The Probabilistic Method

exercise solutions by

MARCEL K. GOH

14 FEBRUARY 2022

1. The Basic Method

Exercise 1.1. *Prove that if there is a real p, with $0 \le p \le 1$ such that*

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1,$$

then the Ramsey number $r(k,t)$ satisfies $r(k,t) > n$. Using this, show that

$$r(4,t) \ge \Omega\big(t^{3/2}/(\ln t)^{3/2}\big).$$

Proof. We follow the proof of Proposition 1.1.1 in the book. We consider a random graph on n vertices, where each edge is present with probability p. Let K be the event that there is a clique of size k in the graph, and let I be the event that there is an independent set of size t in the graph. By the union bound,

$$\mathbf{P}\{K \cup I\} \le \mathbf{P}\{K\} + \mathbf{P}\{I\} \le \sum_{|S|=k} p^{\binom{k}{2}} + \sum_{|S|=t} (1-p)^{\binom{t}{2}} = \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1.$$

This means that $\mathbf{P}\{\neg K \cap \neg I\} > 0$ and since the sample space is finite, there exists a graph on n vertices with no clique of size k and no independent set of size t and therefore $r(k,t) > n$.

Next we show that $r(4,t) > \big(t/(e\ln t)\big)^{3/2}$ for large enough t. Note that

$$\binom{n}{4} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} \le n^4p^6 + \frac{e^tn^t}{t^t}(1-p)^{t^2/4},$$

by the inequalities

$$\frac{n^k}{k^k} \le \binom{n}{k} \le \frac{e^kn^k}{k^k}.$$

Setting $n = t^{3/2}/(e\ln t)^{3/2}$, we have

$$\begin{aligned}
n^4p^6 + \frac{e^tn^t}{t^t}(1-p)^{t^2/4} &= \left(\frac{tp}{e\ln t}\right)^6 + \frac{e^t t^{3t/2}}{t^t e^{3t/2}(\ln t)^{3t/2}}(1-p)^{t^2/4} \\
&= \left(\frac{tp}{e\ln t}\right)^6 + \frac{t^{t/2}}{e^{t/2}(\ln t)^{3t/2}}(1-p)^{t^2/4} \\
&\le \left(\frac{tp}{e\ln t}\right)^6 + \left(\frac{t(1-p)^{t/2}}{e(\ln t)^3}\right)^{t/2} \\
&\le \left(\frac{tp}{e\ln t}\right)^6 + \left(\frac{t}{e^{pt/2+1}(\ln t)^3}\right)^{t/2},
\end{aligned}$$

where in the last line we used the inequality $1-p \le e^{-p}$. Choosing $p = 2\ln t/t$, we simply need t large enough such that

$$\left(\frac{t}{e^{\ln t+1}(\ln t)^3}\right)^{t/2} = \left(\frac{1}{e(\ln t)^3}\right)^{t/2} < 1 - \left(\frac{2}{e}\right)^6,$$

which can be done since the left-hand side goes to 0. ▮

Exercise 1.2. *Suppose $n \geq 4$ and let H be an n-uniform hypergraph with at most $4^{n-1}/3^n$ edges. Prove that there is a colouring of the vertices of H by 4 colours so that in every edge all 4 colours are represented.*

Proof. Let each vertex of H be independently given one of the four colours uniformly at random. (If H is infinite, it does not matter what colour we give to vertices that do not appear in any edge, so it suffices to consider H finite, which makes the sample space finite.) Given some edge e of H with n vertices, there are 4^n total ways that e may be coloured, and for each of the four colours, 3^n total ways that e may be coloured using only the other three colours. Let $K(e)$ denote the event that e does not contain all four colours. By the inclusion-exclusion principle,

$$\mathbf{P}\big\{K(e)\big\} = 4 \cdot 3^n - 6 \cdot 2^n + 4$$

Since $6 \cdot 2^n \geq 96 > 4$, the probability that a given edge *does not* contain all four colours is (much) less than $3^n/4^{n-1}$. By the union bound,

$$\mathbf{P}\Big\{\bigcup_{e \in E(H)} K(e)\Big\} \leq \sum_{e \in E(H)} \mathbf{P}\big\{K(e)\big\} < \frac{4^{n-1}}{3^n} \cdot \frac{3^n}{4^{n-1}} = 1.$$

Since the sample space is finite this implies that there is some colouring of the vertices of H in which every edge has all four colours. ▮