Notes on Ergodic Group Theory

by

Marcel K. Goh

Disclaimer. These notes are based on a course given by Prof. Mikaël Pichot at McGill University in Winter 2020. Where I found it necessary, I have supplied extra material (e.g. definitions). My notes are not officially endorsed in any way and I may have introduced errors. Please contact me if you find any such error.

1. PRELIMINARY NOTIONS

Let X be a topological space. The smallest σ-algebra containing all the open sets in X is called the *Borel σ-algebra* and members of this collection are called *Borel sets*. A map f between topological spaces X and Y is called *Borel* provided that for every open set V in Y, $f^{-1}(V)$ is a Borel set in X. The *graph* of a Borel map f is the set

$$\mathrm{graph}(f) = \{(x, f(x)) : x \in X\}.$$

[Borelness and other things.] A topological space X is called a *Polish space* if it is homeomorphic to a separable complete metric space. In other words, it is a separable completely metrisable topological space. Examples of Polish spaces are $[0, 1]$, $(0, 1)$, manifolds, and Hilbert spaces.

Let G be a group and let X be a Polish space. We define a *Borel action* to be a Borel map from $G \times X \to X$ (where pairs (s, x) are usually denoted sx) such that $ex = x$ for all $x \in X$ and $s(tx) = (st)x$ for all $s, t \in G, x \in X$. The set

$$Gx = \{y \in X : y = sx \text{ for some } s \in G\}$$

is called the *orbit* of $x \in X$. The partition of X into orbits is called the *orbit equivalence relation* of the action.

REFERENCES

The contents of this document are heavily based on MATH 596 lectures given by Mikaël Pichot at McGill University in Winter 2020.