Aula 13 - Sinusoidal Steady - State Analysis

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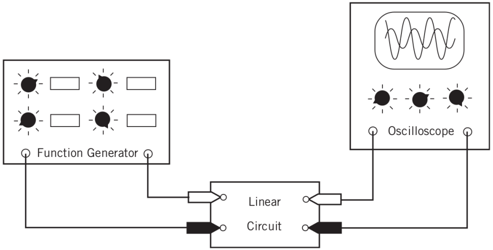
Contents

Introduction to Electric Circuits by James A. Svoboda, Richard C. Dorf, 9th Edition

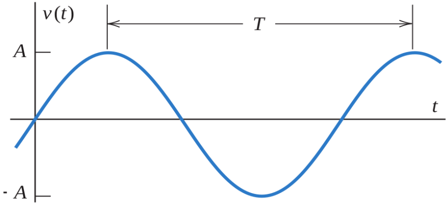
Introduction

Linear circuits with sinusoidal inputs that are at steady state are called ac circuits. The electric power system that provides us with convenient electricity is a very large ac circuit. AC circuits are the subject of this chapter. In particular, we will see that:

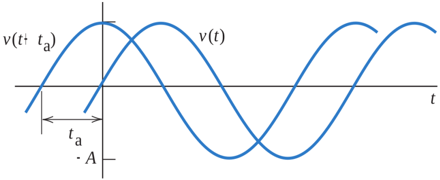
- It's useful to associate a complex number with a sinusoid. Doing so allows us to define phasors and impedances.
- Using phasors and impedances, we obtain a new representation of the linear circuit, called the "frequency-domain representation."
- We can analyze ac circuits in the frequency domain to determine their steady-state response.



Phase Advance and Delay



$$v(t) = \sin(\omega t)$$
$$v(t) = v(t+T)$$
$$f = \frac{1}{T}$$
$$\omega = 2\pi f = \frac{2\pi}{T}$$



$$v(t+t_a) = \sin(\omega(t+t_a)) = \sin(\omega t + \omega t_a)$$
$$v(t+t_a) = \sin(\omega t + \theta)$$

The phase angle in radians is related to the time t_a by

$$\theta = \omega t_a = \frac{2\pi}{T} t_a$$

An advance by time t_a is equivalent $T + t_a$. Similarly, a delay by time t_d is equivalent $T - t_d$.

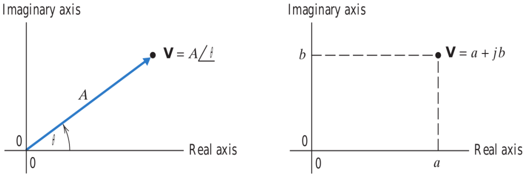
Phasors and Sinusoids

A phasor is a complex number that is used to represent the amplitude and phase angle of a sinusoid. The relationship between the sinusoid and the phasor is described by

$$A\cos(\omega t+\theta) \leftrightarrow A\angle\theta$$

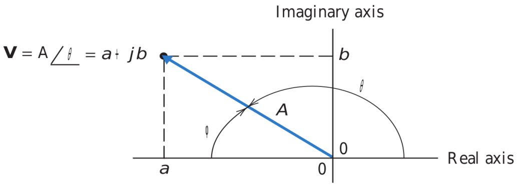
There are a couple of things that we should notice.

- The sinusoid is represented using the cosine rather than the sine function.
- the phasor is a complex number represented in polar form.
- The magnitude of the phasor is equal to the amplitude of the sinusoid, and the angle of the phasor is equal to the phase angle of the sinusoid.



To indicate that A is the magnitude of the phasor V and that y is the angle of V, we write
$A = |\mathbf{V}|$ *and* $\theta = \angle\mathbf{V}$
$a = Re(\mathbf{V})$ *and* $b = Im(\mathbf{V})$
$\mathbf{V} = a + jb$
Since a phasor can be expressed in both rectangular and polar forms, we write
$\mathbf{V} = a + jb = |\mathbf{V}|\angle\mathbf{V} = A\angle\theta$
$\omega = 2\pi f = \dfrac{2\pi}{T}$

Phasors and Sinusoids



The trigonometry provides the following equations for converting between the rectangular and polar forms of phasors.

$a = \cos(\theta),\ b = A\sin(\theta),\ A = \sqrt{a^2 + b^2}$

and $\theta = \begin{cases} \tan^{-1}(\dfrac{b}{a}),\ when\ a > 0 \\ 180^o - \tan^{-1}(\dfrac{b}{-a}),\ when\ a < 0 \end{cases}$

The use of phasors to represent sinusoids is based on Euler's formula. Euler's formula is

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

Consequently,

$$Ae^{j\phi} = A\cos(\phi) + jA\sin(\phi) = A\angle\phi$$

$$Ae^{j\phi} = A\angle\phi$$

$Ae^{j\phi}$ is called the exponential form of a phasor.

The conversion between the polar and exponential forms is immediate. In both, A is the amplitude of the sinusoid and ϕ is the phase angle of the sinusoid.

Phasors and Sinusoids

$\Rightarrow$ **EXAMPLE 10.2-1** - Consider the sinusoids $v_1(t) = 10\cos(200t + 45^o)V$ and $v_2(t) = 8\sin(200t + 15^o)V$. Determine the time by which $v_2(t)$ is advanced or delayed with respect to $v_1(t)$.
Answer: $t_d = -10.47ms$

$\Rightarrow$ **EXAMPLE 10.3-1** - Determine the phasors corresponding to the sinusoids $i_1(t) = 120\cos(400t + 60^o)mA$ and $i_2(t) = 100\sin(400t - 75^o)mA$
Answer: $I_1 = 120\angle 60^o$ and $I_2 = 100\angle -165^o$

$\Rightarrow$ **EXAMPLE 10.3-2** - Consider the phasors $\mathbf{V}_1 = 4.25\angle 115^o$ and $\mathbf{V}_2 = -4 + j3$. Convert $\mathbf{V}_1$ to rectangular form and $\mathbf{V}_2$ to polar form.
Answer: $\mathbf{V}_1 = -1.796 + j3.852$ and $\mathbf{V}_2 = 5\angle 143^o$

$\Rightarrow$ **EXERCISE 10.3-3** - Consider the phasors $\mathbf{V}_1 = -1.796 + j3.852 = 4.25\angle 115^o$ and $\mathbf{V}_2 = -4 + j3 = 5\angle 143^o$. Determine $\mathbf{V}_1 + \mathbf{V}_2$, $\mathbf{V}_1.\mathbf{V}_2$ and $\frac{\mathbf{V}_1}{\mathbf{V}_2}$
Answer: $\mathbf{V}_1 + \mathbf{V}_2 = -5.796 + j6.852$, $\mathbf{V}_1.\mathbf{V}_2 = 21.25\angle -102^o$ and $\frac{\mathbf{V}_1}{\mathbf{V}_2} = 0.85\angle -28^o$

KVL or KCL equation from an ac circuit

Consider

$$e^{\omega t+\theta} = \cos(\omega t + \theta) + j\sin(\omega t + \theta)$$

Taking the real part of both sides gives

$$A\cos(\omega t + \theta) = Re\{e^{\omega t+\theta}\} = Re\{e^{\omega t}e^{\theta}\}$$

Consider a sinusoid and corresponding phasor

$v(t) = A\cos(\omega t + \theta)V$ and $\mathbf{V}(\omega) = A\angle\theta = Ae^{j\theta}$

$$v(t) = Re\{\mathbf{V}(\omega)e^{j\omega t}\}$$

Next, consider a KVL or KCL equation from an ac circuit, for example,

$$0 = \sum_i v_i(t) = \sum_i Re\{\mathbf{V}_i(\omega)e^{j\omega t}\} = \sum_i Re\{e^{j\omega t}\mathbf{V}_i(\omega)\}$$

Eq. before is required to be true for all values of time t. Let t = 0.

$$0 = \sum_i Re\{\mathbf{V}_i(\omega)\}$$

Next,

$$0 = \sum_i Re\{j\mathbf{V}_i(\omega)\} = \sum_i Im\{\mathbf{V}_i(\omega)\}$$

Together,

$$0 = \sum_i \{\mathbf{V}_i(\omega)\}$$

In summary, if a set of sinusoidal voltages $v_i(t)$ satisfy KVL for an ac circuit, the corresponding phasor voltages $\mathbf{V}_i(\omega)$ satisfy the same KVL equation. Similarly, if a set of sinusoidal currents $i_i(t)$ satisfy KCL for an ac circuit, the corresponding phasor currents $\mathbf{I}_i(\omega)$ satisfy the same KCL equation.

KVL or KCL equation from an ac circuit

EXAMPLE 10.3-4 Kirchhoff's Laws for AC Circuits - The input to the circuit shown in Figure is the voltage source voltage,

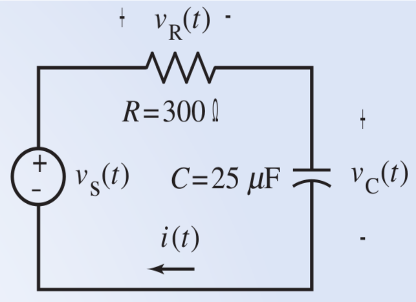
$$v_s(t) = 25\cos(100t + 15^o)$$

The output is the voltage across the capacitor,

$$v_C(t) = 20\cos(100t - 22^o)$$

Determine the resistor voltage $v_R(t)$.

Answer: $V_R(t) = 15\cos(100t + 68.1^o)$



Impedances

The element voltage and element current are labeled as v(t) and i(t). We can write

$v(t) = V_m \cos(\omega t + \theta)\ V$ and $i(t) = I_m \cos(\omega t + \phi)\ A$

The corresponding phasors are

$\mathbf{V}(\omega) = V_m \angle \theta\ V$ and $\mathbf{I}(\omega) = I_m \angle \phi\ A$

The impedance is denoted as $\mathbf{Z}(\omega)$ so

$$\mathbf{Z}(\omega) = \frac{\mathbf{V}(\omega)}{\mathbf{I}(\omega)} = \frac{V_m \angle \theta}{I_m \angle \phi} = \frac{V_m}{I_m} \angle(\theta - \phi)$$

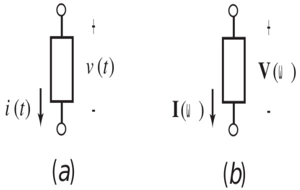
The admittance is denoted as $\mathbf{Y}(\omega)$ so

$$\mathbf{Y}(\omega) = \frac{\mathbf{I}(\omega)}{\mathbf{V}(\omega)}$$

Consequently,

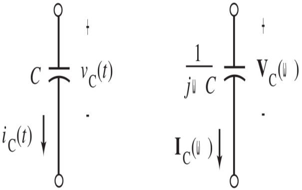
$\mathbf{V}(\omega) = \mathbf{Z}(\omega)\mathbf{I}(\omega)$

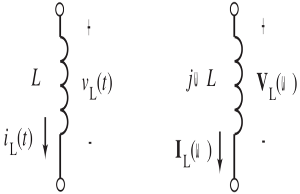
which is **Ohm's law for ac circuits**.

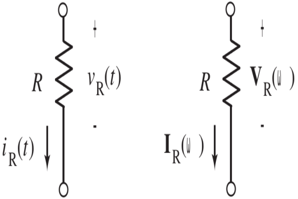


An element of an ac circuit represented (a) in the time domain and (b) in the frequency domain.

Impedances







$v_C(t) = A\cos(\omega t + \theta)$
$\Rightarrow \mathbf{V}_C(\omega) = A\angle\theta$
$i_C(t) = C\frac{d}{dt}v_C(t) =$
$A\omega C\cos(\omega t + \theta + 90^o)$
$\Rightarrow \mathbf{I}_C(\omega) = jA\omega C\angle\theta$

$$\mathbf{Z}(\omega) = \frac{\mathbf{V}(\omega)}{\mathbf{I}(\omega)} = \frac{1}{j\omega C}\ \Omega$$

$i_L(t) = A\cos(\omega t + \theta)$
$\Rightarrow \mathbf{I}_L(\omega) = A\angle\theta$
$v_L(t) = L\frac{d}{dt}i_L(t) =$
$A\omega L\cos(\omega t + \theta + 90^o)$
$\Rightarrow \mathbf{V}_L(\omega) = jA\omega L\angle\theta$

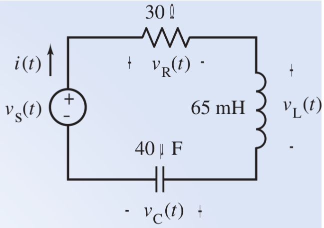
$$\mathbf{Z}(\omega) = \frac{\mathbf{V}(\omega)}{\mathbf{I}(\omega)} = j\omega L\ \Omega$$

$i_R(t) = A\cos(\omega t + \theta)$
$\Rightarrow \mathbf{I}_R(\omega) = A\angle\theta$
$v_R(t) = R\ i_R(t) = R\ A\cos(\omega t + \theta)$
$\Rightarrow \mathbf{V}_R(\omega) = AR\ \angle\theta$

$$\mathbf{Z}(\omega) = \frac{\mathbf{V}(\omega)}{\mathbf{I}(\omega)} = R\ \Omega$$

Impedances

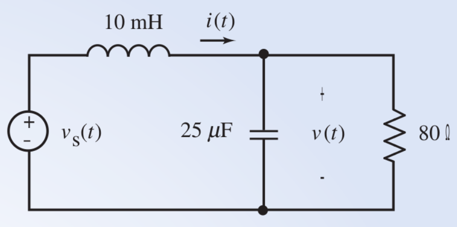
EXAMPLE 10.4-1 - The input to the ac circuit shown in Figure below is the source voltage $v_s(t) = 12\cos(1000t + 15^o)\ V$. Detemine (a) the impedances of the capacitor, inductor, and resistance and (b) the corrent i(t).



Answer: $\mathbf{Z}_c(\omega) = -j25\ \Omega$, $\mathbf{Z}_L(\omega) = j65\ \Omega$, $\mathbf{Z}_R(\omega) = 30\ \Omega$, and $i(t) = 0.24\cos(1000t - 38.13^o)\ A$

Impedances

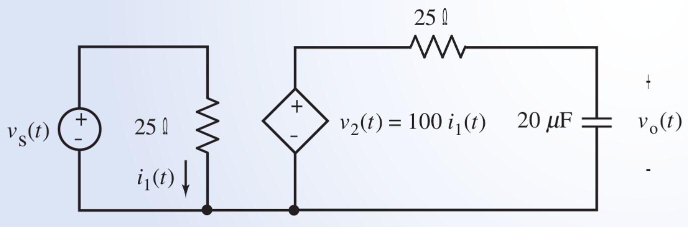
EXAMPLE 10.4-2 - The input to the ac circuit shown in Figure below is the source voltage $v_s(t) = 48\cos(500t + 75^o)\ V$. Detemine the voltage v(t).



Answer: $v(t) = 65.9\cos(500t + 16^o)\ V$

Impedances

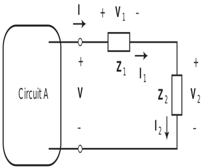
EXAMPLE 10.4-2 - The input to the ac circuit shown in Figure below is the source voltage $v_s(t) = 12\cos(1000t + 45^o)\ V$. Detemine the voltage $v_o(t)$.



Answer: $v_o(t) = 42.933\cos(1000t + 18.44^o)\ V$

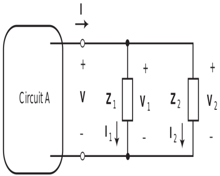
Series, Parallel Impedances and Voltage, Current Division

Series Impedances



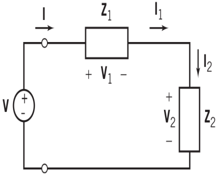
$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2$$

Parallel Impedances



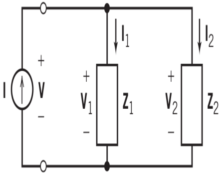
$$\mathbf{Z}_{eq} = \frac{\mathbf{Z}_1 \mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

Voltage Division



$$\mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} V$$

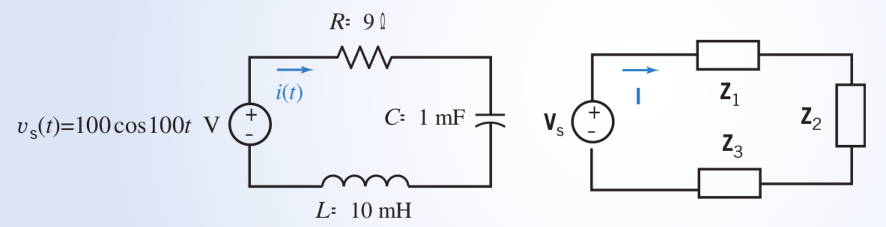
Current Division



$$\mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2} I$$

Impedances

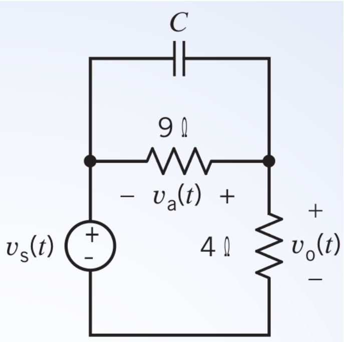
EXAMPLE 10.5-1 - Determine the steady-state current $i(t)$ in the RLC circuit shown in Figure below, using phasors and impedances.



Answer: $i(t) = 7.86\cos(100t + 45^o)\ A$

Impedances

EXAMPLE 10.5-4 - Consider the circuit shown in Figure below. The input to the circuit is the voltage of the voltage source $v_s(t)$, and the output is the voltage across the $4\ \Omega$ resistor, $v_o(t)$. When the input is $v_s(t) = 8.93\cos(2t + 54^o)\ V$, the corresponding output is $v_o(t) = 3.83\cos(2t + 83^o)\ V$. Determine the voltage across the $9\ \Omega$ resistor $v_a(t)$ and the value of the capacitance C of the capacitor.



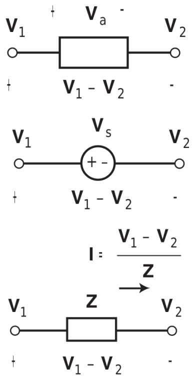
Answer: $v_a(t) = 5.88\cos(2t + 216^o)\ V$ and $C = 60mF$.

Node Equations

The node equations are a set of simultaneous equations in which the unknowns are the node voltages. We write the node equations by

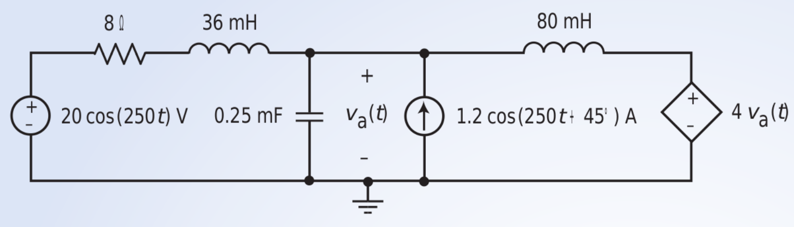
1. Expressing the element voltages and currents (for example, the current and voltage of an impedance) in terms of the node voltages.
2. Applying KCL at the nodes of the ac circuit.

After writing and solving the node equations, we can determine all of the voltages and currents of the ac circuit using Ohm's and Kirchhoff's laws.



Node Equations

EXAMPLE 10.6-1 - Determine the voltage $v_a(t)$ for the circuit shown in Figure below.



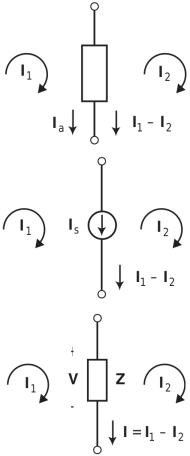
Answer: $v_a(t) = 12.43\cos(250t + 81.2^o)\ V$

Mesh Equation

The mesh equations are a set of simultaneous equations in which the unknowns are the mesh currents. We write the mesh equations by

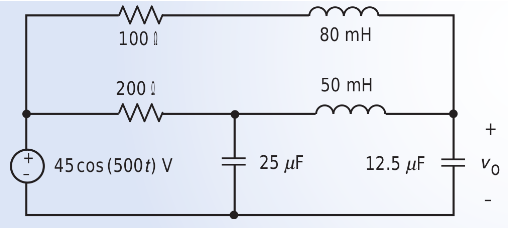
1. Expressing the element voltages and currents (for example, the current and voltage of an impedance) in terms of the mesh currents.
2. Applying KVL to the meshes of the ac circuit.

After writing and solving the mesh equations, we can determine all of the voltages and currents of the ac circuit using Ohm's and Kirchhoff's laws.



Mesh Equations

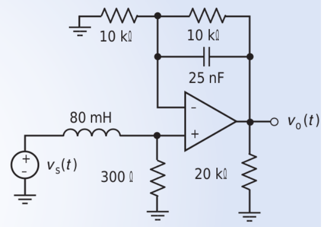
EXAMPLE 10.6-2 - Determine the mesh currents for the circuit shown in Figure below.



Answer: $i_1(t) = 374\cos(500t + 15^o)\ V$, $i_2(t) = 575\cos(500t + 25^o)\ V$ and $i_3(t) = 171\cos(500t + 28^o)\ V$

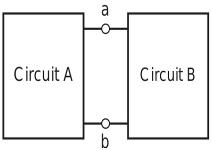
Mesh Equations

EXAMPLE 10.6-4 - The input to the ac circuit shown in Figure below is the voltage source voltage $v_s(t) = 125\cos(500t + 15^o)\ mV$. Determine the output voltage $v_o(t)$.

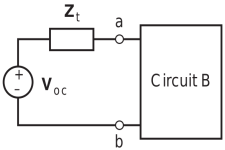


Answer: $v_o(t) = 174\cos(500t + 69.79^o)\ mV$

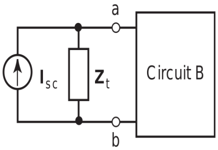
The open-circuit voltage $\mathbf{V}_{oc}$, the short-circuit current $\mathbf{I}_{sc}$, and the Thevenin impedance $\mathbf{Z}_t$, are related by the equation $\mathbf{V}_{oc} = \mathbf{Z}_t\ \mathbf{I}_{sc}$.



Circuit A and Circuit B.

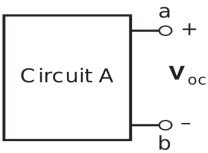


Thevenin equivalent circuit.

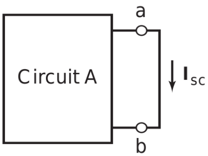


Norton equivalent circuit.

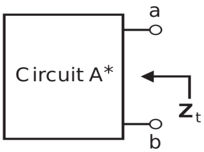
The Thevenin equivalent circuit involves three parameters:



The open-circuit voltage $\mathbf{V}_{oc}$



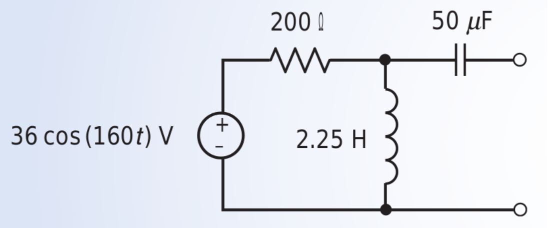
The short-circuit current $\mathbf{I}_{sc}$



The Thevenin impedance $\mathbf{Z}_t$

Thevenin Equivalent Circuit

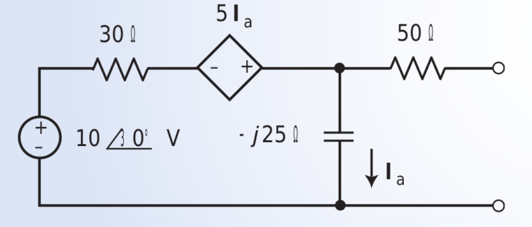
EXAMPLE 10.7-1 - Find the Thevenin equivalent circuit of the ac circuit in Figure shown below.



Answer: $\mathbf{V}_{oc} = 31.47\angle 29.1^o\ V$ and $\mathbf{Z}_t = 152.83 - j40.094\ \Omega$

Norton Equivalent Circuit

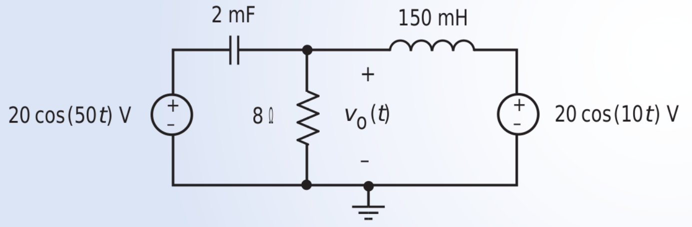
EXAMPLE 10.7-2 - Find the Norton equivalent circuit of the ac circuit in Figure shown below.



Answer: $\mathbf{I}_{sc} = 0.11\angle{-2.01^o}\ A$ and $\mathbf{Z}_n = 66\angle{-13^0}\ \Omega$

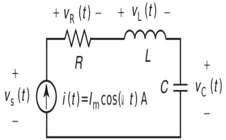
Superposition

EXAMPLE 10.8-1 - Determine the voltage $v_o(t)$ across the 8Ω resistor in the circuit shown below.

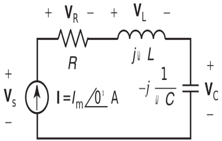


Answer: $v_o(t) = v_{o1}(t) + v_{o2}(t) = 15.46\cos(50t + 104.0^o) + 20.24\cos(10t - 10.94^o)$

A **phasor diagram** is a graphical representation of phasors and their relationship on the complex plane.



The time domain

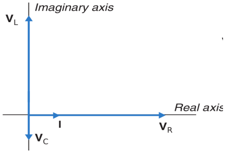


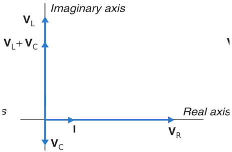
The frequency domain

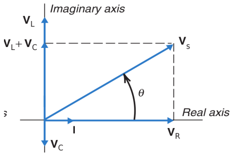
Equations:

$$\mathbf{I}_m = I_m\angle 0^o$$
$$\mathbf{V}_R = RI_m\angle 0^o$$
$$\mathbf{V}_L = j\omega L(I_m\angle 0) = \omega L I_m\angle 90^o$$
$$\mathbf{V}_C = \frac{1}{j\omega C}(I_m\angle 0) = \frac{I_m}{\omega C}\angle -90^o$$
$$\mathbf{V}_S = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C$$

Phasor diagrams for the RLC circuit in Figure above

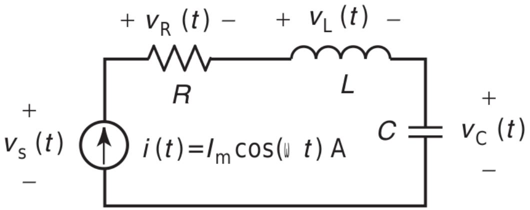






Phasor Diagrams

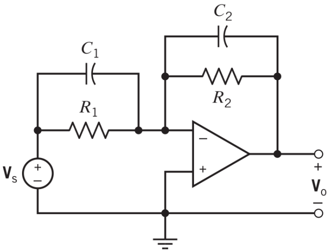
EXAMPLE 10.9-1 - Consider the circuit shown in Figure below when $R = 80\Omega$, $L = 8H$, $C = 5mF$ and $i(t) = 0.25\cos(10t)\ A$. Determine the phasor diagram.



Answer: $\mathbf{V}_R = 20\angle 0^o\ V$, $\mathbf{V}_L = 20\angle 90^o\ V$, $\mathbf{V}_C = 5\angle -90^o\ V$ and $\mathbf{V}_S = 25\angle 36.9^o\ V$

Op Amps

EXAMPLE 10.10-1 - Find the ratio $\mathbf{V}_o/\mathbf{V}_i$ for the circuit of Figure below when $R_1 = 1\ kV$, $R_2 = 10\ kV$, $C_1 = 0\ F$, and $C_2 = 0.1\ \mu F$ for $\omega = 1000\ rad/s$.



Answer: $\mathbf{V}_o/\mathbf{V}_i = 7.07\angle 135^o$

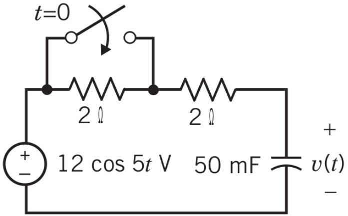
The Complete Response

Next, we consider circuits with sinusoidal inputs that are subject to abrupt changes, as when a switch opens or closes. To find the complete response of such circuits, we:

- Represent the circuit by a differential equation.
- Find the general solution of the homogeneous differential equation.
- Find a particular solution of the differential equation.
- Represent the response of the circuit as $v(t) = v_n(t) + v_f(t)$.
- Use the initial conditions to evaluate the unknown constants.

The Complete Response

EXAMPLE 10.11-1 - Determine $v(t)$, the voltage across the capacitor in Figure below, both before and after the switch closes.



Answer: $v(t) = \begin{cases} 8.49\cos(5t - 45^o), & when\ t < 0 \\ -3.6e^{-10t} + 10.74\cos(5t - 26.6^o), & when\ t > 0 \end{cases}$