Parallel Programming SS21 Final Project

02 Gaussian Elimination

Group 218

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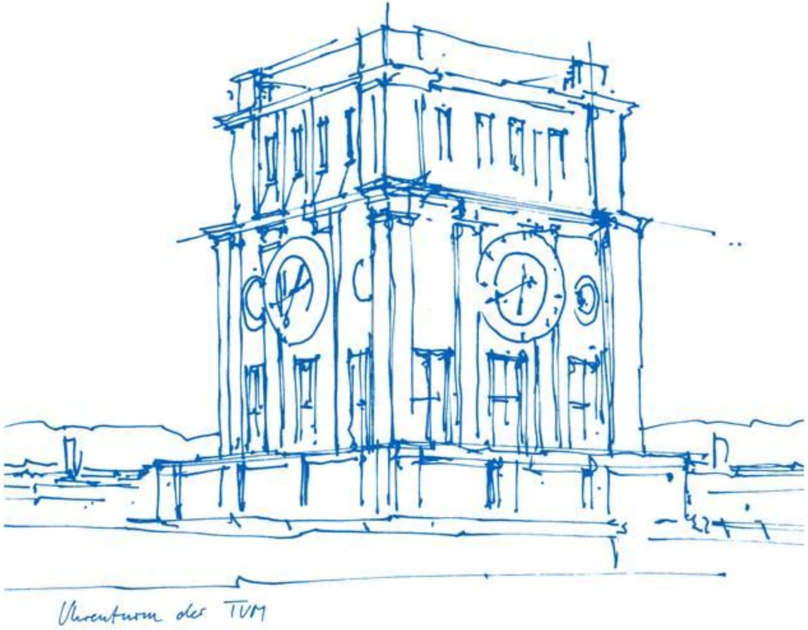


Outline

1. Sequential code analysis
2. OpenMP
3. MPI
4. Hybrid
5. Bonus
6. Conclusion

Sequential code analysis - Forward Elimination

When calculating the forward elimination, for every row, we take the diagonal element d, calculate an elimination factor f=d/x such that for all lower elements in the column r*x-r=0. This diagonal element stays constant with respect to x. Thus we do not need to calculate it over and over again.

Graphically, that is

$d_1 = 1$
$x_{11} = 6$
$f_1 = \frac{1}{6}$
$x_{11} = x_{11} \cdot f_1 - x_{11} = 0$
Repeat for all x_{1j}.

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 1 \\ 6 & -4 & 5 & 0 \\ 2 & 3 & 5 & 0 \\ 5 & -10 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 9 \\ 0 \end{bmatrix}$$

⇒ We can see that d_1 stays constant for all eliminations. The given algorithm constantly calculates d_i, thus we moved this calculation outside. All further analyses apply to the algorithm with this change.

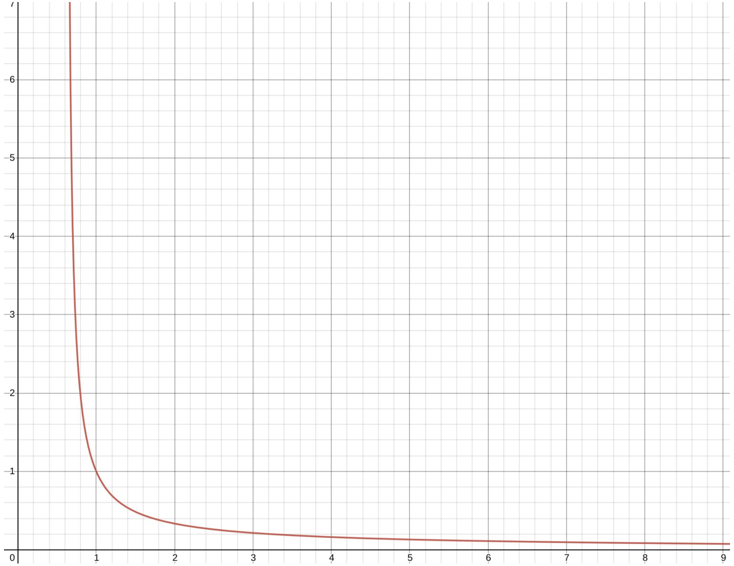
Sequential code analysis - Theoretical Analysis

For the Forward Elimination we have
$n(n+1)/2$ division
$(2n^3+3n^2-5n)/6$ multiplications
$(2n^3+3n^2-5n)/6$ subtractions
$= n(n+1)/2 + (2n^3+3n^2-5n)/3 =$ **FE(n)** operations

For the Backward Elimination we have
$n(n-1)/4$ multiplications
$n(n-1)/4$ additions
$= n(n-1)/2 =$ **BE(n)** operations

Let's look at the ratio for **BE(n)/FE(n)**!

⇒ Quickly converges to 0
⇒ FO outweighs BO by **a lot**
⇒ E.g., for n = 1024, FO causes 99,85% of the computations
⇒ Gives rise to <u>only optimizing **FO**</u> due to high overhead



The graph f(n)=BE(n)/FE(n) showing the ratio of BE and FE indicating the importance of BE

Sequential code analysis - Profiling and Backward

- Perf output roughly coincides with the theoretical calculation
- FE percentage slightly lower since other operations cause overhead
- Backward elimination not even visible

The backward algorithm is

1. doing a negligible amount of work
2. has strong sequential data dependency

⇒ Neglect the backward algorithm throughout all our future optimizations and the rest of the presentation

```
Samples: 520  of event 'cache-references', Event count (approx.): 122283612
Overhead  Command   Shared Object        Symbol
  97,42%  serialge  serialge             [.] Serial::ForwardEliminationOptimized
   0,36%  serialge  serialge             [.] Serial::SerialSolve
   0,35%  serialge  libstdc++.so.6.0.28  [.] std::num_put<char, std::ostreambuf_iterator
   0,19%  serialge  libstdc++.so.6.0.28  [.] std::num_get<char, std::istreambuf_iterator
   0,14%  serialge  libstdc++.so.6.0.28  [.] std::istream::sentry::sentry
   0,13%  serialge  libstdc++.so.6.0.28  [.] std::num_get<char, std::istreambuf_iterator
   0,13%  serialge  libc-2.31.so         [.] __GI_____strtod_l_internal
   0,08%  serialge  libstdc++.so.6.0.28  [.] std::string::reserve
   0,07%  serialge  libc-2.31.so         [.] malloc
   0,06%  serialge  libstdc++.so.6.0.28  [.] std::string::_Rep::_M_clone
   0,05%  serialge  libc-2.31.so         [.] __strlen_avx2
   0,05%  serialge  libstdc++.so.6.0.28  [.] std::__convert_to_v<double>
   0,05%  serialge  [kernel.kallsyms]    [k] clear_page_erms
   0,04%  serialge  serialge             [.] ReadLine
```

Figure: Perf profiling output for serial implementation

Sequential code analysis - Amdahl‘s law

Q: What portion of the algorithm is parallelizable?

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 6 & -4 & 5 & 0 \\ 2 & 3 & 5 & 0 \\ 5 & -10 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 9 \\ 0 \end{bmatrix}$$

- Low data dependency
- Each of i+1th rows is only dependent on the element A_{ii}
- Therefore, propagating down the diagonal element can be nicely parallelized
- As this is all the algorithm is doing (besides a few other operation), **we expect roughly 97%** (according to Perf) to be parallelizable

Sequential code analysis - Amdahl‘s law

Q: What portion of the algorithm is parallelizable?

So let’s evaluate this, given our OMP speed up of L = 14.

Amdahl’s Law is given by

$$L_p(s) = 1 / (1 - p + p/s)$$

where L is the latency (i.e., the speed up), p is the portion to be parallelizable and s the amount of workers.

Plugging in all the values yields

$$14 = 1 / (1 - p + p/s)$$
$$\Leftrightarrow \quad p \approx 0.96$$

⇒ According to a speed up of factor 14, approximately 96% of the program is parallelizable

Similarly, we can ask ...

Sequential code analysis - Amdahl‘s law

Q: Given the portion of the algorithm to parallelizable (p=0.97), what’s the theoretical speed up maximum?

Plotting the $L_p(s)$ yields

⇒ Assuming 97% parallelizability using 32 CPUs theoretical speed up of factor 16.5

⇒ Clearly sublinear but still very nice curve.

⇒ Theoretical max speed up: $L_p(s)$ converges towards 33.3 for $s \rightarrow \infty$.

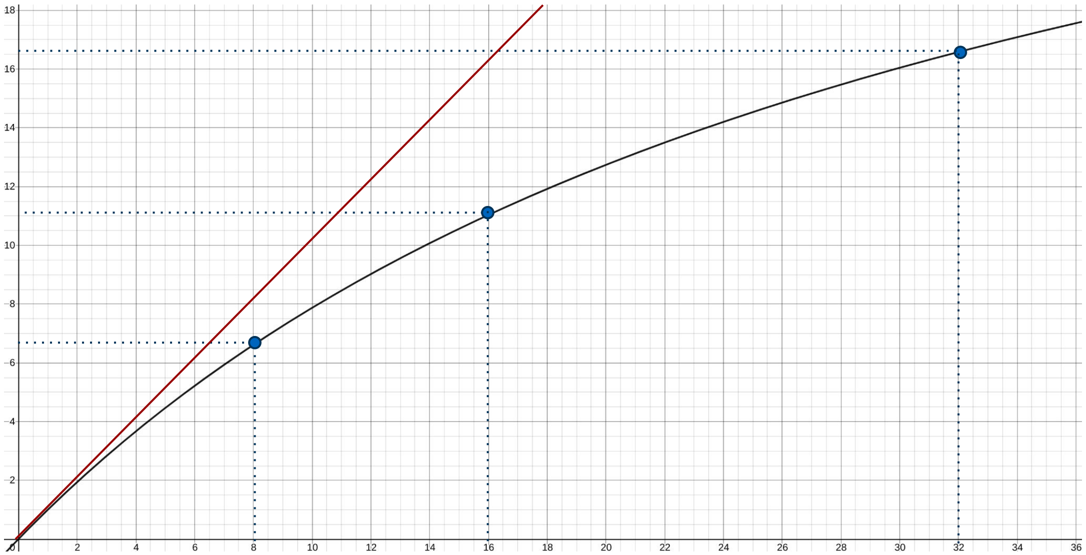


Figure: Amdahl’s law with p=0.97.

OpenMP - Parallelized implementation and approach

Initial approach

- Used **OMP task** to parallelize program
- Resulted in **no speed up at all**
- Taking a closer look we recognized **not the correct approach**
- OMP Tasks are (loosely speaking) a way to dynamically distribute workload
- Use case would be recursion with no interdependence between tree branches
- GE has low data dependence in forward algorithm, however,
- **Dynamically distributing matrix rows very inefficient**

Current approach

- For every pivot row from step i=1, we have to modify the following n-i rows
- All following n-i rows are only dependent on i-th row’s pivot element
- Statically spread the workload across all workers minimizing overhead

OpenMP - Parallelized implementation and approach

We found that

1. Out of the explicit scheduling schemes (static, dynamic, guided) static is the fastest
2. Runtime system using schedule(auto) picks static too →static seems like a good strategy

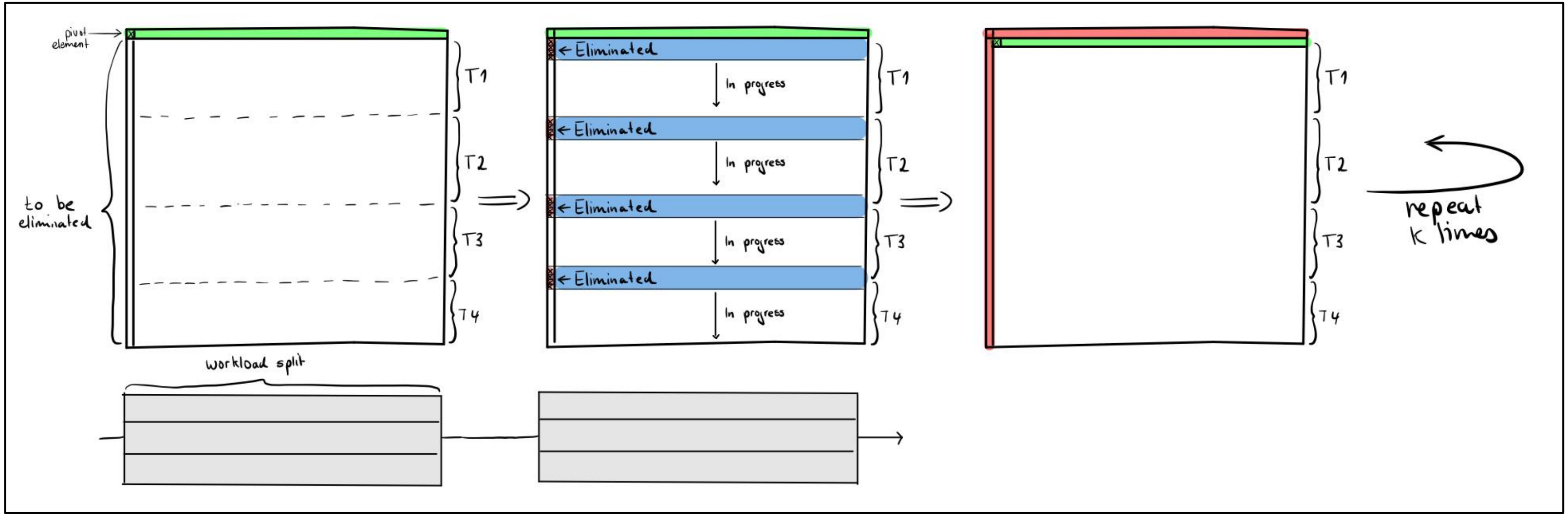


Figure: Schematic workload spread for k × k in forward algorithm with n=4 and static scheduling.

OpenMP - Overhead and when to parallelize

Q: At what problem size does parallelization make sense?

- Ran every problem size 10 times and averaged over computation vs. overhead
- For small problems overhead takes almost 100% of computing time ⇒ parallelization makes no sense
- For the sample problems given, starting from size 512 overhead starts to decrease (but still, 91% overhead)
- Size 2024 fist point very computation time > overhead

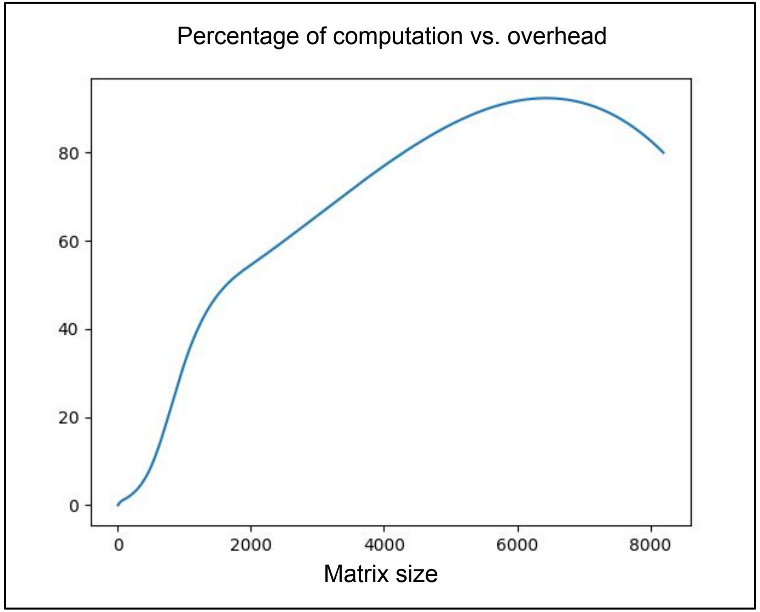


Figure: Spline (with k=3) displaying ratio computation vs. overhead.

OpenMP - Perf and possible bottlenecks

- **No change** between actual work across the programs
- Makes sense as the program **semantics do not change** at all - only the way work is being executed

- **65% of program are arithmetics**

⇒ Straightforward element-wise operations
⇒ Theoretically suitable SIMD use case
⇒ Compiler is already taking care of it

- **30% of program are reading**, writing and moving of elements

⇒ Use matrix instead of element-wise operations
⇒ Again, compiler is already taking care of it

Serial

```
Samples: 4K of event 'cycles', 4000 Hz, Event count (approx.): 4661322483
Serial::ForwardEliminationOptimized  /home/marcelbraasch/CLionProjects/openmp/serialge [Percent: local period]
  2,55 │       vinsertf128  $0x1,0x10(%rbx,%rax,1),%ymm7,%ymm1
  7,37 │       vmovupd      (%rdx,%rax,1),%xmm6
 12,54 │       vinsertf128  $0x1,0x10(%rdx,%rax,1),%ymm6,%ymm0
 17,44 │       vmulpd       %ymm3,%ymm1,%ymm1
 16,41 │       vsubpd       %ymm1,%ymm0,%ymm0
  2,45 │       vmovups      %xmm0,(%rdx,%rax,1)
  6,23 │       vextractf128 $0x1,%ymm0,0x10(%rdx,%rax,1)
 31,91 │       add          $0x20,%rax
  2,21 │       cmp          %rsi,%rax
```

Tasking

```
Samples: 13K of event 'cycles', 4000 Hz, Event count (approx.): 11385415991
OMP::ForwardElimination  /home/marcelbraasch/CLionProjects/openmp/ompge [Percent: local period]
  8,20 │       vinsertf128  $0x1,0x10(%r14,%r13,1),%ymm5,%ymm0
 12,77 │       vmulpd       %ymm3,%ymm1,%ymm1
 16,99 │       vsubpd       %ymm1,%ymm0,%ymm0
  3,67 │       vmovups      %xmm0,(%r14,%r13,1)
  7,45 │       vextractf128 $0x1,%ymm0,0x10(%r14,%r13,1)
 33,84 │       add          $0x20,%r13
  0,87 │       cmp          %rsi,%r13
```

Parallel for

```
Samples: 27K of event 'cycles', 4000 Hz, Event count (approx.): 25442538013
OMP::ForwardElimination  /home/marcelbraasch/CLionProjects/openmp/ompge [Percent: local period]
  0,05 │       vmovupd      (%rax,%r13,1),%xmm6
  3,21 │       vinsertf128  $0x1,0x10(%rax,%r13,1),%ymm6,%ymm1
  8,45 │       vmovupd      (%r14,%r13,1),%xmm5
  8,24 │       vinsertf128  $0x1,0x10(%r14,%r13,1),%ymm5,%ymm0
 13,75 │       vmulpd       %ymm3,%ymm1,%ymm1
 16,93 │       vsubpd       %ymm1,%ymm0,%ymm0
  3,82 │       vmovups      %xmm0,(%r14,%r13,1)
  7,52 │       vextractf128 $0x1,%ymm0,0x10(%r14,%r13,1)
 32,14 │       add          $0x20,%r13
  0,84 │       cmp          %rsi,%r13
```

Figure: Perf output for the respective OMP vs. serial programmes

MPI - Parallelized Implementation and Approach

- Initial parallelization approach
  - Calculate distribution based on current status
  - Send rows from root to other ranks
  - Get calculated rows back

loop

- Motivation of implementation
  - Easy to implement / intuitive solution
  - Organization only done in rank 0

- Bottleneck due to high amount of communication

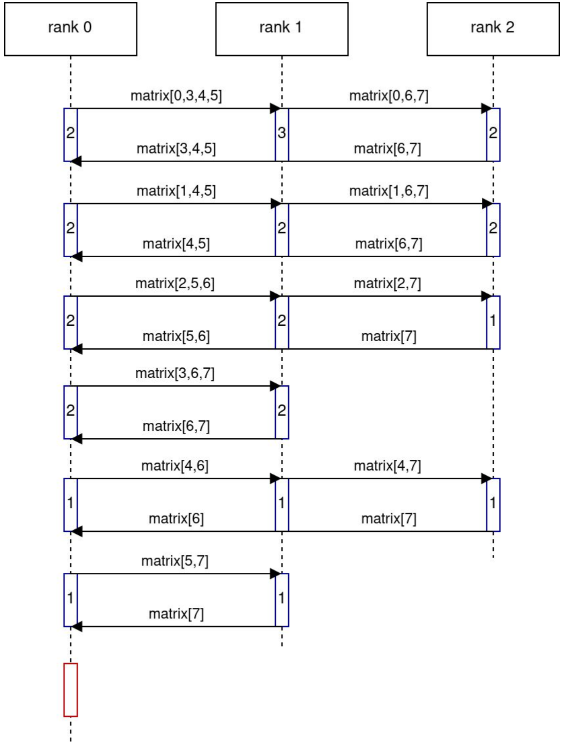


Figure: Schematic sketch of our first approach.

MPI - Parallelized Implementation and Approach

- Bottleneck is removed by:
  - Distribute in the beginning
  - Only share current master row

- Improvements
  - Less communication overhead
  - Worse distribution of calculation load
  - More complex implementation

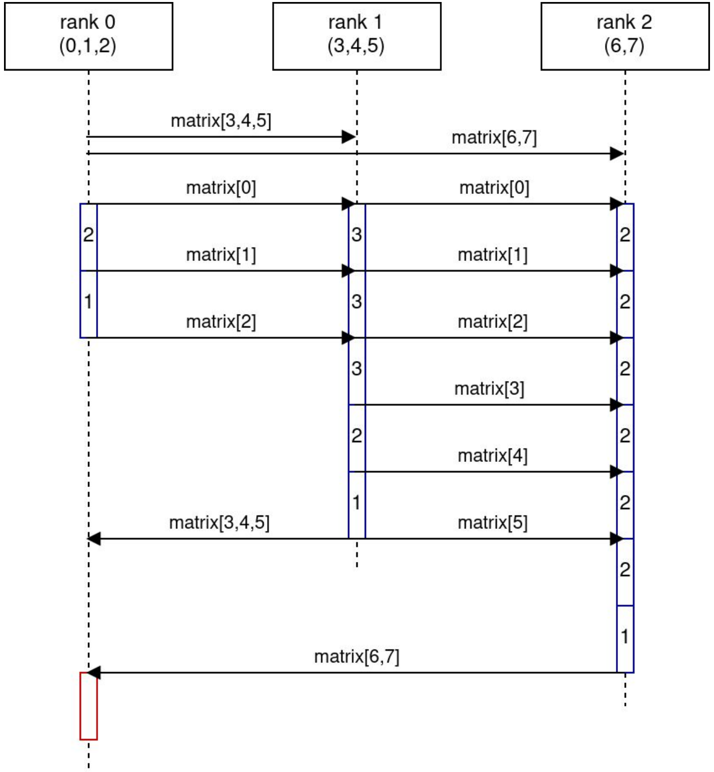
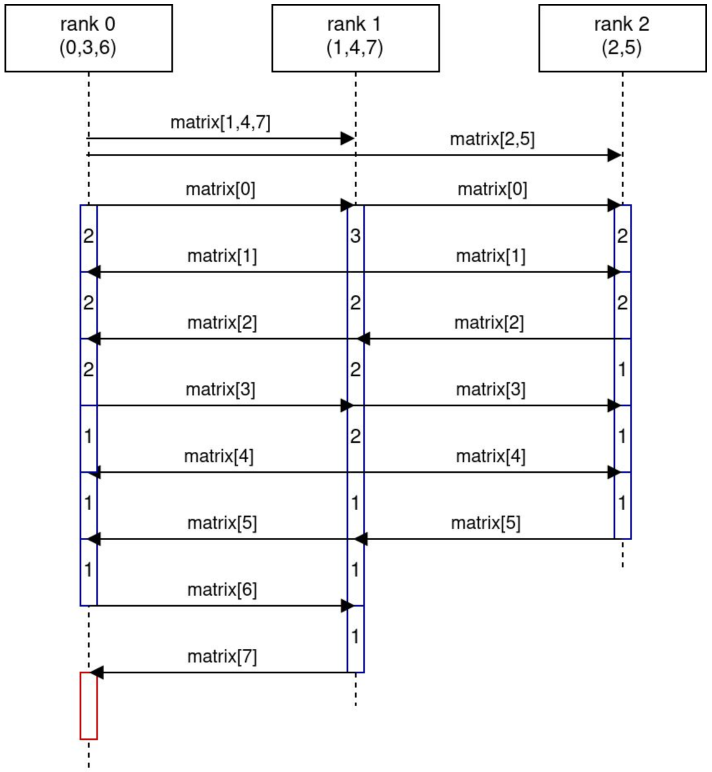


Figure: Schematic sketch of the improved approach.

MPI - Final Implementation improvements and new speed-up

- When starting with hybrid, we developed an even better approach
  - Distribute in the beginning in round robin sequence
  - Matrix is worked off evenly by all processes



MPI - Speed up comparison

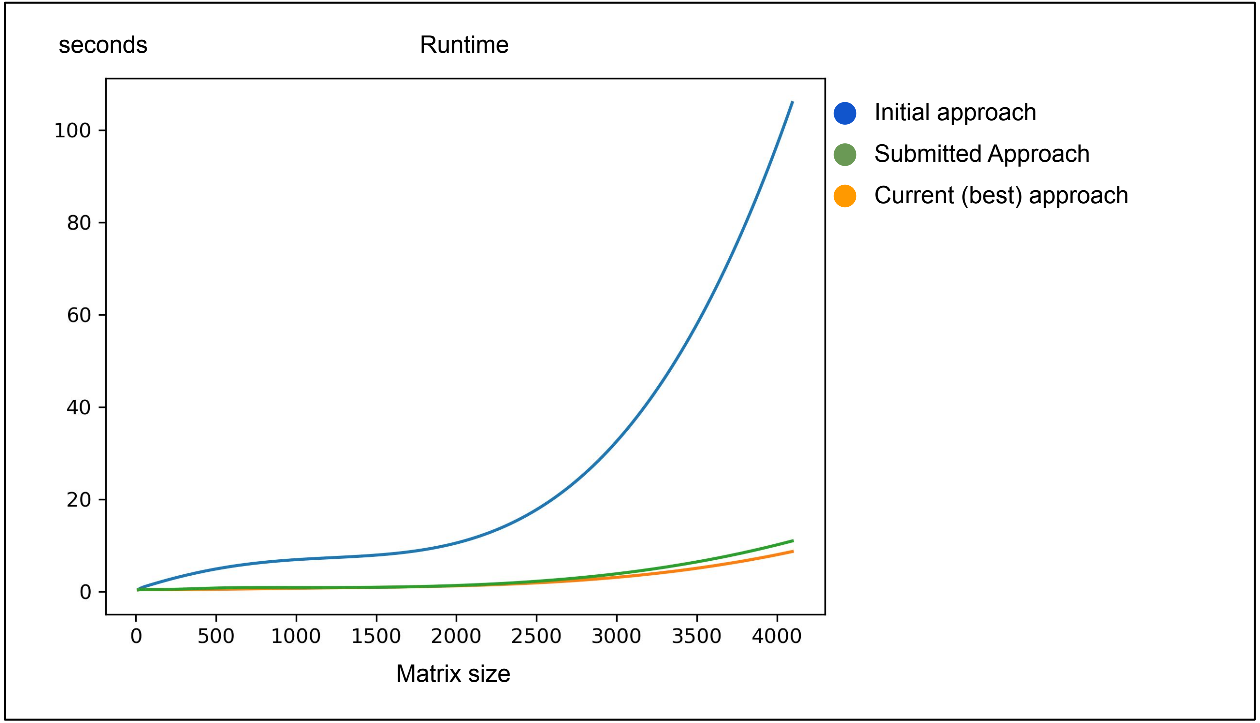


Figure: Compiler already vectorizing element-wise operations.

MPI - Communication vs. Calculation

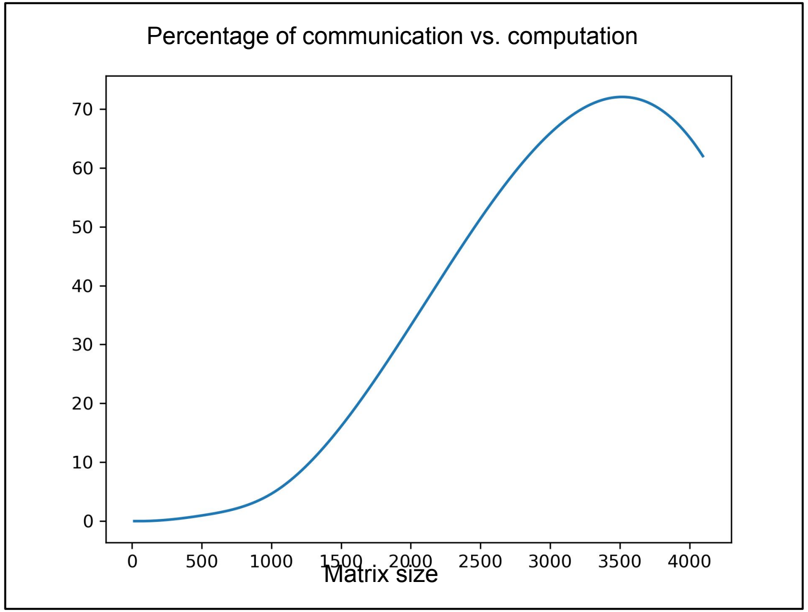
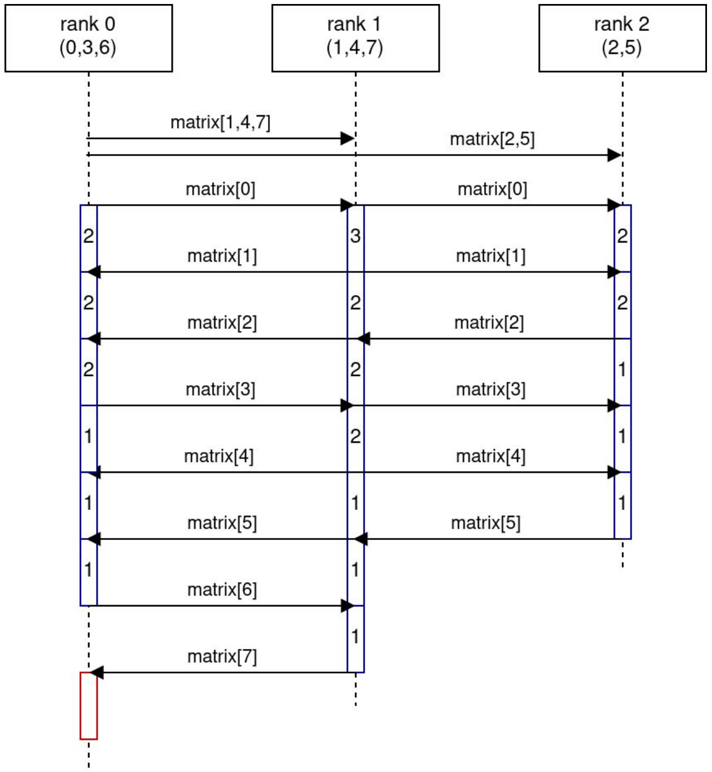


Figure: Spline (with k=3) displaying ratio communication vs. computation using the same procedure as in the OMP part.

Hybrid - Parallelized implementation and approach

- Built on our current best approach
- Addition of OMP similar to OMP part



Hybrid - Final Performance Results

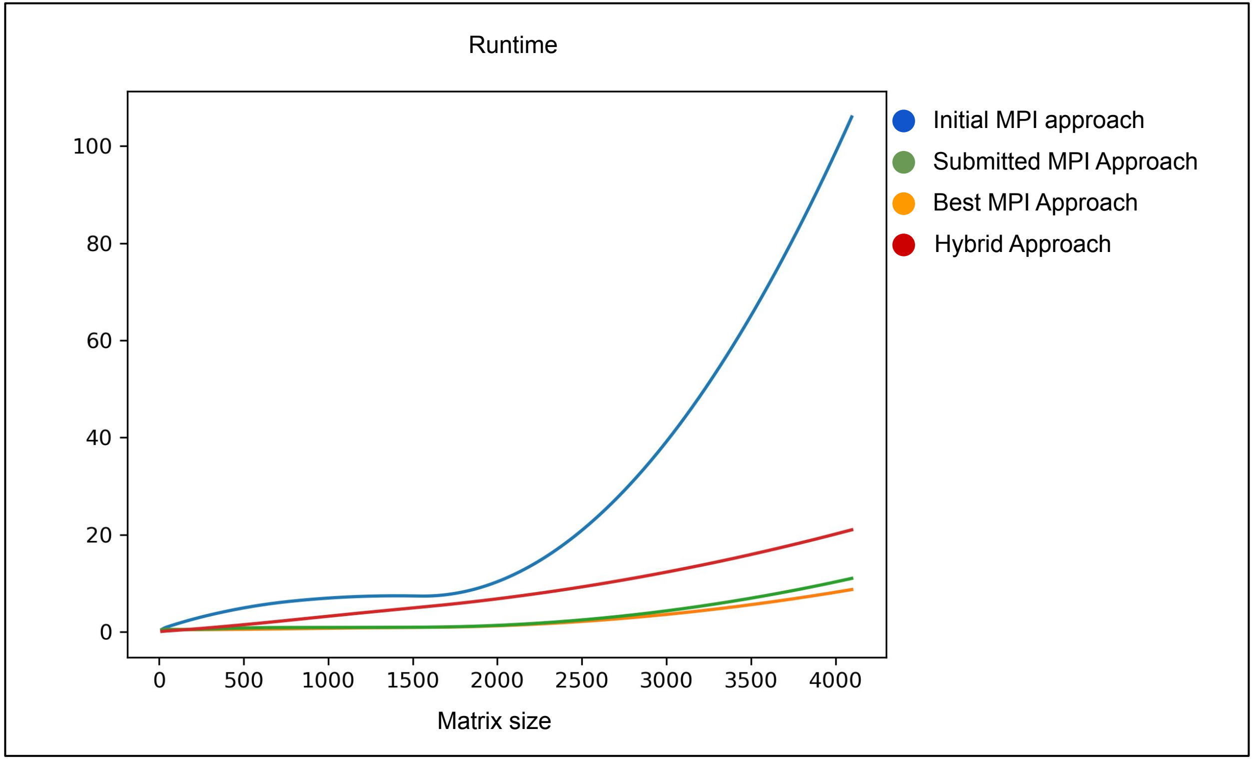


Figure: Runtime comparison between MPI

Hybrid - Communication vs. Calculation

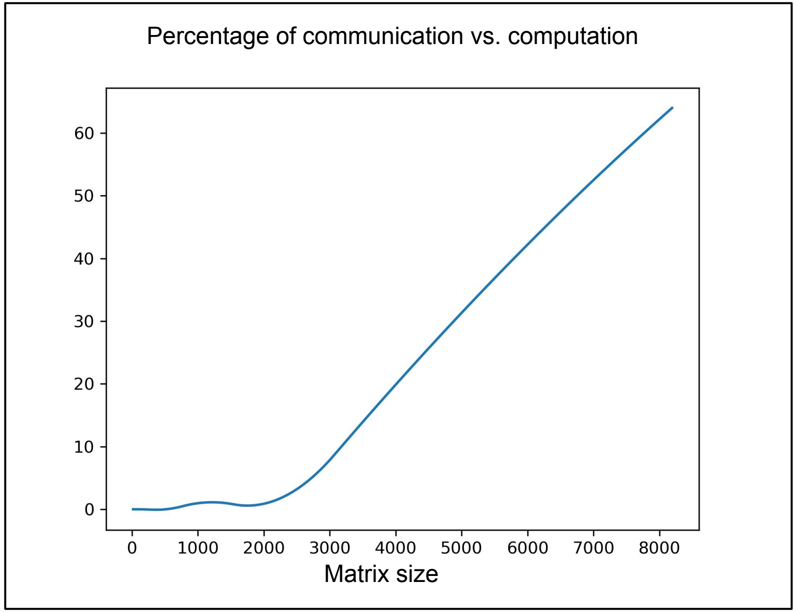


Figure: Spline (with k=3) displaying ratio communication vs. computation using the same procedure as in the OMP part.

Bonus - Parallelized implementation and approach

- Parallelized the inner loop using SIMD intrinsics
- Using `double`, we can calculate 4 values at once
- Achieved no speed-up
- Using -O3 flag compiler already figures out what to vectorize

```
Samples: 4K of event 'cycles', 4000 Hz, Event count (approx.): 4661322483
Serial::ForwardEliminationOptimized  /home/marcelbraasch/CLionProjects/openmp/serialge [Percent: local period]
  2,55 │       vinsertf128  $0x1,0x10(%rbx,%rax,1),%ymm7,%ymm1
  7,37 │       vmovupd      (%rdx,%rax,1),%xmm6
 12,54 │       vinsertf128  $0x1,0x10(%rdx,%rax,1),%ymm6,%ymm0
 17,44 │       vmulpd       %ymm3,%ymm1,%ymm1
 16,41 │       vsubpd       %ymm1,%ymm0,%ymm0
  2,45 │       vmovups      %xmm0,(%rdx,%rax,1)
  6,23 │       vextractf128 $0x1,%ymm0,0x10(%rdx,%rax,1)
 31,91 │       add          $0x20,%rax
  2,21 │       cmp          %rsi,%rax
```

Figure: Compiler already vectorizing element-wise operations.

- Using 32 threads, vectorizing rows by a factor of 4
- Theoretically, should yield a work split of 4 × 32 = 128
- By Amdahl's law, theoretical max speed up is then
  - $1 / (1 - 0.97 + 0.97/(4 \times 32)) = 26.61$

Conclusion

- OMP outperforms all other up to 2048
- MPI is faster for matrices with n ≥ 4092
- Hybrid did not work very well compared to MPI
- The best we got was a speed up of 14
- Amdahl’s law indicates a theoretical max of 17
- The compiler is already vectorizing matrix operations

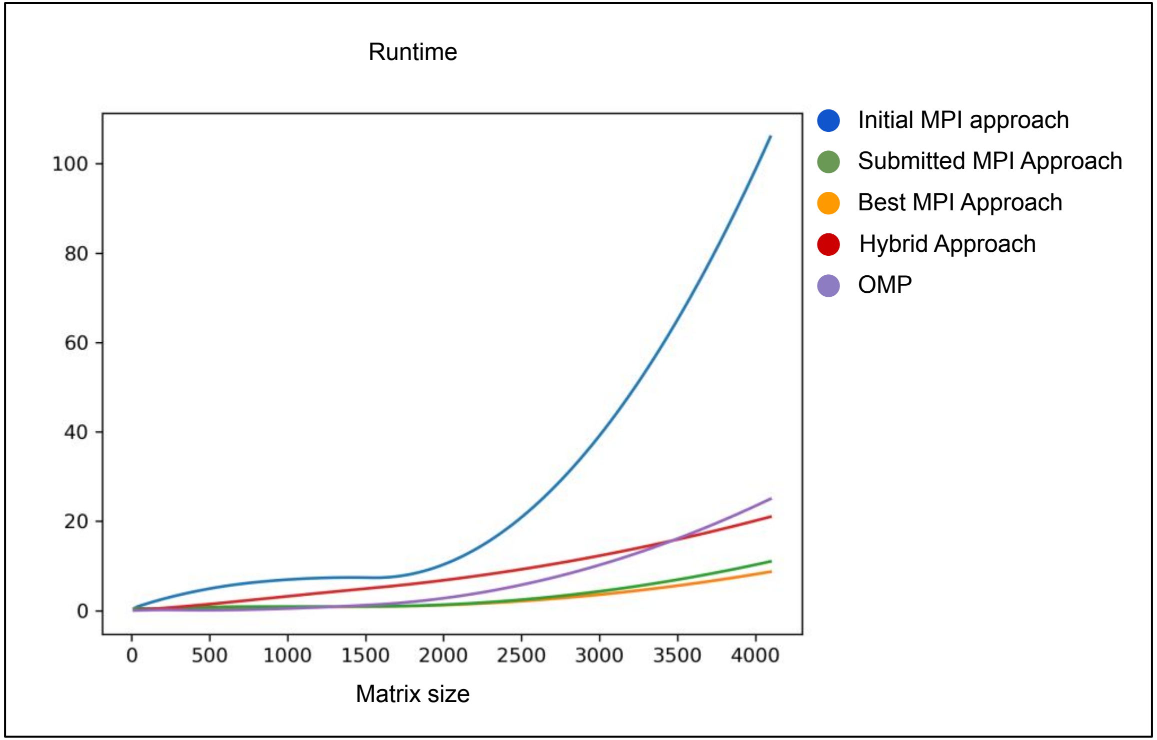


Figure: Runtime comparison between MPI

Additional

System specifications we performed the tests on:

- OS: Ubuntu 20.04.2 LTS
- RAM: 32 GB
- CPU: AMD Ryzen 9 5950X
- GPU: Nvidia GeForce RTX 3090