Time Series Cheat Sheet

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The Econometrics Cheat Sheet Project

Basic concepts

Definitions

Time series - succession of observations ordered in time with a fixed frequency.

Given the format of a time series:

- **Point-in-time (stock)** - a single value is recorded for each period.
- **Aggregated (flow)** - values represent totals or averages over the period.
- **Range/interval (OHLC)** - each period records multiple summary statistics, such as min, max, open, close.

Stochastic process - a sequence of random variables that are indexed in time.

Components of a time series

- **Trend** - the long-term general movement of a series.
- **Seasonal variations** - periodic oscillations that are produced in a period equal to or inferior than a year, and can be easily identified across different years (usually the result of climatology).
- **Cyclical variations** - periodic oscillations that are produced in a period greater than a year (are the result of the economic cycle).
- **Residual variations** - movements that do not follow a recognizable periodic oscillation (irregular events).

Type of time series models

- **Static models** - the relation between y and x is contemporary. Conceptually:
$$y_t = \beta_0 + \beta_1 x_t + u_t$$
- **Distributed-lag models** - the relation between y and x is not contemporary. Conceptually:
$$y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \cdots + \beta_s x_{t-(s-1)} + u_t$$
The long-term cumulative effect in y when Δx is:
$$\beta_1 + \beta_2 + \cdots + \beta_s$$
- **Dynamic models** - lags of the dependent variable (endogeneity). Conceptually:
$$y_t = \beta_0 + \beta_1 y_{t-1} + \cdots + \beta_s y_{t-s} + u_t$$
- Combinations of the above, like the rational distributed-lag models (distributed-lag + dynamic).

Assumptions and properties

OLS model assumptions under time series

Under these assumptions, the OLS estimator will present good properties. **Gauss-Markov assumptions** extended for time series:

t1. **Parameters linearity and weak dependence**.
   a. y_t must be a linear function of the β's.
   b. The stochastic $\{(x_t, y_t) : t = 1, 2, \ldots, T\}$ is stationary and weakly dependent.

t2. **No perfect collinearity**.
   - There are no independent variables that are constant: $\text{Var}(x_j) \neq 0,\ \forall j = 1, \ldots, k$
   - There is no exact linear relation between independent variables.

t3. **Conditional mean zero and correlation zero**.
   a. There are no systematic errors: $\text{E}(u \mid x_1, \ldots, x_k) = \text{E}(u) = 0 \rightarrow$ **strong exogeneity** (a implies b).
   b. There are no relevant variables left out of the model: $\text{Cov}(x_j, u) = 0,\ \forall j = 1, \ldots, k \rightarrow$ **weak exogeneity**.

t4. **Homoscedasticity**. The variability of the residuals is the same for any x: $\text{Var}(u \mid x_1, \ldots, x_k) = \sigma_u^2$

t5. **No autocorrelation**. Residuals do not contain information about any other residuals: $\text{Corr}(u_t, u_s \mid x_1, \ldots, x_k) = 0,\ \forall t \neq s$

t6. **Normality**. Residuals are independent and identically distributed (**i.i.d.**): $u \sim \mathcal{N}(0, \sigma_u^2)$

t7. **Data size**. The number of observations available must be greater than $(k+1)$ parameters to estimate. (It is already satisfied under asymptotic situations)

Asymptotic properties of OLS

Under the econometric model assumptions and the Central Limit Theorem:

- Hold t1 to t3a: OLS is **unbiased**. $\text{E}(\hat{\beta}_j) = \beta_j$
- Hold t1 to t3: OLS is **consistent**. $\text{plim}(\hat{\beta}_j) = \beta_j$ (to t3b left out t3a, weak exogeneity, biased but consistent)
- Hold t1 to t5: **Asymptotic normality** of OLS (then, t6 is necessarily satisfied): $u \underset{a}{\sim} \mathcal{N}(0, \sigma_u^2)$
- Hold t1 to t5: **Unbiased estimate** of σ_u^2. $\text{E}(\hat{\sigma}_u^2) = \sigma_u^2$
- Hold t1 to t5: OLS is BLUE (Best Linear Unbiased Estimator) or **efficient**.
- Hold t1 to t6: Hypothesis testing and confidence intervals can be done reliably.

Trends and seasonality

Spurious regression - is when the relation between y and x is due to factors that affect y and have a correlation with x, $\text{Corr}(x_j, u) \neq 0$. Is the **non-fulfilment of t3**.

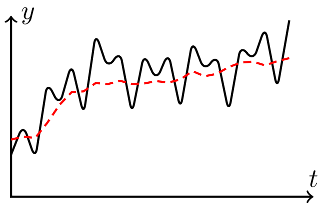
Trends

Two time series can have the same (or contrary) trend, which should lead to a high level of correlation. This can provoke a false appearance of causality; the problem is **spurious regression**. Given the model:
$$y_t = \beta_0 + \beta_1 x_t + u_t$$
where:
$$y_t = \alpha_0 + \alpha_1 \text{Trend} + v_t$$
$$x_t = \gamma_0 + \gamma_1 \text{Trend} + v_t$$
Adding a trend to the model can solve the problem:
$$y_t = \beta_0 + \beta_1 x_t + \beta_2 \text{Trend} + u_t$$
The trend can be linear or non-linear (quadratic, cubic, exponential, etc.)

Another way is to make use of the **Hodrick-Prescott filter** to extract the trend and the cyclical component.

Seasonality

A time series with can manifest seasonality. That is, the series is subject to a seasonal variations or patterns, usually related to climatology conditions.



For example, GDP is usually higher in summer and lower in winter (seasonally adjusted series in dashed red).

- Regressing time series that present seasonality can lead to **spurious results**.

There are different **seasonal adjustment** methods:

a. Include seasonal binary variables in the model. For example, for quarterly series (Sq_t are binary variables): $y_t = \beta_0 + \beta_1 S2_t + \beta_2 S3_t + \beta_3 S4_t + \beta_4 x_{1t} + \cdots + \beta_k x_{kt} + u_t$
b. Seasonally adjust the variables and then perform the regression with the adjusted variables.
c. Take seasonal differences, this can also help with the removal of trend (s is the seasonal period):
$$\nabla_s y_t = y_t - y_{t-s}$$
d. Apply **X-13ARIMA-SEATS** (better and more complicated method than the previous).

Autocorrelation

The residual of any observation, u_t, is correlated with the residual of any other observation. The observations are not independent. Is the **non-fulfilment** of **t5**.

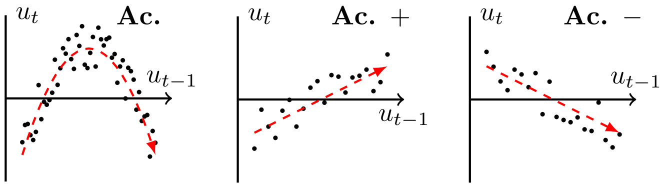
$$\mathrm{Corr}(u_t, u_s \mid x_1, \ldots, x_k) = \mathrm{Corr}(u_t, u_s) \neq 0, \ \forall t \neq s$$

Consequences

- OLS estimators are still unbiased.
- OLS estimators are still consistent.
- OLS is **not efficient** any more, but still a LUE (Linear Unbiased Estimator).
- **Variance estimations** of the estimators are **biased**: the construction of confidence intervals and the hypothesis testing is not reliable.

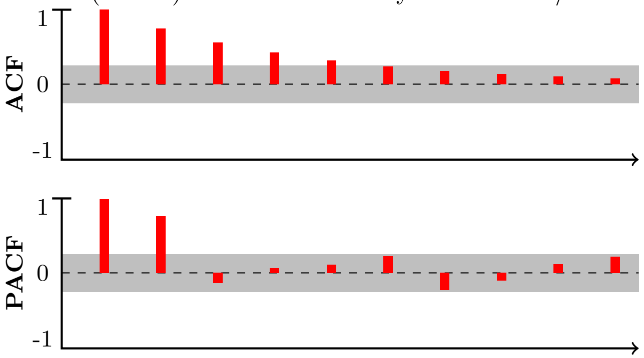
Detection

Scatter plots - look for scatter patterns on u_{t-1} vs. u_t.



Correlogram - autocorrelation function (ACF) and partial ACF (PACF).

- Y axis: correlation.
- X axis: lag number.
- Grey area: $\pm 1.96/T^{0.5}$



- **MA**(q) **process**. <u>ACF</u>: only the first q coefficients are significant, the remaining are abruptly cancelled. <u>PACF</u>: attenuated exponential fast decay or sine waves.
- **AR**(p) **process**. <u>ACF</u>: attenuated exponential fast decay or sine waves. <u>PACF</u>: only the first p coefficients are significant, the remaining are abruptly cancelled.
- **ARMA**(p, q) **process**. <u>ACF</u> and <u>PACF</u>: the coefficients are not abruptly cancelled and present a fast decay.

If the ACF coefficients do not decay rapidly, there is a clear indicator of a lack of stationarity in mean.

Formal tests - Generally, H_0: No autocorrelation.

Supposing that u_t follows an AR(1) process:

$$u_t = \rho_1 u_{t-1} + \varepsilon_t$$

where ε_t is white noise.

- **AR(1) t test** (exogenous regressors):

  $$t = \frac{\hat{\rho}_1}{\mathrm{se}(\hat{\rho}_1)} \sim t_{T-k-1, \alpha/2}$$

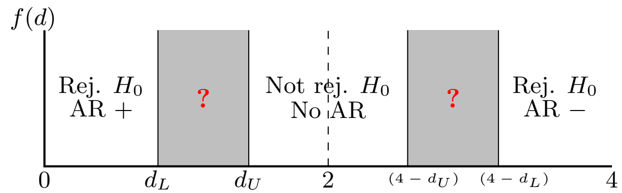
  H_1: Autocorrelation of order one, AR(1).
- **Durbin-Watson statistic** (exogenous regressors and residual normality):

  $$d = \frac{\sum_{t=2}^{n} (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^{n} \hat{u}_t^2} \approx 2 \cdot (1 - \hat{\rho}_1)$$

  Where $0 \leq d \leq 4$

  H_1: Autocorrelation of order one, AR(1).

$d =$	0	2	4
$\rho \approx$	1	0	-1

- **Durbin's h** (endogenous regressors):

  $$h = \hat{\rho} \cdot \sqrt{\frac{T}{1 - T \cdot v}}$$

  where v is the estimated variance of the coefficient associated with the endogenous variable.

  H_1: Autocorrelation of order one, AR(1).
- **Breusch-Godfrey test** (endogenous regressors): it can detect MA(q) and AR(p) processes (ε_t is w. noise):
  - MA(q): $u_t = \varepsilon_t - m_1 u_{t-1} - \cdots - m_q u_{t-q}$
  - AR(p): $u_t = \rho_1 u_{t-1} + \cdots + \rho_p u_{t-p} + \varepsilon_t$

  Under H_0: No autocorrelation:

  $$T \cdot R^2_{\hat{u}_t} \underset{a}{\sim} \chi^2_q \qquad \text{or} \qquad T \cdot R^2_{\hat{u}_t} \underset{a}{\sim} \chi^2_p$$

  H_1: Autocorrelation of order q (or p).
- **Ljung-Box Q test**:

  H_1: Autocorrelation up to lag h.

Correction

- Use OLS with a variance-covariance matrix estimator that is **robust to heteroscedasticity and autocorrelation** (HAC), for example, the one proposed by **Newey-West**.
- Use **Generalized Least Squares** (GLS). Supposing $y_t = \beta_0 + \beta_1 x_t + u_t$, with $u_t = \rho u_{t-1} + \varepsilon_t$, where $|\rho| < 1$ and ε_t is <u>white noise</u>.
  - If ρ is **known**, use a **quasi-differentiated model**:

    $$y_t - \rho y_{t-1} = \beta_0(1 - \rho) + \beta_1(x_t - \rho x_{t-1}) + u_t - \rho u_{t-1}$$
    $$y_t^* = \beta_0^* + \beta_1' x_t^* + \varepsilon_t$$

    where $\beta_1' = \beta_1$; and estimate it by OLS.
  - If ρ is **not known**, estimate it by -for example- the **Cochrane-Orcutt iterative method** (Prais-Winsten's method is also good):
    1. Obtain $\hat{u}_t$ from the original model.
    2. Estimate $\hat{u}_t = \rho \hat{u}_{t-1} + \varepsilon_t$ and obtain $\hat{\rho}$.
    3. Create a quasi-differentiated model:

       $$y_t - \hat{\rho} y_{t-1} = \beta_0(1 - \hat{\rho}) + \beta_1(x_t - \hat{\rho} x_{t-1}) + u_t - \hat{\rho} u_{t-1}$$
       $$y_t^* = \beta_0^* + \beta_1' x_t^* + \varepsilon_t$$

       where $\beta_1' = \beta_1$; and estimate it by OLS.
    4. Obtain $\hat{u}_t^* = y_t - (\hat{\beta}_0^* + \hat{\beta}_1' x_t) \neq y_t - (\hat{\beta}_0^* + \hat{\beta}_1' x_t^*)$.
    5. Repeat from step 2. The algorithm ends when the estimated parameters vary very little between iterations.
- If not solved, look for **high dependence** in the series.

Exponential smoothing

Given $\{y_t\}$, the smoothed series $\{f_t\}$:

$$f_t = \alpha y_t + (1 - \alpha) f_{t-1}$$

where $0 < \alpha < 1$ is the smoothing factor and $f_0 = y_0$.

Forecasts

Two types of forecasts:

- Of the mean value of y for a specific value of x.
- Of an individual value of y for a specific value of x.

Theil's U statistic - compares the forecast results with the ones of forecasting with minimal historical data.

$$U = \sqrt{\frac{\sum_{t=1}^{T-1} \left(\frac{\hat{y}_{t+1} - y_{t+1}}{y_t} \right)^2}{\sum_{t=1}^{T-1} \left(\frac{y_{t+1} - y_t}{y_t} \right)^2}}$$

- < 1: The forecast is better than guessing.
- $= 1$: The forecast is about as good as guessing.
- > 1: The forecast is worse than guessing.

Stationarity

Stationarity allows to correctly identify relations –that stay unchanged with time– between variables.

- **Stationary process** (strict stationarity) - the joint probability distribution of the process remains unchanged when shifted h periods.
- **Non-stationary process** - for example, a series with trend, where at least the mean changes with time.
- **Covariance stationary process** - it is a weaker form of stationarity:
  - $\mathrm{E}(x_t)$ is constant. – $\mathrm{Var}(x_t)$ is constant.
  - For any $t, h \geq 1$, $\mathrm{Cov}(x_t, x_{t+h})$ depends only of h, not of t.

Weak dependence

Weak dependence replaces the random sampling assumption for time series.

- An stationary process $\{x_t\}$ is **weakly dependent** when x_t and x_{t+h} are almost independent as h increases without a limit.
- A covariance stationary process is **weakly dependent** if the correlation between x_t and x_{t+h} tends to 0 fast enough when $h \to \infty$ (they are not asymptotically correlated).

Weakly dependent processes are known as **integrated of order zero**, I(0). Some examples:

- **Moving average** - $\{x_t\}$ is a moving average of order q, MA(q):
$$x_t = e_t + m_1 e_{t-1} + \cdots + m_q e_{t-q}$$
where $\{e_t : t = 0, 1, \ldots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.
- **Autoregressive process** - $\{x_t\}$ is an autoregressive process of order p, AR(p):
$$x_t = \rho_1 x_{t-1} + \cdots + \rho_p x_{t-p} + e_t$$
where $\{e_t : t = 1, 2, \ldots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.
Stability condition: if $1 - \rho_1 z - \cdots - \rho_p z^p = 0$ for $|z| > 1$ then $\{x_t\}$ is an AR(p) stable process that is weakly dependent. For AR(1), the condition is: $|\rho_1| < 1$.
- **ARMA process** - is a combination of the previous; $\{x_t\}$ is an ARMA(p, q):
$$x_t = e_t + m_1 e_{t-1} + \cdots + m_q e_{t-q} + \rho_1 x_{t-1} + \cdots + \rho_p x_{t-p}$$

Unit roots

A process is integrated of order d, I(d), if applying differences d times makes the process stationary.
When $d \geq 1$, the process is said to have a **unit root**. A process has a unit root when the stability condition is not met (there are roots on the unit circle).

Strong dependence

Generally, economic series are highly persistent in time. Some examples of **unit root** I(1):

- **Random walk** - an AR(1) process with $\rho_1 = 1$.
$$y_t = y_{t-1} + e_t$$
where $\{e_t : t = 1, 2, \ldots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.
- **Random walk with a drift** - an AR(1) process with $\rho_1 = 1$ and a constant.
$$y_t = \beta_0 + y_{t-1} + e_t$$
where $\{e_t : t = 1, 2, \ldots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.

Unit root tests

Test	H_0	Reject H_0
ADF	I(1)	tau < Critical value
KPSS	I(0) level	mu > Critical value
	I(0) trend	tau > Critical value
Phillips-Perron	I(1)	Z-tau < Critical value
Zivot-Andrews	I(1)	tau < Critical value

From unit root to weak dependence

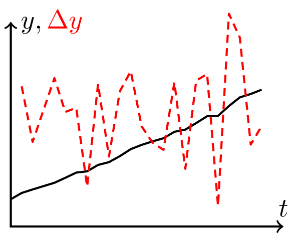
Integrated of **order one**, I(1), means that **the first difference** of the process is **weakly dependent** or I(0) (and usually, stationary). Let $\{y_t\}$ be a random walk:

$$\Delta y_t = y_t - y_{t-1} = e_t$$

where $\{e_t\} = \{\Delta y_t\}$ is *i.i.d.*

Note:

- The first difference of a series removes its trend.
- Logarithms of a series stabilizes its variance.



From unit root to percentage change

When an I(1) series is strictly positive, logs are often used before differencing to approximate percentage changes:

$$\Delta \log(y_t) = \log(y_t) - \log(y_{t-1}) \approx \frac{y_t - y_{t-1}}{y_{t-1}}$$

Ergodicity

A strictly stationary process $\{y_t\}$ is **ergodic** if time averages converge to their ensemble averages (expectations). This is often ensured by **strong mixing**, which implies asymptotic independence of distant events.

$$\frac{1}{T} \sum_{t=1}^{T} y_t \underset{a}{\to} \mathrm{E}(y_t)$$

Without it, sample moments may not reflect population moments. Estimators are inconsistent.

Cointegration

Two I(1) series are **cointegrated** if a linear combination is I(0). In that case, a regression between them is not spurious but reflects a valid **long-run** relationship. Cointegrated variables share a common stochastic trend.
For example, $\{x_t\}$ and $\{y_t\}$ are I(1), but $y_t - \beta x_t = u_t$ where $\{u_t\}$ is I(0). (β is the cointegrating parameter).

Cointegration test

1. Estimate $y_t = \alpha + \beta x_t + \varepsilon_t$ and obtain $\hat{\varepsilon}_t$.
2. Perform an ADF test on $\hat{\varepsilon}_t$ with a modified distribution.
   The result of this test is equivalent to:
   - H_0: $\beta = 0$ (no cointegration)
   - H_1: $\beta \neq 0$ (cointegration)

   if test statistic > critical value, reject H_0.

Heteroscedasticity in time series

The **assumption** affected is **t4**, which leads **OLS to be not efficient**.
Use tests like Breusch-Pagan or White's, where H_0: No heteroscedasticity. For these tests to work, there should be **no autocorrelation**.

ARCH

An autoregressive conditional heteroscedasticity (ARCH) model is used to analyse a form of dynamic heteroscedasticity, where the error variance follows an AR(p) process.
Given the model: $y_t = \beta_0 + \beta_1 z_t + u_t$ where, there is AR(1) and heteroscedasticity:

$$\mathrm{E}(u_t^2 \mid u_{t-1}) = \alpha_0 + \alpha_1 u_{t-1}^2$$

GARCH

A general ARCH (GARCH) model is similar to ARCH, but the error variance follows an ARMA(p, q) process.