Answers to Selected Exercises in *Real and Complex Analysis*[*]

Solutions by

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CHAPTER 1. ABSTRACT INTEGRATION

1. *Does there exist an infinite σ-algebra which has only countably many members?*

The answer is no.

Proof. Let X be a ground set and let $\mathcal{F}$ be a σ-algebra (over X) with infinitely many members. First we claim that we can always find a set $E \neq \emptyset$ such that the set $\{F \cap E^c : F \in \mathcal{F}\}$ is infinite. If this did not hold, then take any $E \neq \emptyset$ in $\mathcal{F}$. Then by assumption, $\mathcal{S}_1 = \{F \cap E^c : F \in \mathcal{F}\}$ is finite and because $E^c \in \mathcal{F}$, $\mathcal{S}_2 = \{F \cap E : F \in \mathcal{F}\}$ is finite as well. Since any member of $\mathcal{F}$ can be expressed as a union of an element of $\mathcal{S}_1$ with an element of $\mathcal{S}_2$, the implies that $\mathcal{F}$ is finite, a contradiction.

Now we may use the claim to construct a countable sequence of pairwise disjoint elements of $\mathcal{F}$. Let G_1 be the set E given by the claim. Now the infinite set $\mathcal{S}_1$ we constructed before is also a σ-algebra, so repeat the argument to get a set G_2, disjoint from G_1. Continuing in this manner, we obtain a sequence $G_1, G_2, \ldots$ where the G_i are pairwise disjoint. Now we see that the map from the power set of the natural numbers to $\mathcal{F}$ given by

$$A \mapsto \bigcup_{i \in A} G_i$$

is injective. So the uncountability of $\mathcal{F}$ follows from the uncountability of $\mathcal{P}(\mathbf{N})$. ▮

3. *Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.*

Proof. We immediately infer that for rational r and s, $f^{-1}\big([r, s)\big)$ is measurable, since

$$f^{-1}\big([r, s)\big) = \{x : f(x) \geq r\} \cap \{x : f(x) \geq s\}^c.$$

Next, let $I \subseteq \mathbf{R}$ be an open interval. Setting $J = I \cap \mathbf{Q}$, we can show that $f^{-1}(I)$ is measurable, since

$$f^{-1}(I) = f^{-1}\left(\bigcup_{r \in J} \bigcup_{s \in J} [r, s)\right) = \bigcup_{r \in J} \bigcup_{s \in J} f^{-1}\big([r, s)\big)$$

is a countable union of measurable sets.

Now let $V \subseteq \mathbf{R}$ be open. By Lindelöf's lemma, we can express V as a countable union of open intervals $V = \bigcup_{n=1}^{\infty} I_n$. Then

$$f^{-1}(V) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$$

is measurable. ▮

5. *Prove the following statements:*

a) If $f : X \to [-\infty, \infty]$ and $g : x \to [-\infty, \infty]$ are measurable, then the sets

$$\{x : f(x) < g(x)\} \text{ and } \{x : f(x) = g(x)\}$$

are measurable.

[*] Walter Rudin. 1987. *Real and complex analysis, 3rd ed.* McGraw-Hill, Inc., USA.

b) *The set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.*

Proof. Observe that $f(x) < g(x)$ if and only if there exists a rational number r such that $f(x) < r < g(x)$. So

$$\{x : f(x) < g(x)\} = \bigcup_{r\in\mathbf{Q}} \big(\{x : f(x) < r\} \cap \{x : g(x) > r\}\big) = \bigcup_{r\in\mathbf{Q}} \Big(f^{-1}\big([-\infty, r)\big) \cap g^{-1}\big((r, \infty]\big)\Big)$$

is a countable union of measurable sets (since f and g are measurable functions). Then

$$\{x : f(x) = g(x)\} = \{x : f(x) < g(x)\}^c \cap \{x : g(x) < f(x)\}^c$$

is measurable as well, proving (a).

Let (f_n) be a sequence of real-valued measurable functions. Then (f_n) converges at a point x if and only if it is Cauchy at x, i.e. for any $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that for all $m, n \geq N$, $|f_m(x) - f_n(x)| < \epsilon$. By the Archimedean property, we can replace ϵ with $1/k$ for some $k \in \mathbf{N}$. So the set of all points at which (f_n) converges can be written thus:

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \{x : |f_m(x) - f_n(x)| < 1/k\}$$

Set $g_{m,n} = f_m - f_n$; this is a measurable function. Hence the set

$$\big\{x : |f_m(x) - f_n(x)| < 1/k\big\} = \big\{x : |g_{m,n}(x)| < 1/k\big\} = {g_{m,n}}^{-1}\big((-1/k, 1/k)\big)$$

is measurable for every m, n, and k. We have thus proved that the set of all points at which (f_n) converges is a countable union of measurable sets. ∎

6. *Let X be an uncountable set, let $\mathcal{M}$ be the collection of all sets $E \subseteq X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathcal{M}$ is a σ-algebra in X and that μ is a measure on $\mathcal{M}$. Describe the corresponding measurable functions and their integrals.*

First we prove that $\mathcal{M}$ is a σ-algebra and that μ is a measure.

Proof. Since $\emptyset$ is countable, $X \in \mathcal{M}$. Then for any set $E \in \mathcal{M}$, either E or E^c is at most countable; in any case $E^c \in \mathcal{M}$. Lastly, let $\{E_n\}$ be a collection of sets in $\mathcal{M}$. If every set E_n is at most countable, then $\bigcup_{n=1}^{\infty} E_n$ is countable and thus in $\mathcal{M}$. Otherwise, there is some k for which E_k is uncountable. Then $E_k^c \in \mathcal{M}$ is at most countable and $\left(\bigcup_{n=1}^{\infty} E_n\right)^c = \bigcap_{n=1}^{\infty} E_n^c \subseteq E_k^c$ is countable, so $\bigcup_{n=1}^{\infty} E_n$ is in $\mathcal{M}$. This shows that $\mathcal{M}$ is a σ-algebra.

It is clear that the range of μ is in $[0, \infty]$. To show that μ is countably additive, let $\{E_n\}$ be a disjoint collection of sets in $\mathcal{M}$. If every E_n is at most countable, then $\bigcup_{n=1}^{\infty} E_n$ is at most countable and $\mu(\bigcup_{n=1}^{\infty} E_n) = 0 = \sum_{n=1}^{\infty} \mu(E_n)$. Otherwise, there exists E_k such that E_k^c is countable and thus we have $\mu(E_k) = 1$ and $\mu(E_k^c) = 0$. Since the E_n are pairwise disjoint, $E_n \subseteq E_k^c$ for all $n \neq k$. So $\mu(E_n) = 0$ for all $n \neq k$. So $\mu(\bigcup_{n=1}^{\infty} E_n) = 1 = \sum_{n=1}^{\infty} \mu(E_n)$ and μ is a measure. ∎

Now we claim that the measurable functions are those that are constant at all but countably many points.

Proof. It suffices to prove this for a real-valued function. Suppose $f : X \to [-\infty, \infty]$ is measurable. For any $a \in \mathbf{R}$, let $E_a = f^{-1}\big([-\infty, a)\big)$. Note that for any a, either E_a is countable or E_a^c is countable. Note also that if $a \leq b$, then $E_a \subseteq E_b$. So let there is a constant c such that

$$k = \sup\{a : E_a \text{ is countable}\}.$$

This supremum is not $-\infty$, since if it were, then E_a^c would be countable for all $a \in \mathbf{R}$, and since $\bigcap_{n=0}^{\infty} E_{-n} = \emptyset$, $X = \bigcap_{n=0}^{\infty} E_{-n}^c$ is countable, a contradiction. By a similar argument, we know that the supremum is not

∞. So $k \in \mathbf{R}$ and there exists a sequence (a_n) all of whose terms are less than k and whose limit is k. Hence $E_k = \bigcup_{n=1}^{\infty} E_{a_n}$ is countable. Now let (b_n) be a sequence that converges to k such that $b_n > k$ for all n. Then note that if $f(x) > k$, then $f(x) \in [b_n, \infty) = E_{b_n}^c$ for some n. So

$$\{x : f(x) \neq k\} = E_k \cup \{x : f(x) > k\} \subseteq E_k \cup \Big(\bigcup_{n=0}^{\infty} E_{b_n}^c \Big)$$

is countable and f equals k at all but countably many points. ∎

Lastly, we show that if f is measurable and takes on the value k at all but countably many points, then $\int_E f\, d\mu = k$ for all uncountable $E \in \mathcal{M}$. (If E is countable then $\int_E f\, d\mu = 0$.)

Proof. Let $E \in \mathcal{M}$ be uncountable; so $\mu(E) = 1$. Let s be a simple measurable function such that $0 \leq s \leq f$. Suppose that s takes values α_i on the n disjoint sets A_i that cover X. We know that one of the α_i, call it α_j, is equal to a constant k_s and that A_i is at most countable for all $i \neq j$. So for all $i \neq j$, $A_i \cap E$ is countable and $A_j \cap E$ must be uncountable. Then we have

$$\int_E s\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) = \alpha_j \mu(A_j \cap E) = k_s \cdot 1 = k_s,$$

where $0 \leq k_s \leq k$. So we have

$$\int_E f\, d\mu = \sup\Big\{ \int_E s\, d\mu : 0 \leq s \leq f \Big\} = \sup\{k_s : 0 \leq s \leq f\} = k,$$

which is what we had to show. ∎

7. *Suppose $f_n : X \to [0,\infty]$ is measurable for $n = 1, 2, 3, \ldots$, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then*

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu$$

and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

Proof. Consider the sequence of functions $(f_1 - f_n)$. This sequence is nonnegative, nondecreasing and for any $x \in X$, $\lim_{n\to\infty}(f_1 - f_n)(x) = f_1(x) - f(x)$. So by the monotone convergence theorem,

$$= \lim_{n\to\infty} \int_X f_1 - f_n\, d\mu = \int_X f_1 - f\, d\mu.$$

Since the integrals of f_n are finite, we can apply Theorem 1.27 to get

$$\int_X f_1\, d\mu - \lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f_1\, d\mu - \int_X f\, d\mu,$$

which gives us what we want after subtracting $\int_X f_1\, d\mu$ (which is finite) and multiplying by -1. ∎

The condition $f_1 \in L^1(\mu)$ is necessary. Let $I = [0,1]$ and $f_n = \frac{1}{nx^2}$, which converges to the constant function 0. Then $\lim_{n\to\infty} \int_I f_n\, d\mu = +\infty$ but $\int_I 0\, d\mu = 0$.

8. *Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?*

This gives an example of strict inequality arising. Let X be a measure space such that $\mu(X) = 1$ and let $E \subseteq X$ be such that $\mu(E) = 2/3$. Then

$$\int_X \liminf_{n\to\infty} f_n\, d\mu = \int_X 0\, d\mu = 0,$$

while on the other hand

$$\liminf_{n\to\infty} \int_X f_n\, d\mu = \int_X 1 - \chi_E\, d\mu = \int_X 1\, d\mu - \int_X \chi_E\, d\mu = \mu(X) - \mu(E) = \frac{1}{3},$$

and we have $0 < 1/3$.