Aula 12 - The Laplace Transform

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Contents

Introduction to Electric Circuits by James A. Svoboda, Richard C. Dorf, 9th Edition

Introduction

The Laplace transform is a very powerful tool for the analysis of circuits and enables the circuit analyst to transform the set of differential equations describing a circuit to the complex frequency domain, where they become a set of linear algebraic equations. In this chapter,

- We find the complete response;
- We analyzed transient part plus steady-state part;
- We analyzed first-order and second-order circuits;
- we analyzed stability of the circuits.

Direct and inverse Laplace transform

The $\mathscr{L}\{f(t)\}$ indicates taking the Laplace transform of $f(t)$. The result, $F(s)$, is called the Laplace transform of $f(t)$.

The (one-sided or unilateral) **Laplace transform** is defined as

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\mathrm{d}t$$

The **inverse Laplace transform** is defined by the complex inversion integral

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\alpha-j\infty}^{\alpha+\infty} F(s)e^{st}\mathrm{d}s$$

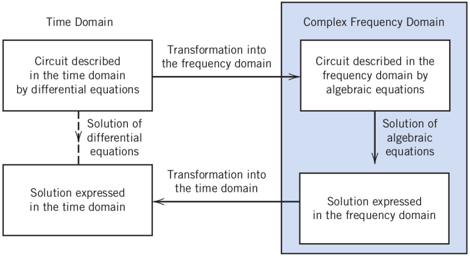
Where s is a complex variable given $s = \sigma + j\omega$. The function $f(t)$ is said to exist in the time domain, where as the function $F(s)$ is said to exist in the s-domain.

Direct and inverse Laplace transform

We say that $f(t)$ and $F(s)$ comprise a Laplace transform pair and denote this fact as

$$f(t) \Leftrightarrow F(s)$$

The transform method is summarized in Figure Below.



EXERCISE 14.2-1 - Laplace Transform Pairs.

(a) Find the Laplace transform of $f(t) = e^{-at}$, where $a > 0$.
(b) Find the Laplace transform of $f(t) = e^{-at}u(t)$, where $a > 0$ and $u(t)$ is the unit step function.

Linearity

Linearity is an important property of the Laplace transform.

$$a_1 f_1(t) + a_2 f_2(t) \Leftrightarrow a_1 F_1(s) + a_2 F_2(s)$$

We have

$$\begin{aligned}
F(s) &= \mathscr{L}\{f(t)\} \\
F(s) &= \mathscr{L}\{a_1 f_1(t) + a_2 f_2(t)\} \\
F(s) &= \mathscr{L}\{a_1 f_1(t)\} + \mathscr{L}\{a_2 f_2(t)\} \\
F(s) &= a_1 \mathscr{L}\{f_1(t)\} + a_2 \mathscr{L}\{f_2(t)\} \\
F(s) &= a_1 F_1(s) + a_2 F_2(s)
\end{aligned}$$

Where $F_1(s)$ and $F_2(s)$ are the Laplace transforms of the time functions $f_1(t)$ and $f_2(t)$, respectively.

EXERCISE 14.2-2 - Linearity.

Find the Laplace transform of $\sin(\omega t)$.

Differentiation in the Time Domain

We can summarize differentiation in the time domain as

$$\frac{df}{dt} \Leftrightarrow sF(s) - f(0^{-1})$$

We have

$$\mathscr{L}\{\frac{\mathrm{d}f}{\mathrm{d}t}\} = \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}t} e^{-st} \mathrm{d}t$$

Integrating by parts

$$\mathscr{L}\{\frac{\mathrm{d}f}{\mathrm{d}t}\} = s\int_0^\infty fe^{-st}\mathrm{d}t + fe^{-st}\Big|_0^\infty$$

$$\mathscr{L}\{\frac{\mathrm{d}f}{\mathrm{d}t}\} = sF(s) - f(0^{-1})$$

Thus, the Laplace transform of the derivative of a function is s times the Laplace transform of the function minus the initial condition.

EXERCISE 14.2-3 - Differentiation in the Time Domain.

Find the Laplace transform of $\cos(\omega t)$.

Direct and inverse Laplace transform

Thus, we use the definition of the Laplace transform to obtain Laplace transform pairs.

Table provides a collection of important Laplace transform pairs.

$f(t)$ for $t > 0$	$F(s) = \mathscr{L}[f(t)u(t)]$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-at}t^n$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Direct and inverse Laplace transform

Thus, we use the definition of the Laplace transform to obtain properties of the Laplace transform.

Table lists important properties of the Laplace transform.

PROPERTY	$f(t),\ t>0$	$F(s)=\mathscr{L}[f(t)u(t)]$
Linearity	$a_1f_1(t)+a_2f_2(t)$	$a_1F_1(s)+a_2F_2(s)$
Time scaling	$f(at)$, where $a>0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Time integration	$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
Time differentiation	$\frac{df(t)}{dt}$	$sF(s)-f(0^-)$
	$\frac{d^2f(t)}{dt^2}$	$s^2F(s)-\left(sf(0^-)+\frac{df(0^-)}{dt}\right)$
	$\frac{d^nf(t)}{dt^n}$	$s^nF(s)-\sum_{k=1}^{n}s^{n-k}\frac{d^{k-1}f(0^-)}{dt^{k-1}}$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{-at}f(t)$	$F(s+a)$
Time convolution	$f_1(t)^*f_2(t)=\int_0^t f_1(\tau)f_2(t-\tau)d\tau$	$F_1(s)F_2(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(\lambda)d\lambda$
Frequency differentiation	$tf(t)$	$-\frac{dF(s)}{ds}$
Initial value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$

Laplace Transform

EXERCISE 14.2-4 - Find the Laplace transform of $5 - 5e^{-2t}(1 + 2t)$.

Answer: $\dfrac{20}{s(s^2 + 4s + 4)}$

EXERCISE 14.2-5 - Find the Laplace transform of $10e^{-4t}cos(20t + 36.9^o)$.

Answer: $\dfrac{8s - 88}{s^2 + 8s + 416}$

EXERCISE 14.2-6 - Find the Laplace transform of $2\delta(t) + 3 + 4u(t)$.

Answer: $2 + \dfrac{7}{s}$

Pulse Inputs

The step function is represented as

$$u(t) = \begin{cases} 0, & when\ t < 0 \\ 1, & when\ t > 1 \end{cases}$$

makes an abrupt transition from 0 to 1 at time $t = 0$. Define the impulse function $\delta(t)$ to be

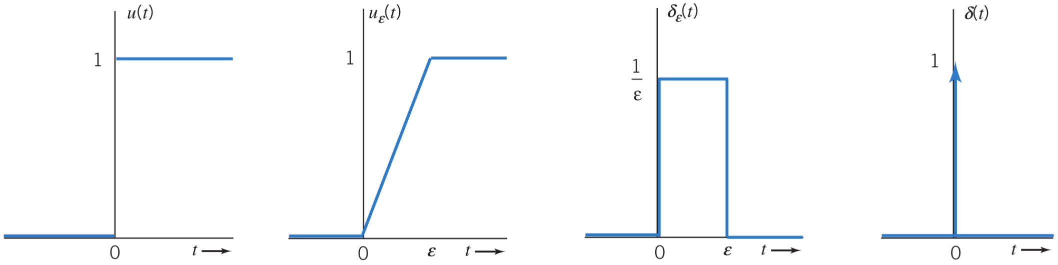
$$\delta(t) = \frac{d}{dt}u(t) \begin{cases} 0, & when\ t < 0 \\ undefined, & when\ t = 0 \\ 0, & when\ t > 0 \end{cases}$$

Because $\delta(t)$ is undefined at time 0, we consider the function $u_\varepsilon(t)$. This function makes the transition from 0 to 1 over the time interval from 0 to ε. Notice that

$$\lim_{\varepsilon \to 0} u_\varepsilon(t) = u(t)$$

$$\delta_\varepsilon(t) = \frac{d}{dt}u_\varepsilon(t) \begin{cases} 0, & when\ t < 0 \\ \frac{1}{\varepsilon}, & when\ 0 < t < \varepsilon \\ 0, & when\ t > \varepsilon \end{cases}$$

Pulse Inputs



$$u(t) = \lim_{\varepsilon \to 0} u_\varepsilon(t)$$

$$u_\varepsilon(t) = \begin{cases} 0, & \text{when } t < 0 \\ \frac{1}{\varepsilon} t, & \text{when } 0 < t < \varepsilon \\ 1, & \text{when } t > \varepsilon \end{cases}$$

$$\delta_\varepsilon(t) = \begin{cases} 0, & \text{when } t < 0 \\ \frac{1}{\varepsilon}, & \text{when } 0 < t < \varepsilon \\ 0, & \text{when } t > \varepsilon \end{cases}$$

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$

Pulse Inputs

Notice that for any value of ε, the area under the pulse is given by

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(t)\ \mathrm{d}t = \int_0^\varepsilon \frac{1}{\varepsilon} t\ \mathrm{d}t = 1$$

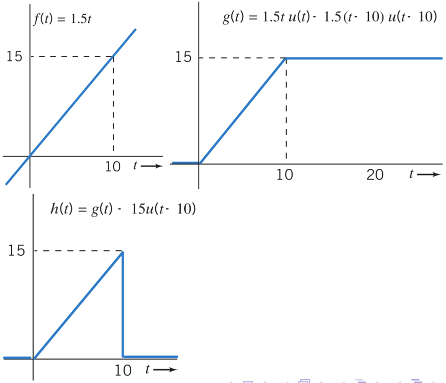
An important property of the impulse function is

$$\int_{-\infty}^{+\infty} f(t)\delta(t)\mathrm{d}t = f(0)$$

Letting $f(t) = 1$ gives

$$\int_{-\infty}^{+\infty} \delta(t)\mathrm{d}t = 1$$

EXERCISE 14.3-1 - Find the Laplace transform of $f(t)$, $g(t)$ and, $h(t)$ shown in Figure below.



Inverse Laplace

We will frequently want to find the inverse Laplace transform of a function represented as a ratio of polynomials in s. Consider:

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0}$$

The roots of the denominator polynomial $D(s)$ are the roots of the equation $D(s) = 0$ and are called the poles of $F(s)$. Factoring $D(s)$, we obtain

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + ... + b_1 s + b_0}{(s - p_1)(s - p_2)...(s - p_n)}$$

The roots of the numerator polynomial $N(s)$ are called the zeros of $F(s)$.

Inverse Laplace

We will find the inverse Laplace transform of a proper rational function $F(s)$ in three steps.

1. First, we perform a partial fraction expansion to express $F(s)$ as a sum of simpler functions, $F_i(s)$;
$$F(s) = F_1(s) + F_2(s) + \ldots + F_i(s) + \ldots + F_n(s)$$
2. Next, we use the transform pairs and properties to find the inverse Laplace transform of each $F_i(s)$;
3. Finally, using linearity, we sum the inverse transforms of the $F_i(s)$ to obtain the inverse Laplace transform of $F(s)$;

Simple Poles

When all of the poles of a proper rational function, $F(s)$, are simple poles, the partial fraction expansion of $F(s)$ is

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

$$F(s) = \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + \ldots + \frac{R_i}{s-p_i} + \ldots + \frac{R_n}{s-p_n}$$

The coefficients R_i are called residues.The values of the residues of simple poles are calulated as

$$R_i = (s-p_i)F(s)\big|_{s=p_i}$$

Complex Conjugate Poles

Suppose $F(s)$ has a pair of simple complex conjugate poles $p_1 = -a + jb$ and $p_2 = -a - jb$. The partial fraction expansion of $F(s)$ is

$$F(s) = \frac{R_1}{s - p_1} + \frac{R_2}{s - p_2} = \frac{c + jd}{s + a - jb} + \frac{c - jb}{s + a + jb}$$

$$F(s) = 2c\frac{s + a}{(s + a)^2 + b^2} - 2d\frac{b}{(s + a)^2 + b^2}$$

Taking the inverse Laplace transform

$$f(t) = 2\ c\ e^{-at}\cos(bt)\ \ - 2\ d\ e^{-at}\sin(bt)$$

Repeated Poles

Next, suppose $F(s)$ has repeated poles, that is,

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{(s - p_1)^q}$$

$$F(s) = \frac{R_1}{s - p_1} + \frac{R_2}{(s - p_1)^2} + \ldots + \frac{R_i}{(s - p_1)^i} + \ldots + \frac{R_n}{(s - p_1)^q}$$

The coefficients R_i are called residues.The values of the residues of repeated poles are calulated as

$$R_{q-k} = \frac{1}{k!} [\frac{d^k}{ds^k} (s - p_i)^q F(s)] \big|_{s=p_1}$$

Inverse Laplace

EXERCISE 14.4-1 - Simple, Real Poles - Find the inverse Laplace transform of $F(s) = \frac{s+3}{s^2+7s+10}$. Answer: $\frac{1}{3}e^{-2t} + \frac{2}{3}e^{-5t}$ *for* $t > 0$

EXERCISE 14.4-2 - Simple Complex Poles - Find the inverse Laplace transform of $F(s) = \frac{10}{(s^2+6s+10)(s+2)}$. Answer: $-5e^{-3t}\cos(t) - 5e^{-3t}\sin(t) + 5e^{-2t}$ *for* $t > 0$

EXERCISE 14.4-3 - Repeated Poles - Find the inverse Laplace transform of $F(s) = \frac{4}{(s+1)^2(s+2)}$. Answer: $-4e^{-t} + 4te^{-t} + 4e^{-2t}$ *for* $t > 0$

EXERCISE 14.4-4 - Improper Rational Function - Find the inverse Laplace transform of $F(s) = \frac{4s^3+15s^2+s+30}{s^2+5s+6}$. Answer: $4\frac{d}{dt}\delta(t) - 5\delta(t) + 6e^{-3t} - 4e^{-2t}$ *for* $t > 0$

Initial Value Theorem

$$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$

To prove the initial value theorem,

$$sF(s) - f(0-) = \mathscr{L}\{\frac{df}{dt}\} = \int_{0-}^{\infty} \frac{df}{dt}\, e^{-st}dt = \int_{0-}^{0+} \frac{df}{dt}\, e^{-st}dt + \int_{0+}^{\infty} \frac{df}{dt}\, e^{-st}dt$$

$$\lim_{s\to\infty} [sF(s) - f(0-)] = \lim_{s\to\infty} \int_{0-}^{0+} \frac{df}{dt}\, e^{-st}dt + \lim_{s\to\infty} \int_{0+}^{\infty} \frac{df}{dt}\, e^{-st}dt$$

$$\lim_{s\to\infty} [sF(s)] - f(0-) = f(0+) - f(0-)$$

$$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$

Final Value Theorem

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

To prove the final value theorem,

$$sF(s) - f(0-) = \mathscr{L}\{\frac{df}{dt}\} = \int_{0-}^{\infty} \frac{df}{dt}\, e^{-st} dt$$

$$\lim_{s\to 0}[sF(s) - f(0-)] = \lim_{s\to 0} \int_{0-}^{\infty} \frac{df}{dt}\, e^{-st} dt$$

$$\lim_{s\to 0}[sF(s)] - f(0-) = f(\infty) - f(0-)$$

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

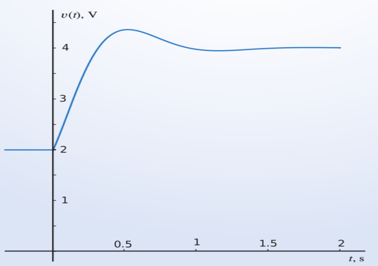
Initial and Final Value Theorems

EXAMPLE 14.5-1 -Consider the situation in which we build a circuit in the laboratory and analyze the same circuit, using Laplace transforms. Figure below shows a plot of the circuit output $v(t)$ obtained by laboratory measurement.

Suppose our circuit analysis gives.

$$V(s) = \mathscr{L}\{v(t)\} = \frac{2s^2 + 30s + 136}{s(s^2 + 9s + 34)}$$

Does the circuit analysis agree with the laboratory measurement?



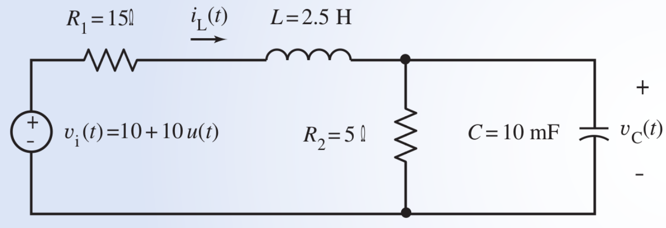
Solution of Differential Equations Describing a Circuit

We can solve a set of differential equations describing an electric circuit, using the Laplace transform of a variable and its derivatives. Here's the procedure:

1. Use Kirchhoff's laws and the element equations to represent the circuit by a differential equation or set of differential equations.
2. Transform each differential equation into an algebraic equation by taking the Laplace transform of both sides of the equation.
3. Solve the algebraic equations to obtain the Laplace transform of the output of the circuit.
4. Take the inverse Laplace transform to obtain the circuit output itself.

Natural Response of an Overdamped Second-Order Circuit

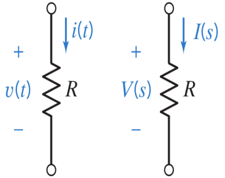
EXERCISE 9.4-2 - Find $v_C(t)$ for the circuit shown in Figure below when $i_L(0-) = 0.5\ A$ and $v_C(0-) = 2.5\ V$.

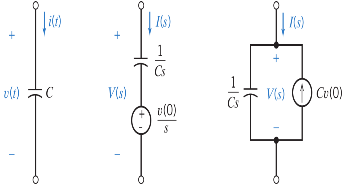


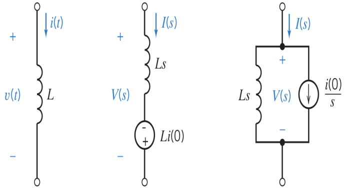
Answer: $v_C(t) = 5 + 4.17e^{-16t} - 6.67e^{-10t}\ V$ for t>0

Circuit Analysis Using Impedance and Initial Conditions

This method will eliminate the need to write differential equations to represent the circuit.







$$v(t) = i(t)R$$
$$V(s) = I(s)R$$
$$Z_R(s) = \frac{V(s)}{I(s)} = R$$

$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v(0)$$
$$V(s) = \frac{1}{sC}I(s) + \frac{v(0)}{s}$$
$$Z_C(s) = \frac{V(s)}{I(s)} = \frac{1}{sC}$$

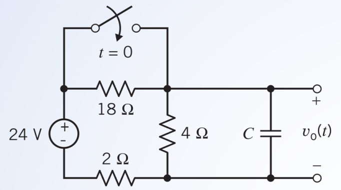
$$v(t) = L\frac{d}{dt}i(t)$$
$$V(s) = LsI(s) - Li(0)$$
$$Z_L(s) = \frac{V(s)}{I(s)} = sL$$

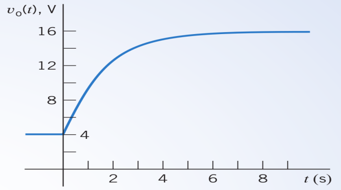
Circuit Analysis Using the Laplace Transform

EXAMPLE 14.7-1 - Consider the circuit shown in Figure below. The input to the circuit is the voltage of the voltage source 24 V. The output of this circuit, the voltage across the capacitor, is given by.

$$v_o(t) = 16 - 12e^{-0.6t}\ V\ when t > 0$$

Determine the value of the capacitance C.





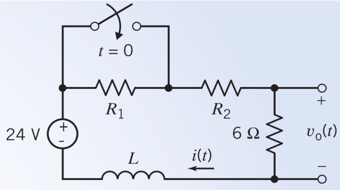
Answer: $C = 1.25\ F$

Circuit Analysis Using the Laplace Transform

EXAMPLE 14.7-2 - Consider the circuit shown in Figure below. The input to the circuit is the voltage of the voltage source, 24 V. The output of this circuit, the voltage across the 6Ω resistor, is given by.

$$v_o(t) = 12 - 6e^{-3.5t}\ V\ when t > 0$$

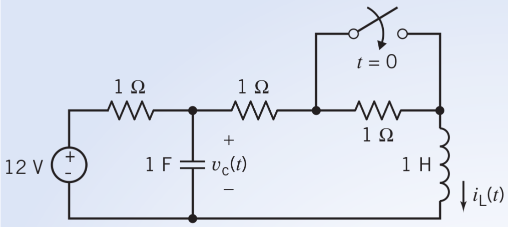
Determine the value of the inductance L and of the resistances R_1 and R_2 .



Answer: $L = 34.29\ H,\ R_1 = 12\ \Omega\, and\ R_2 = 6\ \Omega.$

Circuit Analysis Using the Laplace Transform

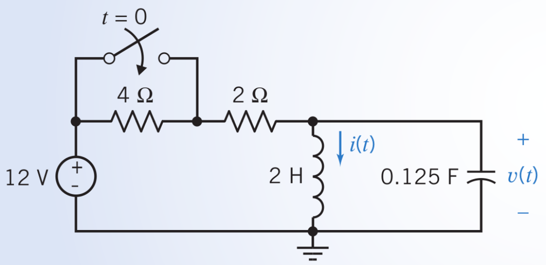
EXAMPLE 14.7-3 - Consider the circuit shown in Figure below. The input to the circuit is the voltage of the voltage source 12 V. The output of this circuit is the current in the inductor $i_L(t)$. Determine the current in the inductor $i_L(t)$, for t > 0.



Answer: $i_L(t) = 6 + 2\sqrt{2}e^{-t}\sin(t - 45^o)\ A\ for\ t > 0$

Circuit Analysis Using the Laplace Transform

EXAMPLE 14.7-4 - The switch in the circuit shown in Figure below closes at time $t = 0$. Determine the voltage $v(t)$ after the switch closes.



Answer: $v(t) = \mathscr{L}^{-1}\{\frac{32}{(s+2)^2}\} = 32te^{-2t}u(t)\ V\ for\ t > 0$

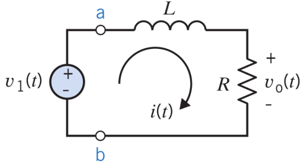
Transfer Function and Impedance

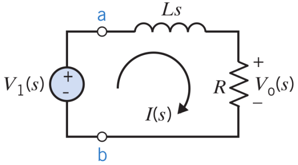
The **transfer function** of a circuit is defined as the ratio of the Laplace transform of the response of the circuit to the Laplace transform of the input to the circuit when the initial conditions are zero.

For the circuit in Figure, the input is the voltage source voltage $v_i(t)$, and the response is the resistor voltage $v_o(t)$. The transfer function of this circuit, denoted by $H(s)$, is then expressed as

$$H(s) = \frac{V_o(s)}{V_i(s)} \text{ or } V_o(s) = H(s)V_i(s)$$

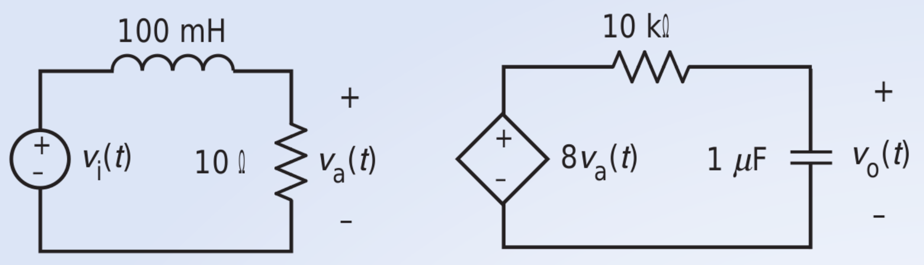
provided all initial conditions are equal to zero. In this case, the only initial condition is the inductor current, so we require $i(0) = 0$.





Circuit Analysis Using the Laplace Transform

EXAMPLE 14.8-1 - The input to the circuit shown in Figure below is the voltage $v_i(t)$, and the output is the voltage $v_o(t)$. Determine the step response of the circuit shown in Figure below.

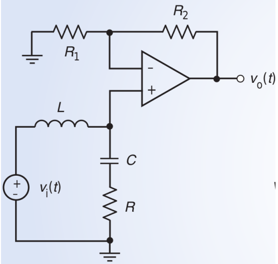


Answer: $v(t) = [8 - 8(1 + 100t)e^{-100t}]u(t)$

Circuit Analysis Using the Laplace Transform

EXAMPLE 14.8-2 - The input to the circuit shown in Figure below is the voltage $v_i(t)$, and the output is the voltage $v_o(t)$. Design the circuit shown to have the step response

$$v_o(t) = [4 - e^{-2t}(4\cos(4t) - 2\sin(4t))]u(t)$$

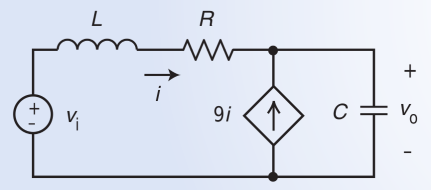


Answer: $L = 0.5\ H, C = 0.1\ F, R = 2\ \Omega, R_1 = 10k\ \Omega,$ $and\ R_2 = 30k\ \Omega.$

Circuit Analysis Using the Laplace Transform

EXAMPLE 14.8-3 - The input to the circuit shown in Figure below is the voltage $v_i(t)$, and the output is the voltage $v_o(t)$. Design the circuit shown to have the step response

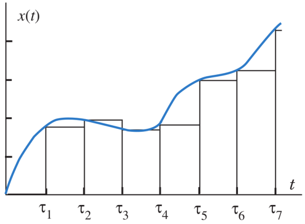
$$v_o(t) = [1 - (10^4 t + 1)e^{-10000t}]u(t) \ V$$



Answer: $R = 200 \ \Omega, \ L = 10 \ mH, and \ C = 10 \ \mu F$

Convolution

In this section, we consider the problem of determining the response of a linear, time-invariant circuit to an arbitrary input, $x(t)$.



Consider:

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & when\ 0 \le t < \Delta \\ 0, & otherwise \end{cases}$$

Thus,

$x(\Delta) = x(\tau_1) = x(\Delta)\delta_\Delta(t-\Delta)\Delta$
$x(k\Delta) = x(\tau_k) = x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$
$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$
$x(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

How

$x(t) \to y(t),\ kx(t) \to ky(t)\ or\ k\delta(t-t_0) \to kh(t-t_0)$

We have

$x(k\Delta) = x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$
$y(k\Delta) = x(k\Delta)h_\Delta(t-k\Delta)\Delta$
$y(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Stability

A circuit is said to be **stable** when the response to a bounded input signal is a bounded output signal. A circuit that is not stable is said to be **unstable**.

Consider a circuit represented by the transfer function H(s). Factoring the denominator of the transfer function gives

$$H(s) = \frac{N(s)}{(s-p_1)(s-p_2)....(s-p_n)}$$

The p_i are the poles of the transfer function, also called the poles of the circuit.

A circuit is stable if, and only if, all of its poles have negative real parts.

We can also use the impulse response $h(t)$ to determine whether a circuit is stable.

$$|y(t)| = |\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau|$$

$$|y(t)| = |\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau|$$

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau$$

If $|x(t-\tau)| \leq B$

$$|y(t)| \leq B\int_{-\infty}^{\infty} |h(\tau)|d\tau$$

Thus, a circuit is said to be **stable** when

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau \leq \infty$$