

Continuous Opinion Dynamics with Multiple Issues: an approximate Bayesian approach

Marcelo V. Maciel

André C. R. Martins

NISC - EACH, Universidade de São Paulo

Av. Arlindo Bétio, 1000, São Paulo, 03828-080, Brazil

Abstract

1 Introduction

Here, we will present an opinion dynamics model [1, 2, 3, 4, 5, 6, 7] where the opinions exist over an one-dimensional axis and each agent has a best estimate on more than one issue over that axis. Describing opinions over a spatial landscape is an usual way to describe policy alternatives and agents' preferences. The geometrical properties of the space are usually defined by mapping from similarity to proximity of the political agents [8, 9]. Such a description can capture the common notion of parties or policies being more "to the left" or "right". If they're similar then they're closer [10, 11]. Major opinions, including political ones, tend to be formed not from only one issue but from how each person feels about a number of them. Locating someone in a left versus right or liberal versus conservative axis, therefore, requires inspecting the opinions of that person in not only one but several issues that constitute the agent ideological positioning [12].

For many problems, it makes sense to consider different issues as having components in more than one single dimension [13]. However, it is often the case that describing the problem as one-dimensional can be justified. We can certainly see this as a first approximation along the most relevant dimension. In that case, the one dimensional case is just a projection along a direction where variation seems especially important of higher-dimensional problems. From an application point of view, it is usual to find discussions to be simplified over only one main disagreement.

One traditional way to model this type of scenario is to use continuous opinions over a fixed interval, as it is done in the Bounded Confidence (BC) models [6, 14]. While discrete models [3, 4, 5] can be very useful at describing choices, they are not easiest way to represent strength of opinion unless a continuous variable is associated to the choice [7]. Discrete models also tend to lack a scale

where we can compare opinions and decide which one is more conservative or more liberal.

On the other hand, continuous models are not particularly well suited for problems involving discrete decisions. As we will not deal with those kinds of problems here, they are a natural choice. Indeed, continuous opinions models have been proposed for several different problems on how opinions spread on a society [15, 16], from questions about the spread of extremism [17, 18, 19, 20, 21, 22] to other issues such as how different networks [23, 24, 25, 26] or the uncertainty of each agent [27] might change how agents influence each other.

Here, we will use a continuous opinion model created by Bayesian-like reasoning [28], inspired by the Continuous Opinions and Discrete Actions (CODA) model [7, 29]. That model was shown previously [28] to provide the same qualitative results as BC models. While a little less simple, the Bayesian basis makes for a clearer interpretation of the meaning of the variables. That makes extending the model and interpreting new results simple. That approach is also consistent with a bounded rationality variant interpretation of the spatial model of political decision making [30, 31].

Variations of how each agent estimate how trustworthy other agents are will also be introduced. Here, we use a function of trust p^* that plays a role similar to the threshold value in BC models. While p^* is not a simple discontinuous cut-off, it is a function of the distance between the opinions of the agent and the neighbor. If only one issue is debated, that distance is uniquely defined. However, if we have multiple issues represented on the same one-dimensional line, it is not clear which distance we should use. That happens because we have the opinion of agent i on a specific issue o_i and we can compare it to o_j to estimate how much i trusts j . However, as there are several issues we could also use the mean opinion of i over all its issues and the same goes for j . Therefore, we will study two additional cases: in the first alternative, we will change p^* to p^{**} , determined by the distance between the neighbor and the agent average opinions. In the second alternative, we will use a p^{***} calculated from the opinion of the neighbor and the average opinion of the agent. The idea here is to make the behavior of our agents closer to what experiments show about human reasoning. We have observed that our reasoning about political problems can be better described as ideologically motivated [32, 33, 34]. Indeed, our opinions tend to come in blocks even when the issues are logically independent [35]. Our reasoning abilities seem to exist more to defend our main point of views [36, 37] and our cultural identity [38] than to find the best answer. In that context, evaluating others by how they differ from us as a whole, instead of in each issue, is a model variation worth exploring.

2 The Model

Here, the population will consist of N agents fully connected (an agent i can interact with any other agent j). Each agent i will have an opinion $0 \leq o_{ik} \leq 1$, where $k = 1, \dots, n$ is a specific issue. We assume each agent opinion

about issue k can be represented as a value o_{ik} at the range of possible values for o s. Agents also have an uncertainty σ associated to their average estimate o . The uncertainty σ could be different for each agent and also updated during the interactions [28]. For the sake of simplicity, however, we will assume the uncertainty $\sigma_i = \sigma$ is identical for (almost) all agents and it does not change. The set of opinions for each agent on all possible issues will define its ideological profile $I_i = ((o_{i1}, \sigma), \dots, (o_{in}, \sigma))$, where n is the number of issues, o is the opinion about the issue and σ is the global uncertainty [29]. The arithmetic mean x_i of agent i opinions in each issue will be called here the ideal point for each agent, that is, it defines the agent ideological position at the dimension of interest [39]. Obviously $x_i = \frac{1}{n} \sum_{k=1}^n o_{ik}$.

In order to have agents with initial ideal points well distributed over the possible range, the initial values for each o_{ik} was randomly drawn using a Beta $Be(\alpha, \beta)$ with random parameters α_i and β_i . Those parameters were drawn for each agent i from the ranges ($\alpha \in [1.1, 100]$, $\beta \in [1.1, 100]$) by dividing those ranges by N equally spaced values and then assigning them to each agent after mixing the values as in a Latin hypercube sampling [40]. That is done to allow agents with a diverse value of initial ideal points and, at the same time, to keep the initial o_{iks} of each agent i correlated.

While we could have σ as a measurement of each agent uncertainty and have it evolve with time, here we will keep it as a fixed parameter of the model. However, a proportion p_{intran} of agents will be stubborn about one single issue k , so that $\sigma_{ik} = 1e-20$. That behavior is kept constant as the simulation unfolds, that is σ_{ik} is also not updated by the model. A proportion of agents who are stubborn is introduced here so we can check if inflexible [41, 42, 43] have a significant impact on the outcomes of the model.

In each iteration of the simulation, two procedures are applied: the opinion update through social influence, and a random opinion update (noise). In the social influence procedure we draw two agents, i and j , from the population, where i will observe j opinion about a randomly drawn issue $k \in (1, \dots, n)$. Agent i updates its opinion (o_{ik}) following an approximate Bayesian rule. That rule is obtained by assuming each agent has a Normal prior $f_i(\theta) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(\theta-o_i)^2}{2\sigma_i^2}}$ compatible with its parameters o_i and σ_i , where the index related to the issue k was omitted for the sake of simplifying the equation. The agents also assume a mixture likelihood where there is an initial chance p , updated to p^* that the agent j has information and a chance $1 - p$ it knows nothing and its opinion is just a random non-informative draw. That is, the likelihood is given by $f(o_j|\theta) = pN(\theta, \sigma_j^2) + (1 - p)U(0, 1)$. While a full Bayesian treatment would produce a posterior distribution that is a mixture of two normals, here we will assume each agent only updates its expected value and it does not carry the full posterior information to the next iteration. That leads to the following update rule for the expected value o_{ik} [29]:

$$o_{ik}(t+1) = p^* \frac{o_{ik}(t) + o_{jk}(t)}{2} + (1 - p^*)o_{ik}(t), \quad (1)$$

where p^* is given by

$$p^* = \frac{p \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(\Delta_{ij})^2}{2\sigma_i^2}}}{p \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(\Delta_{ij})^2}{2\sigma_i^2}} + (1-p)}. \quad (2)$$

Here $\Delta_{ij} = o_{ik}(t) - o_{jk}(t)$ is the distance between the opinions on the k issue. Δ_{ij} plays a similar role to the threshold parameter in the Bounded Confidence models by making distant opinions less influential. As Δ_{ij} increases, it is easy to see that Equation 2 causes p^* to tend to zero. And, as that happens, the weights in average update Equation 1 change so that the previous value $o_{i,k}(t)$ remains almost unchanged.

The threshold role of Δ_{ij} suggests we might change its definition to check how it might better reflect the actual behavior of humans. In particular, people tend to trust better those who have a similar ideology. That means trust might not depend on the specific issue alone, but on the average over all issues. While that is not the model we obtain from applying Bayes rule, such a change makes sense as an attempt to have a better model. In order to check that possibility we will also implement two other cases by changing the way Δ_{ij} is calculated. In the first variation, p^* will be substituted by p^{**} , where p^{**} means $\Delta_{ij} = x_i(t) - x_j(t)$, that is, agent i will observe the average ideological position of j in order to estimate its trust. That assumed i has more information than only the value $o_{jk}(t)$. To check what happens when only $o_{jk}(t)$ can be observed by agent i , we will introduce a second variant case where i compares $o_{jk}(t)$ to its own average. That is, $\Delta_{ij} = x_i(t) - o_{jk}(t)$ and we represent that case by p^{***} .

In order to better understand the model dynamics we also introduce noise as the second procedure. At each iteration, another agent i is randomly chosen and its opinion changed due to random noise. That is, i opinion becomes $o_{ik}(t+1) = o_{ik}(t) + r$ where r is drawn from a Normal distribution with mean 0 and standard deviation ρ . If agent i is intransigent in issue k it won't its o_{ik} opinion when chosen by the noise algorithm. Moreover, if $o_{ik}(t) + r > 1$ then $o_{ik}(t+1) = 1$. Likewise, if $o_{ik}(t) + r < 0$ then $o_{ik}(t+1) = 0$. Noise is introduced here as a way of accounting for the effect of factors not related to the modeled social influence [44] and it is interesting to verify if small noises can lead to drastic changes in systemic properties [45].

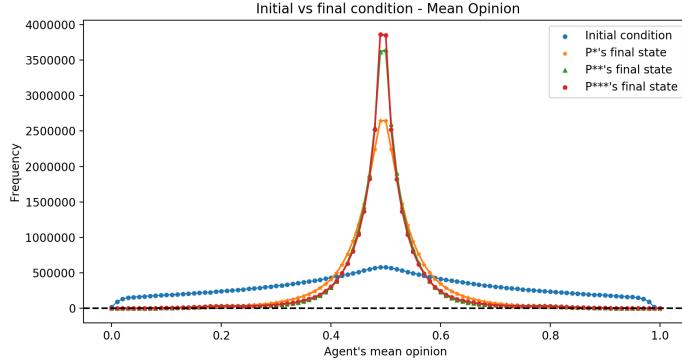
3 Model Results

To better understand the model behavior, we ran simulations using as range for the parameters the values:

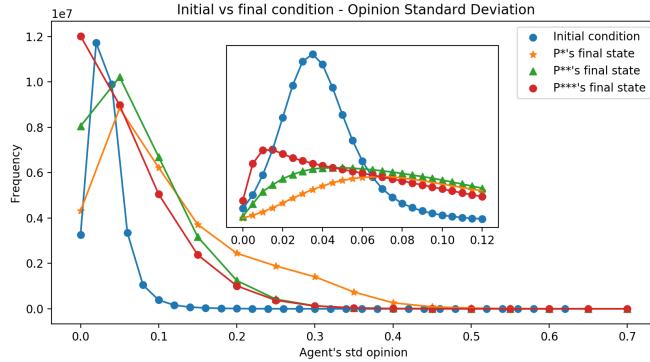
σ	n	p	p_intran	N	ρ
[0.01, 0.5]	[1, 10]	[0.1, 0.99]	[0.0, 0.1]	[500, 5000]	(0.0, 0.1)

Table 1: Parameters' Bounds

The parameter space was explored by two sweeps of its parameters: one sampling of 70,000 times using quasi-random low-discrepancy sequences on all parameters [46], that generate evenly spaced points, and another of 60,000 times keeping $N = 500$ so that we can compare different runs. An interesting question in any opinion dynamics model is if agents can reach consensus, if they diverge, or something between those two states. That can be observed by comparing the initial mean and standard deviation of each agent's opinions. Figure 1 shows histograms for those variables at the initial condition and also for the final opinions corresponding to the (p^*, p^{**}, p^{***}) cases. The upper graphic shows the distribution of the mean opinions x_i , and the lower one the distribution of the standard deviation of the opinions of each agent $s_i^2 = \frac{1}{n} \sum_{k=1}^n (o_{ik} - x_i)^2$.



(a) Initial x Final state:
Distribution of mean opinions



(b) Initial x Final state:
Distribution of the standard deviations of opinions

Figure 1: Difference between initial and final opinion distributions for 60.000 parameterizations. In all cases, $N = 500$.

The histograms show that, in general, opinions have a tendency to move towards the middle value. This seems to suggest most parameter values lead to consensus or near it. However, consensus is not always achieved, as shown by the tails of the mean distribution at Figure 1, even if the final distribution shows that extreme opinions become less common. And we can also observe in the distribution of standard deviations s_i that the opinions of each agent on each issue tend to become more diverse, as, for most cases, s_i seems to increase from its initial values. The only exception is the p^{***} scenario. There, we still see an increase in the frequency of large values of s_i . But there are also many cases where there is a strong tendency for s_i to become smaller. Or, in other words, in several scenarios, the agents' opinions on the issues tended to gather

much closer to their own mean opinions x_i as a consequence of the dynamics.

The cases where the average opinions remain a little spread, while rare, correspond to scenarios where bi-partisanship survives, with agents opinions surviving at both extreme positions. This suggests that, in general, the model can be described as one with similarity biased influence [44]. This tendency to consensus seems to be a little weaker at the p^* case when compared to the two other cases, p^{**} and p^{***} .

Figure 1 (b) also shows that in all cases (p^*, p^{**}, p^{***}) the standard deviation of the opinions distribution becomes more spread. While there are situations where s_i tends to become smaller, signaling the agent opinions become closer to its mean, the reverse also happens. The tails for larger values of s_i show that the intercation with other agents quite often led to a more diverse set of opinions over the different issues. As a matter of fact, except for the p^{***} scenarios, a larger internal spread than in the initial conditions seem to be the usual result. For p^{***} , while we also observe that strengthenig of the spread for many parameter values, we also see that an increase in cases with small s_i . That is, the dynamics can lead to stronger internal consistency.

The distributions for the mean and standard deviation might look contradictory at a first glance. But they are information about different quantities. A tendency of the x_i towards central value while s_i increases is simple to understand. It just shows that, while opinions on different issues might be spreading more, including towards more extreme values, the mean of every issue opinion of the agents show a clear central tendency. That the mean would have a central tendency should not be seen as a surprise. But that does not necessarily correspond to what happens to the agents opinions on specific issues o_{ik} .

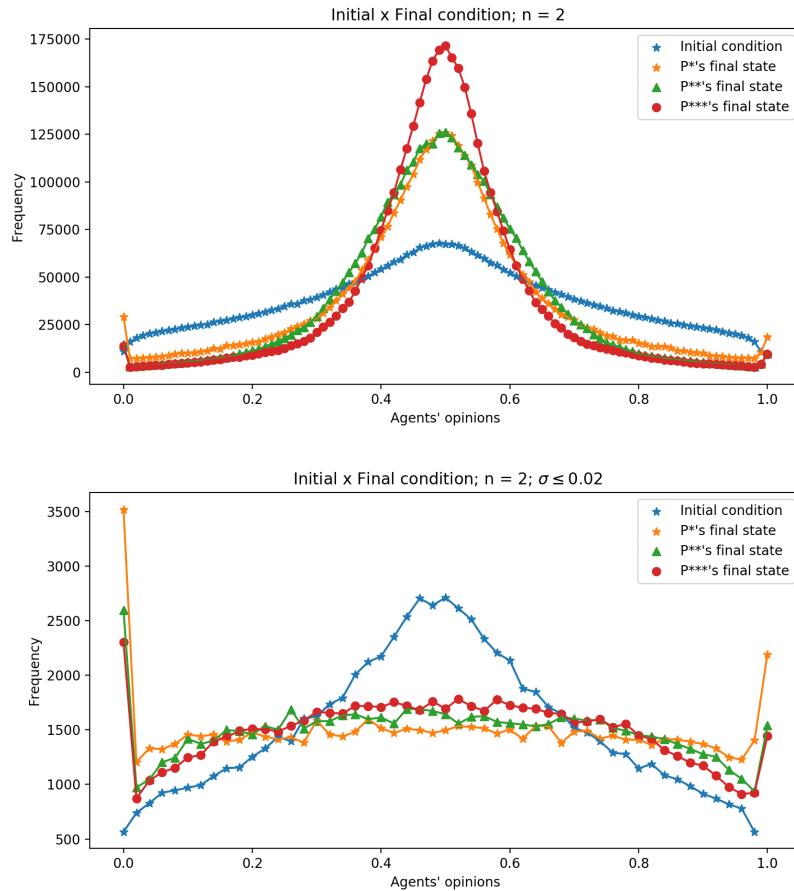


Figure 2: Histogram for the final distribution of values of o_{ik} . Initial distribution is also shown for comparison.

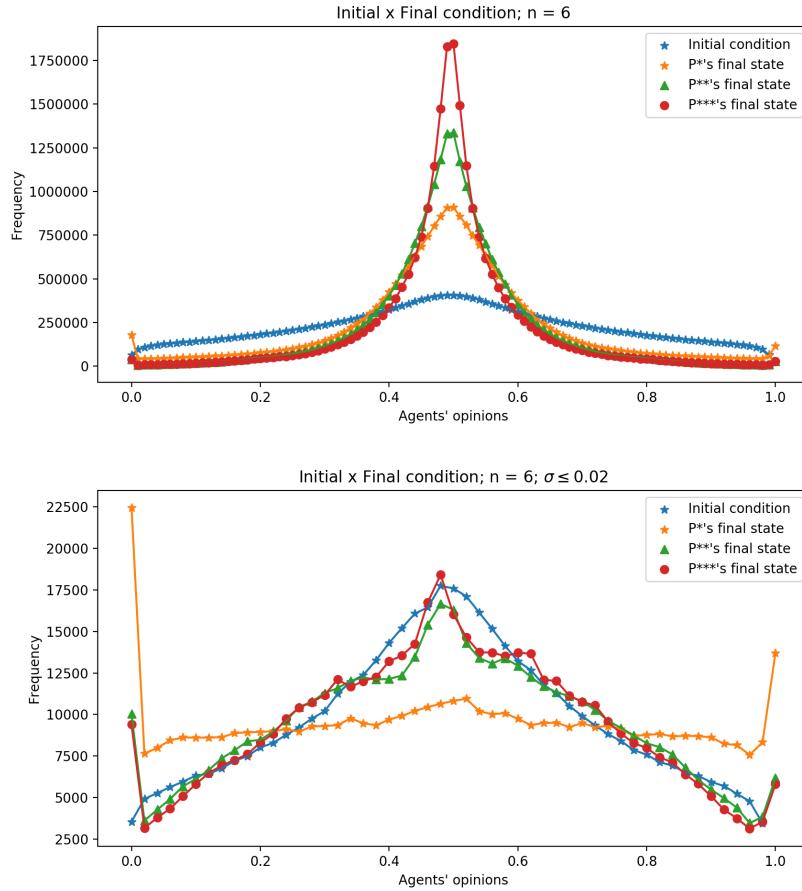


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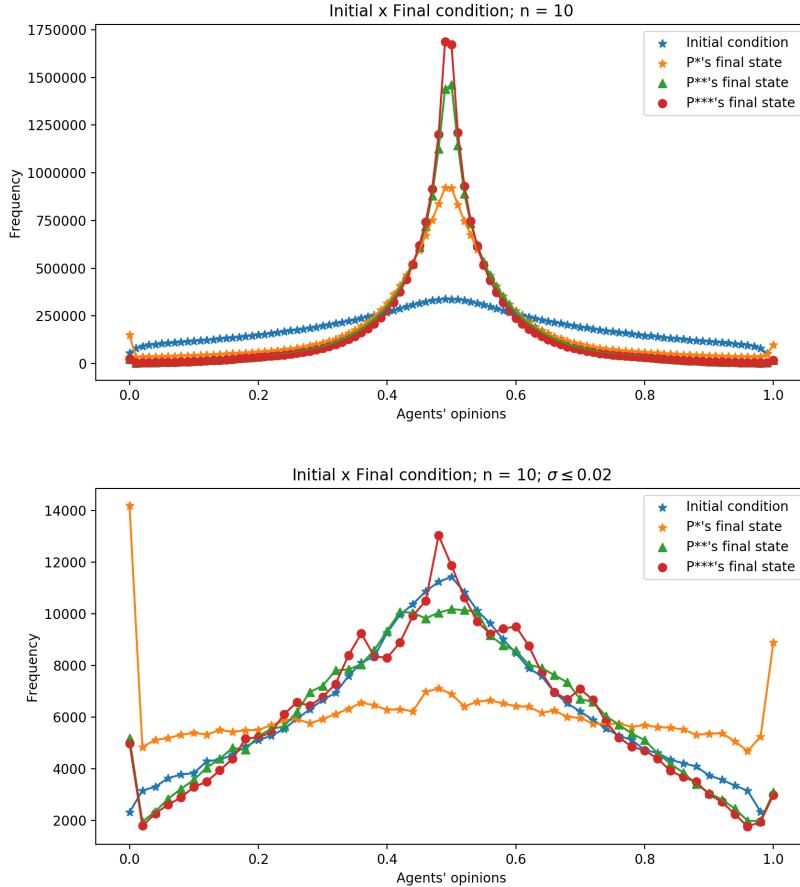


Figure 2: Histogram for the final distribution of values of o_{ik} . Initial distribution is also shown for comparison.

The behavior of the opinions about specific issues o_{ik} values when we have small values of σ can be observed in the Figure 2. The graphics show separate cases for different number n of issues and distinct small values of σ . The stronger central tendency. While the most important peak is still around 0.5, the distribution of values is now much more spread. That reflects the fact that, as σ decreases, consensus becomes much harder and a fraction of the agents find agreement at other values. Indeed, in every graphic, for the p^* scenario, central opinions become less common than in initial conditions, while more extreme values become predominant, showing a clear tendency away from consensus. The same is not true for the more centralizing versions, p^{**} and p^{***} , that seems to show a more varied behavior. For $\sigma \leq 0.5$, both cases show some tendency to consensus. As for $\sigma \leq 0.2$, we see very small changes in the initial distribution, when $n = 6$ or $n = 10$, and a clear drive to disagreement when $n = 2$. In all

scenarios, however, it is clear that the tendency to disagreement is weaker when update is done following the p^{**} and p^{***} rules than what we observe for p^* .

The histograms, however, don't show the whole story of which parameters influence the system behavior. With that in mind we performed a Sobol sensitivity analysis [47] using as outcome both the observed standard deviation of agent's mean opinions $Y \text{std}^2 = \frac{1}{n} \sum_{k=1}^n (x_i - \bar{x}_i)^2$ and the standard deviation S_s of the collection of N internal standard deviations s_i . The Sobol indexes decompose the impact of parameters on the variance of the output. The higher the value of index the bigger the impact of the parameter on the output. First order Sobol indexes include linear and non-linear contributions of the parameters, while total Sobol indexes also include all the interaction effects between parameters [48]. If there are only three parameters, the total effect S_{T1} of the first parameter X_1 is given by the equation $S_{T1} = S_1 + S_{12} + S_{13} + S_{123}$ where $S_i = \frac{V[E(Y|X_i)]}{V(Y)}$; S_{12} is the impact at the variance of the output Y of the interaction of X_1 and X_2 ; that is, their combined effect minus their first order effects: $S_{ij} = \frac{V[E(Y|X_i, X_j)] - V[E(Y|X_i)] - V[E(Y|X_j)]}{V(Y)}$ [46]. For our simulations, we obtain the estimates:

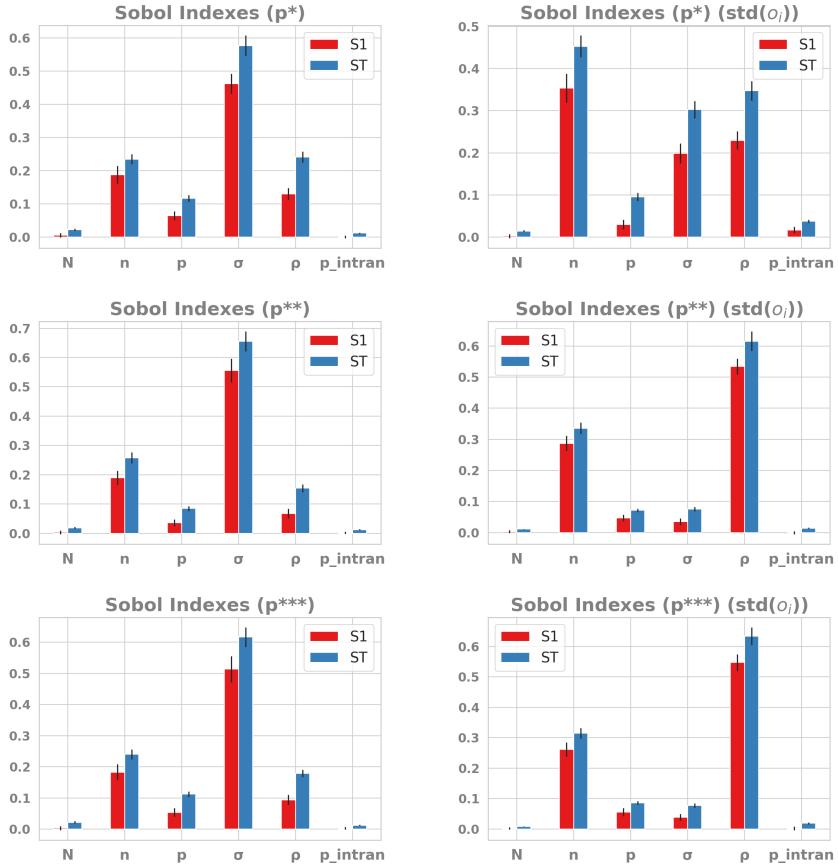


Figure 3: Sobol Indexes for the three cases (p^*, p^{**}, p^{***}). First column shows the analysis for Y_{std} and the second column, for S_s .

The sensitivity analysis in Figure 3 shows that σ, n , and ρ have the most influence on the final values of Y_{std} and S_s . p seems to still have some smaller influence and both N and p_{intran} seem to make no difference for the range of values we used. It is also interesting to notice that the same general behavior appears in the three scenarios for Y_{std} . σ being the parameter that explains most of the variance was expected and it is consistent with [29]. It is interesting to see that two of the new parameters, the number of issues n and the noise ρ , also play an important role in explaining the total variance of Y_{std} .

When we look at the standard deviation of the agent's opinions, S_s , however, we see that σ is the most relevant parameter only for the p case. For p^{**} and p^{***} , however, σ seems to matter very little, basically as much as p . And the noise ρ seems to have most of the impact on S_s . This change is not so hard to understand. As we are speaking of how much change we observe in the standard deviations of each agent opinions s_i , there is a fundamental difference between

the p^* scenario and the p^{**} and p^{***} ones. On the first one, the opinion of i in each issue does not depend on its own opinion on other issues. However, as in both p^{**} and p^{***} agent i uses its own mean to decide how much to trust other agents, it is natural i opinions will tend to a mean value. So, while for p^* , s_i is driven by how much the opinions get spread in general and, therefore, by σ , p^{**} and p^{***} scenarios suggest that the tendency to keep consistent opinions predominates and the variation in the final results is driven mostly by the noise. This is compatible with what we observed Figure 1 for the distribution of s_i . There, we had that the p^* showed it was common to observe agents with very little internal consistency, equivalent to large amounts of s_i . The tail for high s_i , however, was significantly smaller for p^{**} and p^{***} , with p^{***} even showing cases where very small s_i became more frequent than in the initial distribution.

A sensitivity analysis, however, does not show the direction of the impact. Figure 4 shows scatter plots for Y_{std} as a function of the main parameters. Each point corresponds to the outcome of one implementation of the model for the value of the parameter at the x -axis and the observed value of Y_{std} . Similar scatter plots for S_s showed similar behaviors and, therefore, are not shown here.

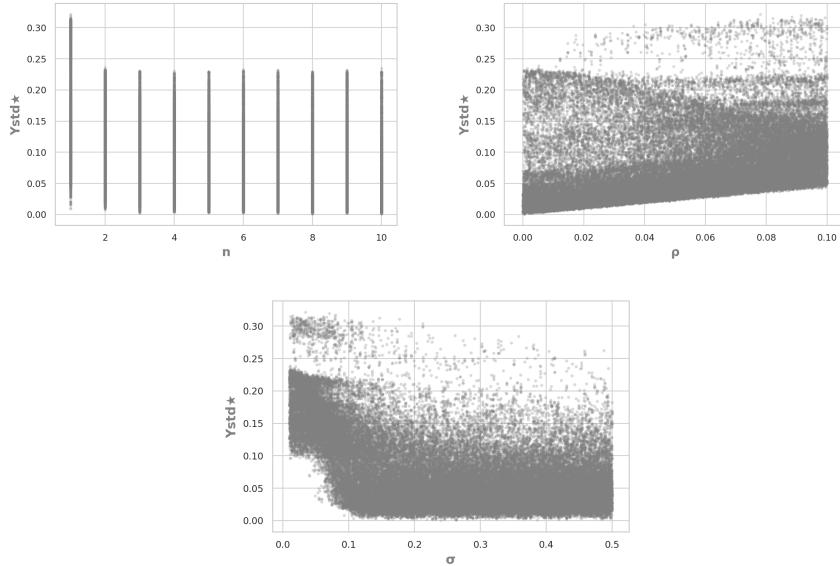


Figure 4: Scatterplots for the parameters with highest impact. Each dot corresponds to the result obtained in a single run.

The negative impact of σ in the population' opinion dispersion is expected: a higher σ means agents are easier to influence. As they are connected to all the other agents, the more uncertain they are, the more centralized the agents' mean opinions will tend to become. The plots also show that the exact value of σ seems to matter little above approximately 0.1. Therefore we restrict our

following analysis to σ in the $(0.0, 0.1]$ range. The effect of ρ is also expected: the bigger the noise, the more dispersed the final state of the system is. The number of issues n seems to have little influence unless it is $n = 1$. When there is only one issue, we see scenarios where the opinions clearly move away to stronger polarizations, with (Y_{std}) around 0.3. Those cases are no longer observed as we have at least two issues. That is probably an artifact of initial conditions. As initial conditions were randomly drawn, extreme values in all issues become more and more rare with increasing number of issues n and, therefore, there are very few agents to push others to the more extreme regions.

Now we have a general picture, looking at typical behaviors for specific parameter values helps us understand the model better. We ran cases where we kept $\rho = 0.05$; $N = 500$; $p_intran = 0.0$ fixed, for 500.000 iterations, and test combinations of $\sigma = (0.02, 0.04, 0.1)$ and $n = (1, 5)$. The time series at Figure 5 show the time evolution of the opinions of all 500 agents for a typical run of the the $p***$ case. Only results for $p***$ are shown here because the time series for p^* and p^{**} are visually almost identical to the ones in the figures. We see that larger values σ seem to lead opinions towards the center: the bigger the σ is, the more agents will tend to move closer to the mean.

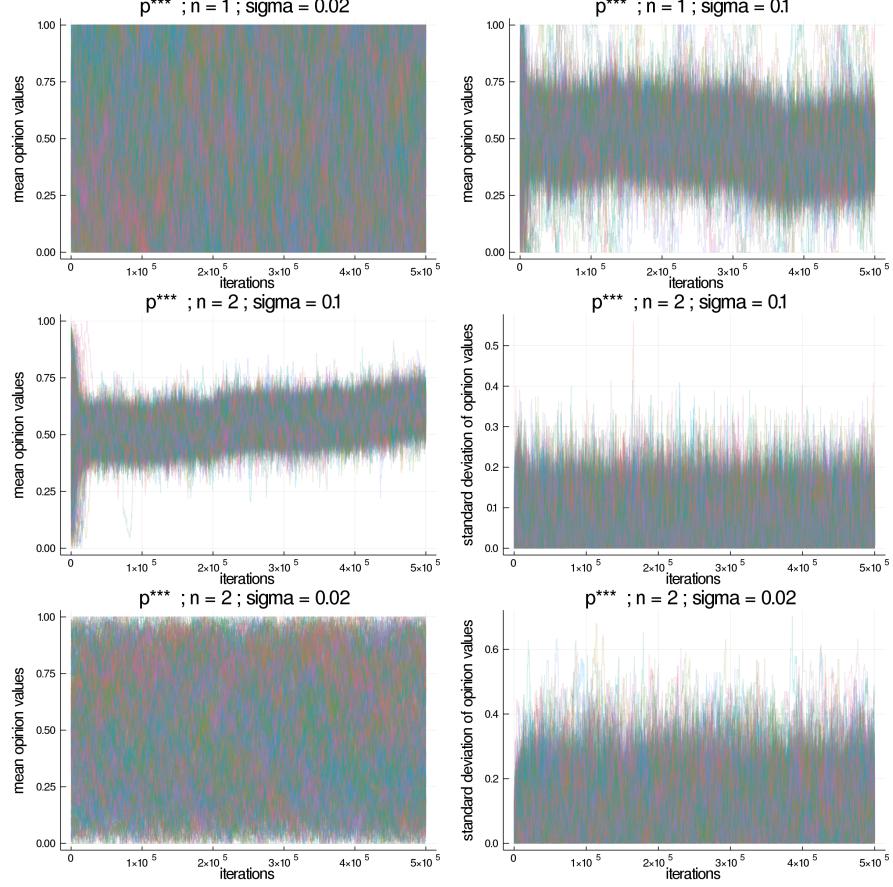


Figure 5: Time series for the parameterization: $\rho = 0.05, N = 500, p_intran = 0.0, n = 1$ or 2

Figure 6 shows a similar time evolution series for $n = 5$ issues. Here, even for small values of σ we observe the opinions tend to avoid the more extreme values. As commented before, that seems to be an artifact of initial random draws. While it is easy to draw the most extreme values with only one draw, for $n = 5$, as we are close to the end of the range, no values outside that range are possible. So, the most extreme values require that, at the start, agents should have all their five issues drawn as extreme. As that is rare, those few that do start there tend to be attracted to still extreme positions, but a little less so. However, there actually seems to be a stronger tendency towards more central opinions. That can be observed as the increase in σ leads to a position closer to consensus than we observed in Figure 5 for the $n = 1$ case.

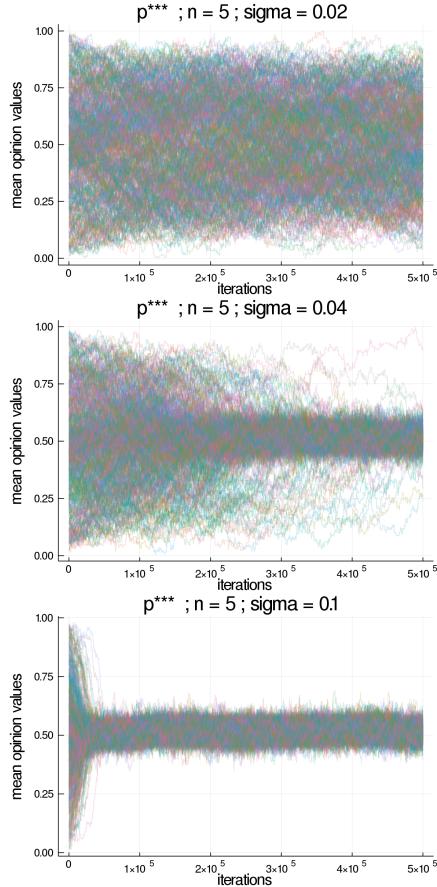


Figure 6: Time series for the parameterization: $\rho = 0.05, N = 500, p_intran = 0.0$

A possible reason we need to check is the fact we are measuring the mean opinion values x_i . As ρ changes a single o_i at each iteration, that means a higher n should imply a lesser impact of ρ on the mean opinion of the agent. That could explain why, for a larger n , consensus might come easier. To test if that was what was actually happening, we ran the same scenarios but increasing ρ with n , such that $\rho_2 = \sqrt{n} * \rho_1$. The effects of increasing ρ that way in the case of Figure 6 can be seen at Figure 7. A comparison with Figure 5 shows a larger spread around the consensus, suggesting there might be other effects here other than random fluctuations.

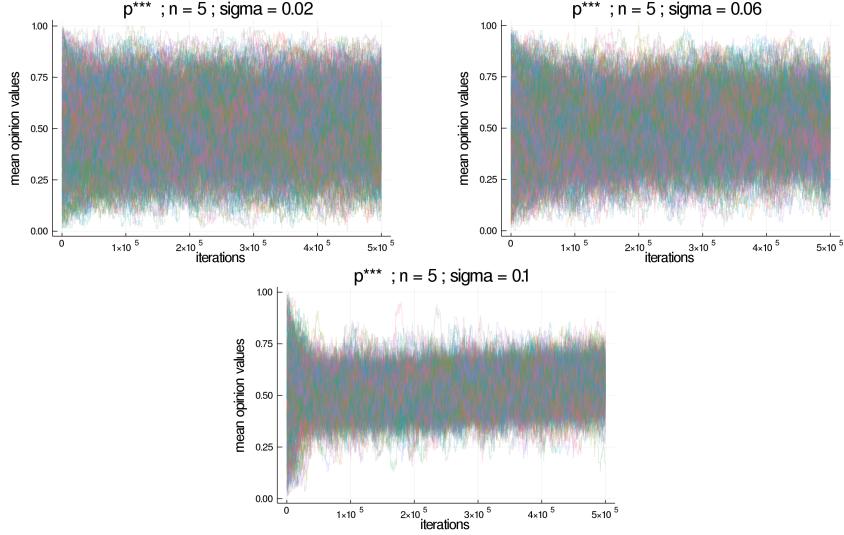


Figure 7: Time series for the parameterization: $\rho \approx 0.12, N = 500, p_intran = 0.0$

Heretofore we've tested parameterizations with noise. To verify the influence of the existence of noisem, we also ran cases with ρ close to zero, that is $\rho = 10^{-5}$. The most obvious difference we can see is that the population mean opinion values converge to well defined values. In parameter combinations in which $\sigma = 0.1$ the tendency is convergence to values close to 0.5. An interesting distinction between the cases in this parameterization is that p^{**} and p^{***} always converge to 0.5, independently of the number of issues. Alternatively, in the p^* case this happens when $n = 1$, but when we have $n = 5$ or 10 there are other values of convergence, more as we increase n , even though the centralizing tendency remains.

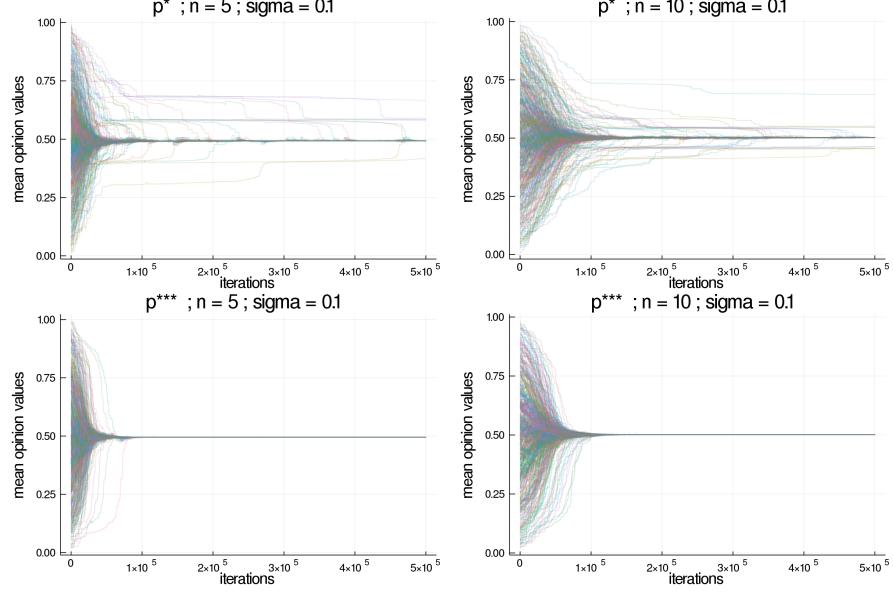


Figure 8: Time series for the parameterization: $\rho = 1e-5$, $N = 500$, $p_intran = 0.0$

When σ is around 0.02 or 0.04, we observe another difference between cases: p^* has more convergence values than p^{**} and p^{***} . Figure 9 illustrates this system behavior for $\sigma = 0.02$. We can also see that the standard deviations s_i are very different for the p^* and p^{***} cases. While for p^* , the values of s_i range from small values around 0.02 to much larger values, up to 0.2, in the p^{***} scenario we see that s_i tends to diminish with time, the majority of the observed values getting close to zero.

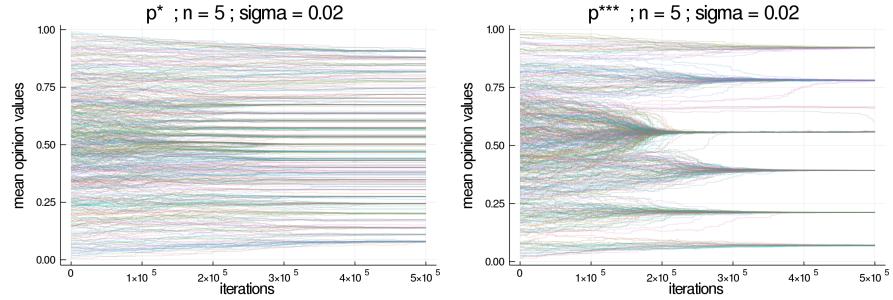


Figure 9: Time series for the parameterization: $\rho = 1e-5$, $N = 500$, $p_intran = 0.0$

The reason for that lies in how Δ (that controls how much an agent will trust others) is calculated in each case: in the p^{**} and p^{***} cases the update

rules make use of mean opinion values. That introduces a tendency for distinct issues opinions to move towards each other. The p^* update rule, however, works with single issue opinions. As the issues are uncoupled, that allows the agents to keep an ideologically more diverse profile.

Another impact of the number of issues, as shown in Figure 10, is that a higher n leads to a longer time for the convergence. The reason is that we're only changing one opinion by iteration, so naturally a higher n means the agents will take longer to be influenced. The relationship here is roughly linear such that the plot the region at 5×10^5 iterations when $n = 10$ is very similar to the corresponding region at 0.5×10^5 when $n = 1$.

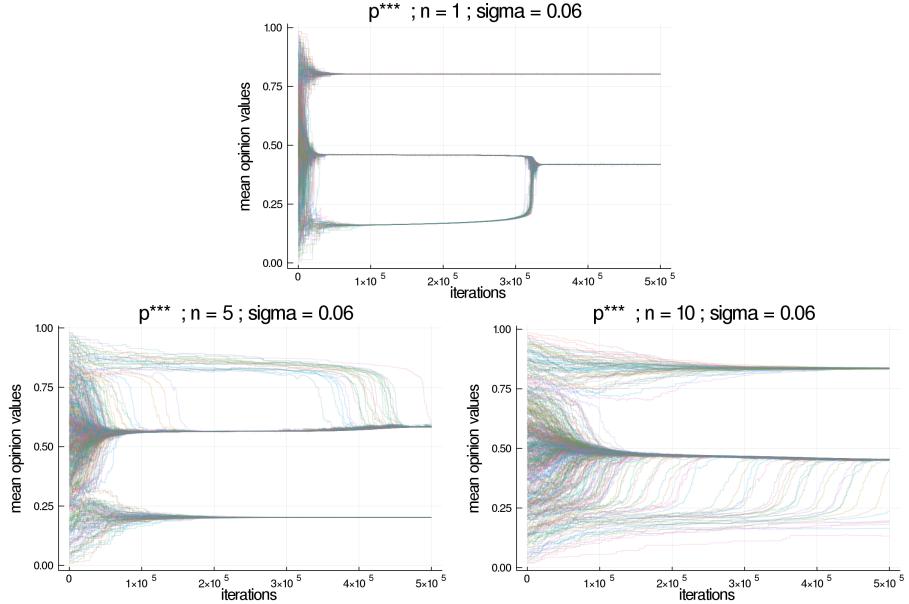


Figure 10: Time series for the parameterization: $\rho = 1e-5$, $N = 500$, $p_intran = 0.0$.

4 Conclusions

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