# The Probabilistic Method

exercise solutions by

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14 February 2022

## 1. The Basic Method

**Exercise 1.1.** *Prove that if there is a real $p$, with $0 \le p \le 1$ such that*

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

*then the Ramsey number $r(k,t)$ satisfies $r(k,t) > n$. Using this, show that*

$$r(4,t) \ge \Omega\bigl(t^{3/2}/(\ln t)^{3/2}\bigr).$$

*Proof.* We follow the proof of Proposition 1.1.1 in the book. We consider a random graph on $n$ vertices, where each edge is present with probability $p$. Let $K$ be the event that there is a clique of size $k$ in the graph, and let $I$ be the event that there is an independent set of size $t$ in the graph. By the union bound,

$$\mathbf{P}\{K \cup I\} \le \mathbf{P}\{K\} + \mathbf{P}\{I\} \le \sum_{|S|=k} p^{\binom{k}{2}} + \sum_{|S|=t} (1-p)^{\binom{t}{2}} = \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1.$$

This means that $\mathbf{P}\{\neg K \cap \neg I\} > 0$ and since the sample space is finite, there exists a graph on $n$ vertices with no clique of size $k$ and no independent set of size $t$ and therefore $r(k,t) > n$.

Next we show that $r(4,t) > \bigl(t/(e \ln t)\bigr)^{3/2}$ for large enough $t$. Note that

$$\binom{n}{4} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} \le n^4 p^6 + \frac{e^t n^t}{t^t} (1-p)^{t^2/4},$$

by the inequalities

$$\frac{n^k}{k^k} \le \binom{n}{k} \le \frac{e^k n^k}{k^k}.$$

Setting $n = t^{3/2}/(e \ln t)^{3/2}$, we have

$$\begin{aligned}
n^4 p^6 + \frac{e^t n^t}{t^t} (1-p)^{t^2/4} &= \left(\frac{tp}{e \ln t}\right)^6 + \frac{e^t t^{3t/2}}{t^t e^{3t/2} (\ln t)^{3t/2}} (1-p)^{t^2/4} \\
&= \left(\frac{tp}{e \ln t}\right)^6 + \frac{t^{t/2}}{e^{t/2} (\ln t)^{3t/2}} (1-p)^{t^2/4} \\
&\le \left(\frac{tp}{e \ln t}\right)^6 + \left(\frac{t(1-p)^{t/2}}{e (\ln t)^3}\right)^{t/2} \\
&\le \left(\frac{tp}{e \ln t}\right)^6 + \left(\frac{t}{e^{pt/2+1} (\ln t)^3}\right)^{t/2},
\end{aligned}$$

where in the last line we used the inequality $1 - p \le e^{-p}$. Choosing $p = 2 \ln t / t$, we simply need $t$ large enough such that

$$\left(\frac{t}{e^{\ln t + 1} (\ln t)^3}\right)^{t/2} = \left(\frac{1}{e (\ln t)^3}\right)^{t/2} < 1 - \left(\frac{2}{e}\right)^6,$$

which can be done since the left-hand side goes to 0. ∎

**Exercise 1.2.** *Suppose $n \geq 4$ and let $H$ be an $n$-uniform hypergraph with at most $4^{n-1}/3^n$ edges. Prove that there is a colouring of the vertices of $H$ by 4 colours so that in every edge all 4 colours are represented.*

*Proof.* Let each vertex of $H$ be independently given one of the four colours uniformly at random. (If $H$ is infinite, it does not matter what colour we give to vertices that do not appear in any edge, so it suffices to consider $H$ finite, which makes the sample space finite.) Given some edge $e$ of $H$ with $n$ vertices, there are $4^n$ total ways that $e$ may be coloured, and for each of the four colours, $3^n$ total ways that $e$ may be coloured using only the other three colours. Let $K(e)$ denote the event that $e$ does not contain all four colours. By the inclusion-exclusion principle,

$$\mathbf{P}\big\{K(e)\big\} = 4 \cdot 3^n - 6 \cdot 2^n + 4$$

Since $6 \cdot 2^n \geq 96 > 4$, the probability that a given edge *does not* contain all four colours is (much) less than $3^n/4^{n-1}$. By the union bound,

$$\mathbf{P}\Big\{\bigcup_{e \in E(H)} K(e)\Big\} \leq \sum_{e \in E(H)} \mathbf{P}\big\{K(e)\big\} < \frac{4^{n-1}}{3^n} \cdot \frac{3^n}{4^{n-1}} = 1.$$

Since the sample space is finite this implies that there is some colouring of the vertices of $H$ in which every edge has all four colours. ∎

## 4. The Second Moment

**Exercise 4.1.** *Let $X$ be a random variable taking integral nonnegative values, let $\mathbf{E}\{X^2\}$ denote the expectation of its square, and let $\mathbf{V}\{X\}$ denote its variance. Prove that*

$$\mathbf{P}\{X = 0\} \leq \frac{\mathbf{V}\{X\}}{\mathbf{E}\{X^2\}}.$$

*Proof.* Since $X$ is integer and nonnegative, we have $\mathbf{P}\{X = 0\} = 1 - \mathbf{P}\{X \geq 1\}$ and since $\mathbf{V}\{X\} = \mathbf{E}\{X^2\} - \mathbf{E}\{X\}^2$, to get our result it suffices to show that

$$\mathbf{P}\{X \geq 1\} \geq \frac{\mathbf{E}\{X\}^2}{\mathbf{E}\{X^2\}}.$$

We start by noting that

$$\mathbf{E}\{X\} = \sum_{k=0}^{\infty} k\,\mathbf{P}\{X = k\} = \mathbf{P}\{X \geq 1\} \sum_{k=1}^{\infty} \frac{k\,\mathbf{P}\{X = k\}}{\mathbf{P}\{X \geq 1\}} = \mathbf{P}\{X \geq 1\}\,\mathbf{E}\{X \mid X \geq 1\}.$$

Since the function $x \mapsto x^2$ is convex, we have, by Jensen's inequality,

$$\mathbf{E}\{X\}^2 = \mathbf{P}\{X \geq 1\}^2\,\mathbf{E}\{X \mid X \geq 1\}^2 \leq \mathbf{P}\{X \geq 1\}^2\,\mathbf{E}\{X^2 \mid X \geq 1\} = \mathbf{P}\{X \geq 1\}\,\mathbf{E}\{X^2\}.$$

Dividing both sides by $\mathbf{E}\{X^2\}$ gives us what we want. ∎