

# Answers to Selected Exercises in *Principles of Mathematical Analysis*\*

Solutions by

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## CHAPTER 2. BASIC TOPOLOGY

1. Prove that the empty set is a subset of every set.

*Proof.* Let  $S$  be an arbitrary set. Then every element of  $\emptyset$  is an element of  $S$ . So  $\emptyset \subseteq S$ . ■

2. A complex number  $z$  is said to be algebraic if there are integers  $a_0, \dots, a_n$ , not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer  $N$ , there are only finitely many equations with

$$n + |a_0| + |a_1| + \dots + |a_n| = N$$

*Proof.* Let  $A$  denote the set of all algebraic numbers and partition  $A$  as follows: For each  $z \in A$ , calculate the positive integer  $N$  that corresponds to its equation and place it in a set  $E_N$ . So

$$A = \bigcup_{N \in \mathbf{N}} E_N,$$

where each  $E_N$  is finite. Then apply the corollary of Theorem 2.12 to find that  $A$  is countable. ■

3. Prove that there exist real numbers which are not algebraic.

*Proof.* Let  $A$  denote the set of all algebraic numbers and suppose, towards a contradiction, that all real numbers are algebraic. Then  $\mathbf{R} \subset A$ . But the set of real numbers is uncountable and we know from Problem 2 that  $A$  is countable. The contradiction completes the proof. ■

4. Is the set of all irrational real numbers countable?

The answer is no.

*Proof.* If  $\mathbf{R} \setminus \mathbf{Q}$  is countable, then  $\mathbf{R} = \mathbf{Q} \cup (\mathbf{R} \setminus \mathbf{Q})$ , is countable, which we know to be false. ■

5. Construct a bounded set of real numbers with exactly three limit points.

$\{\frac{1}{n} : n \in \mathbf{N}\} \cup \{\frac{1}{n} + 2 : n \in \mathbf{N}\} \cup \{\frac{1}{n} + 4 : n \in \mathbf{N}\}$  has limit points 0, 2, and 4. It is bounded above by 6 and below by 0.

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\* Walter Rudin. 1986. *Principles of Mathematical Analysis*, 3rd ed. McGraw-Hill, Inc., USA.