Answers to Selected Exercises in *Real and Complex Analysis*[*]

Solutions by

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CHAPTER 1. ABSTRACT INTEGRATION

1. *Does there exist an infinite σ-algebra which has only countably many members?*

The answer is no.

Proof. Let X be a ground set and let $\mathcal{F}$ be a σ-algebra (over X) with infinitely many members. First we claim that we can always find a set $E \neq \emptyset$ such that the set $\{F \cap E^c : F \in \mathcal{F}\}$ is infinite. If this did not hold, then take any $E \neq \emptyset$ in $\mathcal{F}$. Then by assumption, $\mathcal{S}_1 = \{F \cap E^c : F \in \mathcal{F}\}$ is finite and because $E^c \in \mathcal{F}$, $\mathcal{S}_2 = \{F \cap E : F \in \mathcal{F}\}$ is finite as well. Since any member of $\mathcal{F}$ can be expressed as a union of an element of $\mathcal{S}_1$ with an element of $\mathcal{S}_2$, the implies that $\mathcal{F}$ is finite, a contradiction.

Now we may use the claim to construct a countable sequence of pairwise disjoint elements of $\mathcal{F}$. Let G_1 be the set E given by the claim. Now the infinite set $\mathcal{S}_1$ we constructed before is also a σ-algebra, so repeat the argument to get a set G_2, disjoint from G_1. Continuing in this manner, we obtain a sequence $G_1, G_2, \ldots$ where the G_i are pairwise disjoint. Now we see that the map from the power set of the natural numbers to $\mathcal{F}$ given by

$$A \mapsto \bigcup_{i \in A} G_i$$

is injective. So the uncountability of $\mathcal{F}$ follows from the uncountability of $\mathcal{P}(\mathbf{N})$. ▮

[*] Walter Rudin. 1987. *Real and complex analysis, 3rd ed.* McGraw-Hill, Inc., USA.