Time Series Cheat Sheet

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The Econometrics Cheat Sheet Project

Basic concepts

Definitions

Time series - is a succession of quantitative observations of a phenomena ordered in time.

There are some variations of time series:

- **Panel data** - consist of a time series for each observation of a cross section.
- **Pooled cross sections** - combines cross sections from different time periods.

Stochastic process - sequence of random variables that are indexed in time.

Components of a time series

- **Trend** - is the long-term general movement of a series.
- **Seasonal variations** - are periodic oscillations that are produced in a period equal or inferior to a year, and can be easily identified on different years (usually are the result of climatology reasons).
- **Cyclical variations** - are periodic oscillations that are produced in a period greater than a year (are the result of the economic cycle).
- **Residual variations** - are movements that do not follow a recognizable periodic oscillation (are the result of eventual non-permanent phenomena that can affect the studied variable in a given moment).

Type of time series models

- **Static models** - the relation between y and x's is contemporary. Conceptually:
$$y_t = \beta_0 + \beta_1 x_t + u_t$$
- **Distributed-lag models** - the relation between y and x's is not contemporary. Conceptually:
$$y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \cdots + \beta_s x_{t-(s-1)} + u_t$$
The long term cumulative effect in y when Δx is:
$$\beta_1 + \beta_2 + \cdots + \beta_s$$
- **Dynamic models** - a temporal drift of the dependent variable is part of the independent variables (endogeneity). Conceptually:
$$y_t = \beta_0 + \beta_1 y_{t-1} + \cdots + \beta_s y_{t-s} + u_t$$
- Combinations of the above, like the rational distributed-lag models (distributed-lag + dynamic).

Assumptions and properties

OLS model assumptions under time series

Under this assumptions, the OLS estimator will present good properties. **Gauss-Markov assumptions** extended applied to time series:

t1. **Parameters linearity and weak dependence**.
  a. y_t must be a linear function of the β's.
  b. The stochastic $\{(x_t, y_t) : t = 1, 2, \ldots, T\}$ is stationary and weakly dependent.

t2. **No perfect collinearity**.
  - There are no independent variables that are constant: $\mathrm{Var}(x_j) \neq 0, \ \forall j = 1, \ldots, k$
  - There is not an exact linear relation between independent variables.

t3. **Conditional mean zero and correlation zero**.
  a. There are no systematic errors: $\mathrm{E}(u \mid x_1, \ldots, x_k) = \mathrm{E}(u) = 0 \rightarrow$ **strong exogeneity** (a implies b).
  b. There are no relevant variables left out of the model: $\mathrm{Cov}(x_j, u) = 0, \ \forall j = 1, \ldots, k \rightarrow$ **weak exogeneity**.

t4. **Homoscedasticity**. The variability of the residuals is the same for any x: $\mathrm{Var}(u \mid x_1, \ldots, x_k) = \sigma_u^2$

t5. **No auto-correlation**. Residuals do not contain information about any other residuals:
$\mathrm{Corr}(u_t, u_s \mid x_1, \ldots, x_k) = 0, \ \forall t \neq s$

t6. **Normality**. Residuals are independent and identically distributed (**i.i.d.** so on): $u \sim \mathcal{N}(0, \sigma_u^2)$

t7. **Data size**. The number of observations available must be greater than $(k+1)$ parameters to estimate. (It is already satisfied under asymptotic situations)

Asymptotic properties of OLS

Under the econometric model assumptions and the Central Limit Theorem:

- Hold t1 to t3a: OLS is **unbiased**. $\mathrm{E}(\hat{\beta}_j) = \beta_j$
- Hold t1 to t3: OLS is **consistent**. $\mathrm{plim}(\hat{\beta}_j) = \beta_j$ (to t3b left out t3a, weak exogeneity, biased but consistent)
- Hold t1 to t5: **asymptotic normality** of OLS (then, t6 is necessarily satisfied): $u \underset{a}{\sim} \mathcal{N}(0, \sigma_u^2)$
- Hold t1 to t5: **unbiased estimate** of σ_u^2. $\mathrm{E}(\hat{\sigma}_u^2) = \sigma_u^2$
- Hold t1 to t5: OLS is BLUE (Best Linear Unbiased Estimator) or **efficient**.
- Hold t1 to t6: hypothesis testing and confidence intervals can be done reliably.

Trends and seasonality

Spurious regression - is when the relation between y and x is due to factors that affect y and have correlation with x, $\mathrm{Corr}(x_j, u) \neq 0$. Is the **non-fulfillment of t3**.

Trends

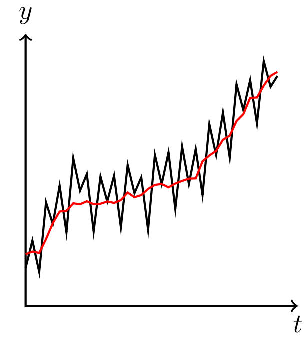
Two time series can have the same (or contrary) trend, that should lend to a high level of correlation. This, can provoke a false appearance of causality, the problem is **spurious regression**. Given the model:
$$y_t = \beta_0 + \beta_1 x_t + u_t$$
where:
$$y_t = \alpha_0 + \alpha_1 \mathrm{Trend} + v_t$$
$$x_t = \gamma_0 + \gamma_1 \mathrm{Trend} + v_t$$
Adding a trend to the model can solve the problem:
$$y_t = \beta_0 + \beta_1 x_t + \beta_2 \mathrm{Trend} + u_t$$
The trend can be linear or non-linear (quadratic, cubic, exponential, etc.)

Another way is make use of the **Hodrick-Prescott filter** to extract the trend and the cyclical component.

Seasonality

A time series with can manifest seasonality. That is, the series is subject to a seasonal variations or pattern, usually related to climatology conditions.

For example, GDP (black) is usually higher in summer and lower in winter. Seasonally adjusted series (red) for comparison.



- This problem is **spurious regression**. A seasonal adjustment can solve it.

A simple **seasonal adjustment** could be creating stationary binary variables and adding them to the model. For example, for quarterly series (Qq_t are binary variables):
$$y_t = \beta_0 + \beta_1 Q2_t + \beta_2 Q3_t + \beta_3 Q4_t + \beta_4 x_{1t} + \cdots + \beta_k x_{kt} + u_t$$
Another way is to seasonally adjust (sa) the variables, and then, do the regression with the adjusted variables:
$$z_t = \beta_0 + \beta_1 Q2_t + \beta_2 Q3_t + \beta_3 Q4_t + v_t \rightarrow \hat{v}_t + \mathrm{E}(z_t) = \hat{z}_t^{sa}$$
$$\hat{y}_t^{sa} = \beta_0 + \beta_1 \hat{x}_{1t}^{sa} + \cdots + \beta_k \hat{x}_{kt}^{sa} + u_t$$
There are much better and complex methods to seasonally adjust a time series, like the **X-13ARIMA-SEATS**.

Auto-correlation

The residual of any observation, u_t, is correlated with the residual of any other observation. The observations are not independent. Is the **non-fulfillment** of **t5**.

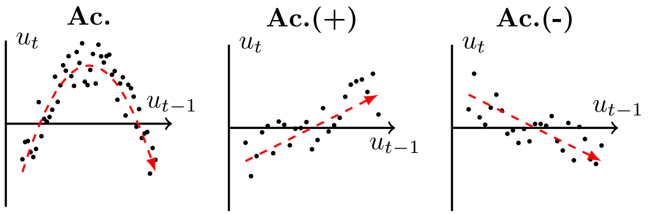
$$\text{Corr}(u_t, u_s \mid x_1, \dots, x_k) = \text{Corr}(u_t, u_s) \neq 0, \ \forall t \neq s$$

Consequences

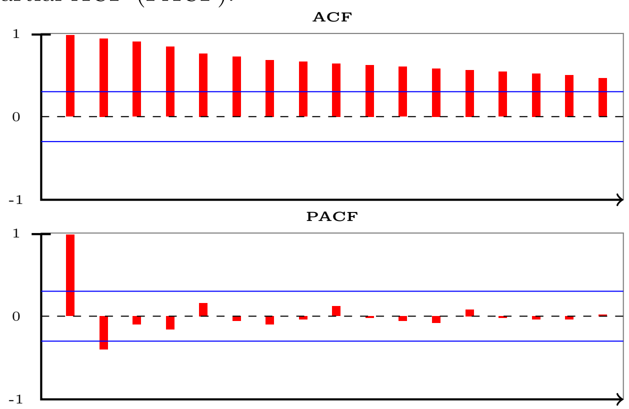
- OLS estimators still are unbiased.
- OLS estimators still are consistent.
- OLS is **not efficient** anymore, but still a LUE (Linear Unbiased Estimator).
- **Variance estimations** of the estimators are **biased**: the construction of confidence intervals and the hypothesis testing is not reliable.

Detection

- **Scatter plots** - look for scatter patterns on u_{t-1} vs. u_t.



- **Correlogram** - composed of the auto-correlation function (ACF) and the partial ACF (PACF).
  - Y axis: correlation [-1, 1].
  - X axis: lag number.
  - Blue lines: $\pm 1.96/T^{0.5}$



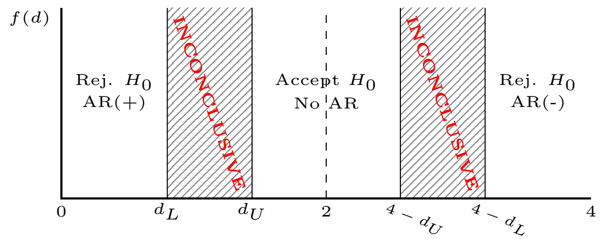
Conclusions differ between auto-correlation processes.

  - **MA(q) process**. **ACF**: only the first q coefficients are significant, the remaining are abruptly canceled. **PACF**: attenuated exponential fast decay or sine waves.
  - **AR(p) process**. **ACF**: attenuated exponential fast decay or sine waves. **PACF**: only the first p coefficients are significant, the remaining are abruptly canceled.
  - **ARMA(p, q) process**. **ACF** and **PACF**: the coefficients are not abruptly canceled and presents a fast decay.

If the ACF coefficients do not decay rapidly, there is a clear indicator of lack of stationarity in mean, which would lead to take first differences in the original series.

- **Formal tests** - Generally, H_0: No auto-correlation.
  Supposing that u_t follows an AR(1) process:
  $$u_t = \rho_1 u_{t-1} + \varepsilon_t$$
  where ε_t is white noise.
  - **AR(1) t test** (exogenous regressors):
    $$t = \frac{\hat{\rho}_1}{\text{se}(\hat{\rho}_1)} \sim t_{T-k-1, \alpha/2}$$
    * H_1: Auto-correlation of order one, AR(1).
  - **Durbin-Watson statistic** (exogenous regressors and residual normality):
    $$d = \frac{\sum_{t=2}^{n} (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^{n} \hat{u}_t^2} \approx 2 \cdot (1 - \hat{\rho}_1), \ 0 \leq d \leq 4$$
    * H_1: Auto-correlation of order one, AR(1).

$d =$	0	2	4
$\rho \approx$	1	0	-1



  - **Durbin's h** (endogenous regressors):
    $$h = \hat{\rho} \cdot \sqrt{\frac{T}{1 - T \cdot v}}$$
    where v is the estimated variance of the coefficient associated to the endogenous variable.
    * H_1: Auto-correlation of order one, AR(1).
  - **Breusch-Godfrey test** (endogenous regressors): it can detect MA(q) and AR(p) processes (ε_t is w. noise):
    * MA(q): $u_t = \varepsilon_t - m_1 u_{t-1} - \cdots - m_q u_{t-q}$
    * AR(p): $u_t = \rho_1 u_{t-1} + \cdots + \rho_p u_{t-p} + \varepsilon_t$

    Under H_0: No auto-correlation:
    $$T \cdot R^2_{\hat{u}_t} \underset{a}{\sim} \chi^2_q \quad \textbf{or} \quad T \cdot R^2_{\hat{u}_t} \underset{a}{\sim} \chi^2_p$$
    * H_1: Auto-correlation of order q (or p).
  - **Ljung-Box Q test**:
    * H_1: There is auto-correlation.

Correction

- Use OLS with a variance-covariance matrix estimator that is **robust to heterocedasticity and auto-correlation** (HAC), for example, the one proposed by **Newey-West**.
- Use **Generalized Least Squares** (GLS). Supposing $y_t = \beta_0 + \beta_1 x_t + u_t$, with $u_t = \rho u_{t-1} + \varepsilon_t$, where $|\rho| < 1$ and ε_t is white noise.
  - If ρ is **known**, use a **quasi-differentiated model**:
    $$y_t - \rho y_{t-1} = \beta_0(1 - \rho) + \beta_1(x_t - \rho x_{t-1}) + u_t - \rho u_{t-1}$$
    $$y_t^* = \beta_0^* + \beta_1' x_t^* + \varepsilon_t$$
    where $\beta_1' = \beta_1$; and estimate it by OLS.
  - If ρ is **not known**, estimate it by -for example- the **Cochrane-Orcutt iterative method** (Prais-Winsten method is also good):
    1. Obtain $\hat{u}_t$ from the original model.
    2. Estimate $\hat{u}_t = \rho \hat{u}_{t-1} + \varepsilon_t$ and obtain $\hat{\rho}$.
    3. Create a quasi-differentiated model:
       $$y_t - \hat{\rho} y_{t-1} = \beta_0(1 - \hat{\rho}) + \beta_1(x_t - \hat{\rho} x_{t-1}) + u_t - \hat{\rho} u_{t-1}$$
       $$y_t^* = \beta_0^* + \beta_1' x_t^* + \varepsilon_t$$
       where $\beta_1' = \beta_1$; and estimate it by OLS.
    4. Obtain $\hat{u}_t^* = y_t - (\hat{\beta}_0^* + \hat{\beta}_1' x_t) \neq y_t - (\hat{\beta}_0^* + \hat{\beta}_1' x_t^*)$.
    5. Repeat from step 2. The algorithm ends when the estimated parameters vary very little between iterations.
- If not solved, look for **high dependence** in the series.

Stationarity and weak dependence

Stationarity means stability of the joint distributions of a process as time progresses. It allows to correctly identify the relations –that stay unchange with time– between variables.

Stationary and non-stationary processes

- **Stationary process** (strong stationarity) - is the one in that the probability distributions are stable in time: if any collection of random variables is taken, and then, shifted h periods, the joint probability distribution should stay unchanged. It's easier to analyze and model.

- **Non-stationary process** - is, for example, a series with trend, where at least the mean changes with time.
- **Covariance stationary process** - is a weaker form of stationarity:
  - $\mathrm{E}(x_t)$ is constant.
  - $\mathrm{Var}(x_t)$ is constant.
  - For any t, $h \geq 1$, the $\mathrm{Cov}(x_t, x_{t+h})$ depends only of h, not of t.

Weakly dependent time series

It is important because it replaces the random sampling assumption, giving for granted the validity of the Central Limit Theorem (requires stationarity and a form of weak dependence). Weakly dependent processes are also known as **integrated of order zero**, I(0).

- **Weakly dependent** - restricts how close the relationship between x_t and x_{t+h} can be as the time distance between the series increases (h).

An **stationary time process** $\{x_t : t = 1, 2, \ldots, T\}$ is weakly dependent when x_t and x_{t+h} are almost independent as h increases without a limit.

A **covariance stationary time process** is weakly dependent if the correlation between x_t and x_{t+h} tends to 0 fast enough when $h \to \infty$ (they are not asymptotically correlated).

Some examples of stationary and weakly dependent time series are:

- **Moving average** - $\{x_t\}$ is a moving average of order one MA(q):
$$x_t = e_t + m_1 e_{t-1} + \cdots + m_q e_{t-q}$$
where $\{e_t : t = 0, 1, \ldots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.
- **Auto-regressive process** - $\{x_t\}$ is an auto-regressive process of order one AR(p):
$$x_t = \rho_1 x_{t-1} + \cdots + \rho_p x_{t-p} + e_t$$
where $\{e_t : t = 1, 2, \ldots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.
If $|\rho_1| < 1$, then $\{x_t\}$ is an AR(1) stable process that is weakly dependent. It is stationary in covariance, $\mathrm{Corr}(x_t, x_{t-1}) = \rho_1$.
- **ARMA process** - is a combination of the two above. $\{x_t\}$ is an ARMA(p, q):
$$x_t = e_t + m_1 e_{t-1} + \cdots + m_q e_{t-q} + \rho_1 x_{t-1} + \cdots + \rho_p x_{t-p}$$

A series with a trend cannot be stationary, but can be weakly dependent (and stationary if the series is detrended).

Strongly dependent time series

Most of the time, economics series are strongly dependent (or high persistent in time). Some special cases of **unit root** processes, I(1):

- **Random walk** - an AR(1) process with $\rho_1 = 1$.
$$y_t = y_{t-1} + e_t$$
where $\{e_t : t = 1, 2, \ldots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.
- **Random walk with a drift** - an AR(1) process with $\rho_1 = 1$ and a constant.
$$y_t = \beta_0 + y_{t-1} + e_t$$
where $\{e_t : t = 1, 2, \ldots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.

I(1) detection

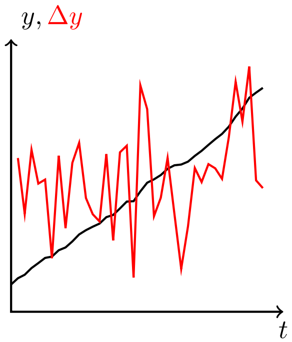
- **Augmented Dickey-Fuller** (ADF) **test** - where H_0: the process is unit root, I(1).
- **Kwiatkowski–Phillips–Schmidt–Shin** (KPSS) **test** - where H_0: the process have no unit root, I(0).
- **Phillips-Perron** (PP) **test** - where H_0: the process is unit root, I(1).

Transforming unit root to weak dependent

Unit root processes are **integrated of order one**, I(1). This means that **the first difference** of the process is **weakly dependent** or I(0) (and usually, stationary). For example, a random walk:
$$\Delta y_t = y_t - y_{t-1} = e_t$$
where $\{e_t\} = \{\Delta y_t\}$ is *i.i.d.*

Getting the first difference of a series also deletes its trend.

For example, a series with a trend (black), and it's first difference (red).



When an I(1) series is strictly positive, it is usually converted to logarithms before taking the first difference. That is, to obtain the (approx.) percentage change of the series:
$$\Delta \log(y_t) = \log(y_t) - \log(y_{t-1}) \approx \frac{y_t - y_{t-1}}{y_{t-1}}$$

Cointegration

When **two series are I(1), but a linear combination of them is I(0)**. If the case, the regression of one series over the other is not spurious, but expresses something about the long term relation. Variables are called cointegrated if they have a common stochastic trend.

For example: $\{x_t\}$ and $\{y_t\}$ are I(1), but $y_t - \beta x_t = u_t$ where $\{u_t\}$ is I(0). (β get the name of cointegration parameter).

Heterocedasticity on time series

The **assumption** affected is **t4**, which leads **OLS to be not efficient**.

Some tests that work could be the Breusch-Pagan or White's, where H_0: No heterocedasticity. It is **important** for the tests to work that there is **no auto-correlation** (so first, it is imperative to test for it).

ARCH

An auto-regressive conditional heterocedasticity (ARCH), is a model to analyze a form of dynamic heterocedasticity, where the error variance follows an AR(p) process.

Given the model: $y_t = \beta_0 + \beta_1 z_t + u_t$ where, there is AR(1) and heterocedasticity:
$$\mathrm{E}(u_t^2 \mid u_{t-1}) = \alpha_0 + \alpha_1 u_{t-1}^2$$

GARCH

A general auto-regressive conditional heterocedasticity (GARCH), is a model similar to ARCH, but in this case, the error variance follows an ARMA(p, q) process.

Exponential smoothing

$$f_t = \alpha y_t + (1 - \alpha) f_{t-1}$$
where $0 < \alpha < 1$ is the smoothing parameter.

Predictions

Two types of prediction:

- Of the mean value of y for a specific value of x.
- Of an individual value of y for a specific value of x.

If the values of the variables (x) approximate to the mean values ($\overline{x}$), the confidence interval amplitude of the prediction will be shorter.