Tsirelson's space and other exotic constructions

Marcel K. Goh

16 December 2021

Abstract. This set of expository notes was written as a final report for the class MATH 567 Introduction to Functional Analysis, taught by Prof. Gantumur Tsogtgerel at McGill University in the Fall 2021 semester. It builds up the necessary theory to describe Tsirelson's example of an infinite-dimensional Banach space with no subspace isomorphic to c_0 or l_p for $1 \le p < \infty$, and modifications thereof.

1. Preliminary notions

In this section we state definitions and well-known results that we shall require later on. Proofs are not given for many of the facts given here; they can be found in introductory textbooks on linear analysis (see, e.g., [4], [12], [13]). The reader comfortable with the terminology of Banach space theory may wish to skip this section, returning only upon finding an unfamiliar concept or definition.

Let K denote either the real or complex field and let V be a vector space over K. A *norm* is a function $\|\cdot\| : V \to \mathbf{R}$ satisfying

i) $\|v\| \ge 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$;
ii) $\|\alpha \cdot v\| = |\alpha| \cdot \|v\|$ for all $\alpha \in K$ and $v \in V$; and
iii) $\|v + w\| \le \|v\| + \|w\|$ for all $v, w \in V$.

A vector space equipped with such a function is called a *normed vector space*, and one can define a metric d on the space by setting $d(x, y) = \|x - y\|$. Thus one has a notion of Cauchy sequences in these spaces, and if a normed vector space X is complete (every Cauchy sequence in X has a limit in X), we say that X is a *Banach space*. Any two norms on a finite-dimensional normed vector space are equivalent, in that they induce the same topology. This fact can be used to show that any finite-dimensional normed vector space over $\mathbf{R}$ or $\mathbf{C}$ is a Banach space, since these fields are complete.

The field $\mathbf{R}$ or $\mathbf{C}$ under the norm given by the absolute value $|\cdot|$ is of course the simplest Banach space. Other basic examples include the following.

i) Consider an n-dimensional vector space over K. Writing an element x of this space as $x = (x_1, \ldots, x_n)$, this vector space can be endowed with the *p-norm*

$$\|x\|_p = \Bigl(\sum_{i=1}^{n} |x_i|^p\Bigr)^{1/p}.$$

for $1 \leq p < \infty$. This defines a Banach space which we shall denote by l_p^n. As $p \to \infty$, this approaches the *maximum norm*

$$\|x\|_\infty = \max_{1\leq i\leq n} |x_i|,$$

and we denote this Banach space by l_∞^n.

ii) For $1 \leq p < \infty$, the p-norm for infinite sequences $x = (x_n)_{n=1}^\infty$ is given by

$$\|x\|_{l_p} = \Big(\sum_{i=1}^\infty |x_i|^p\Big)^{1/p}.$$

As $p \to \infty$, this approaches the *supremum norm*

$$\|x\|_{l_\infty} = \sup_{i\geq 1} |x_i|.$$

For $1 \leq p \leq \infty$, the set l_p of all infinite sequences x with $\|x\|l_p$ finite is a Banach space; when $p = \infty$, these are the bounded sequences.

iii) Let c_0 be the normed vector subspace of l_∞ consisting of sequences that tend to zero. It can be shown that this subspace is closed and thus itself a Banach space.

The latter two examples are infinite-dimensional, meaning that in these cases one can exhibit an infinite sequence of vectors that are all linearly independent.

Hamel bases. Assuming the axiom of choice, one can find a basis for any vector space. This is a set of elements $\{e_i\}_{i\in I}$ such that any $x \in X$ can be expressed uniquely as $\sum_{i\in F} a_i e_i$ for a finite set $F \subseteq I$ and scalars a_i, both depending on x. When dealing with infinite-dimensional spaces, such a basis is called a *Hamel basis* and are not awfully useful. To see why, we first need to establish (a variant of) the Baire category theorem, as well as simple lemma.

Theorem B (*Baire category theorem*). *Let X be a metric space that equals the union of countably many closed sets. If X is complete, then at least one of the closed sets has nonempty interior.* ▮

Lemma 1. *Let X be a normed vector space and let Y be a subspace of X. If Y has nonempty interior, then $Y = X$.*

Proof. Suppose that for some $r > 0$ and $y \in Y$, the ball $B_r(y) = \{x \in X : \|x - y\| < r\}$ is contained in Y. Now let $x \in X$ and note that the vector

$$z = y + \frac{r}{2} \cdot \frac{x}{\|x\|}$$

is at distance $r/2$ from y, meaning that $z \in B_r(y) \subseteq Y$. Since Y is a subspace, we find that

$$x = \frac{2\|x\|}{r}(z - y)$$

is in Y, completing the proof. ▮

We now show that Hamel bases of Banach spaces cannot be countably infinite.

Lemma 2. *Let X be an infinite-dimensional normed vector space. If X is also complete (and thus a Banach space), then any Hamel basis of X is uncountable.*

Proof. Suppose that $\{e_1\}_{i=1}^{\infty}$ is a countable Hamel basis for X. For integers $n \geq 1$, let X_n denote the linear span of $\{e_1, \ldots, e_n\}$. Each normed vector space X_n is finite-dimensional and thus complete, which implies that each X_n is closed in X. We now show that every X_n has empty interior, which completes the proof. ∎

Since many infinite-dimensional Banach spaces have the cardinality of the continuum to begin with, the above lemma tells us that Hamel bases can essentially be as complicated as the underlying space. (In fact, it has been shown that the cardinality of the Hamel basis of a Banach space always equals the cardinality of the space itself [10].)

Completions. Consider now the infinite-dimensional vector space c_{00} of sequences that are eventually zero. For all integers $i \geq 1$, let e_i denote the sequence that is 1 at index i and 0 elsewhere. The set $\{e_i\}_{i=1}^{\infty}$ is countable and clearly a Hamel basis for c_{00}, so we conclude that c_{00}, under any norm, cannot be a Banach space. However, for every normed vector space V, we can always find a Banach space X such that V is dense in X. We do this by letting X be the completion of V as a metric space and then defining scalar multiplication and addition as follows. If $x, y \in X$ are such that $x_n \to y$ and $y_n \to y$, then $\lambda x + \mu y$ shall simply be defined as $\lim_{n\to\infty}(\lambda x_n + \mu y_n)$. We leave it as an easy exercise for the reader to check that this makes X a normed vector space under the norm $\|x\| = \lim_{n\to\infty} \|x_n\|$, and that X is complete with respect to this norm.

Dual spaces and the weak topology. Let X be a normed vector space. A linear operator from X to its base field is called a *linear functional*, and is continuous if and only if it is bounded. The set of bounded linear functionals on a normed vector space X is a Banach space under the operator norm

$$\|T\| = \sup_{x\in X} \frac{\|Tx\|}{\|x\|}.$$

This space is called the *dual space of* X and is denoted X^*. If (x_n) is a sequence of vectors in X such that for some $x \in X$, $x^*(x_n) \to x^*(x)$ for all $x^* \in X^*$, then we say that (x_n) *converges weakly to* x. If (x_n) converges weakly to 0, then we call (x_n) *weakly null.* The topology that this definition of convergence induces on X is called the *weak topology*; it is the coarsest topology such that every element of X^* is still a continuous function.

2. Schauder bases

We saw in the previous section that Hamel bases are not very useful for performing analysis on infinite-dimensional normed vector spaces. A better notion in the infinite-dimensional setting is that of a *Schauder basis*. This is a countable sequence of vectors $(e_i)_{i=1}^{\infty}$ such that every $x \in X$ has a unique representation

$$x = \sum_{i=1}^{\infty} a_i e_i$$

for some sequence $(a_i)_{i=1}^{\infty}$ of scalars, where convergence of the infinite sum is defined in terms of the metric induced by the norm. From here on out, we shall sometimes write "basis" to mean "Schauder basis", for brevity's sake. A basis (e_i) for which there exists C such that $1/C \leq \|e_i\| \leq C$ for all i is called *seminormalised* and a seminormalised basis whose corresponding constant C equals 1 is said to be *normalised*. A sequence of vectors (x_i) that is a basis for the closure of its linear span is called a *basic sequence*.

Equivalent bases. A basis (x_i) for a Banach space X and a basis (y_i) for a Banach space Y are said to be *equivalent* if the map T sending x_i to y_i extends to a linear homeomorphism between X and Y. Of course, this relation is reflexive, symmetric, and transitive. One can also show this to be the same as saying that there are constants $0 < m \leq M$ such that for any sequence (a_i) of scalars,

$$m \cdot \Big\| \sum_{i=1}^{\infty} a_i x_i \Big\| \leq \Big\| \sum_{i=1}^{\infty} a_i y_i \Big\| \leq M \cdot \Big\| \sum_{i=1}^{\infty} a_i x_i \Big\|.$$

To every Banach space X with a basis $(e_i)_{i=1}^{\infty}$, we can associate a sequence $(P_n)_{n=1}^{\infty}$ of bounded linear operators given by

$$P_n\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = \sum_{i=1}^{n} a_i e_i.$$

The quantity $\sup_{n \geq 1} \|P_n\|$ is finite and called the *basis constant* of (e_i). A basis (e_i) with basis constant equal to 1 is said to be *monotone*. Every Schauder basis is monotone with respect to the norm $|||x||| = \sup_n \|P_n x\|$, and this norm happens to be equivalent to the original one. Hence, given a basis, we can always pass to a norm under which the basis is monotone.

Block basic sequences. Let (e_i) be a basic sequence in a Banach space. A sequence of vectors (x_j), each of the form

$$x_j = \sum_{i=n_j+1}^{n_{j+1}} a_i e_i$$

where the a_i are scalars and $n_1 < n_2 < \cdots$ is an increasing sequence of positive integers, is called a *block basic sequence* or *block basis* of (e_i). Defining the *support* $\operatorname{supp} x$ of a vector $x = \sum_{i=1}^{\infty} a_i e_i$ with respect to a basis (e_i) to be the set of indices i for which a_i is nonzero, a block basis is a sequence of vectors x_j for which $\operatorname{supp} x_j$ is finite for all j and $\max \operatorname{supp} x_j < \min \operatorname{supp} x_{j+1}$ for all $j \geq 1$.

Unconditional bases. Note that the definition of a Schauder basis may rely very much on the ordering imposed on its elements. There are some bases for which this is not the case. The following conditions are equivalent for a basis (e_i) of a Banach space X.

i) For every permutation π of the positive integers, $\big(e_{\pi(i)}\big)_{i=1}^{\infty}$ is a basis of X.
ii) If a sum of the form $\sum_{i=1}^{\infty} a_i e_i$ converges, then any reordering of its terms produces a sum which converges to the same value.
iii) There exists a constant C such that for all pairs of scalar sequences (a_i) and (b_i) with $|a_i| \leq |b_i|$ for all i,

$$\Big\| \sum_{i=1}^{\infty} a_i e_i \Big\| \leq C \Big\| \sum_{i=1}^{\infty} b_i e_i \Big\|.$$

The proof of this equivalence can be found in Section 1.c of [12]. A basis satisfying any (and thus all) of these properties is called an *unconditional basis*, and a basis is said to be *C-unconditional* if it satisfies (iii) for some specific constant C.

Symmetric and subsymmetric bases. In the sequence spaces c_0 or l_p for $1 \leq p < \infty$, the canonical basis consists of vectors that have a 1 at some index and zeroes everywhere else. Since each coordinate is weighted equally in l_∞ or l_p norm, this basis is equivalent to any of its permutations. Such a basic sequence in a Banach space is said to be *symmetric*. Note that this is a bit stronger than the condition we saw above for a basic sequence (e_i) to be unconditional. In that case, we only needed $\big(e_{\pi(i)}\big)$ to be a basic sequence for all permutations π.

An unconditional basis (e_i) of a Banach space X is called *subsymmetric* if for all $i_1 < i_2 < \cdots$, the subsequence $\big(e_{i_k}\big)$ is equivalent to (e_i). Note that this time, the requirement that the basis be unconditional is not redundant. It can be shown, however, that every symmetric basis is subsymmetric (see Prop. 3.a.3 of [12]); we shall use this fact in the next section.

Another fact that we'll need later regards a certain dichotomy in the family of subsymmetric basic sequences.

Lemma 3. *Let (x_i) be a bounded subsymmetric basic sequence. Then (x_i) is either equivalent to the standard unit vector basis for l_1 or else (x_i) is weakly null.*

Proof. If (x_i) is weakly null, then we are done. If not, then we know that there exists some $f \in X^*$ and $\delta > 0$ such that $f(x_i) \geq \delta$ for all i. (If not, then (x_i) would have a subsequence that is weakly null, and thus by subsymmetry the sequence itself is weakly null.) By scaling f we can assume that it has operator norm equal to 1. For any n, and scalars $(b_1, \dots, b_n)$ we have

$$\Big\| \sum_{i=1}^{n} |b_i| x_i \Big\| = \|f\| \cdot \Big\| \sum_{i=1}^{n} |b_i| x_i \Big\| \geq f\Big(\sum_{i=1}^{n} |b_i| x_i \Big) = \sum_{i=1}^{n} |b_i| f(x_i) \geq \delta \sum_{i=1}^{n} |b_i|.$$

Now let C be the constant given by the unconditionality of (x_i) and setting $a_i = |b_i|$, we have

$$\Big\| \sum_{i=1}^{n} |b_i| x_i \Big\| \leq C \cdot \Big\| \sum_{i=1}^{n} b_i x_i \Big\|.$$

Putting everything together establishes

$$\Big\| \sum_{i=1}^{n} b_i x_i \Big\| \geq \frac{1}{C} \Big\| \sum_{i=1}^{n} |b_i| x_i \Big\| \geq \frac{\delta}{C} \sum_{i=1}^{n} |b_i|$$

for all n, and taking n to infinity gives a corresponding lower bound on all infinite sequences (b_i).

The upper bound is a simple consequence of the triangle inequality. Let M be the constant bounding (x_i) from above. Then since $\|x_i\| \leq M$ for all i, for an arbitrary infinite sequence (b_i) of scalars we have

$$\Big\| \sum_{i=1}^{\infty} b_i x_i \Big\| \leq \sum_{i=1}^{\infty} |b_i| \cdot \|x_i\| \leq M \sum_{i=1}^{\infty} |b_i|,$$

which combines with the lower bound to give

$$\frac{\delta}{C} \sum_{i=1}^{\infty} |b_i| \leq \Big\| \sum_{i=1}^{\infty} b_i x_i \Big\| \leq M \sum_{i=1}^{\infty} |b_i|.$$

Hence (x_i) is equivalent to the unit vector basis of l_1. ∎

3. Tsirelson's space and its dual

For many decades, the infinite-dimensional Banach spaces known to functional analysts all contained a subspace linearly homeomorphic to c_0 or l_p for $1 \leq p < \infty$. The first infinite-dimensional space without such a subspace was constructed by B. S. Tsirelson in 1974 [15], and that same year T. Figiel and W. B. Johnson gave a more explicit characterisation of its dual [6]. In time, the dual of the original space has come to be known as T, with Tsirelson's original space denoted T^*. Properties of Tsirelson's space and variations thereof are collected in a book by P. G. Casazza and T. J. Shura [5]. We shall describe the space T of Figiel and Johnson in this section, following Casazza and Shura's exposition but supplying some of the proofs that are omitted there.

Let E and E' be subsets of the positive integers. We write $E < F$ to mean that the maximum element of E is less than the minimum element of F and likewise we write $E \le F$ to mean the same thing, with "less than" replaced by "at most". A sequence of finitely many sets $E_1, E_2, \ldots, E_k$ of positive integers is said to be *admissible* if

$$\{k\} \le E_1 < E_2 < \cdots < E_k.$$

For $x \in c_{00}$ and a subset E of positive integers, we write Ex to denote the sequence that equals x_i at index i for all $i \in E$ and 0 elsewhere. For a fixed $x \in c_{00}$ there are only finitely many E such that Ex is nonzero, since x is finitely supported. Fixing a norm $\|\cdot\|$ on c_{00} now, we say that

$$\sum_{i=1}^{k} \|E_i x\|$$

is an *admissible sum for* x if $E_1, \ldots, E_k$ is admissible.

The following proposition establishes a sequence of norms on c_{00} in terms of admissible sequences of sets.

Proposition 4. *For $x \in c_{00}$, we let $\|x\|_0 = \|x\|_{l_\infty}$ and for $n \ge 1$, we inductively set*

$$\|x\|_n = \max\left\{ \|x\|_{n-1}, \frac{1}{2} \max \sum_{i=1}^{k} \|E_i x\|_{n-1} \right\},$$

where the inner maximum is taken over the finitely many admissible sequences $E_1, E_2, \ldots, E_k$ for which the admissible sum for x is nonzero. Then the following statements hold.

i) For all $n \ge 0$, $\|\cdot\|_n$ is a norm.

ii) For all $n \ge 0$ and $x \in c_{00}$, $\|x\|_n \le \|x\|_{l_1}$.

iii) The limit $\lim_{n\to\infty} \|x\|_n$ exists and defines a norm on c_{00}.

Proof. The proof of (i) is inductive; the base case $n = 0$ corresponds to $\|\cdot\|_{l_\infty}$, which we already know to be a norm. Now assume the statement proven for $n-1$. The only property that isn't quite immediate is the triangle inequality. Note that for $x, y \in c_{00}$ nonzero, if $E(x+y)$ is nonzero and $Ex = 0$, then $E(x+y) = Ey$. So

$$\max^{(x+y)} \sum_{i=1}^{k} \|E_i(x+y)\|_{n-1} \le \max^{(x)} \sum_{i=1}^{k} \|E_i x\|_{n-1} + \max^{(y)} \sum_{i=1}^{k} \|E_i y\|_{n-1},$$

where $\max^{(x)}$ ranges over all admissible sequences whose corresponding admissible sum for x is nonzero. Then we have

$$\begin{aligned}
\|x+y\|_n &= \max\Big\{\|x+y\|_{n-1}, \frac{1}{2}\overset{(x+y)}{\max}\sum_{i=1}^{k}\|E_i(x+y)\|_{n-1}\Big\} \\
&\le \max\Big\{\|x\|_{n-1}+\|y\|_{n-1}, \frac{1}{2}\overset{(x+y)}{\max}\sum_{i=1}^{k}\big(\|E_i x\|_{n-1}+\|E_i y\|_{n-1}\big)\Big\} \\
&\le \max\Big\{\|x\|_{n-1}, \frac{1}{2}\overset{(x)}{\max}\sum_{i=1}^{k}\|E_i x\|_{n-1}\Big\} \\
&\qquad + \max\Big\{\|y\|_{n-1}, \frac{1}{2}\overset{(y)}{\max}\sum_{i=1}^{k}\|E_i y\|_{n-1}\Big\} \\
&= \|x\|_n + \|y\|_n,
\end{aligned}$$

by linearity of the maximum function.

The claim (ii) is rather obvious, and again the proof is by induction, with the base case $\|x\|_{l_\infty} \le \|x\|_{l_1}$. Now let $n \ge 1$ and suppose the statement proven for $n-1$. If $\|x\|_n = \|x\|_{n-1}$ we are done, and if not, then

$$\|x\|_n = \frac{1}{2}\overset{(x+y)}{\max}\sum_{i=1}^{k}\|E_i(x+y)\|_{n-1} \le \frac{1}{2}\overset{(x+y)}{\max}\sum_{i=1}^{k}\|E_i(x+y)\|_{l_1}$$

and since any admissible sequence consists of disjoint sets, each coordinate x_i in x contributes $|x_i|$ to the sum on the right-hand side at most once. Hence in this case $\|x\|_n \le \|x\|_{l_1}$ as well.

Since the norms $\|x\|_n$ are nondecreasing in n for any fixed x and bounded above by $\|x\|_{l_1}$, this value must converge to a limit as $n \to \infty$. The fact that $\|x\| = \lim_{n\to\infty}\|x\|_n$ is a norm is easy to see (the triangle inequality follows from linearity of the limit). We have thus proven (iii). ▮

The above proposition asserts that c_{00} endowed with the norm $\|\cdot\|$ is a normed vector space. We then define Tsirelson's space T to be the completion of c_{00} with respect to this norm. For each index i, let t_i be the vector whose ith coordinate is 1 and whose other coordinates are all 0. The sequence (t_i) forms a normalised 1-unconditional Schauder basis for T. This can be proven by inductively verifying that it holds for the completion of c_{00} with respect to $\|\cdot\|_n$ for all n.

We also have the identity

$$\|x\| = \max\Big\{\|x\|_{l_\infty}, \frac{1}{2}\sup\sum_{i=1}^{k}\|E_i x\|\Big\},$$

in which the supremum is taken over all k and all possible admissible sequences $E_1, \ldots, E_k$. This could actually have been taken as the *definition* of the Tsirelson norm (it is the unique norm with this property), which might appear to be nonsensical, because the norm appears on both sides of the definition. However, it is actually perfectly reasonable, since the E_ix on the right hand side have $|\operatorname{supp} E_ix| < |\operatorname{supp} x|$ and we can assume by induction that the norm is already defined for these vectors. The base cases are the vectors that have only one nonzero coordinate, for which we must take the l_∞-norm. In any case, let us prove that this definition is equivalent to the norm that we constructed above.

Proposition 5. *For any $x \in T$, we have*

$$\|x\| = \max\biggl\{ \|x\|_{l_\infty}, \frac{1}{2} \sup \sum_{i=1}^{k} \|E_ix\| \biggr\},$$

where the supremum is taken over all k and all admissible sequences $E_1, \ldots, E_k$.

Proof. Since c_{00} is dense in T, it suffices to prove the proposition for $x \in c_{00}$. If $\|x\| = \|x\|_{l_\infty}$, then for every $n \in \mathbf{N}$ and admissible sequence $E_1, \ldots, E_k$, we have

$$\frac{1}{2} \sum_{i=1}^{k} \|E_ix\|_n \le \|x\|_n.$$

Then by linearity of the supremum, we have

$$\frac{1}{2} \sum_{i=1}^{k} \|E_ix\| \le \frac{1}{2} \sum_{i=1}^{k} \sup_{n\ge 0} \|E_ix\|_n = \sup_{n\ge 0} \frac{1}{2} \sum_{i=1}^{k} \|E_ix\|_n \le \sup_{n\ge 0} \|x\|_n = \|x\|,$$

which shows that the identity holds in this case. If $\|x\| \ne \|_|l_\infty$, then for all $\epsilon > 0$ there is $n \ge 1$ such that $\|x\| < \|x\|_n + \epsilon$ and $\|x\|_n > \|x\|_{n-1}$. This means there is some k' and $E_1, \ldots, E_{k'}$ such that

$$\|x\|_n = \frac{1}{2} \sum_{i=1}^{k'} \|E_ix\|_{n-1}.$$

It follows that

$$\|x\| - \epsilon < \|x\|_n = \frac{1}{2} \sum_{i=1}^{k'} \|E_ix\|_{n-1} \le \frac{1}{2} \sum_{i=1}^{k'} \|E_ix\| \le \sup \frac{1}{2} \sum_{i=1}^{k} \|E_ix\|.$$

For an upper bound on the supremum, we pick k'' and an admissible sequence $E_1, \ldots, E_{k''}$ such that

$$\sup \frac{1}{2} \sum_{i=1}^{k} \|E_ix\| < \frac{1}{2} \sum_{i=1}^{k''} \|E_ix\| + \epsilon.$$

We then pick n' such that

$$\sup n \geq 1 \frac{1}{2} \sum_{i=1}^{k''} \|E_i x\|_n < \frac{1}{2} \sum_{i=1}^{k''} \|E_i x\|_{n'} + \epsilon.$$

This yields the upper bound

$$\begin{aligned}
\frac{1}{2} \sum_{i=1}^{k} \|E_i x\| &< \frac{1}{2} \sum_{i=1}^{k''} \|E_i x\| + \epsilon \\
&= \frac{1}{2} \sum_{i=1}^{k''} \sup_{n \geq 1} \|E_i x\|_n + \epsilon \\
&= \sup_{n \geq 1} \frac{1}{2} \sum_{i=1}^{k''} \|E_i x\|_n + \epsilon \\
&< \frac{1}{2} \sum_{i=1}^{k''} \|E_i x\|_{n'} + 2\epsilon \\
&\leq \|x\|_{n'+1} + 2\epsilon \\
&\leq \|x\| + 2\epsilon.
\end{aligned}$$

We have shown that

$$\|x\| - \epsilon < \sup \frac{1}{2} \sum_{i=1}^{k} \|E_i x\| < \|x\| + 2\epsilon,$$

which, since ϵ was arbitrary, gives us equality. ∎

This recursive identity is the only definition one really uses when performing any serious computations with Tsirelson's norm. In particular, it asserts that we have, for any $x \in T$ and admissible sequence $E_1, \ldots, E_k$,

$$\sum_{i=1}^{k} \|E_i x\| \leq 2\|x\|.$$

As a demonstration of the utility of the identity, consider the proof of the following proposition, which gives us an idea of how Tsirelson's norm behaves on subsets of a seminormalised block basis.

Proposition 6. *Let (t_i) denote the usual unit vector basis of Tsirelson's space and let k be a positive integer and let $M \geq 1$. For any k blocks $(y_j)_{j=1}^{k}$, where each y_j satisfies $1/M \leq \|y_j\| \leq M$ and is of the form*

$$y_j = \sum_{i=n_j+1}^{n_{j+1}} a_i t_i$$

for some scalars a_i *and* $k-1 \le n_1 < n_2 < \cdots < n_{k+1}$*, we have*

$$\frac{1}{2M}\sum_{j=1}^{k}|b_j| \le \Bigl\|\sum_{j=1}^{k} b_j y_j\Bigr\| \le M\sum_{j=1}^{k}|b_j|$$

for all k*-tuples of scalars* $(b_1,\ldots,b_k)$.

Proof. By the triangle inequality and the fact that $\|y_j\| \le M$ for all j, we have

$$\Bigl\|\sum_{j=1}^{k} b_j y_j\Bigr\| \le \sum_{j=1}^{k}\|b_j y_j\| \le M\sum_{j=1}^{k}|b_j|.$$

For the lower bound, we let $x = \sum_{j=1}^{k} b_j y_j$ and let

$$E_j = \{n_j+1, n_j+2, \ldots, n_{j+1}-1\}$$

for $1 \le j \le k$. Then

$$\|x\| \ge \frac{1}{2}\sum_{j=1}^{k}\|E_j x\| = \frac{1}{2}\sum_{j=1}^{k}\|b_j y_j\| \ge \frac{1}{2M}\sum_{j=1}^{k}|b_j|. \qquad \blacksquare$$

We are almost ready to prove that T does not contain a copy of l_1. We just need one more lemma, which is well-known and due to R. C. James [11].

Lemma J (*James,* 1964). *If a normed linear space contains a subspace isomorphic to* l_1*, then for any* $\delta > 0$*, there is a normalised block basic sequence* $(u_j)_{j=0}^{\infty}$ *such that*

$$(1-\delta)\sum_{j=0}|a_j| \le \Bigl\|\sum_{j=0}^{\infty} a_j u_j\Bigr\| \le \sum_{j=0}^{\infty}|a_j|$$

for every sequence $(a_j)_{j=0}^{\infty}$ *of scalars such that at least one* a_j *is nonzero.*

Proof. The statement holds trivially when $\delta \ge 1$, so let $0 < \delta < 1$. The subspace B isomorphic to l_1 contains a sequence $(x_i)_{i=0}^{\infty}$ for which there exist constants m and M such that

$$m\sum_{i=0}^{\infty}|a_i| \le \Bigl\|\sum_{i=0}^{\infty} a_i x_i\Bigr\| \le M\sum_{i=0}^{\infty}|a_i|$$

for all sequences $(a_i)_{i=0}^{\infty}$ of scalars. For $n \ge 1$, let

$$K_n = \inf\Bigl\{\Bigl\|\sum_{i=n}^{\infty} a_i x_i\Bigr\| : \sum_{i=0}^{\infty}|a_i| = 1\Bigr\}$$

and set K to $\lim_{n\to\infty} K_n$. We of course have $m \le K \le M$. Let $\phi > 1$ and $0 < \theta < 1$ be such that $1-\delta \le \theta/\phi$. Choose $n_0 < n_1 < n_2$ such that $K_{n_0+1} \ge \theta K$ and for which there is a block basic sequence $(y_j)_{j=0}^{\infty}$ with each y_j of the form

$$y_j = \sum_{j=n_j+1}^{n_{j+1}} a_i x_i$$

for some scalars a_i (depending on j) and with $\|y_j\| \le \phi K$ for all j. Let $(a_j)_{j=0}^{\infty}$ be an arbitrary sequence of scalars. The normalised block basis consisting of $u_j = y_j/\|y_j\|$ satisfies

$$\Big\| \sum_{j=0}^{\infty} a_j u_j \Big\| \le \sum_{j=0}^{\infty} |a_j|$$

by the triangle inequality. By the construction of the block basis, each y_j has $\operatorname{supp} y_j > \{n_0\}$, so by the definition of K_{n_0+1},

$$\Big\| \sum_{j=0}^{\infty} a_j y_j \Big\| = \Big\| \sum_{j=n_0+1}^{\infty} a_j y_j \Big\| \ge K_{n_0+1} \sum_{j=0}^{\infty} |a_j| \ge \theta K \sum_{j=0}^{\infty} |a_j|.$$

Then since $\|y_j\| \le \phi K$ for all K and $\theta/\phi \ge 1-\delta$, we have

$$\Big\| \sum_{j=0}^{\infty} a_j u_j \Big\| = \Big\| \sum_{j=0}^{\infty} a_j \frac{y_j}{\|y_j\|} \Big\| \ge \frac{1}{\phi K} \Big\| \sum_{j=0}^{\infty} a_j y_j \Big\| \ge (1-\delta) \sum_{j=0}^{\infty} |a_j|,$$

which proves the lower bound. ∎

Note that in James' original statement of the lemma, the u_j are just vectors contained in the closed unit ball. However, the construction in his proof actually produces a block basic sequence, and we normalised it in our own version of the proof above. We are now set to show that l_1 does not embed into T.

Theorem 7. *Tsirelson's space T does not contain a subspace isomorphic to l_1.*

Proof. Suppose, towards a contradiction, that T did contain a subspace isomorphic to l_1. Then, applying Lemma J with $\delta = 1/9$, there exists a normalised block basic sequence $(y_j)_{j=0}^{\infty}$ such that for all sequences $(a_j)_{j=0}^{\infty}$ of scalars,

$$\frac{8}{9} \sum_{j=0}^{\infty} |a_j| \le \Big\| \sum_{j=0}^{\infty} a_j y_j \Big\| \le \sum_{j=0}^{\infty} |a_j|.$$

In particular, if we let $r \ge 1$ be an integer and let $(a_j)_{j=0}^{\infty}$ be the sequence with

$$a_j = \begin{cases} 1, & \text{if } j = 0; \\ 1/r, & \text{if } 1 \le j \le r; \\ 0, & \text{if } j > r, \end{cases}$$

then we have $\sum_{j=0}^\infty |a_j| = 2$. This yields

$$\Big\|y_0 + \frac{1}{r}\sum_{j=1}^r y_j\Big\| = \Big\|\sum_{j=0}^\infty a_j y_j\Big\| \ge \frac{8}{9}\sum_{j=0}^\infty |a_j| = \frac{16}{9}$$

for all integers $r \ge 1$.

Now let $\{k\} \le E_1 < E_2 < \cdots < E_k$ be an arbitrary admissible sequence of sets, and let $r_0 = \max\operatorname{supp} y_0$. If $k > r_0$, then

$$\sum_{i=1}^k \Big\|E_i\Big(y_0 + \frac{1}{r}\sum_{j=1}^r y_j\Big)\Big\| = \sum_{i=1}^k \Big\|E_i\Big(\frac{1}{r}\sum_{j=1}^r y_j\Big)\Big\| \le 2\cdot\Big\|\frac{1}{r}\sum_{j=1}^r y_j\Big\| \le 2.$$

If $k \le r_0$, then we set

$$S = \big\{1 \le j \le r : \|E_i y_j\| \ne 0 \text{ for at least two values of } i\big\}$$

and

$$T = \big\{1 \le j \le r : \|E_i y_j\| \ne 0 \text{ for at most one value of } i\big\}.$$

Note that if $j \in S$, then y_j straddles the border between E_i and E_{i+1} for some i, and no other $y_{j'}$ can do so, since the supports of the y_j are disjoint. So the cardinality of S is at most $k-1$. It is also clear that S and T are disjoint and $|S| + |T| = r$, so

$$2|S| + |T| \le 2(k-1) + r - k + 1 = r + k - 1.$$

Since the y_j are unit vectors, we have

$$\begin{aligned}\sum_{i=1}^k \Big\|E_i\Big(\frac{1}{r}\sum_{j=1}^\infty y_i\Big)\Big\| &= \sum_{j\in S}\sum_{i=1}^k \|E_i y_j\| + \sum_{j\in T}\sum_{i=1}^k \|E_i y_j\| \\ &\le \sum_{j\in S} 2\cdot\|y_j\| + \sum_{j\in T}\|y_j\|, \\ &\le r + k - 1\end{aligned}$$

by rearranging sums. Dividing by r and adding the unit vector y_0 into the mix, we have

$$\begin{aligned}\sum_{i=1}^k \Big\|E_i\Big(y_0 + \frac{1}{r}\sum_{j=1}^r y_j\Big)\Big\| &\le \sum_{i=1}^k \|E_i y_0\| + \frac{1}{r}\sum_{i=1}^k \Big\|E_i\Big(\frac{1}{r}\sum_{j=1}^\infty y_i\Big)\Big\| \\ &\le 2 + \frac{r+k-1}{r} \\ &\le 3 + \frac{n_0}{r}.\end{aligned}$$

Selecting some $r \geq 2n_0$, the right-hand side is at most $7/2$, meaning that

$$\Big\| y_0 + \frac{1}{r}\sum_{j=1}^{r} y_j \Big\| \leq 7/4,$$

and we have $7/4 \geq 16/9$, the contradiction we sought. ▮

In the proof that c_0 and the other l_p spaces do not embed into T either, we shall make use of the following theorem, a version of what is called the Bessaga–Pełczyński Selection Principle (see [3], Theorem 3).

Theorem S (*Bessaga–Pełczyński,* 1958). *Let X be a Banach space with basis (e_i) and let (e_i^*) be the coefficient functionals given by*

$$e_j^*\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = a_j.$$

If $(y_i)_{i=1}^{\infty}$ is a sequence of vectors in X such that

i) $\lim_{i\to\infty} \|y_i\| > 0$*; and*

ii) $\lim_{i\to\infty} e_j^*(y_i) = 0$ *for all indices $j \geq 1$,*

then there exists a subsequence of (y_i) which is equivalent to a block basis with respect to the original basis (e_i). ▮

To prove that c_0 and l_p do not embed into T, we shall actually prove the following stronger result.

Theorem 8. *Tsirelson's space T does not contain a seminormalised subsymmetric basic sequence.*

Proof. Towards a contradiction, suppose that (y_i) is a seminormalised subsymmetric basic sequence in T. Since the sequence is seminormalised, it is bounded from above, so by Lemma 3, either (y_i) is equivalent to the canonical basis for l_1 or (y_i) is weakly null. We already proved Theorem 7 above, which deems the first scenario impossible, so (y_i) must be weakly null and thus in particular satisfies condition (ii) of Theorem S. In addition, since (y_i) is seminormalised, it is uniformly bounded away from zero, so we are in the position to conclude from Theorem S that there is a subsequence of (y_i) which is equivalent to a block basic sequence (x_i) against the unit vector basis (t_i) of T. We shall now pass to subsequences both in (x_i) and in (y_i). First, pass to a subsequence and reindex (x_i) to ensure that $\operatorname{supp} x_i > \{i\}$ for all i. Now we pass to a subsequence of (y_i) that is equivalent to this new x_i and reindex; this is equivalent to the original sequence by subsymmetry.

Since y_i is seminormalised and there is a linear homeomorphism mapping each $y_i \mapsto x_i$, (x_i) is seminormalised as well; that is, there is $M \geq 1$ such that $1/M \leq \|x_i\| \leq M$. Now since $\operatorname{supp} x_i > \{i\}$ for all i, we can apply Proposition 6 with the constant M on any n blocks $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ of (x_i) to find that for all n-tuples $(b_1, \ldots, b_n)$ of scalars,

$$\frac{1}{2M}\sum_{k=1}^{n}|b_k| \le \Big\|\sum_{k=1}^{n} b_k x_{i_k}\Big\| \le M\sum_{k=1}^{n}|b_k|.$$

We thus find that for any n, (x_i) (and hence (y_i)) contains a subsequence of length n which is equivalent to the standard basis of l_1^n. Complete each of these finite sequences to infinite subsequences. Recall that (y_i) is equivalent to all of them, so for all n, the first n elements of (y_i) are equivalent to the standard basis of l_1^n. Thus (y_i) is equivalent to the standard basis of l_1 and we know this cannot happen. ∎

Corollary 9. *Tsirelson's space T does not contain c_0 or l_p for $1 < p < \infty$.*

Proof. The unit vector bases of these spaces are normalised (hence seminormalised) subsymmetric basic sequences. ∎

4. Banach spaces inspired by Tsirelson's space

Tsirelson's example marked a bit of a turning point in the theory of Banach spaces. It is important not only because it settled a long-standing open problem, but because the construction that underlied it turned out to be modifiable in various ways to tackle other problems. In this section we shall give a brief survey of some "Tsirelson-inspired" Banach spaces. The aim of this section is breadth rather than rigour, so we shall not give proofs as we did in the previous section for Tsirelson's space. Instead, we are more interested in the constructions themselves, how they build upon each other, and the properties they satisfy.

Schlumprecht's arbitrarily distortable space. Let X be an infinite-dimensional Banach space under the norm $\|\cdot\|$ For $\lambda > 1$, we say that X is λ-*distortable* if there exists an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X such that for each infinite-dimensional subspace Y of X, we have

$$\sup\left\{\frac{|\!|\!|y_1|\!|\!|}{|\!|\!|y_2|\!|\!|} : y_1, y_2 \in Y,\ \|y_1\| = \|y_2\| = 1\right\} \ge \lambda.$$

The norm $|\!|\!|\cdot|\!|\!|$ is called a λ-*distortion* if this is true. The space X is called *distortable* if it is λ-distortable for some $\lambda > 1$, and *arbitrarily distortable* if it is distortable for every $\lambda > 1$. In 1991, T. Schlumprecht proved the existence of an arbitrarily distortable Banach space [14].

Schlumprecht's construction relies on a function $f : [1,\infty) \to [1,\infty)$ satisfying

i) $f(1) = 1$ and $f(x) < x$ for all $x > 1$;
ii) f is strictly increasing to infinity;

iii) $lim_{x\to\infty} f(x)/x^q = 0$ for all $q > 0$;
iv) the function $g(x) = x/f(x)$ is concave for $x \geq 1$; and
v) $f(x)f(y) \geq f(xy)$ for all $x, y \geq 1$.

Of course, we need such a function to actually exist, but it is easily seen that $f(x) = \log_2(x+1)$ does. We then define a sequence $\|\!| \cdot |\!\|_n$ of norms on c_{00} by setting $\|\!|x|\!\|_0 = \|x\|_{l_\infty}$ and for $n \geq 1$ we put

$$\|\!|x|\!\|_n = \max_{k\geq 2} \max \frac{1}{f(k)} \sum_{i=1}^{k} \|\!|E_i x|\!\|_{n-1},$$

where the inner maximum is over all subsets $E_1 < E_2 < \cdots < E_k$ of positive integers (unlike in Tsirelson's example, we do not need $k \leq E_1$ this time) for which $E_i x$ is nonzero for some $1 \leq i \leq k$. Schlumprecht's space X is then defined to be the completion of c_{00} with respect to the limit $\| \cdot \|$ of these norms.

Let (e_i) denote the sequence of vectors in c_{00}, where e_i has a 1 at the ith coordinate and zeroes elsewhere. It is an unconditional basis for X just as it was for T. But whereas T contained no seminormalised subsymmetric basic sequences, (e_i) is actually a normalised subsymmetric basis of X. Just as in Tsirelson's example, one can show that the norm on X satisfies the recursive identity

$$\|x\| = \max\Big\{\|x\|_{l_\infty}, \sup_{k\geq 2} \sup \frac{1}{f(k)} \sum_{i=1}^{k} \|E_i x\|\Big\},$$

where the inner supremum runs over all subsets $E_1 < E_2 < \cdots < E_k$ of positive integers. Note that X also does not contain c_0 or l_p for $1 \leq p < \infty$, and the fact that X is arbitrarily distortable is established by the following theorem.

Theorem D. *Let X denote Schlumprecht's space and $\| \cdot \|$ its norm. For any positive integer k and $x \in X$, let*

$$\|x\|_k = \sup \frac{1}{f(k)} \sum_{i=1}^{k} \|E_i x\|.$$

For every k, every $\epsilon > 0$, and every infinite-dimensional subspace Y of X, there are unit vectors $y_1, y_2 \in Y$ such that

$$\|y_1\|_k \geq 1 - \epsilon \qquad \text{and} \qquad \|y_2\|_k \leq \frac{1+\epsilon}{f(k)}.$$

In particular, this means that $\| \cdot \|_k$ is an $f(k)$-distortion on X. ▮

The unconditional basic sequence problem. In Section 2 we defined what it means for a basis or a basic sequence to be unconditional. Both Tsirelson's space and Schlumprecht's space have unconditional bases, and so do all classical Banach spaces. It was unknown for many years whether every infinite-dimensional

Banach space contains an unconditional basic sequence. Soon after Schlumprecht presented his example in the summer of 1991, W. T. Gowers and B. Maurey independently constructed Banach spaces without an unconditional basic sequences. They subsequently discovered that both examples were fundamentally the same, and ended up publishing jointly [7].

Recall that we say that X is the *direct sum* of Y and Z and write $X = Y \oplus Z$ if the map $Y \times Z \to X$ sending $(y, z) \mapsto y + z$ is a linear homeomorphism. An infinite-dimensional Banach space X is said to be *decomposable* if X can be written as $X = Y \oplus Z$ where both Y and Z are also infinite-diimensional. An infinite-dimensional Banach space X without any decomposable subspaces is called *hereditarily indecomposable*. If a space X has an unconditional basic sequence $(x_i)_{i=1}^\infty$, then letting Y be the closed linear span of $(x_{2i-1})_{i=1}^\infty$ and Z be the closed linear span of $(x_{2i})_{i=1}^\infty$, we see that X is not hereditarily indecomposable. So to produce an infinite-dimensional space without an unconditional basic sequence, it suffices to produce a hereditarily indecomposable one, and that is what Gowers and Maurey did.

Before describing their construction, we need to establish a fair bit of terminology and notation. Let J be a set of positive integers such that if $m, n \in J$ with $m < n$, then $\log\log\log n \ge 4m^2$. We also assume that the smallest element of J is greater than 256. We write J in increasing order as $\{j_1, j_2, \ldots\}$, let K be the subset of elements with odd index, and let L be the subset of elements with even index. We recall here that a sequence $y = (y_i) \in l_1$ defines a linear functional f_y on c_{00} with $\|f_y\| = \|y\|_{l_1}$ given by

$$f_y\bigl((x_i)\bigr) = \sum_{i=1}^k x_i y_i.$$

Any bounded linear functional f on c_{00} can be associated to an element of l_1 by evaluating f at each of the standard basis elements of c_{00}, so the topological dual of c_{00} is l_1.

We say that a sequence $x_1, x_2, \ldots, x_n$ of scalar sequences is *successive* and write $x_1 < x_2 < \ldots < x_n$ if $\operatorname{supp} x_i < \operatorname{supp} x_{i+1}$ for all $1 \le i < n$. Let $f = \log_2(1 + x)$, as in Schlumprecht's example. Let $Q \subseteq$ denote the set of finitely-supported sequences whose coordinates are all rational numbers in the range $[-1, 1]$. Let σ be an injection from the set of finite successive sequences of elements of Q to L such that for all such sequences $x_1, \ldots, x_s$ is such a sequence, $S = \sigma(x_1, \ldots, x_s)$, and $x = x_1 + \cdots + x_s$, then

$$|\operatorname{supp} z| \le \frac{\sqrt{f\bigl(S^{1/40}\bigr)}}{20}.$$

This injection is used to define functionals on any normed space with c_{00} as the underlying set. Let $X = (c_{00}, \|\cdot\|)$ be such a normed space. For $m \in \mathbf{N}$, we let $A_m^*(X)$ be the set of functionals of the form

$$\frac{1}{f(m)} \sum_{i=1}^m x_i^*,$$

where the x_i^* are functionals with $\|x_i^*\| \le 1$ for all i and $x_1^* < \cdots < x_m^*$ (as sequences). Note that members in A_m^* have operator norm at most 1 for any m. For $k \in \mathbf{N}$, let Γ_k^X be the set of sequences $(g_1, \ldots, g_k)$ such that $g_i \in Q$ for all i, $g_1 \in A_{j_{2k}}^*(X)$, and $g_{i+1} \in A_{\sigma(g_1,\ldots,g_i}^*$ for all $1 \le i < k$. We call the elements of Γ_k^X *special sequences*. Now let $B_k^*(X)$ be the set of functionals of the form

$$\frac{1}{\sqrt{f(k)}} \sum_{j=1}^{k} g_j$$

such that $(g_1, \ldots, g_k)$ is a special sequence. An element of $B_k^*(X)$ is called a *special functional*.

The space without an unconditional basic sequence that Gowers and Maurey exhibit is also constructed via a sequence of norms, as Tsirelson's and Schlumprecht's spaces were. Let $\|\cdot\|_0 = \|\cdot\|_{l_\infty}$ and set $X_0 = (c_{00}, \|\cdot\|_0)$. For $n \ge 1$, we let $X_n = (c_{00}, \|\cdot\|_n)$, where $\|x\|_n$ be the maximum of

$$\sup_{m \in \mathbf{N}} \sup \frac{1}{f(m)} \sum_{i=1}^{m} \|E_i x\|_{n-1},$$

where the second supremum is over all sequences of subsets $E_1 < \cdots < E_m$, and

$$\sup_{k \in K} \sup_{g \in B_k^*(X_{N-1})} \sup_{E \subseteq \mathbf{N}} \big|g(Ex)\big|.$$

This is a nondecreasing sequence of norms that is bounded above by the l_1-norm, so it has a limit, which we call $\|\cdot\|$. The completion X of $(c_{00}, \|\cdot\|)$ is the space we're after, though the proof that X is hereditarily indecomposable would run us aground.

An operator $T : X \to Y$ is *strictly singular* if there is some $c > 0$ such that $\|Tx\| \ge c\|x\|$ for all $x \in X$. Gowers and Maurey show that every operator from X to X can be written as a scalar multiple of the identity, plus a strictly singular operator. This implies that there are, in some sense, very few operators on X. A subsequent paper of Gowers and Maurey, published in 1997, describes a more general method of producing Banach spaces whose spaces of operators are small [9]. Recall that an operator is said to be *compact* if it sends bounded sets in the domain to sets with compact closure in the image. Every compact operator is strictly singular. In their original 1993 paper, the authors ask if there exists a Banach space X such that every operator from X to X is a scalar multiple of the identity, plus a compact operator. This question was answered positively S. A. Argyros and R. G. Haydon in 2009; the paper was published in 2011 [1].

The Banach hyperplane problem. In a book published in 1932, S. Banach [2] asked whether an infinite-dimensional Banach space X over the real numbers is always isomorphic to $X \oplus \mathbf{R}$. This amounts to asking whether every infinite-dimensional Banach space X is such that that every subspace of codimension 1 is isomorphic to X itself. Since a subspace of codimension 1 is sometimes called a *hyperplane*, this question came to be known as the hyperplane problem. It turns out that the answer to the question is no, and there is an infinite-dimensional Banach space that is not isomorphic to any of its hyperplanes.

This problem was solved by W. T. Gowers soon after he solved the unconditional basic sequence problem. His construction, which appeared in print in 1994, is extremely similar [8]. We shall thus retain the definitions from the previous subsection and immediately define this space. As the reader probably suspects at this point, the construction is inductive. Set $X_0 = (c_{00}, \|\cdot\|_0)$. For $n \geq 1$, we let $X_n = (c_{00}, \|\cdot\|_n)$, where $\|x\|_n$ be the maximum of

$$\sup_{m\in\mathbf{N}} \sup \frac{1}{f(m)} \sum_{i=1}^{m} \|E_i x\|_{n-1},$$

where the inner supremum is over all sequences of subsets $E_1 < \cdots < E_m$, and

$$\sup_{k\in K} \sup_{g\in B_k^*(X_{N-1})} \bigl|g(x)\bigr|.$$

The notation we use here is not exactly the same as in the original paper, but using the same letters as before makes it clear that the only change from the space without an unconditional basic sequence is the replacement of $\sup_{E\subseteq\mathbf{N}}\bigl|g(Ex)\bigr|$ has changed to the simpler $\bigl|g(x)\bigr|$. The space X is the completion of c_{00} with respect to the limit of the norms $\|x\|_n$ above. Perhaps surprisingly, this new space X has an unconditional basis. It is noted in Gowers' paper that the Gowers–Maurey space of the previous section is also a counterexample to the hyperplane problem, though is is more difficult to prove. The proof that X is not isomorphic to any of its hyperplanes is done by showing that X satisfies the hypothesis of the following lemma, which is attributed to P. G. Casazza.

Lemma C. *If X is a Banach space in which no two equivalent sequences (y_i) and (z_i) satisfy $y_i < z_i < y_{i+1}$ for all i, then X is not isomorphic to any proper subspace.* ▮

References

[1] Spiros Argyros and Richard Geoffrey Haydon, "A hereditarily indecomposable $\mathcal{L}_\infty$ space that solves the scalar-plus compact problem," *Acta Mathematica* **206** (2011), 1–54.

[2] Stefan Banach, *Théorie des opérations linéaires* (Warsaw: Monografje Matematyczne, 1932).

[3] Czesław Bessaga and Aleksander Pełczyński, "On bases and unconditional convergence of series in Banach spaces," *Studia Mathematica* **17** (1958), 151–164.

[4] Béla Bollobás, *Linear Analysis: An Introductory Course,* 2nd edition (Cambridge, England: Cambridge University Press, 1999).

[5] Peter George Casazza and Thaddeus Joseph Shura, *Tsirelson's Space* (New York: Springer-Verlag, 1989).

[6] Tadeusz Figiel and William Buhmann Johnson, "A uniformly convex Banach space which contains no l_p," *Compositio Mathematica* **29** (1974), 179–190.

[7] William Timothy Gowers and Bernard Maurey, "The unconditional basic sequence problem," *Journal of the American Mathematical Society* **6** (1993), 851–874.

[8] William Timothy Gowers, "A solution to Banach's hyperplane problem," *Bulletin of the London Mathematical Society* **26** (1996), 523–530.

[9] William Timothy Gowers and Bernard Maurey, "Banach spaces with small spaces of operators," *Mathematische Annalen* **307** (1997), 543–568.

[10] Lorenz Halbeisen and Norbert Hüngerbuhler, "The cardinality of Hamel bases of Banach spaces," *East-West Journal of Mathematics* **2** (2000), 153–159.

[11] Robert Clarke James, "Uniformly non-square Banach spaces," *Annals of Mathematics* **80** (1964), 542–550.

[12] Joram Lindenstrauss and Lior Tzafriri, *Classical Banach Spaces, Vol.* I (New York: Springer-Verlag, 1977).

[13] Martin Schechter, *Principles of Functional Analysis,* 2nd edition (Providence, Rhode Island: American Mathematical Society, 2002).

[14] Thomas Schlumprecht, "An arbitrarily distortable Banach space," *Israel Journal of Mathematics* **76** (1991), 81–95.

[15] Boris Semyonovich Tsirelson, "Not every Banach space contains an imbedding of l_p or c_0," *Functional Analysis and Applications* **8** (1974), 138–141.