FMAN-45 Machine Learning, Spring 2019 Assignment 4

April 1, 2019

Solve the problems and write down the solutions. This assignment consists of 8 questions with a maximum total score of 100. The first 5 questions constitute 50 points and are more theoretical. The last 3 questions constitute the remaining 50 points and are more programmatical/experimental. The only way to pass is to score at least 70% in this assignment. Write a detailed report. All solutions, plots and figures should be in one pdf. It should be possible to understand all material presented in the report without running any code. Submit your solutions and code using your individual Moodle account as two files (a pdf and a single archive with all the code) at http://moodle.maths.lth.se/course/ by the deadline; note that there will be no extensions.

1 Introduction

If you feel like there's too much text in this introduction, feel free to try playing the game Snake yourself to see how it works, e.g. at http://playsnake.org/. However, you'll probably be interested in some specifics of the particular version of Snake used in this assignment.

In this assignment you will train reinforcement learning agents¹ to play the classic video game *Snake*. There exist several variants of this game, but in most variants the player is controlling a connected set of pixels known as the *snake* on a two-dimensional grid. Some pixels are *apples*. The goal of the game is to steer the snake to the apples - each apple gives a number of points - while at the same time avoiding colliding with the body of the snake and/or the walls of the grid. The crux is that each time the snake eats an apple, the snake becomes one pixel longer, making it increasingly difficult to control the snake so as to not hit itself or the walls, especially since the snake automatically moves with a certain frequency, even if the player is not making any active decisions. The game ends when the snake hits itself or a wall; the final score is typically proportional to the number of apples "eaten" by the snake.

¹Agents will sometimes be referred to as *players*.

1.1 Details of the game

The following *Snake* variant is used in this assignment; see Figure 1 for some visualizations.

- The game is played on a two-dimensional grid of size $M \times N$ pixels. In this assignment, M = N = 30.
- The border pixels constitute *walls*, through which no movement is possible. In this assignment, it means that the player can only move around on the interior 28×28 grid, and apples can only exist in the interior of the grid.
- The player will be controlling a connected set of pixels known as the *snake*.
- The snake has a *head* (one pixel) and a body (the remaining pixels of the connected set of pixels). The head is at one end of the snake, but looking solely on one fixed state ("screen") of the game, it is not possible to determine at which end of the snake the head is located.
- However, the state of the game is automatically updated at some fixed frequency (i.e., the duration for which the game is "paused" is fixed). What happens from a state to the next state is that the snake moves one pixel in the direction (N/E/S/W) of the head of the snake. Thus, the complete state of the game can be inferred by considering two consecutive frames, and this is already implemented in the code you will use.
- At each time-step, the player must perform precisely one of three possible actions (movements): left, forward, or right (relative to the direction of the head). To be more precise, the player enforces a direction for the head of the snake, so that at the next time-step the snake moves in the chosen direction (see above item). Again, also see Figure 1 for visualizations.
- As mentioned, the state of the game is automatically updated at some fixed frequency. The higher the frequency, the harder the game becomes, as the player needs to choose one of the three actions above in a shorter amount of time.
- Initially, the length of the snake is 10 pixels. It forms a straight line of pixels, with its head located at the center of the grid, facing a random direction (N/E/S/W).
- Each time-step there is exactly one apple in the grid, occupying exactly one pixel.
- When the snake eats an apple, a new apple is simultaneously placed at uniform random on an unoccupied pixel in the grid (meaning a pixel in the interior of the grid, which is not occupied by any pixel of the snake).

²4-connectivity is considered here, i.e., two pixels are connected only if they are horizontal or vertical neighbours, *not* diagonal.

- The game ends as soon as the head of the snake moves into a wall or the body of the snake.
- Eating an apple yields 1 point; no other points (positive nor negative) are awarded throughout the game. The final score is thus the total number of apples eaten by the snake.

2 Reinforcement learning for playing Snake

2.1 Tabular methods

Before considering the Snake game described so far, let's consider a much smaller example in which the Snake length does not increase by eating apples. Specifically, the grid is 7×7 (so the interior grid in which the snake can move is 5×5) and the snake has constant length 3. Everything else is as for the full game (see Section 1). In this small, special case, it is possible³ to maintain a table which stores⁴ the state-action value Q(s,a) for each state-action pair (s,a). This table can be represented by a $K \times 3$ matrix Q, where K is the number of states in the game, and the 3 columns correspond to the three possible actions left, forward and right. Note that while it is possible to store state-action values in terminal states (in the small Snake game this corresponds to the snake having its head in a wall location), it is not necessary to do so, since no action can be taken in a terminal state. In practice, the state-action value of such terminal state-action pairs are set to zero, or they can simply be ignored. We will do the latter, so K refers to the number of non-terminal states.

Exercise 1 (10 points):

Derive the value of K above.

Hint: How many apple locations are possible for a given configuration of the snake? How many configurations of the snake are there, including its movement direction (or to phrase differently: how many configurations of the snake are there, if one also knows which pixels is the head)?

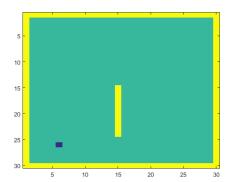
2.1.1 Bellman optimality equation for the Q-function

The optimal Q-value $Q^*(s, a)$ for a given state-action pair (s, a) is given by the state-action value Bellman optimality equation

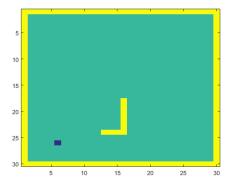
$$Q^*(s, a) = \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma \max_{a'} Q^*(s', a') \right]. \tag{1}$$

³In principle, this is possible even for much larger state-action spaces, but it quickly becomes practically difficult as the size of the state-action space increases.

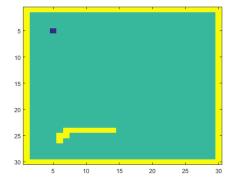
⁴Note that Q(s, a) assumes an underlying policy $\pi(a|s)$ being followed; Q(s, a) is then interpreted as the state-action value of taking action a in state s and thereafter following policy π .



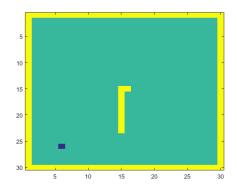
(a) Initial state. The length of the snake is 10 pixels. It forms a straight line of pixels, with its head located at the center of the grid, facing a random direction (N/E/S/W); in this case the direction is north (N). An apple is located at a random location, in this case in (m, n) = (26, 6).



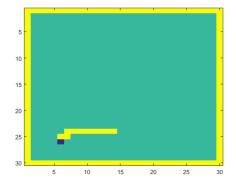
(c) After a few more time-steps, this is where the snake is located. Its direction is west. The initial apple has not yet been eaten.



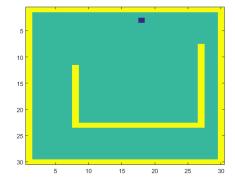
(e) Given the state shown in Figure 1d, the player chooses to go left; the new snake direction is south. The head of the snake eats the apple, causing the length of the snake to increase by 1; the body remains in the same place during the next movement, growing one pixel in the movement direction. The apple disappears and a new one is randomly placed on an empty pixel.



(b) At each time-step, the player must perform precisely one of three possible actions (movements): *left*, *forward*, or *right* (relative to the direction of the head). In this example, the player chooses to go right as their first action. The body of the snake moves along, keeping the snake intact.



(d) After a few more time-steps, this is where the snake is located. Its direction is west. The initial apple has not yet been eaten.



(f) After several more time-steps, the snake has gotten significantly longer by eating apples. Each apple eaten gives one point. The game ends when the player chooses an action such that the head ends up in an occupied space (meaning the body of the snake or a border). The final score is the total number of apples eaten during the game.

Figure 1: A few situations, in chronological order, in a game of Snake on a 30×30 -grid. The snake can only occupy any space in the interior 28×28 grid.

Once we have $Q^*(s,a)$, an optimal policy π^* is given by⁵

$$\pi^*(s) = a^* = \arg\max_a Q^*(s, a).$$
 (2)

Note that in the tabular case we could store all $Q^*(s, a)$ in a matrix Q^* , which would be $K \times 3$ in the small Snake game.

Exercise 2 (10 points):

- a) (1 point) Rewrite eq. (1) as an expectation using the notation $\mathbb{E}[\cdot]$, where $\mathbb{E}[\cdot]$ denotes expected value.
- b) (4 points) Explain what eq. (1) is saying. Your answer must reference all the various components and concepts of the equation $(Q^*, \Sigma, T, R, \gamma, \max, s, a, s')$.
- c) (5 points) Discuss the effect γ has on the optimal policy $\pi^*(s)$ in eq. (2) by considering the following cases: i) $\gamma = 0$, ii) $\gamma = 1$ and iii) $\gamma \in (0,1)$.

Note: All your answers to this exercise should be with respect to general Markov decision processes, not with respect to the Snake game in particular. It is however ok to give some examples referring to the Snake game, if that makes your points clearer.

Exercise 3 (10 points): Explain what T(s, a, s') in eq. (1) is for the small version of *Snake*. You should give values for T(s, a, s'), e.g. "T(s, a, s') = 0.5 if ... and T(s, a, s') = 0.66 if ..." (the correct values are likely different, though).

Hint: There are two cases for a state-action pair (s,a) which must be considered. How many possibilities are there for s' in each of these cases?

Now, the question is how we find such an optimal policy π^* . That is, how should an algorithm look like for computing (1)? One could try to implement the Bellman equation (1) directly through dynamic programming.⁶ The dynamic programming procedure would first fill in all entries (i,j) of \mathbf{Q}^* terminal state-action pairs with zeros, and then go "backwards" (i.e. further and further away from terminal state-action pairs). Once the dynamic programming procedure is done, we would have \mathbf{Q}^* . Then, at a given state s, we would simply look at the corresponding row in \mathbf{Q}^* and pick the action a corresponding to the largest element of the row - this corresponds precisely to $\pi^*(s) = \arg\max_a Q^*(s, a)$.

⁵Sometimes it's interesting to consider stochastic policies π , in which case we use the notation $\pi(a|s)$ rather than just $\pi(s)$.

⁶For a good reference on dynamic programming have a look at Ch4 in Sutton's book or http://www.cs.princeton.edu/~wayne/kleinberg-tardos/ or http://jeffe.cs.illinois.edu/teaching/algorithms/notes/05-dynprog.pdf

To highlight the issues of following the above dynamic programming procedure for finding an optimal policy - which thus motivates the use of more sophisticated methods such as *policy iteration* (see Exercise 6) - we shall now consider applying it for the small version of *Snake*. It will turn out that it can be done by using some specific assumptions about the reward signal, but we would like to avoid using such highly task specific ideas to solve general reinforcement learning problems if possible.

Exercise 4 (10 points): Consider the small *Snake* game. Assume we have a reward signal which gives -1 it the snake dies, +1 if the snake eats an apple, and 0 otherwise. Also assume that $\gamma \in (0,1)$.

- a) (2 points) Consider a version of the game in which the game ends as soon as the snake eats the first apple (yielding a total score of +1 for the game). Let us call this game the *truncated game*. Explain the connection between the truncated game and the normal game (in which the snake can keep eating apples forever). In particular, explain in what sense an optimal agent to the truncated game will be optimal also for the normal game.
- b) (1 point) What are the terminal states in the dynamic programming problem for the truncated game?
- c) (5 points) Beginning from terminal state(s) in b), perform the first step of the dynamic programming procedure, by using eq. (1). Explain what is the problem. Draw a figure (you may draw by hand, take a picture, and attach that picture of your drawing in the report) of the game with the snake one step from the terminal state(s) to motivate your answer. Your answers to Exercise 2 and 3 may help you.
- d) (2 points) What would be a solution to the issue found in c)? Hint: What values in the dynamic programming table are relevant for finding the optimal solution?

2.1.2 Policy iteration

As is hopefully apparent from your solution and discussion to the previous problem, the straightforward dynamic programming approach based on (1) seems unnecessarily convoluted, and depends much on the particular structure of the reward signal. Fortunately, it turns out there exists a general approach which can be used to find the optimal policy π^* without taking into account any particular structures of the problem at hand: policy iteration. Before discussing policy iteration, recall the Bellman optimality equation for the state value function given by

$$V^*(s) = \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^*(s') \right]. \tag{3}$$

Exercise 5 (10 points):

- a) (1 point) Rewrite eq. (3) using the notation $\mathbb{E}[\cdot]$, where $\mathbb{E}[\cdot]$ denotes expected value.
- b) (4 points) Explain what eq. (3) is saying; your answer must reference all the various components and concepts of the equation $(V^*, \Sigma, T, R, \gamma, \max, s, a, s')$.
- c) (1 point) For $Q^*(s, a)$, we have the relation $\pi^*(s) = \arg \max_a Q^*(s, a)$. What is the relation between $\pi^*(s)$ and $V^*(s)$?
- d) (4 points) Why is the relation between π^* and V^* not as simple as that between π^* and Q^* ? You don't need to explain with formulas here; instead, the answer should explain the qualitative difference between V^* and Q^* in your own words.

Policy improvement consists consists of two phases (we will now explicitly write the policy dependencies such as V^{π} instead of just V).

1. **Policy evaluation:** Given an arbitrary policy π (thus not necessarily optimal nor deterministic), the Bellman equation for V^{π} says that

$$V^{\pi}(s) = \sum_{a} \pi(a|s) \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi}(s') \right],$$

where existence and uniqueness of V^{π} are guaranteed as long as $\gamma \in [0, 1)$ or eventual termination is guaranteed from all states under the policy π . Note that $\pi(a|s)$ may be stochastic above⁷, which explains the outer summation over actions a and the conditional notation a|s. By turning the above Bellman equation into an update rule as

$$V_{k+1}(s) = \sum_{a} \pi(a|s) \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V_k(s') \right],$$

it follows that $V_k = V^{\pi}$ is a fixed-point, as guaranteed by the Bellman equality. The sequence $\{V_k\}$ can be shown to converge in general to V^{π} as $k \to \infty$ under the same conditions that guarantee existence of V^{π} . In practice, the iterative policy evaluation is stopped once $\max_s |V_{k+1}(s) - V_k(s)| < \epsilon$ for some threshold $\epsilon > 0$.

2. **Policy improvement:** Suppose we have determined the value function $V^{\pi}(s)$ for an arbitrary deterministic policy π (so instead of $\pi(a|s)$ we simply have $\pi(s)$). We now ask ourselves: should we deterministically take some action $a \neq \pi(s)$ in state s? The answer can be found by considering $Q^{\pi}(s,a)$; if it holds that $Q^{\pi}(s,a) > Q^{\pi}(s,\pi(s))$, then we should take action a instead of action $\pi(s)$. More

⁷We look at deterministic policies for the *Snake* game however.

generally, according to the policy improvement theorem, we should in any state s take action $\max_a Q^{\pi}(s,a)$ instead of action $\pi(s)$. Doing this over all states s will gives us a new, improved, policy $\tilde{\pi}(s)$. Due to the state value Bellman optimality equation (3), we are done if we ever find a new policy $\tilde{\pi}(s)$ satisfying $V^{\tilde{\pi}}(s) = V^{\pi}(s)$. Instead of involving the Q-function, one instead uses $\tilde{\pi}(s) = \arg\max_a Q^{\pi}(s,a) = \arg\max_a T(s,a,s')[R(s,a,s') + \gamma V^{\pi}(s')]$ (convince yourself of the validity of this last equality - no motivation is however required in your report).

The two phases above are run back and forth to give the policy improvement algorithm; see Lecture 9 slides for details.

Exercise 6 (15 points): Look at the code in rl_project_to_students/ small_snake_tabular_pol_iter/. Begin by investigating snake.m. Fill in the blanks to run policy iteration in order to find an optimal policy $\pi^*(s)$ - look at policy_iteration.m. You don't need to think about the internal game state; nor will you need to think about how to implement the state representation, as this is already done for you. However, you of course need to understand how this state representation works in order to implement policy iteration. You are allowed to change settings only where it states so in snake.m and policy_iteration.m.

- a) (3 points) Attach a printout of your policy iteration code. Hint: To check that your code seems to be correct, try running it with $\gamma = 0.5$ and $\epsilon = 1$, and check that you get 6 policy iterations and 11 policy evaluations.
- b) (6 points) What is the effect of γ (in this part, set $\epsilon = 1$)? Try $\gamma = 0$, $\gamma = 1$ and $\gamma = 0.95$ and explain what happens, and **why you get those results**, by answering the following. What happens in the algorithm (i.e., how does γ affect the number of iterations of policy evaluation and policy iteration)? How does the final snake playing agent act for the various γ ? Is the final agent playing optimally?
- c) (6 points) What is the effect of ϵ in the policy evaluation (in this part, set $\gamma = 0.95$)? Try ϵ ranging from 10^{-4} to 10^4 (with exponent step size 1, i.e. $10^{-4}, 10^{-3}, \dots, 10^4$) and explain what happens, and **why you get those results**, by answering the following. What happens in the algorithm (i.e., how does ϵ affect the number of iterations of policy evaluation and policy iteration)? How does the final snake playing agent act for the various ϵ ? Is the final agent playing optimally?

2.1.3 Tabular Q-learning

One of the problems with the policy iteration approach described in Section 2.1.2 is that it requires full knowledge about the environment dynamics. That is, it assumes T(s,a,s') and the corresponding rewards to be fully known. If the environment dynamics are unknown, trying things out and seeing what happens can be used to estimate the entries of Q^* . There exist several approaches to such "trial-and-error" estimation; here we will be considering a particular algorithm known as Q-learning, which arguably is the most well-known algorithm in all of reinforcement learning.

Exercise 7 (15 points): Look at the code in rl_project_to_students/small_snake_tabular_q/. Begin by investigating snake.m. Fill in the blanks to implement the tabular Q-updates (see lecture slides). You are allowed to change settings only where it states so in snake.m, so you can change the number of iterations, ε , α , γ , decrease learning rate, etc. Write down and think about any observations you make as you try different settings - you will be discussing this in b), c) below.

Once you have trained the agent for 5000 episodes, you should test its performance (the code automatically saves your Q-values). It is clear in the code of snake.m how to perform testing (set test_agent = true). This will set α and ε to 0 (meaning no Q-value updates will occur and the policy is completely greedy), load the automatically saved Q-values, and run the game.

- a) (3 points) Attach a printout of your Q-update code (both for terminal and non-terminal updates). This should be 8 lines of Matlab code in total.
- b) (3 points) Explain 3 different attempts (parameter configurations note that the 3 attempts don't need to be completely different, but change some settings in between them) you did in order to train a snake playing agent. Comment, for each three of the attempts, if things worked well and why/why not.
- c) (6 points) Write down your final settings. Were you able to train a good, or even optimal, policy for the small Snake game via tabular Q-learning? When running this final test, you should be able to get a score of at least 250 to pass this task (write down the score you get prior to dying / getting stuck in a loop), but hopefully you can get something even better. For example, it should be possible to obtain a snake playing agent which keeps eating apples forever, although it doesn't necessarily reach them in the shortest amount of steps. An optimal snake player is one which keeps eating apples forever and reaching each apple as fast as possible. Note that, if your test run never terminates and the score just keeps increasing, you should write this in your report (this is a good thing).
- d) (3 points) Independently on how it went, explain why it can be difficult to achieve optimal behavior in this task.

2.2 Q-learning with linear function approximation

Now we are ready to have a look at the full version of the game as described in Section 1. If it is not immediately clear, you should convince yourself why tabular methods are more or less intractable for the full game. The bottleneck of course is the vast amount of

states. As in most other areas of machine learning, this issue is tackled by introducing features, which summarize important concepts of the state. This means that Q(s,a) is approximated by some (possibly nonlinear) function $Q_{\boldsymbol{w}}(s,a)$, parametrized by weights \boldsymbol{w} . In this task we study linear function approximation. That is, Q(s,a) is approximated as

$$Q(s,a) \approx Q_{\mathbf{w}}(s,a) = w_1 f_1(s,a) + w_2 f_2(s,a) + \dots + w_n f_n(s,a), \tag{4}$$

where $\mathbf{w} = [w_1, w_2, \dots, w_n]^{\top}$ is the weight vector and $f_i(s, a)$ are state-action feature functions.

Valid assumptions:

The rule of thumb is that the learning agent(s) are assumed to have the same kind of knowledge about the game state as would a human being with perfect vision. As humans, the input signal which we use to control the snake cleverly is only the visual input from the game screens that we observe. In particular, the game screens are images with precisely three different colors; one background color (for unoccupied pixels), one snake/wall color and one color for the apple; see Figure 1. In analogy, the learning agent "sees" the game screen, which means that it gets as input the matrix representing the current game state. In the code, unoccupied pixels are represented by zeros, walls and the snake are represented by ones, and the apple pixel is represented by -1.

In addition to the above rule of thumb, the following assumptions are valid (of which the two first are implemented already).

- The initial head location, i.e. the center of the screen, can be assumed to be known.
- The initial direction (N/E/S/W) can be assumed to be *known* (since the head location is known, and the snake is initially a straight line of pixels, the movement direction can be inferred).
- You don't need to think about the automatic game state updates, discussed in Section 1. The reason is that the computations that the agent perform when determining what action to take go very quickly. So even if there was a notion of automatic game state updates (in which the snake moves even without player actively telling it to do so), the agent would figure out its next action long before its "decision time" runs out. This is true only to some extent, of course, as you can imagine very complicated ways of figuring out what to do next. So as long as your agent is able to make a decision within, say, 1/20 second, it's fine, although it likely will go much faster.

Exercise 8 (20 points): Look at the code in rl_project_to_students/linear_q/. Begin by investigating snake.m. It contains most things you need to run the game and to train a reinforcement learning agent based on Q-learning using linear function approximation. You are allowed to change settings only where it states so in snake.m and extract_state_action_features.m. You need to do a few things in order to train the agent:

- Fill in the blanks in **snake.m** to implement the Q weight updates (see lecture slides).
- Engineer state-action features $f_i(s, a)$ and implement these in the given function extract_state_action_features.m.
- Tune learning parameters in the main file snake.m, including reward signal, discount γ , learning rate α , greediness parameter ε , and so on (similar to what you did in the previous exercise). Experimentation is necessary to train a successful agent.
- We have a weight vector weights, which needs to be initialized. We will initialize the weights in a bad way. Doing this "adversarial" initialization will force the agent to actually learn something, rather than just work perfectly right away. Hence do as follows:
 - You should set each entry w_i of w to 1 or -1, where the sign is set based on what you believe is bad. So if you believe that a good weight for $f_2(s,a)$ is positive, then in initialization set $w_2 = -1$. Clearly motivate for each w_i why you believe the sign is bad. Of course, during experimentation, you can do sanity checks by using good initial weights, but in the end you should make it work for bad initial weights.

Once you have trained the agent for 5000 episodes, you should test its performance (the code automatically saves your Q-weights). It is clear in the code of snake.m how to perform testing (set $test_agent = true$). This will set α and ε to 0 (meaning no weight updates will occur and the policy is completely greedy), load the automatically saved Q-weights, and run the game for 100 episodes.

- a) (3 points) Attach a printout of your Q weight update code (both for terminal and non-terminal updates). This should be 8 lines of Matlab code.
- b) (6 points) Write down at least 3 different attempts (parameter configurations and/or state-action feature functions note that the 3 attempts don't need to be completely different, but change some settings in between them) you did in order to train a good snake playing agent. Comment on if your attempts worked well and why/why not.
- c) (11 points) Report what average test score you get after 100 game episodes with your final settings; you must get at least 35 on average and not get stuck in an infinite loop (it is possible to get over 100 points with only 3 state-action features). Also report your final weight vector \boldsymbol{w} for $Q_{\boldsymbol{w}}$ and associated final state-action feature functions f_i . Comment on why it works well or why it doesn't work so well.