Aula 11 - The Complete Response of Circuits with Two Energy Storage Elements

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Contents

Introduction to Electric Circuits by James A. Svoboda, Richard C. Dorf, 9th Edition

Introduction

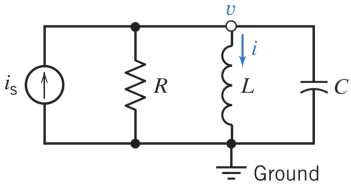
In this chapter, we consider second-order circuits. To find the response of the second-order circuit, we:

- Represent the circuit by a second-order differential equation.
- Find the general solution of the homogeneous differential equation.
- Find a particular solution of the differential equation.
- Represent the response of the second-order circuit as $x(t) = x_n(t) + x_f(t)$.
- Use the initial conditions to evaluate the unknown constants.

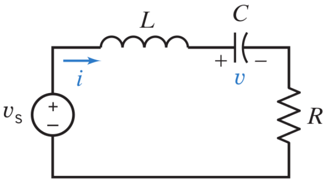
The Direct Method

The Direct Method for Obtaining the Second-Order Differential Equation of a Circuit

Step 1: Identify the first and second variables, x_1 and x_2. These variables are capacitor voltages and/or inductor currents.



A parallel RLC circuit.



A series RLC circuit.

The Direct Method

Step 2: Write one first-order differential equation, obtaining $\frac{dx_1}{dt} = f(x_1, x_2)$.

$$-i_s + \frac{v}{R} + i + C\frac{dv}{dt} = 0 \qquad -v_s + L\frac{di}{dt} + v + Ri = 0$$

Step 3: Obtain an additional first-order differential equation in terms of the second variable so that $\frac{dx_2}{dt} = Kx_1$ or $x_1 = \frac{1}{K}\frac{dx_2}{dt}$.

$$v = L\frac{di}{dt} \qquad i = C\frac{dv}{dt}$$

Step 4: Substitute the equation of step 3 into the equation of step 2, thus obtaining a second-order differential equation in terms of x_2.

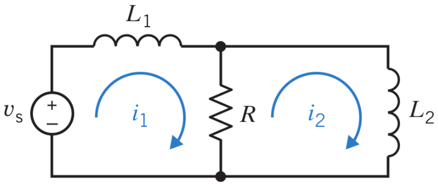
$$\frac{L}{R}\frac{di}{dt} + i + CL\frac{d^2i}{dt^2} = i_s \qquad LC\frac{d^2v}{dt^2} + v + RC\frac{dv}{dt} = v_s$$

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{CL}i = \frac{1}{CL}i_s \qquad \frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{RC}v = \frac{1}{LC}v_s$$

The Operator Method

Operator Method for Obtaining the Second-Order Differential Equation of a Circuit

Step 1: Identify the variable x_1 for which the solution is desired.



i_2?

Step 2: Write one differential equation in terms of the desired variable x_1 and a second variable, x_2.

$$L_1 \frac{di_1}{dt} + Ri_1 - Ri_2 = v_s$$

Step 3: Obtain an additional equation in terms of the second variable and the first variable.

$$L_2 \frac{di_2}{dt} + Ri_2 - Ri_1 = 0$$

.

The Operator Method

Step 4: Use the operator $s = \frac{d}{dt}$ and $\frac{1}{s} = \int dt$ to obtain two algebraic equations in terms of s and the two variables x_1 and x_2.

$$L_1 S i_1 + R i_1 - R i_2 = v_s$$
$$L_2 S i_2 + R i_2 - R i_1 = 0$$

Step 5: Using Cramer's rule, solve for the desired variable so that $x_1 = \frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials in s.

$$i_2 = \frac{R v_s}{L_1 L_2 S^2 + (L_1 R + L_2 R) S}$$

Step 6: Rearrange the equation of step 5 so that $Q(s) x_1 = P(s)$.

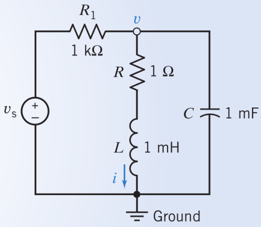
$$[L_1 L_2 S^2 + (L_1 R + L_2 R) S] i_2 = R v_s$$

Step 7: Convert the operators back to derivatives for the equation of step 6 to obtain the second-order differential equation.

$$L_1 L_2 \frac{d^2 i_2}{dt} + (L_1 R + L_2 R) \frac{d i_2}{dt} = R v_s$$

Differential Equation

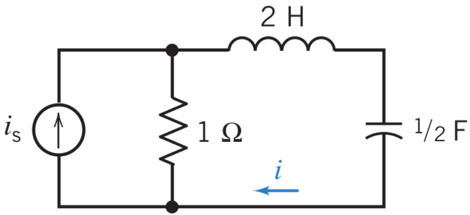
EXAMPLE 9.2-2 - Find the differential equation for the voltage v for the circuit of Figure Below.



Answer: $\frac{d^2v}{dt^2} + 1001\frac{dv}{dt} + 1001x10^3 v = \frac{dv_s}{dt} + 1000v_s$

Differential Equation

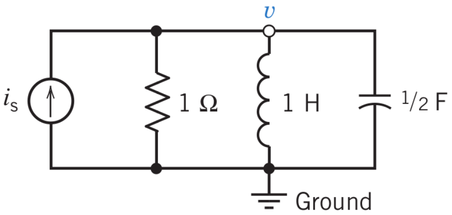
EXERCISE 9.2-1 - Find the second-order differential equation for the circuit shown in Figure Below in terms of i, using the direct method.



Answer: $\frac{d^2i}{dt^2} + \frac{1}{2}\frac{di}{dt} + i = \frac{1}{2}\frac{di_s}{dt}$

Differential Equation

EXERCISE 9.2-2 - Find the second-order differential equation for the circuit shown in Figure Below in terms of v using the operator method.



Answer: $\frac{d^2v}{dt^2} + 2\frac{dv}{dt} + 2v = 2\frac{di_s}{dt}$

The Natural Response

In the preceding section, we found that a circuit with two irreducible energy storage elements can be represented by a second-order differential equation of the form.

$$a_2 \frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t)$$

where the constants a_2 , a_1 , a_0 are known and the forcing function $f(t)$ is specified. The complete response $x(t)$ is given by

$$x(t) = x_n(t) + x_f(t)$$

where $x_n(t)$ is the natural response and $x_f(t)$ is a forced response.

The natural response satisfies the unforced differential equation when $f(t) = 0$. The forced response $x_f(t)$ satisfies the differential equation with the forcing function present.

The Natural Response

The natural response of a circuit x_n will satisfy the equation

$$a_2 \frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = 0$$

Because x_n and its derivatives must satisfy the equation, we postulate the exponential solution

$$x_n(t) = Ae^{st}$$

where A and s are to be determined. Thus, we have

$$\begin{gathered} a_2 A s^2 e^{st} + a_1 A s e^{st} + a_0 A e^{st} = 0 \\ (a_2 s^2 + a_1 s + a_0) A e^{st} = 0 \\ a_2 s^2 + a_1 s + a_0 = 0 \end{gathered}$$

The Natural Response

The **characteristic equation** is derived from the governing differential equation for a circuit by setting all independent sources to zero value and assuming an exponential solution.

$$a_2 s^2 + a_1 s + a_0 = 0$$

The solution of the characteristic equation has two roots, s_1 and s_2, where

$$s_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \; and \; s_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

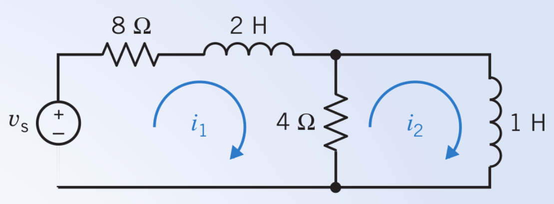
When there are two distinct roots, the natural response is of the form

$$x_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Where A_1 and A_2 are unknown constants that will be evaluated later. The **roots** of the characteristic equation contain all the information necessary for determining the character of the natural response.

The Natural Response

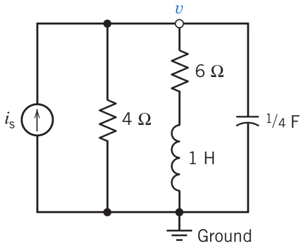
EXAMPLE 9.3-1 - Find the natural response of the circuit current i_2 shown in Figure Below. Use operators to formulate the differential equation and obtain the response in terms of two arbitrary constants.



Answer: $x_n(t) = A_1e^{-2t} + A_2e^{-8t}$

The Natural Response

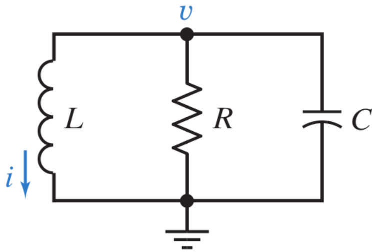
EXERCISE 9.3-1 - Find the characteristic equation and the natural frequencies for the circuit shown in Figure Below.



Answer: $s^2 + 7s + 10 = 0$

Parallel RLC Circuit

In this section, we consider the (unforced) natural response of the parallel RLC circuit shown in Figure Below



The circuit shown does not contain any independent sources, so the input $f(t)$ is zero.

The differential equation is called a homogeneous.

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

The coefficients of this differential equation have names: α is called the damping coefficient, and ω_0 is called the resonant frequency.

Parallel RLC Circuit

The differential equation:

$$\frac{v}{R} + \frac{1}{L}\int_0^t v d\tau + i(0) + C\frac{dv}{dt} = 0$$

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$s^2v + \frac{1}{RC}sv + \frac{1}{LC}v = 0$$

Characteristic Equation: $s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$

We can see

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

$$\frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$$

Thus

$\alpha = \frac{1}{2RC}$ and $\omega_o^2 = \frac{1}{LC}$

Parallel RLC Circuit

The Characteristic Equation:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

The two roots of the characteristic equation are

$$s_1 = -\frac{1}{2RC} + \sqrt{(\frac{1}{2RC})^2 - \frac{1}{LC}}$$
$$s_2 = -\frac{1}{2RC} - \sqrt{(\frac{1}{2RC})^2 - \frac{1}{LC}}$$

The solution to the second-order differential for t > 0 is

$$v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The roots of the characteristic equation may be rewritten as

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \text{ and } s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

The damped resonant frequency, ω_d, is defined to be

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

When $\omega_0 > \alpha$, the roots of the characteristic equation are complex and can be expressed as

$$s_1 = -\alpha + j\omega_d \text{ and } s_1 = -\alpha - j\omega_d$$

Parallel RLC Circuit

The roots of the characteristic equation assume three possible conditions:

1. Two real and distinct roots when $\alpha^2 > \omega_0^2$.
   - When the two roots are real and distinct, the circuit is said to be ***overdamped***.

     .
2. Two real equal roots when $\alpha^2 = \omega_0^2$.
   - When the roots are both real and equal, the circuit is ***critically damped***.

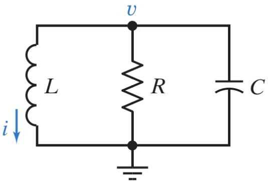
     .
3. Two complex roots when $\alpha^2 < \omega_0^2$.
   - When the two roots are complex conjugates, the circuit is said to be ***underdamped***.

     .

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2},\ \omega_d = \sqrt{\omega_0^2 - \alpha^2} \text{ and } s_{1,2} = -\alpha \pm j\omega_d$$

Natural Response of an Overdamped Second-Order Circuit

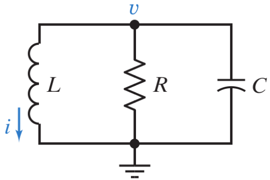
EXAMPLE 9.4-1 - Find the natural response of $v(t)$ for $t > 0$ for the parallel RLC circuit shown in Figure Below when $R = \frac{2}{3}\Omega$,$L = 1H$, $C = \frac{1}{2}F$, $v(0) = 10V$, and $i(0) = 2A$.



Answer: $v_n = -14e^{-t} + 24e^{-2t}V$

Natural Response of an Overdamped Second-Order Circuit

EXERCISE 9.4-2 - Find the natural response of RLC circuit Below when $R = 6\Omega$, $L = 7H$, and $C = \frac{1}{42}F$. The initial conditions are $v(0) = 0$ and $i(0) = 10A$.



Answer: $v_n = -84(e^{-t} - e^{-6t})V$

Natural Response of the Critically Damped Unforced

Again we consider the parallel RLC circuit, and here we will determine the special case when the characteristic equation has two equal real roots. Two real, equal roots occur when $\alpha^2 = \omega^2$.

Let us assume that $s_1 = s_2$ and proceed to find $v_n(t)$. We write the natural response as the sum of two exponentials as

$$v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_3 e^{s_1 t} \tag{1}$$

Because the two roots are equal, we have only one undetermined constant, but we still have two initial conditions to satisfy.

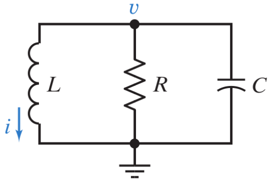
Clearly, Eq. (1) is not the total solution for the natural response of a critically damped circuit.

Thus, we need the solution that will contain two arbitrary constants, so with some foreknowledge, we try the solution

$$v_n = (A_1 t + A_2) e^{s_1 t}$$

Natural Response of the Critically Damped Unforced

EXERCISE 9.5-1 - A parallel RLC circuit has $R = 10\Omega$, $C = 1mF$, $L = 0.4H$, $v(0) = 8V$, and $i(0) = 0$. Find the natural response $v_n(t)$ for $t < 0$.



Answer: $v_n(t) = (8 - 400t)e^{-50t}V$

Natural Response of an Underdamped Unforced

The characteristic equation of the parallel RLC circuit will have two complex conjugate roots when $\alpha^2 < \omega_0^2$.

This condition is met when

$$LC < (2RC)^2$$
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$
$$s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

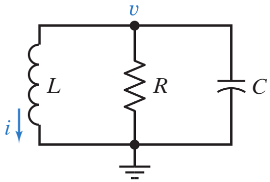
The complex roots lead to an oscillatory-type response.

We define the square root $\sqrt{\omega_0^2 - \alpha^2}$ as ω_d, which we will call the damped resonant frequency. The factor α, called the damping coefficient, determines how quickly the oscillations subside. Then the roots are

$$s_{1,2} = -\alpha \pm j\omega_d$$
$$v_n = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$
$$v_n = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$$
$$v_n = e^{-\alpha t}(B_1 \cos \omega_d + B_2 \sin \omega_d t)$$

Natural Response of an Underdamped Unforced

EXERCISE 9.6-1 - Consider the parallel RLC circuit when $R = 25/3\Omega$, $L = 0.1H$, $C = 1mF$, $v(0) = 10V$, and $i(0) = 0.6A$. Find the natural response $v_n(t)$ for $t > 0$.



Answer: $v_n(t) = 10e^{-60t} \cos(80t)\ V$

Forced Response

The response to a forcing function will often be of the same form as the forcing function.

Forced Responses	
Forcing Function	Assumed Response
K	A
Kt	$At+B$
Kt^2	At^2+Bt+C
$K\sin\omega t$	$A\sin\omega t + B\cos\omega t$
Ke^{at}	Ae^{-at} or Ate^{-at}

Again, we consider the differential equation for the second-order circuit as

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = f(t)$$

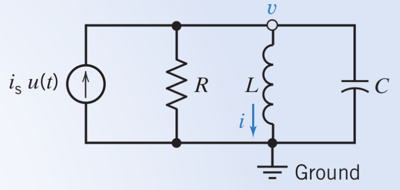
Therefore, substituting x_f, we have

$$\frac{d^2x_f}{dt^2} + a_1\frac{dx_f}{dt} + a_0x_f = f(t) \qquad (2)$$

We need to determine x_f so that x_f and its first and second derivatives all satisfy Eq. 2.

Forced Response to an Exponential Input

EXERCISE 9.7-1 - Find the forced response for the inductor current i_f for the parallel RLC circuit shown in Figure Below when $i_s = 8e^{-2t}A$. Let $R = 6\Omega$, $L = 7H$, and $C = 1/42F$.



Answer: $i_f(t) = -12e^{-2t}\ A$

Complete Response of a Second-Order Circuit

The **complete response** is the sum of the natural response and the forced response; thus,

$$x = x_n + x_f$$

EXERCISE 9.8-1 - Find the complete response $v(t)$ for $t > 0$ for the circuit of Figure. Assume the circuit is at steady state at $t = 0$.