

# The Probabilistic Method

exercise solutions by

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**Note.** A star indicates a starred problem in the text; these are supposed to be harder. I have decided to put my work here even when I haven't solved the problem yet, so that when I come back later I can see what I've tried. I won't put the black square ■ until I think the proof is actually done. Also I wrote  $\log$  everywhere to mean  $\log_e$  even though the text uses  $\ln$ .

## 1. The Basic Method

**Exercise 1.1.** Prove that if there is a real  $p$ , with  $0 \leq p \leq 1$  such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

then the Ramsey number  $r(k, t)$  satisfies  $r(k, t) > n$ . Using this, show that

$$r(4, t) \geq \Omega(t^{3/2}/(\log t)^{3/2}).$$

*Proof.* We follow the proof of Proposition 1.1.1 in the book. We consider a random graph on  $n$  vertices, where each edge is present with probability  $p$ . Let  $K$  be the event that there is a clique of size  $k$  in the graph, and let  $I$  be the event that there is an independent set of size  $t$  in the graph. By the union bound,

$$\mathbf{P}\{K \cup I\} \leq \mathbf{P}\{K\} + \mathbf{P}\{I\} \leq \sum_{|S|=k} p^{\binom{k}{2}} + \sum_{|S|=t} (1-p)^{\binom{t}{2}} = \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1.$$

This means that  $\mathbf{P}\{\neg K \cap \neg I\} > 0$  and since the sample space is finite, there exists a graph on  $n$  vertices with no clique of size  $k$  and no independent set of size  $t$  and therefore  $r(k, t) > n$ .

Next we show that  $r(4, t) > (t/(e \log t))^{3/2}$  for large enough  $t$ . Note that

$$\binom{n}{4} p^6 + \binom{n}{t} (1-p)^{t^2/4} \leq n^4 p^6 + \frac{e^t n^t}{t^t} (1-p)^{t^2/4},$$

by the inequalities

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{e^k n^k}{k^k}.$$

Setting  $n = t^{3/2}/(e \log t)^{3/2}$ , we have

$$\begin{aligned} n^4 p^6 + \frac{e^t n^t}{t^t} (1-p)^{t^2/4} &= \left( \frac{tp}{e \log t} \right)^6 + \frac{e^t t^{3t/2}}{t^t e^{3t/2} (\log t)^{3t/2}} (1-p)^{t^2/4} \\ &= \left( \frac{tp}{e \log t} \right)^6 + \frac{t^{t/2}}{e^{t/2} (\log t)^{3t/2}} (1-p)^{t^2/4} \\ &\leq \left( \frac{tp}{e \log t} \right)^6 + \left( \frac{t(1-p)^{t/2}}{e (\log t)^3} \right)^{t/2} \\ &\leq \left( \frac{tp}{e \log t} \right)^6 + \left( \frac{t}{e^{pt/2+1} (\log t)^3} \right)^{t/2}, \end{aligned}$$

where in the last line we used the inequality  $1 - p \leq e^{-p}$ . Choosing  $p = 2 \log t / t$ , we simply need  $t$  large enough such that

$$\left( \frac{t}{e^{\log t + 1} (\log t)^3} \right)^{t/2} = \left( \frac{1}{e (\log t)^3} \right)^{t/2} < 1 - \left( \frac{2}{e} \right)^6,$$

which can be done since the left-hand side goes to 0.  $\blacksquare$

**Exercise 1.2.** Suppose  $n \geq 4$  and let  $H$  be an  $n$ -uniform hypergraph with at most  $4^{n-1}/3^n$  edges. Prove that there is a colouring of the vertices of  $H$  by 4 colours so that in every edge all 4 colours are represented.

*Proof.* Let each vertex of  $H$  be independently given one of the four colours uniformly at random. (If  $H$  is infinite, it does not matter what colour we give to vertices that do not appear in any edge, so it suffices to consider  $H$  finite, which makes the sample space finite.) Given some edge  $e$  of  $H$  with  $n$  vertices, there are  $4^n$  total ways that  $e$  may be coloured, and for each of the four colours,  $3^n$  total ways that  $e$  may be coloured using only the other three colours. Let  $K(e)$  denote the event that  $e$  does not contain all four colours. By the inclusion-exclusion principle,

$$\mathbf{P}\{K(e)\} = 4 \cdot 3^n - 6 \cdot 2^n + 4$$

Since  $6 \cdot 2^n \geq 96 > 4$ , the probability that a given edge *does not* contain all four colours is (much) less than  $3^n/4^{n-1}$ . By the union bound,

$$\mathbf{P}\left\{ \bigcup_{e \in E(H)} K(e) \right\} \leq \sum_{e \in E(H)} \mathbf{P}\{K(e)\} < \frac{4^{n-1}}{3^n} \cdot \frac{3^n}{4^{n-1}} = 1.$$

Since the sample space is finite this implies that there is some colouring of the vertices of  $H$  in which every edge has all four colours.  $\blacksquare$

**\*Exercise 1.3.** Prove that for two independent, identically distributed real random variables  $X$  and  $Y$ ,

$$\mathbf{P}\{|X - Y| \leq 2\} \leq 3 \mathbf{P}\{|X - Y| \leq 1\}.$$

*Proof.* [Not done. Need to use independence somehow. I think by symmetry it is enough to show that  $\mathbf{P}\{1 < X - Y \leq 2\} \leq \mathbf{P}\{|X - Y| \leq 1\}$  or  $\mathbf{P}\{X - Y > 1 \mid |X - Y| \leq 2\} \leq 1/3$ . In the second expression,  $X$  and  $Y$  are no longer independent.]

**\*Exercise 1.4.** Let  $G = (V, E)$  be a graph with  $n$  vertices and minimum degree  $\delta > 10$ . Prove that there is a partition of  $V$  into two disjoint subsets  $A$  and  $B$  so that  $|A| \leq O((n \log n)/\delta)$ , and each vertex of  $B$  has at least one neighbour in  $A$  and at least one neighbour in  $B$ .

*Proof.* We follow the construction of the dominating set from the proof of Theorem 1.2.2, since the required  $A$  here is a dominating set with some extra conditions. Let  $p$  be chosen later and let  $X$  be a random set of vertices obtained by independently selecting each  $v \in V$  with probability  $p$ . Then as in the proof from the textbook, we let  $Y$  be the set of all  $v \in V \setminus X$  that have no neighbours in  $X$ . At this point  $X \cup Y$  is a dominating set, but we are not yet done constructing  $A$ , since there may still be elements in  $V \setminus (X \cup Y)$  all of whose neighbours belong to  $X \cup Y$ . Let  $Z \subseteq V \setminus (X \cup Y)$  denote all of these elements. For any given  $v \in V$ , we have  $\mathbf{P}\{v \in X\} = p$  and

$$\mathbf{P}\{v \in Y\} = (1 - p)^{\deg(v)+1} \leq (1 - p)^{\delta+1} \leq e^{-p(\delta+1)},$$

since for  $v \in Y$  we need  $v$  itself as well as all  $\deg(v)$  of its neighbours to not be in  $X$ . Lastly,

$$\mathbf{P}\{v \in Z\} = (1 - p) \prod_{w \in N(v)} (p + (1 - p)^{\deg(w)+1}) \leq (1 - p)(p + (1 - p)^{\delta+1})^\delta \leq (1 - p)(p + e^{-p(\delta+1)})^\delta.$$

Now we compute

$$\begin{aligned} \mathbf{E}\{|A|\} &= \sum_{v \in V} \mathbf{P}\{v \in \{X \cup Y \cup Z\}\} \\ &\leq n(p + e^{-p(\delta+1)} + (1 - p)(p + e^{-p(\delta+1)})^\delta). \end{aligned}$$

Since  $p + e^{-p(\delta+1)} < 1$ , we can remove the  $\delta$  from its exponent for the looser but simpler bound

$$\begin{aligned}\mathbf{E}\{|A|\} &\leq np + ne^{-p(\delta+1)} + np - np^2 + ne^{-p(\delta+1)} - npe^{-p(\delta+1)} \\ &= (2-p)(np + ne^{-p(\delta+1)}).\end{aligned}$$

Setting  $p = \log(\delta+1)/(\delta+1)$  just as in the dominating set proof, we have

$$e^{-p(\delta+1)} = \frac{1}{\delta+1}$$

and

$$\mathbf{E}\{|A|\} \leq \left(2 - \frac{\log(\delta+1)}{\delta+1}\right) \left(n \frac{\log(\delta+1)}{\delta+1} + \frac{n}{\delta+1}\right) = 2n \frac{\log(\delta+1)}{\delta+1} + o\left(\frac{n \log \delta}{\delta}\right),$$

which has the required asymptotics. It remains to choose an  $A$  with  $|A|$  at least this average.  $\blacksquare$

**\*Exercise 1.5.** Let  $G = (V, E)$  be a graph on  $n \geq 10$  vertices and suppose that if we add to  $G$  any edge not in  $G$ , the number of copies of a complete graph on 10 vertices in it increases. Show that the number of edges of  $G$  is at least  $8n - 36$ .

*Proof.* The figure  $8n - 36$  in the conclusion obscures the fact that we should use Theorem 1.3.3. In fact,

$$8n - 36 = \frac{n^2 - n - n^2 + 17n - 72}{2} = \frac{n(n-1) - (n-8)(n-9)}{2} = \binom{n}{2} - \binom{n-8}{2}.$$

Let  $F = V^{(2)} \setminus E = \{A_1, A_2, \dots, A_h\}$  be the set of non-edges in  $G$ . Since the addition of any non-edge causes the number of  $K_{10}$  subgraphs to increase, we know that for each  $A_i$  there exists a set  $X_i$  of size 8 and disjoint from  $A_i$  such that the subgraph induced on the vertices  $A_i \cup X_i$  would be a  $K_{10}$  if  $A_i$  happened to be in  $E$  (which it isn't). Set  $B_i = V \setminus (A_i \cup X_i)$ . Clearly,  $A_i \cap B_i = \emptyset$  for all  $i$ . Also, since  $A_i \cup X_i$  is complete except for one missing edge  $A_i$ , for any  $j \neq i$ , we have  $A_j \not\subseteq A_i \cup X_i$ . This means that  $A_j \cap B_i \neq \emptyset$  for all  $j \neq i$ . Hence  $\{(A_i, B_i)\}_{i=1}^h$  is a  $(2, n-10)$ -system as defined in the chapter, and by Theorem 1.3.3,  $|F| = h \leq \binom{2+n-10}{2}$ . We conclude that  $|E| \geq \binom{n}{2} - \binom{n-8}{2}$ .  $\blacksquare$

**Exercise 1.8.** Let  $F$  be a finite collection of binary strings of finite lengths and assume no member of  $F$  is a prefix of another one. Let  $N_i$  denote the number of strings of length  $i$  in  $F$ . Prove that

$$\sum_{i \geq 1} \frac{N_i}{2^i} \leq 1.$$

*Proof.* This inequality looks an awful bit like the LYM inequality, so our proof is informed by the proof of that statement. Let  $m$  be the maximum length of any string in  $F$  (such a maximum exists because  $F$  is finite). Let  $b_1, b_2, \dots, b_m$  be a sequence of independent random variables with  $\mathbf{P}\{b_i = 0\} = \mathbf{P}\{b_i = 1\} = 1/2$  for all  $1 \leq i \leq m$ . Consider the set of strings

$$C = \{b_1, b_1b_2, \dots, b_1b_2 \cdots b_m\}.$$

This is a random set, so we may study the random quantity  $|C \cap F|$ . Note that  $C$  contains exactly one bitstring of each length, and every string of length  $i$  has an equal chance of being in  $C$ , namely  $2^{-i}$ . So

$$\mathbf{E}\{|C \cap F|\} = \sum_{s \in F} \mathbf{P}\{s \in C\} = \sum_{s \in F} 2^{-|s|} = \sum_{i \geq 1} \frac{N_i}{2^i},$$

where  $|s|$  is the length of the string  $s$ . In the last equality we simply grouped strings of the same length together. On the other hand,  $|C \cap F|$  cannot possibly be greater than 1, since for any two members of  $C$ , one must be a prefix of the other. So  $\mathbf{E}\{|C \cap F|\} \leq 1$  as well, and this observation completes the proof.  $\blacksquare$

## 2. Linearity of Expectation

**Exercise 2.1.** Suppose  $n \geq 2$  and let  $H = (V, E)$  be an  $n$ -uniform hypergraph with  $|E| = 4^{n-1}$  edges. Show that there is a colouring of  $V$  by 4 colours so that no edge is monochromatic.

*Proof.* Let  $\phi$  be a random function from  $V$  to  $[4]$ , so that all  $4^n$  possible colourings are equally likely. Let  $X$  be the number of monochromatic edges in  $H$  under this colouring; that is,

$$X = \sum_{e \in E} \mathbf{1}_{[e \text{ is monochromatic}]}.$$

The probability that a given edge  $e = \{v_1, \dots, v_n\}$  is monochromatic is  $1/4^{n-1}$ , since  $\phi(v_1)$  can be anything, but then  $\phi(v_2)$  through to  $\phi(v_n)$  each have a  $1/4$  chance of matching  $\phi(v_1)$ . So

$$\mathbf{E}\{X\} = \sum_{e \in E} \mathbf{P}\{e \text{ is monochromatic}\} = \frac{4^{n-1}}{4^{n-1}} = 1.$$

We can produce a colouring with  $4^{n-1}$  monochromatic edges by setting  $\phi(v) = 1$  for all  $v \in V$ , so  $\mathbf{P}\{X > 1\} > 0$ . Thus for  $\mathbf{E}\{X\}$  to equal 1, we must also have  $\mathbf{P}\{X < 1\} > 0$ , and since  $X$  is nonnegative, this means that  $\mathbf{P}\{X = 0\} > 0$  and there exists some colouring with no monochromatic edges. ■

**Exercise 2.7.** Let  $\mathcal{F}$  be a family of subsets of  $[n] = \{1, 2, \dots, n\}$  and suppose there are no  $A, B \in \mathcal{F}$  satisfying  $A \subseteq B$ . Let  $\sigma \in S_n$  be a random permutation of the elements of  $[n]$  and consider the random variable

$$X = |\{i : \{\sigma(1), \sigma(2), \dots, \sigma(i)\} \in \mathcal{F}\}|.$$

By considering the expectation of  $X$ , prove that  $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ .

*Proof.* This is the proof that I said Exercise 1.8 reminded me of. Let  $S_i = \{\sigma(1), \dots, \sigma(i)\}$  and let

$$C = \{\emptyset, S_1, S_2, \dots, S_n\}.$$

If there are  $i < j$  such that  $S_i$  and  $S_j$  are both in  $\mathcal{F}$ , then  $\mathcal{F}$  violates the condition that there are no  $A, B \in \mathcal{F}$  with  $A \subseteq B$ . So  $X \leq 1$ . On the other hand,  $C$  contains exactly one set of size  $i$  for each  $0 \leq i \leq n$  and the probability that  $S_i$  is a given element of  $C^{(i)}$  is the same. So letting  $N_i$  be the number of sets of size  $i$  in  $\mathcal{F}$ , we have

$$1 \geq \mathbf{E}\{X\} = \sum_{A \in \mathcal{F}} \binom{n}{|A|}^{-1} = \sum_{i=0}^n N_i \binom{n}{i}^{-1} \geq \frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}},$$

where the last inequality holds because  $\binom{n}{k}$  is maximised when  $k = \lfloor n/2 \rfloor$  and because the sum of the  $N_i$  is  $|\mathcal{F}|$ . ■

### 3. Alterations

**Exercise 3.1.** As shown in Section 3.1, the Ramsey number  $R(k, k)$  satisfies

$$R(k, k) > n - \binom{n}{k} 2^{1 - \binom{k}{2}}$$

for every integer  $n$ . Conclude that

$$R(k, k) \geq (1 - o(1)) \frac{k}{e} 2^{k/2}.$$

*Proof.* We set  $n = (k-1)2^{k/2}/e$  in the above, to get the bound

$$R(k, k) > \frac{k}{e} 2^{k/2} - \frac{1}{e} 2^{k/2} - \binom{(k-1)2^{k/2}/e}{k} 2^{1 - \binom{k}{2}}.$$

Now we apply the inequality  $\binom{n}{k} \leq e^k n^k / k^k$ , which yields

$$\begin{aligned} R(k, k) &> \frac{k}{e} 2^{k/2} - \frac{1}{e} 2^{k/2} - \frac{e^k (k-1)^k 2^{k^2/2} / e^k}{k^k} 2^{1-k^2/2+k/2} \\ &= \frac{k}{e} 2^{k/2} - \frac{1}{e} 2^{k/2} - \left(1 - \frac{1}{k}\right)^k 2^{1+k/2}. \end{aligned}$$

Since  $(1 - 1/k)^k \rightarrow 1/e$  as  $k \rightarrow \infty$ ,

$$R(k, k) > \frac{k}{e} 2^{k/2} \left(1 - \frac{1}{k} - \frac{2e(1 - 1/k)^k}{k}\right) = \frac{k}{e} 2^{k/2} (1 - o(1)). \quad \blacksquare$$

**Exercise 3.3.** Prove that every 3-uniform hypergraph with  $n$  vertices and  $m \geq n/3$  edges contains an independent set (i.e., a set of vertices containing no edges) of size at least

$$\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}.$$

*Proof.* We mirror the proof of Theorem 3.2.1. Let  $(X \cap \mathcal{F})$  be a 3-uniform hypergraph with  $|X| = n$  and  $|\mathcal{F}| = m \geq n/3$ . Let  $S \subseteq X$  be a random subset obtained by independently putting each  $x \in X$  in  $S$  with probability  $p$ , to be chosen later. Of course,  $\mathbf{E}\{|X|\} = np$ . What is the expected number of edges contained in  $S$ ? Well, each edge  $\{x, y, z\} \in \mathcal{F}$  has probability  $p^3$  of being contained in  $S$ , so we expect  $mp^3$  edges in the subgraph induced by  $S$ . If  $Y$  is the set of these edges, we now want to choose  $p$  to minimise the quantity  $\mathbf{E}\{|X| - |Y|\} = np - mp^3$ . The derivative is  $n - 3mp^2$  so we ought to set  $p = \sqrt{n/3m}$ , obtaining

$$\mathbf{E}\{|X| - |Y|\} = \frac{n^{3/2} - m\sqrt{n}}{\sqrt{3m}}.$$

Since  $m \geq n/3$ , we have

$$\mathbf{E}\{|X| - |Y|\} \geq \frac{n^{3/2}}{\sqrt{3m}} - \frac{n^{3/2}}{3\sqrt{3m}} = \frac{2n^{3/2}}{3\sqrt{3m}}.$$

This means there exists some set  $S \subseteq X$  with the number of vertices minus the number of edges at least  $2n^{3/2}/(3\sqrt{3m})$ . Removing one vertex from each edge, we arrive at an independent set of vertices  $S'$  with cardinality at least this quantity.  $\blacksquare$

#### 4. The Second Moment

**Exercise 4.1.** Let  $X$  be a random variable taking integral nonnegative values, let  $\mathbf{E}\{X^2\}$  denote the expectation of its square, and let  $\mathbf{V}\{X\}$  denote its variance. Prove that

$$\mathbf{P}\{X = 0\} \leq \frac{\mathbf{V}\{X\}}{\mathbf{E}\{X^2\}}.$$

*Proof.* Since  $X$  is integer and nonnegative, we have  $\mathbf{P}\{X = 0\} = 1 - \mathbf{P}\{X \geq 1\}$  and since  $\mathbf{V}\{X\} = \mathbf{E}\{X^2\} - \mathbf{E}\{X\}^2$ , to get our result it suffices to show that

$$\mathbf{P}\{X \geq 1\} \geq \frac{\mathbf{E}\{X\}^2}{\mathbf{E}\{X^2\}}.$$

We start by noting that

$$\mathbf{E}\{X\} = \sum_{k=0}^{\infty} k \mathbf{P}\{X = k\} = \mathbf{P}\{X \geq 1\} \sum_{k=1}^{\infty} \frac{k \mathbf{P}\{X = k\}}{\mathbf{P}\{X \geq 1\}} = \mathbf{P}\{X \geq 1\} \mathbf{E}\{X \mid X \geq 1\}.$$

Since the function  $x \mapsto x^2$  is convex, we have, by Jensen's inequality,

$$\mathbf{E}\{X\}^2 = \mathbf{P}\{X \geq 1\}^2 \mathbf{E}\{X \mid X \geq 1\}^2 \leq \mathbf{P}\{X \geq 1\}^2 \mathbf{E}\{X^2 \mid X \geq 1\} = \mathbf{P}\{X \geq 1\} \mathbf{E}\{X^2\}.$$

Dividing both sides by  $\mathbf{E}\{X^2\}$  gives us what we want. **■**

**Exercise 4.4.** Let  $X$  be a random variable with expectation  $\mathbf{E}\{X\} = 0$  and variance  $\mathbf{V}\{X\} = \sigma^2$ . Prove that for all  $\lambda > 0$ ,

$$\mathbf{P}\{X \geq \lambda\} \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

*Proof.* Let  $\lambda > 0$ . Note that if  $X \geq \lambda$ , then  $X$  is also positive so in particular, for any  $a > 0$ ,  $X + a$  and  $\lambda + a$  are both positive. This implies that  $(X + a)^2 \geq (\lambda + a)^2$ . Note that since  $\mathbf{V}\{X\} = \mathbf{E}\{X^2\} - \mathbf{E}\{X\}^2 = \sigma^2$  and  $\mathbf{E}\{X\} = 0$ , we have  $\mathbf{E}\{X^2\} = \sigma^2$ , which in turn implies that

$$\mathbf{E}\{(X + a)^2\} = \mathbf{E}\{X^2\} + 2a \mathbf{E}\{X\} + \mathbf{E}\{a^2\} = \sigma^2 + a^2.$$

Now we take the infimum over all  $a > 0$  and apply Markov's inequality to get

$$\mathbf{P}\{X \geq \lambda\} \leq \inf_{a>0} \mathbf{P}\{(X + a)^2 \geq (\lambda + a)^2\} \leq \inf_{a>0} \frac{\sigma^2 + a^2}{(\lambda + a)^2}.$$

We now optimise over  $a > 0$ . We have

$$\frac{d}{da} \frac{\sigma^2 + a^2}{(\lambda + a)^2} = \frac{(\lambda + a)^2 2a - (\sigma^2 + a^2) 2(\lambda + a)}{(\lambda + a)^4},$$

which is zero when  $a = \sigma^2/\lambda > 0$ . Plugging in this value of  $a$  gives

$$\mathbf{P}\{X \geq \lambda\} \leq \frac{\sigma^2 + \sigma^4/\lambda^2}{(\lambda + \sigma^2/\lambda)^2} = \frac{\sigma^2(\lambda + \sigma^2/\lambda)}{\lambda(\lambda + \sigma^2/\lambda)^2} = \frac{\sigma^2}{\sigma^2 + \lambda^2},$$

exactly what we need. **■**