ELEC 4700 Assignment 2
Finite Difference Method

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I. Introduction

This assignment is concerned with solutions to Laplace's equation using the finite difference method. Laplace's equation describes the behaviour of static electric potentials in space, it is given as

$$\nabla^2 V = 0 \tag{1}$$

However, expression is usable only in the continuous case. It can be re-written with finite differences replacing the derivatives as follows

$$\frac{V_{m-1,n} + V_{m+1,n} + V_{m,n-1} + V_{m,n+1} + 4V_{m,n}}{\Delta s^2} = 0 \tag{2}$$

Where Δs is the distance between nodes, which is assumed to be equal in the x- and y-axes. The relationships between adjacent potentials described by this expression can be implemented as a matrix G, such that

$$GV = F \tag{3}$$

Allowing us to solve for V knowing only the matrix G and F, which contains fixed values at nodes which are held at a specific potential, and zero everywhere else.

II. Laplace's Equation with Simple Boundary Conditions

For all computations done in this assignment, the area of interest is a rectangular 2D space 1 unit wide and 1.5 units long. The units are unspecified, but this has no impact for our purposes.

The first scenario examined is a simple case where $V = V_0$ at $x = 0$, and $V = 0$ at $x = L$, with the top and bottom boundaries free ($\frac{dV}{dy} = 0$). Setting $V_0 = 1V$, V is computed by the finite difference method described above yields the potential surface plot shown in figure 1.

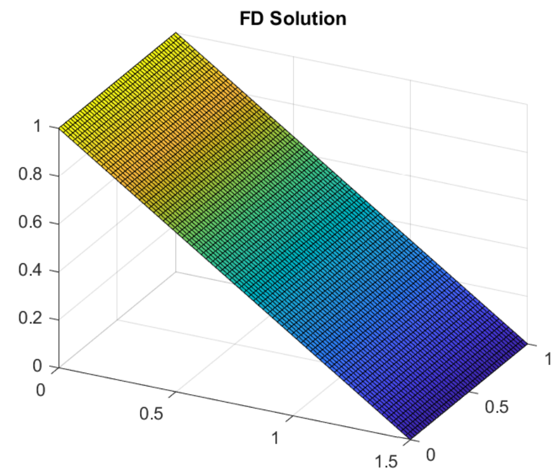

Fig. 1: Simple Potential Drop FD Solution

From figure 1 a constant slope along the x-axis from V = 1 to V = 0 is observed. This agrees with the expected solution, as a separation of parallel charged surfaces like this will have a constant electric field between the surfaces, which is equivalent to a constant slope in V ($\vec{E} = -\nabla V$).

Next we examine a scenario where $V = V_0$ at $x = 0$, $x = L$ and $V = 0$ at $y = 0$, $y = W$. Again setting $V_0 = 1V$, V is computed by the finite difference method with a 50×75 mesh as shown in figure

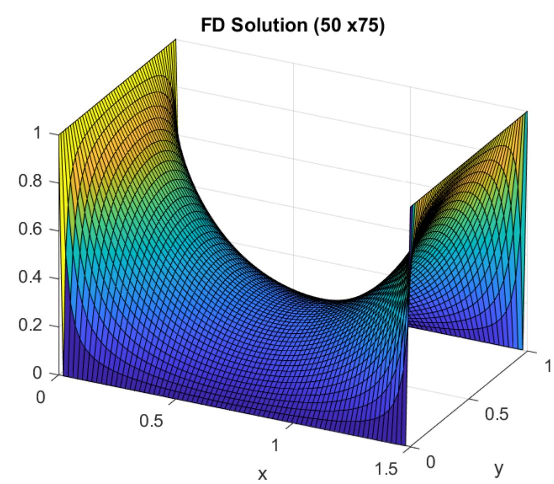

Fig. 2: Fixed Potential Edges FD Solution

In addition to the 50×75 mesh shown in figure 2, several other mesh sizes were computed. It was found that at much smaller mesh sizes the slopes were more shallow, especially near the scene boundaries. As a result, small mesh sizes were not able to accurately model the expected behaviour of potential in this scenario. Conversely, it was found that as mesh size increased, the improvement in accuracy did too. Care must then be taken to choose a mesh size large enough to achieve a reasonably accurate solution, but not so large that computational resources are wasted for a negligible improvement. The 50x75 mesh used in figure 2 was chosen for this reason.

This is also a scenario for which an analytical solution exists, given by the expression

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5...}^{\infty} \frac{1}{n} \frac{cosh\left(\frac{n\pi x}{a}\right)}{cosh\left(\frac{n\pi b}{a}\right)} sin\left(\frac{n\pi y}{a}\right) \tag{4}$$

For comparison, this was also plotted as shown in figure 3. In this instance the series was truncated once the percent change in potential at each node was less than 0.01% between terms. It should also be noted that the analytic solution defines the x-boundaries at $x = \pm 0.75$, but the scenes are otherwise identical.

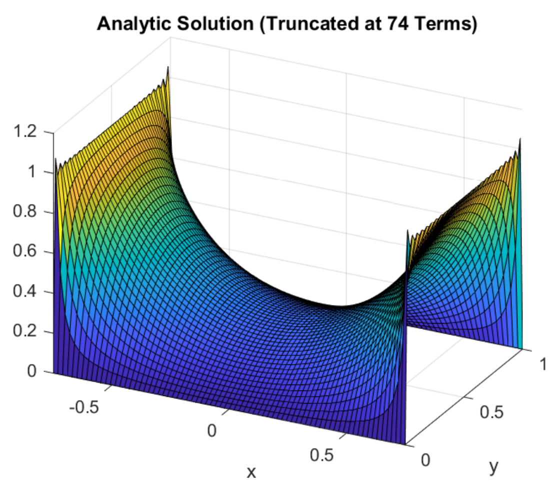

Fig. 3: Fixed Potential Edges Analytic Solution

At the boundaries where $V = V_0$ in the analytic solution, there is still some visible oscillation resulting from the sinusoidal nature of the series solution and were it possible to sum the entire infinite series, these oscillations would disappear. Otherwise the truncated analytic, and finite-difference solutions are very close.

III. Effects of a Bottleneck

The constant potential drop scenario explored in section I can be complicated by the addition of a resistive bottleneck as shown in figure 4.

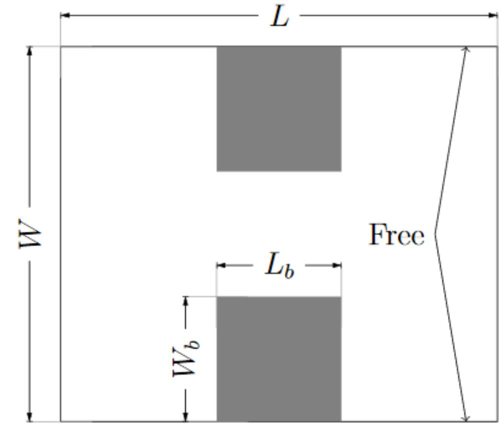

Fig. 4: Rectangular region with isolated conducting sides and bottleneck [1]

For solving Laplace's equation in inhomogenous scenarios such as this, the equation is rewritten as

$$\nabla(\sigma(x,y)\nabla V) = 0 \tag{5}$$

This change is implemented in the G matrix and we let $\sigma = 1E-2$ within the bottleneck regions and $\sigma = 1$ everywhere else. To start, the bottlenecks are sized at $W_b = L_b = 0.2$, as shown in the conductivity map shown in figure 5.

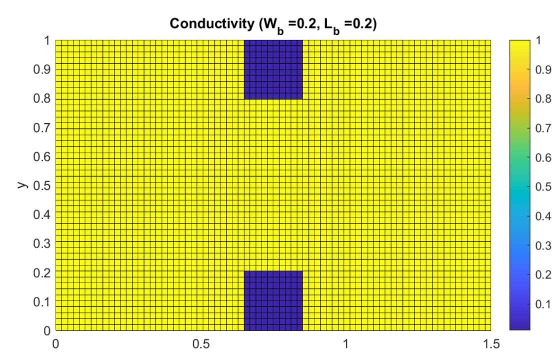

Fig. 5: Conductivity Map

The potential in this scenario is then computed by finite difference as shown in figure 6. The effects of the resistive bottleneck can be observed here as the potential slopes more steeply across the bottleneck, sloping inward as current would flow around these regions through the conductive center.

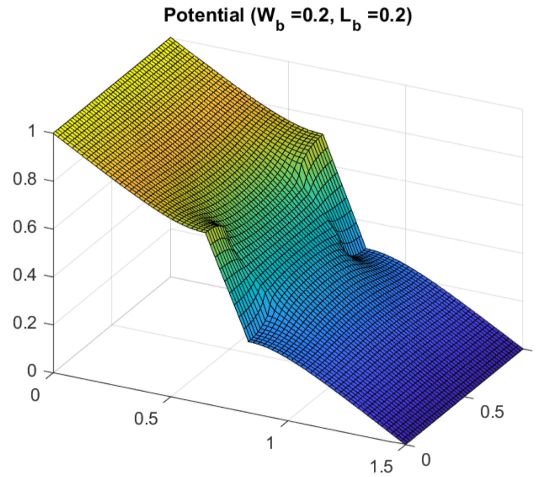

Fig. 6: Bottleneck Potential FD Solution

Knowing potential, the electric field and current density throughout the scene can also be computed, where electric field is given as

$$\vec{E} = -\nabla V \tag{6}$$

and current density is given by the expression

$$\vec{J} = \sigma \vec{E} \tag{7}$$

Computing these quantities from the potential field shown in figure 6 yields the plots shown in figures 7 and 8.

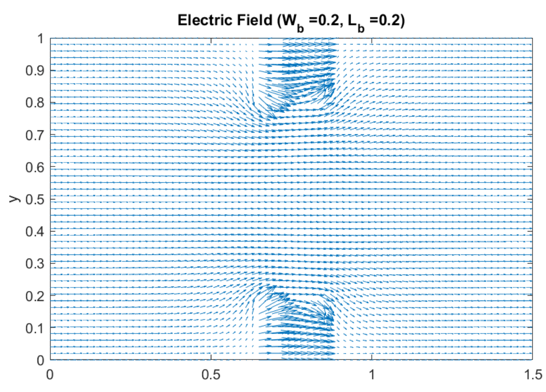

Fig. 7: Bottleneck Electric Field FD Solution

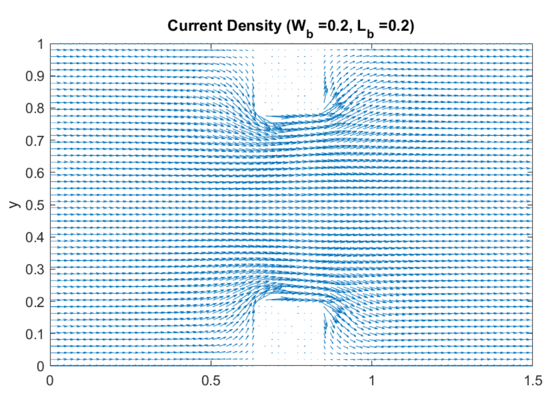

Fig. 8: Bottleneck Current Density FD Solution

These results agree with the expected solution to this problem, the electric field shown in figure 7 is relatively constant outside of the bottleneck, where it is much stronger through the resistive sections. The current in figure 8 also appears to flow around the bottleneck, taking the path of least resistance from high to low potential as expected. Finally, the total current entering the left edge and leaving the right edge can be computed from current density. For a continuous current density field, this would be computed as

$$I = \iint_A \vec{J} \cdot \vec{ds} \tag{8}$$

But for this discrete case, it is computed from the x-components of J (J_x) as

$$I_x = \sum J_x \Delta s \tag{9}$$

Computing current in this way at the input plane ($x = 0$) and output plane ($x = 1.5$) yields $I_{x,in} = 0.5726A$ and $I_{x,out} = 0.5727A$. Because the top and bottom boundaries are free ($\frac{dV}{dy} = 0$) current should only flow through the left and right boundaries, and by Kirchoff's current law the current in and out of this system should be identical. For this reason, the small difference between I_{in} and I_{out} must be an error, but it is sufficiently small to be ignored.

These solutions represent only one possible bottleneck and mesh size. Because it is likely that changes in these parameters would impact the results, the effect of changes in mesh size, bottleneck dimensions, and conductivity were explored.

A. Mesh Size

The solutions above used a mesh size of 75x50 or "50 divisions per unit". Because it had been previously observed that mesh size has a significant impact on the the accuracy of results, input and output current were computed for varying mesh sizes as shown in figure 9.

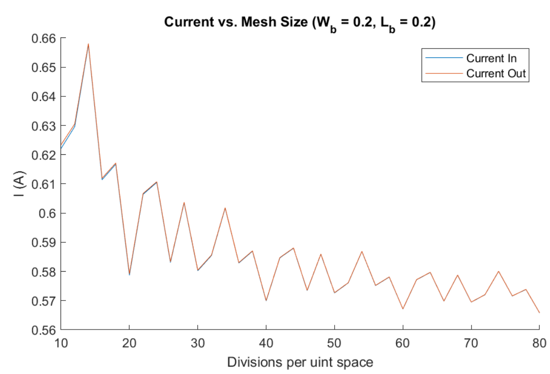

Fig. 9: Current vs. Mesh Size

These results further support the observation that increasing mesh size improves accuracy with diminishing returns as it increases further. The difference between current at 80 divisions per unit is very close to the current at 50 divisions per unit. It is also observed that the difference between input and output current becomes increasingly small as mesh size increases, providing further confirmation that higher mesh size produces results more representative of reality.

B. Bottleneck Dimensions

The dimensions of the bottleneck should intuitively have a significant impact on current, as a larger bottleneck will increase the effective resistance of the system, reducing current as a result by Ohm's law. To confirm this, input current was computed for several different bottleneck dimensions as shown in figure 10.

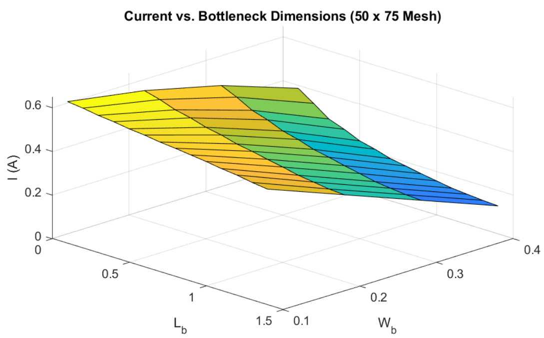

Fig. 10: Current vs. Bottleneck Dimensions

As expected, current decreases with any increase in either dimension of the bottleneck. It decreases more quickly as width increases, which is intuitive as this represents narrowing of the conductive path through the system.

C. Bottleneck Conductivity

Finally, the conductivity of the bottleneck region should determine the degree to which current flow is impeded. To determine the nature of this relationship, current was computed for a bottleneck with $W_b = L_b = 0.2$ on a 50×75 mesh at various bottleneck conductivities as shown in figure 11.

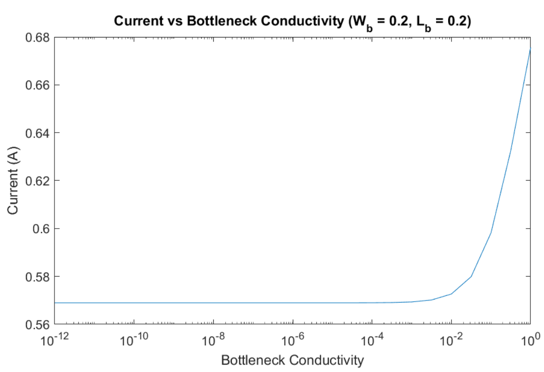

Fig. 11: Current vs. Bottleneck Conductivity

As expected, lower bottleneck conductivity lowers current flow. When $\sigma = 1$ everywhere, current flows uniformly from high to low impedance, but as the bottleneck's conductivity decreases, current flow in those regions is impeded, forcing the majority of current flow into the conductive center channel. At first, decreasing bottleneck conductivity decreases total current flow, but as it approaches zero, there is already effectively zero current flowing in the high conductivity regions, meaning that further decreases in conductivity have little effect, as current is already effectively restricted to flowing through the middle.

References

[1] Tom Smy *ELEC 4700 Assignment-2 Finite Difference Method* Carleton University, 2021