Posets and incidence algebras

by

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1. Preliminaries

A *partially ordered set* or *poset* is a pair $(P, \le)$ where P is a ground set and $\le$ is a binary relation on P that satisfies the following three properties.

P1. [*Reflexivity.*] For all $x \in P$, $x \le x$.

P2. [*Antisymmetry.*] For all $x, y \in P$, $x \le y$ and $y \le x$ implies that $x = y$.

P3. [*Transitivity.*] For all $x, y, z \in P$, if $x \le y$ and $y \le z$, then $x \le z$.

When $x \le y$, we write $y \ge x$; when $x \le y$ and $x \ne y$, we may write $x < y$ and $y > x$. Since it is cumbersome to always to refer to a poset as a pair, will also allow ourselves to denote the entire poset by its ground set P when no confusion can arise.

For our purposes, a *subposet* of a poset $(P, \le)$ is obtained by taking a subset $S \subseteq P$ and declaring that $x \le y$ in $(S, \le)$ if and only if $x \le y$ in $(P, \le)$ (this is sometimes called an *induced* subposet). An *order ideal* is a subset I of P such that if $x \in I$ and $y < x$, then $y \in I$. Dually, a *filter* is a subset F of P such that if $x \in I$ and $y > x$, then $y \in I$. The *principal order ideal* $\downarrow x$ of an element $x \in P$ is the set of all $y \in P$ such that $y \le x$; similarly, the *order ideal generated by* $X = \{x_1, x_2, \ldots, x_k\} \subseteq P$ is the set $\downarrow X$ of all $y \in P$ such that $y \le x_i$ for some $1 \le i \le k$. (One defines the dual notion of a *principal filter* $\uparrow x$ and *filter* $\uparrow X$ *generated by* X in the obvious way.)

Whenever $x \le y$, we can define the *interval* $[x, y]$ to be the subposet $\{z \in P : x \le z \le y\}$. If the interval $[x, y]$ is the two-element set $\{x, y\}$, then we say that y *covers* x. Given two elements $x, y \in P$, we define an *upper bound* to be an element $z \in P$ such that $z \ge x$ and $z \ge y$. Similarly, we can define a *lower bound* of two elements, and by induction, we obtain a notion of upper and lower bounds of any finite subset of P. If an upper bound z of x and y is such that for any upper bound w of x and y, we have $z \le w$, then z is said to be the *least upper bound* or *join* of x and y and we may denote $z = x \vee y$. Dually, we define the *greatest lower bound* or *meet* $x \wedge y$ of two elements x and y. In either case, the existence of such an element implies its uniqueness.