

Time Series Cheat Sheet

By Marcelo Moreno - Universidad Rey Juan Carlos
The Econometrics Cheat Sheet Project

Basic concepts

Definitions

Time series - is a succession of quantitative observations of a phenomena ordered in time.

There are some variations of time series:

- **Panel data** - consist of a time series for each observation of a cross section.
- **Pooled cross sections** - combines cross sections from different time periods.

Stochastic process - sequence of random variables that are indexed in time.

Components of a time series

- **Trend** - is the long-term general movement of a series.
- **Seasonal variations** - are periodic oscillations that are produced in a period equal or inferior to a year, and can be easily identified on different years (usually are the result of climatology reasons).
- **Cyclical variations** - are periodic oscillations that are produced in a period greater than a year (are the result of the economic cycle).
- **Residual variations** - are movements that do not follow a recognizable periodic oscillation (are the result of eventual non-permanent phenomena that can affect the studied variable in a given moment).

Type of time series models

- **Static models** - the relation between y and x 's is contemporary. Conceptually:

$$y_t = \beta_0 + \beta_1 x_t + u_t$$

- **Distributed-lag models** - the relation between y and x 's is not contemporary. Conceptually:

$$y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \dots + \beta_s x_{t-(s-1)} + u_t$$

The long term cumulative effect in y when Δx is:

$$\beta_1 + \beta_2 + \dots + \beta_s$$

- **Dynamic models** - a temporal drift of the dependent variable is part of the independent variables (endogeneity). Conceptually:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \dots + \beta_s y_{t-s} + u_t$$

- Combinations of the above, like the rational distributed-lag models (distributed-lag + dynamic).

Assumptions and properties

OLS model assumptions under time series

Under this assumptions, the OLS estimator will present good properties. **Gauss-Markov assumptions** extended applied to time series:

- t1. **Parameters linearity and weak dependence.**
 - a. y_t must be a linear function of the β 's.
 - b. The stochastic $\{(x_t, y_t) : t = 1, 2, \dots, T\}$ is stationary and weakly dependent.
- t2. **No perfect collinearity.**
 - There are no independent variables that are constant: $\text{Var}(x_j) \neq 0, \forall j = 1, \dots, k$
 - There is not an exact linear relation between independent variables.
- t3. **Conditional mean zero and correlation zero.**
 - a. There are no systematic errors: $E(u | x_1, \dots, x_k) = E(u) = 0 \rightarrow$ **strong exogeneity** (a implies b).
 - b. There are no relevant variables left out of the model: $\text{Cov}(x_j, u) = 0, \forall j = 1, \dots, k \rightarrow$ **weak exogeneity**.
- t4. **Homoscedasticity.** The variability of the residuals is the same for any x : $\text{Var}(u | x_1, \dots, x_k) = \sigma_u^2$
- t5. **No auto-correlation.** Residuals do not contain information about any other residuals: $\text{Corr}(u_t, u_s | x_1, \dots, x_k) = 0, \forall t \neq s$
- t6. **Normality.** Residuals are independent and identically distributed (**i.i.d.** so on): $u \sim \mathcal{N}(0, \sigma_u^2)$
- t7. **Data size.** The number of observations available must be greater than $(k + 1)$ parameters to estimate. (It is already satisfied under asymptotic situations)

Asymptotic properties of OLS

Under the econometric model assumptions and the Central Limit Theorem:

- Hold t1 to t3a: OLS is **unbiased**. $E(\hat{\beta}_j) = \beta_j$
- Hold t1 to t3: OLS is **consistent**. $\text{plim}(\hat{\beta}_j) = \beta_j$ (to t3b left out t3a, weak exogeneity, biased but consistent)
- Hold t1 to t5: **asymptotic normality** of OLS (then, t6 is necessarily satisfied): $u \sim_a \mathcal{N}(0, \sigma_u^2)$
- Hold t1 to t5: **unbiased estimate** of σ_u^2 . $E(\hat{\sigma}_u^2) = \sigma_u^2$
- Hold t1 to t5: OLS is **BLUE** (Best Linear Unbiased Estimator) or **efficient**.
- Hold t1 to t6: hypothesis testing and confidence intervals can be done reliably.

Trends and seasonality

Spurious regression - is when the relation between y and x is due to factors that affect y and have correlation with x , $\text{Corr}(x_j, u) \neq 0$. Is the **non-fulfillment of t3**.

Trends

Two time series can have the same (or contrary) trend, that should lend to a high level of correlation. This, can provoke a false appearance of causality, the problem is **spurious regression**. Given the model:

$$y_t = \beta_0 + \beta_1 x_t + u_t$$

where:

$$y_t = \alpha_0 + \alpha_1 \text{Trend} + v_t$$

$$x_t = \gamma_0 + \gamma_1 \text{Trend} + v_t$$

Adding a trend to the model can solve the problem:

$$y_t = \beta_0 + \beta_1 x_t + \beta_2 \text{Trend} + u_t$$

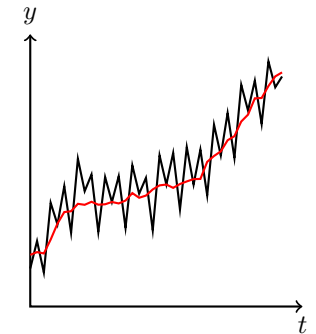
The trend can be linear or non-linear (quadratic, cubic, exponential, etc.)

Another way is make use of the **Hodrick-Prescott filter** to extract the trend and the cyclical component.

Seasonality

A time series with can manifest seasonality. That is, the series is subject to a seasonal variations or pattern, usually related to climatology conditions.

For example, GDP (black) is usually higher in summer and lower in winter. Seasonally adjusted series (red) for comparison.



- This problem is **spurious regression**. A seasonal adjustment can solve it.

A simple **seasonal adjustment** could be creating stationary binary variables and adding them to the model. For example, for quarterly series (Qq_t are binary variables):

$$y_t = \beta_0 + \beta_1 Q2_t + \beta_2 Q3_t + \beta_3 Q4_t + \beta_4 x_{1t} + \dots + \beta_k x_{kt} + u_t$$

Another way is to seasonally adjust (sa) the variables, and then, do the regression with the adjusted variables:

$$z_t = \beta_0 + \beta_1 Q2_t + \beta_2 Q3_t + \beta_3 Q4_t + v_t \rightarrow \hat{v}_t + E(z_t) = \hat{z}_t^{sa}$$

$$\hat{y}_t^{sa} = \beta_0 + \beta_1 \hat{x}_{1t}^{sa} + \dots + \beta_k \hat{x}_{kt}^{sa} + u_t$$

There are much better and complex methods to seasonally adjust a time series, like the **X-13ARIMA-SEATS**.

Auto-correlation

The residual of any observation, u_t , is correlated with the residual of any other observation. The observations are not independent. Is the **non-fulfillment** of **t5**.

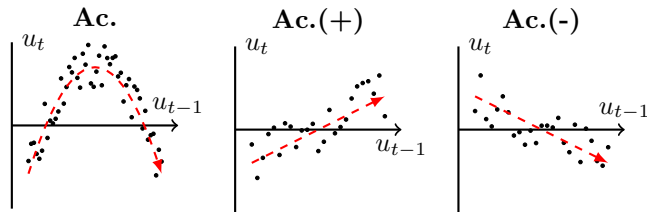
$$\text{Corr}(u_t, u_s \mid x_1, \dots, x_k) = \text{Corr}(u_t, u_s) \neq 0, \forall t \neq s$$

Consequences

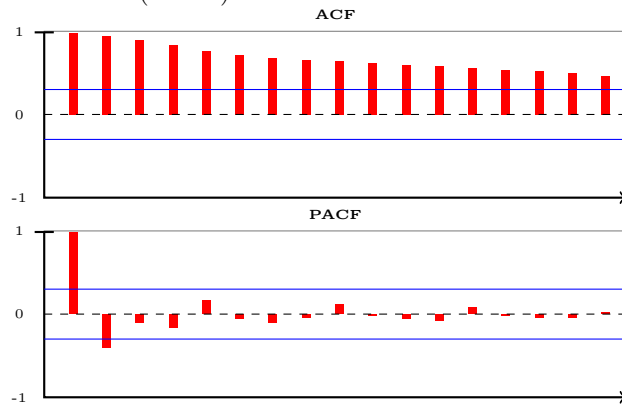
- OLS estimators still are unbiased.
- OLS estimators still are consistent.
- OLS is **not efficient** anymore, but still a LUE (Linear Unbiased Estimator).
- **Variance estimations** of the estimators are **biased**: the construction of confidence intervals and the hypothesis testing is not reliable.

Detection

- **Scatter plots** - look for scatter patterns on u_{t-1} vs. u_t .



- **Correlogram** - composed - Y axis: correlation [-1, 1]. of the auto-correlation - X axis: lag number. function (ACF) and the - Blue lines: $\pm 1.96/T^{0.5}$ partial ACF (PACF).



Conclusions differ between auto-correlation processes.

- **MA(q) process.** **ACF**: only the first q coefficients are significant, the remaining are abruptly canceled.

- **AR(p) process.** **ACF**: attenuated exponential fast decay or sine waves. **PACF**: only the first p coefficients are significant, the remaining are abruptly canceled.

- **ARMA(p, q) process.** **ACF** and **PACF**: the coefficients are not abruptly canceled and presents a fast decay.

If the ACF coefficients do not decay rapidly, there is a clear indicator of lack of stationarity in mean, which would lead to take first differences in the original series.

- **Formal tests** - Generally, H_0 : No auto-correlation.

Supposing that u_t follows an AR(1) process:

$$u_t = \rho_1 u_{t-1} + \varepsilon_t$$

where ε_t is white noise.

- **AR(1) t test** (exogenous regressors):

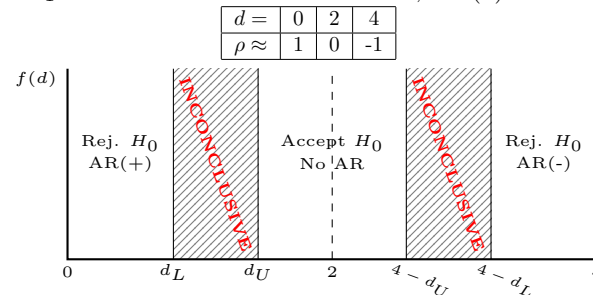
$$t = \frac{\hat{\rho}_1}{\text{se}(\hat{\rho}_1)} \sim t_{T-k-1, \alpha/2}$$

- * H_1 : Auto-correlation of order one, AR(1).

- **Durbin-Watson statistic** (exogenous regressors and residual normality):

$$d = \frac{\sum_{t=2}^n (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^n \hat{u}_t^2} \approx 2 \cdot (1 - \hat{\rho}_1), 0 \leq d \leq 4$$

- * H_1 : Auto-correlation of order one, AR(1).



- **Durbin's h** (endogenous regressors):

$$h = \hat{\rho} \cdot \sqrt{\frac{T}{1-T \cdot v}}$$

where v is the estimated variance of the coefficient associated to the endogenous variable.

- * H_1 : Auto-correlation of order one, AR(1).

- **Breusch-Godfrey test** (endogenous regressors): it can detect MA(q) and AR(p) processes (ε_t is w. noise):

- * MA(q): $u_t = \varepsilon_t - m_1 u_{t-1} - \dots - m_q u_{t-q}$

- * AR(p): $u_t = \rho_1 u_{t-1} + \dots + \rho_p u_{t-p} + \varepsilon_t$

Under H_0 : No auto-correlation:

$$T \cdot R_{\hat{u}_t}^2 \underset{a}{\sim} \chi_q^2 \quad \text{or} \quad T \cdot R_{\hat{u}_t}^2 \underset{a}{\sim} \chi_p^2$$

- * H_1 : Auto-correlation of order q (or p).

- **Ljung-Box Q test**:

- * H_1 : There is auto-correlation.

Correction

- Use OLS with a variance-covariance matrix estimator that is **robust to heterocedasticity and auto-correlation** (HAC), for example, the one proposed by **Newey-West**.

- Use **Generalized Least Squares** (GLS). Supposing $y_t = \beta_0 + \beta_1 x_t + u_t$, with $u_t = \rho u_{t-1} + \varepsilon_t$, where $|\rho| < 1$ and ε_t is white noise.

- If ρ is **known**, use a **quasi-differentiated model**:

$$y_t - \rho y_{t-1} = \beta_0(1 - \rho) + \beta_1(x_t - \rho x_{t-1}) + u_t - \rho u_{t-1}$$

$$y_t^* = \beta_0^* + \beta_1^* x_t^* + \varepsilon_t$$

where $\beta_1' = \beta_1$; and estimate it by OLS.

- If ρ is **not known**, estimate it by -for example the **Cochrane-Orcutt iterative method** (Prais-Winsten method is also good):

1. Obtain \hat{u}_t from the original model.

2. Estimate $\hat{u}_t = \rho \hat{u}_{t-1} + \varepsilon_t$ and obtain $\hat{\rho}$.

3. Create a quasi-differentiated model:

$$y_t - \hat{\rho} y_{t-1} = \beta_0(1 - \hat{\rho}) + \beta_1(x_t - \hat{\rho} x_{t-1}) + u_t - \hat{\rho} u_{t-1}$$

$$y_t^* = \beta_0^* + \beta_1^* x_t^* + \varepsilon_t$$

where $\beta_1' = \beta_1$; and estimate it by OLS.

4. Obtain $\hat{u}_t^* = y_t - (\hat{\beta}_0^* + \hat{\beta}_1^* x_t) \neq y_t - (\hat{\beta}_0^* + \hat{\beta}_1^* x_t^*)$.

5. Repeat from step 2. The algorithm ends when the estimated parameters vary very little between iterations.

- If not solved, look for **high dependence** in the series.

Stationarity and weak dependence

Stationarity means stability of the joint distributions of a process as time progresses. It allows to correctly identify the relations -that stay unchange with time- between variables.

Stationary and non-stationary processes

- **Stationary process** (strong stationarity) - is the one in that the probability distributions are stable in time: if any collection of random variables is taken, and then, shifted h periods, the joint probability distribution should stay unchanged. It's easier to analyze and model.

- **Non-stationary process** - is, for example, a series with trend, where at least the mean changes with time.
- **Covariance stationary process** - is a weaker form of stationarity:
 - $E(x_t)$ is constant.
 - $\text{Var}(x_t)$ is constant.
 - For any $t, h \geq 1$, the $\text{Cov}(x_t, x_{t+h})$ depends only of h , not of t .

Weakly dependent time series

It is important because it replaces the random sampling assumption, giving for granted the validity of the Central Limit Theorem (requires stationarity and a form of weak dependence). Weakly dependent processes are also known as **integrated of order zero**, $I(0)$.

- **Weakly dependent** - restricts how close the relationship between x_t and x_{t+h} can be as the time distance between the series increases (h).

An **stationary time process** $\{x_t : t = 1, 2, \dots, T\}$ is weakly dependent when x_t and x_{t+h} are almost independent as h increases without a limit.

A **covariance stationary time process** is weakly dependent if the correlation between x_t and x_{t+h} tends to 0 fast enough when $h \rightarrow \infty$ (they are not asymptotically correlated).

Some examples of stationary and weakly dependent time series are:

- **Moving average** - $\{x_t\}$ is a moving average of order one $MA(q)$:

$$x_t = e_t + m_1 e_{t-1} + \dots + m_q e_{t-q}$$

where $\{e_t : t = 0, 1, \dots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.

- **Auto-regressive process** - $\{x_t\}$ is an auto-regressive process of order one $AR(p)$:

$$x_t = \rho_1 x_{t-1} + \dots + \rho_p x_{t-p} + e_t$$

where $\{e_t : t = 1, 2, \dots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.

If $|\rho_1| < 1$, then $\{x_t\}$ is an $AR(1)$ stable process that is weakly dependent. It is stationary in covariance, $\text{Corr}(x_t, x_{t-1}) = \rho_1$.

- **ARMA process** - is a combination of the two above. $\{x_t\}$ is an $ARMA(p, q)$:

$$x_t = e_t + m_1 e_{t-1} + \dots + m_q e_{t-q} + \rho_1 x_{t-1} + \dots + \rho_p x_{t-p}$$

A series with a trend cannot be stationary, but can be weakly dependent (and stationary if the series is de-trended).

Strongly dependent time series

Most of the time, economics series are strongly dependent (or high persistent in time). Some special cases of **unit root** processes, $I(1)$:

- **Random walk** - an $AR(1)$ process with $\rho_1 = 1$.

$$y_t = y_{t-1} + e_t$$

where $\{e_t : t = 1, 2, \dots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.

- **Random walk with a drift** - an $AR(1)$ process with $\rho_1 = 1$ and a constant.

$$y_t = \beta_0 + y_{t-1} + e_t$$

where $\{e_t : t = 1, 2, \dots, T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.

Unit root tests

Test	H_0	Reject H_0
ADF	$I(1)$	$\tau < \text{Critical value}$
KPSS	$I(0)$ level	$\mu > \text{Critical value}$
	$I(0)$ trend	$\tau > \text{Critical value}$
Phillips-Perron	$I(1)$	$Z\text{-}\tau < \text{Critical value}$
Zivot-Andrews	$I(1)$	$\tau < \text{Critical value}$

Transforming unit root to weak dependent

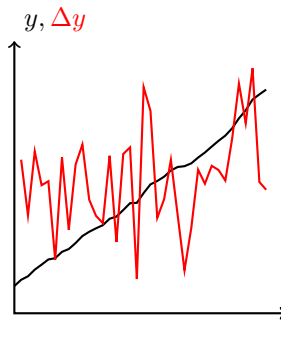
Unit root processes are **integrated of order one**, $I(1)$. This means that the **first difference** of the process is **weakly dependent** or $I(0)$ (and usually, stationary). For example, a random walk:

$$\Delta y_t = y_t - y_{t-1} = e_t$$

where $\{e_t\} = \{\Delta y_t\}$ is *i.i.d.*

Getting the first difference of a series also deletes its trend.

For example, a series with a trend (black), and its first difference (red).



When an $I(1)$ series is strictly positive, it is usually converted to logarithms before taking the first difference. That is, to obtain the (approx.) percentage change of the series:

$$\Delta \log(y_t) = \log(y_t) - \log(y_{t-1}) \approx \frac{y_t - y_{t-1}}{y_{t-1}}$$

Cointegration

When **two series are $I(1)$** , but a **linear combination of them is $I(0)$** . If the case, the regression of one series over the other is not spurious, but expresses something about the long term relation. Variables are called cointegrated if they have a common stochastic trend.

For example: $\{x_t\}$ and $\{y_t\}$ are $I(1)$, but $y_t - \beta x_t = u_t$ where $\{u_t\}$ is $I(0)$. (β get the name of cointegration parameter).

Heterocedasticity on time series

The **assumption** affected is **t4**, which leads **OLS to be not efficient**.

Some tests that work could be the Breusch-Pagan or White's, where H_0 : No heterocedasticity. It is **important** for the tests to work that there is **no auto-correlation** (so first, it is imperative to test for it).

ARCH

An auto-regressive conditional heterocedasticity (ARCH), is a model to analyze a form of dynamic heterocedasticity, where the error variance follows an $AR(p)$ process.

Given the model: $y_t = \beta_0 + \beta_1 z_t + u_t$ where, there is $AR(1)$ and heterocedasticity:

$$E(u_t^2 | u_{t-1}) = \alpha_0 + \alpha_1 u_{t-1}^2$$

GARCH

A general auto-regressive conditional heterocedasticity (GARCH), is a model similar to ARCH, but in this case, the error variance follows an $ARMA(p, q)$ process.

Exponential smoothing

$$f_t = \alpha y_t + (1 - \alpha) f_{t-1}$$

where $0 < \alpha < 1$ is the smoothing parameter.

Predictions

Two types of prediction:

- Of the mean value of y for a specific value of x .
 - Of an individual value of y for a specific value of x .
- If the values of the variables (x) approximate to the mean values (\bar{x}), the confidence interval amplitude of the prediction will be shorter.