

MATH 240 Fall 2024

notes by

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A note on these notes. After each class, this document will be updated with the new material that was just covered. The timestamps in the left margin indicate when the notes from each day start. Subsections labelled with a * are optional. This document is heavily based on notes by Jeremy Macdonald, but any errors are likely my own. Please email me if you find any.

1. Set theory

29.VIII A *set* is a collection of distinct objects, called its *elements* or its *members*. If x is a member of set A , then we write $x \in A$, and if x is not an element of A , then we write $x \notin A$. Sets can be written by listing out its elements. For example,

$$\{1, 4, 7, 10, \sqrt{782}\}$$

and

$$\{\{1, 2\}, \pi, \{4\}\}$$

are both sets (the second example shows that sets can themselves contain other sets). The order of the elements is not important, and duplicate elements are ignored (so $\{1, 2\} = \{2, 1\} = \{1, 1, 2\}$). The notation using $\{$ and $\}$ is useful for defining small, concrete examples, but expressing large sets can become very cumbersome. The first way one can describe larger sets is to use the \dots symbol and the power of suggestion. For instance, anyone faced with the notation

$$A = \{1, 3, 5, 7, 9, \dots\}$$

can quickly guess that this set is supposed to contain all the positive odd integers. We can also use \dots to define finite sets. Most Canadians will be able to tell you that the set

$$\{\text{Alberta, British Columbia}, \dots, \text{Yukon}\}$$

of provinces and territories contains 13 elements. But this notation inherently produces some ambiguity. For example, since the sequence of positive palindromic binary numbers starts 1, 3, 5, 7, 9, 15, 17, 21, 27, \dots , we are left with some doubt as to whether the set A above should be the set of odd numbers or the set of palindromic binary numbers.

But there is another, less ambiguous way to define large sets. It is called set-builder notation and it refers to any construction of the form

$$\{x \in U : P(x)\},$$

where x is a variable, U is a set, and P is a statement about x . The resulting set contains *all x such that $P(x)$ holds*. For example, letting \mathbf{N} denote the set $\{0, 1, 2, 3, \dots\}$ of counting numbers (more on this later), to define the set of all odd numbers, we can write

$$A = \{x \in \mathbf{N} : \text{there exists } k \in \mathbf{N} \text{ such that } x = 2k + 1\}.$$

Note that the statement $P(x)$ must contain x , but it may also contain other previously defined symbols, as well as new symbols defined within the statement (such as k in the example above).

Special sets of numbers. There are certain infinite sets of numbers that are used so often as to be given special bold notation. Back in elementary school, you learned to use the counting numbers $0, 1, 2, 3, \dots$. We already saw this set in the previous paragraph; in the business, this set is known as the *natural numbers*, because if you go on a nature hike you can use them to count the number of bluebells, donkeys, etc. that you see. (Many mathematicians use the symbol \mathbf{N} to denote this set without zero. When in doubt, clarify with the person you're talking to; in this class, $0 \in \mathbf{N}$.)

Sometime towards the start of junior high you were introduced to the concept of natural numbers. The set

$$\mathbf{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

is called the set of *integers* or *whole numbers*. (The word *integer* just means “whole” in Latin; cf. French *entier*. We use the letter \mathbf{Z} because of the German word *Zahl* meaning “number.”)

Even before you learned about negative numbers, you probably learned about fractions. They can be defined in set-builder notation as a collection of ratios of integers, where the denominator is not zero:

$$\mathbf{Q} = \{p/q : p, q \in \mathbf{Z}, q \neq 0\}.$$

(The nitpicky reader will notice that p/q is not stipulated to be a member of any set here. This is because a rigorous definition of \mathbf{Q} involves quotienting out by an equivalence relation, which we don't know how to do yet.) This is the set of *rational numbers*. Remember that sets only contain distinct elements, so $2/3$, $4/6$, $6/9$, etc. are all considered the same rational number.

Lastly, we have the set \mathbf{R} of real numbers. Constructing this set using only notions from set theory and logic is quite the byzantine task and well outside the scope of this course, but you can think of \mathbf{R} as the set of decimal numbers with

a finite number of digits to the left of the decimal point, and a possibly infinite number of digits to the right of the decimal point.

A set of numbers near and dear to many mathematicians' hearts is the set of *prime* numbers P , defined by

$$P = \{p \in \mathbf{N} : p \geq 2, \text{ and if } p = ab \text{ then } \{a, b\} = \{1, p\}\}.$$

(You may have met a different definition of prime numbers in the past. Pause a moment and convince yourself that the statement above defines the set of prime numbers as you know it.)

Set inclusion. When we use set builder notation $B = \{x \in A : P(x)\}$, every element of B is necessarily a member of the set A as well, since B is defined to be the set of all x in A satisfying $P(x)$. This is one way in which we can obtain a *subset* of another set. More generally, we write $B \subseteq A$ if every element of B is also an element of A , and $B \supseteq A$ if every element of A is an element of B . Sometimes, if $B \supseteq A$, we say that B *contains* A , or B *includes* A . For example, we have the chain

$$\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$$

for the special sets of numbers defined earlier.

Symbols like \subseteq , \supseteq , and $=$ (that are used to produce statements) can be negated with a slash; for example, if there is some element of B that is not an element of A , then we write $B \not\subseteq A$.

The concept of set inclusion is important, because the most common way to prove that two sets A and B are equal is to show that A is a subset of B , then show that B is a subset of A . We illustrate this with the following example.

Proposition 1. *The sets*

$$A = \{x \in \mathbf{Z} : \text{there exists } k \in \mathbf{Z} \text{ such that } x = 2k + 1\}$$

and

$$B = \{x \in \mathbf{Z} : \text{there exists } l \in \mathbf{Z} \text{ such that } x = 2l + 5\}$$

are equal. (Both are different ways of expressing the set of all odd integers.)

Proof. Let $x \in A$. Then there exists $k \in \mathbf{Z}$ such that $x = 2k + 1$. Letting $l = k - 2$, we find that l is an integer (since k was). Furthermore,

$$x = 2k + 1 = 2k - 4 + 5 = 2(k - 2) + 5 = 2l + 5.$$

We have found l such that $x = 2l + 5$, so $x \in B$. This shows that $A \subseteq B$.

On the other hand, let $x \in B$, so that there exists $l \in \mathbf{Z}$ such that $x = 2l + 5$. Now we let $k = l + 2$; $k \in \mathbf{Z}$ since $l \in \mathbf{Z}$. We have

$$x = 2l + 5 = 2l + 4 + 1 = 2(l + 2) + 1 = 2k + 1.$$

This shows that $x \in A$, and we have proved that $B \subseteq A$. This combined with the previous paragraph shows that $A = B$. ■

The above result is not so important, but pay attention to the structure of this proof. It is called a “proof by double inclusion,” since we have shown that A includes B and B includes A .

Set operations. Now we describe a number of operations that may be performed on sets to produce other sets. They can all be built up from the following two operations.

- The *union* $A \cup B$ of two sets A and B is the set of all elements that are either in A or in B (or both).
- The *intersection* $A \cap B$ of A and B is the set of all elements that are in both A and B .

As an example, if $A = \{1, 2, 4\}$ and $B = \{1, 3, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$ and $A \cap B = \{1\}$.

To avoid logical difficulties, we always assume that the sets we’re working with are a subset of some larger ambient set U , often called the *universe*. Once we know what U is, we may define the *complement* of a set A to be the set \bar{A} of all the elements in U except those that are in A . So if $U = \{1, 2, 3, 4, 5\}$ in the example above, then $\bar{A} = \{3, 5\}$ and $\bar{B} = \{2, 4\}$. What about $\overline{A \cup B}$? Well since $A \cup B$ is all of U , its complement must be empty, and we can denote it $\{\}$. This is one valid notation for the *empty set*. The other is \emptyset .

Now is a good time to define the cardinality $|A|$ of a set A . This is the number of elements in it, so $|A| = |B| = 3$ in our example, and $|A \cup B| = 5$, etc. We have $|\emptyset| = 0$, and it is possible for the cardinality of a set to be infinity; for example, $|\mathbf{N}| = \infty$. We also have $|\mathbf{R}| = \infty$, but this infinity is, in some sense, larger than $|\mathbf{N}|$. (More on that later.)

Next, we define the *difference* $B \setminus A$ (sometimes $B - A$) of two sets. This is the set of all elements in B that are *not* in A . So, using the complement notation we just learned about, we can express $B \setminus A = B \cap \bar{A}$. It is not necessary that A be a subset of B . In the small example above, we have $A \setminus B = \{2, 4\}$ and $B \setminus A = \{3, 5\}$.

Lastly, we define the *symmetric difference* $A \triangle B$ of two sets A and B to be the set of all elements that are either in A or in B *but not both*. Invoking the above example one last time, we have $A \triangle B = \{2, 3, 4, 5\}$. We can express it as using unions, intersections, and complements as

$$A \triangle B = (A \cup B) \cap \overline{A \cap B}. \quad (1)$$

To practise using all the different operations we just learned, convince yourself that the following are three more valid ways to express the symmetric difference:

$$A \triangle B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A) = \overline{\overline{A \cup B} \cap (A \cap B)}$$

2. More set identities abound. We state the following proposition without proof; you should try going through this list and convincing yourself that each identity holds, for all sets A , B , and C . (This is a great way of practising proofs by double inclusion.)

Proposition 2. *Let A , B , and C be subsets of a universe U . Then*

- i) $A \cap U = A$ and $A \cup \emptyset = A$;
- ii) $A \cup U = U$ and $A \cap \emptyset = \emptyset$;
- iii) $A \cup A = A$ and $A \cap A = A$;
- iv) $\overline{\overline{A}} = A$;
- v) $A \cup B = B \cup A$ and $A \cap B = B \cap A$;
- vi) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
- vii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
- viii) $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$; and
- ix) $A \cup \overline{A} = U$ and $A \cap \overline{A} = \emptyset$. ■

These laws have names, some of which we will use more often than others: (i) is called the identity law, (ii) the domination law, (iii) the idempotent law, (iv) the law of double negation, (v) the commutative law, (vi) the associative law, (vii) the distributive law, (viii) the absorption law, and (ix) the complement law.