Additional Cheat Sheet

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The Econometrics Cheat Sheet Project

OLS matrix notation

The general econometric model:

$$y_i = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_k x_{ki} + u_i$$

Can be written in matrix notation as:

$$y = X\beta + u$$

Let's call $\hat{u}$ the vector of estimated residuals ($\hat{u} \neq u$):

$$\hat{u} = y - X\hat{\beta}$$

The **objective** of OLS is to **minimize** the SSR:

$$\min \text{SSR} = \min \sum_{i=1}^n \hat{u}_i^2 = \min \hat{u}^\top \hat{u}$$

- Defining $\hat{u}^\top \hat{u}$:

$$\hat{u}^\top \hat{u} = (y - X\hat{\beta})^\top (y - X\hat{\beta})$$
$$= y^\top y - 2\hat{\beta}^\top X^\top y + \hat{\beta}^\top X^\top X \hat{\beta}$$

- Minimizing $\hat{u}^\top \hat{u}$:

$$\frac{\partial \hat{u}^\top \hat{u}}{\partial \hat{\beta}} = -2X^\top y + 2X^\top X\hat{\beta} = 0$$
$$\hat{\beta} = (X^\top X)^{-1}(X^\top y)$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} n & \sum x_1 & \dots & \sum x_k \\ \sum x_1 & \sum x_1^2 & \dots & \sum x_1 x_k \\ \vdots & \vdots & \ddots & \vdots \\ \sum x_k & \sum x_k x_1 & \dots & \sum x_k^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sum y \\ \sum y x_1 \\ \vdots \\ \sum y x_k \end{bmatrix}$$

The second derivative $\frac{\partial^2 \hat{u}^\top \hat{u}}{\partial \hat{\beta}^2} = X^\top X > 0$ (is a min.)

Variance-covariance matrix of $\hat{\beta}$

Has the following shape:

$$\text{Var}(\hat{\beta}) = \hat{\sigma}_u^2 \cdot (X^\top X)^{-1}$$
$$= \begin{bmatrix} \text{Var}(\hat{\beta}_0) & \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) & \dots & \text{Cov}(\hat{\beta}_0, \hat{\beta}_k) \\ \text{Cov}(\hat{\beta}_1, \hat{\beta}_0) & \text{Var}(\hat{\beta}_1) & \dots & \text{Cov}(\hat{\beta}_1, \hat{\beta}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(\hat{\beta}_k, \hat{\beta}_0) & \text{Cov}(\hat{\beta}_k, \hat{\beta}_1) & \dots & \text{Var}(\hat{\beta}_k) \end{bmatrix}$$

where: $\hat{\sigma}_u^2 = \frac{\hat{u}^\top \hat{u}}{n-k-1}$

The standard errors are on the diagonal of:

$$\text{se}(\hat{\beta}) = \sqrt{\text{Var}(\hat{\beta})}$$

Error measurements

- $\text{SSR} = \hat{u}^\top \hat{u} = y^\top y - \hat{\beta}^\top X^\top y = \sum (y_i - \hat{y}_i)^2$
- $\text{SSE} = \hat{\beta}^\top X^\top y - n\overline{y}^2 = \sum (\hat{y}_i - \overline{y})^2$
- $\text{SST} = \text{SSR} + \text{SSE} = y^\top y - n\overline{y}^2 = \sum (y_i - \overline{y})^2$

Variance-covariance matrix of u

Has the following shape:

$$\text{Var}(u) = \begin{bmatrix} \text{Var}(u_1) & \text{Cov}(u_1, u_2) & \dots & \text{Cov}(u_1, u_n) \\ \text{Cov}(u_2, u_1) & \text{Var}(u_2) & \dots & \text{Cov}(u_2, u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(u_n, u_1) & \text{Cov}(u_n, u_2) & \dots & \text{Var}(u_n) \end{bmatrix}$$

Under no heteroscedasticity and no autocorrelation, the variance-covariance matrix:

$$\text{Var}(u) = \sigma_u^2 \cdot I_n = \begin{bmatrix} \sigma_u^2 & 0 & \dots & 0 \\ 0 & \sigma_u^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_u^2 \end{bmatrix}$$

where I_n is an identity matrix of $n \times n$ elements.

Under **heteroscedasticity** and **autocorrelation**, the variance-covariance matrix:

$$\text{Var}(u) = \sigma_u^2 \cdot \Omega = \begin{bmatrix} \sigma_{u_1}^2 & \sigma_{u_{12}} & \dots & \sigma_{u_{1n}} \\ \sigma_{u_{21}} & \sigma_{u_2}^2 & \dots & \sigma_{u_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{u_{n1}} & \sigma_{u_{n2}} & \dots & \sigma_{u_n}^2 \end{bmatrix}$$

where $\Omega \neq I_n$.

- Heteroscedasticity: $\text{Var}(u) = \sigma_{u_i}^2 \neq \sigma_u^2$
- Autocorrelation: $\text{Cov}(u_i, u_j) = \sigma_{u_{ij}} \neq 0, \ \forall i \neq j$

Variable omission

Most of the time, it is hard to get all relevant variables for an analysis. For example, a true model with all variables:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + v$$

where $\beta_2 \neq 0$, v is the error term and $\text{Cov}(v \mid x_1, x_2) = 0$.

The model with the available variables:

$$y = \alpha_0 + \alpha_1 x_1 + u$$

where $u = v + \beta_2 x_2$.

Relevant variable omission can cause OLS estimators to be **biased** and **inconsistent**, because there is no weak exogeneity, $\text{Cov}(x_1, u) \neq 0$. Depending on the $\text{Corr}(x_1, x_2)$ and the sign of β_2, the bias on $\hat{\alpha}_1$ could be:

	$\text{Corr}(x_1, x_2) > 0$	$\text{Corr}(x_1, x_2) < 0$
$\beta_2 > 0$	$(+)$ bias	$(-)$ bias
$\beta_2 < 0$	$(-)$ bias	$(+)$ bias

- $(+)$ bias: $\hat{\alpha}_1$ will be higher than it should be (it includes the effect of x_2) $\rightarrow \hat{\alpha}_1 > \beta_1$
- $(-)$ bias: $\hat{\alpha}_1$ will be lower than it should be (it includes the effect of x_2) $\rightarrow \hat{\alpha}_1 < \beta_1$

If $\text{Corr}(x_1, x_2) = 0$, there is no bias on $\hat{\alpha}_1$, because the effect of x_2 will be fully captured by the error term, u.

Variable omission correction

Proxy variables

Is the approach when a relevant variable is not available because it is non-observable, and no data is available.

- A **proxy variable** is something **related** to the non-observable variable that has data available.

For example, the GDP per capita is a proxy variable for life quality (non-observable).

Instrumental variables

When the variable of interest (x) is observable but **endogenous**, the proxy variables approach is no longer valid.

- An **instrumental variable** (IV) **is an observable variable** (z) that is **related** to the variable of interest that is endogenous (x), and meets the **requirements**:

$$\text{Cov}(z, u) = 0 \rightarrow \text{instrument exogeneity}$$
$$\text{Cov}(z, x) \neq 0 \rightarrow \text{instrument relevance}$$

Instrumental variables let the omitted variable into the error term, but instead of estimating the model by OLS, it utilizes a method that recognizes the presence of an omitted variable. It can also solve error measurement problems.

- **Two-Stage Least Squares** (TSLS) is a method to estimate a model with multiple instrumental variables. The $\text{Cov}(z, u) = 0$ can be relaxed, but there has to be a minimum number of variables that satisfy it.

  The TSLS **estimation procedure** is as follows:

  1. Estimate a model regressing x by z using OLS, obtaining $\hat{x}$:
  $$\hat{x} = \hat{\pi}_0 + \hat{\pi}_1 z$$
  2. Replace x by $\hat{x}$ in the final model and estimate it by OLS:
  $$y = \beta_0 + \beta_1 \hat{x} + u$$

  There are some important things to know about TSLS:

  - TSLS estimators are less efficient than OLS when the explanatory variables are exogenous; the **Hausman test** can be used to check this:

    H_0: OLS estimators are consistent.

    If H_0 is not rejected, the OLS estimators are better than TSLS and vice versa.
  - When more instruments than endogenous variables are used, the model may be over-identified; the **Sargan test** can be used to check this:

    H_0: All instruments are valid.

Information criterion

Compare models with different numbers of parameters (p). The general formula:

$$\text{Cr}(p) = \log(\tfrac{\text{SSR}}{n}) + c_n \varphi(p)$$

where:

- SSR from a model of order p.
- c_n is a sequence indexed by the sample size.
- $\varphi(p)$ is a function that penalizes large p orders.

It is interpreted as the relative amount of information lost by the model. The order p that min. the criterion is chosen.

There are different $c_n \varphi(p)$ functions:

- Akaike: $\text{AIC}(p) = \log(\frac{\text{SSR}}{n}) + \frac{2}{n} p$
- Hannan-Quinn: $\text{HQ}(p) = \log(\frac{\text{SSR}}{n}) + \frac{2\log(\log(n))}{n} p$
- Schwarz / Bayesian: $\text{BIC}(p) = \log(\frac{\text{SSR}}{n}) + \frac{\log(n)}{n} p$

$\text{BIC}(p) \leq \text{HQ}(p) \leq \text{AIC}(p)$

The non-restricted hypothesis test

An alternative to the F test when there are few hypothesis to test on the parameters. Let β_i, β_j be parameters, $a, b, c \in \mathbb{R}$ are constants.

- $H_0 : a\beta_i + b\beta_j = c$
- $H_1 : a\beta_i + b\beta_j \neq c$

$$\text{Under } H_0: \quad t = \frac{a\hat{\beta}_i + b\hat{\beta}_j - c}{\text{se}(a\hat{\beta}_i + b\hat{\beta}_j)}$$

$$= \frac{a\hat{\beta}_i + b\hat{\beta}_j - c}{\sqrt{a^2 \operatorname{Var}(\hat{\beta}_i) + b^2 \cdot \operatorname{Var}(\hat{\beta}_j) + 2ab \operatorname{Cov}(\hat{\beta}_i, \hat{\beta}_j)}}$$

If $|t| > |t_{n-k-1, \alpha/2}|$, there is evidence to reject H_0.

ANOVA

Decompose SST:

Variation origin	Sum Sq.	df	Sum Sq. Avg.
Regression	SSE	k	SSE$/k$
Residuals	SSR	$n-k-1$	SSR$/(n-k-1)$
Total	SST	$n-1$	

- $H_0 : \beta_1 = \beta_2 = \cdots = \beta_k = 0$
- $H_1 : \beta_1 \neq 0$ and/or $\beta_2 \neq 0 \ldots$ and/or $\beta_k \neq 0$

Under H_0:

$$F = \frac{\text{SSA of SSE}}{\text{SSA of SSR}} = \frac{\text{SSE}}{\text{SSR}} \cdot \frac{n-k-1}{k} \sim F_{k, n-k-1}$$

If $F > F_{k,n-k-1}$, there is evidence to reject H_0.

Panel data

Observations on n entities over T periods.

$$y_{it} = X_{it}\beta + \alpha_i + u_{it}$$

α_i represents the time-invariant unobserved heterogeneity.

Pooled OLS model

- Apply OLS to the data directly.
- Assumption: α_i is constant.

Fixed effects model (within estimator)

$$y_{it} - \overline{y}_i = (X_i - \overline{X}_{it})\beta + (\alpha_i - \overline{\alpha}_i) + (u_{it} - \overline{u}_i)$$

- Demeaning is performed to remove α_i.
- Control for unobserved entity-specific effects.
- Assumption: $\text{Corr}(X_{it}, \alpha_i) \neq 0$.

Least square dummy variable model (LSDV)

Dummy variables are added for each entity and/or time period to capture the fixed effects.

First difference model

$$y_{it} - y_{i,t-1} = (X_{it} - X_{i,t-1})\beta + (\alpha_i - \alpha_i) + (u_{it} - u_{i,t-1})$$

- First differences are performed to remove α_i.
- Assumption: $\text{Corr}(u_{it} - u_{i,t-1} \mid X_{it} - X_{i,t-1}) = 0$.

Random effects model

$$y_{it} = X_{it}\beta + \alpha_i + \epsilon_{it} \text{ where } u_{it} = \alpha_i + \epsilon_{it}$$

- Assumption: $\text{Corr}(X_{it}, \alpha_i) = 0$.

Logistic regression

Binary (0, 1) dependent variable. **Logit model**:

$$P_i = \frac{1}{1 + e^{-(\beta_0 + \beta_1 x_i + u_i)}} = \frac{e^{\beta_0 + \beta_1 x_i + u_i}}{1 + e^{\beta_0 + \beta_1 x_i + u_i}}$$

where $P_i = \text{E}(y_i = 1 \mid x_i)$ and $(1 - P_i) = \text{E}(y_i = 0 \mid x_i)$

The **odds ratio** (in favor of $y_i = 1$):

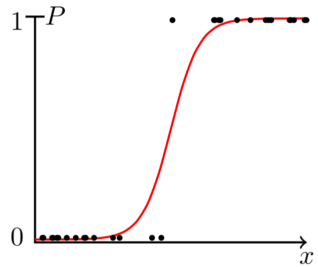
$$\frac{P_i}{1 - P_i} = \frac{1 + e^{\beta_0 + \beta_1 x_i + u_i}}{1 + e^{-(\beta_0 + \beta_1 x_i + u_i)}} = e^{\beta_0 + \beta_1 x_i + u_i}$$

Taking the natural logarithm of the odds ratio, the **logit**:

$$L_i = \ln\left(\frac{P_i}{1 - P_i}\right) = \beta_0 + \beta_1 x_i + u_i$$

P_i is between 0 and 1, but L_i goes from $-\infty$ to $+\infty$.

If L_i is positive, it means that when x_i increases, the probability of $y_i = 1$ increases, and vice versa.



Incorrect functional form

Ramsey's RESET (Regression Specification Error Test).

H_0: The model is correctly specified.

1. Estimate the original model and obtain $\hat{y}$ and R^2:
$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \cdots + \hat{\beta}_k x_k$$
2. Estimate a model adding powers of $\hat{y}$ and obtain R^2_{new}:
$$\tilde{y} = \hat{y} + \tilde{\gamma}_2 \hat{y}^2 + \cdots + \tilde{\gamma}_l \hat{y}^l$$
3. Test statistic, under $\gamma_2 = \cdots = \gamma_l = 0$ as H_0:
$$F = \frac{R^2_{\text{new}} - R^2}{1 - R^2_{\text{new}}} \cdot \frac{n-(k+1)-l}{l} \sim F_{l, n-(k+1)-l}$$

If $F > F_{l,n-(k+1)-l}$, there is evidence to reject H_0.

Statistical definitions

Let ξ, η be random variables, $a, b \in \mathbb{R}$ be constants, and P denotes probability.

Mean $\quad E(\xi) = \sum_{i=1}^n \xi_i \cdot P[\xi = \xi_i]$

Sample mean: $\quad \text{E}(\xi) = \dfrac{1}{n} \sum_{i=1}^n \xi_i$

Properties of the mean:

- $\text{E}(a) = a$
- $\text{E}(\xi + a) = \text{E}(\xi) + a$
- $\text{E}(a \cdot \xi) = a \cdot \text{E}(\xi)$
- $\text{E}(\xi \pm \eta) = \text{E}(\xi) + \text{E}(\eta)$
- $\text{E}(\xi \cdot \eta) = \text{E}(\xi) \cdot \text{E}(\eta)$ only if ξ and η are independent.
- $\text{E}(\xi - \text{E}(\xi)) = 0$
- $\text{E}(a \cdot \xi + b \cdot \eta) = a \cdot \text{E}(\xi) + b \cdot \text{E}(\eta)$

Variance $\quad \operatorname{Var}(\xi) = \text{E}\left[(\xi - \text{E}(\xi))^2\right]$

Sample variance: $\quad \operatorname{Var}(\xi) = \dfrac{\sum_{i=1}^n (\xi_i - \text{E}(\xi))^2}{n-1}$

Properties of the variance:

- $\operatorname{Var}(a) = 0$
- $\operatorname{Var}(\xi + a) = \operatorname{Var}(\xi)$
- $\operatorname{Var}(a \cdot \xi) = a^2 \cdot \operatorname{Var}(\xi)$
- $\operatorname{Var}(\xi \pm \eta) = \operatorname{Var}(\xi) + \operatorname{Var}(\eta) \pm 2 \cdot \operatorname{Cov}(\xi, \eta)$
- $\operatorname{Var}(a \cdot \xi \pm b \cdot \eta) = a^2 \cdot \operatorname{Var}(\xi) + b^2 \cdot \operatorname{Var}(\eta) \pm 2ab \cdot \operatorname{Cov}(\xi, \eta)$

Covariance $\quad \operatorname{Cov}(\xi, \eta) = \text{E}\left[(\xi - E(\xi)) \cdot (\eta - E(\eta))\right]$

Sample covariance: $\quad \dfrac{\sum_{i=1}^n (\xi_i - \text{E}(\xi)) \cdot (\eta_i - \text{E}(\eta))}{n-1}$

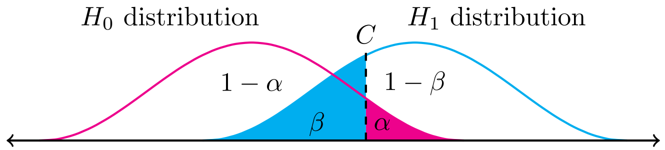
Properties of the covariance:

- $\operatorname{Cov}(\xi, a) = 0$
- $\operatorname{Cov}(\xi + a, \eta + b) = \operatorname{Cov}(\xi, \eta)$
- $\operatorname{Cov}(a \cdot \xi, b \cdot \eta) = ab \cdot \operatorname{Cov}(\xi, \eta)$
- $\operatorname{Cov}(\xi, \xi) = \operatorname{Var}(\xi)$
- $\operatorname{Cov}(\xi, \eta) = \operatorname{Cov}(\eta, \xi)$

Hypothesis testing

	H_0 true	H_0 false
Reject H_0	False positive Type I Error (α)	True positive $(1-\beta)$
Not reject H_0	True negative $(1-\alpha)$	False negative Type II Error (β)

Typical one-tail test:



where $(1-\alpha)$ is the confidence level, α is the significance level, C is the critical value, $(1-\beta)$ is the statistical power.

Bootstrapping

Problem - Asymptotic approximations to the distributions of test statistics do not work on small samples.
Solution - Bootstrap is sampling with replacement. The observed data is treated like a population, and multiple samples are extracted to recalculate an estimator or test statistic multiple times (improves accuracy).

VAR (Vector Autoregressive)

A VAR model captures **dynamic interactions** between time series. The VAR(p):

$$y_t = A_1 y_{t-1} + \cdots + A_p y_{t-p} + Bx_t + CD_t + u_t$$

where:

- $y_t = (y_{1t}, \ldots, y_{Kt})^\top$ is a vector of K observable endogenous time series.
- A_i's are $K \times K$ coefficient matrices.
- $x_t = (x_{1t}, \ldots, x_{Mt})^\top$ is a vector of M observable exogenous time series.
- B is a $K \times M$ coefficient matrix.
- D_t is a vector that contains all deterministic terms: a constant, linear trend, seasonal dummy, and/or any other user-specified dummy variables.
- C is a coefficient matrix of suitable dimension.
- $u_t = (u_{1t}, \ldots, u_{Kt})^\top$ is a vector of K white noise series.

Stability condition:

$$\det(I_K - A_1 z - \cdots - A_p z^p) \neq 0 \quad \text{for} \quad |z| \leq 1$$

this is, there are **no roots** in and on the complex unit circle.

For example, a VAR model with two endogenous variables ($K = 2$), two lags ($p = 2$), an exogenous contemporaneous variable ($M = 1$), a constant (const) and a trend (Trend$_t$):

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} a_{11,1} & a_{12,1} \\ a_{21,1} & a_{22,1} \end{bmatrix} \cdot \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} a_{11,2} & a_{12,2} \\ a_{21,2} & a_{22,2} \end{bmatrix} \cdot \begin{bmatrix} y_{1,t-2} \\ y_{2,t-2} \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \cdot \begin{bmatrix} x_t \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \cdot \begin{bmatrix} \text{const} \\ \text{Trend}_t \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

Visualizing the separate equations:

$$y_{1t} = a_{11,1} y_{1,t-1} + a_{12,1} y_{2,t-1} + a_{11,2} y_{1,t-2} + a_{12,2} y_{2,t-2} + b_{11} x_t + c_{11} + c_{12} \text{Trend}_t + u_{1t}$$
$$y_{2t} = a_{21,1} y_{2,t-1} + a_{22,1} y_{1,t-1} + a_{21,2} y_{2,t-2} + a_{22,2} y_{1,t-2} + b_{21} x_t + c_{21} + c_{22} \text{Trend}_t + u_{2t}$$

If there is a unit root, the determinant is zero for $z = 1$; then some or all variables are integrated, and a VAR model is no longer appropriate (it becomes unstable).

SVAR (Structural VAR)

In a VAR model, causal interpretation is not explicit, and results are sensitive to variable ordering. A SVAR extends VAR by imposing theory-based restrictions on A and/or B matrices. This can enable causal interpretation and shock analysis without reliance on arbitrary ordering.

For example, a basic SVAR(p) model:

$$\mathsf{A} y_t = \mathsf{A}[A_1, \ldots, A_p] y_{t-1} + \mathsf{B} \varepsilon_t$$

where:

- $u_t = \mathsf{A}^{-1} \mathsf{B} \varepsilon_t$
- A, B are $(K \times K)$ matrices.

VECM (Vector Error Correction Model)

If **cointegrating relations** are present in a system of variables, the VAR form is not the most convenient. It is better to use a VECM, that is, the levels VAR, subtracting y_{t-1} from both sides. The VECM($p-1$):

$$\Delta y_t = \Pi y_{t-1} + \textstyle\sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} + Bx_t + CD_t + u_t$$

where:

- $\Delta y_t = (\Delta y_{1t}, \ldots, \Delta y_{Kt})^\top$ is a vector of K observable endogenous time series.
- Πy_{t-1} is the **long-term** part.
  - ◇ $\Pi = -(I_K - A_1 - \cdots - A_p)$ for $i = 1, \ldots, p-1$
  - ◇ $\Pi = \alpha \beta^\top$
  - ◇ α is the **loading matrix** $(K \times r)$. It represents the speed of adjustment.
  - ◇ β is the **cointegration matrix** $(K \times r)$.
  - ◇ $\beta^\top y_{t-1}$ is the **cointegrating equation**. It represents the long-run equilibrium.
  - ◇ $\text{rk}(\Pi) = \text{rk}(\alpha) = \text{rk}(\beta) = r$ is the **cointegrating rank**.
- $\Gamma_i = -(A_{i+1} + \cdots + A_p)$ for $i = 1, \ldots, p-1$ are the **short-term** parameters.
- x_t, B, C, D_t and u_t are as in VAR.

For example, a VECM with three endogenous variables ($K = 3$), two lags ($p = 2$) and two cointegrating relations ($r = 2$):

$$\Delta y_t = \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + u_t$$

where:

$$\Pi y_{t-1} = \alpha \beta^\top y_{t-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{32} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \end{bmatrix} = \begin{bmatrix} \alpha_{11} ec_{1,t-1} + \alpha_{12} ec_{2,t-1} \\ \alpha_{21} ec_{1,t-1} + \alpha_{22} ec_{2,t-1} \\ \alpha_{31} ec_{1,t-1} + \alpha_{32} ec_{2,t-1} \end{bmatrix}$$

$$ec_{1,t-1} = \beta_{11} y_{1,t-1} + \beta_{21} y_{2,t-1} + \beta_{31} y_{3,t-1}$$
$$ec_{2,t-1} = \beta_{12} y_{1,t-1} + \beta_{22} y_{2,t-1} + \beta_{32} y_{3,t-1}$$

and

$$\Gamma_1 \Delta y_{t-1} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \begin{bmatrix} \Delta y_{1,t-1} \\ \Delta y_{2,t-1} \\ \Delta y_{3,t-1} \end{bmatrix} \qquad u_t = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Visualizing the separate equations:

$$\Delta y_{1t} = \alpha_{11} ec_{1,t-1} + \alpha_{12} ec_{2,t-1} + \gamma_{11} \Delta y_{1,t-1} + \gamma_{12} \Delta y_{2,t-1} + \gamma_{13} \Delta y_{3,t-1} + u_{1t}$$
$$\Delta y_{2t} = \alpha_{21} ec_{1,t-1} + \alpha_{22} ec_{2,t-1} + \gamma_{21} \Delta y_{1,t-1} + \gamma_{22} \Delta y_{2,t-1} + \gamma_{23} \Delta y_{3,t-1} + u_{2t}$$
$$\Delta y_{3t} = \alpha_{31} ec_{1,t-1} + \alpha_{32} ec_{2,t-1} + \gamma_{31} \Delta y_{1,t-1} + \gamma_{32} \Delta y_{2,t-1} + \gamma_{33} \Delta y_{3,t-1} + u_{3t}$$

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