1.2. DEMOSTRACIONS

PROVA DIRECTA

$\mathbb{Z}$

1. $a = 2k+1$ per algun k $\overset{?}{\Longrightarrow}$ $a^2 = 2t+1$ per algun t

Dem.

$$(2k+1)^2 = (2k)^2 + 1^2 + 2\cdot 2k =$$

$$= 4k^2 + 1 + 4k = 2\underbrace{(2k^2+2k)}_{t}+1$$

aquest és el t que buscàvem.

qed

(2) $\left.\begin{array}{l} a = 2k \\ b = 2t+1 \end{array}\right\} \overset{?}{\Rightarrow} a+b = 2s+1$

per algun s

Dem.

$a+b = 2k+2t+1 = 2\underbrace{(k+t)}_{= s}+1$

és el què buscàvem

qed

(3) $\left.\begin{array}{l} a = k^2 \\ b = t^2 \end{array}\right\} \overset{?}{\Rightarrow} ab = r^2$

$a \cdot b = k^2 \cdot t^2 = \underbrace{(k \cdot t)}_{= r}^2$

qed

(4.) $a = 2k+1$, $b = 2t+1$ $\Big\} \overset{?}{\Longrightarrow} a+b = 2r$

per algun r

Dem.

$$a+b = 2k+1+2t+1 = 2(k+t)+2 =$$

$$= 2\underbrace{(k+t+1)}_{\overset{\shortparallel}{r}}$$

← aquest és el què buscàvem.

qed

(5) $a = 2k \Rightarrow a^2 = 2r$

Dem.

$$a^2 = (2k)^2 = 2^2 \cdot k^2 = 4k^2 = 2\underbrace{(2k^2)}_{r}$$

(6.) $n \in \mathbb{Z}$ $\qquad n = 2k+1 \Rightarrow 5n^2 - 1 = 2r$

Dem

$$5n^2 - 1 = 5(2k+1)^2 - 1 =$$

$$= 5[4k^2 + 4k + 1] - 1 = 20k^2 + 20k + 5 - 1 =$$

$$= 20k^2 + 20k + 4 = 4 \cdot (5k^2 + 5k + 1) =$$

$$= 2 \cdot \underbrace{[2 \cdot (5k^2 + 5k + 1)]}_{= r}$$

qed

CONTRARECÍPROC

8. $n^2 = 2k \Rightarrow n = 2t$

Farem $n = 2t+1 \Rightarrow n^2 = 2k+1$

$$n^2 = (2t+1)^2 = 4t^2 + 4t + 1 = 2\underbrace{(2t^2+2t)}_{K}+1$$

qed

9. $3n^2 + 6n + 5 = 2k \Rightarrow n+1 = 2t$

Fem $n+1 = 2t+1 \Rightarrow 3n^2 + 6n + 5 = 2k+1$

$n = 2t$

$$3n^2 + 6n + 5 = 3(2t)^2 + 6(2t) + 5 =$$

$$= 12t^2 + 12t + 4 + 1 = 2\underbrace{(6t^2 + 6t + 2)}_{=K}+1$$

qed

10. $c = ab \Rightarrow a \leq \sqrt{c}$ ó $b \leq \sqrt{c}$

$(a, b, c \in \mathbb{R})$

Foren $\left.\begin{array}{l} a > \sqrt{c} \\ b > \sqrt{c} \end{array}\right\} \Rightarrow c \neq a \cdot b$

[Nota: $\sqrt{x} \geq 0 \ \forall x$ (ja que $x = \alpha^2$, $\alpha = \sqrt{x}$ ó $\alpha = -\sqrt{x}$)

$a \cdot b > \sqrt{c} \cdot \sqrt{c} = c$

O sigui, $\left.\begin{array}{l} a > \sqrt{c} \\ b > \sqrt{c} \end{array}\right\} \Rightarrow a \cdot b > c$

és diferent!

qed.

(11) $n \in \mathbb{Z}$ $\quad 5n^2+1 = 2k+1 \Rightarrow n = 2r$

Foren $\quad n = 2r+1 \Rightarrow 5n^2+1 = 2k$

$$5n^2+1 = 5(2r+1)^2+1 = 5[4r^2+4r+1]+1 =$$

$$= 20r^2+20r+5+1 = 20r^2+20r+6 =$$

$$= 2\underbrace{(10r^2+10r+3)}_{k}$$

qed

(12) $n \in \mathbb{Z}$ $\quad n^3 = 2k+1 \Rightarrow n = 2t+1$

Foren $\quad n = 2t \Rightarrow n^3 = 2k$

$$n^3 = (2t)^3 = 8t^3 = 2\cdot\underbrace{(4t^3)}_{k}$$

qed

(13) $x, y \in \mathbb{R}, \quad x, y > 0$

$$xy > 1 \overset{?}{\Longrightarrow} x > 1 \text{ ó } y > 1$$

Forem $x \leq 1$ i $y \leq 1 \Longrightarrow x \cdot y \leq 1$

Podem suposar $x \leq y$

Pertant, $x \cdot y \leq y \cdot y = y^2 \leq 1$

Ja que $y > 0$, $y \leq 1 \Longrightarrow y^2 \leq 1$

Fem-lo pel contrarecíproc

$$y^2 > 1 \overset{?}{\Longrightarrow} y > 1$$

Dem: $y^2 > 1 \Rightarrow y^2 - 1 > 0 \Rightarrow$

$\Rightarrow (y+1)(y-1) > 0 \Rightarrow$

$\Rightarrow$ $y+1 > 0$ i $y-1 > 0 \Rightarrow y > 0$ i $y > +1$

o bé $y+1 < 0$ i $y-1 < 0 \Rightarrow y < -1$ i $y < 1$

NO POT SER

REDUCCIÓ A L'ABSURD

(16) $\sqrt{2} \notin \mathbb{Q}$ $\qquad$ si $\sqrt{2} = \dfrac{a}{b} \Rightarrow$!!

(17) 15 dies diferents $\Rightarrow$ n'hi ha 3 que seran el mateix dia de la setmana

Dem — Principi del colomer

(18) $a, b, c \in \mathbb{Z}$

$a+b = 2k$ ó $b+c = 2t$ ó $a+c = 2r$

Dem

Suposem que no:

$$\left.\begin{array}{l} a+b = 2k+1 \\ b+c = 2t+1 \\ a+c = 2r+1 \end{array}\right\} \underset{\substack{\uparrow \\ \text{sumem} \\ \text{tots}}}{\Longrightarrow} \underbrace{2a+2b+2c}_{\text{parell}} = \underbrace{2k+2t+2r+2+1}_{\text{senar}}$$

$\neq$

!!!

(19) $\nexists\, r \in \mathbb{Q} \mid r^2 + r + 1 = 0$

Dem

Si fos $r = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

$r^2 + r + 1 = 0 \Rightarrow a \mid 1$ i $b \mid 1 \Rightarrow$

$\Rightarrow a, b = \pm 1 \Rightarrow r = \pm 1 \Rightarrow$

$$\Rightarrow \begin{cases} r = 1 \Rightarrow 1^2 + 1 + 1 = 3 \neq 0 \\ r = -1 \Rightarrow (-1)^2 - 1 + 1 = 1 \neq 0 \end{cases}$$!!!

(20) → és el #26

(21) $a, b, c \in \mathbb{R}$

$$a \leq \frac{b+c}{2} \quad ó \quad b \leq \frac{a+c}{2} \quad ó \quad c \leq \frac{a+b}{2}$$

<u>Dem</u>

Si fos
$$\left.\begin{aligned} a &> \frac{b+c}{2} \\ b &> \frac{a+c}{2} \\ c &> \frac{a+b}{2} \end{aligned}\right\} \Rightarrow a+b+c \overset{\text{diferent!}}{>} \frac{b+c+a+c+a+b}{2} = a+b+c$$

!!

(22) $\log_2 3 \notin \mathbb{Q}$

<u>Dem</u>

Si fos $\underbrace{\log_2 3 = \frac{a}{b}}$ $(a, b \in \mathbb{Z},\ b \neq 0)$

Podem suposar $b > 0$ i $a > 0$

$$2^{a/b} = 3 \Rightarrow \left(2^{a/b}\right)^b = 3^b \Rightarrow 2^a = 3^b \Rightarrow$$

$$\underset{\text{ja que } a, b}{\Rightarrow} 2 \cdot 2 \cdot 2 \cdot \overset{a)}{\ldots} \cdot 2 = 3 \cdot 3 \cdot \overset{b)}{\ldots} \cdot 3 \Rightarrow 2 \mid 3 \text{ i } 3 \mid 2$$

!!

(23) $\sqrt{6} \notin \mathbb{Q}$

primers entre si

Si fos $\sqrt{6} = \frac{a}{b} \Rightarrow 6 = \frac{a^2}{b^2} \Rightarrow b^2 \mid a^2 \Rightarrow$

$\Rightarrow b \mid a$!!!

(24) $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$

Si fos $\sqrt{2} + \sqrt{3} = \frac{a}{b} \Rightarrow \underbrace{\frac{a^2}{b^2}}_{\mathbb{Q}} = \underbrace{2+3}_{\mathbb{Q}} + \sqrt{6} \Rightarrow$

$\Rightarrow \sqrt{6} \in \mathbb{Q}$!!!

(25) $a_1 \dots a_n \in \mathbb{R}$

$$\exists a_i \mid a_i \leq \frac{a_1 + \dots + a_n - a_i}{n-1}$$

NOTA: $n \geq 2$

Si no fos així, llavors tindríem:

$$\left.\begin{aligned} a_1 &> \frac{a_2 + \dots + a_n}{n-1} \\ a_2 &> \frac{a_1 + a_3 + \dots + a_n}{n-1} \\ &\vdots \end{aligned}\right\} \Rightarrow$$

els sumarem tots

$$\Rightarrow a_1 + a_2 + \dots + a_n > \frac{a_2 + \dots + a_n}{n-1} + \frac{a_1 + a_3 + \dots + a_n}{n-1} + \dots =$$

$$= \frac{(a_1 + \dots + a_n - a_1) + (a_1 + \dots + a_n - a_2) + \dots + (a_1 + \dots + a_n - a_n)}{n-1} =$$

$$= \frac{(a_1 + \dots + a_n) \cdot (n) - (a_1 + \dots + a_n)}{n-1} = \frac{(a_1 + \dots + a_n)\cancel{(n-1)}}{\cancel{n-1}}$$

!!!

(26) $a_1 \cdots a_n > 0$

$$\exists\, a_i \mid \sqrt[n-1]{\frac{a_1 \cdots a_n}{a_i}} \geq a_i$$

Dem

Si fos $\forall_i \quad a_i > \sqrt[n-1]{\dfrac{a_1 \cdots a_n}{a_i}}$

$$a_1 \cdot \ldots \cdot a_n > \sqrt[n-1]{\frac{a_1 \cdots a_n}{a_1}} \cdot \sqrt[n-1]{\frac{a_1 \cdots a_n}{a_2}} \cdot \ldots \cdot \sqrt[n-1]{\frac{a_1 \cdots a_n}{a_n}} =$$

$$= \sqrt[n-1]{\frac{(a_1 \cdots a_n)^n}{a_1 \cdots a_n}} = \sqrt[n-1]{(a_1 \cdots a_n)^{n-1}} = a_1 \cdots a_n$$

!!!

(27) $a+c = 2k+1$ ó $b-a = 2r+1$ ó $b+c-1 = 2t+1$

Dem

Si fos
$$\left.\begin{aligned} a+c &= 2k \\ b-a &= 2t \\ b+c-1 &= 2r \end{aligned}\right\} \Rightarrow \underbrace{\underbrace{a+c+b-a+b+c-1}_{2b+2c-1}}_{\text{senar}} = \underbrace{2(k+t+r)}_{\text{parell}}$$

sumem les equacions

!!!

(30) $\underset{\in \mathbb{Q}}{a} + \underset{\notin \mathbb{Q}}{b} \notin \mathbb{Q}$

Si fos $a+b = \dfrac{x}{y}$ i $a = \dfrac{s}{t}$ $\Rightarrow$

$$\Rightarrow b = \frac{x}{y} - \frac{s}{t} = \frac{tx - sy}{y \cdot t} = \frac{\alpha}{\beta} \in \mathbb{Q}$$

i això és una contradicció, ja que hem dit que $b \notin \mathbb{Q}$!!

(31) $a+b+c=0 \Rightarrow 2|a$ ó $2|b$ ó $2|c$

Si tots fossin senars,

$$\underbrace{2k+1}_{a}+\underbrace{2r+1}_{b}+\underbrace{2s+1}_{c}=0 \Rightarrow$$

$$\Rightarrow 2(k+r+s)+2+1=0 \Rightarrow$$

$$\Rightarrow 2\underbrace{[k+r+s+1]}_{\alpha}+1=0$$

α i ha de ser $\mathbb{Z}$

Però $2\alpha+1=0 \Rightarrow \alpha=-1/2 \notin \mathbb{Z}$!!!

(32) p primer $\Rightarrow \sqrt{p} \notin \mathbb{Q}$

Si fos $\sqrt{p} \in \mathbb{Q} \Rightarrow \sqrt{p}=\frac{a}{b}$ irreductible $\Rightarrow p=\frac{a^2}{b^2} \Rightarrow$

$\Rightarrow p \cdot b^2=a^2 \Rightarrow p|a \Rightarrow a=k\cdot p$

Però llavors, $\sqrt{p}=p\cdot\frac{k}{b} \Rightarrow p=p^2\frac{k^2}{b^2} \Rightarrow$

$b^2 p = p^2 \cdot K \Rightarrow b^2 = pK \Rightarrow p \mid b \Rightarrow$

$p \mid a$ i $p \mid b \Rightarrow a/b$ no és irreductible !!

(33) $a, b, c \geq 0$

$c = ab \Rightarrow a \leq \sqrt{c}$ ó $b \leq \sqrt{c}$

Si fos $a > \sqrt{c}$ i $b > \sqrt{c}$, com que $a, b, c \geq 0$,

$a \cdot b > \sqrt{c}\sqrt{c} = c$!!

(34) $\left(\frac{a}{b}\right)^2 \cdot \left(\frac{c}{d}\right) = \frac{e}{f}$

$\left(\frac{a}{b}\right)^2 \neq \left(\alpha/\beta\right)^2$, $\frac{e}{f} \neq \left(\gamma/\delta\right)^2$

Si fos $\frac{e}{f} = \left(\gamma/\delta\right)^2 \Rightarrow \frac{a^2}{b^2} \cdot \frac{c}{d} = \frac{\gamma^2}{\delta^2} \Rightarrow$

$\Rightarrow \frac{c}{d} = \frac{b^2 \cdot \gamma^2}{a^2 \cdot \delta^2} = \frac{\alpha^2}{\beta^2}$!!!

(35) $a+3 = b+c \Rightarrow a = 2k+1$ ó $b = 2t+1$ ó $c = 2r+1$

Si fos $\left.\begin{array}{l} a=2k \\ b=2t \\ c=2r \end{array}\right\} \Rightarrow \underbrace{\overset{a}{\widetilde{2k}}+3}_{2(k+1)+1} = \overset{b}{\widetilde{2t}} + \overset{c}{\widetilde{2r}} = 2(t+r)$

!!!

DISJUNCIÓ

(38) $n = 2k+1$ ó $n^2 = 4r$

Dem

Provarem que "$n = 2k \Rightarrow n^2 = 4r$" ✓

(39) $a \leq \frac{a+b}{2}$ ó $b \leq \frac{a+b}{2}$

$a > \frac{a+b}{2} \Rightarrow 2a - a > b \Rightarrow a > b \Rightarrow$

$\Rightarrow b \leq b = \frac{b+b}{2} < \frac{a+b}{2}$ qed

40 $a \leq \frac{a+b+c}{3}$ ó $b \leq \frac{a+b+c}{3}$ ó $c \leq \frac{a+b+c}{3}$

Dem

$$\left.\begin{array}{l} a > \frac{a+b+c}{3} \\ \text{i} \\ b > \frac{a+b+c}{3} \end{array}\right\} \Rightarrow c = \frac{3c}{3} < \frac{2a-b+2b-a+c}{3}$$

41 $a, b \geq 0$ $a \leq \sqrt{ab}$ ó $b \leq \sqrt{ab}$

Demostrarem que $a > \sqrt{ab} \Rightarrow b \leq \sqrt{ab}$

$a > \sqrt{ab} \Rightarrow a^2 > a \cdot b \Rightarrow$ si fos $a=0 \Rightarrow 0>0$!!

$a \neq 0 \Rightarrow a > b$

ja que tot és ≥ 0

Per tant, $b^2 = b \cdot b < a \cdot b$

$\Rightarrow b < \sqrt{a \cdot b}$ q.e.d

De fet, hem demostrat una cosa més estricte!

(42) $n \in \mathbb{Z}$ $\qquad n = 2k+1$ ó $n^2+1 = 2r+1$

Farem $n = 2k \Rightarrow n^2+1 = 2r+1$

Ho és! $\quad ((2k)^2+1 = 4k^2+1 = 2(2k^2)+1$

qed

(43) $a, b, c \geq 0$ $\qquad a \leq \sqrt[3]{abc}$ ó $b \leq \sqrt[3]{abc}$ ó $c \leq \sqrt[3]{abc}$

Farem $a > \sqrt[3]{abc}$ i $b > \sqrt[3]{abc} \Longrightarrow c \leq \sqrt[3]{abc}$

$$\Downarrow \qquad\qquad \Downarrow$$

$$a^3 > abc \qquad\qquad b^3 > abc$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$a^2 > bc \qquad\qquad b^2 > ac$$

$$c < \frac{a^2}{b} \qquad\qquad c < \frac{b^2}{a}$$

Ja que suposem $a, b, c \neq 0$

Llavors,

$$c^3 = c \cdot c \cdot c < c\,\frac{a^2}{b}\,\frac{b^2}{a} = abc$$

$$\Longrightarrow c < \sqrt[3]{abc}$$

(49) $a, b \in \mathbb{Z}$ $\quad a-b = 2k$ ó $b-c = 2t$ ó $c-a = 2r$

Forem $a-b = 2k+1$ $\underset{=}{i}$ $b-c = 2t+1 \Longrightarrow c-a = 2r$

Fem:

$$\begin{array}{l} a-b = 2k+1 \\ b-c = 2t+1 \\ \hline a-c = 2s \end{array} \Longrightarrow c-a = 2 \cdot r \quad \underline{\text{qed}}$$

DISJUNCIÓ AL CONSEQUENT

(52). $a, b, c, d \geq 0$, $\mathbb{R}$

$$d = abc \Longrightarrow a \leq \sqrt[3]{d} \text{ ó } b \leq \sqrt[3]{d} \text{ ó } c \leq \sqrt[3]{d}$$

Forem $a > \sqrt[3]{d}$ $\underset{=}{i}$ $b > \sqrt[3]{d} \Longrightarrow c \leq \sqrt[3]{d}$

$$a \cdot b \cdot c = d = \sqrt[3]{d} \cdot \sqrt[3]{d} \sqrt[3]{d} < a \cdot b \cdot \sqrt[3]{d}$$

Com que $a, b \neq 0$ (ja que $a > \sqrt[3]{d} \geq 0$, $b > \sqrt[3]{d} \geq 0$),

tenim que $c < \sqrt[3]{d}$

(53) $a, b, c \in \mathbb{R}$

$$c = a+b \Rightarrow 2a \leq c \text{ ó } 2b \leq c$$

Farem $\quad 2a > c \Rightarrow 2b \leq c$

$$2a > c \Rightarrow 2a > a+b \Rightarrow a > b$$

$$2b = b+b = b + c-a < a+c-a = c$$

qed

(54) $a, b, c \in \mathbb{Z}$

$$a+c = 2k+1 \Rightarrow a+b = 2r+1 \text{ ó } b+c = 2s+1$$

Farem $\quad a+c = 2k+1$ i $a+b = 2r \Rightarrow b+c = 2s+1$

Sumem:

$$\underbrace{a+c+a+b}_{2a+b+c} = 2k+1+2r = 2t+1$$

$$\Downarrow$$

$$b+c = 2t-2a+1 = 2s+1$$

qed

55. $a+b+c = 2k+1 \implies a-b = 2r$ ó $c = 2s$

Forem $a+b+c = 2k+1$ i $a-b = 2r+1 \implies c = 2s$

Sumem

$$\underbrace{a+b+c+a-b}_{2a+c} = 2k+1+2r+1 = 2\ell$$

$$\implies c = 2s$$ ped

56. $12+a-b-3c = 0 \implies \begin{cases} a+3b-c = 2k \\ \text{ó} \\ a-b-5c = 2r \\ \text{ó} \\ a+b-c = 2s \end{cases}$

Forem
$$\left.\begin{array}{l} 12+a-b-3c = 0 \quad \text{(I)} \\ a+3b-c = 2k+1 \\ a-b-5c = 2r+1 \end{array}\right\} \implies a+b-c = 2s$$

~~Sumem les 2 últimes primeres i restem la 3a~~

(I): $a = b+3c-12$

~~[illegible]~~

$a+b-c = b+3c-12+b-c =$

~~[illegible]~~

$= 2b+2c-12 =$ parell

ped

(57) $a, b, c, d \in \mathbb{Z}$

$$a+b+c+d = 2k+1 \Rightarrow \begin{cases} a+b = 2k_1+1 \\ \text{ó} \\ a+c = 2k_2+1 \\ \text{ó} \\ a+d = 2k_3+1 \end{cases}$$

Fem

$$\left.\begin{array}{l} a+b+c+d = 2k+1 \\ a+b = 2k_1 \\ a+c = 2k_2 \end{array}\right\} \Longrightarrow a+d = 2k_3+1$$

$$a+b = 2k_1 \Rightarrow \begin{cases} a = \dot{2} \text{ i } b = \dot{2} \\ \text{o bé} \\ a = \dot{2}+1 \text{ i } b = \dot{2}+1 \end{cases}$$

$$a+b+c+d = 2k+1$$

$\dot{2}$ $\Rightarrow$ $b+d = 2k+1$

Pot ser:

$a = \dot{2}$ i $b = \dot{2}$ $\Rightarrow$ $d = 2k+1$ $\Rightarrow$ $a+d = 2k_3+1$ ✓

$a = \dot{2}+1$ i $b = \dot{2}+1$ $\Rightarrow$ $d = 2k$ $\Rightarrow$ $a+d = 2k_3+1$ ✓

57

b)

És cert el recíproc?

i.e. $a+b=\dot{2}+1$ ó $a+c=\dot{2}+1$ ó $a+d=\dot{2}+1$ $\Big\}$ $\overset{?}{\Longrightarrow}$ $a+b+c+d=\dot{2}+1$

Contra exemple:

$$\underset{2}{a}+\underset{2}{b}+\underset{1}{c}+\underset{2}{d}=2K$$

Tot i pre $a+b=2t+1$!!

Per tant, el recíproc és fals!

DISJUNCIÓ ANTECEDENT

61

$n \in \mathbb{Z}$

$$\left.\begin{array}{l} n = 10k_1+3 \\ \text{o } n = 10k_2+7 \end{array}\right\} \Rightarrow n^2 = 10t+9$$

Farem que $n = 10k+3 \Rightarrow n^2 = 10t+9$ (I)

i també que

$n = 10k+7 \Rightarrow n^2 = 10t+9$ (II)

(I) $n^2 = (10k+3)^2 = 100k^2 + 60k + 9 =$

$= 10(10k^2 + 6k) + 9$ ✓

(II) $n^2 = (10k+7)^2 = 100k^2 + 70k + 49 =$

$= 100k^2 + 70k + 40 + 9 =$

$= 10(10k^2 + 7k + 4) + 9$ ✓

qed

(62) $m, n \in \mathbb{Z}$

$$m \cdot n = 2k+1 \Longrightarrow \begin{array}{l} m = 2t+1 \\ \text{i } n = 2s+1 \end{array}$$

Farem $\left.\begin{array}{l} m = 2t \\ \text{ó} \\ n = 2s \end{array}\right\} \Rightarrow m \cdot n = 2k$

i això ho farem amb les 2 parts:

(I) $m = 2t \stackrel{?}{\Rightarrow} m \cdot n = 2k$

$$m \cdot n = 2t \cdot n = 2 \cdot \underbrace{(t \cdot n)}_{k}$$ ✓

(II) $n = 2s \stackrel{?}{\Rightarrow} m \cdot n = 2k$

$$m \cdot n = m \cdot 2s = 2 \cdot \underbrace{m \cdot s}_{k}$$ ✓

qed

PROVA PER CASOS

(64) $n \in \mathbb{Z}$ $\qquad n^2 + n = 2k$

Dem Fem $n = 2t \Rightarrow n^2 + n = 2k$ (I)

i

$n = 2t + 1 \Rightarrow n^2 + n = 2k$ (II)

(I) $n = 2t \Rightarrow n^2 + n = (2t)^2 + 2t = 4t^2 + 2t = 2k$ ✓

(II) $n = 2t+1 \Rightarrow n^2 + n = (2t+1)^2 + 2t + 1 =$

$= 4t^2 + 2t + 1 + 2t + 1 = 4t^2 + 4t + 2 = 2k$ ✓

(65) $a, b = \dot{2} \Rightarrow a + b = \dot{2}$ (I)

$a, b = \dot{2} + 1 \Rightarrow a + b = \dot{2} + 1$ (II)

Obvi ✓

(66) $n \in \mathbb{Z}$ $\qquad n^2 \geq n$

Fem $n > 0 \Rightarrow n^2 \geq n$ (I)

i

$n < 0 \Rightarrow n^2 \geq n$ (II)

i

$n = 0 \Rightarrow n^2 \geq n$ (III)

✓

(67) $x, y, z \in \mathbb{R}$

$\max(x, y) + \min(x, y) = x + y$

(I) $x \geq y \Rightarrow \max(x, y) + \min(x, y) = x + y$ ✓

(II) $x \leq y \Rightarrow$ ✓

(68) $a, b \in \mathbb{Z}$

$a + b = 2k \Rightarrow \begin{cases} a = 2t \text{ i } b = 2s \\ \text{ó bé} \\ a = 2t+1 \text{ i } b = 2s+1 \end{cases}$

Fem $\underbrace{(a = 2t+1 \text{ ó } b = 2s+1) \text{ i } (a = 2t \text{ ó } b = 2s)}_{\parallel\parallel} \Rightarrow a+b = 2k+1$

$\left.\begin{array}{l} a = 2t+1 \text{ i } b = 2s \\ \text{ó} \\ a = 2t \text{ i } b = 2s+1 \end{array}\right\} \Rightarrow a+b = 2k+1$

(I) $\left.\begin{array}{l} a = 2t+1 \\ b = 2s \end{array}\right\} \Rightarrow a+b = 2(t+s)+1$ ✓

(II) $\left.\begin{array}{l} a = 2t \\ b = 2s+1 \end{array}\right\} \Rightarrow a+b = 2(t+s)+1$ ✓

69

$n \in \mathbb{Z}$

$$3n^2 + n + 3 = 2k+1$$

<u>Dem</u>

(I) $n = 2t \Rightarrow 3(2t)^2 + 2t + 3 = \underbrace{12t^2 + 2t + 2}_{2k} + 1$ ✓

(II) $n = 2t+1 \Rightarrow 3(2t+1)^2 + (2t+1) + 3 =$

$= 12t^2 + 3 + 12t + 2t + 1 + 3 =$

$= \underbrace{12t^2 + 14t + 6}_{2k} + 1$ ✓

70

$n \in \mathbb{Z}$ $\quad n^3 + 2n = 3k$

Farem 3 casos:

(I) $n = 3t$

(II) $n = 3t+1$

(III) $n = 3t+2$

(I) $n^3 + 2n = (3t)^3 + 2\cdot 3t = 3\cdot 9t^3 + 3\cdot 2t = 3k$ ✓

(II) $n^3 + 2n = (3t+1)^3 + 2(3t+1) = 3\cdot 9t^3 + 3\cdot 9t^2 + 3\cdot 3t + 1 + 3\cdot 2t + 2 =$

$= 3\cdot k$ ✓ $\quad 3$

(III) $n^3 + 2n = (3t+2)^3 + 2(3t+2) = 3\cdot 9t^3 + 3\cdot 18t^2 + 3\cdot 12t + 8 + 3\cdot 2t + 4 = 3\cdot K$ ✓ $\quad =12$

(71) $a = t^2$

$3 \nmid a \Rightarrow a = 3k+1$

Dem

Com que $3 \nmid a$, llavors $3 \nmid t^2$ i $3 \nmid t$

Casos: (I) $t = 3s+1$

(II) $t = 3s+2$

(I) $t^2 = (3s+1)^2 = 9s^2+1+6s = 3(3s^2+2s)+1$ ✓

(II) $t^2 = (3s+2)^2 = 9s^2+4+12s = 3(3s^2+4s+1)+1$ ✓

(72) $n \in \mathbb{Z}$

$3 \nmid n \Rightarrow 3 \mid n^2-1$

Cas I: $n = 3k+1$

Cas II: $n = 3k+2$

(I) $n^2-1 = (3k+1)^2-1 = 9k^2+6k = 3r$ ✓

(II) $n^2-1 = (3k+2)^2-1 = 9k^2+12k+3 = 3r$ ✓

73 $x, y \in \mathbb{R}$

$$|x \cdot y| = |x|\,|y|$$

Farem

I $x \geq 0,\ y \geq 0 \Rightarrow x = |x|,\ y = |y|,\ x \cdot y = |x \cdot y| \Rightarrow$ ✓

II $x < 0,\ y < 0 \Rightarrow -x = |x|,\ -y = |y|,\ |x \cdot y| = x \cdot y \Rightarrow$ ✓

III $x < 0,\ y \geq 0 \Rightarrow -x = |x|,\ y = |y|,\ |x \cdot y| = (-1) \cdot x \cdot y = |x|\,|y|$

IV $x \geq 0,\ y < 0 \Rightarrow x = |x|,\ -y = |y|,\ |x \cdot y| = (-1) \cdot x \cdot y = |x| \cdot |y|$

74 $x, y \in \mathbb{R}$

$$\big|\,|x| - |y|\,\big| \leq |x - y|$$

Farem

I $x \geq 0,\ y \geq 0,\ x \geq y$

II $x < 0,\ y \geq 0,\ x \geq y$

III $x < 0,\ y < 0,\ x \geq y$

IV $x < 0,\ y < 0,\ x < y$

V $x \geq 0,\ y < 0,\ x \geq y$

VI $x \geq 0,\ y < 0,\ x < y$

VII $x \geq 0,\ y \geq 0,\ x < y$

VIII $x < 0,\ y \geq 0,\ x < y$

75. $x, y, z \in \mathbb{R}$

a) $z \geq x$ i $z \geq y \iff z \geq \text{màx}(x,y)$

<u>Dem</u>

(I) $x \geq y$: $z \geq x$ i $z \geq y \iff z \geq x = \text{màx}(x,y)$

(II) $y \leq x$: idem.

b) $\min(\min(x,y), z) = \min(x, \min(y,z)) = \min(x,y,z)$

<u>Casos</u>:

(I) $x \leq y \leq z$

(II) $x \leq z \leq y$

(III) $y \leq x \leq z$

(IV) $y \leq z \leq z$

(V) $z \leq x \leq y$

(VI) $z \leq y \leq x$

(*) Són simètrics!

Només cal fer-ne un. (p.ex., el (I))

$$\underbrace{\min(\underbrace{\min(x,y)}_{x}, z)}_{x} = \underbrace{\min(x, \underbrace{\min(y,z)}_{y})}_{x} = \underbrace{\min(x,y,z)}_{x}$$

✓

c) $\max(\max(x,y), z) = \max(x, \max(y,z)) = \max(x,y,z)$

ídem que b)

d) $\max(z+x, z+y) = z + \max(x,y)$

Cas I $z+x \geq z+y$

Cas II $z+x \leq z+y$

I $\max(z+x, z+y) = z+x$ (A) ✓

$z+x \geq z+y \Rightarrow x \geq y \Rightarrow \max(x,y) = x \Rightarrow$

$\Rightarrow z + \max(x,y) = z+x$ (B) ✓

II ídem

76 $n \in \mathbb{Z}$ $n^2 \neq 10K+8$

Casos:

I $n = 10K \Rightarrow n^2 = 10 \cdot 10K^2$ ✓

II $n = 10K+1 \Rightarrow n^2 = 10 \cdot 10K^2 + 1 + 10 \cdot 2K = 10s+1$ ✓

III $n = 10K+2$

$+3$

$+4$

$\vdots$

X $+9$

Mai surt $10K+8$

(77) $n^3 = 9k$ ó $9k-1$ ó $9k+1$

(I) $n = 3t \Rightarrow n^3 = 27t^3 = 9k$ ✓

(II) $n = 3t+1 \Rightarrow n^3 = 27t^3 + \cancel{} + 27t^2 + 9t + 1 = 9k+1$

(III) $n = 3t+2 \Rightarrow n^3 = 27t^3 + 18t^2 + 36t + 8 = 9\overline{K} + 8 + 9 - 9 =$

$= 9k - 1$

(78) $m, n \in \mathbb{Z}$ $\qquad m+n = 2k+1 \Rightarrow m \cdot n = 2r$

(I) $m = 2t \Rightarrow n = 2s+1$ (ja que si no, $m+n \neq 2k+1$) $\Rightarrow$

$\Rightarrow n \cdot m = 2t(2s+1) = 2r$ ✓

(II) $m = 2t+1 \Rightarrow n = 2s \Rightarrow n \cdot m = 2r$ ✓

EQUIVALÈNCIES

(86) $n \in \mathbb{Z}$ $\qquad$ Són equivalents a), b), c)

a) $n = 3k_1 + 1$

b) $n^2 + n = 9k_2 + 2$

c) $n^2 + n \neq 3k_3$

Fem)

$a \Rightarrow b$) $\quad n = 3k_1 + 1 \Rightarrow n^2 + n = 9k_1^2 + 1 + 6k_1 + 3k_1 + 1 =$

$= 9k_2 + 2$ ✓

$b \Rightarrow c$) Clàrament, $n^2 + n \neq 3k_3$, ja que $n^2 + n = 9k_2 + 2$

$c \Rightarrow a)$ $\quad n^2+n \neq 3k_3 \overset{?}{\Longrightarrow} n = 3k_1+1$

$n^2+n = n\cdot(n+1) \neq 3k_3 \Rightarrow 3 \nmid n$ i $3 \nmid n+1$

$\Rightarrow 3 \mid n+2 \Rightarrow n+2 = 3r \Rightarrow n = 3r-2+3-3 =$
$= 3k_1+1$

qed

(87) $n, m \in \mathbb{Z}$ $\quad$ Són equivalents:

a) $n, m = \dot{2}$ ó $n, m = \dot{2}+1$

b) $n+m = \dot{2}$

c) $n-m = \dot{2}$

$a \Rightarrow b)$ $\quad n+m = \begin{cases} 2k+2r = 2s \checkmark \\ 2k+1+2r+1 = 2s \checkmark \end{cases}$

$b \Rightarrow c)$ $\quad n-m = \underbrace{n+m}_{2l} - 2m = 2l-2m = 2t \checkmark$

$c \Rightarrow a)$ Si $n = 2k+1$ ó $m = 2r+1$, llavors

Si $n = 2k+1$ i $n-m = 2r \Rightarrow m = 2k+1-2r = 2s+1 \checkmark$

Si $m = 2k+1$ i $n-m = 2r \Rightarrow n = 2r+m = 2r+2k+1 = 2s+1 \checkmark$

O sigui, per fer $c \Rightarrow a$ tenim:

$$\underbrace{n - m = \dot{2}}_{\varphi} \Longrightarrow \underbrace{n, m = \dot{2}}_{\psi} \text{ ó } \underbrace{n, m = \dot{2}+1}_{\theta}$$

i recordem que $\varphi \to \psi \vee \theta \equiv \varphi \wedge \neg\psi \to \theta$

Però atenció! $\neg\psi$ és $\underbrace{n = 2k+1}_{\Omega_1}$ ó $\underbrace{m = 2k+1}_{\Omega_2}$

Per tant, caldria fer:

$$\varphi \wedge \neg\psi \to \theta$$

és a dir

$$\varphi \wedge (\Omega_1 \vee \Omega_2) \to \theta$$

equivalent a $\varphi \wedge \Omega_1 \vee \varphi \wedge \Omega_2 \to \theta$

equivalent a $(\varphi \wedge \Omega_1 \to \theta) \wedge (\varphi \wedge \Omega_2 \to \theta)$

que és el què hem fet

(88) $n \in \mathbb{Z}$ Són equivalents:

a) $n = 2k$

b) $n+1 = 2r+1$

c) $3n^2+1 = 2s+1$

$a \Rightarrow b)$ obvi

$b \Rightarrow c)$ $3n^2+1 = 3(2r)^2+1 = 2\cdot 6r^2+1$ ✓

$c \Rightarrow a)$ $3n^2+1 = 2s+1 \Rightarrow 3n^2 = 2s$

Com que $3 \mid 2\cdot s$ i $3 \nmid 2$ llavors $3 \mid s$

Per tant, $n^2 = 2\ell$

i $2 \mid n^2 \Rightarrow 2 \mid n \Rightarrow n = 2k$ qed.

89 $x \in \mathbb{R}$ Són equivalents:

a) $x \in \mathbb{Q}$

b) $3x - 1 \in \mathbb{Q}$

c) $\frac{x+1}{2} \in \mathbb{Q}$

$a \Rightarrow b)$ $x = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

$$3 \cdot \frac{a}{b} - 1 = \frac{\overbrace{3a - b}^{\in \mathbb{Z}}}{\underbrace{b}_{\in \mathbb{Z}}} \quad \checkmark$$

$b \Rightarrow c)$

$$3x - 1 = \frac{a}{b} \Rightarrow x = \frac{a-b}{3b} \Rightarrow$$

$$\Rightarrow \frac{\frac{a-b}{3b} + 1}{2} = \frac{a - b + 3b}{6b} \in \mathbb{Q} \quad \checkmark$$

$c \Rightarrow a)$ $\frac{x+1}{2} = \frac{a}{b} \Rightarrow x = \frac{2a}{b} - 1 = \frac{2a - b}{b} \in \mathbb{Q}$

90 $a, b \in \mathbb{R}$

a) $a^2 + b^2 = 2ab$

b) $a^2 + b^2 \leq 2ab$

c) $a = b$

$a \Rightarrow b)$ obvi

$b \Rightarrow c)$ $0 \geq a^2 + b^2 - 2ab = (a-b)^2 \geq 0 \Rightarrow$

$\Rightarrow (a-b)^2 = 0 \Rightarrow a - b = 0 \Rightarrow a = b$

$c \Rightarrow a)$ $\underbrace{a^2 + a^2}_{2a^2} = \underbrace{2 \cdot a \cdot a}_{2a^2}$ ✓

91 $a, b \geq 0 \in \mathbb{R}$

a) $a > \sqrt{ab}$

b) $a > b$

c) $b < \sqrt{ab}$

$a \Rightarrow b)$ $a > \sqrt{ab} \underset{a,b \geq 0}{\Rightarrow} a^2 > ab \Rightarrow$ $a = 0$: $0 > 0$ no pot ser; $a \neq 0$: $a > b$ ✓

$b \Rightarrow c)$ $b^2 < b \cdot a \underset{a,b \geq 0}{\Rightarrow} b < \sqrt{ba}$ ✓

$c \Rightarrow a)$

$b < \sqrt{ab} \Rightarrow b^2 < ab \Rightarrow$ $\begin{cases} b=0 & 0<0 \text{ no potser} \\ b\neq 0 & b<a \end{cases}$

$\uparrow$
$a,b>0$

Per tant $\sqrt{ab} \leq \sqrt{a\cdot a} = a$ ✓

92 $n \in \mathbb{Z}$

a) $n = 4k+3$

b) $n(n+2)(n+3) \neq 4r$

c) $2n^2+n-1 = 4s$

$a \Rightarrow b)$ $\underbrace{(4k+3)(4k+5)}_{16k^2+20k^2+16k+15}(4k+6) = \overset{\dot{4}}{32k^3} + \overset{\dot{4}}{80k^2} + \overset{\dot{4}}{64k} + \overset{\dot{4}}{60k} +$

$+ \underset{\dot{4}}{96k^3} + \underset{\dot{4}}{120k^2} + \underset{\dot{4}}{96k} + 90$

$4 \nmid 90$

$4 \nmid n(n+2)(n+3)$

$b \Rightarrow c)$ $4 \nmid n(n+2)(n+3) \Rightarrow 4 \nmid n$ i $4 \nmid n+2$ i $4 \nmid n+3$

$\Rightarrow 4 \mid n+1$

$2n^2+n-1 = (2n-1)(n+1) = 4s$ ✓

$c \Rightarrow a)$ ó $4 \mid 2n-1 \Rightarrow 2n-1 = 4k \Rightarrow$ No potser!

ó $4 \mid n+1 \Rightarrow n+1 = 4k \Rightarrow n = 4k-1 = 4s+3$ ✓

ó $2 \mid 2n-1$ i $2 \mid n+1 \rightarrow$ No potser!

93) $n \in \mathbb{Z}$

a) $n = a \cdot (a+1)$

b) $4n+1 = k^2$

$a \Rightarrow b)$ $4n+1 = 4(a^2+a)+1 = 4a^2+4a+1 =$

$= (2a+1)^2$

$b \Rightarrow a)$ $4n = k^2-1 = (k+1)(k-1) \Rightarrow$

$\Rightarrow n = \dfrac{(k+1)(k-1)}{4}$

Però $k^2 = 2r+1 \Rightarrow k = 2s+1 \Rightarrow 2 \mid k+1$

i $2 \mid k-1$

$\Rightarrow n = \underbrace{\dfrac{k+1}{2}}_{\mathbb{Z}?} \cdot \underbrace{\dfrac{k-1}{2}}_{\in \mathbb{Z}}$

i, a més, $\dfrac{k+1}{2} - \dfrac{k+1}{2} = \dfrac{k+1-k+1}{2} = 1$

$\Rightarrow a := \dfrac{k+1}{2}$ compleix $n = a(a+1)$

qed

(97) $(A, \circ)$ associativa

Veure que e neutre, si $\exists$, és únic

Suposem que hi ha e, ν neutres

$$ex = x \cdot e = x \quad \forall x$$

$$x\nu = \nu x = x \quad \forall x$$

$\Downarrow$

$$e \cdot \nu = \nu$$

ja que e és neutre

$\parallel$

e ja que ν és neutre $\Rightarrow e = \nu$

qed

(98) si $\exists$, A^{-1} és única

Suposem que per A tenim A_1, A_2 | $A \cdot A_1 = 1$, $A_1 \cdot A = 1$, $A \cdot A_2 = 1$, $A_2 \cdot A = 1$

Tenim $A \cdot A_1 = 1 = A \cdot A_2$

multiplicant per A_1,

$$A_1 \cdot \underbrace{A \cdot A_1}_{1} = \underbrace{A_1 \cdot A}_{1} \cdot A_2$$

$$\Rightarrow A_1 = A_2$$