THE BETTING INTERPRETATION OF PROBABILITY AND DUTCH BOOKS

MARCELLO DI BELLO

0. PROBABILITIES AS DEGREES OF BELIEF

On the subjective interpretation, probability is not an objective property of events in the world out there, but rather, a property of people's beliefs about events in the world. Specifically, on the subjective interpretation, that event E has, say, a 70% probability of occurring just means that a rational agent would assign a 70% degree of belief (or degree of confidence, degree of sureness) in the truth of E. If probabilities are degrees of beliefs, then what are degrees of belief?

1. PROBABILITIES AS BETTING ODDS

According to the betting interpretation, your degree of belief about an event's is reflected in your willingness to buy or sell bets at specific odds. An example should make this clerar.

Example

You believe there's a 70% probability of rain. What does that mean? These will seem fair to you:

- **Buying the bet:** You pay \$70 to enter a bet that gives you a chance to win \$100 if it rains.
- **Selling the bet:** You receive \$70 by selling a bet, but you agree to pay \$100 if it rains.

This is the **same bet**, which pays \$100 if it rains, and nothing if it does not rain. The bet costs \$70. In one case, you buy it for \$70, and in the other case, you sell it for \$70.

Indifference

Crucially, you are **indifferent** between buying and selling the bet at \$70. This might not be obvious at first, but can be readily seen by working out the expected utilities. The expected utility in both scenarios is zero, so you must be indifferent between buying and selling the bet:

- **Buying:** The expected payout is 0.7 × \$100 = \$70 since for you there is a 70% probability of rain. This expected payout matches the \$70 you pay to enter the bet.[1]
- **Selling:** The \$70 you receive upfront by selling the bet matches the expected loss of 0.7 × -\$100 = -\$70 since for you there is a 70% probability of rain.[2]

[1] Equivalently, you stand to gain \$30 if it rains (70% chance) and lose \$70 if it does not rain (30% chance), so the expected utility of the bet is $30 \times 0.7 - 70 \times .3 = 0$.

[2] Equivalently, you stand to lose \$30 if it rains (70% chance) and gain \$70 if it does not rain (30% chance), so the expected utility of the bet is $-30 \times 0.7 + 70 \times .3 = 0$.

What about different prices?

- **Buying at a lower price:** If you buy the bet for $60, it's even better for you because the expected gain if it rains ($\$100 \times .7 = \$70$) exceeds the cost (-$60). The bet is fair and favorable.
- **Selling at a lower price:** If you sell the bet for $60, that's no good for you because the expected loss ($-\$100 \times .7 = -\$70$) is greater than your gain ($60). The bet is not fair to you.
- **Buying at a higher price:** If you buy the bet for $80, that's no good for you because the expected gain ($\$100 \times .7 = \$70$) is less than the cost gain (-$80). The bet is not fair to you.
- **Selling at a higher price:** If you sell the bet for $80, it's better for you because the expected loss ($-\$100 \times .7 = -\$70$) is less than your gain ($80). The bet is fair and favorable.

Different prices are still fair as long as they reflect your belief about the event's probability, but at higher or lower prices you will no longer be indifferent between buying and selling the bet.

de Finetti's definition

"Let us suppose that an individual is obliged to evaluate the rate p at which he would be ready to exchange the possession of an arbitrary sum S (positive or negative) dependent on the occurrence of a given event E, for the possession of the sum pS; we will say by definition that this number p is the measure of the degree of probability attributed by the individual considered to the event E." (de Finetti (1937), 'Foresight. Its Logical Laws, Its Subjective Sources')

- **Buying:** If you are willing to pay (at most) pS—a p percentage of sum S—for a bet that pays you the sum S if E occurs and 0 otherwise, then your degree of belief in E equals p.
- **Selling:** If you are willing to receive (at least) pS—a p percentage of sum S—for a bet that costs you the sum S if E occurs and 0 otherwise, then your degree of belief in E equals p.

In both cases, you are willing to exchange the sum pS for S if E occurs.

2. Dutch Book

A Dutch book occurs when incoherent probabilities (for example, summing to more than 100%) allow an opponent to create bets that guarantee your loss.

Example: probabilities add up to more than 100%

You assign the following probabilities to a football game:

- **Team A wins:** 60% (0.6)
- **Team B wins:** 50% (0.5)

A clever bookie offers:

1. Bet $60 to win $100 if Team A wins. This is a fair bet if Team A wins with 60% probability.
2. Bet $50 to win $100 if Team B wins. This is a fair bet if Team B wins with 50% probability.

No matter the outcome, you'll lose $10:

- **Team A wins:** You gain $40 but lose $50, so **$-10**.
- **Team B wins:** You gain $50 but lose $60, so **$-10**.