The Probabilistic Method

exercise solutions by

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Note. A star indicates a starred problem in the text; these are supposed to be harder. I have decided to put my work here even when I haven't solved the problem yet, so that when I come back later I can see what I've tried. I won't put the black square ▮ until I think the proof is actually done. Also I wrote log everywhere to mean $\log_e$ even though the text uses ln.

1. The Basic Method

Exercise 1.1. *Prove that if there is a real p, with $0 \le p \le 1$ such that*

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1,$$

then the Ramsey number $r(k,t)$ satisfies $r(k,t) > n$. Using this, show that

$$r(4,t) \ge \Omega\bigl(t^{3/2}/(\log t)^{3/2}\bigr).$$

Proof. We follow the proof of Proposition 1.1.1 in the book. We consider a random graph on n vertices, where each edge is present with probability p. Let K be the event that there is a clique of size k in the graph, and let I be the event that there is an independent set of size t in the graph. By the union bound,

$$\mathbf{P}\{K \cup I\} \le \mathbf{P}\{K\} + \mathbf{P}\{I\} \le \sum_{|S|=k} p^{\binom{k}{2}} + \sum_{|S|=t} (1-p)^{\binom{t}{2}} = \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1.$$

This means that $\mathbf{P}\{\neg K \cap \neg I\} > 0$ and since the sample space is finite, there exists a graph on n vertices with no clique of size k and no independent set of size t and therefore $r(k,t) > n$.

Next we show that $r(4,t) > \bigl(t/(e\log t)\bigr)^{3/2}$ for large enough t. Note that

$$\binom{n}{4} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} \le n^4 p^6 + \frac{e^t n^t}{t^t}(1-p)^{t^2/4},$$

by the inequalities

$$\frac{n^k}{k^k} \le \binom{n}{k} \le \frac{e^k n^k}{k^k}.$$

Setting $n = t^{3/2}/(e\log t)^{3/2}$, we have

$$\begin{aligned} n^4 p^6 + \frac{e^t n^t}{t^t}(1-p)^{t^2/4} &= \left(\frac{tp}{e\log t}\right)^6 + \frac{e^t t^{3t/2}}{t^t e^{3t/2}(\log t)^{3t/2}}(1-p)^{t^2/4} \\ &= \left(\frac{tp}{e\log t}\right)^6 + \frac{t^{t/2}}{e^{t/2}(\log t)^{3t/2}}(1-p)^{t^2/4} \\ &\le \left(\frac{tp}{e\log t}\right)^6 + \left(\frac{t(1-p)^{t/2}}{e(\log t)^3}\right)^{t/2} \\ &\le \left(\frac{tp}{e\log t}\right)^6 + \left(\frac{t}{e^{pt/2+1}(\log t)^3}\right)^{t/2}, \end{aligned}$$

where in the last line we used the inequality $1-p \le e^{-p}$. Choosing $p = 2\log t/t$, we simply need t large enough such that

$$\left(\frac{t}{e^{\log t+1}(\log t)^3}\right)^{t/2} = \left(\frac{1}{e(\log t)^3}\right)^{t/2} < 1 - \left(\frac{2}{e}\right)^6,$$

which can be done since the left-hand side goes to 0. ∎

Exercise 1.2. *Suppose $n \ge 4$ and let H be an n-uniform hypergraph with at most $4^{n-1}/3^n$ edges. Prove that there is a colouring of the vertices of H by 4 colours so that in every edge all 4 colours are represented.*

Proof. Let each vertex of H be independently given one of the four colours uniformly at random. (If H is infinite, it does not matter what colour we give to vertices that do not appear in any edge, so it suffices to consider H finite, which makes the sample space finite.) Given some edge e of H with n vertices, there are 4^n total ways that e may be coloured, and for each of the four colours, 3^n total ways that e may be coloured using only the other three colours. Let $K(e)$ denote the event that e does not contain all four colours. By the inclusion-exclusion principle,

$$\mathbf{P}\big\{K(e)\big\} = 4\cdot 3^n - 6\cdot 2^n + 4$$

Since $6\cdot 2^n \ge 96 > 4$, the probability that a given edge *does not* contain all four colours is (much) less than $3^n/4^{n-1}$. By the union bound,

$$\mathbf{P}\Big\{\bigcup_{e\in E(H)} K(e)\Big\} \le \sum_{e\in E(H)} \mathbf{P}\big\{K(e)\big\} < \frac{4^{n-1}}{3^n}\cdot\frac{3^n}{4^{n-1}} = 1.$$

Since the sample space is finite this implies that there is some colouring of the vertices of H in which every edge has all four colours. ∎

***Exercise 1.3.** *Prove that for two independent, identically distributed real random variables X and Y,*

$$\mathbf{P}\big\{|X-Y|\le 2\big\} \le 3\,\mathbf{P}\big\{|X-Y|\le 1\big\}.$$

Proof. [Not done. Need to use independence somehow. I think by symmetry it is enough to show that $\mathbf{P}\{1 < X-Y \le 2\} \le \mathbf{P}\big\{|X-Y|\le 1\big\}$ or $\mathbf{P}\big\{X-Y>1 \,\big|\, |X-Y|\le 2\big\} \le 1/3$. In the second expression, X and Y are no longer independent.]

***Exercise 1.4.** *Let $G=(V,E)$ be a graph with n vertices and minimum degree $\delta > 10$. Prove that there is a partition of V into two disjoint subsets A and B so that $|A| \le O\big((n\log n)/\delta\big)$, and each vertex of B has at least one neighbour in A and at least one neighbour in B.*

Proof. We follow the construction of the dominating set from the proof of Theorem 1.2.2, since the required A here is a dominating set with some extra conditions. Let p be chosen later and let X be a random set of vertices obtained by independently selecting each $v \in V$ with probability p. Then as in the proof from the textbook, we let Y be the set of all $v \in V\setminus X$ that have no neighbours in X. At this point $X\cup Y$ is a dominating set, but we are not yet done constructing A, since there may still be elements in $V\setminus(X\cup Y)$ all of whose neighbours belong to $X\cup Y$. Let $Z \subseteq V\setminus(X\cup Y)$ denote all of these elements. For any given $v\in V$, we have $\mathbf{P}\{v\in X\} = p$ and

$$\mathbf{P}\{v\in Y\} = (1-p)^{\deg(v)+1} \le (1-p)^{\delta+1} \le e^{-p(\delta+1)},$$

since for $v \in Y$ we need v itself as well as all $\deg(v)$ of its neighbours to not be in X. Lastly,

$$\mathbf{P}\{v\in Z\} = (1-p)\prod_{w\in N(v)}\big(p+(1-p)^{\deg(w)+1}\big) \le (1-p)\big(p+(1-p)^{\delta+1}\big)^{\delta} \le (1-p)(p+e^{-p(\delta+1)})^{\delta}.$$

Now we compute

$$\begin{aligned}\mathbf{E}\big\{|A|\big\} &= \sum_{v\in V} v \in \{X\cup Y\cup Z\} \\ &\le n\big(p+e^{-p(\delta+1)} + (1-p)(p+e^{-p(\delta+1)})^{\delta}\big).\end{aligned}$$

Since $p + e^{-p(\delta+1)} < 1$, we can remove the δ from its exponent for the looser but simpler bound

$$\begin{aligned}\mathbf{E}\{|A|\} &\le np + ne^{-p(\delta+1)} + np - np^2 + ne^{-p(\delta+1)} - npe^{-p(\delta+1)} \\ &= (2-p)(np + ne^{-p(\delta+1)}).\end{aligned}$$

Setting $p = \log(\delta+1)/(\delta+1)$ just as in the dominating set proof, we have

$$e^{-p(\delta+1)} = \frac{1}{\delta+1}$$

and

$$\mathbf{E}\{|A|\} \le \left(2 - \frac{\log(\delta+1)}{\delta+1}\right)\left(n\frac{\log(\delta+1)}{\delta+1} + \frac{n}{\delta+1}\right) = 2n\frac{\log(\delta+1)}{\delta+1} + o\left(\frac{n\log\delta}{\delta}\right),$$

which has the required asymptotics. It remains to choose an A with $|A|$ at least this average. ▮

***Exercise 1.5.** *Let $G = (V, E)$ be a graph on $n \ge 10$ vertices and suppose that if we add to G any edge not in G, the number of copies of a complete graph on 10 vertices in it increases. Show that the number of edges of G is at least $8n - 36$.*

Proof. The figure $8n - 36$ in the conclusion obscures the fact that we should use Theorem 1.3.3. In fact,

$$8n - 36 = \frac{n^2 - n - n^2 + 17n - 72}{2} = \frac{n(n-1) - (n-8)(n-9)}{2} = \binom{n}{2} - \binom{n-8}{2}.$$

Let $F = V^{(2)} \setminus E = \{A_1, A_2, \ldots, A_h\}$ be the set of non-edges in G. Since the addition of any non-edge causes the number of K_{10} subgraphs to increase, we know that for each A_i there exists a set X_i of size 8 and disjoint from A_i such that the subgraph induced on the vertices $A_i \cup X_i$ would be a K_{10} if A_i happened to be in E (which it isn't). Set $B_i = V \setminus (A_i \cup S_i)$. Clearly, $A_i \cap B_i = \emptyset$ for all i. Also, since $A_i \cup S_i$ is complete except for one missing edge A_i, for any $j \ne i$, we have $A_j \not\subseteq A_i \cup S_i$. This means that $A_j \cap B_i \ne \emptyset$ for all $j \ne i$. Hence $\big\{(A_i, B_i)\big\}_{i=1}^{h}$ is a $(2, n-10)$-system as defined in the chapter, and by Theorem 1.3.3, $|F| = h \le \binom{2+n-10}{2}$. We conclude that $|E| \ge \binom{n}{2} - \binom{n-8}{2}$. ▮

Exercise 1.8. *Let F be a finite collection of binary strings of finite lengths and assume no member of F is a prefix of another one. Let N_i denote the number of strings of length i in F. Prove that*

$$\sum_{i\ge1} \frac{N_i}{2_i} \le 1.$$

Proof. This inequality looks an awful bit like the LYM inequality, so our proof is informed by the proof of that statement. Let m be the maximum length of any string in F (such a maximum exists because F is finite). Let $b_1, b_2, \ldots, b_m$ be a sequence of independent random variables with $\mathbf{P}\{b_i = 0\} = \mathbf{P}\{b_i = 1\} = 1/2$ for all $1 \le i \le m$. Consider the set of strings

$$C = \{b_1, b_1b_2, \ldots, b_1b_2\cdots b_m\}.$$

This is a random set, so we may study the random quantity $|C \cap F|$. Note that C contains exactly one bitstring of each length, and every string of length i has an equal chance of being in C, namely 2^{-i}. So

$$\mathbf{E}\{|C \cap F|\} = \sum_{s\in F} \mathbf{P}\{s \in C\} = \sum_{s\in F} 2^{-|s|} = \sum_{i\ge1} \frac{N_i}{2^i},$$

where $|s|$ is the length of the string s. In the last equality we simply grouped strings of the same length together. On the other hand, $|C \cap F|$ cannot possibly be greater than 1, since for any two members of C, one must be a prefix of the other. So $\mathbf{E}\{|C \cap F|\} \le 1$ as well, and this observation completes the proof. ▮

2. Linearity of Expectation

Exercise 2.1. *Suppose $n \geq 2$ and let $H = (V, E)$ be an n-uniform hypergraph with $|E| = 4^{n-1}$ edges. Show that there is a colouring of V by 4 colours so that no edge is monochromatic.*

Proof. Let ϕ be a random function from V to $[4]$, so that all 4^n possible colourings are equally likely. Let X be the number of monochromatic edges in H under this colouring; that is,

$$X = \sum_{e \in E} \mathbf{1}_{[e \text{ is monochromatic}]} \cdot$$

The probability that a given edge $e = \{v_1, \ldots, v_n\}$ is monochromatic is $1/4^{n-1}$, since $\phi(v_1)$ can be anything, but then $\phi(v_2)$ through to $\phi(v_n)$ each have a $1/4$ chance of matching $\phi(v_1)$. So

$$\mathbf{E}\{X\} = \sum_{e \in E} \mathbf{P}\{e \text{ is monochromatic}\} = \frac{4^{n-1}}{4^{n-1}} = 1.$$

We can produce a colouring with 4^{n-1} monochromatic edges by setting $\phi(v) = 1$ for all $v \in V$, so $\mathbf{P}\{X > 1\} > 0$. Thus for $\mathbf{E}\{X\}$ to equal 1, we must also have $\mathbf{P}\{X < 1\} > 0$, and since X is nonnegative, this means that $\mathbf{P}\{X = 0\} > 0$ and there exists some colouring with no monochromatic edges. ∎

Exercise 2.7. *Let $\mathcal{F}$ be a family of subsets of $[n] = \{1, 2, \ldots, n\}$ and suppose there are no $A, B \in \mathcal{F}$ satisfying $A \subseteq B$. Let $\sigma \in S_n$ be a random permutation of the elements of $[n]$ and consider the random variable*

$$X = \left|\left\{i : \{\sigma(1), \sigma(2), \ldots, \sigma(i)\} \in \mathcal{F}\right\}\right|.$$

By considering the expectation of X, prove that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof. This is the proof that I said Exercise 1.8 reminded me of. Let $S_i = \{\sigma(1), \ldots, \sigma(i)\}$ and let

$$C = \{\emptyset, S_1, S_2, \ldots, S_n\}.$$

If there are $i < j$ such that S_i and S_j are both in $\mathcal{F}$, then $\mathcal{F}$ violates the condition that there are no $A, B \in \mathcal{F}$ with $A \subseteq B$. So $X \leq 1$. On the other hand, C contains exactly one set of size i for each $0 \leq i \leq n$ and the probability that S_i is a given element of $X^{(i)}$ is the same. So letting N_i be the number of sets of size i in $\mathcal{F}$, we have

$$1 \geq \mathbf{E}\{X\} = \sum_{A \in \mathcal{F}} \binom{n}{|A|}^{-1} = \sum_{i=0}^{n} N_i \binom{n}{i}^{-1} \geq \frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}},$$

where the last inequality holds because $\binom{n}{k}$ is maximised when $k = \lfloor n/2 \rfloor$ and because the sum of the N_i is $|\mathcal{F}|$. ∎

3. Alterations

Exercise 3.1. *As shown in Section 3.1, the Ramsey number $R(k, k)$ satisfies*

$$R(k, k) > n - \binom{n}{k} 2^{1 - \binom{k}{2}}$$

for every integer n. Conclude that

$$R(k, k) \geq \bigl(1 - o(1)\bigr) \frac{k}{e} 2^{k/2}.$$

Proof. We set $n = (k-1)2^{k/2}/e$ in the above, to get the bound

$$R(k, k) > \frac{k}{e} 2^{k/2} - \frac{1}{e} 2^{k/2} - \binom{(k-1)2^{k/2}/e}{k} 2^{1 - \binom{k}{2}}.$$

Now we apply the inequality $\binom{n}{k} \le e^k n^k / k^k$, which yields

$$\begin{aligned} R(k,k) &> \frac{k}{e}2^{k/2} - \frac{1}{e}2^{k/2} - \frac{e^k(k-1)^k 2^{k^2/2}/e^k}{k^k} 2^{1-k^2/2+k/2} \\ &= \frac{k}{e}2^{k/2} - \frac{1}{e}2^{k/2} - \left(1 - \frac{1}{k}\right)^k 2^{1+k/2}. \end{aligned}$$

Since $(1-1/k)^k \to 1/e$ as $k \to \infty$,

$$R(k,k) > \frac{k}{e}2^{k/2}\left(1 - \frac{1}{k} - \frac{2e(1-1/k)^k}{k}\right) = \frac{k}{e}2^{k/2}\big(1 - o(1)\big). \quad ▮$$

Exercise 3.3. *Prove that every 3-uniform hypergraph with n vertices and $m \ge n/3$ edges contains an independent set (i.e., a set of vertices containing no edges) of size at least*

$$\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}.$$

Proof. We mirror the proof of Theorem 3.2.1. Let $(X \cap \mathcal{F})$ be a 3-uniform hypergraph with $|X| = n$ and $|\mathcal{F}| = m \ge n/3$. Let $S \subseteq X$ be a random subset obtained by independently putting each $x \in X$ in S with probability p, to be chosen later. Of course, $\mathbf{E}\big\{|X|\big\} = np$. What is the expected number of edges contained in S? Well, each edge $\{x, y, z\} \in \mathcal{F}$ has probability p^3 of being contained in S, so we expect mp^3 edges in the subgraph induced by S. If Y is the set of these edges, we now want to choose p to minimise the quantity $\mathbf{E}\big\{|X| - |Y|\big\} = np - mp^3$. The derivative is $n - 3mp^2$ so we ought to set $p = \sqrt{n/3m}$, obtaining

$$\mathbf{E}\big\{|X| - |Y|\big\} = \frac{n^{3/2} - m\sqrt{n}}{\sqrt{3m}}.$$

Since $m \ge n/3$, we have

$$\mathbf{E}\big\{|X| - |Y|\big\} \ge \frac{n^{3/2}}{\sqrt{3m}} - \frac{n^{3/2}}{3\sqrt{3m}} = \frac{2n^{3/2}}{3\sqrt{3m}}.$$

This means there exists some set $S \subseteq X$ with the number of vertices minus the number of edges at least $2n^{3/2}/\big(3\sqrt{3m}\big)$. Removing one vertex from each edge, we arrive at an independent set of vertices S' with cardinality at least this quantity. ▮

4. The Second Moment

Exercise 4.1. *Let X be a random variable taking integral nonnegative values, let $\mathbf{E}\{X^2\}$ denote the expectation of its square, and let $\mathbf{V}\{X\}$ denote its variance. Prove that*

$$\mathbf{P}\{X = 0\} \le \frac{\mathbf{V}\{X\}}{\mathbf{E}\{X^2\}}.$$

Proof. Since X is integer and nonnegative, we have $\mathbf{P}\{X = 0\} = 1 - \mathbf{P}\{X \ge 1\}$ and since $\mathbf{V}\{X\} = \mathbf{E}\{X^2\} - \mathbf{E}\{X\}^2$, to get our result it suffices to show that

$$\mathbf{P}\{X \ge 1\} \ge \frac{\mathbf{E}\{X\}^2}{\mathbf{E}\{X^2\}}.$$

We start by noting that

$$\mathbf{E}\{X\} = \sum_{k=0}^{\infty} k\,\mathbf{P}\{X = k\} = \mathbf{P}\{X \ge 1\} \sum_{k=1}^{\infty} \frac{k\,\mathbf{P}\{X = k\}}{\mathbf{P}\{X \ge 1\}} = \mathbf{P}\{X \ge 1\}\,\mathbf{E}\{X \mid X \ge 1\}.$$

Since the function $x \mapsto x^2$ is convex, we have, by Jensen's inequality,

$$\mathbf{E}\{X\}^2 = \mathbf{P}\{X \geq 1\}^2\,\mathbf{E}\{X \mid X \geq 1\}^2 \leq \mathbf{P}\{X \geq 1\}^2\,\mathbf{E}\{X^2 \mid X \geq 1\} = \mathbf{P}\{X \geq 1\}\,\mathbf{E}\{X^2\}.$$

Dividing both sides by $\mathbf{E}\{X^2\}$ gives us what we want. ▮

Exercise 4.4. *Let X be a random variable with expectation $\mathbf{E}\{X\} = 0$ and variance $\mathbf{V}\{X\} = \sigma^2$. Prove that for all $\lambda > 0$,*

$$\mathbf{P}\{X \geq \lambda\} \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

Proof. Let $\lambda > 0$. Note that if $X \geq \lambda$, then X is also positive so in particular, for any $a > 0$, $X + a$ and $\lambda + a$ are both positive. This implies that $(X + a)^2 \geq (\lambda + a)^2$. Note that since $\mathbf{V}\{X\} = \mathbf{E}\{X^2\} - \mathbf{E}\{X\}^2 = \sigma^2$ and $\mathbf{E}\{X\} = 0$, we have $\mathbf{E}\{X^2\} = \sigma^2$, which in turn implies that

$$\mathbf{E}\big\{(X + a)^2\big\} = \mathbf{E}\{X^2\} + 2a\,\mathbf{E}\{X\} + \mathbf{E}\{a^2\} = \sigma^2 + a^2.$$

Now we take the infimum over all $a > 0$ and apply Markov's inequality to get

$$\mathbf{P}\{X \geq \lambda\} \leq \inf_{a>0} \mathbf{P}\big\{(X + a)^2 \geq (\lambda + a)^2\big\} \leq \inf_{a>0} \frac{\sigma^2 + a^2}{(\lambda + a)^2}.$$

We now optimise over $a > 0$. We have

$$\frac{d}{da}\frac{\sigma^2 + a^2}{(\lambda + a)^2} = \frac{(\lambda + a)^2 2a - (\sigma^2 + a^2)2(\lambda + a)}{(\lambda + a)^4},$$

which is zero when $a = \sigma^2/\lambda > 0$. Plugging in this value of a gives

$$\mathbf{P}\{X \geq \lambda\} \leq \frac{\sigma^2 + \sigma^4/\lambda^2}{(\lambda + \sigma^2/\lambda)^2} = \frac{\sigma^2(\lambda + \sigma^2/\lambda)}{\lambda(\lambda + \sigma^2/\lambda)^2} = \frac{\sigma^2}{\sigma^2 + \lambda^2},$$

exactly what we need. ▮

6. Correlation Inequalities

Exercise 6.1. *Let G be a graph and let P denote the probability that a random subgraph of G obtained by picking each edge of G with probability $1/2$ independently is connected (and spanning). Let Q denote the probability that in a random two-colouring of G, where each edge is chosen, randomly and independently, to be either red or blue, the red graph and the blue graph are both connected (and spanning). Is $Q \leq P^2$?*

Solution. The answer is yes. Let $\mathcal{A}$ be the family of subsets $A \subseteq E$ such that $G' = (V, A)$ is connected. Let $\mathcal{C}$ be the family of subsets $C \subseteq E$ such that $G'' = (V, E \setminus C)$ is connected. Adding any edge to $A \in \mathcal{A}$ keeps the set in $\mathcal{A}$, and removing any edge from $C \in \mathcal{C}$ keeps the set in $\mathcal{C}$, so $\mathcal{A}$ is monotone increasing and $\mathcal{C}$ is monotone decreasing. Note that if S is a random subset of edges obtained by taking each edge independently with probability $1/2$, then $\mathbf{P}\{S = A\} = \mathbf{P}\{S = E \setminus A\}$ for any fixed $A \subseteq E$, so we find that $P = \mathbf{P}\{\mathcal{A}\} = \mathbf{P}\{\mathcal{C}\}$. On the other hand, Q is the probability that both S and $E \setminus S$ induce connected subgraphs; i.e., $Q = \mathbf{P}\{S \in \mathcal{A} \cap \mathcal{C}\}$. Applying Proposition 6.3.1 now gives

$$Q = \mathbf{P}\{\mathcal{A} \cap \mathcal{C}\} \leq \mathbf{P}\{\mathcal{A}\}\,\mathbf{P}\{\mathcal{C}\} = P^2,$$

as desired. ▮

Exercise 6.2. *A family of subsets $\mathcal{G}$ is called intersecting if $G_1 \cap G_2 \neq \emptyset$ for all $G_1, G_2 \in \mathcal{G}$. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ be k intersecting families of subsets of $X = [n]$. Prove that*

$$\left|\bigcup_{i=1}^{k} \mathcal{F}_i\right| \leq 2^n - 2^{n-k}.$$

Proof. We do this by induction on k. If $k = 1$, then the upper bound of 2^{n-1} on the size of an intersecting family $\mathcal{F}$ is clear from the fact that $\mathcal{F}$ may contain at most one element of each pair $(A, X \setminus A)$, and there are 2^{n-1} such pairs. For the induction step, we split

$$\bigcup_{i=1}^{k} \mathcal{F}_i = \bigcup_{i=1}^{k-1} \mathcal{F}_i \cup \mathcal{F}_k;$$

call the first term on the right-hand side $\mathcal{A}$ and let $\mathcal{C} = 2^X \setminus \mathcal{F}_k$, so that

$$\bigcup_{i=1}^{k-1} \mathcal{F}_i \cup \mathcal{F}_k = \mathcal{A} \cup \mathcal{F}_k = (\mathcal{A} \cap \mathcal{C}) \cup \mathcal{F}_k.$$

Since we are trying to form an upper bound on this quantity, we might as well assume that $\mathcal{A}$ is maximal; i.e., adding any set to $\mathcal{F}_i$ for $1 \leq i < k$ causes $\mathcal{F}_i$ to no longer be intersecting. The claim is that $\mathcal{A}$ is monotone increasing. To see this, suppose that $A \supseteq B$ and $B \in \mathcal{A}$. This means that $B \in \mathcal{F}_i$ for some $1 \leq i < k$, and so for any $C \in \mathcal{F}_i$, $A \cap C \supseteq B \cap C \neq \emptyset$, meaning that $A \in \mathcal{F}_i \subseteq \mathcal{A}$ as well, otherwise we could add A to $\mathcal{A}$ and contradict maximality. We have actually shown that the union of any m maximal intersecting families is monotone increasing, so in particular $\mathcal{F}_k$ is as well. But this means that $\mathcal{C} = 2^X \setminus \mathcal{F}_k$ is monotone decreasing, since if $C \notin \mathcal{F}_k$ and $B \subseteq C$ with $B \in \mathcal{F}_k$, then $\mathcal{F}_k$ would not be monotone increasing. We are now in a position to apply the cardinality version of Proposition 6.3.1. It gives

$$\left|\bigcup_{i=1}^{k} \mathcal{F}_i\right| = |\mathcal{A} \cap \mathcal{C}| + |\mathcal{F}_k| \leq \frac{|\mathcal{A}| \cdot |\mathcal{C}|}{2^n} + |\mathcal{F}_k| \leq \left(1 - \frac{1}{2^{k-1}}\right)|\mathcal{C}| + |\mathcal{F}_k|,$$

where in the last inequality we used the induction hypothesis $|\mathcal{A}| \leq 2^n - 2^{n-k+1}$. Since $|\mathcal{C}| + |\mathcal{F}_k| = 2^n$ and the coefficient of the latter is larger than the coefficient of the former in the above, the entire expression is largest when $|\mathcal{F}_k| = |\mathcal{C}| = 2^{n-1}$. In this case, we have

$$\left|\bigcup_{i=1}^{k} \mathcal{F}_i\right| \leq \left(1 - \frac{1}{2^{k-1}}\right)2^{n-1} + 2^{n-1} = 2^n - 2^{n-k},$$

completing the induction and the proof. ▮