Markov Chain Monte Carlo

Homework

Markov Chains and Metropolis Monte Carlo

(i) Consider a game of "ladder climbing".[1] There are 5 levels (states) in the game, level 1 is the lowest (bottom) and level 5 is the highest (top).

A player starts at the bottom (initial state $x = 1$). Each time, a fair coin is tossed. If it turns up heads, the player moves up one rung. If tails, the player moves down to the very bottom. Once at the top level, the player moves to the very bottom if a tail turns up, and stays at the top if head turns up.

- Find the transition probability matrix, $\mathbf{W}$.
- Find the eigenvalues and eigenvectors of $\mathbf{W}$.
- What is the steady-state distribution $\pi(x)$ of the Markov chain?
- What is the expected value $E(x) = \sum x\pi(x)$?

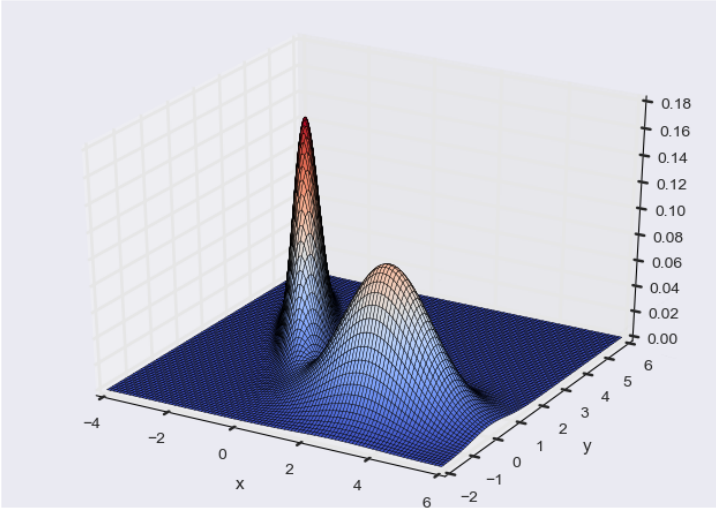
(ii) Consider the following mixture of 2D Gaussian distributions,

$$\pi(x, y) = 0.6\mathcal{N}(\mu_1, \Sigma_1) + 0.4\mathcal{N}(\mu_2, \Sigma_2),$$

where,

$$\mu_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \Sigma_1 = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \text{and}$$

$$\mu_2 = \begin{bmatrix} -1.25 \\ 2.5 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 0.5 & -0.6 \\ -0.6 & 1.0 \end{bmatrix}.$$



[1] Example adapted from `https://www.utdallas.edu/~mbaron/`

This distribution is shown in the figure above.

Holding all other parameters constant, vary the following:

(a) number of MC steps

(b) the choice of δ

(c) use a normal proposal function instead of uniform

(d) initial state

Tune the parameters, until you are "visually" satisfied with the sampling. Report your observations, and try to explain them in terms of (a) the acceptance ratio, and (b) peaks exploration of both peaks?