Idempotent theorems and matrices

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Abstract. These expository notes move at a leisurely pace though background material needed to appreciate the Cohen idempotent theorem and its quantitative version, due to Green and Sanders. We introduce Schur multipliers and investigate a conjectured analogue in this setting.

1. Topological preliminaries

A *topology* on a set S is a collection of subsets of S that contains the empty set and is closed under the formation of unions and finite intersections. If S has a topology defined on it, then S is said to be a *topological space*, and the members of its topology are called *open sets*. The complements of open sets are called *closed sets*.

Proposition 1.1. *A set A is open if and only if for every $x \in A$, there is an open set U_x with $x \in U_x$ and $U_x \subseteq A$.*

Proof. If A is open, then for any $x \in A$ we can simply take $U = A$. Conversely, if for every point $x \in A$ there is some open $U_x \subseteq A$ with $x \in U_x$, then $A = \bigcup_{x\in A} U_x$, so A is open, being a union of open sets. ∎

The largest open set contained in a set A is the *interior* of A, and the smallest closed set containing A is its *closure*, denoted $\overline{A}$. If $A\overline{A}$ is the whole set S, then we say A is *dense* in S, and if some countable set is dense in S, then S is a *separable* space.

A set A is said to be a *neighbourhood* of a point p if p belongs to the interior of A. The space S is called *Hausdorff* if for all points $x, y \in S$ with $x \neq y$, there is some open neighbourhood U of X and open neighbourhood V of y such that $U \cap V = \emptyset$.

If the singleton set $\{p\}$ is open, then p is called an *isolated point* of S, and if p is isolated for all points $p \in S$, then S is a *discrete space*. It is easy to see that every subset of a discrete space is open.

Limit points of nets. A relation $\preceq$ on a set I is a *preorder* if it is reflexive and transitive. A set I is said to be *directed* if there exists a preorder $\preceq$ on it such that any two elements have an upper bound; that is, for all $i, j \in I$, there exists $k \in I$ such that $i \preceq k$ and $j \preceq k$.

Let I be a directed set and S any set. A function $f : I \to S$ is called a *net*. We usually index the elements of the net by elements of I; writing $x_i = f(i)$

for all $i \in I$, the net f can be expressed as $(x_i)_{i\in I}$. When $I = \mathbf{N}$ under the usual ordering $\leq$, we recover the usual definition of a *sequence* of points in S. We say that $(x_i)_{i\in I}$ is *eventually* in A if there exists some $j \in I$ such that for all $i \succeq j$, $x_i \in A$. A point $x \in S$ is called a *limit point* of $(x_i)_{i\in I}$ if for every open neighbourhood U of x, the net $(x_i)_{i\in I}$ is eventually in U. In this case we write $x_i \to x$.

Proposition 1.2. *If S is a Hausdorff space, then every net $(x_i)_{i\in I}$ valued in S has at most one limit point.*

Proof. Let S be a topological space and suppose there is a net $(x_i)_{i\in I}$ with two distinct limit points x and y. Let U be an open neighbourhood of x and V an open neighbourhood of Y. By definition of limit point, there is some $i \in I$ such that $x_k \in U$ for all $k \succeq i$, and similarly there is some $j \in I$ such that $x_i \in V$ for all $k \succeq j$. But since I is a directed set, there must exist some $k \in I$ with $k \succeq i$ and $k \succeq j$, and we find that $x_k \in U \cap V$. Since U and V were arbitrary, we conclude that S is not Hausdorff. ∎

In light of the proposition, so long as the ambient space is Hausdorff it makes sense to write $\lim_{i\in I} x_i = x$ whenever the limit point of $(x_i)_{i\in I}$ is x. We may also choose to write $x_i \to x$.

We now introduce a common choice of directed set that features in some of the proofs below.

Proposition 1.3. *Let x be a point in a topological space S and let $N(x)$ be the set of all open neighbourhoods of x. Define an order on $N(x)$ by declaring $U \preceq V$ if and only if $U \supseteq V$. Then the order $\preceq$ is directed, and x is a limit point of any net $(x_U)_{U\in N(x)}$ satisfying $x_U \in U$ for all $U \in N(x)$.*

Proof. The set $N(x)$ is clearly directed, since for every $U, V \in N(x)$, $U \supseteq U \cap V$ and $V \supseteq U \cap V$. Now let $(x_U)_{U\in N(x)}$ be any net with $x_U \in U$ for all $U \in N(x)$. Pick any open set V containing x. This set V is a member of $N(x)$, and for any $U \in N(x)$ with $U \succeq V$, we must have $x_U \in U \subseteq V$. ∎

One can characterise closed sets in terms of limit points of nets.

Proposition 1.4. *A set A is closed if and only if for every net $(x_i)_{i\in I}$ satisfying $x_i \in A$ for all $i \in I$, every limit point x of $(x_i)_{i\in I}$ must be in A.*

Proof. Suppose that A is closed and let $(x_i)_{i\in I}$ be a net with limit point x and $x_i \in A$ for all $i \in I$. For a contradiction, suppose that $x \notin A$. Then $x \in A^c$, which is open. By the definition of limit point, there exists j such that $x_i \in A^c$ for all $j \succeq i$. This contradicts the assumption that x is a limit point of $(x_i)_{i\in I}$.

Now we assume the second condition and prove that A^c is open. Let $N(x)$ be the set of all open neighbourhoods of x. Towards a contradiction, assume that there is some $x \in A^c$ such that for all $U \in N(x)$, we have $U \cap A \neq \emptyset$. So for all $U \in N(x)$, we may pick $x_U \in U \cap A$, and the net $(x_U)_{U\in N(x)}$ has x as its limit point, by Proposition 1.3. But $x_U \in A$ for all $U \in N(x)$, so by our assumption, we must have $x \in A$. This contradiction shows that any $x \in A^c$ must have some open neighbourhood U that is contained in A^c. Hence by Proposition 1.1, A is closed. ∎

Cluster points and subnets. A point x is a *cluster point* of the net $(x_i)_{i\in I}$ if for every open neighbourhood U of x and any $i \in I$, there is some $j \succeq i$ with $x_i \in U$. Next, consider two directed sets I and J. A map $\phi : J \to I$ is said to be *cofinal* if for every $i \in I$ there is $j \in J$ such that for all $j' \succeq j$, $\phi(j') \succeq i$. Now if $(x_i)_{i\in I}$ is a net (with elements in some topological space X) and $\phi : B \to A$ is cofinal, then the composition $x \circ \phi : B \to X$ is a net; we say that $x \circ \phi$ is a *subnet* of the original net. The following proposition relates subnets to cluster points.

Proposition 1.5. *A point x is a cluster point of the net $(x_i)_{i\in I}$ if and only if there is a subnet of $(x_i)_{i\in I}$ converging to x.*

Proof. Suppose there is a subnet of $(x_i)_{i\in I}$ converging to x. That is, there is a directed set J and a cofinal map $\phi : J \to I$ such that, setting $y_j = x_{\phi(j)}$ for all $j \in J$, the subnet $(y_j)_{j\in J}$ of $(x_i)_{i\in I}$ converges to x. Let U be an open neighbourhood of x and let $i \in I$. There is some $j_0 \in J$ such that $j \succeq j_0$ implies that $y_j \in U$. Since ϕ is cofinal, there is $j_1 \in J$ such that for all $j \succeq j_1$, $\phi(j) \succeq i$. Now since J is directed, there is some $j' \in J$ with $j' \succeq j_0$ and $j' \succeq j_1$. For this choice of j' we have $\phi(j') \succeq i$ and $x_{\phi(j')} = y_{j'} \in U$. Hence x is a cluster point of $(x_i)_{i\in I}$.

Conversely, suppose that x is a cluster point of the net $(x_i)_{i\in I}$. Let $N(x)$ be the set of all neighbourhoods of x as in Proposition 1.3. We define

$$J = \bigl\{(U, i) : U \in N(x), x_i \in U\bigr\}$$

and impose the preorder $(U, i) \preceq (U', i')$ if and only if $U \supseteq U'$ and $i \preceq i'$. For any (U_0, i_0) and (U_1, i_1) in J, we can pick $i' \in I$ such that $i \succeq i_0$ and $i \succeq i_1$. Since x is a cluster point, there is some $i^* \geq i$ such that $x_{i^*} \in U_0 \cap U_1$, and we see that $(U_0, i_0) \preceq (U_0 \cap U_1, i^*)$ and $(U_1, i_1) \preceq (U_0 \cap U_i, i^*)$. This shows that J is a directed set. We define the map $\phi : J \to I$ by sending $(U, i) \mapsto i$. This is certainly cofinal, by our choice of preorder on J. To show that $\bigl(x_{\phi(j)}\bigr)_{j\in J}$ converges to x, let V be any neighbourhood of x. Since x is a cluster point, there is some $i \in I$ with $x_i \in V$; consider the element $j = (V, i) \in J$. For any $j' = (V', i') \in J$ with $(V', i') \succeq (V, i)$, we have $V' \subseteq V$ and $x_{i'} \in V'$, so $x_{\phi(j')} = x_{i'} \in V$. ∎

Compactness. A set A is *compact* if every open cover of A has a finite subcover. If every point of S has some compact neighbourhood, then S is called *locally compact.*

2. The structure of locally compact abelian groups

A topological abelian group is an abelian group G endowed with the topology of a Hausdorff space, such that the map from $G \times G$ to G sending $(x, y) \mapsto x - y$ is continuous. (The topology on $G \times G$ is taken to be the product topology).

If, in addition, G is also locally compact, then G is said to be a *locally compact abelian* group.

The following proposition shows that by passing to closures, we can always assume subgroups to be closed.

Proposition 2.1. *Let G be a locally compact abelian group and H a subgroup of G. Then the closure $\overline{H}$ of H is also a subgroup of G.*

Proof. Let $x, y \in \overline{H}$ and let Z be any neighbourhood of $x-y$. Since multiplication is continuous (and by the definition of the product topology), there is an open neighbourhood X of x and Y of y such that $X - Y \subseteq Z$. By the definition of closure, there are points $a \in X \cap H$ and $b \in Y \cap H$. The point $a - b$ is in H because H is a subgroup, and thus $a - b \in Z \cap H$. ∎

Let G be a locally compact abelian group. The set of all measures μ on G forms a topological semigroup under the convolution operation

$$\mu * \nu(A) = \int_G \chi_A(x + y)\, d\mu(x)\, d\nu(x),$$

where χ_A denotes the characteristic function of A.

References