Aula 8 - Energy Storage Elements

Prof. Marcelino Andrade

Faculdade UnB Gama

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Contents

Introduction to Electric Circuits by James A. Svoboda, Richard C. Dorf, 9th Edition

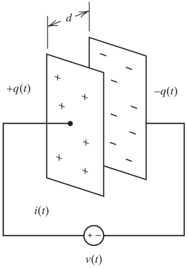
Introduction

This chapter introduces two more circuit elements, the capacitor and the inductor. Consequently:

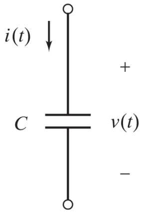
- Electric circuits that contain capacitors and/or inductors are represented by differential equations, able to store energy and have memory;
- In a dc circuit, capacitors act like open circuits, and inductors act like short circuits;
- Series or parallel capacitors and inductors can be reduced to an equivalent capacitor and equivalent inductor, respectively;
- An op amp and a capacitor and/or inductors can be used to make circuits that perform the mathematical operations of integration or differentiation.;

Capacitors

Capacitance is a measure of the ability of a device to store energy in the form of an electric field.



Typical capacitor



Symbol for capacitor

$$C = \epsilon \frac{A}{d} \quad (1)$$

$$C = \frac{q(t)}{v(t)} \quad (2)$$

$$i(t) = \frac{d}{dt} q(t) \quad (3)$$

$$i(t) = C \frac{d}{dt} v(t) \quad (4)$$

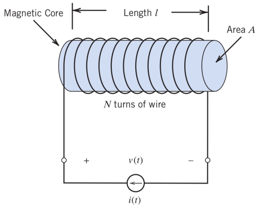
$$v(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau) d\tau \quad (5)$$

C = capacitance, A = area, d = distance,
q = electric charge, v = voltage, i = current,
ϵ = dielectric constant.

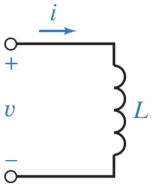
Capacitor equations

Inductors

Inductance is a measure of the ability of a device to store energy in the form of a magnetic field.



Typical inductor



Symbol for inductor

$$L = \mu \frac{N^2 A}{l} \quad (6)$$

$$L = \frac{\phi(t)}{i(t)} \quad (7)$$

$$v(t) = \frac{d}{dt}\phi(t) \quad (8)$$

$$v(t) = L\frac{d}{dt}i(t) \quad (9)$$

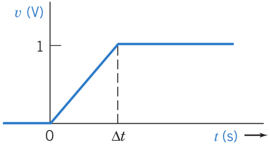
$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau \quad (10)$$

L = indutance, A = area, μ = permeability,
l = length of the winding, N = number of turns,
ϕ = fluxo magnético, v = voltage, i = current,

Inductor equations

Capacitors

The voltage across a **capacitor** cannot change instantaneously.

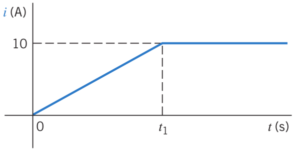


$$i(t) = C\frac{d}{dt}v(t) \qquad i(t) = \begin{cases} 0, \ t < 0 \\ \frac{C}{\Delta t}, \ 0 \leq \ t \leq \Delta t \\ 0, \ t > \Delta t \end{cases}$$

Clearly, Δt cannot decline to zero or we would experience an infinite current. An infinite current is an impossibility because it would require infinite power. Thus, an instantaneous $(\Delta t = 0)$ change of voltage across the capacitor is not possible. In other words, we cannot have a discontinuity in $v(t)$.

Inductors

The current in an **inductance** cannot change instantaneously.



$$v(t) = L\frac{d}{dt}i(t)$$

$L = 0.1\,H$

$$i(t) = \begin{cases} 0, \; t < 0 \\ \frac{1}{t_1}, \; 0 \leq \; t_1 \leq \Delta t \\ 0, \; t > t_1 \end{cases}$$

Clearly, we cannot let $t_1 = 0$ because the voltage required would then become infinite, and we would require infinite power at the terminals of the inductor. Thus, instantaneous changes in the current through an inductor are not possible. In other words, we cannot have a discontinuity in $i(t)$.

Capacitors, Inductors and **initial condition**

Capacitors:

This equation says that the capacitor voltage $v(t)$ can be found by integrating the capacitor current from time $-\infty$ until time t.

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + \frac{1}{C}\int_{-\infty}^{t_0} i(\tau)d\tau$$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$$

The time t_0 is called the **initial time**, and the capacitor voltage $v(t_0)$ is called the **initial condition**.

Inductors:

This equation says that the inductor current $i(t)$ can be found by integrating the inductor voltage from time $-\infty$ until time t.

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

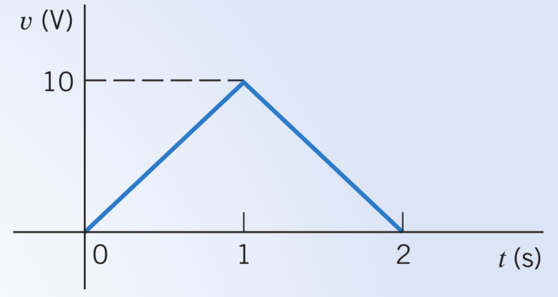
$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + \frac{1}{L}\int_{-\infty}^{t_0} v(\tau)d\tau$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0)$$

The time t_0 is called the **initial time**, and the inductor current $i(t_0)$ is called the **initial condition**.

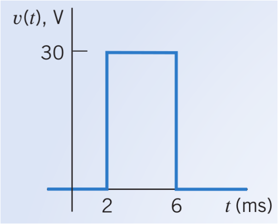
Capacitor Current and Voltage

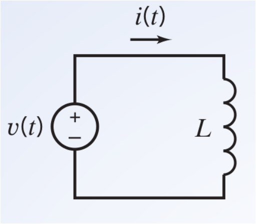
EXAMPLE 7.2-1 - Find the current for a capacitor $C = 1mF$ when the voltage across the capacitor is represented by the signal shown in Figure below.

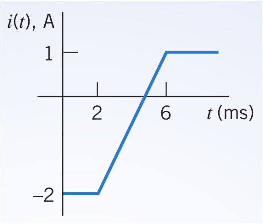


Inductor Current and Voltage

EXAMPLE 7.5-2 - Figure below shows a circuit together with two plots. The plots represent the current and voltage of the inductor in the circuit. Determine the value of the inductance of the inductor.



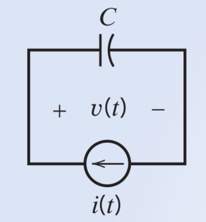




Answer: $L = 40mH$.

Capacitor Current and Voltage

EXAMPLE 7.2-5 - The input to the circuit shown in Figure below is the current



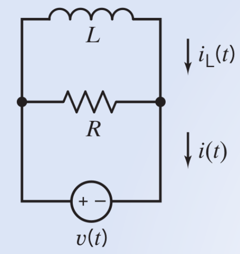
$$i(t) = 3.75e^{-1.2t}\ A\ (t > 0).$$

The output is the capacitor voltage $v(t) = 4 - 1.25e^{-1.2t}\ V\ (t > 0)$. Find the value of the capacitance C.

Answer: $C = 2.5F$.

Inductor Current and Voltage

EXAMPLE 7.5-3 - The input to the circuit shown in Figure below is the voltage



$$v(t) = 4e^{-20t}\ V\ (t > 0).$$

The output is the current $i(t) = -1.2e^{-20t} - 1.5\ A\ (t > 0)$.The initial inductor current is $i_L(0) = -3.5\ A$. Determine the values of the inductance L and resistance R.

Answer: $L = 0.1H\ R = 5\Omega$.

Energy Storage in a Capacitor

The forces acting on the charges stored in a capacitor are said to result from an **electric field**. The energy stored in a capacitor is

$$w_C(t) = \int_{-\infty}^{t} v(\tau) i(\tau) d\tau \ J$$

$$i(\tau) = C\frac{dv(\tau)}{d\tau}$$

$$w_C(t) = \int_{-\infty}^{t} v(\tau) C\frac{dv(\tau)}{d\tau} d\tau$$

$$w_C(t) = C\int_{v(-\infty)}^{v(t)} v(\tau) dv(\tau)$$

$$w_C(t) = C\frac{1}{2}v^2(t)\Big|_{v(-\infty)=0}^{v(t)}$$

$$w_C(t) = \frac{1}{2}Cv^2(t) \ J$$

$$v(t) = \frac{q(t)}{C}$$

$$w_C(t) = \frac{1}{2C}q^2(t) \ J$$

The capacitor is a storage element that stores but does not dissipate energy.

Energy Storage in a Inductor

The energy stored in the inductor is stored in its **magnetic field.** The energy stored in the inductor during the interval $-\infty$ to t is given by

$$w_L(t) = \int_{-\infty}^{t} v(\tau)i(\tau)d\tau \ J$$

$$v(\tau) = L\frac{di(\tau)}{d\tau}$$

$$w_L(t) = \int_{-\infty}^{t} L\frac{di(\tau)}{d\tau}i(\tau)d\tau$$

$$w_L(t) = L\int_{i(-\infty)}^{i(t)} i(\tau)di(\tau)$$

$$w_L(t) = L\frac{1}{2}i^2(t)\Big|_{i(-\infty)=0}^{i(t)}$$

$$w_L(t) = \frac{1}{2}Li^2(t) \ J$$

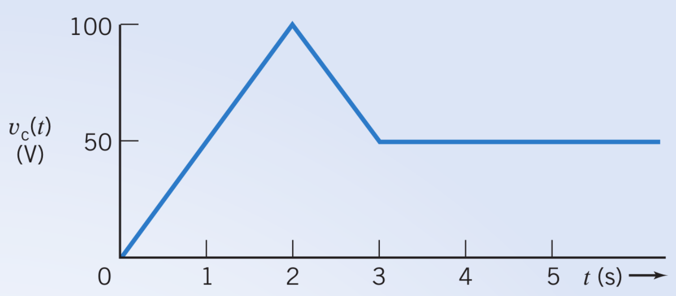
$$i(t) = \frac{\phi(t)}{L}$$

$$w_L(t) = \frac{1}{2L}\phi^2(t) \ J$$

The inductor is a storage element that stores but does not dissipate energy.

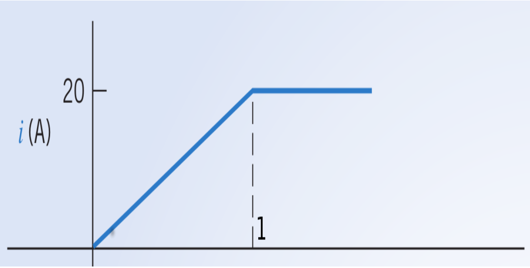
Capacitor Current, Voltage, Power and Energy

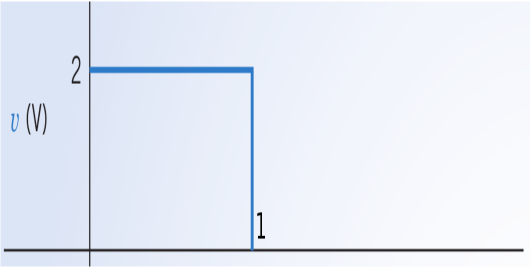
EXAMPLE 7.3-2 - The voltage across a 5mF capacitor varies as shown in Figure below. Determine and plot the capacitor current, power, and energy.



Inductor Current, Voltage, Power and Energy

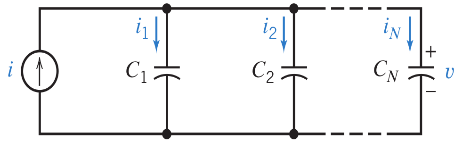
EXAMPLE 7.6-2 - Find the power and energy for an inductor of 0.1H when the current and voltage are as shown in Figures below.



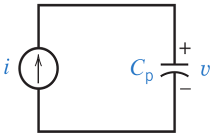


Parallel Capacitor

First, let us consider the parallel connection of N capacitors as shown in Figure below.



Parallel connection of N capacitors



Equivalent circuit for N parallel capacitors.

Using KCL, we have
$i = i_1 + i_2 + i_3 + \ldots + i_N$

Because
$i_n = C_n \frac{dv}{dt}$

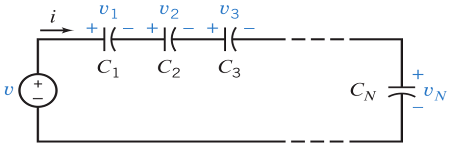
We obtain

$i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \ldots + C_N \frac{dv}{dt}$
$i = (C_1 + C_2 + C_3 + \ldots + C_N) \frac{dv}{dt} = C_p \frac{dv}{dt}$

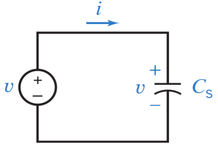
$$C_p = \sum_{n=1}^{N} C_n$$

Series Capacitor

Now let us determine the equivalent capacitance C_s of a set of N series-connected capacitances, as shown in Figure below.



Serie connection of N capacitors



Equivalent circuit for N Serie capacitors.

Using KVL, we have

$v = v_1 + v_2 + v_3 + ... + v_N$

Because

$v_n(t) = \frac{1}{C_n} \int_{t_0}^{t} i d\tau + v_n(t_0)$

We obtain

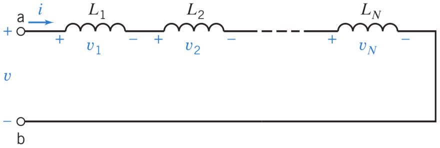
$$v = \frac{1}{C_1} \int_{t_0}^{t} i d\tau + v_1(t_0) + ... + \frac{1}{C_N} \int_{t_0}^{t} i d\tau + v_N(t_0)$$

$$v = \left(\frac{1}{C_1} + \frac{1}{C_2} + ... + \frac{1}{C_N}\right) \int_{t_0}^{t} i d\tau + \sum_{n=1}^{N} v_N(t_0)$$

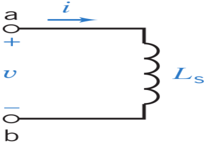
$$\frac{1}{C_s} = \sum_{n=1}^{N} \frac{1}{C_n}$$

Series Inductor

Consider a series connection of N inductors as shown in Figure below.



Serie connection of N inductors



Equivalent circuit for N Serie inductors.

Using KVL, we have
$v = v_1 + v_2 + v_3 + ... + v_N$

Because
$v_n = L_n \frac{di}{dt}$

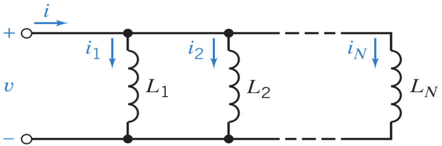
We obtain

$v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} + ... + L_N \frac{di}{dt}$
$v = (L_1 + L_2 + L_3 + ... + L_N) \frac{di}{dt} = L_s \frac{di}{dt}$

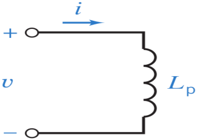
$$L_s = \sum_{n=1}^{N} L_n$$

Parallel Inductor

Now, consider the set of N inductors in parallel, as shown in Figure below.



Parallel connection of N inductors



Equivalent circuit for N parallel inductors.

Using KCL, we have
$i = i_1 + i_2 + i_3 + ... + i_N$

Because
$i_n(t) = \frac{1}{L_n} \int_{t_0}^{t} v d\tau + i_n(t_0)$

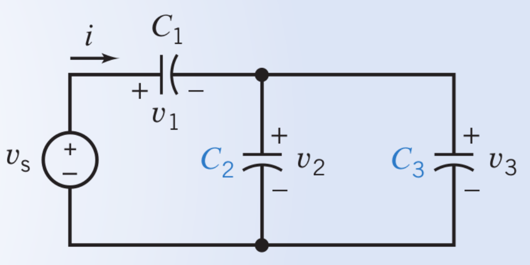
We obtain

$i = \frac{1}{L_1} \int_{t_0}^{t} v d\tau + i_1(t_0) + ... + \frac{1}{L_N} \int_{t_0}^{t} v d\tau + i_N(t_0)$
$i = (\frac{1}{L_1} + \frac{1}{L_2} + ... + \frac{1}{L_N}) \int_{t_0}^{t} v d\tau + \sum_{n=1}^{N} i_n(t_0)$

$$\frac{1}{L_p} = \sum_{n=1}^{N} \frac{1}{L_n}$$

Series and Parallel Capacitor

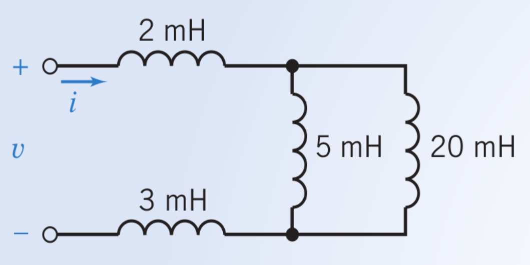
EXAMPLE 7.4-1 - Find the equivalent capacitance for the circuit of Figure below when $C_1 = C_2 = C_3 = 2mF,\ v_1(0) = 10V, and\ v_2(0) = v_3(0) = 20V$.



Answer: $C_{eq} = \frac{8}{6}mF\ and\ v_s(0) = 30V$.

Series and Parallel inductor

EXAMPLE 7.7-1 - Find the equivalent inductance for the circuit of Figure below. All the inductor currents are zero at t_0.



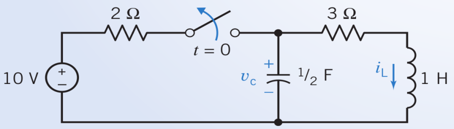
Answer: $L_{eq} = 9mH$.

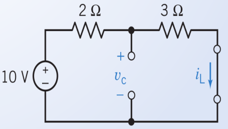
Initial Conditions of Switched Circuits

Our plan to analyze switched circuits has two steps:

- Analyze the dc circuit that exists before time t_0 to determine the capacitor voltages and inductor currents. In doing this analysis, we will take advantage of the fact that capacitors behave as open circuits, $i(t) = C\frac{d}{dt}v(t)$, and inductors behave as short circuits, $v(t) = L\frac{d}{dt}i(t)$, when they are in dc circuits;
- Recognize that capacitor voltages and inductor currents cannot change instantaneously, so the capacitor voltages and inductor currents at time t_0^+ have the same values that they had at time t_0^-.

EXAMPLE 7.8-1 - Consider the circuit Figure below. Prior to $t = 0$, the switch has been closed for a long time. Determine the values of the capacitor voltage and inductor current immediately after the switch opens at time $t = 0$.

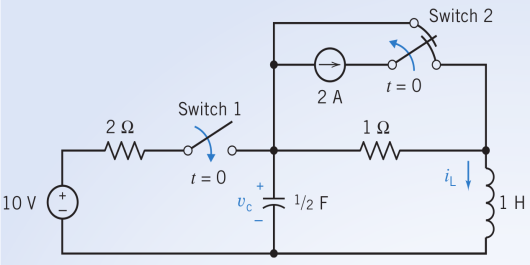




Answer: $v_C(0^-) = v_C(0^+) = 6V \text{ and } i_L(0^-) = i_L(0^+) = 2A$

Initial Conditions of Switched Circuits

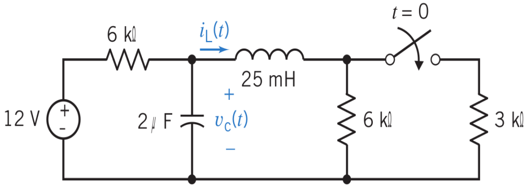
EXAMPLE 7.8-2 - Find $i_L(0^+)$, $v_C(0^+)$, $\frac{dv_C(0^+)}{dt}$, and $\frac{di_L(0^+)}{dt}$ for the circuit of Figure below.



Answer: $i_L(0^+) = 0A$, $v_C(0^+) = -2V$, $\frac{dv_C(0^+)}{dt} = 12V/s$, and $\frac{di_L(0^+)}{dt} = -2A/s$.

Initial Conditions of Switched Circuits

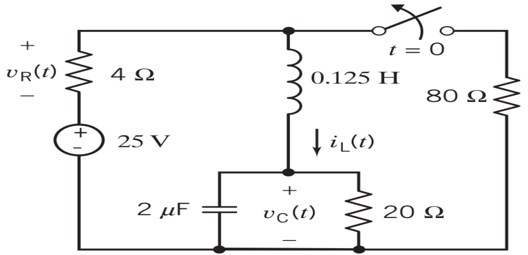
PROBLEM 7.8-2 - The switch in Figure below has been open for a long time before closing at time $t = 0$. Find $v_C(0^+)$ and $i_L(0^+)$, the values of the capacitor voltage and inductor current immediately after the switch closes. Let $v_C(\infty)$ and $i_L(\infty)$ denote the values of the capacitor voltage and inductor current after the switch has been closed for a long time. Find $v_C(\infty)$ and $i_L(\infty)$.



Answer:$v_C(0^+) = 6V$, $i_L(0^+) = 1mA$, $v_C(\infty) = 3V$, and $i_L(\infty) = 1.5mA$.

Initial Conditions of Switched Circuits

PROBLEM 7.8-11 -The circuit shown in Figure below has reached steady state before the switch opens at time $t = 0$. Determine the values of $i_L(t)$, $v_C(t)$, and $v_R(t)$ immediately before the switch opens and the value of $v_R(t)$ immediately after the switch opens.

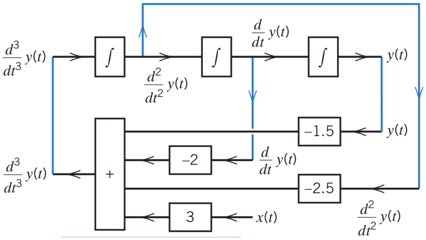


Answer: $i_L(0^-) = 1A$, $v_C(0^-) = 20V$, $v_R(0^-) = -5V$, and $v_R(0^+) = -4V$.

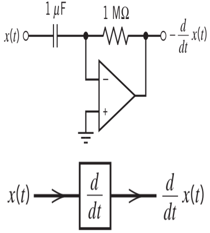
Operational Amplifier Circuits and Linear Differential Equations

This section describes a procedure for designing operational amplifier circuits that implement linear differential equations such as

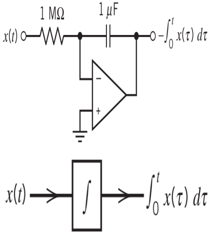
$$2\frac{d^3}{dt^3}y(t) + 5\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = 6x(t)$$



A block diagram that represents



Differentiation

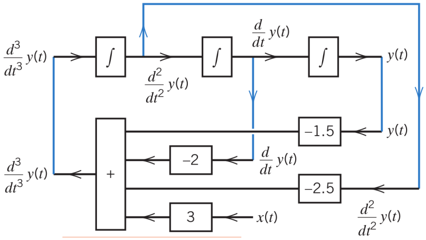


Integration

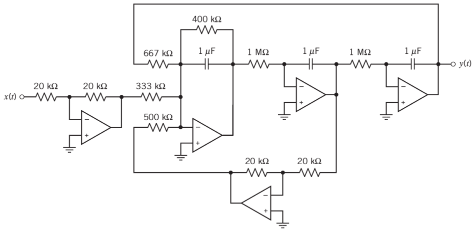
Operational Amplifier Circuits and Linear Differential Equations

This section describes a procedure for designing operational amplifier circuits that implement linear differential equations such as

$$2\frac{d^3}{dt^3}y(t) + 5\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = 6x(t)$$



A block diagram that represents



An operational amplifier circuit that implements

Using MATLAB to Plot Capacitor or Inductor Voltage and Current

The equations below show current and voltage in a 2-F capacitor. The initial capacitor voltage is $v(0) = -5V$;

$$i(t) = \begin{cases} 2, \ when \ t < 2 \\ t + 2, \ when \ 2 \leq t < 6 \\ 20 - 2t, \ when \ 6 \leq t < 14 \\ -8, \ when \ t > 14 \end{cases} \qquad v(t) = \begin{cases} 2t - 5, \ when \ t < 2 \\ \frac{t^2}{4} + t - 4, \ when \ 2 \leq t < 6 \\ -\frac{t^2}{2} + 10t - 31, \ when \ 6 \leq t < 14 \\ 67 - 4t, \ when \ t > 14 \end{cases}$$

where the units of current are A and the units of time are s.

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The Matlab solution:

```
function i = CapCur(t)
  if t<2
    i=4;
  elseif t<6
    i=t+2;
  elseif t<14
    i=20-2*t;
  else
    i=-8;
  end
```

```
function v = CapVol(t)
  if t<2
    v = 2*t - 5;
  elseif t<6
    v = 0.25*t*t + t - 4;
  elseif t<14
    v = -.5*t*t + 10*t - 31;
  else
    v = 67 - 4*t;
  end
```

```
t = 0:1:20;
for k = 1:1:length(t)
  i(k) = CapCur(k-1);
  v(k) = CapVol(k-1);
end
plot(t,i,t,v)
text(12,10,'v(t), V')
text(10,-5,'i(t), A')
title('Capacitor Voltage and Current')
xlabel('time, s')
```