\mathbb{Z}_{parells}) \cup f(\mathbb{Z}_{senars}) =$

$= \mathbb{Z}_{parells}$

$f(\{x \in \mathbb{Z} \mid x = \dot{2}\}) = \{x \in \mathbb{Z} \mid x = \dot{2}\}$

$f(\{x \in \mathbb{Z} \mid x < 0\}) = f(\{2k \mid k \in \mathbb{Z} \wedge k < 0\}) \cup f(\{2k+1 \mid k \in \mathbb{Z}, k \leq 0\})$

$= \{2k \mid k \in \mathbb{Z} \text{ i } k < 0\} \cup \{2k \mid k \in \mathbb{Z} \text{ i } k \leq 0\} =$

$= \{2k \mid k \in \mathbb{Z} \text{ i } k \leq 0\}$

(27) b)

$$f^{-1}(\{1,2,3,4,5,6,7,8,9,10\}) = \{1,2,3,4,5,6,7,8,9,10\}$$

$$f^{-1}(\{x \in \mathbb{Z} \mid x = \dot{2}+1\}) = \emptyset$$

xa vemos que $\nexists\, y = \dot{2}+1 \mid f(x) = y$ para algún x

que é equivalente a ver que

$$\forall x,\ f(x) = \dot{2}$$

$$f^{-1}(\mathbb{N}) = \mathbb{N} \setminus \{1\}$$

$\supseteq]$ $x \in \mathbb{N}\setminus\{1\} \Rightarrow \left\{ \begin{array}{l} x = \dot{2} \Rightarrow \exists\, y = x \\ \text{ó} \\ x = \dot{2}+1 \Rightarrow \exists\, y = \underbrace{x+1}_{\in \mathbb{N}, \text{ xa que } x>1} \end{array} \right\} \Rightarrow x \in f^{-1}(\mathbb{N})$

$\subseteq]$ $x \in f^{-1}(\mathbb{N}) \Rightarrow \exists\, y = \underbrace{f(x)}_{\in \mathbb{N}} \Rightarrow \left\{ \begin{array}{l} x = \dot{2} \Rightarrow y = x \\ x = \dot{2}+1 \Rightarrow y = \underbrace{x+1}_{\in \mathbb{N}} \end{array} \right\} \Rightarrow x \in \mathbb{N} \text{ y } x > 1$

xa que $x > 1$

$f^{-1}(\mathbb{Z}) = \mathbb{Z}$

$\subseteq)$ $x \in f^{-1}(\mathbb{Z}) \Rightarrow \exists y \in \mathbb{Z} \mid y = f(x) \Rightarrow$

$$\Rightarrow \begin{cases} y = x & \text{si } x = \dot{2} \\ y = x+1 & \text{si } x = \dot{2}+1 \end{cases} \Rightarrow x \in \mathbb{Z}$$

$\supseteq)$ $x \in \mathbb{Z} \Rightarrow \begin{cases} x = \dot{2} \Rightarrow \exists y = x \Rightarrow y = \cancel{x} = f(x) \Rightarrow x \in f^{-1}(\mathbb{Z}) \\ x = \dot{2}+1 \Rightarrow \exists y = x+1 \Rightarrow \underset{y}{\cancel{x}}+1 = f(x) \Rightarrow x \in f^{-1}(\mathbb{Z}) \end{cases}$

$f^{-1}\{x \in \mathbb{Z} \mid x \leq 0\} = \cancel{\{x \in \mathbb{N} \cup \{0\}\}}$

$= \{-t \mid t \in \mathbb{N} \cup \{0\} \text{ i } t = \dot{2}\}$

$\subseteq)$ $x \in f^{-1}(x \in \mathbb{Z} \mid x \leq 0\}) \Rightarrow \exists y_{\leq 0} \mid y = f(x) \Rightarrow$

$\Rightarrow f(x) \leq 0 \Rightarrow f(x) = -t$ amb $t \in \mathbb{N} \cup \{0\}$

Però per construcció de f, $f(x) = \dot{2}$

per tant, $x = -t$ amb $t = \dot{2}$ i $t \in \mathbb{N} \cup \{0\}$

$\supseteq)$ $-t$ parell $\Rightarrow \exists y = -t \Rightarrow$ com que $y \leq 0 : y \in \mathbb{Z}$

$\Downarrow$

$f(y) = y = -t$ ✓

$f^{-1}(\{x \in \mathbb{Z} \mid x < 0\}) = \{-t \mid t \in \mathbb{N}, t = \dot{2}\}$

(28) $g: \mathbb{R} \to \mathbb{R}$

$x \longmapsto E(x)$

a) $g^{-1}(\{0\}) = [0, 1)$

b) $g^{-1}(\{-1, 0, 1\}) = [-1, 1)$

c) $g^{-1}(\{x \in \mathbb{R} \mid 0 < x < 1\}) = \emptyset$

És a dir, $\nexists\, x \in g^{-1}(\{x \in \mathbb{R} \mid 0 < x < 1\})$:

Si $\exists\, x \in g^{-1}(\{x \in \mathbb{R} \mid 0 < x < 1\})$ voldria dir que

$\exists\, y \in (0,1) \mid y = g(x)$ però $g(x) \in \mathbb{Z}\ \forall x$!!

(29) $f: A \to B \qquad Y_1, Y_2 \subseteq B$

Demostrar que $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$

$\subseteq$) $x \in f^{-1}(Y_1 \cap Y_2) \Rightarrow \exists y \in Y_1 \cap Y_2 \mid y = f(x) \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} y \in Y_1 \mid y = f(x) \\ \text{i} \\ y \in Y_2 \mid y = f(x) \end{array}\right\} \Rightarrow \begin{array}{l} x \in f^{-1}(Y_1) \\ \text{i} \\ x \in f^{-1}(Y_2) \end{array} \checkmark$$

$\supseteq$) $x \in f^{-1}(Y_1)$ i $x \in f^{-1}(Y_2) \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} \exists y \in Y_1 \mid y = f(x) \\ \text{i} \\ \exists z \in Y_2 \mid z = f(x) \end{array}\right\} \Rightarrow y = f(x) = z$$

ja que f ben definida

$\Downarrow$

$y = z$

⚠ poden ser diferents! a priori

$\Downarrow$

$\exists y \mid y \in Y_1 \cap Y_2 \mid y = f(x)$

$\Downarrow$

$x \in f^{-1}(Y_1 \cap Y_2)$

(30)

$f: A \to B$

$\forall X_1, X_2 \subseteq A \qquad f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$

Demostrar que f és injectiva

Dem

$\forall x,y \quad f(x) = f(y) \overset{?}{\Longrightarrow} x = y$

$\{x\}, \{y\} \subseteq A$ per tant, $f(\{x\} \cap \{y\}) = \underbrace{f(\{x\})}_{\{f(x)\}} \cap \underbrace{f(\{y\})}_{\{f(y)\}}$

Si $f(x) = f(y)$ llavors $f(x) \in \{f(x)\} \cap \{f(y)\}$

Però deshores, $f(x) \in f(\{x\} \cap \{y\})$

O sigui $\exists z \in \{x\} \cap \{y\} \mid f(x) = f(z)$

Però només pot ser $z = x = y$ $\qquad$ qed

(31) $f: A \to B$

$f(A) = B \Longleftrightarrow f$ exhaustiva

f exhaustiva $\overset{(def)}{\Longleftrightarrow} \forall y \in B \; \exists x \in A \mid f(x) = y$

$f(A) = B \Rightarrow B \subseteq f(A) \Longleftrightarrow f$ exhaustiva ✓

Però $f(A) \subseteq B$ perquè f està ben definida ✓

(32) $f: A \to B \qquad Y \subseteq B$

$f(f^{-1}(Y)) \subseteq Y$??

Dem $y \in f(f^{-1}(Y)) \Rightarrow \exists x \mid y = f(x)$ amb $x \in f^{-1}(Y)$

O sigui, $\exists z \in Y \mid z = f(x)$

$\Rightarrow y = f(x) = z \Rightarrow y \in Y$

✓

(33) f exhaustiva $\Rightarrow f(f^{-1}(Y)) = Y$

Dem

Segons (32), tenim $f(f^{-1}(Y)) \subseteq Y$

Vegem ara $\supseteq$

$y \in Y \underset{f \text{ exhaustiva}}{\Rightarrow} \exists x \in A \mid y = f(x)$

Volem que aquest $x \in f^{-1}(Y)$

Si $x \notin f^{-1}(Y)$ llavors $\nexists z \in Y \mid f(x) = z$ però

això és contradicció, ja que acabem de dir que

$\exists y \in Y \mid y = f(x)$!!!

ped

(34) $f: A \to B$

f exhaustive $\iff f(f^{-1}(Y)) = Y \quad \forall\, Y \subseteq B$

Dem

$\Rightarrow$) fet $\to$ (33)

$\Leftarrow$) Vedem ca $\forall\, y \in B \ \exists\, x \in A \mid y = f(x)$

Cum ca tenem ca $f(\underbrace{f^{-1}(\{y\})}) = \{y\} \neq \emptyset$

$\{x \in A \mid f(x) = y\}$

$\Rightarrow \{x \in A \mid f(x) = y\} \neq \emptyset$

$\Rightarrow \exists\, x \in A \mid y = f(x)$

qed

(52) $f: \mathbb{N} \to \mathbb{N}$

$x \longmapsto x$ si $x = \dot{2}$

$x \longmapsto x+1$ si $x = \dot{2}+1$

Veure que $f \circ f = f$

$$f(f(x)) = \begin{cases} f(x) \checkmark & x = \dot{2} \\ f(\underbrace{x+1}_{= \dot{2}}) = x+1 = f(x) \checkmark & x = \dot{2}+1 \end{cases}$$

(53)

(IV) ~~$g \circ f$ injectiva $\Rightarrow f$ injectiva~~

Dem

~~Suposem $g \circ f(x) = g \circ f(y) \Rightarrow g(f(x)) = g(f(y)) \Rightarrow f(x) = f(y)$~~

$f(x) = f(y) \xRightarrow{g \text{ ben definida}} g[f(x)] = g[f(y)] \Rightarrow (g \circ f)(x) = (g \circ f)(y)$

$\xLeftarrow{g \circ f \text{ injectiva}}$ $x = y$ qed

VIII $g \circ f$ bijectiva $\Rightarrow$ f injectiva i g exhaustiva

$g \circ f$ bijectiva $\Rightarrow$ $g \circ f$ injectiva $\Rightarrow$ f injectiva ✓

fet abans

Vegem que g és exhaustiva: $A \xrightarrow{f} B \xrightarrow{g} C$

$\forall z \in C \ \exists y \in B \mid z = g(y)$??

Com que $g \circ f$ exhaustiva,

$\forall z \in C \ \exists x \in A \mid z = (g \circ f)(x)$.

Agafant $y := f(x)$ tenim que $\exists y \in B$

que compleix que $g(y) = g(f(x)) = z$

ped

IX $f: A \to B$ bijectiva $\Rightarrow f^{-1} \circ f = I_A$ i $f^{-1} \circ f = I_B$

Vein $\forall x \in A \quad f^{-1}(f(x)) = x$, ja que si $\exists y \mid f^{-1}(f(x)) = y$

llavors $f(y) = f(x)$ i com que f injectiva, $x = y$

Per tant, $f^{-1} \circ f = I_A$

$f \circ f^{-1} = I_B$, ja que:

$\forall y \in B \quad \exists!\, x \in A \mid y = f(x)$ ← ja que f bijectiva

Per tant, $x = f^{-1}(y)$

$y = f(x) = f(f^{-1}(y)) = (f \circ f^{-1})(y)$

Per tant, $f \circ f^{-1} = I_B$

54) $\left.\begin{array}{l} f: A \to B \text{ bijectiva} \\ g: B \to C \text{ bijectiva} \end{array}\right\} \Rightarrow (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

~~$\forall z \in C\ \exists$~~

f i g bijectives $\Rightarrow g \circ f$ també ho ha de ser ✓

$\forall z \in C \ \exists!\, x \in A \mid z = (g \circ f)(x)$

Però g bijectiva $\Rightarrow \forall z \in C\, \exists!\, y \in B \mid z = g(y)$

i f bijectiva $\Rightarrow \forall y \in B\ \exists!\, x \in A \mid y = f(x)$

Però llavors tenim que:

$$\left((g \circ f)(x) = z = g(y) = g(f(x)) \right)$$

$$(g \circ f)^{-1} z = x = f^{-1}(y) = f^{-1}(g^{-1}(z)) = f^{-1} \circ g^{-1}(z)$$

(55) $f: A \to B$ bijectiva $\Leftrightarrow \exists g: B \to A \mid g \circ f = I_A$, $f \circ g = I_B$

$\Rightarrow$ $\forall y \in B \; \exists! x \in A \mid y = f(x)$

Per tant, podem definir (correctament)

$$g: B \to A$$
$$y \mapsto x$$

aquest únic x

Per construcció, $g \circ f = I_A$

A més, $(f \circ g)(y) = f(g(y)) = f(x) = y$

O sigui $f \circ g = I_B$

$\Leftarrow$)

f injectiva, ja que:

$$\underbrace{f(x)}_{\in B} = \underbrace{f(y)}_{\in B} \Rightarrow g(f(x)) = g(f(y)) \Rightarrow$$

$$\Rightarrow \underbrace{(g \circ f)}_{I_A}(x) = \underbrace{(g \circ f)}_{I_A}(y) \Rightarrow x = y$$

f exhaustiva, ja que

$$\forall y \in B, \quad y = (f \circ g)(y) = f(g(y))$$

Llavors, $\exists\, x := g(y)$ que compleix que $f(x) = y$

(16) $f: A \to B$, $g: B \to C$ $g \circ f$ bijectiva

Són equivalents:

a) f exhaustiva

b) f bijectiva

c) g injectiva

d) g bijectiva

Dem

a ⇒ b) Hem de veure que f és injectiva:

$$f(x) = f(y) \Rightarrow g(f(x)) = g(f(y)) \Rightarrow (g \circ f)(x) = (g \circ f)(y)$$

$$\underset{g \circ f \text{ injectiva}}{\Rightarrow} x = y$$

b ⇒ c)

$$\underset{\in B}{g(x)} = \underset{\in B}{g(y)} \Rightarrow g(f(z)) = g(f(w)) \Rightarrow (g \circ f)(z) = (g \circ f)(w)$$

$\exists! z \in A \mid x = f(z)$

$\exists! w \in A \mid y = f(w)$

f bijectiva

$$\underset{g \circ f \text{ bijectiva}}{\Downarrow} \quad z = w$$

$$\Downarrow$$

$$f(z) = f(w)$$

$$\Downarrow$$

$$x = y$$

c ⇒ d) Cal veure g exhaustiva

$\forall z \in C \ \exists\, y \in B \mid g(y) = z$??

Com que $g \circ f$ bijectiva,

$\forall z \in C \ \exists!\, x \in A \mid z = g(f(x))$

Llavors, $y := f(x) \in B$ existeix i compleix $g(y) = z$ ✓

d ⇒ e)

$\forall y \in B \ \exists\, x \in A \mid y = f(x)$??

$\forall y \in B,\ g(y) \in C \underset{g \circ f \text{ bijectiva}}{\Rightarrow} \exists!\, x \in A \mid (g \circ f)(x) = g(y)$

Llavors $g(f(x)) = g(y)$

Però g injectiva $\Rightarrow f(x) = y$ com volíem

qed

(57) $f: A \to A$ $\quad f \circ f = I_A$

Per la propietat #55, $f = f^{-1}$ i, per tant, és bijectiva

<u>Nota</u>: <u>no</u> vol dir $f = I_A$

Ex. $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto -x$

complex $(f \circ f)(x) = x \quad \forall x$

i en canvi $f \neq I_{\mathbb{R}}$

(58) $\left.\begin{array}{l} g \circ f \text{ injectiva} \\ f \text{ exhaustiva} \end{array}\right\} \Rightarrow g$ injectiva

$(A \xrightarrow{f} B \xrightarrow{g} C)$

<u>Dem</u>. $\underset{\in B}{g(y)} = \underset{\in B}{g(z)} \underset{f \text{ exhaustiva}}{\Rightarrow} \exists x_1 \in A \mid f(x_1) = y$
$\exists x_2 \in A \mid f(x_2) = z$

$\Rightarrow g f(x_1) = g f(x_2) \underset{g \circ f \text{ injectiva}}{\Rightarrow} x_1 = x_2 \Rightarrow y = z$ ✓

59 $\left.\begin{array}{l} h \circ f = h \circ g \\ h \text{ injetiva} \end{array}\right\} \Rightarrow f = g$ $\left(\begin{array}{l} A \xrightarrow{f} B \xrightarrow{h} C \\ A \xrightarrow{g} B \xrightarrow{h} C \end{array}\right)$

Dem $\forall x \in A$

$$h(f(x)) = h(g(x)) \underset{h \text{ injetiva}}{\Longrightarrow} f(x) = g(x)$$

ped

60 $f : A \to A$

$g : A \to B$

g injetiva ; $g \circ f = g \Longrightarrow f = I_A$

Dem

$x \in A \Rightarrow (g \circ f)(x) = g(x) \Longrightarrow g(f(x)) = g(x) \Longrightarrow$

$\underset{g \text{ injetiva}}{\Longrightarrow} f(x) = x$ ped

61 $f: A \to A$

Són equivalents:

a) $f \circ f = f$

b) $\exists B \subseteq A \mid f(A) \subseteq B$ i $f(x) = x \quad \forall x \in B$

Dem

$a \Rightarrow b)$ Fem $B = f(A)$, compleix:

- $B \subseteq A$ (ja que $f: A \to A$)
- $f(A) \subseteq B$ (ja que $f(A) = B \subseteq B$)
- $\forall x \in B, \ \exists y \mid x = f(y)$ (ja que $x \in f(A)$)

Llavors $f(x) = f(\underbrace{f(y)}) \Rightarrow$

$\Rightarrow f(x) = f \circ f(y) \underset{a)}{=} f(y) = x$ com volíem ✓

$b \Rightarrow a)$ Tenim $\exists B \subseteq A \mid f(A) \subseteq B$ i $f(x) = x \quad \forall x \in B$

$x \in A \Rightarrow f(x) \in f(A) \underset{b)}{\subseteq} B \Rightarrow f(\underbrace{f(x)}_{\in B}) = f(x) \Rightarrow f \circ f = f$

qed

(62) $f: A \to B$

$g: B \to A$ g injective ; $g \circ f = I_A$ (*)

Veure que $f \circ g = I_B$

Dem

$\forall y \in B$, $g(y) \in A$

$(g \circ f)(g(y)) \overset{(*)}{=} g(y)$

O sigui $g((f \circ g)(y)) = g(y)$

Però g injectiva $\Rightarrow$ $(f \circ g)(y) = y$ que és el què volíem

(63) $f: A \to B$

a) f injectiva

b) $\exists\, g: B \to A \mid g \circ f = I_A$

$a \Rightarrow b)$

Tenim $f: A \to B$ injectiva

$f(A) \subseteq B$ i, per construcció, $f: A \to f(A)$ és bijectiva

$\Downarrow$

Per tant, podem construir: $\exists f^{-1}: f(A) \to A$

$$g: B \to A$$
$$y \longmapsto f^{-1}(y) \text{ si } y \in f(A)$$
$$y \longmapsto e \text{ si } y \notin f(A)$$

e és un element qualsevol de A

Per construcció,

$x \in A \Rightarrow f(x) \in f(A) \Rightarrow (g \circ f)(x) = f^{-1} \circ f(x) = x$

com volíem ✓

$b \Rightarrow a)$ $f(x) = f(y) \Rightarrow g(f(x)) = g(f(y)) \Rightarrow$

b)

$\Rightarrow \underbrace{(g \circ f)}_{I_A}(x) = \underbrace{(g \circ f)}_{I_A}(y) \Rightarrow x = y$ com volíem ✓

(69) $f: A \to A$ $f \circ f \circ f = I_A$

$x R y \overset{(\text{def})}{\iff} x = y \vee x = f(y) \vee x = f \circ f(y)$

Demostrar:

a) R és d'equivalència

b) $\overline{x} = \{x, f(x), ff(x)\}$

Dem

a) Reflexiva? $x R x$, ja que $x = x$ ✓

Simètrica? $x R y \overset{?}{\Rightarrow} y R x$

$\Downarrow$ ~~yRx~~

$x = y \vee x = f(y) \vee x = f \circ f(y)$

• Cas $x = y$ $x = y \Rightarrow y = x \Rightarrow y R x$ ✓

• Cas $x = f(y)$

- Si $y = x$ ja està
- $y \neq x \Rightarrow ff(x) = fff(y) \Rightarrow y = ff(x)$ ✓

• Cas $x = ff(y)$

- Si $y = x$ ja està
- $y \neq x \Rightarrow f(x) = fff(y) \Rightarrow f(x) = y$ ✓

Transitiva?

$$\left.\begin{array}{l} xRy \\ yRz \end{array}\right\} \overset{?}{\Rightarrow} xRz$$

$$(x=y \vee x=f(y) \vee x=ff(y)) \wedge (y=z \vee y=f(z) \vee y=ff(z)) \equiv$$

$$\equiv \underbrace{(x=y \wedge y=z)}_{x=z} \vee \underbrace{(x=y \wedge y=f(z))}_{x=f(z)} \vee \underbrace{(x=y \wedge y=ff(z))}_{x=ff(z)} \vee$$

$$\vee \underbrace{(x=f(y) \wedge y=z)}_{x=f(z)} \vee \underbrace{(x=f(y) \wedge y=f(z))}_{x=ff(z)} \vee \underbrace{(x=f(y) \wedge y=ff(z))}_{x=fff(z)=z}$$

$$\vee \underbrace{(x=ff(y) \wedge y=z)}_{x=ff(z)} \vee \underbrace{(x=ff(y) \wedge y=f(z))}_{x=fff(z)=z} \vee \underbrace{(x=ff(y) \wedge y=ff(z))}_{x=ffff(z)=f(z)}$$

Per tant hem vist

$$\left.\begin{array}{l} xRy \\ yRz \end{array}\right\} \Rightarrow xRz \vee xRz \vee xRz$$

o sigui, tot va bé ☺

b) $\overline{x} = \{ y \in A \mid y R x \} = \{ y \in A \mid y = x \vee y = f(x) \vee y = ff(x) \}$

$= \{ x, f(x), ff(x) \}$