Aula 10 - The Complete Response of RL and RC Circuits

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Contents

Introduction to Electric Circuits by James A. Svoboda, Richard C. Dorf, 9th Edition

Introduction

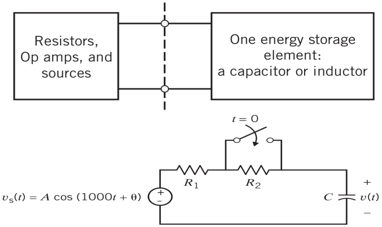
In this chapter, we consider the response of RL and RC circuits to abrupt changes. The abrupt change might be a change to the circuit, as when a switch opens or closes. Consequently, we will do the following:

- Develop vocabulary that will help us talk about the response of a first-order circuit.
- Analyze first-order circuits with inputs that are constant after some particular time, t_0.
- Introduce the notion of a stable circuit and use it to identify stable first-order circuits.
- Analyze first-order circuits that experience more than one abrupt change.
- Introduce the step function and use it to determine the step response of a first-order circuit.
- Analyze first-order circuits with inputs that are not constant.

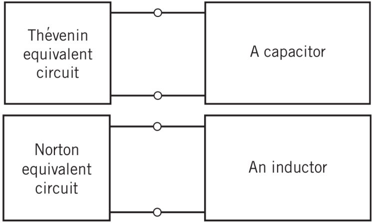
First-Order Circuits

Circuits that contain only one inductor and no capacitors or only one capacitor and no inductors can be represented by a first-order differential equation. These circuits are called **first-order circuits**.

A plan for analyzing first-order circuits.



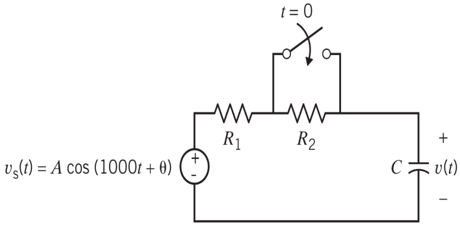
First, separate the energy storage element from the rest of the circuit.

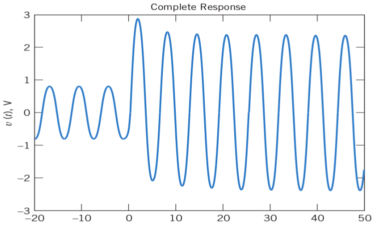


Next, replace the circuit connected to a capacitor by its Thevenin equivalent circuit or replace the circuit connected to an inductor by its Norton equivalent circuit.

First-Order Circuits

After the switch closes, the response will consist of two parts: a transient part that eventually dies out and a steady-state part.





Complete Response $=$ Transient Response
$+$ Steady-State Response

$$v(t) = Ke^{\frac{-t}{\tau}} + M\cos(1000t + \delta)$$

First-Order Circuits

In general, the **complete response** of a first-order circuit can be represented as the sum of two parts, the natural response and the forced response:

$$\textbf{Complete Response} = \textbf{Natural Response} + \textbf{Forced Response}$$

- The **natural response** is the source free solution and the initial condition is not neglected.
- The **forced response** is not the source free solution and the initial condition is neglected.

The complete response of a first-order circuit will depend on an initial condition, usually a capacitor voltage or an inductor current at a particular time.

First-Order Circuits

In general, the **complete response** of a first-order circuit can be represented as the sum of two parts, the natural response and the forced response:

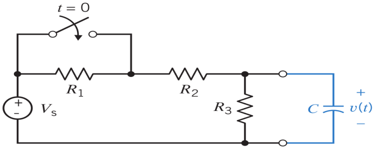
Plan for finding the complete response of first-order circuits

1. Obtain the initial condition of the energy storage element.
2. Find the natural response after the disturbance.
3. Find the forced response after the disturbance.
4. Add the natural response to the forced response to get the complete response.

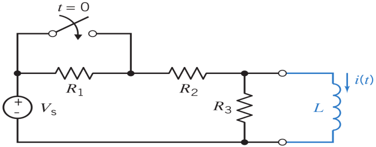
Use the initial condition to obtain natural response.

The Response of a First-Order Circuit to a Constant Input

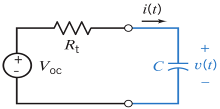
In this section, we find the complete response of a first-order circuit when the input to the circuit is constant after time t_0 .



A first-order circuit containing a capacitor.

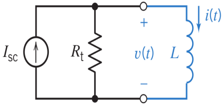


A first-order circuit containing an inductor.



$$V_{oc} = \frac{R_3}{R_3+R_2} V_s$$

$$R_t = \frac{R_2 R_3}{R_2+R_3}$$



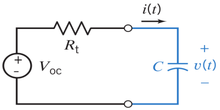
$$I_{sc} = \frac{V_s}{R_2}$$

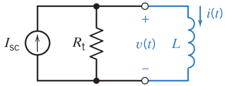
$$R_t = \frac{R_2 R_3}{R_2+R_3}$$

After the switch closes, the circuit connected to the capacitor is replaced by its Thevenin equivalent circuit.

After the switch closes, the circuit connected to the inductor is replaced by its Norton equivalent circuit.

The Response of a First-Order Circuit to a Constant Input





$$
\begin{aligned}
-V_{oc} + R_t i(t) + v(t) &= 0 \\
-V_{oc} + R_t C \frac{d}{dt} v(t) + v(t) &= 0 \\
\frac{d}{dt} v(t) + \frac{v(t)}{R_t C} &= \frac{V_{oc}}{R_t C} \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
-I_{sc} + \frac{v(t)}{R_t} + i(t) &= 0 \\
-I_{sc} + \frac{L}{R_t} \frac{d}{dt} i(t) + i(t) &= 0 \\
\frac{d}{dt} i(t) + \frac{i(t)}{L/R_t} &= \frac{I_{sc}}{L/R_t} \qquad (2)
\end{aligned}
$$

The parameter τ is called time constant and both equations have the same form. That is

$$\frac{d}{dt} x(t) + \frac{x(t)}{\tau} = K$$

The Response of a First-Order Circuit to a Constant Input

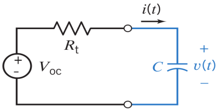
We will solve the differential equation below by separating the variables and integrating. Then we will use the solution of Eqs. 3 to obtain solutions of Eqs. 1 and 2.

$$
\begin{aligned}
\frac{d}{dt}x(t) + \frac{x(t)}{\tau} &= K \qquad (3)\\
\frac{dx}{dt} &= \frac{K\tau - x}{\tau}\\
\frac{dx}{x - K\tau} &= -\frac{dt}{\tau}\\
\int \frac{dx}{x - K\tau} &= -\int \frac{dt}{\tau} + D\\
\int \frac{dx}{x - K\tau} &= -\frac{1}{\tau}\int dt + D
\end{aligned}
$$

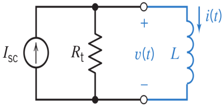
$$
\begin{aligned}
\ln(x - K\tau) &= -\frac{t}{\tau} + D\\
x(t) &= K\tau + Ae^{-\frac{t}{\tau}} \; and \; A = e^{D}\\
To\ find\ A, let\ t = 0.\ Then &\\
x(0) &= K\tau + Ae^{-\frac{0}{\tau}}\\
A &= x(0) - K\tau\\
Because,\ x(\infty) = K\tau &\\
x(t) &= x(\infty) + (x(0) - x(\infty))e^{-\frac{t}{\tau}} \qquad (4)
\end{aligned}
$$

The Response of a First-Order Circuit to a Constant Input

Then we will use the Eqs. 4 to obtain solutions of Eqs. 1 and 2.



$$\frac{d}{dt}v(t) + \frac{v(t)}{R_tC} = \frac{V_{oc}}{R_tC}$$



$$\frac{d}{dt}i(t) + \frac{i(t)}{L/R_t} = \frac{I_{sc}}{L/R_t}$$

$$\begin{aligned} \frac{d}{dt}x(t) + \frac{x(t)}{\tau} &= K \\ x(t) &= x(\infty) + (x(0) - x(\infty))e^{-\frac{t}{\tau}} \\ v(t) &= V_{oc} + (v(0) - v_{oc})e^{-\frac{t}{R_tC}} \end{aligned}$$

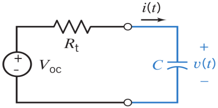
$$\begin{aligned} \frac{d}{dt}x(t) + \frac{x(t)}{\tau} &= K \\ x(t) &= x(\infty) + (x(0) - x(\infty))e^{-\frac{t}{\tau}} \\ i(t) &= I_{sc} + (i(0) - I_{sc})e^{-\frac{t}{L/R_t}} \end{aligned}$$

Complete response $v(t)$, Steady-State Response V_{oc} and Transient Response $(v(0) - v_{oc})e^{-\frac{t}{R_tC}}$

Complete response $i(t)$, Steady-State Response I_{sc} and Transient Response $(i(0) - I_{sc})e^{-\frac{t}{L/R_t}}$

The Response of a First-Order Circuit to a Constant Input

Then we will use the Eqs. 4 to obtain solutions of Eqs. 1 and 2.



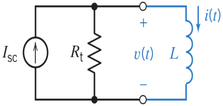
$$\frac{d}{dt}v(t) + \frac{v(t)}{R_tC} = \frac{V_{oc}}{R_tC}$$

$$v(t) = V_{oc} + (v(0) - V_{oc})e^{-\frac{t}{R_tC}}$$

Complete response $v(t)$, Steady-State Response V_{oc} and Transient Response $(v(0) - V_{oc})e^{-\frac{t}{R_tC}}$

$$v(t) = V_{oc}(1 - e^{-\frac{t}{R_tC}}) + v(0)e^{-\frac{t}{R_tC}}$$

Complete response $v(t)$, forced response $V_{oc}(1 - e^{-\frac{t}{R_tC}})$ and natural response $v(0)e^{-\frac{t}{R_tC}}$



$$\frac{d}{dt}i(t) + \frac{i(t)}{L/R_t} = \frac{I_{sc}}{L/R_t}$$

$$i(t) = I_{sc} + (i(0) - I_{sc})e^{-\frac{t}{L/R_t}}$$

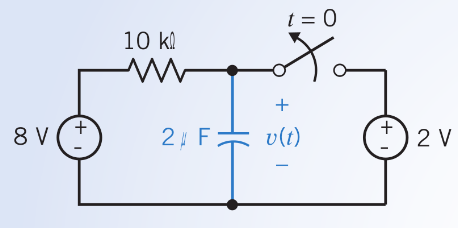
Complete response $i(t)$, Steady-State Response I_{sc} and Transient Response $(i(0) - I_{sc})e^{-\frac{t}{L/R_t}}$

$$i(t) = I_{sc}(1 - e^{-\frac{t}{L/R_t}}) + i(0)e^{-\frac{t}{L/R_t}}$$

Complete response $i(t)$, forced response $I_{sc}(1 - e^{-\frac{t}{L/R_t}})$ and natural response $i(0)e^{-\frac{t}{L/R_t}}$

First-Order Circuit to a Constant Input

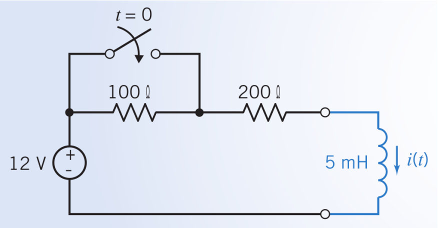
EXAMPLE 8.3-1 - Find the capacitor voltage after the switch opens in the circuit shown in Figure below. What is the value of the capacitor voltage $50ms$ after the switch opens?



Answer: $v(50) = 8 - 6e^{-\frac{50}{20}} = 7.51V \ and \ t[ms]$.

First-Order Circuit to a Constant Input

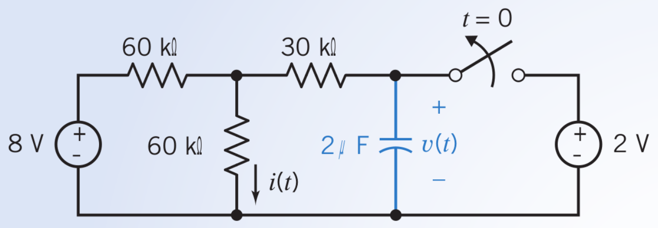
EXAMPLE 8.3-4 - The switch in Figure below has been open for a long time, and the circuit has reached steady state before the switch closes at time $t = 0$. Find the inductor current for $t \geq 0$.



Answer: $i(t) = 60 - 20e^{-t/25} mA \text{ and } t[\mu s]$.

First-Order Circuit to a Constant Input

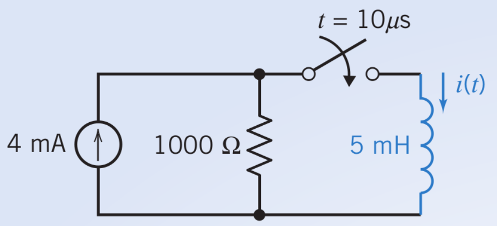
EXAMPLE 8.3-5- The circuit in Figure below is at steady state before the switch opens. Find the current $i(t)$ for $t > 0$.



Answer: $i(t) = 66.7 - 16.7e^{-\frac{t}{120}} \mu A \ and \ t[ms]$.

First-Order Circuit to a Constant Input

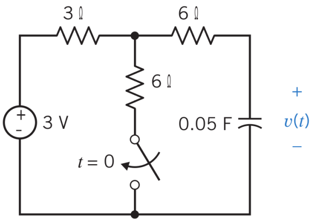
EXAMPLE 8.3-7- Find the inductor current after the switch closes in the circuit shown in Figure below. How long will it take for the inductor current to reach $2\ mA$?



Answer: $t = 13.47\mu s$

First-Order Circuit to a Constant Input

EXERCISE 8.3-1- The circuit shown in Figure below is at steady state before the switch closes at time $t = 0$. Determine the capacitor voltage $v(t)$ for $t \geq 0$.



Answer: $v(t) = 2 - e^{-\frac{t}{0.4}}\ V\ for\ t > 0$

Calculating the time constant

A graphical technique for measuring the time constant of a first-order circuit.

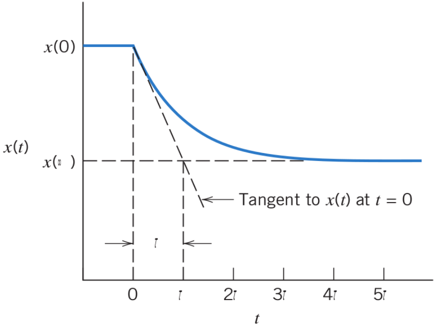
Equation (4),

$$x(t) = x(\infty) + (x(0) - x(\infty))e^{-\frac{t}{\tau}}$$

Derivative of $x(t)$

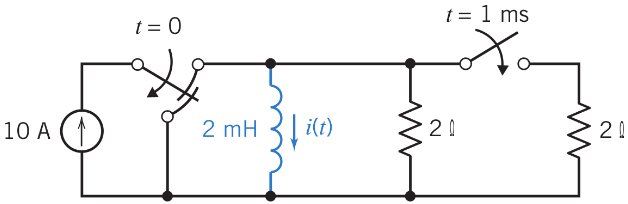
$$\left.\frac{d}{dx}x(t)\right|_{t=0} = \left.-\frac{1}{\tau}(x(0) - x(\infty))e^{-\frac{t}{\tau}}\right|_{t=0}$$

$$\tau = \frac{x(\infty)-x(0)}{\left.\frac{d}{dx}x(t)\right|_{t=0}}$$



Sequential Switching

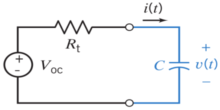
Sequential switching occurs when a circuit contains two or more switches that change state at different instants. As an example of sequential switching, consider the circuit shown in Figure below:



Answer: $i(t) = 10e^{-1}e^{-\frac{t-1}{2}}\ A,\ t > 1ms\ and\ t[ms].$

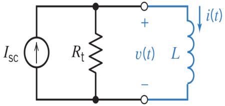
Stability of First-Order Circuits

How can we design first-order circuits to be stable?



$$v(t) = V_{oc} + (v(0) - V_{oc})e^{-\frac{t}{R_t C}}$$

Complete response $v(t)$, Steady-State Response V_{oc} and Transient Response $(v(0) - V_{oc})e^{-\frac{t}{R_t C}}$

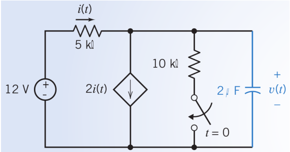


$$i(t) = I_{sc} + (i(0) - I_{sc})e^{-\frac{t}{L/R_t}}$$

Complete response $i(t)$, Steady-State Response I_{sc} and Transient Response $(i(0) - I_{sc})e^{-\frac{t}{L/R_t}}$

Response: Recalling that $\tau = R_t C$ or $\tau = \frac{L}{R_t}$, we see that $R_t > 0$ is required to make a first-order circuit stable.

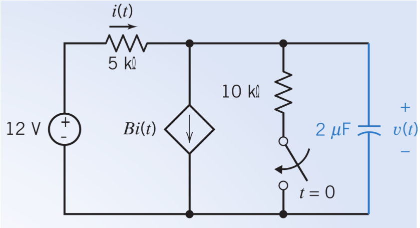
EXAMPLE 8.5-1 - The first-order circuit shown in Figure below is at steady state before the switch closes at $t < 0$. Find the capacitor voltage $v(t)$ for $t > 0$.



Answer: $v(t) = 24 - 12e^{t/20} mA$, *where t has units of ms.*

Stability of First-Order Circuits

EXERCISE 8.5-2- The circuit considered before has been redrawn in Figure below, with the gain of the dependent source represented by the variable B. What restrictions must be placed on the gain of the dependent source to ensure that it is stable? Design this circuit to have a time constant of +20 ms.



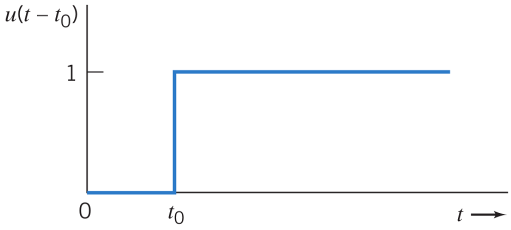
Answer: $B < \frac{3}{2}$ $and\ R_t = 10K\Omega.$

The Unit Step Source

The unit step function provides a convenient way to represent an abrupt change in a voltage or current.

We define the unit step function as a function of time that is zero for $t < t_0$ and unity for $t > t_0$. At $t = t_0$, the value changes from zero to one. We represent the unit step function by $u(t - t_0)$, where

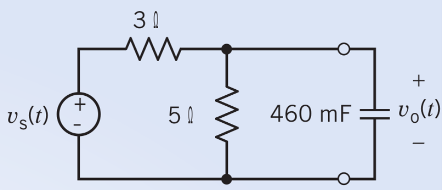
$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$



Unit step forcing function,$u(t - t_0)$.

The Unit Step Source

EXERCISE 8.6-2- Figure below shows a first-order circuit. The input to the circuit is the voltage of the voltage source, $v_s(t)$. The output is the voltage across the inductor, $v_o(t)$. Determine the output of this circuit when the input is $v_s(t) = 7 - 14u(t)$ V.

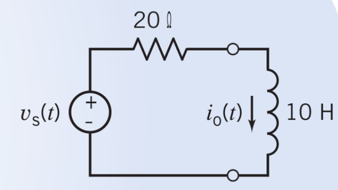


Answer:

$$i_o(t) = \begin{cases} -4.38, \ t < t_0 \\ -4.38 + 8.76e^{-1.16t}V, \ t > t_0 \end{cases}$$

The Unit Step Source

EXERCISE 8.6-1- Figure below shows a first-order circuit. The input to the circuit is the voltage of the voltage source, $v_s(t)$. The output is the current of the inductor, $i_o(t)$. Determine the output of this circuit when the input is $v_s(t) = 4 - 8u(t)$ V.



Answer:

$$i_o(t) = \begin{cases} 0.2A, \ t < t_0 \\ -0.2 + 0.4e^{-2t}A, \ t > t_0 \end{cases}$$

Integrating Factor Method

In this section, we introduce the integrating factor method

$$General\ form\ (GF)$$
$$\frac{d}{dt}y(t) + ay(t) = x(t)$$
$$Multiply\ GF\ by\ e^at$$
$$(\frac{d}{dt}y + ay)e^{at} = xe^{at}$$
$$However$$
$$\frac{d(ye^{at})}{dt} = \frac{dy}{dt}e^{at} + aye^{at}$$

$$\frac{d(ye^{at})}{dt} = (\frac{d}{dy}y + ay)e^{at}$$
$$Therefore$$
$$\frac{d(ye^{at})}{dt} = xe^{at}$$
$$ye^{at} = \int xe^{at}dt + K$$
$$y(t) = e^{-at}\int xe^{at}dt + Ke^{-at} \tag{5}$$

Integrating Factor Method

The solution of the differential equation $\frac{d}{dt}y(t) + ay(t) = x(t)$ is $y(t) = e^{-at} \int xe^{at}dt + Ke^{-at}$.

When the source is a constant so that $x(t) = M$, we have

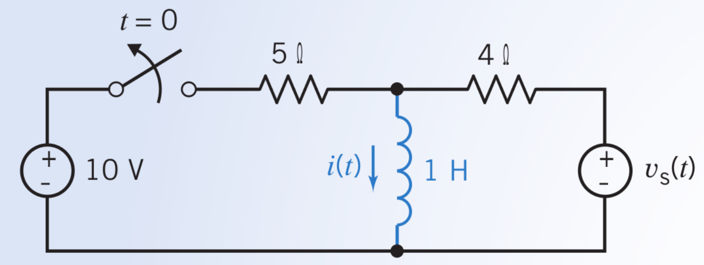
$$y(t) = e^{-at} \int Me^{at}dt + Ke^{-at}$$
$$y(t) = e^{-at}M \int e^{at}dt + Ke^{-at}$$
$$y(t) = \frac{M}{a} + Ke^{-at}$$
$$y(t) = y_f + y_n$$

When the source is an exponential so that $x(t) = Me^{bt}$, we have

$$y(t) = e^{-at} \int Me^{bt}e^{at}dt + Ke^{-at}$$
$$y(t) = Me^{-at} \int e^{(a+b)t}dt + Ke^{-at}$$
$$y(t) = \frac{Me^{bt}}{a+b} + Ke^{-at}$$
$$y(t) = y_f + y_n$$

The Response of a First-Order Circuit to a Nonconstant Source

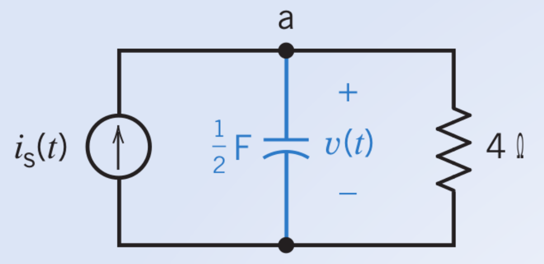
EXERCISE 8.7-1- Find the current i for the circuit of Figure below for $t > 0$ when $v_s = 10e^{-2t}u(t)V$.Assume the circuit is in steady state at $t = 0^-$.



Answer: $i(t) = -3e^{-4t} + 5e^{-2t}A \ t > 0$.

The Response of a First-Order Circuit to a Nonconstant Source

EXERCISE 8.7-2- Find the response $v(t)$ for $t > 0$ for the circuit of Figure below. The initial voltage $v(0) = 0$, and the current source $i(t)_s = 10\sin(2t)u(t)A$.



Answer: $v(t) = \frac{160}{17}e^{-\frac{t}{2}} + \frac{40}{17}\sin(2t) - \frac{160}{17}\cos(2t)V \; t > 0$.