Szemerédi's regularity lemma

notes by

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1. Introduction

[Put some kind of introduction here.]

2. The regularity lemma

In this section we will prove Szemerédi's regularity lemma via a sequence of auxiliary ones. We will first require some definitions. Fix a graph $G = (V, E)$ and let X and Y be subsets of V (not necessarily disjoint). Let $e(X, Y) = \{xy \in E : x \in X,\, y \in Y\}$ denote the number of edges that have an endpoint in each of X and Y. We define the *edge density* between X and Y to be the ratio

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

If X and Y are disjoint, then this is the fraction of all possible edges between X and Y that are actually present in the graph (and in the case that they are not disjoint, it isn't awfully far off anyway).

We say that a pair of vertex subsets (X, Y) is *ϵ-regular* if for all subsets $A \subseteq X$ and $B \subseteq Y$ with $|A| \ge \epsilon|X|$ and $|B| \ge \epsilon|Y|$, we have $|d(A, B) - d(X, Y)| \le \epsilon$. This means that if we zoom in to look at the edges between a subset of X and a subset of Y, we find that the picture is sort of a "scale-model" of the whole of X and the whole of Y in the sense that the number of edges that we see is proportional to the sizes of the subsets, unless the subsets are taken to be very small. If the pair (X, Y) is *not* ϵ-regular, then there must be some $A \subseteq X$ and $B \subseteq Y$, with $|A| \ge \epsilon|X|$ and $|B| \ge \epsilon|Y|$, such that $|d(A, B) - d(X, Y)| > \epsilon$. The pair (A, B) is said to *witness* the irregularity.

A *partition* $\mathcal{P}$ is a collection $V_1, \ldots, V_k$ of disjoint subsets of V whose union is all of V. We will say that a partition is *equitable* if the sizes of any two parts do not differ by more than 1. A partition is said to be *ϵ-regular* if the sum of $|V_i||V_j|$, taken over all pairs (V_i, V_j) that are not ϵ-regular, is less than $\epsilon|V|^2$. If the partition is equitable, then this is equivalent to saying that at most ϵk^2 of the pairs (V_i, V_j) are not ϵ-regular. Szemerédi's regularity lemma says that every graph admits an ϵ-regular equitable partition into a number of parts depending only on ϵ (and not the size of the graph).