PROBABILITY EXAM - CRITICAL REASONING - PHI 169 - SOLUTIONS

(a) What is the difference between $P(A|B)$ and $P(B|A)$? *Please explain by giving a couple of illustrative examples.*

(answer) *Example 1*: Consider
A = "the forecast says there will be rain tomorrow"
B = "there will be rain tomorrow".
It is one thing to say that it is 80% likely there will be rain tomorrow given that the whether forecasts says so, that is, $P(B|A) = 80\%$. It is quite another to say that it is 80% likely that the forecast says there will be rain tomorrow given that there will be rain tomorrow, that is, $P(A|B) = 80\%$.
Example 2: Consider
A = "an individual is in prison"
B = "an individual is a white man".
It is one thing to say that it is likely that an individual is in prison, given that he is a white man. It is quite another to say that it is likely that an individual is a white men, given that he is in prison.

(b) If $P(B) = P(A)$, does it follow that $P(A|B) = P(B|A)$? *Please explain with simple mathematical reasoning.*

(answer) It does follow. We know that

$$P(A|B) = \frac{P(A\&B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

and assuming that $P(A) = P(B)$, then

$$P(A|B) = \frac{P(A\&B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)\cancel{P(A)}}{\cancel{P(B)}} = P(B|A).$$

(c) Suppose you want to know what your chances are to succeed at X, where X is something that matters to you and you want to do (say, graduate, make one billion dollars, write a novel, win the nobel prize, whatever). Suppose you learn from a very reliable source that those who succeed at X are very often of type Y, that is,

$$P(\textit{a person is of type } Y | \textit{a person succeeds at } X)$$

is very high. You are a person of type Y (say, introvert or extrovert or shy or ... whatever). What is your reaction? Could you then be reassured that you are very likely to succeed at X since you are of type Y? *Please explain and show me the logic of your argument*.

(answer) You should not be reassured. After all, even if $P(\textit{person is of type } Y|\textit{person succeeds at } X)$ is high, what you really care is whether $P(\textit{person succeeds at } X|\textit{person is of type } Y)$ is high.

If $P(\textit{person is of type } Y|\textit{person succeeds at } X)$ is high, it does not necessarily follow that $P(\textit{person succeeds at } X|\textit{person is of type } Y)$ must be high as well because

$$P(\textit{person succeeds at } X|\textit{person is of type } Y) = \frac{P(Y|\textit{person succeeds at } X)P(\textit{person succeeds at } X)}{P(\textit{person is of type } Y)}$$

(d) You are deciding whether to go to school A or B. They cost the same, are equally prestigious and you have been accepted to both. You care mostly about a highly paying job and it turns out that most of those who have highly paying jobs in your town studied at school A. That is, $P(\textit{person studied at school } A|\textit{person has highly paying job})$ is higher than $P(\textit{person studied at school } B|\textit{person has highly paying job})$. Is this a good reason to pick school A over B? *Please explain and show me the logic of your argument*.

(answer) It is not a good reason to pick school A over school B. To see why, let's make first the following abbreviations:

- \$\$ means "person has highly paying job"
- $studied(A)$ means "person studied at school A"
- $studied(B)$ means "person studied in school B"

A good reason to pick school A over B would be that \$\$ is more likely given $studied(A)$ than given $studied(B)$. That is,

$$P(\$\$|studied(A)) > P(\$\$|studied(B)),$$

but all we are told is that

$$P(studied(A)|\$\$) > P(studied(B)|\$\$).$$

The latter inequality could be true without the former inequality being true. How so? For the sake of illustration, suppose that

$$P(studied(A)|\$\$) = 90\% and P(studied(b)|\$\$) = 10\%.$$

In this way, the latter inequality above is satisfied. Suppose, again for the sake of illustration, that most people study in school A, say,

$$P(studied(A)) = 99\% and P(studied(B)) = 1\%$$

This is not given in the problem, but it is a possibility. By the stipulations above, we get

$$P(\$\$|studied(A)) = \frac{P(studied(A)|\$\$)P(\$\$)}{P(studied(A))} = \frac{90\% \times P(\$\$)}{99\%}$$

and also

$$P(\$\$|studied(B)) = \frac{P(studied(B)|\$\$)P(\$\$)}{P(studied(B))} = \frac{10\% \times P(\$\$)}{1\%},$$

from which it follows that

$$P(\$\$|studied(A)) < P(\$\$|studied(B)).$$

So, given our illustrative stipulations, going to school A is less likely to lead one to a highly paying job than going to school B despite the fact that most of those who have a highly paying job come from school A. The key here—intuitively—is that a lot of people go to school A and it is natural that among those who have a highly paying job most come from school A.

(e) Luveko and Eresia are happy living together. They do a regular check-up one day and it turns out they both test positive for a rare disease which could cause serious health complications as they get older. They both start to get worried, but they try to remain calm and seek more information. Here is what they find out. The test is 99% reliable, but the disease is quite rare. It affects overall only 1% of the population. However, as it turns out, the disease affects males more frequently than females, and in particular, it affects males with blue eyes forty times as often as females with brown eyes. Luveko is a male with blue eyes, while Eresia is a female with brown eyes. We know the disease in question affects 20% of males with blues eyes. Should Luveko and Eresia still be worried? How likely are they to have the disease given that they tested positive? *Please answer this question using Bayes' theorem and the method of counting cases. Make sure you show me the logic of your argument.*

(answer) Consider the case of Luveko first. He should be rather worried! Let D means that one has the disease and $T(pos)$ that one tested positive and $T(neg)$ one tested negative. We know that

$$P(D|T(pos)) = \frac{P(T(pos)|D)P(D)}{P(T(pos)|D)P(D) + P(T(pos)|\neg D)P(\neg D)}$$

We can plug in the values we are given. That is, in the case of Luveko, as a male with blue eyes, $P(D) = 20\%$ and thus $P(\neg D) = 80\%$. (NB: We are told that in general the disease affects 1% of the population, but we can disregard that because we are interested in the case of a male with blue eyes.) We are also told the test has reliability of 99%, that is, $P(T(pos)|D) = 99\%$ and $P(T(neg)|\neg D) = 99\%$ and thus $P(T(pos)|\neg D) = 1\%$. So we have

$$P(D|T(pos)) = \frac{99\% \times 20\%}{99\% \times 20\% + 1\% \times 80\%} \approx 96\%$$

Consider the case of Eresia. She should not be too worried. In her case, as a female with brown yes, $P(D) = 0.5\%$ and thus $P(\neg D) = 99.5\%$. So we have

$$P(D|T(pos)) = \frac{99\% \times 0.5\%}{99\% \times 0.5\% + 1\% \times 99.5\%} \approx 33\%$$