MATH 240 Fall 2024

notes by

MARCEL GOH

A note on these notes. After each class, this document will be updated with the new material that was just covered. The datestamps in the left margin indicate when the notes from each day start. Subsections labelled with a $*$ are optional. This document is heavily based on notes by Jeremy Macdonald, but any errors are likely my own. Please email me if you find any.

I. FOUNDATIONS

Die Mathematik ist in ihrer Entwickelung völlig frei
und nur an die selbstredende Rücksicht gebunden,
dass ihre Begriffe sowohl in sich widerspruchslos sind,
als auch in festen durch Definitionen geordeneten Beziehungen
zu den vorher gebildeten,
bereits vorhandenen und bewährten Begriffen stehen.

— GEORG CANTOR, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (1883)

1. Set theory

29.VIII A *set* is a collection of distinct objects, called its *elements* or its *members.* If x is a member of set A, then we write $x \in A$, and if x is not an element of A, then we write $x \notin A$. Sets can be written by listing out its elements. For example,

$$\{1, 4, 7, 10, \sqrt{782}\}$$

and

$$\{\{1, 2\}, \pi, \{4\}\}$$

are both sets (the second example shows that sets can themselves contain other sets). The order of the elements is not important, and duplicate elements are ignored (so $\{1, 2\} = \{2, 1\} = \{1, 1, 2\}$). The notation using $\{$ and $\}$ is useful for defining small, concrete examples, but expressing large sets can become very cumbersome. The first way one can describe larger sets is to use the $\ldots$ symbol and the power of suggestion. For instance, anyone faced with the notation

$$A = \{1, 3, 5, 7, 9, \ldots\}$$

can quickly guess that this set is supposed to contain all the positive odd integers. We can also use $\ldots$ to define finite sets. Most Canadians will be able to tell you that the set

$$\{\text{Alberta}, \text{British Columbia}, \ldots, \text{Yukon}\}$$

of provinces and territories contains 13 elements. But this notation inherently produces some ambiguity. For example, since the sequence of positive palindromic binary numbers starts $1, 3, 5, 7, 9, 15, 17, 21, 27, \ldots$, we are left with some doubt as to whether the set A above should be the set of odd numbers or the set of palindromic binary numbers.

But there is another, less ambiguous way to define large sets. It is called set-builder notation and it refers to any construction of the form

$$\{x \in U : P(x)\},$$

where x is a variable, U is a set, and P is a statement about x. The resulting set contains *all x such that $P(x)$ holds.* For example, letting $\mathbf{N}$ denote the set $\{0, 1, 2, 3, \ldots\}$ of counting numbers (more on this later), to define the set of all odd numbers, we can write

$$A = \{x \in \mathbf{N} : \text{there exists } k \in \mathbf{N} \text{ such that } x = 2k + 1\}.$$

Note that the statement $P(x)$ must contain x, but it may also contain other previously defined symbols, as well as new symbols defined within the statement (such as k in the example above).

Special sets of numbers. There are certain infinite sets of numbers that are used so often as to be given special bold notation. Back in elementary school, you learned to use the counting numbers $0, 1, 2, 3, \ldots$. We already saw this set in the previous paragraph; in the business, this set is known as the *natural numbers*, because if you go on a nature hike you can use them to count the number of bluebells, donkeys, etc. that you see. (Many mathematicians use the symbol $\mathbf{N}$ to denote this set without zero. When in doubt, clarify with the person you're talking to; in this class, $0 \in \mathbf{N}$.)

Sometime towards the start of junior high you were introduced to the concept of natural numbers. The set

$$\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

is called the set of *integers* or *whole numbers.* (The word *integer* just means "whole" in Latin; cf. French *entier.* We use the letter $\mathbf{Z}$ because of the German word *Zahl* meaning "number.")

Even before you learned about negative numbers, you probably learned about fractions. They can be defined in set-builder notation as a collection of ratios of integers, where the denominator is not zero:

$$\mathbf{Q} = \{p/q : p, q \in \mathbf{Z}, q \neq 0\}.$$

(The nitpicky reader will notice that p/q is not stipulated to be a member of any set here. This is because a rigorous definition of $\mathbf{Q}$ involves quotienting out by an equivalence relation, which we don't know how to do yet.) This is the set of *rational numbers.* Remember that sets only contain distinct elements, so $2/3$, $4/6$, $6/9$, etc. are all considered the same rational number.

Lastly, we have the set $\mathbf{R}$ of real numbers. Constructing this set using only notions from set theory and logic is quite the byzantine task and well outside the scope of this course, but you can think of $\mathbf{R}$ as the set of decimal numbers with a finite number of digits to the left of the decimal point, and a possibly infinite number of digits to the right of the decimal point.

A set of numbers near and dear to many mathematicians' hearts is the set of *prime* numbers P, defined by

$$P = \bigl\{p \in \mathbf{N} : p \geq 2, \text{ and if } p = ab \text{ then } \{a, b\} = \{1, p\}\bigr\}.$$

(You may have met a different definition of prime numbers in the past. Pause a moment and convince yourself that the statement above defines the set of prime numbers as you know it.)

Set inclusion. When we use set builder notation $B = \bigl\{x \in A : P(x)\bigr\}$, every element of B is necessarily a member of the set A as well, since B is defined to be the set of all x in A satisfying $P(x)$. This is one way in which we can obtain a *subset* of another set. More generally, we write $B \subseteq A$ if every element of B is also an element of A, and $B \supseteq A$ if every element of A is an element of B. Sometimes, if $B \supseteq A$, we say that B *contains* A, or B *includes* A. For example, we have the chain

$$\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$$

for the special sets of numbers defined earlier.

Symbols like $\subseteq$, $\supseteq$, and $=$ (that are used to produce statements) can be negated with a slash; for example, if there is some element of B that is not an element of A, then we write $B \not\subseteq A$.

The concept of set inclusion is important, because the most common way to prove that two sets A and B are equal is to show that A is a subset of B, then show that B is a subset of A. We illustrate this with the following example.

Proposition 1. *The sets*

$$A = \{x \in \mathbf{Z} : \textit{there exists } k \in \mathbf{Z} \textit{ such that } x = 2k+1\}$$

and

$$B = \{x \in \mathbf{Z} : \textit{there exists } l \in \mathbf{Z} \textit{ such that } x = 2l+5\}$$

are equal. (Both are different ways of expressing the set of all odd integers.)

Proof. Let $x \in A$. Then there exists $k \in \mathbf{Z}$ such that $x = 2k+1$. Letting $l = k-2$, we find that l is an integer (since k was). Furthermore,

$$x = 2k+1 = 2k-4+5 = 2(k-2)+5 = 2l+5.$$

We have found l such that $x = 2l+5$, so $x \in B$. This shows that $A \subseteq B$.

On the other hand, let $x \in B$, so that there exists $l \in \mathbf{Z}$ such that $x = 2l+5$. Now we let $k = l+2$; $k \in \mathbf{Z}$ since $l \in \mathbf{Z}$. We have

$$x = 2l+5 = 2l+4+1 = 2(l+2)+1 = 2k+1.$$

This shows that $x \in A$, and we have proved that $B \subseteq A$. This combined with the previous paragraph shows that $A = B$. ▮

The above result is not so important, but pay attention to the structure of this proof. It is called a "proof by double inclusion," since we have shown that A includes B and B includes A.

Set operations. Now we describe a number of operations that may be performed on sets to produce other sets. They can all be built up from the following two operations.

- The *union* $A \cup B$ of two sets A and B is the set of all elements that are either in A or in B (or both).
- The *intersection* $A \cap B$ of A and B is the set of all elements that are in both A and B.

As an example, if $A = \{1,2,4\}$ and $B = \{1,3,5\}$, then $A \cup B = \{1,2,3,4,5\}$ and $A \cap B = \{1\}$.

To avoid logical difficulties, we always assume that the sets we're working with are a subset of some larger ambient set U, often called the *universe.* Once we know what U is, we may define the *complement* of a set A to be the set $\overline{A}$ of all the elements in U except those that are in A. So if $U = \{1,2,3,4,5\}$ in the example above, then $\overline{A} = \{3,5\}$ and $\overline{B} = \{2,4\}$. What about $\overline{A \cup B}$? Well since $A \cup B$ is all of U, its complement must be empty, and we can denote it $\{\}$. This is one valid notation for the *empty set*. The other is $\emptyset$.

Now is a good time to define the cardinality $|A|$ of a set A. This is the number of elements in it, so $|A| = |B| = 3$ in our example, and $|A \cup B| = 5$, etc. We have $|\emptyset| = 0$, and it is possible for the cardinality of a set to be infinity; for example, $|\mathbf{N}| = \infty$. We also have $|\mathbf{R}| = \infty$, but this infinity is, in some sense, larger than $|\mathbf{N}|$. (More on that later.)

Next, we define the *difference* $B \setminus A$ (sometimes $B - A$) of two sets. This is the set of all elements in B that are *not* in A. So, using the complement notation we just learned about, we can express $B \setminus A = B \cap \overline{A}$. It is not necessary that A be a subset of B. In the small example above, we have $A \setminus B = \{2,4\}$ and $B \setminus A = \{3,5\}$.

Lastly, we define the *symmetric difference* $A \mathbin{\triangle} B$ of two sets A and B to be the set of all elements that are either in A or in B *but not both.* Invoking the above example one last time, we have $A \mathbin{\triangle} B = \{2,3,4,5\}$. We can express it as using unions, intersections, and complements as

$$A \mathbin{\triangle} B = (A \cup B) \cap \overline{A \cap B}. \tag{1}$$

To practise using all the different operations we just learned, convince yourself that the following are three more valid ways to express the symmetric difference:

$$A \mathbin{\triangle} B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A) = \overline{\overline{A \cup B} \cup (A \cap B)}$$

More set identities abound. We state the following proposition without proof; you should try going though this list and convincing yourself that each identity holds, for all sets A, B, and C. (This is a great way of practising proofs by double inclusion.)

Proposition 2. *Let A, B, and C be subsets of a universe U. Then*

i) $A \cap U = A$ *and* $A \cup \emptyset = A$;
ii) $A \cup U = U$ *and* $A \cap \emptyset = \emptyset$;
iii) $A \cup A = A$ *and* $A \cap A = A$;
iv) $\overline{\overline{A}} = A$;
v) $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$;
vi) $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C)$;
vii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *and* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
viii) $A \cup (A \cap B) = A$ *and* $A \cap (A \cup B) = A$; *and*
ix) $A \cup \overline{A} = U$ *and* $A \cap \overline{A} = \emptyset$. ▮

These laws have names, some of which we will use more often than others: (i) is called the identity law, (ii) the domination law, (iii) the idempotent law,

(iv) the law of double negation, (v) the commutative law, (vi) the associative law, (vii) the distributive law, (viii) the absorption law, and (ix) the complement law.

***Analogy with addition and multiplication.** Some of these laws bear some resemblance to laws about numbers that you already know. As an exercise, replace $\cup$ with $+$ (addition), $\cap$ with $\cdot$ (multiplication), $\overline{A}$ with $-A$ (negation), U with 1, and $\emptyset$ with 0 in all the formulas above, and now assume that A, B, and C are arbitrary real numbers. Which identities still hold in the number setting, and which ones don't? As a more advanced exercise, try replacing $\cup$ with $\triangle$ in the identities above (some statements will have to be tweaked a bit so that they're actually true, some won't). Now do the same replacement as before, except replace $\triangle$ with $+$. You will find that many more identities carry over.

03.IX **De Morgan's laws.** There are two important laws relating complements with union and intersection. We shall state them as a proposition, this time giving a proof (of one of them).

Proposition 3 (*De Morgan's laws*). *Let A and B be sets. Then*

i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$*; and*

ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proof. Let $x \in \overline{A \cup B}$. This means that x does not belong to the union of A and B, x cannot be in A, nor can it be in B. Since $x \notin A$, $x \in \overline{A}$, and since $x \notin B$, $x \in \overline{B}$. Therefore, $x \in \overline{A} \cap \overline{B}$. This shows that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Now assume that $x \in \overline{A} \cap \overline{B}$. So $x \in \overline{A}$ and $x \in \overline{B}$, meaning that $x \notin A$ and $x \notin B$. Since x is in neither A nor B, it is also not a member of the union $A \cup B$. We conclude that $x \in \overline{A \cup B}$. We have shown that $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$, which fact, combined with the previous paragraph, completes the proof of (i).

The proof of (ii) is similar and left to the reader as an exercise. ▮

Armed with all of these laws, we are able to perform lots of mechanical set manipulations to simplify expressions. For example, consider the expression

$$\big((A \setminus B) \cup A\big) \cap \overline{A \cap \overline{B}}.$$

Since $A \setminus B = A \cap \overline{B}$ and invoking the second De Morgan law on the right of the intersection yields

$$\big((A \cap \overline{B}) \cup A\big) \cap (\overline{A} \cup B).$$

Now, we can use absorption on the left-hand side to obtain

$$A \cap (\overline{A} \cup B),$$

and then distributing gives us

$$(A \cap \overline{A}) \cup (A \cap B) = \emptyset \cup (A \cap B) = A \cap B.$$

We thus see that the nasty expression $\bigl((A \setminus B) \cup A\bigr) \cap \overline{A \cap \overline{B}}$ is simply another way of writing $A \cap B$.

The Cartesian product and power set. From the real line $\mathbf{R}$, we can geometrically construct the Cartesian plane $\mathbf{R}^2$ by lining up parallel copies of $\mathbf{R}$, one for each element of the original line and all parallel to the original line. Notationally, $\mathbf{R}^2$ is the set of all ordered pairs (a, b), where $a, b \in \mathbf{R}$. Generalising this, for any sets A and B we can define the *Cartesian product* $A \times B$ to be the set

$$A \times B = \bigl\{(a, b) : a \in A, b \in B\bigr\}.$$

We sometimes write A^2 for $A \times A$, and more generally A^n for the n-fold Cartesian product of A with itself. (This explains the notation $\mathbf{R}^2$ for the Cartesian plane, and $\mathbf{R}^n$ for the n-dimensional vector space over $\mathbf{R}$.) Note that $A \times B$ is not equal to $B \times A$ in general.

Proposition 4. *If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.*

Proof. The set $A \times B$ consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$. There are $|A|$ choices for a, and for a, there are $|B|$ ways to pair it with a b from B. So there are $|A| \cdot |B|$ pairs in total. ▌

Now we define the *power set*. For a set A, this is the set of all subsets of A, and is commonly denoted by $\mathcal{P}(A)$ or 2^A. (We will use the latter notation in these notes.) In set-builder notation, we have

$$2^A = \{X \subseteq U : X \subseteq A\}.$$

As an example, if $A = \{1, 2, 3\}$, then

$$2^A = \bigl\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\bigr\}.$$

Note that even though, say, $1 \in A$, we do not have $1 \in 2^A$. We do, however, have $\{1\} \in 2^A$.

Another example is $2^{\mathbf{Z}}$, the set of all subsets of integers. If P is the set of primes, then $P \in 2^{\mathbf{Z}}$. Also in $2^{\mathbf{Z}}$ is the set $S = \{n^2 : n \in \mathbf{Z}\}$ of square numbers.

There is a way of encoding subsets with strings of 0s and 1s. Suppose we have a set

$$\{-3, 1, 7, 19, 23\}.$$

Fixing this order of the elements, for any arbitrary subset of this set, we can associate to it a binary string. Consider the subset $\{1, 19, 23\}$. This corresponds to the binary string 01011: since the first element, 3, is not in the subset, we write a 0. Then 1 is in the subset, so we write a 1, and so on. This is a reversible process. Given a binary string of length 5, say, 00101, we can reconstruct the subset that corresponds to it. The first two 0s mean that -3 and 1 do not belong to the set, but 7 does, 19 doesn't, and 23 does. So the subset is $\{7, 23\}$. In this way we see that there is a one-to-one correspondence between the elements of 2^A and binary strings of length $|A|$. We'll use this fact in the proof of the following proposition.

Proposition 5. *If A is finite, then $|2^A| = 2^{|A|}$.*

Proof. We just saw that there is a one-to-one correspondence between elements of 2^A and binary strings of length $|A|$. So it suffices to count the number of binary strings of length $|A|$. Well, each digit can be either 0 or 1, and there are $|A|$ digits, so the number of strings is

$$\underbrace{2 \cdot 2 \cdot \dots \cdot 2}_{|A| \text{ times}} = 2^{|A|}. \qquad \blacksquare$$

Counterexamples in proofs. We finish this subsection with a little example problem. *Is it true that $2^A \cup 2^B = 2^{A \cup B}$ for all sets A and B?*

Let's start by trying to prove the statement is true. As usual, we will attempt a double-inclusion proof. Let $2^A \cup 2^B$. This means X is either a subset of A or it is a subset of B. Either way, X is a subset of $A \cup B$, so $X \in 2^{A \cup B}$. So far so good; we have proved that $2^A \cup 2^B \subseteq 2^{A \cup B}$.

Now we try the other direction. Let $X \in 2^{A \cup B}$. So X is a subset of $A \cup B$. From here we want to say that X must be a subset of A or it must be a subset of B, but is that necessarily true? It is possible that X is contained slightly in A and slightly in B. So we have failed to prove that $2^{A \cup B} \subseteq 2^A \cup 2^B$ in general. But just because we have failed to prove that something is true doesn't mean we have proved it is false!

To actually prove that $2^{A \cup B} \subseteq 2^A \cup 2^B$ doesn't hold in general, we need to find a *counterexample*. That is, we need to construct sets A and B such that the statement is false. In this case, we can let $A = \{1, 2\}$, $B = \{3, 4\}$, so that $A \cup B = \{1, 2, 3, 4\}$. Then the set $\{1, 3\}$ is a subset of $A \cup B$ but is not a subset of A and it is not a subset of B. In other words, $\{1, 3\} \in 2^{A \cup B}$ but $\{1, 3\} \notin 2^A \cup 2^B$, proving that $2^{A \cup B} \not\subseteq 2^A \cup 2^B$.

05.IX **Russell's paradox.** Earlier, we said that the sets we are working with need to be a subset of a universe U, which has already been proved to be a set. We gave lots of ways to make sets out of new sets, such as the union and intersection operations, etc. Starting with the assumption that the empty set $\emptyset$ is a set, it is possible to define the set of natural numbers as follows. We can define $0 = \emptyset$, $1 = \{0\}$, and $2 = \{0, 1\}$, and so on. Now we take the set of all of these, and call this $\mathbf{N}$. (We can also define addition and multiplication on these set-theoretic "numbers" so that they behave like addition and multiplication do on $\mathbf{N}$.) From here we can do more funky stuff to define $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$, and prove that these are all sets (you can see this in a higher-level course on set theory, if you're interested). So there isn't much of a problem with all the sets we have played with so far; they are all subsets of things that are already known to be sets.

Ungodly things can happen if we don't stick by these rules. An example, due to Bertrand Russell, is the "set"

$$R = \{\text{sets } X : X \notin X\}.$$

In plain English, R is defined to be the set of all sets that contain themselves. This is not a subset of any known thing, so by our criterion above we would not consider it a set. But supposing it is, let us ask ourselves the following question. Does R contain itself? If it does not, then $R \notin R$, so R would be a set that satisfied the condition of R, so $R \in R$. But on the other hand, if $R \in R$, then R violates the condition defining R, so $R \notin R$. Round and round we go in a circle of contradiction.

Such is the price of meddling with "sets" that aren't subsets of known sets. This also shows that there is no such thing as "the set of all sets."

2. Propositional logic

A *proposition* is a statement that is true or false. For example "8 is even" is a statement we know to be true, and "8 is prime" is a statement we know to be false. The statement "n is prime" is not a proposition because its truth or falsity depends on what n is. The statement "$2^{2^{240}} - 1$ is prime" is a proposition, because it is either true or false (even though you or I might not know which one it is).

A *propositional variable* or a *boolean variable* is a variable which can take either the value 0 or 1, where 0 means "false" and 1 means "true." Usually we use letters p, q, and r to denote propositional variables. The simplest logical operator is negation, defined by the table

p	$\neg p$	
0	1	
1	0	.

This is also called the NOT operator, since if p is true, then $\neg p$ is false, and vice versa. Next is *conjunction*, which has the table

p	q	$p \wedge q$	
0	0	0	
0	1	0	
1	0	0	
1	1	1	.

This is also called the AND operator, because $p \wedge q$ is true if and only if p and q are both true. The OR operator, also called *disjunction*, has the table

p	q	$p \vee q$	
0	0	0	
0	1	1	
1	0	1	
1	1	1	.

We see that $p \vee q$ is true if p or q is true (or both). The symbol $\vee$ is meant to recall the Latin word *vel*, meaning "or." (One of the most important early treatises on mathematical logic and set theory was *Arithmetices principia, nova methodo exposita*, published in Latin in 1889 by Giuseppe Peano. It established the now-standard axiomatisation of the natural numbers.)

On the other hand, in English, we often use the word "or" to mean an *exclusive* or; that is, either p and q are true but not both. In mathematics, on the other hand, "or" is usually *inclusive*, so both p and q are allowed to hold at the same time. It is possible to express the exclusive disjunction (often called XOR) by a table, however. We will use the symbol $\oplus$ for this operator, and its table looks like this:

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

So $p \oplus q$ is true if p is true or q is true, but not both.

A *formula* is an expression containing propositional variables, 0, 1, logical operators, and parentheses. The formula must syntactically make sense; for instance, $0(\vee \wedge q\neg$ is not a formula. Just as in ordinary mathematical notation, parentheses are used to clarify which operators should be evaluated first. We will assume that negation applies first, but an expression such as $p \vee q \wedge r$ is ambiguous. (In many programming languages, conjunction has higher priority than disjunction, but in this class, just add parentheses to clarify.)

Above we have illustrated the basic logical operators by writing out their *truth tables*. These are tables that give the value of a formula for all possible values of its variables. We can write truth tables for more complex formulas as well:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg(p \wedge q) \oplus q$
0	0	0	1	1
0	1	0	1	0
1	0	0	1	1
1	1	1	0	1

Strictly speaking, the third and fourth columns are not necessary, but these intermediary columns help us verify the accuracy of the following ones.

Two formulas f_1 and f_2 are said to be *logically equivalent* if they have the same truth table; that is, they produce the same output if given the same input. In this case we write $f_1 \equiv f_2$. For example, let $f_1 = \neg(p \wedge q) \oplus q$, the formula whose truth table is illustrated above. Now let $f_2 = p \vee \neg q$. Its truth table is

p	q	$\neg q$	$p \vee \neg q$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	0	1

,

so we conclude that $f_1 \equiv f_2$. As a larger example, suppose we want to find all values of p, q, and r such that

$$f = (p \vee q) \wedge (\neg q \vee \neg r)$$

evaluates to 1. The truth table

p	q	r	$p \vee q$	$\neg q \vee \neg r$	f
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	1	1	1
0	1	1	1	0	0
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	0	0

shows that f evaluates to 1 precisely when

$$(p, q, r) \in \bigl\{(0,1,0), (1,0,0), (1,0,1), (1,1,0)\bigr\}.$$

To write out the truth table for a formula with n variables, we need 2^n rows, so this method is unsuitable for formulas with more than three or four variables.

Simplifying logical formulas. Just as we have rules for simplifying set expressions, there are ways to turn complicated logical formulas into simpler ones that are logically equivalent. What might surprise you is that the rules turn out to be exactly the same! To see the basis for this correspondence, consider the definition of a union of sets A and B. In set-builder notation, this is

$$A \cup B = \{x \in U : x \in A \text{ or } x \in B\}.$$

The "or" in the definition suggests that $\cup$ is intimately related to the $\vee$ operation in propositional logic. Repeating this process, we have the "dictionary"

Set theory	Propositional logic
sets A, B	variables p, q
unions $A \cup B$	disjunctions $p \vee q$
intersections $A \cap B$	conjunctions $p \wedge q$
complements $\overline{A}$	negations $\neg p$
symmetric differences $A \bigtriangleup B$	exclusive disjunctions $p \oplus q$
the empty set $\emptyset$	0
the universe U	1

Exploiting this connection, we have the following analogue of Proposition 2.

Proposition 2. *Let p, q, and r be propositional variables. Then*

i) $p \wedge 1 \equiv p$ *and* $p \vee 0 \equiv p$;
ii) $p \vee 1 \equiv 1$ *and* $p \wedge 0 \equiv 0$;
iii) $p \vee p \equiv p$ *and* $p \wedge p \equiv p$;
iv) $\neg\neg p \equiv p$;
v) $p \vee q \equiv q \vee p$ *and* $p \wedge q \equiv q \wedge p$;
vi) $p \vee (q \vee r) \equiv (p \vee q) \vee r$ *and* $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r)$;
vii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ *and* $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$;
viii) $p \vee (p \wedge q) \equiv p$ *and* $p \wedge (p \vee q) \equiv p$; *and*
ix) $p \vee \overline{p} \equiv 1$ *and* $p \wedge \overline{p} \equiv 0$. ▮

Since they are essentially the same as their set equivalents, the names of these laws are the same as in the realm of sets. We also have the propositional equivalent of De Morgan's law, which states that

x) $\neg(p \vee q) \equiv \neg p \wedge \neg q$ *and* $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Using these rules, we can now show that $\neg(p \wedge q) \oplus q \equiv p \vee \neg q$, which we already saw earlier from their truth tables. First we observe that

$$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q), \tag{2}$$

which is analogous to the identity (1) for symmetric differences. So

$$\begin{aligned}\neg(p \wedge q) \oplus q &\equiv \big(\neg(p \wedge q) \vee q\big) \wedge \neg\big(\neg(p \wedge q) \wedge q\big)\\ &\equiv (\neg p \vee \neg q \vee q) \wedge \big((p \wedge q) \vee \neg q\big)\\ &\equiv (\neg p \vee 1) \wedge \big((p \vee \neg q) \wedge (q \vee \neg q)\big)\\ &\equiv 1 \wedge \big((p \vee \neg q) \wedge 1\big)\\ &\equiv p \vee \neg q,\end{aligned}$$

where in the first line we use (2), in the second line we use De Morgan's law twice, in the third line we use the complement and distributive laws, the fourth line we use the domination and complement laws, and in the last line we use the identity law (twice). (It is not super important to remember the names of all these laws, as long as you remember their statements, but you may actually find it is easier to remember the names along with the statements, just as it might be easier to remember faces of people you've met if you also know their names.)

10.IX **Conditional and biconditional.** We now examine the *conditional* logical relation IF p THEN q. It is denoted by $p \Rightarrow q$ and its truth table is given by

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

.

Within the conditional statement, p is called the *antecedent*; this is the assumption. The statement that is asserted, conditional that the antecendent holds, is called the *consequent* q. You can quickly check that the relation $p \Rightarrow q$ is equivalent to $\neg p \vee q$. This fact is useful when performing mechanical simplifications. In English, the statement "if p then q" asserts a causal relation between p and q. Take a moment and reconcile this idea with the truth table above. It is normal to get a little tripped up if it's your first time seeing this table. In the first row, q doesn't even happen, so it might feel weird to say that p has "caused" q to happen in this case, and in the second row, p is not true and q is true, so it might seem odd that we have set $p \Rightarrow q$ to true, because there seems to be no relation between p and q. But it should make sense if you think of p as a precondition to a promise q, and then considering whether the promise is broken. As an example, suppose your friend says, If it snows tomorrow I'll work in Trottier with you. If it doesn't snow and she doesn't pull up, she hasn't technically broken her promise. If it doesn't snow and she shows up, then she still hasn't broken her promise. The only way she can break her promise is if it snows and she doesn't come to Trottier; this situation corresponds to the only 0-row in the truth table.

The conditional is a very important logical operator to understand, because most theorem statements assume some hypothesis and claim some conclusion. You will be asked to prove statements of this form, so it is important to understand the logical nature of the statements to begin with.

The last relation is called the *biconditional*, and it asserts that variables p and q are logically equivalent. That is, p happens if and only if q happens. It's truth table

p	q	$p \Leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

is pretty self-explanatory; in it, $p \Leftrightarrow q$ is true whenever p and q have the same truth value. The name "biconditional" is suggested by (part of) the following proposition.

Proposition 6. *Let p and q be propositional variables. Then*

$$\begin{aligned} p \Leftrightarrow q &\equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \\ &\equiv \big((\neg p) \vee q\big) \wedge \big((\neg q) \vee p\big) \\ &\equiv (p \wedge q) \vee \big((\neg p) \wedge (\neg q)\big). \end{aligned}$$

Proof. We leave the first equivalence to the reader (writing out the truth table is one way of proving it). The second equivalence follows from our earlier observation that $p \Rightarrow q \equiv \neg p \vee q$, and the third equivalence follows from the distributive, complement, and identity laws. ▮

A formula f is called a

i) *tautology* if $f \equiv 1$, i.e., f always evaluates to true;
ii) *contradiction* if $f \equiv 0$, i.e., f always evaluates to false;
iii) *contingency* if f can evaluate to both 1 and 0, depending on the values of its variables;
iv) *satisfiable* if f evaluates to 1 for at least one input; and
v) *falsifiable* if f evaluates to 0 for at least one input.

An example of a tautology is $p \vee \neg p$ and an example of a contradiction is $p \wedge \neg p$. This follows from the complement laws. To say that something is satisfiable is precisely to say that it is not a contradiction, and to say that something is falsifiable is equivalent to saying that it is not a tautology. Contingencies are those formulas that are both satisfiable and falsifiable (in other words, formulas that are neither tautologies nor contradictions).

Suppose we are asked which of the above definitions the formula

$$f \equiv \big(p \wedge (p \Rightarrow q)\big) \Rightarrow q$$

satisfies. This formula only has two variables, so it is easy enough to use a truth table for this purpose, but we will take the opportunity to practise simplifying the expression symbolically. First of all, let's change all conditionals of the form $r \Rightarrow s$, to disjunctions of the form $\neg r \vee s$. This gives us

$$f \equiv \neg\big(p \wedge (\neg p \vee q)\big) \vee q.$$

Now we use the distributive law to distribute the conjunction over the innermost disjunction, obtaining

$$f \equiv \neg\big((p \wedge \neg p) \vee (p \wedge q)\big) \vee q;$$

by the complement and identity laws in that order, this simplifies to

$$f \equiv \neg(p \wedge q) \vee q.$$

Now De Morgan's law and associativity give

$$f \equiv (\neg p \vee \neg q) \vee q \equiv \neg p \vee (\neg q \vee q),$$

and thus

$$f \equiv \neg p \vee 1 \equiv 1,$$

by the complement and domination laws in that order. We conclude that f is a tautology, which also means that it is satisfiable.

The fact that $\big(p \wedge (p \Rightarrow q)\big) \Rightarrow q$ is a tautology symbolically justifies the argument that of p is true and $p \Rightarrow q$ is true, then we should be able to conclude q. This form of argument is called *modus ponens*, and it dates back to ancient times. You probably use *modus ponens* all the time in everyday life without knowing it, and we will certainly use it in this class a lot.

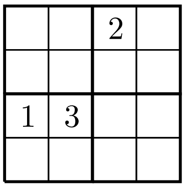
Encoding problems in propositional logic. Many algorithmic and logical problems can be encoded in propositional logic (and then later solved by a computer program). For example, suppose we want to play 4 × 4 Sudoku. In this game, we have a 4 × 4 grid and we want to fill it with the numbers 1 through 4 such that

i) every row contains 1 through 4;

ii) every column contains 1 through 4; and

iii) the four subsquares each contain 1 through 4.

In a given instance of the game, some cells are already filled in. The puzzle is: *Is there a solution and if so, what is it?*



		2	
1	3		

Fig. 1. An example 4 × 4 Sudoku game.

To represent a Sudoku game in propositional logic, we can define boolean variables $p_{i,j,k}$, where i, j, and k range over $\{1,2,3,4\}$. (So there are 4^3 variables in total.) We shall set

$$p_{i,j,k} = \begin{cases} 1, & \text{if the number } k \text{ is in row } i \text{ and column } j; \\ 0, & \text{otherwise.} \end{cases}$$

We'll number the rows increasing from the top and the columns increasing from left to right. For example, in Fig. 1 there is a 2 in row 1 and column 3, so $p_{1,3,2} = 1$.

Now we set to work encoding the conditions of a Sudoku grid in propositional logic:

i) To stipulate that every row contain 1 through 4, we first define auxiliary variables

$$r_{i,k} = p_{i,1,k} \vee p_{i,2,k} \vee p_{i,3,k} \vee p_{i,3,k},$$

for $i, k \in \{1,2,3,4\}$. With these helper variables, we now see that

$$r_{1,1} \wedge r_{1,2} \wedge r_{1,3} \wedge r_{1,4}$$

encodes the requirement that row 1 contains one of each number. We do the same for rows 2, 3, and 4 as well, and then combine with AND.

ii) We do a similar thing as in part (i) for each of the four columns.

iii) Ditto for subsquares.

iv) We need to set the initial values of the grid. For the grid in Fig. 1, we have the formula
$$p_{1,3,2} \wedge p_{3,1,1} \wedge p_{3,2,3}.$$

v) Lastly, we need to make sure that there is not more than one number per cell. To do this, for each cell $(i,j) \in \{1,2,3,4\}^2$ we write
$$p_{i,j,1} \Rightarrow (\neg p_{i,j,2} \wedge \neg p_{i,j,3} \wedge \neg p_{i,j,4}),$$
and so on (four conditionals in total). Of course we'll need to AND all these together.

Now we combine the formulas from each of these five steps into one long formula f such that f is satisfiable if and only if the grid has a solution, and the values for $p_{i,j,k}$ give a solution.

Defining all of these variables was a rather arduous and cumbersome process, and not entirely worth it for a 4×4 game of Sudoku (which can just be solved by eyeballing the grid). But one could imagine writing a general computer program to encode larger and larger grids. In fact, there are lots of problems that can be *reduced* to the problem of determining if a boolean formula is satisfiable. This means that if we have a program capable of taking a formula f as input and spitting out whether or not it is satisfiable (in a reasonable amount of time), then there are lots of real-world problems that this program could be applied to.

This problem is called the *boolean satisfiability problem*, often abbreviated SAT. One way of solving it for any given f is to just compute its truth table. We already know the downside of this approach: if f has n variables, then its truth table will have 2^n rows. Given a few minutes, you are certainly capable of writing down a formula f that has 300 variables, call them $p_1, \dots, p_{300}$. The truth table of f will have 2^{300} rows, which is more than the number of atoms in the observable universe. You can learn a lot more about SAT in a higher-level class on computational complexity (e.g., COMP 360/362).

3. Predicate logic

12.IX Propositional logic allows us to work with simple declarations, but this isn't powerful enough to express some deeper mathematical concepts. For this purpose, we now introduce the notion of a *predicate*. This is a statement involving some number of variables, each of which may take values coming from a universe U, such that the statement evaluates to either true or false *once all variables are assigned values*. The statement $P(n)$ given by "n is prime" is an example of this, where n can take any value in the universe $\mathbf{Z}$. An example with two variables is the predicate $L(x,y)$ defined by "x is less than y." (A more commonly-used notation for this predicate is "$x < y$.")

Predicates contain variables, but at the moment we don't have any way of introducing new variables into a statement. This is done using two different *quantifiers.* The first is the *universal quantifier*, denoted $\forall$ and meaning "for all." The statement $\forall n : P(n)$ is true if and only if $P(n)$ is true for every possible value that n can take. The second quantifier is the *existential quantifier*, written "$\exists$" and with the meaning "there exists." The statement $\exists n : P(n)$ is true if and only if there is (at least) one value that n can take such that $P(n)$ is true. The colon doesn't really have any mathematical meaning in these formulas; they just visually set the quantifiers apart from the predicates that follow.

For instance, taking $P(n)$ to be the statement "n is prime," where the universe U is $\mathbf{N}$, the statement $\exists n P(n)$ is true and the statement $\forall n : P(n)$ is false. What about the statement $\forall x \, \exists y : y < x$? Well, if the universe U is taken to be $\mathbf{N}$, then the statement is false, because setting x equal to 0, there is no element y of $\mathbf{N}$ such that $y < 0$. But if $U = \mathbf{Z}$, then the statement is true, since for every integer x, we can put $y = x - 1$, so that $y < x$.

Now we practise translating converting mathematical statements written in English into formulas in predicate logic. Suppose we want to write, Every integer is even or odd. The universe here is is the set $\mathbf{Z}$ of integers. The word "every" has the same meaning as "for all," so right off the bat, we can reexpress the statement as, For all $n \in \mathbf{Z}$, n is even or n is odd. In symbols, this is

$$\forall n \, (n \text{ even} \vee n \text{ odd}).$$

Lastly, we need to figure out how to express the property of being even or being odd. An integer n is even if an only if it is a multiple of two; that is, if there is some integer k such that $2k = n$. Likewise, an integer is odd if and only if it is one more than a multiple of two. The corresponding formula is $\exists k : 2k + 1 = n$. So our statement can be expressed

$$\forall n \, \bigl((\exists k : n = 2k) \vee (\exists k : n = 2k + 1)\bigr).$$

The variable k appears twice in this formula, but its first instance is independent of its second instance, because of the parentheses. (Readers who write computer programs will be familiar with the concept of a variable "going out of scope.") So there is nothing wrong with this formula, but to be extra clear that the first k is different from the second k, why don't we replace it with a different letter? Thus we arrive at

$$\forall n \, \bigl((\exists k : n = 2k) \vee (\exists l : n = 2l + 1)\bigr),$$

a formula in predicate logic that means, Every integer is even or odd.

Restrictions using quantifiers. It is not true that every real number has a multiplicative inverse, since one cannot divide by zero. However, the statement "every nonzero real number has a multiplicative inverse" is true. How should we write this as a formula in predicate logic? One way is to use the conditional: over the universe $U = \mathbf{R}$, we could write

$$\forall x : (x \neq 0 \Rightarrow \exists y : xy = 1).$$

Another is to use subscripts: in the same universe, we write

$$\forall x_{x \neq 0} \, \exists y : xy = 1.$$

Using subscripts is slightly informal, since we didn't formally define above what a subscript is supposed to mean, but it is something that you might encounter. The last way is to simply restrict the universe itself: in the universe $U = \mathbf{R} \setminus \{0\}$, the statement

$$\forall x \, \exists y : xy = 1$$

is true.

Multiple quantifiers. Withing a formula, quantifiers cannot be interchanged willy-nilly. The order of $\forall$ and $\exists$ matters! They are read from left to right. Consider the following examples, over the universe $U = \mathbf{R}$. The statement

$$\forall x \, \exists y : x + y = 0$$

is true, since for each given x we can take y to be $-x$. On the other hand,

$$\exists y \, \forall x : x + y = 0$$

is false, since it would mean that there is some integer that adds up to zero with *any* integer. Of course, sometimes switching the order of quantifiers, doesn't change the truth value of a statement. Both

$$\exists y \, \forall x : xy = 0$$

and

$$\forall x \, \exists y : xy = 0$$

are true, since in the first case, we can take $y = 0$, and in the second case, we can set y to 0 no matter what x is given.

So we know that the order of $\forall$ and $\exists$ matters in general, but repeated instances of the *same* quantifier *can* be interchanged. For instance,

$$\forall x \, \forall y : x^2 + y^4 \geq 0$$

is the same as

$$\forall y \, \forall x : x^2 + y^4 \geq 0,$$

and we can even write $\forall x, y : x^2 + y^4 \geq 0$, to introduce both variables simultaneously.

Negating quantifiers. The universal quantifier is sort of like a big chain of conjunctions that goes over the whole of the universe. For example, in the universe $\mathbf{N}$, the statement $\forall n : P(n)$ is equivalent to

$$P(1) \wedge P(2) \wedge P(3) \wedge \cdots,$$

if this were a valid propositional formula (it isn't because we don't allow propositional formulas to be infinite). Likewise, the existential quantifier $\exists n : P(n)$ is equivalent to

$$P(1) \vee P(2) \vee P(3) \vee \cdots.$$

We know, by De Morgan's laws, that negating a big series of conjunctions requires us to flip all the ANDs to ORs. So, once again abusing notation somewhat, we expect

$$\neg\bigl(P(1) \wedge P(2) \wedge P(3) \wedge \cdots\bigr) \equiv \neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \cdots.$$

Thus we conclude that

$$\neg\bigl(\forall n : P(n)\bigr) \equiv \exists n : \neg P(n).$$

You can play the same game with the other De Morgan's law to show that

$$\neg\bigl(\exists n : P(n)\bigr) \equiv \forall n : \neg P(n).$$

Going back to our example of $P(n)$ denoting "n is prime," the statement $\neg\bigl(\forall n : P(n)\bigr)$ is true, since not all integers n are prime, and we have just shown that this is equivalent to the statement $\exists n : \neg P(n)$; that is, there exists n such that n is not prime.

We end this section with a longer example. Let's express the statement, "There is a nonzero real number such that every real number is not its inverse or is negative." In the universe $\mathbf{R}$, the formula corresponding to this statement is

$$\exists x : (x \neq 0 \wedge (\forall y : xy \neq 1 \vee y < 0)\bigr).$$

(Work it out yourself!) Is this statement true or false? It turns out that it is true. You might be able to stare at the formula long enough to convince yourself of this fact, but another way to see that it's true is to note that its negation is false. Let's do this now (it's a good excuse to practise negating a formula). We have

$$\begin{aligned}
&\neg\Bigl(\exists x : (x \neq 0 \wedge (\forall y : xy \neq 1 \vee y < 0))\Bigr)\\
&\qquad\qquad \equiv \forall x : \neg(x \neq 0 \wedge (\forall y : xy \neq 1 \vee y < 0)\bigr)\\
&\qquad\qquad \equiv \forall x : \bigl(x = 0 \vee \neg(\forall y : xy \neq 1 \vee y < 0)\bigr)\\
&\qquad\qquad \equiv \forall x : \bigl(x = 0 \vee \exists y : \neg(xy \neq 1 \vee y < 0)\bigr)\\
&\qquad\qquad \equiv \forall x : \bigl(x = 0 \vee \exists y : (xy = 1 \wedge y \geq 0)\bigr).
\end{aligned}$$

This negated statement is false, since if $x = -2$, then $x = 0$ doesn't hold, so the left-hand side of the OR isn't true, and there is no y such that $-2y = 1$ and $y \geq 0$ are both true, since the only y satisfying $-2y = 1$ is $-1/2$.

Negating a formula in predicate logic is entirely mechanical. The $\neg$ symbol moves from left to right like a bulldozer that flips quantifiers and negates predicates it finds along the way, until eventually its job is done and it disappears. More broadly, we write mathematical statements in formal logic to make things more precise and mechanical. This can be useful to humans, since English is often ambiguous whereas the notation we just established is not. It can be useful to machines as well, since, as we just saw, manipulating a formula is something that can very easily be automated.

4. Proofs

We're getting to the fun part of the course now. Further back in these notes, we already proved a few statements about sets. This was just a taste of what's to come, as the main focus of this course is to teach you how to prove mathematical statements. These will all be statements that can ultimately be stated in predicate logic, so using the tools from the previous section, you can boil them down to their logical skeleton. In this section, we will formally define what it means to prove a statement in predicate logic.

To prove a statement, we always process the quantifiers from left to right. Whenever we encounter something of the form $\exists x : P(x)$, we are allowed to choose the value for x (from the given universe), and we just need to show that $P(x)$ holds for that value of x. Here's an example.

Proposition 7. *There exists an integer $m > 0$ and an integer $n < 0$ such that $m^2 + n^2 = 25$.*

We've written the statement in words, and we will continue to do so for all the propositions in these notes because we are humans and not cyborgs, but notice that the underlying predicate logical formula here is

$$\exists m \, \exists n : m > 0 \wedge n < 0 \wedge m^2 + n^2 = 25,$$

with $U = \mathbf{Z}$. For the remainder of this section, we'll continue to write out the formulaic equivalents of propositions, to practise converting between the two worlds.

Proof. After a moment's contemplation, we notice that $9 + 16 = 25$. So we need a positive integer that squares to to 9 and a negative number that squares to 16. Hence we may pick $m = 3$ and $n = -4$, and $m^2 + n^2 = 25$. ▌

On the other hand, to prove a statement of the form $\forall x : P(x)$, we are not allowed to choose the value of x. Instead, we imagine it is given to us, and we still have to prove $P(x)$, no matter what the x might be. Here's what we mean.

Proposition 8. *For all $x \in \mathbf{Q}$, there exists $y \in \mathbf{Z}$ such that $xy \in \mathbf{Z}$.*

The formula this time is

$$\forall x\, \exists y : y \in \mathbf{Z} \wedge xy \in \mathbf{Z},$$

over $U = \mathbf{Q}$.

Proof. Let $x \in \mathbf{Q}$. Then x can be expressed as the ratio of two integers; write $x = m/n$ with $m, n \in \mathbf{Z}$. Then, setting $y = n \in \mathbf{Z}$, we have $xy = (m/n) \cdot n = m \in \mathbf{Z}$. ▌

Notice how we introduced the variable x in the above proof. Because we're trying to prove a "for all" statement, we use the word "let," to indicate that x is given to us by some higher power. In fact, we should sort of view this higher power as possibly being malicious. A very useful way to think about writing a proof is to imagine a game in which you are trying to prove a statement in predicate logic, and a supernatural adversary is attempting to thwart you. Let's say this predicate has four variables, so the statement is

$$\exists x\, \forall y\, \forall z\, \exists w : P(x, y, z, w).$$

The variables in the statement are introduced from left to right. Each time you see a $\exists$ symbol, it's your turn. In the example above, you get to set the variable x to any element of the given universe (keeping in mind that your eventual goal is to prove $P(x, y, z, w)$). Each time there is a $\forall$ symbol, it's the adversary's turn, and you should be prepared for whatever he throws at you. So in our example, the adversary may set y and z to anything in the universe, and he knows you picked x. Lastly, you get to pick w. If $P(x, y, z, w)$ is true, you win, and if not, the adversary wins. Writing a proof is equivalent to describing a winning strategy for the player against the adversary.

17.IX **Proving conditional statements.** Many mathematical statements introduce some hypotheses, then assert some conclusion. Thus they are some kind of statement of the form $p \Rightarrow q$. To prove this kind of statement, we assume that p holds, then prove that q is true. This is because the only way a statement for $p \Rightarrow q$ to be false is if p is true and q is false, so we're showing this cannot happen. Take a look at the following example.

Proposition 9. *If n is an odd integer, then n^2 is also odd.*

The underlying formula is $\forall n\big((\exists k : n = 2k + 1) \Rightarrow (\exists l : n^2 = 2l + 1)\big)$, in the universe $U = \mathbf{Z}$.

Proof. Let n be an arbitrary odd integer, so that there exists k such that $n = 2k + 1$. Then

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

so setting $l = 2k^2 + 2k$, which is an integer, we have $n^2 = 2l + 1$. Hence n^2 is odd. ▌

To disprove a statement, we simply prove that its negation is true. For example, suppose we want to disprove the statement $\exists x\, \forall y : x + y = 0$ (this was a statement we encountered in the previous section). Its negation is

$$\forall x\, \exists y : x + y \neq 0,$$

and to prove it we let x be given, set $y = -x + 1$, and observe that $x + y = x + (-x + 1) = 1$, which is not equal to 0.

Now we define two words that sound similar but are very different, logically speaking. The *converse* of a conditional statement $p \Rightarrow q$ is the statement $q \Rightarrow p$. The statement $p \Rightarrow q$ and its converse both hold if and only if the biconditional statement $p \Leftrightarrow q$ holds. On the other hand, the *contrapositive* to $p \Rightarrow q$ is the statement $\neg q \Rightarrow \neg p$. What's the deal with this silly-looking conditional? Well, it turns out to actually be equivalent to $p \Rightarrow q$. Check it out:

$$\begin{aligned} p \Rightarrow q &\equiv \neg p \vee q \\ &\equiv q \vee \neg p \\ &\equiv \neg(\neg q) \vee \neg p \\ &\equiv \neg q \Rightarrow \neg p \end{aligned}$$

So to prove a statement of the form $p \Rightarrow q$, it is sufficient to prove $\neg q \Rightarrow \neg p$. Sometimes, this can make the proof a lot easier. Here's an example. (It is the converse of Proposition 9.)

Proposition 10. *If n is an integer such that n^2 is odd, then n is odd.*

Proof attempt. Assume n^2 is odd, so that $n^2 = 2k + 1$ for some integer k. Then $n = \sqrt{2k + 1}$.

Where do we go from here? It's not even clear that $\sqrt{2k+1}$ is an integer, let alone an odd one. The contrapositive comes to the rescue.

Proof of Proposition 10. We proceed by contraposition. Suppose that n is not odd; that is, n is even. So there exists an integer k such that $n = 2k$. Then $n^2 = (2k)^2 = 4k^2$. Setting $l = 2k^2$, we have $n^2 = 2l$, so n^2 is even. Hence n^2 is not odd. ▌

Proofs by contradiction. We now discuss a powerful method of proof, which dates back to at least the ancient Greeks and referred to for much of Western history by its Latin name, *reductio ad absurdum.* It proceeds to prove a statement p by assuming its negation $\neg p$ holds, then deriving a contradiction, i.e., a statement whose truth value is 0. By doing this, one will have proved that $\neg p \Rightarrow 0$, which in turn shows that p is true, since

$$\neg p \Rightarrow 0 \equiv \neg(\neg p) \vee 0 \equiv p \vee 0 \equiv p.$$

Here is an example of a proof by contradiction.

Proposition 11. *There is no least positive rational number.*

In $U = \mathbf{Q}$, one way to formulate this is

$$\neg\bigl(\exists x : x > 0 \land (\forall y : y > 0 \Rightarrow x \le y)\bigr).$$

Proof. Suppose, towards a contradiction, that there exists some rational $x > 0$ such that for all $y \in \mathbf{Q}$ with $y > 0$, $x \le y$. Then, we can apply this property with $y = x/2$ to see that $x \le x/2$. Since x is positive, we can divide this inequality by x on both sides to obtain $1 \le 1/2$. But this is absurd, since $1/2 < 1$. This contradiction completes the proof. ▮

Here is another classical example of a proof by contradiction. It was known to the Pythagoreans.

Theorem 12. *The number $\sqrt{2}$ is irrational.*

Proof. Suppose, towards a contradiction, that $\sqrt{2}$ is rational. Then there are integers p and q such that $\sqrt{2} = p/q$, and furthermore, we can assume that p and q do not have any common factors, since if they did, we could divide them out and the ratio would remain the same. Squaring both sides of the equation, we have $2 = p^2/q^2$, or in other words, $2q^2 = p^2$. This implies that p^2 is even, so by (the contrapositive of) Proposition 10, p is even as well. That means we can write $p = 2r$ for some integer r, and substitute this new information into the above equation to obtain $2q^2 = (2r)^2 = 4r^2$. Dividing out by 2 yields $q^2 = 4r^2$, so q^2 is even, and so is q.

We have deduced that p and q are both even. On the other hand, we assumed they had no common factors. This is a contradiction. ▮

Earlier we described the composition of a proof as playing a game with a supernatural adversary. To prove something by contradiction, then is is akin to starting the game by letting adversary believe he has already won, and then working from that assumption to derive an impossibility. Quite the devious strategem.

> Reductio ad absurdum, *which Euclid loved so much,*
> *is one of a mathematicians finest weapons.*
> *It is a far finer gambit than any chess gambit:*
> *a chess player may offer the sacrifice of a pawn*
> *or even a piece,*
> *but a mathematician offers the* game.
>
> — G. H. HARDY, *A Mathematician's Apology* (1940)

Case analysis. For all $n \geq 2$ and all propositions $p_1, \ldots, p_n, q$, we have the equivalence

$$(p_1 \vee p_2 \vee \cdots \vee p_n) \Rightarrow q \equiv (p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q) \wedge \cdots \wedge (p_n \Rightarrow q).$$

(As an exercise, prove this when $n = 2$. Later on, when we learn about mathematical induction, you'll be able to prove it in general.)

Why is this useful? Well, if we have two or more propositions $p_1, p_2, \ldots, p_n$ such that at least one of them must hold, that is,

$$p_1 \vee p_2 \vee \cdots \vee p_n \equiv 1,$$

then if we are able to show that

$$p_1 \Rightarrow q, \quad p_2 \Rightarrow q, \quad \ldots, \quad \text{and} \quad p_n \Rightarrow q$$

all hold, then we will have shown

$$1 \Rightarrow q \equiv \neg 1 \vee q \equiv 0 \vee q \equiv q.$$

Let's see an example of this in action.

Proposition 13. *For all integers $n \geq 0$, $1 + (-1)^n(2n - 1)$ is a multiple of 4.*

Proof. Let an integer $n \geq 0$ be given. We know that n is even or n is odd.

If n is even, then $n = 2k$ for some integer k, and

$$1 + (-1)^{2k}\big(2(2k) - 1\big) = 1 + 1^k(4k - 1) = 4k,$$

which is a multiple of 4.

If n is odd, then $n = 2k + 1$ for some integer k, and

$$1 + (-1)^{2k+1}\big(2(2k + 1) - 1\big) = 1 - (4k + 2 - 1) = 1 - (4k + 1) = -4k,$$

which is also a multiple of 4. ▮

19.IX Sometimes, the cases into which one might split the proof are not so obvious.

Proposition 14. *There exist irrational numbers a and b such that a^b is rational.*

Proof. Consider the number $(\sqrt{2})^{\sqrt{2}}$. It is either rational or irrational.

If it is rational, then we can set $a = b = \sqrt{2}$, which we already know to be irrational, by Theorem 12.

If it is irrational, then we can set $a = (\sqrt{2})^{\sqrt{2}}$ and $b = \sqrt{2}$. We compute

$$a^b = \big((\sqrt{2})^{\sqrt{2}}\big)^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2}\cdot\sqrt{2}} = (\sqrt{2})^2 = 2,$$

which is rational. ▮

Note that in this proof, we did not need to know whether $(\sqrt{2})^{\sqrt{2}}$ is rational or not; we just know that it must either be rational or irrational. (In fact, it is known that $(\sqrt{2})^{\sqrt{2}}$ is irrational, but the usual proof of this requires some notions from complex analysis as well as Galois theory, subjects that are typically not encountered until at least the second or third year of a mathematics degree.)

Mathematical induction. Sometimes we want to prove an infinite number of statements, indexed by the natural numbers. If the complexity of the statements sort of "grows" in n (a vague notion to be made precise soon), the following theorem holds.

Theorem 15 (*Principle of mathematical induction*). *Let $P(n)$ be a family of predicates indexed by $n \in \mathbf{N}$. Let $m \in \mathbf{N}$. If*

i) $P(m)$ holds; and

ii) for all $n \geq m$,

$$\bigl(P(m) \wedge P(m+1) \wedge \cdots \wedge P(n)\bigr) \Rightarrow P(n+1)$$

then $P(n)$ holds for all $n \geq m$.

In the following proof, we use the *well-ordering principle*, which is the fact that every nonempty subset of the natural numbers has a least element. You can see the proof of the well-ordering principle in a higher-level set theory course.

Proof. Assume that (i) and (ii) both hold. Our goal is to prove that $P(n)$ is true for all $n \geq m$, so we shall suppose, towards a contradiction, that there is some $n \geq m$ such that $\neg P(n)$ holds. (As an exercise, convince yourself, via symbolic manipulations, that

$$\neg \forall n : n \geq m \Rightarrow P(n) \equiv \exists n : n \geq m \wedge \neg P(n).)$$

In other words, the set of $n \geq m$ such that $\neg P(n)$ holds is nonempty, so it has a least element, call it k. So $k \geq m$, and we cannot have $k = m$ due to (i), so $k > m$. Furthermore, by the minimality of k, the statements $P(m)$, $P(m+1)$, all the way up to $P(k-1)$ are true. By (ii) though, this implies that $P(k)$ holds: a contradiction. ▌

When proving something by induction, we need to prove two things: one of the form (i) and another of the form (ii). The former is called the *base case* and the latter the *induction step* or *inductive step*. The assumption to the left of the $\Rightarrow$ symbol in (ii) is called the *induction hypothesis* or *inductive hypothesis*. Technically, the theorem above is called the principle of *strong* induction; the principle of *weak* induction has

ii)$'$ for all $n \geq m$, $P(n) \Rightarrow P(n+1)$,

instead of the stronger (ii). It turns out that both types of induction are actually equivalent, so we'll use both interchangeably. More often than not, the hypothesis (ii)$'$ is perfectly sufficient, and in these cases we'll simply use weak induction so as not to clutter our proofs with lots of unused hypotheses.

Here is an example. Suppose we want to find (and prove) a formula for the sum of the first n odd numbers. The first thing to do is to try some small cases. When $n = 1$, the sum is just 1. When $n = 2$, the sum is $1 + 3 = 4$, when $n = 3$, we have $1 + 3 + 5 = 9$, and for $n = 4$, we compute $1 + 3 + 5 + 7 = 16$. So the pattern goes $1, 4, 9, 16, \ldots$, which leads us to conjecture that the sum of the first n odd numbers might equal n^2. In fact, this is true, and we shall prove it by induction.

Proposition 16. *For all integers $n \geq 1$,*

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Proof. By induction on n. First we prove the base case, $n = 1$. We have

$$\sum_{i=1}^{1}(2i-1) = 2 - 1 = 1 = 1^2.$$

Now for the inductive step. Let $n \geq 1$ and assume that

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Then

$$\begin{aligned}\sum_{i=1}^{n+1}(2i-1) &= \sum_{i=1}^{n}(2i-1) + \big(2(n+1)-1\big)\\ &= n^2 + 2n + 2 - 1\\ &= n^2 + 2n + 1\\ &= (n+1)^2,\end{aligned}$$

where it is in the second line that we used the inductive hypothesis. ∎

The first thing to notice about doing a proof by induction is that the proof method itself doesn't tell you what it is you should prove. You have to guess at the correct statement first. Also, proofs by induction are often "unenlightening," in that they often don't reveal the fundamental reasons why something might be true. (The previous proposition can be illustrated by a rather simple picture, which is not a proof, but is somewhat more elucidating that the induction proof.)

A longer example now. Suppose we have a pile of n stones, with $n \geq 1$. We have a job, which can be described by a pseudo-algorithm.

Algorithm S (*Divide stones*). The input to this algorithm is an integer $n \geq 1$, representing a number of stones. We have a list `PILES` of integers representing a collection of piles of stones, as well as an integer variable `BALANCE`. Initialise `PILES` $\Leftarrow 1$ (one pile with n stones), and set `BALANCE` $\leftarrow 0$. This algorithm splits the stones into n piles of 1 stone each, accumulating profits into `BALANCE` along the way.

S1. [Done?] If every element of `PILES` is 1, terminate the algorithm and output `BALANCE`.

S2. [Choose a pile.] Select some element $m > 2$ from `PILES` and remove it from the list. (The variable m is the number of stones in this pile.)

S3. [Split.] Let k and l be two numbers with $k+l=m$. We append $\texttt{PILES} \Leftarrow k$ and $\texttt{PILES} \Leftarrow l$, and increment $\texttt{BALANCE} \leftarrow \texttt{BALANCE} + k \cdot l$. (We have split pile n into two piles of size k and l, and the payout for doing so is $k \cdot l$ dollars.)

S4. [Loop.] Go to step S1. ▮

This is not, strictly speaking, an algorithm, since we didn't specify how the algorithm should choose the integers k and l that add up to m in step S3. However, running through a few instances with, say, $n = 6$, on paper, using whatever choices of split you like in every iteration of step S3, you'll find that the algorithm always terminates with $\texttt{BALANCE} = 15$. Trying a few different starting values of n might lead you to conjecture the following proposition, which we will prove by (strong) induction.

Proposition 17. *For a given input n, Algorithm S always outputs $\texttt{BALANCE} = n(n-1)/2$, regardless of the choice of split at any given iteration of step S3.*

Proof. By induction on n. For the base case $n = 1$, note that we immediately output $\texttt{BALANCE} = 0$ in the very first step of the algorithm, and $0 = 1(1-1)/2$.

Now for the inductive step, let $n \geq 1$ and suppose that for $1 \leq k \leq n$, the payout for running algorithm S on input l is $l(l-1)/2$. Suppose we have a pile of $n+1$ stones. We shall divide it into piles of size k and size $n+1-k$. The total payout will be the pay for this division, namely $k(n+1-k)$, plus the pay for further subdividing the two piles. Thus by the induction hypothesis applied twice, the total payout will be

$$\begin{aligned} k(n+1-k) + {} & \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2} \\ &= \frac{2nk + 2k - 2k^2 + k^2 - k + n^2 + n - 2nk - k + k^2}{2} \\ &= \frac{n^2+n}{2} \\ &= \frac{(n+1)\big((n+1)-1\big)}{2}, \end{aligned}$$

which is the expected formula for $n+1$. ▮

5. Functions

A *function* f from a set A to a set B is a subset $f \subseteq A \times B$ such that for every $a \in A$, there is exactly one $b \in B$ such that $(a,b) \in f$. (If there is no $b \in B$, or more than one, we say that f is not well-defined. If $(a,b) \in f$, we write $f(a) = b$. The set A is called the *domain* and the set B is the *codomain*. The notation $f : A \to B$ is a way of concisely writing "f is a function with domain A and codomain B."

Here are some examples and non-examples.

i) The function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$ is

$$f = \big\{(a,b) \in \mathbf{R}^2 : b = a^2\big\},$$

when written in set-builder notation.

ii) On the other hand, the set

$$g = \big\{(a,b) \in \mathbf{R}^2 : a = b^2\big\}$$

is not a function, since $(1,1)$ and $(1,-1)$ are both in g. Furthermore, there is no element of g with -1 as its first coordinate.

iii) If $X = \{1,2,5\}$ and $Y = \{0,1,2,3,4,5\}$, then

$$\big\{(1,0),(2,4),(5,5)\big\}$$

is a function from X to Y.

iv) The set

$$h = \big\{(x,y) \in \mathbf{R}^2 : y = 1/x\big\}$$

is not a function, since there is no y such that $(0,y) \in h$. However if we amend the domain and consider

$$h = \big\{(x,y) \in \big(\mathbf{R} \setminus \{0\}\big) \times \mathbf{R} : y = 1/x\big\},$$

then in fact, $h : \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is a function.

24.IX **Injective and surjective functions.** The *range* or *image* of a function $f : A \to B$ is

$$f(A) = \big\{b \in B : \text{there exists } a \in A \text{ such that } b = f(a)\big\}.$$

These are all the values f actually outputs. For instance, if we let $f : \mathbf{Z} \to \mathbf{N}$ be given by $f(n) = n^2$, then

$$\begin{aligned} f(\mathbf{Z}) &= \{\dots, f(-2), f(-1), f(0), f(1), f(2), \dots\} \\ &= \{\dots, 9, 4, 1, 01, 4, 9, \dots\} \\ &= \{0, 1, 4, 9, 16, 25, \dots\}. \end{aligned}$$

A function $f : A \to B$ is called *surjective* or *onto* if $f(A) = B$, that is, for every $b \in B$ there exists some $a \in A$ such that $f(a) = b$. An example of a surjective function is $f : \mathbf{Q} \to \mathbf{Q}$ sending $x \mapsto x/2$. This is because for any $q \in \mathbf{Q}$, we can set $r = 2q$, and

$$f(r) = \frac{r}{2} = \frac{2q}{2} = q.$$

A function $f : A \to B$ is called *injective* or *one-to-one* if for all $a_1, a_2 \in A$ with $a_1 \neq a_2$, we also have $f(a_1) \neq f(a_2)$. Equivalently, f is injective if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$ for all $a_1, a_2 \in A$. For instance, the function $f : \mathbf{Z} \to \mathbf{N}$ that sends $n \mapsto n^2$ is not injective since $f(-1) = f(1)$, but $-1 \neq 1$. On the other hand, if we modify the domain, considering $f : \mathbf{N} \to \mathbf{N}$ sending $n \mapsto n^2$, then now f is injective, since if $f(m) = f(n)$, then $m^2 = n^2$, and there is only one positive integer that squares to any given integer, so $m = n$.

Hence we see that any function can be transformed into an injective one, in principle, by shrinking its domain (though this new function might no longer have the properties you liked in the original one), and any function $f : A \to B$ can be made surjective by changing its codomain to its range, i.e., letting $B = f(A)$.

The pigeonhole principle. We now take a brief pause to introduce one of the most fundamental laws in discrete mathematics, called the pigeonhole principle. We begin with the following intuitive theorem.

Theorem 18. *Let $a_1, a_2, \ldots, a_n$ be a finite sequence (repeats allowed) of real numbers. Let*

$$a = \frac{1}{n}\sum_{i=1}^{n} a_i$$

be the average value of the sequence and let m be the maximum value the sequence attains. Then $m \geq a$.

Proof. We have

$$a = \frac{1}{n}\sum_{i=1}^{n} a_i \leq \frac{1}{n}\sum_{i=1}^{n} m = \frac{mn}{n} = m. \quad ▮$$

This theorem can be summed up in one sentence: *The maximum is at least the average.* Don't underestimate this theorem even though its proof was a one-liner! It is often used to prove highly nontrivial results. (As an exercise, prove the similar statement: *The minimum is at most the average.*) From here we are now equipped to prove (a general version of) the pigeonhole principle.

Theorem 19. *Let A and B be finite sets with $|A| = m$ and $|B| = n$. For every function $f : A \to B$, then there is some $b \in B$ such that there are at least $\lceil m/n \rceil$ elements $a \in A$ with $f(a) = b$.*

Proof. Enumerate $B = \{b_1, b_2, \ldots, b_n\}$. For $1 \leq i \leq n$, let r_i be the number of $a \in A$ such that $f(a) = b_i$. This is a sequence that adds up to m, since every a in A maps to exactly one element of B. So the sequence has average m/n, and by the previous theorem, there must be some j such that $r_j \geq m/n$. But the r_i are all actually integers (since cardinalities of finite sets are integers), meaning that $r_j \geq \lceil m/n \rceil$. Letting $b = b_j$ completes the proof. ▮

The reason this is called the pigeonhole principle is because of the following special case.

Corollary 20 (*Pigeonhole principle*). *Let $n \geq 2$. If n pigeons nest in $n-1$ holes, there is at least one hole that contains at least two pigeons.*

Proof. Let A be the set of pigeons and B the set of pigeonholes. Let f be any function sending the set of pigeons to the set of holes. By the previous theorem, there is some hole with at least $\lceil n/(n-1) \rceil = 2$ pigeons in it. (This is because $1 < n/(n-1) < 2$ for all integers $n \geq 2$.) ∎

Bijections. A function f is called *bijective* (or a *bijection*, or a *one-to-one correspondence*) if it is injective and surjective. Bijections are important because of the following proposition.

Proposition 21. *Let A and B be finite sets. Then*

i) there exists a bijection $f : A \to B$ if and only if $|A| = |B|$; and

ii) if $|A| = |B|$ and $f : A \to B$ then f is injective if and only if f is surjective.

Proof. Suppose $|A| = |B| = n$. Then choose an ordering $a_1, \ldots, a_n$ of A and an ordering $b_1, \ldots, b_n$ of B. Let $f(a_i) = b_i$ for all $1 \leq i \leq n$. By construction, this is a bijection, proving one direction of (i).

On the other hand, suppose $|A| \neq |B|$ (so we prove this direction by contraposition). If $A < B$, then f cannot be surjective, since the image of f has size at most $|A| < |B|$ (at least one element of B must be missed). If $A > B$, then $|A|/|B| > 1$, so by Theorem 19, there is some element $b \in B$ such that the number of $a \in A$ mapping to b is at least $\lceil |A|/|B| \rceil \geq 2$. This means that f is not injective. We have proved part (i).

To prove part (ii), let $|A| = |B|$ and let $f : A \to B$. First we assume that f is injective. We enumerate $A = \{a_1, a_2, \ldots, a_n\}$. Then

$$f(A) = \big\{f(a_1), f(a_2), \ldots, f(a_n)\big\} \subseteq B.$$

All of the $f(a_i)$ are distinct, since if $f(a_i) = f(a_j)$, then $a_i = a_j$. So $\big|f(A)\big| = |A| = n$, and $f(A)$ is a size n subset of B, which also has size n. Hence $f(A) = B$; that is, f is surjective.

Lastly, suppose f is not injective (again we are using contraposition). So there are a_i and a_j such that $a_i \neq a_j$ but $f(a_i) = f(a_j)$. So

$$\big|f(A)\big| = \big|\big\{f(a_1), f(a_2), \ldots, f(a_n)\big\}\big| < n = |B|,$$

so $f(A) \neq B$ and f is not surjective. ∎

Item (i) of the previous proposition should be entirely intuitive, especially if we use the alternative nomenclature "one-to-one correspondence" instead of "bijection." (In fact, we already implicitly used (i) in these notes, in the proof of Proposition 5.) Item (ii) is perhaps not as immediate, but should become clear if you work it out with a picture.

Bijections. A function $f : A \to B$ is called *invertible* if there exists $g : B \to A$ such that

i) for all $b \in B$, $f\bigl(g(b)\bigr) = b$; and

ii) for all $a \in A$, $g\bigl(f(a)\bigr) = a$.

If g exists, it can be shown that g must be unique, so we write $g = f^{-1}$ and speak of *the* inverse of f.

Proposition 22. *Let $f : A \to B$ be a function. Then f is invertible if and only if f is bijective.*

Proof. First we assume that f is invertible. So there exists an inverse g of f. For each $b \in B$, setting $a = g(b)$ we have

$$f(a) = f\bigl(g(b)\bigr) = b.$$

This proves that f is surjective. To show that f is injective, suppose that $f(a_1) = f(a_2)$. By applying g on both sides, we have $g\bigl(f(a_1)\bigr) = g\bigl(f(a_2)\bigr)$, whence $a_1 = a_2$, by definition of g.

Now assume that f is bijective. We need to define $g : B \to A$. Well, gven any $b \in B$, there is some a such that $f(a) = b$, from surjectivity of f, and this a is unique, since f is injective. So set $g(b) = a$ (and repeat this process for every $b \in B$). We have $f\bigl(g(b)\bigr) = f(a) = b$, and for every $a \in A$, by definition of g the element $g\bigl(f(a)\bigr)$ is the unique element in A that gets brought to $f(a)$ by f, has to be a itself. ▮

Sometimes to prove that two sets have the same cardinality, it is easier to prove that there exists a bijection (as we already saw in the example of Proposition 5), and sometimes to prove that a function is a bijection, it is easier to show that it has an inverse, rather than messing around with the definitions of injective and surjective. Here's an example.

Proposition 23. *Let X be a finite nonempty set. Let E be the set of all subsets of X with even cardinality, and let F be the set of all subsets of X with odd cardinality. Then $|E| = |F|$.*

Proof. We shall construct a function $f : E \to F$. Fix one specific $x \in X$; we can do this because $X \neq \emptyset$. Now, for all $A \in E$, let

$$f(A) = \begin{cases} A \setminus \{x\}, & \text{if } x \in A; \\ A \cup \{x\}, & \text{if } x \notin A. \end{cases}$$

Note that since $|A|$ is even for all $A \in E$, the cardinality of $f(A)$ is odd (in the first case it is $|A| - 1$ and in the second case it is $|A| + 1$. This shows that f is indeed a function with codomain F. Now to prove $|E| = |F|$ we will show that f is bijective, which we do by showing that f is an inverse (as an exercise, you might instead try to prove bijectivity from the definitions of injective and surjective).

We define $g : F \to E$ by

$$g(A) = \begin{cases} A \setminus \{x\}, & \text{if } x \in A; \\ A \cup \{x\}, & \text{if } x \notin A. \end{cases}$$

As before, $\lvert g(A)\rvert$ is even, since A is assumed to be a member of F now. Now for any $A \in E$,

$$\begin{aligned} g\big(f(A)\big) &= \begin{cases} g\big(A \setminus \{x\}\big), & \text{if } x \in A; \\ g\big(A \cup \{x\}\big), & \text{if } x \notin A \end{cases} \\ &= \begin{cases} \big(A \setminus \{x\}\big) \cup \{x\}, & \text{if } x \in A; \\ \big(A \cup \{x\}\big) \setminus \{x\}, & \text{if } x \notin A \end{cases} \\ &= \begin{cases} A, & \text{if } x \in A; \\ A, & \text{if } x \notin A \end{cases} \\ &= A. \end{aligned}$$

The proof that $f\big(g(A)\big) = A$ is similar. Thus g is the inverse of f. ∎

6. Cardinality

26.IX Earlier, we defined the cardinality of a set to be the number of elements it contains. What, then, is the cardinality of $\mathbf{N}$? How about $\mathbf{R}$? You might say ∞, but this is not a number (at least, it's not an element of $\mathbf{N}$, the way all cardinalities of finite sets are). So perhaps we should amend our question to the following: *When do infinite sets have the same size?* Our experience with functions leads us to the answer: *When there exists a bijection between them.* We shall say that A and B are *equipotent* (or *equinumerous*, or *have the same cardinality*) if there exists a bijection between A and B. In this case we write $|A| = |B|$.

As an example, the sets $\mathbf{N}$ and $\mathbf{N} \setminus \{0\}$ are equipotent, since f given by $n \mapsto n+1$ is a bijection $\mathbf{N} \to \mathbf{N} \setminus \{0\}$. (Check that $f^{-1}(m) = m - 1$ is its inverse.)

It is even possible to remove an infinite number of elements from $\mathbf{N}$ and end up with something still equipotent with $\mathbf{N}$. To see this, let E be the set of nonnegative even integers, and consider the function $f : \mathbf{N} \to E$ sending $n \mapsto 2n$. This is injective because if $2m = 2n$, then dividing out by 2 on both sides yields $m = n$. It is surjective because if $n \in E$, then $n = 2k$ for some $k \in \mathbf{N}$, by definition, and $f(k) = 2k = n$.

So we can find subsets of $\mathbf{N}$ equipotent with it. It turns out we can also find supersets of $\mathbf{N}$ with the same property.

Theorem 24. *We have* $|\mathbf{N}| = |\mathbf{Z}|$.

Proof. We define $f : \mathbf{N} \to \mathbf{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even;} \\ -\frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

We shall show that f is a bijection.

Note first that if n is even, then $f(n) \geq 0$, and if n is odd, then $f(n) < 0$. So if $f(m) = f(n)$ for some $m, n \in \mathbf{N}$, then $f(m)$ and $f(n)$ must either both be negative, or both be nonnegative. Either way, m and n are either both even or they are both odd. If m and n are both even, then from $f(m) = f(n)$ we derive

$$\frac{m}{2} = \frac{n}{2},$$

whence multiplying by 2 on both sides we see that $m = n$. If m and n are both odd, then

$$-\frac{m+1}{2} = -\frac{n+1}{2},$$

so, multiplying by -2 and subtracting 1 from both sides we have $m = n$ in this case as well.

Now we show that f is surjective. Let $k \in \mathbf{Z}$. If $k \geq 0$, then consider $n = 2k$. We have

$$f(n) = f(2k) = \frac{2k}{2} = k.$$

If $k < 0$, then consider $n = -2k-1$. (Check that this is an element of $\mathbf{N}$.) Then

$$f(n) = f(-2k-1) = -\frac{-2k-1+1}{2} = -\frac{-2k}{2} = k.$$

This shows that f is surjective, so f is in fact bijective and we conclude that $|\mathbf{N}| = |\mathbf{Z}|$. ∎

We say that a set A is *countably infinite* if there exists a bijection $f : \mathbf{N} \to A$, that is, if $|\mathbf{N}| = |A|$. A set is said to be *countable* if it is either finite or countably infinite. Otherwise it is called *uncountable*. The previous theorem shows that $\mathbf{Z}$ is countably infinite.

Sometimes it is difficult to come up with a bijection directly. Instead, we would like to find an injection from A to B (which, in some sense, shows that $|A| \leq |B|$), and then an injection from B to A. This is made possible by the following useful theorem, named for Ernst Schröder and Felix Bernstein, who independently proved it in 1898. The proof is a bit difficult, so it's technically outside the scope of the course. For fun, you might try to do it as an exercise.

Theorem 25. (*Schröder–Bernstein theorem*). *If there exists an injective function $f : A \to B$ and another injective function $g : B \to A$, then there is a bijection $h : A \to B$.*

**Proof.* We present the proof as a (difficult) exercise. Here is the roadmap. Call $b \in B$ *unattached* if there is no $a \in A$ such that $f(a) = b$. Let $h : B \to B$ be given by $h(b) = f\bigl(g(b)\bigr)$. Given $b, b' \in B$, we shall say that b is a *peer* of b' if either $b = b'$ or there exists some $n \in \mathbf{N}$ such that

$$b = \underbrace{h(h(\cdots(h}_{n\text{ times.}}(b'))\cdots))$$

Say that $b \in B$ is a *PAE* if it is the peer of an unattached element. (So unattached elements are automatically PAEs, by setting $n = 0$.)

a) Show that if $a \in A$ is such that $f(a)$ is a PAE, then there is a unique element $b^* \in B$ such that $g(b^*) = a$, and that this element is a PAE.

By part (a), if $f(a)$ is a PAE, it makes sense to speak of $g^{-1}(a)$. It is the element $b^* \in B$ such that $g(b^*) = a$. From here we define

$$r(a) = \begin{cases} g^{-1}(a), & \text{if } f(a) \text{ is a PAE;} \\ f(a), & \text{otherwise.} \end{cases}$$

b) Show that if $b \in B$ is a PAE then so is $f\big(g(b)\big)$.

c) Show that r is surjective. [*Hint*: Do a proof by cases. Every $b \in B$ is either a PAE or it is not a PAE.]

d) We already proved in part (a) that if $f(a)$ is a PAE, then so is $r(a)$. Prove the converse of this statement.

e) Show that r is injective, and therefore bijective. [*Hint*: Assume $r(a_1) = r(a_2)$, and do a proof by cases again. Part (d) will be useful here.] ▮

Once again, this proof is outside the scope of the course. Don't lose sleep over it if you can't do it. Feel free to ask questions at office hours if you get stuck.

Using the Schröder–Bernstein theorem, we now show that the Cartesian product of two countable sets is also countable. In the proof, we shall employ the fact that if a set A is countable, then there exists an enumeration

$$A = \{a_0, a_1, a_2, \ldots\}.$$

(If f is the bijection given by the definition, we can set $a_0 = f(0)$, $a_1 = f(1)$, and so on.)

In the following proof, we'll also use the Fundamental Theorem of Arithmetic, which we state now, and prove later in the course (in the number theory section). It says that any integer can be factored as a product of primes, a theorem you should have learned in grade school.

Theorem 26 (*Fundamental Theorem of Arithmetic*). *Every positive integer $n \geq 2$ can be factored into a product*

$${p_1}^{v_1} {p_2}^{v_2} \cdots {p_m}^{v_m},$$

where $m \geq 0$ is an integer, $p_1, p_2, \ldots, p_m$ are distinct primes, and $v_1, v_2, \ldots, v_m$ are positive integers. This factorisation is unique up to the order of the primes. ▮

Theorem 27. *If A and B are countably infinite sets, then $A \times B$ is also countably infinite.*

Proof. Enumerate $A = \{a_0, a_1, \ldots\}$ and $B = \{b_0, b_1, \ldots\}$. Now given $(a_i, b_j) \in A \times B$ for some $i, j \in \mathbf{N}$, we can let

$$f(a_i, b_j) = 2^i 3^j.$$

We show that this defines an injective function. Suppose that $f(a_i, b_j) = f(a_{i'}, b_{j'})$. Then $2^i 3^j = 2^{i'} 3^{j'}$, and by the uniqueness of prime factorisations in the Fundamental Theorem of Arithmetic, we see that $i = i'$ and $j = j'$.

Now we produce an injection $g : \mathbf{N} \to A \times B$ by simply setting

$$g(n) = (a_n, b_0).$$

It is clear this is injective, since if $g(m) = g(n)$, then $(a_m, b_0) = (a_n, b_0)$, and that implies that $m = n$. ▮

Corollary 28. *We have* $|\mathbf{Z} \times \mathbf{Z}| = |\mathbf{N}|$.

Proof. We proved earlier that $\mathbf{Z}$ is countably infinite, so we may apply the previous theorem with $A = B = \mathbf{Z}$. ▮

This corollary concerning $\mathbf{Z} \times \mathbf{Z}$ allows us to prove that $\mathbf{Q}$ is countable as well.

Theorem 29. *The set* $\mathbf{Q}$ *of rational numbers is countable.*

Proof. We define $f : \mathbf{Q} \to \mathbf{Z} \times \mathbf{Z}$ as follows. For an element $q \in \mathbf{Q}$, we let $q = a/b$ the fraction q written in lowest terms (where $a \in \mathbf{Z}$ and $b \in \mathbf{N} \setminus \{0\}$). Then we set $f(q) = (a, b)$. This is an injective function because if $f(q) = f(q')$, then writing $q = a/b$ and $q' = a'/b'$ in lowest terms, we have $(a, b) = (a', b')$, so $q = a/b = a'/b' = q'$.

To define an injection $g : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Q}$, we recycle the injection we had from the proof of Theorem 27. We already know that $\mathbf{Z}$ is countable, so fix an enumeration $\mathbf{Z} = \{a_0, a_1, a_2, \ldots\}$. Then let $g(a_i, a_j) = 2^i 3^j$. The range of g is a subset of $\mathbf{N}$, so *a fortiori* it is a subset of $\mathbf{Q}$, and we already showed before that it is an injection.

We have shown that $|\mathbf{Q}| = |\mathbf{Z} \times \mathbf{Z}|$, which in turn shows that $|\mathbf{Q}| = |\mathbf{N}|$, after applying Corollary 28. ▮

So far, we have just given lots of examples of countably infinite sets. Finally, we give an example of a set that is not countably infinite.

Theorem 30. *The set* A *of all infinite binary strings is uncountable.*

Proof. Certainly A is not finite, so we need to show that $|\mathbf{N}| \neq A$.

Let $f : \mathbf{N} \to A$. We shall show that f is not surjective. For all $m, n \in \mathbf{N}$, let $a_{m,n}$ be the nth bit of $f(m)$. Consider the infinite binary string

$$s = (1 - a_{0,0}, 1 - a_{1,1}, 1 - a_{2,2}, 1 - a_{3,3}, \ldots).$$

This string s cannot equal $f(m)$ for any $m \in \mathbf{N}$, since given an arbitrary m, the mth bit of $f(m)$ is $a_{m,m}$, whereas the mth bit of s is $1 - a_{m,m}$. Hence there is some $s \in A$ such that $f(m) \neq s$ for all $m \in \mathbf{N}$. So f is not surjective.

Since there can be no surjection $f : \mathbf{N} \to A$, *a fortiori* we cannot have a bijection $\mathbf{N} \to A$. We conclude that $|A| \neq |\mathbf{N}|$. ▮

The proof above was published by Georg Cantor in 1891, and hence is known as *Cantor's diagonal argument.* The technique has since been used to prove many other things.

The set of all infinite binary strings is in bijection with the set of all real numbers in the interval $[0,1)$. If we're being extra pedantic, we need to forbid an infinite trailing string of all 1s, since, e.g., $0.0111\ldots = 0.1000\ldots$ after carrying the 1s (akin to the fact that, e.g., $0.0999\ldots = 0.1$). After dealing with this detail, one will have shown that $\bigl|[0,1]\bigr| \neq |\mathbf{N}|$.

7. Relations

01.X A *relation* on a set X is a subset $R \subseteq X \times X$. If $(a,b) \in R$ we write aRb and say "a is related to b." Here are some examples.

i) The set $L = \bigl\{(a,b) \in \mathbf{R} \times \mathbf{R} : a < b\bigr\}$ is a relation. For instance, we can write $(2,\pi) \in L$, or $2L\pi$, or $2 < \pi$ (which is the more common notation for this relation).

ii) The set $E = \bigl\{(a,b) \in \mathbf{R} \times \mathbf{R} : a = b\bigr\}$ is a relation (it is the "equals" relation).

iii) On $\mathbf{Z}$, the set

$$R = \bigl\{(-1,4),(8,-3),(0,0),(0,1)\bigr\}$$

is a relation.

iv) For any set A, a function $f : A \to A$ is by definition a subset of $A \times A$, and hence is an example of a relation.

v) Let H be the set of all humans, define $M \subseteq H \times H$ by setting (h_1,h_2) if and only if h_1 is married to h_2. For instance,

$$(\text{Michelle Obama}, \text{Barack Obama}) \in M.$$

On the other hand, it is unfortunately the case that

$$(\text{Marcel Goh}, \text{Taylor Swift}) \notin M.$$

Properties of relations. Let $R \subseteq A \times A$ be a relation. R is called

i) *reflexive* if for all $a \in A$, aRa;

ii) *symmetric* if for all $a,b \in A$ with aRb, we also have bRa; and

iii) *transitive* if for all $a,b,c \in A$ with aRb and bRc, we also have aRc.

Going back to the examples we had above, the relation L is transitive but neither reflexive nor symmetric, the relation E satisfies all three properties, the relation R satisfies none of them, and the relation M is symmetric but neither reflexive nor transitive. (Convince yourself of all of these facts.)

As a more involved example, let X be any set with $|X| \geq 2$. On 2^X, define a relation R by

$$(A, B) \in R \Leftrightarrow A \cap B \neq \emptyset$$

for all $A, B \subseteq X$. Which of the three properties above does R satisfy?

Is it reflexive? Well, is it true that for all $A \subseteq X$, $A \cap A \neq \emptyset$? The answer is no, since we have $\emptyset \cap \emptyset = \emptyset$, so we have $(\emptyset, \emptyset) \notin R$.

Is R symmetric? Well, if $A \cap B \neq \emptyset$, then since $\cap$ is commutative, we have $B \cap A \neq \emptyset$ as well, so $(A, B) \in R$ implies that $(B, A) \in R$. In other words, yes, R is symmetric.

Is R transitive? If $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$, does that necessarily mean that $A \cap C \neq \emptyset$? The answer is no. Here's the proof. Since $|X| \geq 2$ we can find $x, y \in X$ with $x \neq y$. Let $A = \{x\}$, $B = \{x, y\}$, and $C = \{y\}$. Then $A \cap B = \{x\} \neq \emptyset$, $B \cap C = \{y\} \neq \emptyset$, but alas $A \cap C = \emptyset$.

Equivalence relations and classes. If R is reflexive, symmetric, and transitive, then we say that R is an *equivalence relation*. Here are two examples (as an exercise, prove that both of these are equivalence relations).

i) Let F be the set of all formulas in propositional logic with the variables p, q, and r, and operations $\neg$, $\wedge$, and $\vee$. The relation $\equiv$ defined by $f_1 \equiv f_2$ if and only if f_1 and f_2 have the same truth table is an equivalence relation.
ii) On the set $\mathbf{R}^2$, define the relation $R \subseteq \mathbf{R}^2 \times \mathbf{R}^2$ by letting $(x, y)R(w, z)$ if and only if $\sqrt{x^2 + y^2} = \sqrt{w^2 + z^2}$. This is also an equivalence relation

If R is an equivalence relation, often we shall write $x \sim y$ to mean xRy. Sometimes we might even just say that $\sim$ *is* the equivalence relation.

Let $R \subseteq A \times A$ be an equivalence relation. Define the *equivalence class* of $a \in A$ to be the set

$$[a] = \{b \in A : a \sim b\}.$$

This is the set of all $b \in A$ that are related to a.

Take for instance the example F above of all formulas under the relation $\equiv$. The equivalence class of the formula $p \vee \neg p$ is the set of all formulas whose truth table contains only 1s, that is, the set of all tautologies. And for the example R above, the equivalence class of the point $(1, 3)$ is the set

$$\big[(1, 3)\big] = \big\{(x, y) \in \mathbf{R}^2 : \sqrt{10} = \sqrt{x^2 + y^2}\big\};$$

that is, the circle of radius $\sqrt{10}$ centred about the origin.

We have the following proposition concerning equivalence relations.

Proposition 31. *Let R be an equivalence relation on A. Then*

i) for all $x \in A$, $x \in [x]$;
ii) for all $x, y \in A$, $x \sim y$ if and only if $[x] = [y]$; and
iii) for all $x, y \in A$, $x \not\sim y$ if and only if $[x] \cap [y] = \emptyset$.

Proof. Let $x \in A$. Since R is reflexive, $x \sim x$, so we have $x \in [x]$. This proves part (i).

For part (ii), first we prove the "only if" direction. Let $x, y \in A$ be such that $x \sim y$. To prove that $[x] \subseteq [y]$, we let $z \in [x]$; so $x \sim z$. By symmetry, we have $z \sim x$, and this combined with $x \sim y$ allow us to deduce that $z \sim y$, by transitivity. Then by symmetry again, we have $y \sim z$, so $z \in [y]$. *Mutatis mutandis*, i.e., by swapping the roles of x and y, we also have $[y] \subseteq [x]$. Hence $[x] = [y]$.

Now for the "if" direction of (ii). Let $x, y \in A$ be such that $[x] = [y]$. By (i), we have $x \in [x]$, but then this means that $x \in [y]$. By definition of $[y]$, this means that $y \sim x$, so $x \sim y$ by symmetry.

On to part (iii). We prove both implications by contraposition (that is, we negate both sides of the statement). Let $x, y \in A$ be such that $[x] \cap [y] \neq \emptyset$. This means there is some $z \in A$ such that $z \in [x]$ and $z \in [y]$; so $x \sim z$ and $y \sim z$. By symmetry, $z \sim y$, so by transitivity, we have $x \sim y$.

On the other hand, suppose that $x, y \in A$ satisfy $x \sim y$. By (ii), we have $[x] = [y]$, so $[x] \cap [y] = [x] \neq \emptyset$, where we know that $[x] \neq \emptyset$ because it contains at least x (again, using (i)). ▮

Let A be a set. A *partition* of A is a set P of subsets of A (i.e., $P \subseteq 2^A$), such that

i) for all $x \in A$ there exists $S \in P$ such that $x \in S$;
ii) for all $S_1, S_2 \in P$ with $S_1 \neq S_2$, the intersection $S_1 \cap S_2$ is empty; and
iii) $\emptyset \notin P$.

Here are some examples.

i) Let E be the set of all even integers and F the set of all odd integers. Then $\{E, F\}$ is a partition of $\mathbf{Z}$.
ii) The set $\big\{(-\infty, 0), \{0\}, (0, \infty)\big\}$ is a partition of $\mathbf{R}$.

03.x If $\sim$ is an equivalence relation on a set A, then we can define the *quotient* of A by $\sim$ as the set of all equivalence classes of A under $\sim$. We denote this set by

$$A/{\sim} = \big\{[x] : x \in A\big\}.$$

We use the previous proposition to prove that quotients of sets by equivalence relations are partitions.

Proposition 32. *Let A be a set and $\sim$ an equivalence relation on A. Then $A/{\sim}$ is a partition of A.*

Proof. By part (i) the previous proposition, every $x \in A$ belongs to the equivalence class $[x]$. Then, by part (ii) of the previous proposition, we know that for any equivalence classes $[x]$ and $[y]$ such that $[x] \neq [y]$, we must have $x \not\sim y$, and by part (iii) of the previous proposition, we deduce that $[x] \cap [y] = \emptyset$. This shows that $A/{\sim}$ satisfies the second part of the definition of partition. Lastly, we note that $\emptyset \notin A/{\sim}$, since every element of $A/{\sim}$ is equal to $[x]$ for some $x \in A$, and must thus contain at least the element x. ▮

Let us now revisit the examples of equivalence relations from last class, and see what partitions they give rise to. In the example F of propositional formulas with variables p, q, and r, under the equivalence relation $\equiv$, the set of equivalence classes is the set of all possible truth tables on three variables. Each such truth table has 8 rows, so there are 2^8 equivalence classes. In other words, $|F/{\equiv}| = 2^8$.

How about the relation R defined on $\mathbf{R}^2$ where

$$(x,y) \sim (w,z) \qquad \text{if and only if} \qquad \sqrt{x^2+y^2} = \sqrt{w^2+z^2}?$$

Well, each $\bigl[(x,y)\bigr]$ is the circle of radius $\sqrt{x^2+y^2}$ centred around the origin, so $\mathbf{R}^2/{\sim}$ is the set of all circles in the plane centred at $(0,0)$.

II. NUMBER THEORY

Die Mathematik ist die Königin der Wissenschaften
und die Zahlentheorie ist die Königin der Mathematik.

— *Attributed to* C. F. GAUSS *(1777–1855)*

8. Division

Let $a, b \in \mathbf{Z}$. We say that a *divides* b (or b is a *multiple* of a, or b is *divisible* by a, or a is a *factor* of b) if there exists $n \in \mathbf{Z}$ such that $b = na$. In this case we write $a \mid b$. For example, $2 \mid 10$, since $10 = 5 \cdot 2$, but 3 does not divide 10, since there does not exist $n \in \mathbf{Z}$ such that $10 = 3n$. This defines a relation on $\mathbf{Z}$.

Note that for all $n \in \mathbf{Z}$, $n \mid 0$, since $0 = 0 \cdot n$. We also have $1 \mid n$ for all $n \in \mathbf{Z}$¡ since $n = n \cdot 1$. It is true that $0 \mid 0$, since we have, say, $0 = 1 \cdot 0$, but for all nonzero $n \in \mathbf{N}$, 0 does not divide n, since all multiples of 0 equal 0 (and thus cannot equal n). Further properties of the "divides" relation are given by the next proposition.

Proposition 33. *For all $a, b, c, d \in \mathbf{Z}$,*

i) if $a \mid b$, then $a \mid bc$;

ii) if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$;

iii) if $a \mid b$ and $b \mid c$, then $a \mid c$;

iv) if $a \mid b$ and $b \neq 0$, then $|a| \leq |b|$; and

v) if $a \mid b$ and $b \mid a$, then $|a| = |b|$.

Proof. We leave parts (i) and (ii) as exercises to the reader.

For part (iii), if $a \mid b$ and $b \mid c$, then there are integers k and l such that $b = ka$ and $c = lb$. Then $c = kla$, so $a \mid c$ (since kl is also an integer).

For part (iv), suppose that $a \mid b$ and $b \neq 0$. Then $b = ka$ for some $k \in \mathbf{Z}$, and $k \neq 0$ since $b \neq 0$. This means that $|b| = |k| \cdot |a|$, but $|k| \geq 1$, so $|b| \geq |a|$.

For part (v), assume that $a \mid b$ and $b \mid a$. If $a \neq 0$ and $b \neq 0$, then we may apply part (iv) twice to get $|a| \leq |b|$ and $|b| \leq |a|$, which together imply $|a| = |b|$. If $b = 0$, then $0 \mid a$ so $a = 0$ as well, and $|a| = |b|$. Likewise, if $a = 0$, then $0 \mid b$, so $b = 0$ and in this case as well, $|a| = |b|$. ▮

In grade school, you learned how to divide an integer by another one, obtaining a quotient and a remainder. We state this as a theorem. Its proof (which we shall consider outside the scope of the course, but which we include as optional reading for those interested) relies on the well-ordering principle (which we've used already in these notes) and the *Archimedean property* of $\mathbf{R}$, which states that for every $x \in \mathbf{R}$ there exists $n \in \mathbf{N}$ such that $n > x$. (One can learn the proof of the Archimedean property from an introductory course in analysis, e.g., MATH 242/254.)

Theorem 34 (*Division algorithm*). *Let $a, b \in \mathbf{Z}$ with $b > 0$. Then there exist unique integers q and r such that $a = bq + r$ and $0 \leq r \leq |b|$.*

**Proof.* First we show that such integers q and r exist. If $b \mid a$ then $a = kb$ for some $k \in \mathbf{Z}$, and we can set $q = k$ and $r = 0$.

If a is not divisible by b, then consider the numbers

$$\ldots, a - 3b, a - 2b, a - b, a, a + b, a + 2b, a + 3b, \ldots.$$

Let S be the set of these integers that are positive. Symbolically, we have

$$S = \{a - kb : k \in \mathbf{Z} \text{ and } a - kb \geq 0\}.$$

By the Archimedean property, there is some $n \in \mathbf{N}$ such that $n > -a$, which implies that $nb \geq n > -a$ (here we use the fact that if $b > 0$ and b is an integer, then $b \geq 1$). From this we derive $a > -nb$, and hence $a + nb = a - (-n)b > 0$. This shows that S is nonempty.

Since S is a nonempty subset of $\mathbf{N}$, by the well-ordering principle it has a least element, call it r. By definition of S, there must be some integer q such that $r = a - qb$, so $a = bq + r$. We now claim that $0 < r < b$.

We know that $r > 0$, since all elements of S are positive by definition. Suppose, for a contradiction, that $r \geq b$. Then $a - bq = r \geq b$, and so

$$0 \leq r - b = a - qb - b = a - (q + 1)b.$$

Since $q + 1$ is an integer, by definition of S, either $r - b$ is an element of S, or $r - b = 0$. Since r was defined to be the minimal element of S, it cannot be the case that $r - b$ is in S. So $r - b = 0$. But this means that $0 = a - (q + 1)b$; that is, $a = (q+1)b$, contradicting our assumption that b does not divide a. The contradiction allows us to conclude that $0 < r < b$ (in the case that b does not divide a). In general, we have shown that $0 \leq r < b$.

Lastly, we need to prove that q and r are uniquely determined by the integers a and b. Suppose that

$$a = bq_1 + r_1 \qquad \text{and} \qquad 0 \leq r_1 < b$$

and

$$a = bq_2 + r_2 \qquad \text{and} \qquad 0 \leq r_2 < b,$$

for some integers q_1, q_2, r_1, and r_2. We shall show that $q_1 = q_2$ and $r_1 = r_2$. Well, suppose that $r_1 \neq r_2$, for a contradiction. Without loss of generality we can assume that $r_1 < r_2$. Then subtracting the two equations, we obtain

$$0 = a - a = (bq_1 + r_1) - (bq_2 + r_2) = b(q_1 - q_2) + (r_1 - r_2).$$

This means that

$$r_2 - r_1 = b(q_1 - q_2),$$

so we find that $b \mid (r_2 - r_1)$. By part (iv) of the previous proposition, we obtain $|b| \leq |r_2 - r_1|$, and we can simply write $b \leq r_2 - r_1$, since both of these quantities are positive. But this is a contradiction, since

$$0 \leq r_1 < r_2 < b,$$

yields $r_2 - r_1 < b$. The contradiction shows that that $r_2 = r_1$. Substituting this into the relation $r_2 - r_1 = b(q_1 - q_2)$, we get $0 = b(q_1 - q_2)$ and conclude that $q_1 - q_2 = 0$, since $b > 0$. ▮

Let a and b be integers, not both zero. Their *greatest common divisor*, $\gcd(a,b)$ is defined to be the greatest positive integer d such that $d \mid a$ and $d \mid b$. Note that $\gcd(0,0)$ is not defined, since all positive integers d satisfy $d \mid 0$. On the other hand $\gcd(x,0)$ is simply $|x|$, and $\gcd(x,1) = 1$. Lastly, we don't need to worry about negative signs when computing greatest common divisors; i.e., $\gcd(\pm x, \pm y) = \gcd\bigl(|x|, |y|\bigr)$.

Euclid's algorithm. Now we ask ourselves, How do we compute greatest common divisors in general? The answer lies in one of the oldest algorithms known to humankind. It appears in Euclid's *Elements*, written around 300 B.C.

Algorithm E (*Euclid's algorithm*). Given two nonnegative integers a and b, not both zero, this algorithm outputs $\gcd(a,b)$.

E1. If $b = 0$, then output a and terminate the algorithm.

E2. Since $b \neq 0$, by the division algorithm we my write $a = qb + r$, where $0 \le r < b$. Set $a \leftarrow b$, $b \leftarrow r$, and return to step E1. ▌

The algorithm will eventually terminate, since the stopping criterion is that b be equal to 0, and in step E2 we replace b with a number that is strictly closer to 0. But will it terminate with the correct answer? Well, we know step E1 is correct, because of our earlier observation that $\gcd(a,0) = a$ (whenever a is positive). On the other hand, it is not at all evident that step E2 will eventually output $\gcd(a,b)$, since we actually overwrite the values of a and b in the step! The following lemma clarifies the situation.

Lemma 35. *Let $a, b, q, r \in \mathbf{Z}$ be integers such that $a = qb+r$. Then* $\gcd(a,b) = \gcd(b,r)$.

Proof. We shall show that for any $d \in \mathbf{Z}$,

$$d \mid a \text{ and } d \mid b \qquad \text{if and only if} \qquad d \mid b \text{ and } d \mid r.$$

For the forward implication, suppose that $a = kd$ and $b = ld$ for some $k, l \in \mathbf{Z}$. Substituting this into the identity $a = qb + r$ yields $kd = ldq + r$, whence $r = c(k - lq)$, so we conclude that $c \mid r$. (This is because $k - lq \in \mathbf{Z}$.)

For the reverse implication, suppose that $b = ld$ and $r = md$ for some $k, l \in \mathbf{Z}$. Substituting this into $a = qb+r$, we have $a = ldq+md$, so $a = d(lq+m)$, which means that $c \mid a$.

We have proved that a and b have the same common factors as b and r, so they must have the same greatest common divisor. ▌

Now that we are secure in the fact that Euclid's algorithm will indeed terminate with the correct output, let us now see it in action. Suppose we want to find $\gcd(30, 112)$. We perform successive divisions replacing the pair (a,b) with a new pair (b, r) each time:

$$\begin{aligned} 30 &= 0 \cdot 112 + 30 \\ 112 &= 3 \cdot 30 + 22 \\ 30 &= 1 \cdot 22 + 8 \\ 22 &= 2 \cdot 8 + 6 \\ 8 &= 1 \cdot 6 + 2 \\ 6 &= 3 \cdot 2 + 0 \end{aligned} \tag{3}$$

We stop once the remainder r equals 0, and the answer is $\gcd(b, r) = \gcd(b, 0) = b$. (So in the example above, the final answer is 2.)

Bézout's identity. The following theorem allows us to express the greatest common divisor as a linear combination of the two integers in question.

Theorem 36 (*Bézout's identity*). *Let a and b be nonzero integers with greatest common divisor $\gcd(a, b)$. Then there exist integers s and t such that*

$$\gcd(a, b) = sa + tb.$$

Moreover, $\gcd(a, b)$ is the least positive integer that can be expressed as an integer linear combination of a and b.

08.X *Proof.* Let

$$S = \{s'a + t'b : s', t' \in \mathbf{Z},\ ax + by > 0\}.$$

This set is nonempty, since if a is negative then $(-1)a + 0b \in S$ and if a is positive, then $1a + 0b \in S$. Since S is a nonempty set of positive integers, it has a least element, by the well-ordering principle. Call this integer $d = sa + tb$ (for some specific choices of $s, t \in \mathbf{Z}$); the claim is that $d = \gcd(a, b)$.

By the division algorithm, we may write

$$a = dq + r$$

where q and r are integers with $0 \le r < d$. But we can write

$$\begin{aligned} r &= a - qd \\ &= a - q(sa + tb) \\ &= (1 - qs)a + (qt)b, \end{aligned}$$

so $r \in S \cup \{0\}$. But $r < d$, so if $r \in S$, then d would not be the smallest element of S. So we must have $r = 0$. This implies that d is a divisor of a. Repeating this argument with b instead of a, we find that d divides b as well.

We have shown that d is a common divisor of a and b. It remains to show that it is the greatest one. That is, we must show that if $c \mid a$ and $c \mid b$, then $c \le d$. Well, if $a = kc$ and $b = lc$, then the identity

$$d = sa + tb = skc + tlc = (sk + tl)c,$$

shows that d is a multiple of c as well. Since $d > 0$, this means that $c \leq d$. ▌

This theorem is named for Étienne Bézout, who proved an analogous result (with polynomials instead of integers) in 1779, but the result above for integers has been known since at least the 1600s.

To actually find the integers s and t such that $\gcd(a, b) = sa + tb$, we first perform the Euclidean algorithm, keeping track of all our intermediate steps. Then we combine all the information from each step to work out what s and t are. For example, in the earlier example showing that $2 = \gcd(112, 30)$, we start with

$$2 = 8 - 1 \cdot 6,$$

which is (a rearrangement of) the fifth line of (3). Then the fourth line of (3) says that $6 = 22 - 2 \cdot 8$, so

$$2 = 8 - 1 \cdot (22 - 2 \cdot 8) = 8 - 22 + 2 \cdot 8 = 3 \cdot 8 - 22.$$

The third line of (3) tells us that $8 = 30 - 1 \cdot 22$, which gives

$$2 = 3 \cdot (30 - 1 \cdot 22) - 22 = 3 \cdot 30 - 3 \cdot 22 - 22 = 3 \cdot 30 - 4 \cdot 22.$$

We now have the number 30 appearing in the expression, we just need to get rid of the 22 and replace it with 112. To do this we use the second line of (3), which says that $22 = 112 - 3 \cdot 30$. We end up with

$$2 = 3 \cdot 30 - 4(112 - 3 \cdot 30) = 3 \cdot 30 - 4 \cdot 112 + 12 \cdot 30 = (-4) \cdot 112 + 15 \cdot 30.$$

So in the case that $a = 112$ and $b = 30$, we have $d = 2$, $s = -4$, and $t = 15$.

We now summarise this section on Bézout's identity with a little scenario. Imagine a frog that lives on a doubly-infinite line of lilypads, indexed by the integers. It starts at the point 0 and can hop in steps of a or b (in either direction). Theorem 36 tells us that the lilypad $d = \gcd(a, b)$ is reachable by the frog. The next proposition characterises the set of *all* lilypads that the frog can get to.

Proposition 37. *Let a and b be nonzero integers. The set*

$$X = \{s'a + t'b : s', t' \in \mathbf{Z}\}$$

is exactly the set of multiples of $d = \gcd(a, b)$.

Proof. By Bézout's identity, there exist integers s and t such that $d = sa + tb$. First let $n \in \mathbf{Z}$ be a multiple of d. Then there is $k \in \mathbf{Z}$ such that $n = kd$, and we have

$$n = kd = d(sa + tb) = (ds)a + (dt)b,$$

which means that $n \in X$ (since ds and dt are both integers).

Conversely, suppose that $n \in X$, so $n = s'a + t'b$ for some $s', t' \in \mathbf{Z}$. Then since d divides a and d divides b, we can write $a = ld$ and $b = md$ for some integers $l, m \in \mathbf{Z}$. So we have

$$n = s'a + t'b = s'ld + t'md = (s'l + t'm)d,$$

which shows that $d \mid n$, since $s'l + t'm$ is an integer. ∎

As a corollary, if $\gcd(a, b) = 1$, then it is possible for the robot to reach every integer! The situation in which $\gcd(a, b) = 1$ is very special, so much so that we have a name for it. We say that integers a and b are *relatively prime* or *coprime* if $\gcd(a, b) = 1$. By Bézout's identity, a and b are relatively prime if and only if there are integers s and t such that $1 = sa + tb$. This gives a very quick and easy way to check if certain numbers are relatively prime. For example, we have the following proposition.

Proposition 38. *For all integers $n > 1$, n and $n + 1$ are relatively prime.*

Proof. We have $1 = 1(n + 1) + (-1)n$. ∎

9. Primes

An integer p is *prime* if $p \geq 2$ and for all $d \in \mathbf{N}$ with $d \mid p$, we either have $d = 1$ or $d = p$. An integer n is *composite* if $n \geq 2$ and n is not prime. (By negating the definition of prime, we see that $n \geq 2$ is composite if and only if there exist $a, b \in \{2, \ldots, n-1\}$ such that $n = ab$.) Note that the integers 0 and 1 are neither prime nor composite.

The following theorem gives another characterisation of prime numbers.

Theorem 39. *An integer p with $p \geq 2$ is prime if and only if for all $a, b \in \mathbf{N}$, $p \mid ab$ implies that $p \mid a$ or $p \mid b$.*

Proof. First we prove the forward implication. Suppose that p is prime and let $a, b \in \mathbf{N}$ be such that $p \mid ab$. So there exists an integer k such that $ab = kp$. Consider $\gcd(a, p)$. Since the only divisors if p are 1 and p, this must be 1 or p. If it is p, then $p \mid a$ and we are done. So we restrict our attention to the case that $\gcd(a, p) = 1$, and our goal is to prove $p \mid b$. By Bézout's identity, there are integers s and t such that $1 = sa + tp$, so multiplying both sides by b, we have

$$b = bsa + btp = skp + btp = (sk + bt)p,$$

so $p \mid b$.

We prove the reverse implication by contraposition. Now suppose that p is composite. So there exist integers $2 \leq a, b \leq p - 1$ such that $p = ab$. We want to show that p does not divide a and p does not divide b. We shall do this by showing that $u = a/p$ and $v = b/p$ are both not integers. Well since $p = ab = upb$, by dividing through by p we arrive at $1 = ub$, and since $b \geq 2$, $u = 1/b$ is between 0 and 1. This shows that p does not divide a. On the other hand, since $p = ab = apv$, we have $1 = av$, and since $a \geq 2$, this means that $v = a/1$ is between 0 and 1. Hence p does not divide b. ▮

To illustrate that p really does have to be prime for this theorem to hold, consider $p = 6$, $a = 2$ and $b = 15$. We have $6 \mid 30 = 2 \cdot 15$, but 6 divides neither 2 nor 15. By induction, the theorem extends to arbitrary finite products.

Corollary 40. *Let p be prime and n be a positive integer. If $a_1, a_2, \ldots, a_n$ are integers such that $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_i$ for some $1 \leq i \leq n$.*

Proof. By induction on n. When $n = 1$, there is nothing to prove, for if $p \mid a_1$, then $p \mid a_i$ for $i = 1$.

Now suppose the statement holds for n. Assume that $p \mid a_1 a_2 \ldots a_n a_{n+1}$. By setting $a = a_1 a_2 \cdots a_n$ and $b = a_{n+1}$, we have $p \mid ab$, so by the previous theorem, either $p \mid a$ or $p \mid b$. If $p \mid b$, then $p \mid a_i$ for $i = n + 1$, and if $p \mid a$, then $p \mid a_1 \cdots a_n$, so by the induction hypothesis, $p \mid a_i$ for some $1 \leq i \leq n$. ▮

Back in Section 6, we used the prime factorisation of integers in a proof, but didn't prove that statement itself. It's finally time to do so. Recall that we statement we used is that any integer n can be factored into a product

$$n = {p_1}^{v_1} {p_2}^{v_2} \cdots {p_k}^{v_k}$$

where $p_1 < p_2 < \cdots < p_k$ are primes and $v_1, v_2, \ldots, v_n$ are positive integers. By renumbering the primes, allowing them to possibly equal one another, we have the following equivalent statement.

Theorem 26′ (*Fundamental Theorem of Arithmetic, again*). *Every integer $n \geq 2$ can be expressed as a product*

$$n = p_1 p_2 \cdots p_k$$

where $p_1 \leq p_2 \leq \cdots \leq p_k$ are prime numbers. Furthermore, this factorisation is unique.

Proof. The proof that such a decomposition of n exists is by (strong) induction. The base case is $n = 2$. This is already a prime factorisation, since 2 is prime.

For the inductive step, let $n \geq 2$ and assume such a decomposition exists for all $2 \leq i \leq n$. Now consider $n + 1$. If $n + 1$ is prime, then by setting $p_1 = n + 1$, we have a prime factorisation of $n + 1$ without getting out of bed. If $n + 1$ is not prime, then $n + 1 = ab$ for some integers $2 \leq a, b \leq n$. By the induction hypothesis, a and b can be factored into primes; that is, $a = p_1, \ldots, p_l$ and $b = q_1, \ldots, q_m$ for some primes $p_1, \ldots, p_l, q_1, \ldots, q_m$. So

$$n + 1 = ab = p_1 \cdots p_l q_1 \cdots q_m$$

is a factorisation of $n + 1$ into primes. It remains to arrange the ps and qs in nondecreasing order.

Now we prove that the prime factorisation of an integer is unique. In other words, $n \geq 2$ decomposes into

$$n = p_1 p_2 \cdots p_s$$

and

$$n = q_1 q_2 \cdots q_t$$

for primes $p_1, \ldots, p_s, q_1, \ldots, q_t$, then $s = t$ and $p_i = q_i$ for all $1 \leq i \leq s$. We also prove this statement by induction, but this time it is on the integer s. If $s = 1$, then

$$p_1 = n = q_1 q_2 \cdots q_t.$$

Note that t has to equal 1 here, since otherwise q_1 and q_2 would both divide p and satisfy $2 \leq q_1, q_2 \leq p_1$, contradicting the fact that p_1 is prime. So $p_1 = q_1$ and we are done.

Next, suppose that the uniqueness statement holds for s (i.e., for any integer that decomposes into a product of s primes, the decomposition is unique) and we want to show that it holds for $s+1$. We assume that

$$n = p_1 p_2 \cdots p_{s+1}$$

and

$$n = q_1 q_2 \cdots q_t,$$

where $p_1 \leq p_2 \leq \cdots \leq p_{s+1}$ and $q_1 \leq q_2 \leq \cdots \leq q_t$. We have $p_{s+1} \mid n = q_1 q_2 \cdots q_t$, so by the preceding corollary, there exists $1 \leq i \leq t$ such that $p_{s+1} \mid q_i$, and since q_i is prime, this means that $p_{s+1} = q_i \leq q_t$. Similarly, q_t divides $n = q_1 \ldots p_{s+1}$, so it divides p_j for some $1 \leq j \leq s+1$. This means that $q_t = p_j \leq p_{s+1}$. Hence $q_t = p_{s+1}$. But

$$p_1 p_2 \cdots p_s p_{s+1} = q_1 q_2 \cdots q_t,$$

so we may divide out by $p_{s+1} = q_t$ on both sides to get

$$\frac{n}{p_{s+1}} = p_1 p_2 \cdots p_s = q_1 q_2 \cdots q_{t-1}.$$

By the induction hypothesis, the decomposition of n/p_{s+1} is unique, so $t - 1 = s$ and $p_i = q_i$ for all $1 \leq i \leq t-1$. This proves that $s + 1 = t$ and $p_i = q_i$ for all $1 \leq i \leq s+1$. ▌

10.X We can use the Fundamental Theorem of Arithmetic to prove the following generalisation of Theorem 12.

Theorem 41. *Let k and n be positive integers. Then either $\sqrt[k]{n}$ is an integer or it is irrational.*

Proof. We prove this by contraposition, showing that if $\sqrt[k]{n}$ is rational, then it must be an integer. Suppose that $\sqrt[k]{n}$ is rational; so there exist integers a and b with $b \neq 0$ such that $\sqrt[k]{n} = a/b$. Without loss of generality, we may choose a and b such that $\gcd(a,b) = 1$. Writing

$$a = {p_1}^{v_1}{p_2}^{v_2}\cdots{p_r}^{v_r}$$

and

$$b = {q_1}^{w_1}{q_2}^{w_2}\cdots{q_s}^{w_s},$$

for primes $p_1 < p_2 < \cdots < p_r$, primes $q_1 < q_2 < \cdots < q_s$, and positive integers $v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_s$, all the primes p_i and q_j must be different, since if there was a prime in common, call it $p = p_i = q_j$, we would have $p \mid \gcd(a,b)$, contradicting the fact that $\gcd(a,b) = 1$.

Taking the identity $\sqrt[k]{n} = a/b$ to the power of k yields

$$n = \frac{a^k}{b^k} = \frac{{p_1}^{kv_1}{p_2}^{kv_2}\cdots{p_r}^{kv_r}}{{q_1}^{kw_1}{q_2}^{kw_2}\cdots{q_s}^{kw_s}}.$$

Since there do not exist $1 \leq i \leq r$ and $1 \leq j \leq s$ such that $p_i = q_j$, this fraction is in reduced form. But since n is an integer, that means the denominator equals 1. In other words, $n = a^k$, so $\sqrt[k]{n} = a$ is an integer. ▮

Let's see how to use this theorem. Suppose we want to know if the number $\sqrt[6]{18}$ is irrational or not. Well, since $1^6 = 1$ and $2^6 = 64$, we have $1^6 < 18 < 2^6$, meaning that $1 < \sqrt[6]{18} < 2$. (This is because the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^6$ is increasing on the interval $[1,2]$.) Hence $\sqrt[6]{18}$ is not an integer, and by the theorem it must be irrational.

Next, we show that there are infinitely many primes, a theorem first proved by Euclid.

Theorem 42. *There are infinitely many prime numbers.*

Proof. Suppose, towards a contradiction, that there are finitely many prime numbers, call them $p_1, p_2, \ldots, p_m$. Let $n = p_1p_2\cdots p_m$, and consider the integer $n+1$. Either it is prime or it is not. If $n+1$ is prime, then we already have a contradiction, since $n+1 > p_i$ for all $1 \leq i \leq m$, and we assumed that the $p_1, \ldots, p_m$ were all the primes. If $n+1$ is not prime, then it is divisible by some prime in our list, call it p_i. Hence we can write $n = kp_i$ for some integer k, and we have

$$kp_i = n + 1 = p_1p_2\cdots p_m + 1.$$

Rearranging this a bit, we have

$$1 = kp_i - p_1p_2\cdots p_m = p_i(k - p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_m);$$

in other words, p_i divides 1. This is a contradiction as 1 is not divisible by any integer greater than 1. ▮

Though the set P of prime numbers is infinite, it does sort of get "sparser" as one heads off towards infinity. This is quantified by the following theorem, proved independently in 1896 by Jacques Hadamard and Charles Jean de la Vallée Poussin.

Theorem 43 (*Prime number theorem*). *For $x \in \mathbf{R}$, let*

$$\pi(x) = \big|\{p \le x : p \text{ prime}\}\big|.$$

Then $\pi(x) \sim x/\ln x$, in the sense that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1.$$

The proof is long and arduous, requiring a lot of background in complex analysis. It is often taught in a first-year graduate class in analytic number theory. Here is an easy corollary of the prime number theorem.

Corollary 44. *Let n be a positive integer and let m be chosen uniformly at random from the set $\{1, 2, \dots, n\}$. Then*

$$(\ln n)\,\mathbf{P}\{m \text{ prime}\} \to 1$$

as $n \to \infty$. In other words, the probability that m is prime is asymptotically $1/\ln n$.

Proof. Since m is chosen uniformly at random from $\{1, \dots, n\}$, the probability $\mathbf{P}\{m \text{ prime}\}$ equals $\pi(n)/n$. So

$$\lim_{n\to\infty} (\ln n)\,\mathbf{P}\{m \text{ prime}\} = \lim_{n\to\infty} \frac{\pi(n)\ln n}{n} = 1,$$

by the prime number theorem. ▮

As an example, if we choose a 30-digit number at random (so $n = 10^{30} - 1$), the probability that this number is prime is roughly $1/\ln(10^{30}) = 1/(30\ln 10) = 0.0145$ or 1.45%.

10. Modular arithmetic

Fix $n \ge 1$, and let $a, b \in \mathbf{Z}$. We say a is congruent to b modulo n if $n \mid a - b$, i.e., if $a - b = kn$ for some $k \in \mathbf{Z}$. Write $a \equiv b \pmod{n}$ or $a \equiv_n b$.

Take for example $n = 12$. We have, e.g., $4 \equiv 16 \pmod 14$, since $4 - 16 = -12 = (-1)12$, but, e.g., $7 \not\equiv 17$, since $7 - 17 = -10$, and 12 does not divide -10. This situation should be a familiar one, since we are used to working with numbers modulo 12 when telling the time.

For any fixed n, the set of all $(a, b) \in \mathbf{Z}^2$ with $a \equiv_n b$ is a relation on $\mathbf{Z}$. In fact, we have the following proposition.

Proposition 45. *For all fixed n, the relation $\equiv_n$ is an equivalence relation on the set $\mathbf{Z}$.*

Proof. We must show that $\equiv_n$ is reflexive, symmetric, and transitive.

Let $a \in \mathbf{Z}$. We have $a - a = 0 = 0 \cdot n$, so $a \equiv a \pmod{n}$, proving reflexivity.

Let $a, b \in \mathbf{Z}$ and suppose that $a \equiv_n b$, so there exists k such that $a - b = kn$. Then $b - a = (-k)n$, so $b \equiv_n a$. This proves symmetry.

Lastly, let $a, b, c \in \mathbf{Z}$ be such that $a \equiv_n b$ and $b \equiv_n c$. So there exist integers $k, l \in \mathbf{Z}$ such that $a - b = kn$ and $b - c = ln$. Then

$$a - c = (a - b) + (b - c) = kn + ln = (k + l)n,$$

which shows that $a \equiv_n c$. ▮

Since $\equiv_n$ is an equivalence relation, it partitions $\mathbf{Z}$ into equivalence classes. We shall denote by $[a]_n$ the equivalence class of a modulo n. This is the set

$$[a]_n = \{b \in \mathbf{Z} : a - b = kn \text{ for some } k \in \mathbf{Z}\} = \{a + ln : l \in \mathbf{Z}\}.$$

For instance, when $n = 3$, the set

$$[0]_3 = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$$

is just the set of all multiples of 3, and we also have

$$[1]_3 = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$$

and

$$[2]_3 = \{\ldots, -4, -1, 2, 5, 8, \ldots\}.$$

These are all of the equivalence classes, since, for instance,

$$[4]_3 = \{\ldots, -2, 1, 4, 7, 10, \ldots\} = [1]_3.$$

Lastly, we shall touch upon the *modulo* operator, which is a feature of many programming languages. Let $a \in \mathbf{Z}$ and $b \ge 1$. Let $a \,\%\, b = r$ where q and r are the integers given by the division algorithm. (That is, $a = qb + r$ where $0 \le r \le b$.)

Proposition 46. *Fix an integer $n \ge 2$. Let $a, b \in \mathbf{Z}$. Then $a \equiv b \pmod{n}$ if and only if*

$$a \,\%\, n = b \,\%\, n.$$

Proof. Suppose that $a \equiv_n b$, so that there exists $k \in \mathbf{N}$ such that $a - b = kn$. Then, let $a = nq_1 + r_1$ and $b = nq_2 + r_2$ from the division algorithm. (So $r_1 = a \,\%\, n$ and $r_2 = b \,\%\, n$, and we have $0 \le r_1, r_2 < n$.) Write

$$kn = a - b = (q_1 - q_2)n + (r_1 - r_2),$$

which we can rearrange to

$$r_1 - r_2 = kn - (q_1 - q_2)n = (k - q_1 + q_2)n.$$

This implies that $r_1 - r_2$ divides n, but since $r_1, r_2 \in [0, n)$, the quantity $r_1 - r_2$ is in the range $[-n-1, n)$. Hence the only way it can divide n is for $r_1 - r_2 = 0$. We conclude that $a \,\%\, n = b \,\%\, n$.

For the other direction, we once again let $a = nq_1 + r_1$ and $b = nq_2 + r_2$ from the division algorithm. Now the assumption is that $r_1 = a \,\%\, n = b \,\%\, n = r_2$. So

$$a - b = (q_1 - q_2)n + (r_1 - r_2) = (q_1 - q_2)n.$$

This shows that $a \equiv b \pmod{n}$. ▮

22.X This proposition is useful in practice. For example, suppose we want to know whether $22 \equiv 8 \pmod 3$. We calculate $22 = 7 \cdot 3 + 1$, so $22 \,\%\, 3 = 1$; meanwhile $8 = 2 \cdot 3 + 2$, so $8 \,\%\, 3 = 2$. By the proposition, this means that $22 \not\equiv 8 \pmod 3$.

The proposition also implies that for all $a \in \mathbf{Z}$ and $n \geq 2$ one has $[a]_n = [a \,\%\, n]$. From the division algorithm, we know that $a \,\%\, n$ is an element in the range $[0, n)$; it is equal to the integer r in that range such that we may write $a = qn + r$ for some integer q. We may choose to denote the whole equivalence class by this element r. The set

$$\mathbf{Z}/n\mathbf{Z} = \mathbf{Z}/{\equiv_n} = \big\{[0], [1], \ldots, [n-1]\big\}$$

is called the *ring of integers modulo n* or the *cyclic group on n elements.*

Computations modulo *n*. We can do addition and multiplication modulo n by first doing addition and multiplication and then taking the remainder with respect to n. For example, when $n = 4$ we have the addition table

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

and the multiplication table

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1 .

For $x, y \in \mathbf{Z}$, if $xy = 0$, then either $x = 0$ or $y = 0$. On the other hand, from the assumption $xy \equiv 0 \pmod{n}$ we cannot conclude in general that $x \equiv 0 \pmod{n}$ or $y \equiv 0 \pmod{n}$. As an example, when $n = 4$ and $x = y = 2$, neither x nor y is congruent to 0 modulo 4, but their product is 0 modulo 4. An element $a \in \mathbf{Z}$ with $a \not\equiv 0 \pmod{n}$ is said to be a *zero divisor* if there exists $b \in \mathbf{Z}$ with $b \not\equiv 0 \pmod{n}$ such that $ab \equiv 0 \pmod{n}$.

Remember that the numbers in the tables above are elements of $\mathbf{Z}/n\mathbf{Z}$ (in the case $n = 4$), hence they are not really integers so much as equivalence classes of integers, which are sets! So it is not immediate that addition and multiplication are well-defined modulo some fixed integer n. For instance, to say that $[3]_4 \cdot [1]_4 = [3]_4$, it is not enough to just observe that $3 \cdot 1 = 3$. We need to show that for any integers $a \in [3]_4$ and $b \in [1]_4$, we have $a \cdot b \in [3]_4$. This is the essence of the following proposition.

Proposition 47. *Let* $a \equiv c \pmod{n}$ *and* $b \equiv d \pmod{n}$. *Then*

i) $a + b \equiv c + d \pmod{n}$*;*

ii) $ab \equiv cd \pmod{n}$*; and*

iii) $a^m \equiv c^m \pmod{n}$ *for all* $m \in \mathbf{N}$.

Proof. By assumption, there are integers k and l such that $a - c = kn$ and $b - d = ln$.

To prove (i), we compute

$$(a + b) - (c + d) = a - c + b - d = kn + ln = (k + l)n,$$

so $a + b \equiv c + d \pmod{n}$.

For part (ii), we compute

$$\begin{aligned} ab - cd &= ab - cb + cb - cd \\ &= b(a - c) + c(b - d) \\ &= bkn + cln \\ &= (bk + cl)n, \end{aligned}$$

so $ab \equiv cd \pmod{n}$.

We prove part (iii) by induction. The base case is $m = 0$; we have $a^0 = 1 = c^0$, so $a^0 \equiv c^0 \pmod{n}$. Now let $m \geq 0$ and assume that $a^m \equiv c^m \pmod{n}$. Combining this with the hypothesis $a \equiv c \pmod{n}$, we use part (ii) to conclude that $a \cdot a^m \equiv c \cdot c^m \pmod{n}$, which simplifies to $a^{m+1} \equiv c^{m+1} \pmod{n}$. ∎

We can use this to perform the modulo operation on large numbers without necessarily knowing the large numbers themselves. For example, let $m = 21^{10} - 3 \cdot 13 + 14$ and suppose we want to know $m \,\%\, 6$.

First we observe that $14 \equiv 2 \pmod{6}$ and $13 \equiv 1 \pmod{6}$. So

$$m = 21^{10} - 3 \cdot 13 + 14 \equiv_6 21^{10} - 3 \cdot 1 + 2 \equiv_6 21^{10} - 1.$$

Now we deal with the exponent by first taking the base of the exponent modulo 6. We have $21 \equiv 3 \pmod 6$; hence

$$m \equiv_6 3^{10} - 1.$$

To reduce the exponent, we factorise $10 = 2 \cdot 5$, yielding

$$m \equiv_6 (3^2)^5 - 1.$$

But again we can reduce the base here, since $9 \equiv_6 3$, which gives us

$$m \equiv_6 3^5 - 1.$$

Now since the exponent 5 is prime, we aren't able to reduce it by factorising. However, we do have $5 = 2{+}2{+}1$, and, using once again the fact that $3^2 = 9 \equiv_6 3$, we obtain

$$m \equiv_6 3^2 \cdot 3^2 \cdot 3 - 1 \equiv_6 3 \cdot 3 \cdot 3 - 1 \equiv_6 3 \cdot 3 - 1 \equiv_6 3 - 1 \equiv_6 2.$$

Inverses modulo n. Considering 2 as a rational number, we say that the inverse of 2 is $1/2$, since $2 \cdot (1/2) = 1$. We write $2^{-1} = 1/2$. We shall use the same terminology in the ring of integers modulo n. An element $a \in \mathbf{Z}$ is said to be *invertible modulo n* if there exists $b \in \mathbf{Z}$ such that $ab \equiv 1 \pmod n$. In this case we shall say that b is an *inverse* of a. In fact, inverses are unique (in $\mathbf{Z}/n\mathbf{Z}$).

Proposition 48. *Let $a, n \in \mathbf{Z}$ with $n \geq 2$. Then if $ab \equiv 1 \pmod n$ and $ac \equiv 1 \pmod n$, then $b \equiv c \pmod n$.*

Proof. We have

$$b \equiv b \cdot 1 \equiv b(ac) \equiv (ab)c \equiv 1 \cdot c \equiv c \pmod n. \quad ▮$$

Hence we may speak of *the* inverse of an element a, which we shall denote by a^{-1}. We have $a^{-1}a \equiv 1 \equiv aa^{-1} \pmod n$.

For instance, when $n = 5$, the fact that

$$2 \cdot 3 \equiv_5 6 \equiv_5 1$$

tells us that 2 (and 3, for that matter) are invertible modulo 5. When $n = 6$, is 2 still invertible? Here n is small enough that we can just try all the cases. We have $2 \cdot 0 = 0$, $2 \cdot 1 = 2$, $2 \cdot 2 = 4$, $2 \cdot 3 = 6 \equiv_6 = 0$ $2 \cdot 4 = 8 \equiv_6 2$, and $2 \cdot 5 = 10 \equiv_6 4$. Since none of these were equivalent to 1 modulo 6, we see that 2 is not invertible modulo 6.

It was quite a tedious process to exhaustively show that 2 is not invertible modulo 6. The following theorem gives a much easier-to-use criterion for invertibility.

Theorem 49. *Let $a, n \in \mathbf{Z}$ with $n \geq 2$. Then*

i) a is invertible modulo n if and only if $\gcd(a,n) = 1$; and

ii) if a is invertible, then there is a unique integer $b \in [0, n-1]$ such that $ab \equiv 1 \pmod{n}$. Namely, if

$$1 = sa + tn,$$

then we can set $b = s \,\%\, n$.

Proof. The proof of (i) is a fairly simple corollary of Bézout's theorem. f a is invertible, then $ab \equiv 1 \pmod{n}$, so there exists $k \in \mathbf{Z}$ such that $1 - ab = kn$. Rearranging this, we have $1 = ab + kn$, and by Bézout's theorem (specifically, the final statement of Theorem 36, this implies that $\gcd(a,n) = 1$. On the other hand, if $\gcd(a,n) = 1$, then there exists $k \in \mathbf{Z}$ such that $1 = ab + kn$. This means that $1 - ab = kn$, so $ab \equiv 1 \pmod{n}$.

For part (ii), suppose that $s, t \in \mathbf{Z}$ are such that $1 = sa + tn$. Then taking this equation modulo n, we have

$$1 = sa + tn \equiv_n sa + 0t \equiv_n sa.$$

Letting $b = s \,\%\, n$, we have $1 = ba$, since s and b belong to the same equivalence class modulo n, and b is an integer in $[0, n-1]$ by the division algorithm. This proves existence of such a b. To show uniqueness, we suppose that b and c are both elements of $[0, n-1]$ with $ab \equiv 1 \pmod{n}$ and $ac \equiv 1 \pmod{n}$. By Proposition 48, we have $b \equiv c \pmod{n}$; in other words, $n \mid b - c$. But since $b, c \in [0, n-1]$, we have $b - c \in [-n+1, n-1]$, and the only divisor of n in this range is 0. Hence $b - c = 0$ and we conclude that $b = c$. ▮

We can use this theorem to systematically find inverses of integers modulo other integers. For instance, suppose we want to find the inverse of 17 modulo 20. First we find $\gcd(20, 17)$ by Euclid's algorithm:

$$\begin{aligned} 20 &= 1 \cdot 17 + 3 \\ 17 &= 5 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 1 &= 1 \cdot 1 + 0 \end{aligned}$$

The fact that $\gcd(20, 17) = 1$ tells us that 17 indeed has an inverse. Now we go backwards to express 1 as an integer linear combination of 20 and 17:

$$\begin{aligned} 1 &= 3 - 1 \cdot 2 \\ &= 3 - (17 - 5 \cdot 3) \\ &= -17 + 6 \cdot 3 \\ &= -17 + 6(20 - 17) \\ &= 6 \cdot 20 - 7 \cdot 17 \end{aligned}$$

So the inverse of 17 modulo 20 is $-7 \equiv_{20} 13$. (We can verify that

$$17 \cdot 13 \equiv (-3)(-7) \equiv 21 \equiv 1 \pmod{20}.$$

As a trick for computations, use the range $-n/2$ to $n/2$ instead of 0 to $n-1$ to keep the numbers a bit smaller. It might help you avoid errors.)

Let's find the full list of invertible integers modulo 20, as well as their inverses. First, we list the integers in the range $[0, 19]$ that are relatively prime with 20:

$$1, 3, 7, 9, 11, 13, 17, 19$$

If some number is in this list, then its inverse must also be in this list (since inverses to a given element must themselves be invertible). The elements 1 and 19 are their own inverses, since $1 \cdot 1 = 1$ and $19 \cdot 19 \equiv_{20} (-1)(-1) = 1$. We already saw above that 13 and 17 are inverses to each other. That leaves 3, 7, 9, and 11. First we see that $3 \cdot 7 = 21 \equiv_{20} 1$, so 3 and 7 are inverses to each other. Then we note that $9 \cdot 9 = 81 \equiv_{20} 1$, so 9 is its own inverse, leaving us to conclude that 11 must also be its own inverse as well—either by the fact that we've ruled out all other possibilities, or by the computation

$$11 \cdot 11 \equiv_{20} (-9)(-9) = 81 \equiv_{20} 1.$$

We now investigate what happens in the particular case where we the modulus n is prime. In this situation we have the following proposition.

Proposition 50. *Let p be prime. Then*

i) every $x \in \mathbf{Z}$ with $x \not\equiv 0 \pmod{p}$ is invertible modulo p; and

ii) for all $a, b \in \mathbf{Z}$ with $ab \equiv 0 \pmod{p}$ one has $a \equiv 0 \pmod{p}$ or $b \equiv 0 \pmod{p}$.

Proof. Since p only has factors 1 and p, $\gcd(x, p)$ must either be 1 or p. But the condition $x \not\equiv 0 \pmod{p}$ implies that p does not divide x. Hence $\gcd(x, p) = 1$, proving part (i).

For part (ii), assume that $ab \equiv 0 \pmod{p}$, so $p \mid ab$. But p is prime, so by Theorem 39, either $p \mid a$ or $p \mid b$. In the first case we have $a \equiv 0 \pmod{p}$, and in the second case, $b \equiv 0 \pmod{p}$. ▮

24.X **Solving congruences modulo *n*.** Suppose we want to solve for all integers x satisfying $x^2 \equiv x \pmod{n}$, first for $n = 6$, then for $n = 7$.

For the case where $n = 6$, it suffices to consider $x \in \{0, 1, \dots, n-1\}$, since if $x^2 \equiv x \pmod{n}$ and $x \equiv a \pmod{n}$, then $a^2 \equiv a \pmod{n}$. So we simply try all $x \in \{0, 1, \dots, 5\}$. We have $0^2 = 0$, $1^2 = 1$, $2^2 = 4 \not\equiv_6 2$, $3^2 = 9 \equiv_6 3$, $4^2 = 16 \equiv_6 4$, and $5^2 \equiv_2 (-1)^2 = 1 \not\equiv_6 5$. We conclude that 0, 1, 3, and 4 are solutions to this congruence modulo 6, and more broadly, any integer y of the form

$$y = x + 6k$$

where $k \in \mathbf{Z}$ and $x \in \{0, 1, 3, 4\}$, is a solution $x^2 \equiv x \pmod{6}$.

Now we tackle the case where $n = 7$. In fact, the solution would be no different should n be any other prime. First we subtract x from both sides, obtaining the congruence $x^2 - x \equiv 0 \pmod{7}$, then we factorise the left side to get $x(x-1) \equiv 0 \pmod{7}$. Now by the previous proposition, since 7 is prime if $x(x-1) \equiv 0 \pmod{7}$, we must have either $x \equiv 0 \pmod{7}$ or $x - 1 \equiv 0 \pmod{7}$, so the only solutions are $x \equiv_7 0$ and $x \equiv_7 1$ (and anything in their equivalence classes).

Replacing 7 with any other prime p in the above paragraph yields a proof of the following proposition.

Proposition 51. *Let p be a prime. Then $a^2 \equiv a \pmod{p}$ if and only if a is either congruent to* 0 *or* 1 *modulo p.* ▮

Here is a similar proposition.

Proposition 52. *Let p be a prime and let $a \not\equiv 0 \pmod{p}$ (so that a is invertible. Then $a \equiv a^{-1} \pmod{p}$ if and only if a is either congruent to* 1 *or* -1 *modulo p.*

Proof. If $a = 1$ or $a = -1$, then $a^2 = 1$, so $a^2 \equiv 1$, and multiplying by a^{-1} on both sides, we have $a \equiv a^{-1} \pmod{p}$.

Now suppose that $a \equiv a^{-1} \pmod{p}$. Multiplying by a on both sides, we get $a^2 \equiv 1 \pmod{p}$; then, subtracting 1 from both sides and factoring the resulting polynomial, we get

$$(a+1)(a-1) \equiv 0 \pmod{p}.$$

This implies that either $a + 1 \equiv 0 \pmod{p}$ or $a - 1 \equiv 0 \pmod{p}$. In the first case, $a \equiv_p -1$, and in the second case, $a \equiv_p 1$. ▮

We finish off this section with an important theorem, first stated by Pierre de Fermat in 1640. He did not supply a proof; the first published proof of this theorem was given by Leonhard Euler in 1736.

Theorem 53 (*Fermat's little theorem*). *Let a and p be integers with p prime. If $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$.*

Before proving this theorem, we first state and prove a lemma. Recall that we define the *factorial* of $n \in \mathbf{N}$ to be the product $n! = 1 \cdot 2 \cdots (n-1)n$.

Lemma 54. *For all prime numbers p, the integer $(p-1)!$ is congruent to -1 modulo p.*

Proof. If $p = 2$, we have $(p-1)! = 1 \equiv -1 \pmod{2}$, and if $p = 3$, then $(p-1)! = 2 \equiv -1 \pmod{3}$.

For the rest of the proof, assume that $p \geq 5$. By Proposition 52, each element in the set $S = \{2, 3, \ldots, p-3, p-2\}$ is not its own inverse modulo p, so for each element $s \in S$, there is some other element $s' \in S$ with $s' \neq s$ such that $ss' \equiv 1 \pmod{p}$. This means that

$$2 \cdot 3 \cdots (p-3)(p-2) \equiv 1 \pmod{p}.$$

From this, we see that

$$(p-1)! = 1\cdot 2 \dots (p-2)(p-1) = \big(2\cdot 3 \cdots (p-3)(p-2)\big)(p-1) \equiv_p (p-1) \equiv_p -1. \quad ▮$$

We are now able to prove Fermat's little theorem.

Proof of Theorem 53. Since $a \not\equiv 0 \pmod{p}$, it is invertible modulo p by Proposition 50. Denote its inverse by a^{-1}. Let $G = (\mathbf{Z}/p\mathbf{Z}) \setminus \{0\}$ for short. Define a function $f : G \to G$ by letting $f(x) = a \cdot x$, where we consider the result modulo p (and hence the result is an element of $\mathbf{Z}/p\mathbf{Z}$). This function is well defined: since a and x are both not zero modulo p, their product will be an element of G. If $f(x_1) = f(x_2)$, then

$$a \cdot x_1 \equiv a \cdot x_2 \pmod{p},$$

so multiplying both sides by a^{-1}, we have $x_1 \equiv x_2 \pmod{p}$. This proves that f is injective. Now let $y \in G$. We want to find x with $f(x) \equiv y \pmod{p}$. To do so, simply set $x \equiv a^{-1}y \pmod{p}$. Then

$$f(x) = f(a^{-1}y) = a(a^{-1}y) \equiv y \pmod{p}.$$

This proves that f is surjective, and hence bijective.

The bijection f shows that the set

$$\big\{a, 2a, \dots, (p-2)a, (p-1)a\big\},$$

taken as a subset of $\mathbf{Z}/p\mathbf{Z}$ (i.e., we take each element modulo p, in the range $[0, \dots p-1]$), is exactly the same as the set

$$\{1, 2, \dots, p-2, p-1\},$$

just that the order of elements might be permuted. Hence we have

$$a(2a) \cdots \big((p-2)a\big)\big((p-1)a\big) \equiv (p-1)! \pmod{p}.$$

Combining all the a factors on the left-hand side, we get

$$a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}.$$

But by the previous lemma, -1 is the inverse of $(p-1)!$ modulo p, so multiplying both sides of this congruence by -1, we get

$$a^{p-1} \equiv 1 \pmod{p},$$

which is what we wanted to show. ▮

As a matter of interest, the intermediary lemma we proved is one direction of Wilson's theorem, proved by John Wilson in 1770.

Theorem 55 (*Wilson's theorem*). *For all integers $n \geq 2$, the congruence*

$$(n-1)! \equiv -1 \pmod{n}$$

holds if and only if n is prime.

Proof. We already proved the "if" direction earlier, as Lemma 54. We leave the "only if" direction of the proof as an exercise for the reader. ▮

> *Tout nombre premier mesure infalliblement*
> *une des puissances - 1 de quelque progression que ce soit,*
> *& l'exposant de ladite puissance est soûs-multiple du nombre premier donné - 1.*
> *Et aprés qu'on a trouvé la premiere puissance qui satisfait à la question,*
> *toutes celles dont les exposans sont multiples de l'exposant de la premiere*
> *satisfont de méme à la question.*
>
> — PIERRE DE FERMAT, *in a letter to Bernard Frénicle de Bessy* (1640)

11. Applications of number theory

In this section we present a potpourri of interesting ways number theory is applied to make your life better.

ISBN book identifiers. Every published book has an ISBN code that serves as its unique identifier. This code is 10 digits long for books published before 2007, but three new digits have been added for books published after 2007. In this section we'll deal with the simpler case of 10 digits.

What we want to do is to bake some redundancy into the codes, so that if a single digit is typed wrong, a computer will be able to tell the user that the code is invalid. This allows the user to correct their mistake. The way we do this, in a ten-digit code $d_{10}d_9d_8 \cdots d_2d_1$, is to have only the first nine digits encode the book. The last digit is a *check digit*, chosen so that

$$\sum_{i=1}^{10} id_i \equiv 0 \pmod{11}. \tag{4}$$

In other words,

$$d_1 \equiv -\sum_{i=2}^{10} id_i \pmod{11}.$$

If d_1 needs to equal 10, we use the symbol 'X'.

How does this solve our problem? Well, if a hapless person takes a valid ISBN code and mangles it by either getting one digit wrong, or by swapping two adjacent digits, the result is an invalid ISBN code, and the computer will be able to flag it. We formalise this statement as the following theorem.

Theorem 56. *Let $d_{10}d_9d_8\cdots d_2d_1$ be a valid* ISBN *code; that is, a code that satisfies (4). Then any code obtained by either*

i) changing exactly one digit d_i, for some $1 \leq i \leq 10$, or

ii) swapping distinct adjacent digits d_i and d_{i-1}, for some $2 \leq i \leq 10$,

is not a valid ISBN *codes (it does not satisfy (4)).*

Proof. First we prove (i). Suppose that $c_{10}c_9c_8\cdots c_2c_1$ is a code that is equal to $d_{10}d_9d_8\cdots d_2d_1$ except at one place $1 \leq j \leq 10$. So $c_j \neq d_j$, but for all $1 \leq i \leq 10$ with $i \neq j$, we have $c_i = d_i$. Then

$$\begin{aligned}\sum_{i=1}^{10} ic_i &\equiv_{11} \sum_{i=1}^{10} ic_i - 0\\ &\equiv_{11} \sum_{i=1}^{10} ic_i - \sum_{i=1}^{10} id_i\\ &= \sum_{i=1}^{10} i(c_i - d_i)\\ &= j(c_j - d_j).\end{aligned}$$

But $j \in \{1, \ldots, 10\}$ and $(c_j - d_j) \in [-10, 10]$. So neither j nor $c_j - d_j$ are congruent to 0 modulo 11. Since 11 is prime, their product cannot be congruent to 0 modulo 11. We conclude that $\sum_{i=1}^{10} ic_i \not\equiv 0 \pmod 1)1$; i.e., this is not a valid ISBN code.

To prove part (ii), suppose that $c_{10}c_9c_8\cdots c_2c_1$ is a code that is equal to $d_{10}d_9d_8\cdots d_2d_1$, except that for some $2 \leq j \leq 10$, we have $c_j = d_{j-1}$ and $c_{j-1} = d_j$ (the adjacent digits are swapped). Furthermore, assume that $d_j \neq d_{j-1}$, since swapping adjacent digits that are the same doesn't do anything to the code. Then

$$\begin{aligned}\sum_{i=1}^{10} ic_i &\equiv_{11} \sum_{i=1}^{10} ic_i - 0\\ &\equiv_{11} \sum_{i=1}^{10} ic_i - \sum_{i=1}^{10} id_i\\ &= \sum_{i=1}^{10} i(c_i - d_i)\\ &= (j-1)c_{j-1} + jc_j - (j-1)d_{j-1} - jd_j\\ &= (j-1)d_j + jd_{j-1} - (j-1)d_{j-1} - jd_j\\ &= d_{j-1} - d_j.\end{aligned}$$

But since d_{j-1} and d_j are both in the range $\{0, \ldots, 10\}$, their difference is in the range $[-10, 10]$, and we assumed that $d_{j-1} \neq d_j$, so this difference is not zero. We conclude that $d_{j-1} - d_j \not\equiv 0 \pmod{11}$, so the code is not valid. ∎

Something similar is done with credit card numbers, though we don't take digits modulo 11 (or else we'd have to use the digit 'X', which is cumbersome). Instead, a slightly different method called Luhn's algorithm is used. The calculation used to find the check digit is a bit different (let's not get into it), but the result is that we can still detect single-digit errors, and we can detect all swapping errors except for the transposition 09 $\leftrightarrow$ 90.

Divisibility tests. In grade school you might have learned the following divisiblity rule: *To tell if a number is divisible by* 3*, you sum up all of its digits. If this is a multiple of* 3*, then the original number was a multiple of* 3*, and if not, then the original number wasn't.* The same holds with 3 replaced by 9.

Using the number theory we've learned, we are now able to prove this fact.

Proposition 57. *Suppose that $n \in \mathbf{N}$ is written in base-*10 *digits as*

$$n = d_k d_{k-1} d_{k-2} \cdots d_2 d_1 d_0,$$

for some integer $k \geq 0$ and $d_i \in \{0, \ldots, 9\}$ for $0 \leq i \leq k$. Then n is divisible by 3 *if and only if*

$$\sum_{i=0}^{k} d_i$$

is also divisible by 3.

Proof. Since $10 \equiv 1 \pmod 3$, any power of 10 is also congruent to 1 modulo 3. So we have

$$n = \sum_{i=0}^{k} d_i 10^i \equiv_3 \sum_{i=0}^{k} d_i,$$

and we see that n is divisible by 3 if and only if the sum on the right-hand side is. ▮

This works with 3 replaced by 9 since we also have $10 \equiv 1 \pmod 9$.

Computing large powers modulo n. The next application is not a precise mathemathical statement; rather, it's an observation that Fermat's little theorem can allow us to compute what large powers are modulo a prime without having to compute the number itself. We'll illustrate this by two examples.

First we compute $25^{134} \pmod{11}$. We begin by noting that

$$25 \equiv 3 \pmod{11},$$

so what we really want is $3^{134} \pmod{11}$. By Fermat's little theorem, $3^{10} \equiv 1 \pmod{11}$, so we may cast out multiples of 10 from 134 to arrive at the easy computation

$$3^{134} \equiv 3^{10\cdot 13+4} \equiv (3^{10})^{13} \cdot 3^4 \equiv 3^2 \cdot 3^2 \equiv 9 \cdot 9 \equiv (-2)(-2) \equiv 4 \pmod{11}.$$

Now let's try 25134 (mod 14). Since 14 is not prime, we can no longer use Fermat's little theorem. The trick we shall use is to square 25 repeatedly. First note that $25 \equiv -3 \pmod{14}$, so $25^2 \equiv 9 \pmod{14}$. Then, squaring again, we get

$$25^4 \equiv 9^2 \equiv (-5)^2 \equiv 25 \equiv -3 \pmod{14}.$$

This means that repeated squaring of 25 cycles between -3 and 9 modulo 14, so we immediately have

$$25^8 \equiv 9, \qquad 25^{16} \equiv -3, \qquad 25^{32} \equiv 9, \qquad 25^{64} \equiv -3,$$

and

$$25^{128} \equiv 9$$

modulo 14. Now writing the exponent 134 as a sum of powers of two allows us to conclude that

$$25^{134} = 25^{128+4+2} = 25^{128} \cdot 25^4 \cdot 25^2 \equiv 9 \cdot (-3) \cdot 9 \equiv (-3)(-3) \equiv 9 \pmod{14}.$$

Fermat's primality test. Fermat's little theorem says that if n is prime, then for all integers a with $a \not\equiv 0 \pmod{n}$ one has $a^{n-1} \equiv 1 \pmod{n}$. The contrapositive of this statement is that if there is some integer a with $a \not\equiv 0 \pmod{n}$ and $a^{n-1} \not\equiv 1 \pmod{n}$, then n is not prime. This leads us to the following algorithm that can (usually) tell us when a number is not prime.

Algorithm F (*Fermat's primality test*). Given an integer $n \geq 4$, this algorithm will attempt to declare that n is not prime. If the algorithm fails to do so, then the test is inconclusive (n may either be prime or not).

F1. [Initialise.] Set $a \leftarrow 2$. (We can skip the step $a = 1$ since a^{n-1} will always equal 1 in this case.)

F2. [Compute power.] Set $b \leftarrow a^{n-1} \,\%\, n$. (So $0 \leq b < n$.)

F3. [Not prime?] If $b \neq 1$, output "Not prime," and terminate the algorithm.

F4. [Loop?] Set $a \leftarrow a + 1$. If $a = n$, output "Inconclusive," and terminate the algorithm. Otherwise, go to step F2. ▮

29.x Let's see a couple of examples. Suppose we want to find out if 9 is prime or not (we both know it isn't but play along for a second). In the very first step loop, when $a = 2$, we have

$$2^{9-1} = 2^8 = (2^4)^2 = 16^2 \equiv (-2)^2 = 4 \pmod{9},$$

so since $4 \not\equiv 1 \pmod{9}$, we conclude right away that 9 is not prime.

A harder one. Is 341 prime? We start with $a = 2$. Noting that $3 \cdot 341 = 1023$, we have

$$2^{340} = (2^{10})^{34} = 1024^{34} \equiv 1^{34} \equiv 1 \pmod{341},$$

so the test is inconclusive after the first loop. Fine. If we try $a = 3$ next, we will find that $3^{340} \equiv 56 \not\equiv 1 \pmod{341}$, so we conclude that 341 is not prime. (Those wishing to practise computing large powers modulo n might like to verify the fact that $3^{340} \equiv 56 \pmod{341}$. Use the squaring trick.)

If n causes Algorithm F to output "Inconclusive," it is very likely that n is prime. However, there are rare cases of composite numbers n that cause Algorithm F to output "Inconclusive." These are called *Carmichael numbers*. The smallest one is 561.

Vernam's one-time pad. Alice and Bob want to exchange secret messages over a public channel without anyone being able to decipher them. A message can be encoded as a binary string M. Here is one way that Alice and Bob can securely send encrypted messages to each other. It is called Vernam's one-time pad, named for G. Vernam, an AT&T engineer who patented the method in 1917.

First, they need to meet in person and generate a long sequence of random bits. (How can one generate random numbers? This is an interesting quasi-philosophical question.) Each of them keeps a copy of this sequence, call it S.

Now suppose Alice and Bob separate and Alice wants to send the message $M = 0110$ to Bob. Of course, she shouldn't send this string directly, as it could be intercepted by foes. She needs to use the sequence S. Suppose the first four bits of S are 1101. To encrypt her message, Alice adds each bit of M to the corresponding bit of S, modulo 2. We will denote this by the $\oplus$ operation. The encrypted message is $M' = 0110 \oplus 1101 = 1011$. She sends it to Bob.

Bob receives the string $M' = 1011$. To decode it and recover the message M, Bob simply has to perform the $\oplus$ operation using the (same) first four bits of S that Alice used to encode the message in the first place. Indeed, we see that $1101 \oplus 1101 = 0110 = M$. The reason that this works is that for any binary string S, we have $S \oplus S = 0$, so performing the $\oplus$ operation twice always returns the original string.

If they want to send more messages, they simply discard the bits they used before and continue using the next few bits of S.

The advantage to this approach is that an adversarial third-party, reading M', gains no information about the original string M, provided they do not know the sequence S. In other words, it is entirely secure. The disadvantages of Vernam's one-time pad are that Alice and Bob must meet in person beforehand, they need to find a way to generate random bits securely, and they need to periodically repeat this process to replenish the bits once they run out.

The RSA cryptosystem. Now we move on to a more sophisticated method of encrypting and decrypting messages. It is called the RSA *cryptosystem*, named for R. Rivest, A. Shamir, and L. Adleman, who described the algorithm in 1977.

Here's the scenario. Alice wants people to be able to send her messages such that only she can decode them. To do so, she needs to create a *public key*, which allows people to encrypt their messages before sending them to her, and then a *private key* that will allow her to decrypt messages. (Only she knows the private key.)

To set things up, Alice needs two large primes, call them p and q. (Something like 200 bits is sufficient.) Then she computes $n = pq$. Now, she picks some integer k with

$$\gcd\big(k, (p-1)(q-1)\big) = 1.$$

By Theorem 49, there is some integer s such that $ks \equiv 1 \pmod{(p-1)(q-1)}$. Alice publishes the integer n and the public key k. The private key is s; this she keeps to herself.

Now Bob wants to send a secret message M to Alice. Considering M as an integer (in binary), a requirement for the algorithm to work is $M < n$. Bob computes $\overline{M} = M^k \,\%\, n$ and sends $\overline{M}$ to A. (We have $0 \le \overline{M} < n$ by the division algorithm.)

Alice receives $\overline{M}$. She computes $\overline{M}^s$. It turns out that this is M. To prove this, we need a lemma.

Lemma 58. *Let a and b be integers.*

i) For all positive integers m and n, if $a \equiv b \pmod{mn}$, then $a \equiv b \pmod{m}$ and $a \equiv b \pmod{n}$.

In the case that m and n are different primes, the converse holds; that is, the following holds.

ii) For all primes p and q with $p \neq q$, if $a \equiv b \pmod{p}$ and $a \equiv b \pmod{q}$, then $a \equiv b \pmod{pq}$.

Proof. Assume that $a \equiv b \pmod{mn}$, so there exists $k \in \mathbf{Z}$ such that $a-b = kmn$. This means that $a - b \equiv 0 \pmod{m}$ and $a - b \equiv 0 \pmod{n}$, so $a \equiv b \pmod{m}$ and $a \equiv b \pmod{n}$.

To prove (ii), suppose that there exist $k, l \in \mathbf{Z}$ such that $a - b = kp$ and $a - b = lq$. We have $kp = lq$, so $p \mid lq$. Because p is prime, p must divide either l or q. But q is prime as well, so the only divisors of q are 1 and q. Hence $p \mid l$, and we may write $l = rp$ for some $r \in \mathbf{Z}$. So we have

$$a - b = lq = rpq,$$

implying that $a - b \equiv 0 \pmod{pq}$; that is, $a \equiv b \pmod{pq}$. ∎

Now we shall prove that the encrypted message $\overline{M} = M^k$ can be decrypted into the original message.

Theorem 59 (RSA *encryption*). *Let p and q be distinct primes and let $n = pq$. Let k and s be such that $ks \equiv 1 \pmod{n}$. Then for all integers $0 \le M < n$, $(M^k)^s \equiv M \pmod{n}$.*

Proof. Without loss of generality, we can take k and s both positive. From $ks \equiv 1 \pmod{(p-1)(q-1)}$, we know there is an integer l such that $ks-1 = l(p-1)(q-1)$.

Since $k, s > 0$, this identity means that $l > 0$ as well. Now, since $n = pq$, we have

$$(M^k)^s \equiv M^{ks} \equiv M^{1+l(p-1)(q-1)} \equiv M(M^{p-1})^{l(q-1)} \pmod{pq}.$$

By part (i) of the previous lemma, this means we have the same congruence modulo p; i.e.,

$$(M^k)^s \equiv M(M^{p-1})^{l(q-1)} \pmod{p},$$

and by Fermat's little theorem $M^{p-1} \equiv 1$ for all $M \not\equiv 0 \pmod{p}$, so this simplifies to

$$\begin{aligned}(M^k)^s &\equiv_p \begin{cases} M, & \text{if } M \not\equiv 0 \pmod{p}; \\ 0, & \text{otherwise} \end{cases} \\ &\equiv_p M.\end{aligned}$$

Mutatis mutandis (by the roles of p and q in the above), we have $(M^k)^s \equiv_q M$ as well, so part (ii) of the previous lemma can now be applied to give

$$(M^k)^s \equiv M \pmod{pq}. \quad \blacksquare$$

Here is an example of the RSA algorithm in action. Alice sets $p = 3$ and $q = 11$, so that $n = 33$. This means that $(p-1)(q-1) = 2 \cdot 10 = 20$. Using $k = 7$ (which we can do since $\gcd(7, 20) = 1$, we have $s = 3$, since $3 \cdot 7 = 21 \equiv 1 \pmod{20}$. Alice publishes the integer $n = 33$ as well as the public key $k = 7$. She keeps the number $s = 3$ private.

The possible messages M one can send to Alice are $0, 1, \ldots, 32$, since we need $0 \le M < n$. Suppose Bob wants to send the message 00010 to Alice; that is, we have $M = 2$. Bob computes

$$M^k = 2^7 = 2^5 \cdot 2^2 = 32 \cdot 4 \equiv (-1) \cdot 4 \equiv 29 \pmod{33},$$

and sends Alice $\overline{M} = 29$.

Now Alice computes

$$\overline{M}^s = 29^3 \equiv (-4)^3 = -64 \equiv -64 + 66 = 2 \pmod{33}$$

and recovers the original message $M = 2$.

Now imagine that there is a third person, Charlie, who wants to thwart Alice and Bob. The integer n and the public key n are known to all, including Charlie. If he wants to decrypt Bob's message, all Charlie needs to know is the the private key s. In this case, since n is very small, the Adversary can simply factor $33 = 3 \cdot 11$ to find out that $(p-1)(q-1) = 20$, and then use Euclid's Algorithm to find the inverse of 7 modulo 20, and he will be able to decrypt all messages between Alice and Bob.

In real life, RSA is implemented using very large primes p and q. This makes it difficult, but not impossible, to break the encryption. To put things in perspective, it took a team of researchers 900 core-years of computing time to factorise the 240-digit (795-bit) modulus n that is used in RSA-240 encryption.

These researchers estimated that increasing the number of bits to 1024 would make their factorisation program take 500 times as long to crack the encryption.

However, this does not mean there doesn't exist a fast algorithm for factorising integers. We just haven't found one yet (or if people have, they are keeping the algorithm to themselves). Most computer scientists believe that this is impossible though. It is conjectured that integer factorisation is a difficult problem—that is, there is no algorithm to factorise a b-bit integer in time polynomial in b. Until this conjecture is proved, all of the data we rely on encryption to protect could theoretically be in jeopardy.

III. GRAPH THEORY

Quamobrem, cum nuper problematis cuiusdam mentio esset facta,
quod quidem ad geometriam pertinere videbatur,
at ita erat comparatum, ut neque determinationem quantitatum requieret,
neque solutionem calculi quantitatum ope admitteret,
id ad geometriam situs referre haud dubitavi:
praesertim quod in eius solutione solus situs in considerationem veniat,
calculus vero nullius prorsus sit usus.

— LEONHARD EULER *"Solutio problematis ad geometriam situs pertinentis"* (1741)

12. Definitions and basic notions

A *graph* is a pair $G = (V, E)$ where V is a nonempty set and

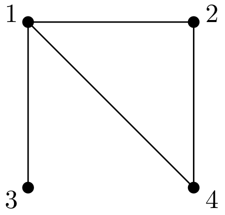
$$E \subseteq \big\{\{u, v\} : u, v \in V, u \neq v\big\}.$$

The graph G is said to be *finite* if and only if V is finite. The elements of V are called *vertices* and the elements of E are called edges. If $e = \{u, v\} \in E$, we say that u and v are *adjacent*, and e is *incident on* u and v. For brevity, we often write uv instead of $\{u, v\}$. Note that an edge is defined to be a 2-element set, not an ordered pair, so $uv = vu$.

Let's start with a small example. Let $V = \{1, 2, 3, 4\}$ and

$$E = \{12, 13, 14, 24\}.$$

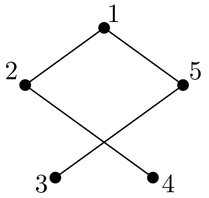
We can draw the graph $G = (V, E)$ as follows:



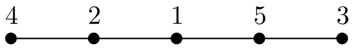
Instead of listing out all the vertices, we can define graphs by setting conditions for when a pair of vertices should be connected by an edge. For instance, let $G = (V, E)$ be the graph with $V = \mathbf{Z}/5\mathbf{Z}$ and, for $a, b \in V$ with $a \neq b$, we put $ab \in E$ if and only if

$$a + b \equiv 1 \pmod 5 \qquad \text{or} \qquad a + b \equiv 3 \pmod 5.$$

Here is a drawing of G:



Turns out this graph was simpler than it might have seemed at first glance. Here is an alternate drawing of G:



Note that when we refer to G, we are talking about a set of vertices and edges, not any particular drawing! There are infinitely many valid drawings of any graph.

Suppose $G = (V, E)$ has a finite number of vertices. Then we can put the elements of V in bijection with $\{1, \ldots, n\}$. Without loss of generality, we can simply assume that $\{1, \ldots, n\}$ *is* the vertex set of G. The *adjacency matrix* of G is then defined to be the symmetric $n \times n$ matrix obtained by setting

$$A_{ij} = \begin{cases} 1, & \text{if } ij \in E, \\ 0, & \text{otherwise.} \end{cases}$$

For instance, the adjacency matrix of the graph on $\mathbf{Z}/5\mathbf{Z}$ defined earlier is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We now define certain important families of graphs. For any integer $n \geq 1$, we define the *complete graph* K_n to be the graph on n vertices with every possible edge present. Small examples are illustrated in Fig. 2.

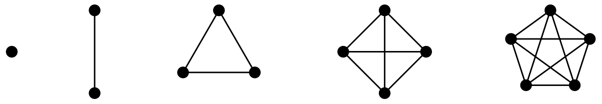


Fig. 2. Complete graphs K_n for small n.