Math 242 Tutorial 4

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Problem 1. *For any sets $A, B \subseteq \mathbf{R}$, we define the difference set to be the set*

$$A - B = \{a - b : a \in A, b \in B\}.$$

Prove that for all sets $U, V, W \subseteq \mathbf{R}$, we have the inequality

$$|U - W| \le \frac{|U - V| \cdot |V - W|}{|V|}.$$

This is a very fundamental inequality in the field of additive combinatorics. It is called the *Ruzsa triangle inequality*, for its superficial resemblance to the ordinary triangle inequality. (When you learn about metric spaces, you'll learn that the resemblance is actually more than superficial.)

Proof. What we have to show can be rearranged to

$$|U - W| \cdot |V| \le |U - V| \cdot |V - W|.$$

We are done then, if we can furnish an injective function from $(U - W) \times V$ to $(U - V) \times (V - W)$.

First we need to do some bookkeeping. For each $x \in U - W$, let $u(x) \in U$ and $w(x) \in W$ be such that $u(x) - w(x) = x$. (If there are multiple choices for this pair $\big(u(x), w(x)\big)$, just pick one.) Now we define a function. For $(x, v) \in (U - W) \times V$, we let

$$f(x, v) = \big(u(x) - v, w(x) - v\big).$$

By construction, the output of this function will be an element of $(U - V) \times (V - W)$, so we have $f : (U - W) \times V \to (U - V) \times (V - W)$.

We now show that f is injective. Suppose that $(x_1, v_1) \in (U - W) \times V$ and $(x_2, v_2) \in (U - W) \times V$ are such that $f(x_1, v_1) = f(x_2, v_2)$. Then we have

$$\big(u(x_1) - v_1, w(x_1) - v_1\big) = \big(u(x_2) - v_2, w(x_2) - v_2\big),$$

so subtracting the second element of each pair from the first element, we have

$$u(x_1) - w(x_1) = u(x_2) - w(x_2).$$

But by definition of the functions $u(x)$ and $w(x)$, we have $u(x_1) - w(x_1) = x_1$ and $u(x_2) - w(x_2) = x_2$. So we have found that $x_1 = x_2$. Lastly, note that

$$u(x_1) - v_1 = u(x_2) - v_2,$$

so since we have found out that $x_1 = x_2$, we must also have $v_1 = v_2$. ∎

Students of algebra will be delighted to realise that this proof did not use anything particular about $\mathbf{R}$. The Ruzsa triangle inequality is valid in any group.

Problem 2. *In class we took the least upper bound property as an axiom (the completeness axiom) and proved the nested intervals property as well as the Archimedean property. Assuming the nested intervals property and the Archimedean property as axioms, prove the least upper bound property.*

Proof. Let $S \subseteq \mathbf{R}$ be a nonempty set that is bounded from above. Let $a \in S$ and $b \in \mathbf{R}$ be an upper bound of S. We have $a \leq b$, so we can let $a_1 = a$, $b_1 = b$, and call $I_1 = [a_1, b_1]$. Note that the length of this interval is $|I_1| = a_1 - b_1 = a - b$.

We now define a sequence of nested intervals, starting with I_1. For $n \geq 1$, we assume that a_n and b_n are already defined, and let x_n be the midpoint $(b_n - a_n)/2$. There are two possibilities.

i) If x_n is an upper bound of S, then we let $a_{n+1} = a_n$ and $b_{n+1} = x_n$.
ii) If not, then let $a_{n+1} = x_n$ and $b_{n+1} = b_n$.

Since I_{n+1} is half the length of I_n for all $n \in \mathbf{N}$ and we had $|I_1| = a - b$, we have $|I_n| = (a - b)/2^{n-1}$. Next, since we only set $b_{n+1} = x_n$ if x_n is an upper bound of S and otherwise we have $b_{n+1} = b_n$, we see that b_n is an upper bound of S for all $n \in \mathbf{N}$. Lastly, note that I_n always contains some element of S. This we show by induction. It is true for $n = 1$, so for the induction step, suppose it's true for some $n \in \mathbf{N}$. We want to show that I_{n+1} contains some element of S. If x_n is an upper bound of S, then $(x_n, b_n]$ does not contain any element of S, so $I_n = [a_n, x_n]$ must. If x_n not an upper bound of S, then there must be some $y \in S$ with $y > x$. But b_n is an upper bound of S, so then in this case $I_n = [x_n, b_n]$ and y is in this range.

By the nested interval property, there is an element $s \in \bigcap_{n\in\mathbf{N}} I_n$. The claim is that s is the supremum of S. There are two things to show: that it is an upper bound of S, and that no smaller element can be an upper bound of S.

Suppose, towards a contradiction, that there is some $y \in S$ with $y > s$. Let $\epsilon = y - s > 0$. By the Archimedean property, there is some $N \in \mathbf{N}$ with $N > \log_2\big((b - a)/\epsilon\big) + 1$. This can be rearranged to get

$$\epsilon > \frac{b - a}{2^{N-1}}.$$

Consider $I_N = [a_N, b_N]$, which has $b_N - a_N = (b - a)/2^{N-1}$. Since $s \in I_N$, we have $s \geq a_N$, and

$$y = s + \epsilon \geq a_N + \epsilon > a_N + \frac{b - a}{2^{N-1}} = a_N + (b_N - a_N) = b_N.$$

This contradicts the fact that b_N is an upper bound of S. So we see that s is an upper bound of S.

Now let $t < s$. We must show that t is not an upper bound of S. This time, set $\epsilon = s - t > 0$. By the Archimedean property again, we can pick $N \in \mathbf{N}$ large enough so that

$$\epsilon > \frac{b - a}{2^{N-1}} = b_N - a_N.$$

Again we note that $s \in I_n = [a_N, b_N]$. Hence $s \leq b_N$ and we have

$$t = s - \epsilon \leq b_N - \epsilon < b_N - (b_N - a_N) = a_N.$$

But since I_N contains an element of S, there is some element of S that is at least a_N, which implies that it is greater than t. We conclude that t is not an upper bound of S, which settles the proof that $s = \sup S$. ∎

Problem 3. *Show that the set* $[0, 1)$ *is uncountable.*

Proof. If we disallow infinite trailing sequences of 9s, then every element $x \in [0, 1)$ has a unique decimal representation as

$$x = 0.d_1 d_2 d_3 \cdots,$$

where, for all $n \in \mathbf{N}$, d_n is a digit from the set $\{0, 1, \ldots, 9\}$.

Now we show that there is no surjective function $f : \mathbf{N} \to [0, 1)$. Let f be any function from $f : \mathbf{N} = [0, 1)$. For all $n \in \mathbf{N}$, let

$$f(n) = 0.d_{n,1} d_{n,2} d_{n,3} \cdots,$$

where we always use a representation of $f(n)$ that doesn't have an infinite trailing sequence of 9s. The claim is that there is some $x \in [0, 1)$ that differs from every $f(n)$ by at least one digit.

Consider the diagonal sequence $d_{1,1}, d_{2,2}, d_{3,3}, \ldots$. For all n, define

$$c_n = \begin{cases} d_{n,n} - 1, & \text{if } d_{n,n} > 0; \\ 1, & \text{if } d_{n,n} = 0. \end{cases}.$$

Let x be the element of $[0, 1)$ with decimal expansion

$$x = 0.c_1 c_2 c_3 \cdots.$$

Note that none of these digits are equal to 9, so there is no infinite trailing sequence of 9s in the decimal expansion of x. Furthermore, $f(n) \neq x$ for all $n \in \mathbf{N}$, since $f(n)$ has the digit $d_{n,n}$ in the nth decimal place, whereas x has the digit $c_n \neq d_{n,n}$. So f is not surjective.

We have proved that there is no surjection from $\mathbf{N}$ to $[0, 1)$, so *a fortiori* there is no bijection from $\mathbf{N}$ to $[0, 1)$. We conclude that $[0, 1)$ is uncountable. ∎

Problem 4. *Let x be a real number. Prove that for every $\epsilon > 0$ there exist two rational numbers q and q' such that $q < x < q'$ and $|q - q'| < \epsilon$.*

Proof. We know from class that $\mathbf{Q}$ is dense in $\mathbf{R}$. We can apply this with $x - \epsilon/2$ and x to get $q \in \mathbf{Q}$ with

$$x - \frac{\epsilon}{2} < q < x.$$

Then we can apply it with x and $x + \epsilon/2$ to get $q' \in \mathbf{Q}$ with

$$x < q' < x + \frac{\epsilon}{2}.$$

By construction, we have $q < x < q'$, as desired. It remains to show that $|q - q'| < \epsilon$. Well, we know that $q' < x + \epsilon/2$ and $q > x - \epsilon/2$. Negating the latter, we get $-q < -x + \epsilon/2$, and adding this to the former, we obtain

$$|q - q'| = q' - q < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So we are done. ∎