The Minimum Snap Algorithm and its Properties

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I. The Minimum Snap Algorithm

This section briefly reviews the solution of [1] for generating minimum snap trajectories. This algorithm solves the problem of starting at an initial point $\boldsymbol{P}_0$ at time $t_0 = 0$ and passing through m *waypoints* $\boldsymbol{P}_k, k \in \{1, 2, \cdots, m\}$, reaching the final state $\boldsymbol{P}_m$ at time $t_m = T$.

The minimum snap algorithm solves separately for each cartesian degree of freedom x, y and z. The trajectory for each of these states, denoted as $\sigma_i(t), i \in \{x, y, z\}$, is written as piecewise polynomials of order n over m time intervals:

$$\sigma_i(t) = \begin{cases} \sum_{j=0}^{n} a_{j,1} t^j, & t_0 \le t < t_1 \\ \sum_{j=0}^{n} a_{j,2} t^j, & t_1 \le t < t_2 \\ \vdots \\ \sum_{j=0}^{n} a_{j,m} t^j, & t_{m-1} \le t \le t_m \end{cases}, \tag{1}$$

This problem can be written in the following quadratic programming form [1]:

$$\begin{cases} \min\limits_{\boldsymbol{a}, \boldsymbol{\delta t}} & \mathbb{J}(\boldsymbol{\delta t}) = \boldsymbol{a}^T \boldsymbol{Q} \boldsymbol{a} \\ s.t. & \boldsymbol{A}_{eq} \boldsymbol{a} = \boldsymbol{b}_{eq} \\ & \boldsymbol{A}_{ineq} \boldsymbol{a} \le \boldsymbol{b}_{ineq} \end{cases}, \tag{2}$$

where $\boldsymbol{a} \triangleq \begin{bmatrix} \boldsymbol{a}_1^T & \cdots & \boldsymbol{a}_m^T \end{bmatrix}^T$ is the vector of solution variables, and $\boldsymbol{a}_i \triangleq \begin{bmatrix} a_{0,k} & \cdots & a_{n,k} \end{bmatrix}^T$ is the vector of polynomial coefficients for the interval $t_{k-1} \le t < t_k$. The matrices $\boldsymbol{Q}$, $\boldsymbol{A}_{eq}$ and $\boldsymbol{A}_{ineq}$ are function of the time allocation $\boldsymbol{\delta t} \triangleq \begin{bmatrix} \delta t_1 & \cdots & \delta t_m \end{bmatrix}^T$, where $\delta t_k \triangleq t_k - t_{k-1}$. Equality constraints in Eq. 2 are used to constrain position, velocity, orientation, or angular velocity through waypoints, as well as enforcing continuity of $\sigma_i(t)$ and its derivatives. Inequality constraints are added to enforce safety constraints, such as collision avoidance, maximum velocity, or maximum acceleration.

For a fixed time allocation $\boldsymbol{\delta t}$, the problem in Eqs. 2 reduces to a linear quadratic programming problem, which can be solved in polynomial time. On the other hand, if $\boldsymbol{dt}$ is free, the optimal time allocation problem can be stated as:

$$\begin{cases} \min\limits_{\boldsymbol{\delta t}} & \mathbb{J}(\boldsymbol{\delta t}) \\ s.t. & \sum \delta t_k = t_m \\ & \delta t_k \ge 0 \end{cases} \tag{3}$$

Assuming that (i) the problem has only position equality constraints in the waypoints, (ii) it has no inequality constraints, and (iii) that the vehicle starts from rest and finishes at rest:

Property 1 - If we solve Eq. 3 with $t_m = T$ and find a trajectory $\sigma_i(t)$ with derivatives $\dot{\sigma}_i(t)$, $\ddot{\sigma}_i(t)$, ..., $\sigma_i^{(N)}(t)$, then the optimal trajectory for $t_m = T/\alpha$, for $\alpha > 0$, is $\sigma_i(\alpha \cdot t)$ with derivatives $\alpha \dot{\sigma}_i(\alpha \cdot t)$, $\alpha^2 \ddot{\sigma}_i(\alpha \cdot t)$, ..., $\alpha^N \sigma_i^{(N)}(\alpha \cdot t)$, for $0 \le t \le T/\alpha$ (see [1]).

Property 2 - If we solve Eq. 3 with $t_m = T$ and find a solution $\boldsymbol{\delta t} = \boldsymbol{\delta t}^*$ and optimal cost $\mathbb{J} = \mathbb{J}^*$, then the solution for the same problem with $t_m = \alpha T$ is $\boldsymbol{\delta t} = \alpha \cdot \boldsymbol{\delta t}^*$ and has optimal cost $\mathbb{J} = \alpha^{-7} \mathbb{J}^*$. In other words, the optimal time allocation scales linearly with the final time t_m, but the optimal cost scales with the inverse of the 7-th power of the final time (the proof is in Appendix II-A).

Property 3 - If we solve Eq. 3 for waypoints $\boldsymbol{P}_k, k \in \{0, 1, \cdots, m\}$ and find a solution $\boldsymbol{\delta t} = \boldsymbol{\delta t}^*$, then the solution for the problem with waypoints $s\boldsymbol{T}\boldsymbol{P}_k, k \in \{0, 1, \cdots, m\}$, where $s > 0$ is a scaling factor and $\boldsymbol{T}$ is a homogeneous transformation, will also have optimal time allocation $\boldsymbol{\delta t} = \boldsymbol{\delta t}^*$. In other words, the optimal time allocation is invariant to scaling and homogeneous transformations on the waypoints (the proof is in Appendix II-B).

II. Appendix

A. Proof of Property 2

Solving for $t_m = T$ leads to a trajectory $\sigma_i^*(t)$ for $0 \le t \le T$ that passes through the waypoints $\boldsymbol{P}_k$ at times $t_k^*, k \in \{0, 1, \cdots, m\}$. According with Property 1, the optimal trajectory for $t_m = \alpha T$ ($\alpha > 0$) is $\sigma_i(t) = \sigma_i^*\left(\frac{t}{\alpha}\right)$ for $0 \le t \le \alpha T$. This implies that $\sigma_i(t)$ passes through the waypoints $\boldsymbol{P}_k$ at times $t_k = \alpha t_k^*, k \in \{0, 1, \cdots, m\}$. Therefore, it follows that $\boldsymbol{\delta t} = \alpha \boldsymbol{\delta t}^*$.

In order to prove that the optimal cost scales with the inverse 7-th power of the final time, we start with the definition of the optimal cost for the solution to $t_m = T$:

$$\mathbb{J}^* = \int_0^T \left\| \sigma^{*(4)}(t) \right\|^2 dt \tag{4}$$

According with Property 1, the optimal snap for $t_m = \alpha T$ is given by $\sigma^{(4)}(t) = \frac{1}{\alpha^4} \sigma^{*(4)}\left(\frac{t}{\alpha}\right)$ for $0 \le t \le \alpha T$. Then, the optimal cost for $t_m = \alpha T$ is given by:

$$\mathbb{J} = \int_0^{\alpha T} \left\| \tfrac{1}{\alpha^4} \sigma^{*(4)}\left(\tfrac{t}{\alpha}\right) \right\|^2 dt = \frac{1}{\alpha^8} \int_0^{\alpha T} \left\| \sigma^{*(4)}\left(\tfrac{t}{\alpha}\right) \right\|^2 dt$$

Substituting $t = \alpha \tau$ into the integral, we get:

$$\mathbb{J} = \frac{1}{\alpha^7} \int_0^T \left\| \sigma^{*(4)}(\tau) \right\|^2 d\tau = \frac{1}{\alpha^7} \mathbb{J}^*. \tag{5}$$

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B. Proof of Property 3

In the absence of inequality constraints, the nonlinear optimization from Eqs. 3-2 can be rewritten as follows:

$$\begin{cases} \min\limits_{\boldsymbol{a}_1, \boldsymbol{\delta t}^*} & \mathbb{J}(\boldsymbol{\delta t}) = \boldsymbol{a}_1^T \boldsymbol{Q} \boldsymbol{a}_1 \\ s.t. & \sum \delta t_k = t_m \\ & \delta t_k \geq 0 \\ & \boldsymbol{A}_{eq} \boldsymbol{a}_1 = \boldsymbol{b}_{eq} \end{cases} \tag{6}$$

If we scale all the waypoints $\boldsymbol{P}_{sk} = s\boldsymbol{P}_k$ by the scaling factor $s > 0$, the new problem can be rewritten as follows:

$$\begin{cases} \min\limits_{\boldsymbol{a}_2, \boldsymbol{\delta t}} & \mathbb{J}(\boldsymbol{\delta t}) = \boldsymbol{a}_2^T \boldsymbol{Q} \boldsymbol{a}_2 \\ s.t. & \sum \delta t_k = t_m \\ & \delta t_k \geq 0 \\ & \boldsymbol{A}_{eq} \boldsymbol{a}_2 = s\boldsymbol{b}_{eq} \end{cases} \tag{7}$$

Note that the only modification to the problem statement is in the last equality constraint. Taking the variable transformation $s\boldsymbol{a}_3 = \boldsymbol{a}_2$, the problem in Eq. 7 becomes:

$$\begin{cases} \min\limits_{\boldsymbol{a}_3, \boldsymbol{\delta t}} & \mathbb{J}(\boldsymbol{\delta t}) = s^2 \boldsymbol{a}_3^T \boldsymbol{Q} \boldsymbol{a}_3 \\ s.t. & \sum \delta t_k = t_m \\ & \delta t_k \geq 0 \\ & \boldsymbol{A}_{eq} \boldsymbol{a}_3 = \boldsymbol{b}_{eq} \end{cases} \tag{8}$$

We can note that the problem in Eq. 8 is the same as in 6, except for a constant cost scaling. This implies that the solution to the problem in Eq. 8 is $\boldsymbol{a}_3 = \boldsymbol{a}_1$ and $\boldsymbol{\delta t} = \boldsymbol{\delta t}^*$. Therefore, the optimal time allocations is invariant with respect to scaling of the waypoints. In addition, the solution to the minimum snap problem does not change with respect to the frame of reference. Thus, the optimal time allocations also have to be invariant with respect to homogeneous transformations.

References

[1] D. Mellinger and V. Kumar, "Minimum snap trajectory generation and control for quadrotors," in *Robotics and Automation (ICRA), 2011 IEEE International Conference on*. IEEE, 2011, pp. 2520–2525.