Math 242 Tutorial 6

prepared by

Marcel Goh

23 October 2025

Problem 1. *Determine (with proof) the limits of*

a) n^{1/n^2}*; and*

b) $(n!)^{1/n^2}$

as $n \to \infty$*. You may use the fact that* $\sqrt[n]{n} \to 1$ *as* $n \to \infty$*; this you shall prove on your next assignment.*

Proof. Note that for all $n \in \mathbf{N}$, $n \ge 1 = 1^{n^2}$, so we have $n^{1/n^2} \ge 1$. On the other hand, since $n \le n^n$, taking n^2th roots we have $n^{1/n^2} \le \sqrt[n]{n}$. But $\sqrt[n]{n} \to 1$, so by the squeeze theorem $n^{1/n^2} \to 1$ too, as $n \to \infty$.

For part (b), observe first that for all $n \in \mathbf{N}$, $n! \ge 1 \ge 1^{n^2}$, so $(n!)^{1/n^2} \ge 1$. On the other hand, $n! = 1 \cdot 2 \cdot 3 \cdots n \le n^n$, so taking n^2th roots again we have $(n!)^{1/n^2} \le \sqrt[n]{n}$. Hence by the squeeze theorem, we see that $(n!)^{1/n^2} \to 1$. ▮

Problem 2. *Let* (a_n) *be a sequence with* $\lim_{n\to\infty} a_n = a$*, and let* (b_n) *be a sequence with* $b_n \ne 0$ *for all* $n \in \mathbf{N}$ *and* $\lim_{n\to\infty} b_n = b \ne 0$*.*

a) Prove that $\lim_{n\to\infty} 1/b^n = 1/b$*.*

b) Conclude that $\lim_{n\to\infty}(a_n/b_n) = a/b$*.*

Proof. Let $\epsilon > 0$. Since $b_n \to b \ne 0$, there exists $N_1 \in \mathbf{N}$ such that

$$|b - b_n| < \frac{|b|^2\epsilon}{2}$$

for all $n \ge N_1$. For the same reason, there exists $N_2 \in \mathbf{N}$ such that

$$|b - b_n| < \frac{|b|}{2}$$

for all $n \ge N_2$. This second bound implies that for all $n \ge N_2$

$$|b_n| > \frac{|b|}{2}.$$

So, letting $N = \max\{N_1, N_2\}$, we have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b_n| \cdot |b|} < \frac{2}{|b|^2} \cdot \frac{|b|^2\epsilon}{2} = \epsilon.$$

We see, then, that $1/b_n \to 1/b$ as $n \to \infty$, since ϵ was arbitrary.

For part (b), simply let $c_n = 1/b_n$, with limit $c = 1/b$. By the product law for limits (shown in class),

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n c_n = ac = \frac{a}{b},$$

and we are done. ▮

Problem 3. *Let* (a_n) *be a sequence, let* (b_n) *be a null sequence, and let* $L \in \mathbf{R}$*. Show that if there exists* $K \in \mathbf{N}$ *such that* $|a_n - L| \le |b_n|$ *for all* $n \ge K$*, then* (a_n) *converges to* L*.*

Proof. Let $\epsilon > 0$. Since (b_n) is a null sequence, there is some $N_1 \in \mathbf{N}$ such that for all $n \ge N_1$ we have $|b_n| = |b_n - 0| < \epsilon$. Let $N = \max\{N_1, K\}$. Then for all $n \ge N$, we have

$$|a_n - L| \le |b_n| < \epsilon,$$

so $a_n \to L$ as $n \to \infty$. ▮

Problem 4. *Let (x_n) be a sequence of nonnegative real numbers that is subadditive in the sense that for all $m, n \in \mathbf{N}$,*

$$x_{m+n} \leq x_m + x_n.$$

a) Prove that $x_{qk} \leq qx_k$ for all integers $k, q \in \mathbf{N}$.

b) Explain why $\inf\{x_n/n : n \in \mathbf{N}\}$ exists.

c) Letting $L = \inf\{x_n/n : n \in \mathbf{N}\}$, show that $x_n/n \to L$ as $n \to \infty$.

Proof. For part (a), we perform induction on q. The base case $q = 1$ is trivial. For $q \geq 1$, suppose that $n = (q+1)k + r$. Then

$$x_{(q+1)k} = x_{qk+k} \leq x_{qk} + x_k \leq qx_k + x_k = (q+1)x_k,$$

where the first inequality is a consequence of subadditivity, and the second inequality follows from the induction hypothesis.

For part (b), we simply note that $x_n \geq 0$ for all $n \in \mathbf{N}$. Hence $x_n/n \geq 0$ for all $n \in \mathbf{N}$, and the infimum exists by completeness of the reals.

Part (c) is where the real work comes in. Let $\epsilon > 0$. Since L is the infimum of the set $\{x_n/n : n \in \mathbf{N}\}$, there exists some $k \in \mathbf{N}$ such that

$$\frac{x_k}{k} \leq L + \frac{\epsilon}{2}.$$

Let $M = \max\{x_1, x_2, \ldots, x_{k-1}\}$, and pick $N = \max\{k, 2M/\epsilon\}$. Now let $n \geq N$ be given. Since $n \geq k$, by the division algorithm we can write $n = qk + r$ for some integers $q \geq 1$ and $r \in \{0, \ldots, k-1\}$. Then we have

$$x_n = x_{qk+r} \leq x_{qk} + x_r \leq qx_k + x_r,$$

by subadditivity and part (a). (Technically we didn't define x_0, but we can just say $x_0 = 0$, and the above holds.) This implies that

$$\frac{x_n}{n} - \frac{x_k}{k} \leq \frac{qx_k + x_r}{n} - \frac{x_k}{k} = x_k\left(\frac{q}{kq+r} - \frac{1}{k}\right) + \frac{x_r}{n}.$$

But since $r \in \{0, \ldots, k-1\}$, we have $x_r \leq M$, by definition of M. This, along with our assumption that $n \geq 2M/\epsilon$, implies that

$$\frac{x_r}{n} \leq \frac{M}{n} < \frac{M}{2M/\epsilon} = \frac{\epsilon}{2}.$$

Furthermore, we have

$$\frac{q}{qk+r} \leq \frac{q}{qk} = \frac{1}{k},$$

so plugging these inequalities in above, we see that

$$\frac{x_n}{n} - \frac{x_k}{k} < \frac{\epsilon}{2}.$$

Lastly, we bound

$$\left|\frac{x_n}{n} - L\right| = \frac{x_n}{n} - L = \frac{x_n}{n} - \frac{x_k}{k} + \frac{x_k}{k} - L < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which completes the proof that $x_n/n \to L$ as $n \to \infty$.