Managing currency risk: An application of copula-based multivariate dynamic models

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Abstract

In order to access foreign markets, global investors often need to post collateral in currencies that are different from their benchmark currency, resulting in undesired foreign exchange risk. Investors might therefore be tempted to strictly post the minimum margins required, however doing so maximizes the probability of margin calls and their associated operational risk and cost. This article finds the posted margins that represent the best equilibrium between foreign exchange risk and probability of a margin call. In order to do so, a robust dynamic model of the asset prices and exchange rates dynamics is proposed, and an optimization problem seeking the optimal equilibrium is solved. The proposed model overperformed both a naive and a Gaussian alternative model on a dataset of five futures contracts in four different currencies over a period of six years for the metrics of interest.

1 Introduction

With the globalization of financial markets, investors have access to a larger investment universe giving them the opportunity to better diversify their portfolio. A subclass of financial transactions (short selling and entering into derivative contracts for example) requires the posting of margins, that is, the deposit of

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collateral to attenuate the credit risk for the counterparty. If such a transaction is performed in a foreign market, the required collateral will most likely be in a currency different from the portfolio's benchmark currency, creating foreign exchange risk for the global investor. This investor might therefore want to maintain the posted collateral to the strict minimum. This policy will however maximize the probability that the counterparty, either the investor's broker or exchange, will require the posting of additional collateral following an adverse price change, so-called a margin call.

Margin calls sometimes result in undesired operational costs for investors, they are therefore to be avoided if possible. Not only counterparties may impose penalties on margin calls, but the regulatory framework agreed upon by members of the Basel Committee on Banking Supervision in the Basel III accords impose tougher capital buffers and liquidity coverage on derivatives transactions (BCBS, 2011a,b).

The purpose of this work is to use recent developments in the field of econometrics to come up with a rigorous solution to the problem of multicurrency collateral management. In addition, the methodology proposed here can be extended to many financial and risk management applications where an optimization problem needs to be constrained by practical considerations (transaction costs for example).

The quest to a better understanding and forecasting of financial time series resulted in a constant increase in the level of sophistication of econometrics models. For univariate series, the canonical Brownian process (Brown, 1828, Bachelier, 1900, Merton, 1969) has often been substituted with more recent models capturing empirical properties of the financial time series such as heteroskedasticity and leptokurticity (fat tails). The most popular of these models are AutoRegressive Conditionnal Heteroskedasticity (ARCH) (Engle, 1982) and its extension the generalized ARCH (GARCH) (Bollerslev, 1986). In these models, the variance, and hence the volatility, of the time series for a given period is a direct function of the variance in the previous period, thus accounting for volatility clustering. Furthermore, the processes generate data with fatter tails than the normal density. Thanks to its parsimonious notation (see equations (3) and (4) in Section 2), GARCH has become the most popular ARCH model in practice, including this work.

Proper financial modeling requires the consideration of dependence between time series. For this reason the development of multivariate models quickly followed the one of univariate models. It is now widely accepted (Cont, 2001, Peng and Ng, 2012) that robust risk management requires the inclusion of higher-order moments and co-moments in the time series of financial assets. Not only do univariate distributions capturing higher moments need to be picked to model single financial time series, but a model of dependence between the different time series allowing for higher co-moments is primordial. Indeed, traditional linear correlation between the returns of financial assets fails to capture the "correlation breakdown" or "assets boom alone but bust together" asymmetry observed in the markets. Models have been proposed to extend the ARCH-type framework described in the previous paragraphs to multivariate settings. They all face the challenge of balancing sophistication with parsimony, of being flexible enough to capture co-moment dynamics while avoiding the curse of dimensionality. For a comprehensive survey of multivariate GARCH models, refer to Silvennoinen and Teräsvirta (2008) or Laurent et al. (2006).

A promising family of multivariate models combines the tractability of univariate processes with the power of copulas to capture the dependence between the marginal process in a robust manner. A copula, in combination with the univariate marginal distributions, is sufficient to fully specify a multivariate distribution function, as proven by Sklar (1959). The copula C underlying the random variables X_1, X_2, ..., X_D is the joint cumulative distribution function of the transformed variables $F_1(x_1), F_2(x_2), \ldots, F_D(x_D)$, where $F_i(x) = \mathbb{P}[X_i \leq x]$ are the marginal cumulative distribution functions, and

$$C(u_1,\ u_2,\ \ldots\ ,u_D) = \mathbb{P}[F_1(X_1) \leq u_1,\ F_2(X_2) \leq u_2,\ \ldots\ ,F_D(X_D) \leq u_D].$$

This work owes much to the findings of Chen Xiaohong and Fan Yanqin (Chen and Fan, 2006), Andrew Patton (Patton and Kearney, 2000, Patton, 2006) and Bruno Rémillard (Rémillard, 2010). The basis of these articles is that the dependence between the error terms of multivariate time series is described not in term of traditional correlation but by a given copula.

Unfortunately, copulas have often been used prior to the existence of proper goodness-of-fit tests, that is, without properly testing whether the data being modeled belongs to the chosen copula or not in a statistically significant manner. One of the first instance of copulas goodness-of-fit test for multivariate time series can be found in Chen and Fan (2006), however this test can only rank the appropriateness of different copulas relative to each other, not in a statistically absolute sense. Fortunately, Genest et al. (2009) and Rémillard (2010) applied the parametric bootstrapping technique to copulas allowing one to do so in an intuitive, powerful manner. Its use with copulas evolved naturally from previous applications to univariate and multivariate distributions. The application of this test prevents the use of a copula where inappropriate.

The tools developed in this work can also be applied to related problems: balancing tracking error and transaction costs (see Chan and Ramkumar (2011)), foreign currency risk and hedging costs (see Campbell et al. (2010)), sales and marketing costs, etc. In the case of balancing foreign exchange risk on posted collateral and the costs associated to margin calls, global investors have historically used heuristics or more sophisticated policies to mitigate the two inconveniences. For literature on the subject, one can refer to Miller and Orr (1966), Higson et al. (2010) for solutions to the similar problem of optimal inventory management, to Cotter (2001), Lam et al. (2004), Kao and Lin (2010), Longin (1999) for solutions to the problem of setting the margins requirements by a central counterparties (e.g. clearing houses) and to Fujii et al. (2010) for an example where the market participant has the choice of collateral currency.

The remainder of this article is organized as follows. Section 2 exhibits the framework used to model the dynamics of the asset prices and exchange rates. Section 3 overviews the optimization problem at hand. Section 4 presents the results of a backtest of the proposed solution performed on a real data set. Section 5 concludes and proposes avenues of future research.

2 Copula-based multivariate dynamic models

In order to develop a cash management strategy which strikes an optimal balance between the opposing goals of minimizing foreign exchange risk and minimizing the cost associated with margin calls, it is first necessary to choose a model of underlying asset prices and exchange rates movements. It is now widely accepted that proper modeling is robust to volatility clustering and excess kurtosis. That is, the volatility in financial time series is not constant across time but periods of relatively high volatility tend to cluster together. Furthermore, the probability of extreme events in the joint distributions of multivariate financial time series innovations is higher than predicted under a Gaussian model. The latter phenomenon is also known as “fat tails”. Refer to Pagan (1996), Cont (2001) for evidence of stylized facts of financial time series and more precisely to Vries and Leuven (1994), Guillaume et al. (1997) for evidence of such facts in exchange rate returns.

We propose the use of a dynamic multivariate discrete stochastic volatility model with a copula-based dependence structure. This model has been successfully applied to exchange rate returns in the literature (Chen and Fan, 2006, Rémillard, 2010, Patton, 2006). The multivariate time series $\mathbf{X}_t$, $t \geq 1$ has D dimensions and is given by

$$X_{i,t} = \mu_t(\boldsymbol{\theta}_i) + h_t(\boldsymbol{\theta}_i)^{1/2}\ \epsilon_{i,t}, \tag{1}$$

where $i = 1, \ldots, D$ and innovations $\epsilon_{1,t}, \ldots, \epsilon_{D,t}$ are *i.i.d.* with copula distribution function C.

We know from Sklar theorem (Sklar, 1959) that, given that K is continuous, there exists a unique copula C such that

$$K(x_1, \ldots, x_D) = C_{\boldsymbol{\theta}}(F_1(x_1), \ldots, F_D(x_D)), \tag{2}$$

where the F_i are the cumulative distribution functions of the marginal distributions X_i and $C_{\boldsymbol{\theta}}$ is the copula function with parameter(s) $\boldsymbol{\theta}$.

An interesting property of copulas is that the dependence between the variables is encapsulated in the copula function and is independent of the marginal distribution functions chosen.

We model the marginal distributions of the financial time series with AR(k)-GARCH(p,q) models (Bollerslev, 1986). We chose this model because we believe it offers the best equilibrium between sophistication and parsimony. Their formulation is given by

$$\mu_t(\boldsymbol{\theta}_i) = \kappa_{\mu,i} + \sum_{j=1}^{k} \gamma_{i,j}\ x_{i,t-j}, \tag{3}$$

and

$$h_t(\boldsymbol{\theta}_i) = \kappa_{h,i} + \sum_{j=1}^{q} \alpha_{i,j}\ \epsilon_{i,t-j}^2 + \sum_{j=1}^{p} \beta_{i,j}\ h_{t-j}(\boldsymbol{\theta}_i). \tag{4}$$

We tested the goodness-of-fit of common elliptical (Gaussian and Student) and Archimedean (Clayton, Frank, Gumbel) copulas on the standardized residuals of the AR-GARCH margins processes emerging from the dataset described in section 4.1. Their distributions are given below, and the recipe of the parametric bootstrapping goodness-of-fit tests is given in Appendix A.

The Gaussian copula with dependence parameter matrix Σ is given by

$$C_\Sigma(\mathbf{u}) = \Phi_\Sigma\left(\Phi^{-1}(u_1),\ \dots\ \Phi^{-1}(u_D)\right), \tag{5}$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of a standard normal, $\Phi_\Sigma(\cdot)$ is the joint cumulative distribution function of a multivariate normal distribution with zero means and covariance matrix Σ. Its density is given by

$$c_\Sigma(\mathbf{u}) = |\Sigma|^{-\frac{1}{2}}\ \exp\left(-\frac{1}{2}\begin{pmatrix}\Phi^{-1}(u_1)\\ \vdots \\ \Phi^{-1}(u_D)\end{pmatrix}^T (\Sigma^{-1}-\mathbf{I})\begin{pmatrix}\Phi^{-1}(u_1)\\ \vdots \\ \Phi^{-1}(u_D)\end{pmatrix}\right).$$

The D-dimensional Student copula distribution as given by (Demarta and McNeil, 2005):

$$C_{\Sigma,\nu}(\mathbf{u}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_D)} \frac{\Gamma\left(\frac{\nu+D}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^D|\Sigma|}} \left(1+\frac{\mathbf{x}'\Sigma^{-1}\mathbf{x}}{\nu}\right)^{-\frac{\nu+D}{2}} d\mathbf{x}, \tag{6}$$

where $t_\nu^{-1}()$ is the quantile function of a standard univariate Student distribution with ν degrees of freedom. Its density can be derived to be

$$c_{\Sigma,\nu}(\mathbf{u}) = \frac{f_{\Sigma,\nu}(t_\nu^{-1}(u_1),\dots,t_\nu^{-1}(u_D))}{\prod_{i=1}^D f_\nu(t_\nu^{-1}(u_i))}, \tag{7}$$

where $f_{\Sigma,\nu}$ is the joint density of a D-dimensional random vector from a multivariate Student distribution with ν degrees of freedom and covariance matrix Σ and f_ν is the density of a univariate Student distribution with ν degrees of freedom. This copula not only captures excess kurtosis but was also shown to accurately model the dependence between financial time series in recent literature (Chen and Fan, 2006, Fischer et al., 2009).

Archimedean copulas are characterized by a single dependence parameter and the following representation:

$$C(u_1,\ u_2,\ \ldots\ ,u_D) = \psi(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_D)),$$

where $\psi(\cdot)$ is the *generator* of the copula. The generators for the three Archimedean copulas tested in the applied section of this thesis are displayed in table 1.

Family	**Generator $\psi(\mathbf{t})$**	**Parameter**	**G-distribution**
Clayton	$(1+t)^{-1/\theta}$	$0<\theta<\infty$	$\text{Gamma}(1/\theta, 1)$
Frank	$-\frac{1}{\theta}\log(1-(1-e^{-\theta})e^{-t})$	$0<\theta<\infty$	Log series with $\alpha = (1-e^{-\theta})$
Gumbel	$\exp(-t^{1/\theta})$	$1\leq\theta<\infty$	$\text{Stable}(1/\theta, 1, (\cos(\pi/(2\theta)))^{\theta}, 0)$

Table 1: Generators of selected Archimedean copulas. G-distribution refers to the distribution that has as Laplace transform the generator $\psi(\cdot)$.

The parameters of the marginal processes and dependence copula can be estimated with the maximum likelihood estimation (MLE) method. An alternative two-stage estimation method for the Student copula is proposed by McNeil et al. (2005).

Chen and Fan (2006) showed the important result that applying the copula dependence structure from equation (2) to the innovations $\epsilon_{i,t}$ from equation (1) rather than to the realizations $X_{i,t}$ yields the same results in terms of estimation of the parameters of the copula. Rémillard (2010) showed that the empirical copula and most dependence measures are unaffected as well.

3 Constrained optimization

One difficulty in the problem at hand is to find a way to balance foreign exchange risk on one hand and the costs associated to margin calls on the other. This exercise proves difficult given that these two forces have different units. We posited that the costs associated to margin calls are an increasing function of their frequency, and chose to minimize the foreign exchange risk on the posted collateral conditionally to an arbitrary upper bound on the probability of a margin call. Chan and Ramkumar (2011) offers an elegant framework to a problem similar in nature to ours. Their goal is to balance trading costs associated with the rebalancing of a given portfolio and tracking error. Their solution is to minimize the trading costs subject to an imposed upper ceiling on the forecasted tracking error. Our objective function described below is inspired from this approach.

We formulate the issue at hand as a one-period constrained optimization problem. The first step consists in choosing a risk measure to assess the foreign exchange risk. This work uses the expected tracking error (ETE), the Value-at-Risk (VaR) and the Tail Conditional Expectation (TCE). The method consists in finding the cash balances that minimize the (foreign exchange) risk measure subject to an arbitrary tolerance for the probability of a margin call. A mathematical formulation of the optimization objective is

$$\min_{\lambda_{1,t},\ldots,\lambda_{D,t}} R^{\alpha}\left(\sum_{i=1}^{D}\left(\lambda_{i,t}-\lambda_{i,t}^{*}\right)\cdot Y_{i,t+1}\right), \tag{8}$$

where

$$R^{\alpha}(X)=\begin{cases}\mathbb{E}\left[|X|\right] & \text{for expected tracking error,}\\ -\mathbb{Q}_{\alpha}(X) & \text{for Value-at-Risk,}\\ -\mathbb{E}[X|X\leq\mathbb{Q}_{\alpha}(X)] & \text{for Tail Conditional Expectation.}\end{cases} \tag{9}$$

where $\mathbb{Q}_{\alpha}(X)$, $\alpha \in [0,1]$ is a quantile of X such that $P(X\leq\mathbb{Q}_{\alpha}(X))=\alpha$. In words, equation (8) finds the collateral in each currency $\lambda_{1,t},\ldots,\lambda_{D,t}$ above the minimum margins requirements imposed by the counterparty $\lambda_{1,t}^{*}$, …, $\lambda_{D,t}^{*}$ that minimizes one of the risk measures $R^{\alpha}(\cdot)$ described in equation (9) against the changes in the exchange rates $Y_{1,t+1}$, …, $Y_{D,t+1}$. The first constraint of the optimization is

$$\lambda_{i,t}^{*}\leq\lambda_{i,t}\leq\infty,\ i=1,\ldots,D,$$

that is, each posted collateral $\lambda_{i,t}$ must be between the minimum margin requirement $\lambda_{i,t}^{*}$ and infinity. The second constraint of the optimization is

$$\mathbb{P}\left(\left(\prod_{i=1}^{D}\mathbb{1}\left(\lambda_{i,t}+\mathrm{PnL}_{i,t+1}\geq\lambda_{i,t}^{*}\right)\right)=0\right)\leq P_{\mathrm{tol}},$$

where

$$\mathrm{PnL}_{i,t+1}=\sum_{j=1}^{n_{i,t}}{}_{i}\omega_{j,t}\cdot{}_{i}W_{j,t+1}.$$

Here, P_{tol} is an arbitrary tolerance for the probability of a margin call, $n_{i,t}$ is the number of different assets of the i^{th} currency held, ${}_i\omega_{j,t}$ is the number of the j^{th} asset of the i^{th} currency held and ${}_iW_{j,t+1}$ is the change in the j^{th} asset of the i^{th} currency between time t and $t+1$. To make things clear, the only random variables in the model are the $Y_{i,t}$ and the ${}_iW_{j,t+1}$. Furthermore, these random variables are not independent of each other. That is, the spot exchange rates are not independent of the futures contracts prices and vice versa.

The proposed methodology for the optimization is as follow: First, univariate process are fitted to each time-series of log returns of exchange rates and future prices. This step often involves the calibration of parameters with the maximum likelihood estimation method. Second, goodness-of-fit tests of the selected model are performed on each time-series. If the test result is negative for a given time-series of log returns, then another univariate model is chosen for that specific time-series and the process is started anew from the first step. Again, different time-series can have different univariate models. Third, a parametric multivariate copula is fitted to the normalized ranks of the standardized residuals obtained in the first step. Fourth, a goodness-of-fit test as put forward in Appendix A is performed on the copula-based dynamic model with respect to the sample of time-series. If the test result is negative, another copula is chosen and the process is restarted from the third step. At this point, we know that the model is appropriate for the data (at least in the statistical sense of the term). Here is where Monte Carlo methods come into play. Fifth, a large number N of D-dimensional $U[0,1]$ realizations is drawn from the parametric copula, were D is the number of time-series modeled. Most modern statistical software packages provide tools to perform this step for elliptical copulas. One can refer to Marshall and Olkin (1988) and to Melchiori (2006) for Archimedean copula random variable gerneration. Sixth, the simulated standardized residuals of each univariate process are obtained by entering the uniform drawings from the previous step into their respective inverse cumulative distribution function. Seventh, the simulated realizations of the log returns are obtained by combining of the random standardized residuals and the computed deterministic part of each univariate process. The final step consists in using constrained numerical optimization to find the values of the control variables that optimize the objective function while satisfying the constraint function. For us, the control variables are the amount of collateral held in each currency, the objective function is the risk measure (tracking error, Var and TCE) and the constraint function is the probability of a margin call. Notice that both the objective and constraint are functions of the control variables (the amount of collateral held in each foreign currency) and the realizations from the Monte Carlo simulation (the log returns of exchange rates and future prices). The objective and constraint functions being nonlinear, we obtained the best results using the interior point optimization method, closely followed by the active set algorithm.

4 Backtesting

4.1 Dataset

In order to demonstrate the validity of the model proposed in the previous sections, we compare its performance over a data set against alternative strategies described in Section 4.2. The data set consists of daily holdings of five futures contracts denominated in non-USD currencies from November 2003 to December 2012, for a total of 2115 observations. Figure 1 exhibits the future contracts prices and the number of each contract held at any moment. These holdings are exogeneous to our control; our aim is to find the optimal cash margins given certain holdings of contracts denominated in foreign currencies.

A graphical test proposed by Gnanadesikan (1977) to test joint normality was run on the data. If $\mathbf{X} = X_1, \ \dots \ , X_d$ is from a multivariate Gaussian distribution, then

$$(\mathbf{X}_i - \bar{\mathbf{X}})' \ \Sigma \ (\mathbf{X}_i - \bar{\mathbf{X}}), \ i = 1, \ \dots \ , N$$

where Σ is the covariance matrix of $\mathbf{X}$ have a χ^2_d-distribution. A QQ plot can then be used as a quick, "litmus" test of joint normality, as seen in Figure (2). Evidently joint normality has to be rejected.

4.2 Alternative strategies

In order to access the validity of our model, we backtested it on the dataset presented in the previous section. We also backtested two other strategies on the same dataset to contrast their performances.

The first, "naive" strategy simply consists in keeping twice the compulsory collateral in the cash accounts at all times. That is, the collateral posted in the accounts of the different currencies is brought back to twice the mandatory

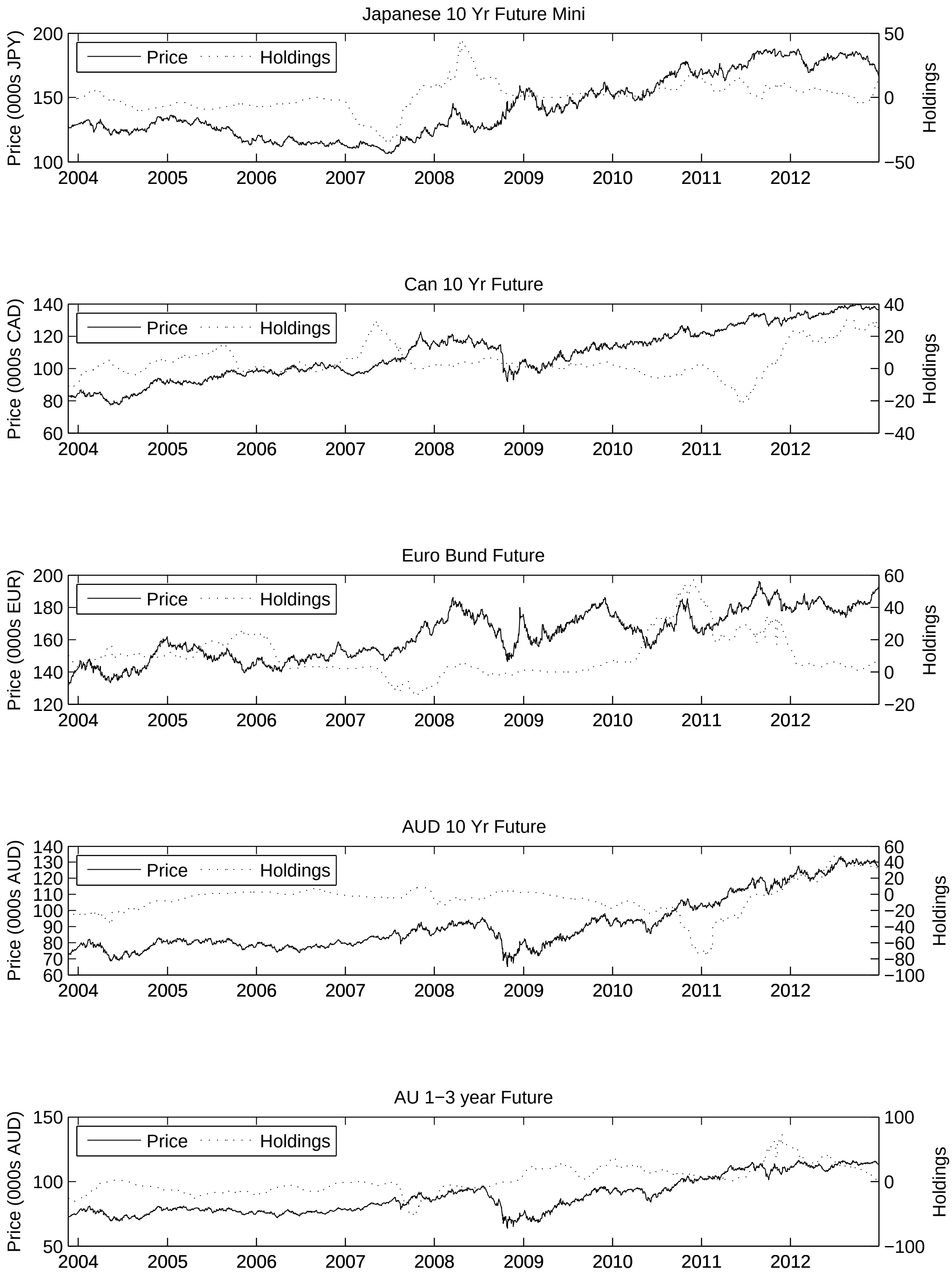

Figure 1: Futures contracts prices and holdings

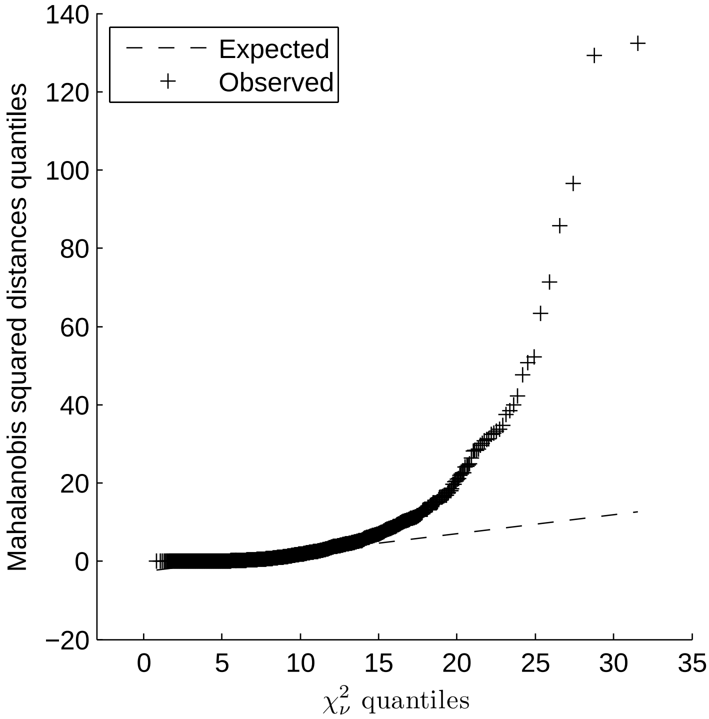

Figure 2: Graphical test of joint normality - QQ plot

minimum every day according to that day's movements in futures contracts prices and holdings.

The second strategy consists in modeling the returns of the futures contracts and exchange rates with a static multivariate normal distribution as in the Markowitz framework. Doing so allows us to gauge the accuracy added by factoring in stochastic volatility and higher co-moments as done by the model proposed in Section 2.

4.3 Calibration

A buffer of 500 business days (approximately 2 years) is used at the beginning of the sample to calibrate the marginal processes and the dependence copula parameters. The means and covariance matrix for the static multivariate strategy is also computed with this buffer. Both models are recalibrated every day using all data from the beginning of the sample (i.e. with an extending window).

The tolerance for the probability of a margin call P_{tol} was arbitrarily set to 0.05, which was also the level chosen for the quantile α of the VaR and TCE measures.

Once at the beginning and then every 250 business days, with an extending window, the goodness-of-fit tests described in Appendix A were performed on both the marginal processes and the dependence copulas. The results of the goodness-of-fit tests for the AR(1)-GARCH(1,1) with Gaussian residuals, AR(2)-GARCH(2,2) with Gaussian residuals, AR(1)-GARCH(1,1) with Student residuals and different dependence copulas are displayed in tables 2, 3, 4 and 5 respectively. Clearly, a combination of AR(1)-GARCH(1,1) with Student residuals and the Student copula is the only appropriate model from a statistical point of view for the time-series at hand.

4.4 Results

Tables 6, 7 and 8 exhibit the results of the backtests over the entire sample for the three risk measures of interest. The copula-based multivariate dynamic approach always had the lowest frequency of margin calls, while maintaining similar risk measures realizations. The quality of a risk management strategy is known during rough, volatile markets environments. We thus observed how our proposed model fared during the 2008-2009 financial crisis. We chose the period form March 17^{th} 2008, the fall of Bear Stearns, followed not long after by Lehman Brothers and American Insurance Group (AIG), to February 17^{th}

	Feb06	Mar07	Apr08	Apr09	May10	Jun11	Jun12
Japanese 10 Yr Future Mini	0.04	0.09	0.00	0.00	0.00	0.00	0.00
Can 10 Yr Future	0.11	0.55	0.55	0.03	0.02	0.01	0.01
Euro Bund Future	0.00	0.00	0.01	0.00	0.00	0.00	0.00
AUD 10 Yr Future	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AU 1-3 year Future	0.07	0.07	0.00	0.01	0.00	0.00	0.00
AUD/USD	0.44	0.13	0.00	0.00	0.00	0.00	0.00
CAD/USD	0.15	0.26	0.08	0.00	0.00	0.00	0.00
EUR/USD	0.00	0.01	0.00	0.00	0.00	0.00	0.00
JPY/USD	0.05	0.02	0.00	0.00	0.00	0.00	0.00

Table 2: p-values from the goodness-of-fit tests of AR(1)-GARCH(1,1) with Gaussian residuals on the marginal processes. The number of bootstrapped samples is $N = 100$.

	Feb06	Mar07	Apr08	Apr09	May10	Jun11	Jun12
Japanese 10 Yr Future Mini	0.06	0.10	0.02	0.00	0.00	0.00	0.00
Can 10 Yr Future	0.12	0.48	0.39	0.01	0.00	0.00	0.00
Euro Bund Future	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AUD 10 Yr Future	0.01	0.00	0.00	0.00	0.00	0.00	0.00
AU 1-3 year Future	0.04	0.01	0.00	0.00	0.00	0.00	0.00
AUD/USD	0.33	0.09	0.00	0.00	0.00	0.00	0.00
CAD/USD	0.09	0.35	0.11	0.00	0.00	0.00	0.00
EUR/USD	0.01	0.00	0.00	0.00	0.00	0.00	0.00
JPY/USD	0.07	0.01	0.00	0.00	0.00	0.00	0.00

Table 3: p-values from the goodness-of-fit tests of AR(2)-GARCH(2,2) with Gaussian residuals on the marginal processes. The number of bootstrapped samples is $N = 100$

	Feb06	Mar07	Apr08	Apr09	May10	Jun11	Jun12
Japanese 10 Yr Future Mini	0.42	0.57	0.62	0.51	0.55	0.58	0.41
Can 10 Yr Future	0.55	0.70	0.80	0.72	0.81	0.80	0.88
Euro Bund Future	0.34	0.42	0.43	0.28	0.33	0.45	0.56
AUD 10 Yr Future	0.58	0.54	0.54	0.49	0.63	0.51	0.43
AU 1-3 year Future	0.56	0.70	0.59	0.45	0.60	0.45	0.60
AUD/USD	0.71	0.70	0.55	0.49	0.51	0.42	0.48
CAD/USD	0.46	0.55	0.50	0.46	0.56	0.61	0.71
EUR/USD	0.32	0.53	0.36	0.26	0.44	0.47	0.57
JPY/USD	0.56	0.57	0.49	0.51	0.49	0.50	0.47

Table 4: p-values from the goodness-of-fit tests of AR(1)-GARCH(1,1) with Student residuals on the marginal processes. The number of bootstrapped samples is $N = 100$.

	Feb06	**Mar07**	**Apr08**	**Apr09**	**May10**	**Jun11**	**Jun12**
MV Gaussian	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AR(1)-GARCH(1,1) & Gaussian copula	0.13	0.04	0.00	0.00	0.00	0.00	0.00
AR(1)-GARCH(1,1) & Student copula	0.64	0.17	0.34	0.21	0.17	0.23	0.19
AR(1)-GARCH(1,1) & Clayton copula	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AR(1)-GARCH(1,1) & Frank copula	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AR(1)-GARCH(1,1) & Gumbel copula	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 5: p-values from the goodness-of-fit tests of copulas. The marginal processes were modeled with AR(1)-GARCH(1,1) with Student residuals. The number of bootstrapped samples is $N = 100$.

2009, when the American Recovery and Reinvestment Act was passed. As we now know this date did not mark the end of the crisis, but it does represent a milestone when volatility in the futures and exchange rates markets declined. Tables 9, 10 and 11 exhibit the results of the backtests over this critical period. During the crisis, the copula-based multivariate dynamic model clearly outperformed the two other strategies. Though it did breach the limits on the frequency of margin calls, it did so in a way much less drastic than the naive and Gaussian strategies, while realizing similar risk statistics.

	Naive	MV Gaussian	AR-GARCH & t copula
Avg. daily tracking error (% collateral)	0.52	0.52	0.52
# of margin calls (out of 1614 days)	110	99	85
Frequency of margin call	0.07	0.06	0.05

Table 6: Results of the backtests for the three strategies for the complete dataset period (November 2003 to March 2012) when the optimization objective is set to minimize the tracking error.

	Naive	MV Gaussian	AR-GARCH & t copula
Realized daily VaR (% collateral)	1.12	1.20	1.13
# of margin calls (out of 1614 days)	110	90	89
Frequency of margin call	0.07	0.06	0.06

Table 7: Results of the backtests for the three strategies for the complete dataset period (November 2003 to March 2012) when the optimization objective is set to minimize the Value-at-Risk.

	Naive	MV Gaussian	AR-GARCH & t copula
Avg. daily tail loss (% collateral)	-1.75	-1.79	-1.79
# of margin calls (out of 1614 days)	110	99	83
Frequency of margin call	0.07	0.06	0.05

Table 8: Results of the backtests for the three strategies for the complete dataset period (November 2003 to March 2012) when the optimization objective is set to maximize the Tail Conditional Expectation

	Naive	MV Gaussian	AR-GARCH & t copula
Avg. daily tracking error (% collateral)	0.75	0.78	0.79
# of margin calls (out of 220 days)	38	47	20
Frequency of margin call	0.17	0.21	0.09

Table 9: Results of the backtests for the three strategies for the crisis subperiod (March 2008 to February 2009) when the optimization objective is set to minimize the tracking error.

	Naive	MV Gaussian	AR-GARCH & t copula
Realized daily VaR (% collateral)	2.09	2.01	2.13
# of margin calls (out of 220 days)	38	38	22
Frequency of margin call	0.17	0.17	0.10

Table 10: Results of the backtests for the three strategies for the crisis subperiod (March 2008 to February 2009) when the optimization objective is set to minimize the Value-at-Risk.

	Naive	MV Gaussian	AR-GARCH & t copula
Avg. daily tail loss (% collateral)	-2.81	-2.70	-2.79
# of margin calls (out of 220 days)	38	43	15
Frequency of margin call	0.17	0.20	0.07

Table 11: Results of the backtests for the three strategies for the crisis sub-period (March 2008 to February 2009) when the optimization objective is set to maximize the Tail Conditional Expectation.

5 Conclusion

We met three objectives in this work; first, we proposed a model that better fits high-dimensional multivariate financial time-series than the classical Gaussian model with the copula-based multivariate dynamic model. Second, we used recent advances in absolute goodness-of-fit tests to show the appropriateness of the chosen model from a statistical point of view. Third, we used the proposed model to solve a problem encountered by managers whose portfolio contains multiple assets on different exchanges and/or in different currencies, that is, how to balance the opposing nuisances of margin calls and foreign exchange risk on collateral posted in a currency different from the participant's benchmark currency. The solution lies in calibrating the model proposed in this work (or another robust model encapsulating stochastic volatility, excess kurtosis and higher co-moments) to the financial time series of interest. Then the desired constrained optimization is solved using Monte Carlo methods.

Two potential avenues of future study are improvements in model sophistication and the development of rigorous models for problems with high numbers of random variables. On the first front, models including time-varying copula parameters, regime-switching and/or correlation asymmetry may yield a better fit to financial time series. On the second front, portfolios often have a large number of assets in many different currencies, however the use of copula becomes exponentially harder when the number of dimension is above 10. Methods for dimensionality reduction that conserve higher moments in the factors or efficiency improvements in algorithms for copula use would help toward accomplishing this objective.

A Goodness-of-fit tests

Responsible modeling requires proper testing as to whether the modeled data do in fact belong to the chosen model. When it comes to dynamic models, many authors either omit goodness-of-fit (GOF) tests, or use only relative tests that ranks the fit of different models. The problem with the latter approach is that picking the model with the best fit out of a set of incorrect models will still yield an incorrect model. Fortunately, absolute GOF tests for dynamic models based on parametric bootstrapping have recently been made available in the literature (Genest et al., 2009, Rémillard, 2011).

The null hypothesis of the GOF test for a general dynamic univariate process

X can be stated as follows:

> H_0: The conditional distribution of X_t given $\mathcal{F}_{t-1}$ is $F_{t,\theta}$, for some parameter $\theta \subseteq \mathcal{O}$.

Under the null hypothesis, it can be shown that $U_1 = F_{1,\theta}(X_1), \ldots, U_T = F_{T,\theta}$ are i.i.d. uniform variates on $(0,1)$. A general recipe to use parametric bootstrapping for the hypothesis is as follow:

(i) Estimate the parameter θ on the process $X_1, \ \ldots \ , X_T$ by $\hat{\theta}$.

(ii) Compute a distance statistic S_T between the uniform distribution function and the distribution function F_T of the pseudo-observations $u_1 = F_{1,\hat{\theta}}(X_1), \ldots, u_T = F_{T,\hat{\theta}}(X_T)$. A good candidate is the Cramér-von Mises criterion:

$$S_T = \int_0^1 \{F_T(u) - u\}^2 \, du.$$

(iii) Generate a large number $k = 1, \ \ldots \ , N$ of random sequences $X_1^{(k)}, \ldots, X_T^{(k)}$ from the dynamic model with parameters $\hat{\theta}$.

(iv) For each k from step (iii):

   (a) Estimate the parameter θ by $\theta^{(k)}$ for the sample $X_1^{(k)}, \ \ldots \ , X_T^{(k)}$.

   (b) Compute the same distance statistic $S_T^{(k)}$ as in step (ii) for the sample $X_1^{(k)}, \ \ldots \ , X_T^{(k)}$.

(v) The p-value of the test is approximated by the fraction of values $S_T^{(k)}$ greater than the S_T computed in step (ii):

$$p = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1} \left(S_T^{(k)} > S_T \right).$$

As for the null hypothesis of the GOF test of a copula-based multivariate dynamic model, it is of the form

> H_0: The copula associated with the innovations $\epsilon_t = (\epsilon_{1,t}, \ldots, \epsilon_{D,t})$, $t = 1, \ \ldots \ , T$, belongs to a parametric family $C_{\mathbf{\Theta}}$.

The procedure for the parametric bootstrap is:

(i) Estimate the parameters of each univariate marginal process and compute the associated standardized residuals $e_t = (e_{1,t}, \dots, e_{D,t})$, $t = 1, \dots, T$.

(ii) Compute the normalized ranks $u_{i,t},\ i = 1,\ \dots\ , D,\ t = 1,\ \dots\ , T$ of the standardized residuals resulting from step (i):

$$u_{i,t} = \frac{1}{T+1} \sum_{k=1}^{T} \mathbb{1}(e_{i,t} \geq e_{i,k}).$$

(iii) Estimate the dependence parameter $\mathbf{\Theta}$ of the parametric copula by $\hat{\mathbf{\Theta}}$, using the normalized ranks resulting from step (ii).

(iv) Compute a distance statistic S_T between the empirical copula C_T of the normalized ranks and the parametric copula $C_{\hat{\mathbf{\Theta}}}$, where the empirical copula is given by

$$C_T(x_1,\ \dots\ , x_D) = \frac{1}{T} \sum_{t=1}^{T} \prod_{i=1}^{D} \mathbb{1}(u_{i,t} \leq x_i).$$

A good candidate for S_T is the Cramér-von Mises statistic

$$S_T = \frac{1}{T} \sum_{t=1}^{T} \left\{ C_T(u_{1,t},\ \dots\ , u_{D,t}) - C_\theta(u_{1,t},\ \dots\ , u_{D,t}) \right\}^2 .$$

(v) For some large integer N, repeat the following steps for each k in $[1, \dots, N]$:

(a) Generate random vectors $\mathbf{U}_1^{(k)}, \dots, \mathbf{U}_T^{(k)}$ with distribution $C_{\hat{\mathbf{\Theta}}}$. Most existing statistical packages does not currently support the generation of multivariate Archimedean copulas random variables; recipes to do so are proposed in Marshall and Olkin (1988).

(b) Repeat steps (ii) to (iv) on trajectories generated in (a) to obtain $S_T^{(k)},\ k = 1,\ \dots\ , N$.

(vi) The approximate p-value for the test is given by

$$p = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}\left(S_T^{(k)} > S_T \right).$$

Using the above procedure to test for the goodness-of-fit of elliptical copulas can prove tedious because there exists no explicit form for their cumulative distribution functions and Monte Carlo integration of equations (5) or (6) requires excessive computational resources for any non-small number of dimensions. An ingenious alternative is described by Genest et al. (2009). The method uses a critical property of Rosenblatt's probability integral transform $\mathcal{T}$ (Rosenblatt, 1952), namely that a multivariate vector $\mathbf{U} \in [0,1]$ has distribution function C if and only if the Rosenblatt transform of $\mathbf{U}$ has the independence copula as distribution function $C_{\perp}$:

$$\mathbf{U} \sim C \;\Leftrightarrow\; \mathcal{T}(\mathbf{U}) \sim C_{\perp}$$

The algorithm of the parametric bootstrap test applied to the elliptical copulas is as follow:

1. Estimate the parameters of each univariate marginal process and compute the associated standardized residuals $e_t = (e_{1,t}, \ldots, e_{D,t})$, $t = 1, \ldots, T$.

2. Compute the normalized ranks $\mathbf{u}_t = (u_{1,t}, \ldots, u_{D,t})$, where for $i = 1, \ldots, D$, and $t = 1, \ldots, T$,

$$u_{i,t} = \frac{1}{T+1} \sum_{k=1}^{T} \mathbb{1}(e_{i,t} \geq e_{i,k}).$$

3. Estimate the dependence parameter $\boldsymbol{\Theta}$ of the parametric copula by $\hat{\boldsymbol{\Theta}}$, using the normalized ranks resulting from step (2).

4. Compute Rosenblatt transforms $\mathbf{v}_t = (v_{1,t}, \ldots, v_{D,t}) = \mathcal{T}_{\hat{\boldsymbol{\Theta}}}(\mathbf{u}_t)$, $t = 1, \ldots, T$.

5. Compute a distance statistic S_T between the empirical copula C_T of the Rosenblatt transforms and the independence copula $C_{\perp}$. The empirical copula is given by

$$C_T(x_1, \ldots, x_D) = \frac{1}{T} \sum_{t=1}^{T} \prod_{i=1}^{D} \mathbb{1}(v_{i,t} \leq x_i).$$

The Cramér-von Mises criterion in this case is given by:

$$
\begin{aligned}
S_T &= T \int_{[0,1]^D} \{F_T(\mathbf{v}) - C_\perp(\mathbf{v})\}^2 d\mathbf{v} \\
&= \frac{T}{3^D} - \frac{1}{2^{D-1}} \sum_{t=1}^{T} \prod_{i=1}^{D} \left(1 - v_{i,t}^2\right) + \frac{1}{T} \sum_{t=1}^{T} \sum_{k=1}^{T} \prod_{i=1}^{D} (1 - \max(v_{i,t}, v_{i,k})).
\end{aligned}
$$

6. For some large integer N, repeat the following steps for each k in $[1, \ldots, N]$:
   (a) Generate random vectors $\mathbf{U}_1^{(k)}, \ldots, \mathbf{U}_T^{(k)}$ with distribution $C_{\hat{\boldsymbol{\Theta}}}$.
   (b) Repeat steps (2) to (4) on trajectories generated in (a) to obtain $S_T^{(k)},\ k = 1,\ \ldots\ , N$.
7. The approximate p-value for the test is given by

$$
p = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}\left(S_T^{(k)} > S_T\right).
$$

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