

# Additional Cheat Sheet

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The Econometrics Cheat Sheet Project

## OLS matrix notation

The general econometric model:

$$y_i = \beta_0 + \beta_1 x_{1i} + \dots + \beta_k x_{ki} + u_i$$

Can be written in matrix notation as:

$$y = X\beta + u$$

Let's call  $\hat{u}$  the vector of estimated residuals ( $\hat{u} \neq u$ ):

$$\hat{u} = y - X\hat{\beta}$$

The **objective** of OLS is to **minimize** the SSR:

$$\min \text{SSR} = \min \sum_{i=1}^n \hat{u}_i^2 = \min \hat{u}^\top \hat{u}$$

- Defining  $\hat{u}^\top \hat{u}$ :

$$\begin{aligned} \hat{u}^\top \hat{u} &= (y - X\hat{\beta})^\top (y - X\hat{\beta}) = \\ &= y^\top y - 2\hat{\beta}^\top X^\top y + \hat{\beta}^\top X^\top X \hat{\beta} \end{aligned}$$

- Minimizing  $\hat{u}^\top \hat{u}$ :

$$\frac{\partial \hat{u}^\top \hat{u}}{\partial \hat{\beta}} = -2X^\top y + 2X^\top X \hat{\beta} = 0$$

$$\hat{\beta} = (X^\top X)^{-1} (X^\top y)$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} n & \sum x_1 & \dots & \sum x_k \\ \sum x_1 & \sum x_1^2 & \dots & \sum x_1 x_k \\ \vdots & \vdots & \ddots & \vdots \\ \sum x_k & \sum x_k x_1 & \dots & \sum x_k^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum y \\ \sum y x_1 \\ \vdots \\ \sum y x_k \end{bmatrix}$$

The second derivative  $\frac{\partial^2 \hat{u}^\top \hat{u}}{\partial \hat{\beta}^2} = X^\top X > 0$  (is a min.)

## Variance-covariance matrix of $\hat{\beta}$

Has the following form:

$$\begin{aligned} \text{Var}(\hat{\beta}) &= \hat{\sigma}_u^2 \cdot (X^\top X)^{-1} = \\ &= \begin{bmatrix} \text{Var}(\hat{\beta}_0) & \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) & \dots & \text{Cov}(\hat{\beta}_0, \hat{\beta}_k) \\ \text{Cov}(\hat{\beta}_1, \hat{\beta}_0) & \text{Var}(\hat{\beta}_1) & \dots & \text{Cov}(\hat{\beta}_1, \hat{\beta}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(\hat{\beta}_k, \hat{\beta}_0) & \text{Cov}(\hat{\beta}_k, \hat{\beta}_1) & \dots & \text{Var}(\hat{\beta}_k) \end{bmatrix} \end{aligned}$$

where:  $\hat{\sigma}_u^2 = \frac{\hat{u}^\top \hat{u}}{n-k-1}$

The standard errors are in the diagonal of:

$$\text{se}(\hat{\beta}) = \sqrt{\text{Var}(\hat{\beta})}$$

## Error measurements

- $\text{SSR} = \hat{u}^\top \hat{u} = y^\top y - \hat{\beta}^\top X^\top y = \sum (y_i - \hat{y}_i)^2$
- $\text{SSE} = \hat{\beta}^\top X^\top y - n\bar{y}^2 = \sum (\hat{y}_i - \bar{y})^2$
- $\text{SST} = \text{SSR} + \text{SSE} = y^\top y - n\bar{y}^2 = \sum (y_i - \bar{y})^2$

## Variance-covariance matrix of $u$

Has the following shape:

$$\text{Var}(u) = \begin{bmatrix} \text{Var}(u_1) & \text{Cov}(u_1, u_2) & \dots & \text{Cov}(u_1, u_n) \\ \text{Cov}(u_2, u_1) & \text{Var}(u_2) & \dots & \text{Cov}(u_2, u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(u_n, u_1) & \text{Cov}(u_n, u_2) & \dots & \text{Var}(u_n) \end{bmatrix}$$

When there is no heterocedasticity and no auto-correlation, the variance-covariance matrix of  $u$  has the form:

$$\text{Var}(u) = \sigma_u^2 \cdot I_n = \begin{bmatrix} \sigma_u^2 & 0 & \dots & 0 \\ 0 & \sigma_u^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_u^2 \end{bmatrix}$$

where  $I_n$  is an identity matrix of  $n \times n$  elements.

When there is **heterocedasticity** and **auto-correlation**, the variance-covariance matrix of  $u$  has the shape:

$$\text{Var}(u) = \sigma_u^2 \cdot \Omega = \begin{bmatrix} \sigma_{u1}^2 & \sigma_{u12} & \dots & \sigma_{u1n} \\ \sigma_{u21} & \sigma_{u2}^2 & \dots & \sigma_{u2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{un1} & \sigma_{un2} & \dots & \sigma_{un}^2 \end{bmatrix}$$

where  $\Omega \neq I_n$ .

- Heterocedasticity:  $\text{Var}(u) = \sigma_{u_i}^2 \neq \sigma_u^2$
- Auto-correlation:  $\text{Cov}(u_i, u_j) = \sigma_{u_{ij}} \neq 0, \forall i \neq j$

## Variable omission

Most of the time, is hard to get all relevant variables for an analysis. For example, a true model with all variables:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + v$$

where  $\beta_2 \neq 0$ ,  $v$  is the error term and  $\text{Cov}(v|x_1, x_2) = 0$ .

The model with the available variables:

$$y = \alpha_0 + \alpha_1 x_1 + u$$

where  $u = v + \beta_2 x_2$ .

Relevant variable omission causes OLS estimators to be **biased** and **inconsistent**, because there is no weak exogeneity,  $\text{Cov}(x_1, u) \neq 0$ . Depending on the  $\text{Corr}(x_1, x_2)$  and the sign of  $\beta_2$ , the bias on  $\hat{\alpha}_1$  could be:

	$\text{Corr}(x_1, x_2) > 0$	$\text{Corr}(x_1, x_2) < 0$
$\beta_2 > 0$	(+) bias	(-) bias
$\beta_2 < 0$	(-) bias	(+) bias

- (+) bias:  $\hat{\alpha}_1$  will be higher than it should be (it includes the effect of  $x_2$ )  $\rightarrow \hat{\alpha}_1 > \beta_1$
- (-) bias:  $\hat{\alpha}_1$  will be lower than it should be (it includes the effect of  $x_2$ )  $\rightarrow \hat{\alpha}_1 < \beta_1$

If  $\text{Corr}(x_1, x_2) = 0$ , there is no bias on  $\hat{\alpha}_1$ , because the effect of  $x_2$  will be fully picked up by the error term,  $u$ .

## Variable omission correction

### Proxy variables

Is the approach when a relevant variable is not available because it is non-observable, and there is no data available.

- A **proxy variable** is something **related** with the non-observable variable that has data available.

For example, the GDP per capita is a proxy variable for the life quality (non-observable).

### Instrumental variables

When the variable of interest ( $x$ ) is observable but **endogenous**, the proxy variables approach is no longer valid.

- An **instrumental variable** (IV) is an **observable variable** ( $z$ ) that is **related** with the variable of interest that is endogenous ( $x$ ), and meet the **requirements**:

$$\text{Cov}(z, u) = 0 \rightarrow \text{instrument exogeneity}$$

$$\text{Cov}(z, x) \neq 0 \rightarrow \text{instrument relevance}$$

Instrumental variables let the omitted variable in the error term, but instead of estimate the model by OLS, it utilizes a method that recognizes the presence of an omitted variable. It can also solve error measurement problems.

- Two-Stage Least Squares** (TSLS) is a method to estimate a model with multiple instrumental variables. The  $\text{Cov}(z, u) = 0$  requirement can be relaxed, but there has to be a minimum of variables that satisfies it.

The TSLS **estimation procedure** is as follows:

- Estimate a model regressing  $x$  by  $z$  using OLS, obtaining  $\hat{x}$ :

$$\hat{x} = \hat{\pi}_0 + \hat{\pi}_1 z$$

- Replace  $x$  by  $\hat{x}$  in the final model and estimate it by OLS:

$$y = \beta_0 + \beta_1 \hat{x} + u$$

There are some **important** things to know about TSLS:

- TSLS estimators are less efficient than OLS when the explanatory variables are exogenous. The **Hausman test** can be used to check it:

$$H_0: \text{OLS estimators are consistent.}$$

If  $H_0$  is accepted, the OLS estimators are better than TSLS and vice versa.

- There could be some (or all) instrument that are not valid. This is known as over-identification, **Sargan test** can be used to check it:

$$H_0: \text{all instruments are valid.}$$

## Information criterion

It is used to compare models with different number of parameters ( $p$ ). The general formula:

$$\text{Cr}(p) = \log\left(\frac{\text{SSR}}{n}\right) + c_n \varphi(p)$$

where:

- SSR is the Sum of Squared Residuals from a model of order  $p$ .
- $c_n$  is a sequence indexed by the sample size.
- $\varphi(p)$  is a function that penalizes large  $p$  orders.

Is interpreted as the relative amount of information lost by the model. The  $p$  order that min. the criterion is chosen.

There are different  $c_n \varphi(p)$  functions:

- Akaike:  $\text{AIC}(p) = \log\left(\frac{\text{SSR}}{n}\right) + \frac{2}{n}p$
  - Hannan-Quinn:  $\text{HQ}(p) = \log\left(\frac{\text{SSR}}{n}\right) + \frac{2 \log(\log(n))}{n}p$
  - Schwarz:  $\text{Sc}(p) = \log\left(\frac{\text{SSR}}{n}\right) + \frac{\log(n)}{n}p$
- $\text{Sc}(p) \leq \text{HQ}(p) \leq \text{AIC}(p)$

## The non-restricted hypothesis test

Is an alternative to the F test when there are few hypothesis to test on the parameters. Let  $\beta_i, \beta_j$  be parameters,  $a, b, c \in \mathbb{R}$  are constants.

- $H_0 : a\beta_i + b\beta_j = c$
- $H_1 : a\beta_i + b\beta_j \neq c$

$$\begin{aligned} \text{Under } H_0: \quad t &= \frac{a\hat{\beta}_i + b\hat{\beta}_j - c}{\sqrt{\text{Var}(a\hat{\beta}_i + b\hat{\beta}_j)}} \\ &= \frac{a\hat{\beta}_i + b\hat{\beta}_j - c}{\sqrt{a^2 \text{Var}(\hat{\beta}_i) + b^2 \text{Var}(\hat{\beta}_j) \pm 2ab \text{Cov}(\hat{\beta}_i, \hat{\beta}_j)}} \end{aligned}$$

If  $|t| > |t_{n-k-1, \alpha/2}|$ , there is evidence to reject  $H_0$ .

## ANOVA

Decompose the total sum of squared in sum of squared residuals and sum of squared explained:  $\text{SST} = \text{SSR} + \text{SSE}$

Variation origin	Sum Sq.	df	Sum Sq. Avg.
Regression	SSE	$k$	$\text{SSE}/k$
Residuals	SSR	$n - k - 1$	$\text{SSR}/(n - k - 1)$
Total	SST	$n - 1$	

The F statistic:

$$F = \frac{\text{SSA of SSE}}{\text{SSA of SSR}} = \frac{\text{SSE}}{\text{SSR}} \cdot \frac{n - k - 1}{k} \sim F_{k, n-k-1}$$

If  $F > F_{k, n-k-1}$ , there is evidence to reject  $H_0$ : There is no difference among group means.

## Incorrect functional form

To check if the model **functional form** is correct, we can use **Ramsey's RESET** (Regression Specification Error Test). It test the original model vs. a model with variables in powers.

$H_0$ : the model is correctly specified.

Test procedure:

1. Estimate the original model and obtain  $\hat{y}$  and  $R^2$ :

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k$$

2. Estimate a new model adding powers of  $\hat{y}$  and obtain the new  $R_{\text{new}}^2$ :

$$\tilde{y} = \hat{y} + \tilde{\gamma}_2 \hat{y}^2 + \dots + \tilde{\gamma}_l \hat{y}^l$$

3. Define the test statistic, under  $\gamma_2 = \dots = \gamma_l = 0$  as null hypothesis:

$$F = \frac{R_{\text{new}}^2 - R^2}{1 - R_{\text{new}}^2} \cdot \frac{n - (k+1) - l}{l} \sim F_{l, n - (k+1) - l}$$

If  $F > F_{l, n - (k+1) - l}$ , there is evidence to reject  $H_0$ .

## Logistic regression

When there is a binary (0, 1) dependent variable, the linear regression model is no longer valid, we can use logistic regression instead. For example, a **logit model**:

$$P_i = \frac{1}{1 + e^{-(\beta_0 + \beta_1 x_i + u_i)}} = \frac{e^{\beta_0 + \beta_1 x_i + u_i}}{1 + e^{\beta_0 + \beta_1 x_i + u_i}}$$

where  $P_i = E(y_i = 1 | x_i)$  and  $(1 - P_i) = E(y_i = 0 | x_i)$

The **odds ratio** (in favor of  $y_i = 1$ ):

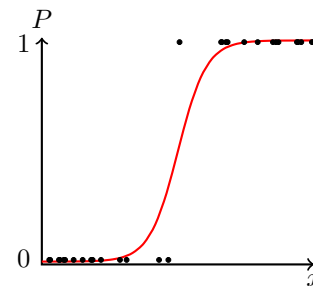
$$\frac{P_i}{1 - P_i} = \frac{1 + e^{\beta_0 + \beta_1 x_i + u_i}}{1 + e^{-(\beta_0 + \beta_1 x_i + u_i)}} = e^{\beta_0 + \beta_1 x_i + u_i}$$

Taking the natural logarithm of the odds ratio, we obtain the **logit**:

$$L_i = \ln\left(\frac{P_i}{1 - P_i}\right) = \beta_0 + \beta_1 x_i + u_i$$

$P_i$  is between 0 and 1, but  $L_i$  goes from  $-\infty$  to  $+\infty$ .

If  $L_i$  is positive, it means that when  $x_i$  increments, the probability of  $y_i = 1$  increases, and vice versa.



## Statistical definitions

Let  $\xi, \eta$  be random variables,  $a, b \in \mathbb{R}$  constants, and  $P$  denotes probability.

### Mean

Definition:  $E(\xi) = \sum_{i=1}^n \xi_i \cdot P[\xi = \xi_i]$

Population mean:

$$E(\xi) = \frac{1}{N} \sum_{i=1}^N \xi_i$$

Sample mean:

$$E(\xi) = \frac{1}{n} \sum_{i=1}^n \xi_i$$

Some properties:

- $E(a) = a$
- $E(\xi + a) = E(\xi) + a$
- $E(a \cdot \xi) = a \cdot E(\xi)$
- $E(\xi \pm \eta) = E(\xi) \pm E(\eta)$
- $E(\xi \cdot \eta) = E(\xi) \cdot E(\eta)$  only if  $\xi$  and  $\eta$  are independent.
- $E(\xi - E(\xi)) = 0$
- $E(a \cdot \xi + b \cdot \eta) = a \cdot E(\xi) + b \cdot E(\eta)$

### Variance

Definition:  $\text{Var}(\xi) = E(\xi - E(\xi))^2$

Population variance:

$$\text{Var}(\xi) = \frac{\sum_{i=1}^N (\xi_i - E(\xi))^2}{N}$$

Sample variance:

$$\text{Var}(\xi) = \frac{\sum_{i=1}^n (\xi_i - E(\xi))^2}{n - 1}$$

Some properties:

- $\text{Var}(a) = 0$
- $\text{Var}(\xi + a) = \text{Var}(\xi)$
- $\text{Var}(a \cdot \xi) = a^2 \cdot \text{Var}(\xi)$
- $\text{Var}(\xi \pm \eta) = \text{Var}(\xi) + \text{Var}(\eta) \pm 2 \cdot \text{Cov}(\xi, \eta)$
- $\text{Var}(a \cdot \xi \pm b \cdot \eta) = a^2 \cdot \text{Var}(\xi) + b^2 \cdot \text{Var}(\eta) \pm 2ab \cdot \text{Cov}(\xi, \eta)$

### Covariance

Definition:  $\text{Cov}(\xi, \eta) = E[(\xi - E(\xi)) \cdot (\eta - E(\eta))]$

Population covariance:

$$\frac{\sum_{i=1}^N (\xi_i - E(\xi)) \cdot (\eta_i - E(\eta))}{N}$$

Sample covariance:

$$\frac{\sum_{i=1}^n (\xi_i - E(\xi)) \cdot (\eta_i - E(\eta))}{n - 1}$$

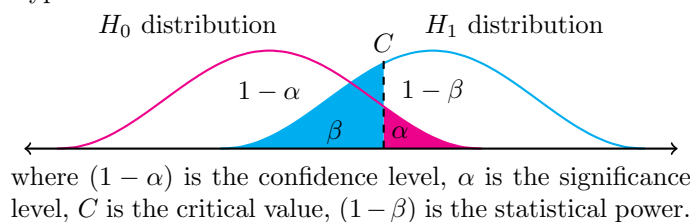
Some properties:

- $\text{Cov}(\xi, a) = 0$
- $\text{Cov}(\xi + a, \eta + b) = \text{Cov}(\xi, \eta)$
- $\text{Cov}(a \cdot \xi, b \cdot \eta) = ab \cdot \text{Cov}(\xi, \eta)$
- $\text{Cov}(\xi, \xi) = \text{Var}(\xi)$
- $\text{Cov}(\xi, \eta) = \text{Cov}(\eta, \xi)$

## Hypothesis testing (extra)

	$H_0$ true	$H_0$ false
Reject $H_0$	False positive Type I Error ( $\alpha$ )	True positive ( $1 - \beta$ )
Accept $H_0$	True negative ( $1 - \alpha$ )	False negative Type II Error ( $\beta$ )

Typical one-tail test:



## Bootstrapping

**Problem** - Asymptotic approximations to the distributions of test statistics do not work on small samples.

**Solution** - Bootstrap is basically sampling with replacement. The observed data is treated like a population, and multiple samples are extracted to recalculate an estimator or test statistic multiple times (improve accuracy).

## VAR (Vector Autoregressive)

A VAR model captures **dynamic interactions** between time series variables. The VAR( $p$ ):

$$y_t = A_1 y_{t-1} + \dots + A_p y_{t-p} + B_0 x_t + \dots + B_q x_{t-q} + C D_t + u_t$$

where:

- $y_t = (y_{1t}, \dots, y_{Kt})^\top$  is a vector of  $K$  observable endogenous time series variables.
- $A_i$ 's are  $K \times K$  coefficient matrices.
- $x_t = (x_{1t}, \dots, x_{Mt})^\top$  is a vector of  $M$  observable exogenous time series variables.
- $B_j$ 's are  $K \times M$  coefficient matrices.
- $D_t$  is a vector that contains all deterministic terms, that may be a: constant, linear trend, seasonal dummy, and/or any other user specified dummy variables.
- $C$  is a coefficient matrix of suitable dimension.
- $u_t = (u_{1t}, \dots, u_{Kt})^\top$  is a vector of  $K$  white noise series.

The process is **stable** if:

$$\det(I_K - A_1 z - \dots - A_p z^p) \neq 0 \quad \text{for } |z| \leq 1$$

this is, there are **no roots** in and on the complex unit circle.

For example, a VAR model with two endogenous variables ( $K = 2$ ), two lags ( $p = 2$ ), an exogenous contemporaneous variable ( $M = 1$ ), a constant (const) and a trend (Trend $_t$ ):

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} a_{11,1} & a_{12,1} \\ a_{21,1} & a_{22,1} \end{bmatrix} \cdot \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} a_{11,2} & a_{12,2} \\ a_{21,2} & a_{22,2} \end{bmatrix} \cdot \begin{bmatrix} y_{1,t-2} \\ y_{2,t-2} \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \cdot [x_t] + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \cdot \begin{bmatrix} \text{const} \\ \text{Trend}_t \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

Visualizing the separate equations:

$$y_{1t} = a_{11,1}y_{1,t-1} + a_{12,1}y_{2,t-1} + a_{11,2}y_{1,t-2} + a_{12,2}y_{2,t-2} + b_{11}x_t + c_{11} + c_{12}\text{Trend}_t + u_{1t}$$

$$y_{2t} = a_{21,1}y_{1,t-1} + a_{22,1}y_{2,t-1} + a_{21,2}y_{1,t-2} + a_{22,2}y_{2,t-2} + b_{21}x_t + c_{21} + c_{22}\text{Trend}_t + u_{2t}$$

If there is an unit root, the determinant is zero for  $z = 1$ , then some or all variables are integrated and a VAR model is no longer appropriate (is unstable).

## VECM (Vector Error Correction Model)

If **cointegrating relations** are present in a system of variables, the VAR form is not the most convenient. It is better to use a VECM, that is, the levels VAR subtracting  $y_{t-1}$  from both sides. The VECM( $p - 1$ ):

$$\Delta y_t = \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + \dots + \Gamma_{p-1} \Delta y_{t-p+1} + B_0 x_t + \dots + B_q x_{t-q} + C D_t + u_t$$

where:

- $y_t$ ,  $x_t$ ,  $D_t$  and  $u_t$  are as specified in VAR.
  - $\Pi = -(I_K - A_1 - \dots - A_p)$  for  $i = 1, \dots, p - 1$ ;  $\Pi y_{t-1}$  is referred as the **long-term** part.
  - $\Gamma_i = -(A_{i+1} + \dots + A_p)$  for  $i = 1, \dots, p - 1$  is referred as the **short-term** parameters.
  - $A_i$ ,  $B_j$  and  $C$  are coefficient matrices of suitable dimensions.
- If the VAR( $p$ ) process is unstable (there are roots),  $\Pi$  can be written as a product of  $(K \times r)$  matrices  $\alpha$  (**loading matrix**) and  $\beta$  (**cointegration matrix**) with  $\text{rk}(\Pi) = \text{rk}(\alpha) = \text{rk}(\beta) = r$  (**cointegrating rank**) as follows  $\Pi = \alpha\beta^\top$ .
- $\beta^\top y_{t-1}$  contains the cointegrating relations.

For example, if there are three endogenous variables ( $K = 3$ ) with two cointegrating relations ( $r = 2$ ), the long term part of the VECM:

$$\Pi y_{t-1} = \alpha \beta^\top y_{t-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{32} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \end{bmatrix} = \begin{bmatrix} \alpha_{11}ec_{1,t-1} + \alpha_{12}ec_{2,t-1} \\ \alpha_{21}ec_{1,t-1} + \alpha_{22}ec_{2,t-1} \\ \alpha_{31}ec_{1,t-1} + \alpha_{32}ec_{2,t-1} \end{bmatrix}$$

where:

$$ec_{1,t-1} = \beta_{11}y_{1,t-1} + \beta_{21}y_{2,t-1} + \beta_{31}y_{3,t-1}$$

$$ec_{2,t-1} = \beta_{12}y_{1,t-1} + \beta_{22}y_{2,t-1} + \beta_{32}y_{3,t-1}$$